This is a PRACTICE EXAM!

Final verstion (July 19, 2014)

1. Let $T : \mathbb{P}_3 \to \mathbb{P}_2$ be a transformation that maps $\mathbf{p}(t)$ to $\mathbf{p}'(t)$ (the derivative of $\mathbf{p}$).

   a. Show that it is a linear transformation.

   b. Find the matrix for T relative to $\{1, t, \frac{1}{2}t^2, \frac{1}{3}t^3\}$ and the standard basis for $\mathbb{P}_2$.

   c. Show that

   $$S = \{\mathbf{p}(t) \in \mathbb{P}_3 : \mathbf{p}'(1) = 0\}$$

   is a subspace of $\mathbb{P}_3$.

2. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

   a. If T is one-to-one, what would the dimension of $\text{im } T$ be? Explain.

   b. If T is onto, what would the dimension of $\ker T$ be? Explain.

   c. Are above problems still true for general vector spaces V and W? Explain.

3. Let A be an $m \times n$matrix, P be an $m \times m$ invertible matrix, and Q be an $n \times n$ invertible matrix. Prove that

$$\text{rank } PA = \text{rank } A \qquad \text{and} \qquad \text{rank } AQ = \text{rank } A$$

following 4 steps below.

a. Let B be an $n \times p$ matrix. Show that

$$\text{rank } AB \leq \text{rank } A$$

(Hint. Compare Col AB with Col A.)

b. Under the same condition, show that

$$\text{rank } AB \leq \text{rank } B$$

(Hint. Take the transpose function and use the Rank Theorem.)

c.

$$\text{rank } PA = \text{rank } A$$

d.

$$\text{rank } AQ = \text{rank } A$$

4. Let V be the vector space of 3×3 matrices. (Precisely, V equipped with two operations (addition, scalar multiplication) such that

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix} + \begin{pmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \\ b_7 & b_8 & b_9 \end{pmatrix} = \begin{pmatrix} a_1+b_1 & a_2+b_2 & a_3+b_3 \\ a_4+b_4 & a_5+b_5 & a_6+b_6 \\ a_7+b_7 & a_8+b_8 & a_9+b_9 \end{pmatrix}$$

and

$$c\begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix} = \begin{pmatrix} ca_1 & ca_2 & ca_3 \\ ca_4 & ca_5 & ca_6 \\ ca_7 & ca_8 & ca_9 \end{pmatrix}$$

is a vector space.)

Let W be the set of matrices $A \in V$ such that $A^T = -A$. Is W a subspace? If so, find a basis for W. In other words, is

$$\{A \in V : A^T = -A\}$$

a subspace of V? If so, find a basis for W.

5. Let $\mathcal{D} = \{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$ and $\mathcal{F} = \{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ be bases for a vector space V, and suppose $\mathbf{f}_1 = 2\mathbf{d}_1 - \mathbf{d}_2 + \mathbf{d}_3$, $\mathbf{f}_2 = 3\mathbf{d}_2 + \mathbf{d}_3$, and $\mathbf{f}_3 = -3\mathbf{d}_1 + 2\mathbf{d}_3$.

   a. Find the change-of-coordinates matrix from $\mathcal{F}$ to $\mathcal{D}$.

   b. Find $[\mathbf{x}]_{\mathcal{D}}$ for $\mathbf{x} = \mathbf{f}_1 - 2\mathbf{f}_2 + 2\mathbf{f}_3$.

6. Let $A = \begin{pmatrix} 1 & -4 \\ -2 & 7 \end{pmatrix}$.

   a. Is A diagonalizable? If so, find an invertible 2×2 matrix P and a diagonal matrix D satisfying

$$P^{-1}AP = D$$

   b. Compute $D^3 - D^2 + 6D$. Then, compute $A^3 - A^2 + 6A$ using the previous result.

7. Recall the formula for the orthogonal projection of v onto W given an orthogonal basis $\{u_1, \cdots, u_r\}$;

$$\mathrm{proj}_W(v) = \frac{u_1 \cdot v}{u_1 \cdot u_1} u_1 + \frac{u_2 \cdot v}{u_2 \cdot u_2} u_2 + \cdots + \frac{u_r \cdot v}{u_r \cdot u_r} u_r$$

Prove that the orthogonal projection of v onto W does not depend on which orthogonal bases we choose, following 2 steps below.

a. Prove that $W^{\perp}$ is a subspace.

b. Let there be two bases $\mathcal{U} = \{u_1, \cdots, u_r\}$ and $\mathcal{W} = \{w_1, \cdots, w_r\}$. Let the orthogonal projections be denoted by $\mathrm{proj}_{\mathcal{U}}(v)$ and $\mathrm{proj}_{\mathcal{W}}(v)$. Show that $\mathrm{proj}_{\mathcal{W}}(v) - \mathrm{proj}_{\mathcal{U}}(v) \in W^{\perp}$.

c. Prove the fact that every vector that belongs to W and $W^{\perp}$ at the same time is the zero vector. Use this fact to get the conclusion.

8. Under the condition that $x_1^2 + x_2^2 + x_3^2$, find the maximum and minimum value of

$$f(x_1, x_2, x_3) = 2x_1^2 + 12x_2^2 + 10x_3^2 - 16x_2x_3 + 6x_3x_1$$

following 3 steps below

a. Find the matrix A of the quadratic form $f(\mathbf{x})$.

b. Find the eigenvalues and corresponding eigenspaces of A.

c. Find the maximum and minimum of $f(\mathbf{x})$.

9. Prove Cayley-Hamilton Theorem, that is,
$$\chi_A(A) = 0$$
for a specific case when A is a diagonal matrix.

(Note. Given a polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$,
we define $f(A)$ where A is a square matrix as $a_n A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I_n$.)

10. Let A and B be two $n \times n$ matrices such that $I - AB$ is invertible. Prove that $I - BA$ is invertible.