1. Mark "T" if the statement is always true, "F" if it is sometimes false. ***No** explanations are needed.*

1) T | If all of the eigenvalues of a matrix are not real but complex, then it must be invertible.

2) T | If A is an $n \times n$ orthogonal matrix then the RREF of A must have n pivots.

3) T | If λ is an eigenvalue of A then λ^2 must be an eigenvalue of A^2.

4) F | The set of diagonalizable 2×2 matrices is a subspace of the vector space $M_{2\times 2}$.

5) T | If the 3×3 matrix A has two rows that are the same, then $\det A = 0$.

6) T | Let A be an $n \times n$ matrix. If A^9 is the zero matrix, then the only eigenvalue of A is 0.

7) T | If A is a square matrix and $A^5 = I$ then A is invertible.

8) F | If A is a 5×5 matrix such that $\det(2A) = \det A$ then $A = 0$.

9) T | If T is a one-to-one linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$ then T is onto.

10) F | Let A be an $n \times n$ matrix such that $A^2 = A$, then A is invertible.

11) T | Every symmetric $n \times n$ matrix with real entries is similar to a diagonal matrix with real entries.

2. Select the correct answers. Be aware that there might be more than one answer to each problem.

1) The exponential of a square matrix A is

(a) The sum of the series $I + A + A^2 + A^3 + \cdots$
(b) The sum of the series $I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots$
(c) The matrix whose (i,j)-entry is $\exp(A_{ij})$, where A_{ij} is the (i,j)-entry of A
(d) The diagonal matrix with entries e^{λ_i}, where the λ_i are the eigenvalues of A

2) The exponential of the matrix $\begin{bmatrix} t & t \\ 0 & -t \end{bmatrix}$ is

(a) $\begin{bmatrix} e^t & e^t \\ t & e^{-t} \end{bmatrix}$
(b) $\begin{bmatrix} e^t & te^t \\ 0 & e^{-t} \end{bmatrix}$
(c) $\begin{bmatrix} e^t & (e^t - e^{-t})/2 \\ 0 & e^{-t} \end{bmatrix}$
(d) $\begin{bmatrix} e^t & (e^t + e^{-t})/2 \\ 0 & e^{-t} \end{bmatrix}$

3) Pick the matrix on the list which is NOT diagonalizable over $\mathbb{C}$, if any; else, pick option (e).

(a) $\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$
(b) $\begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}$
(c) $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$
(d) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$
(e) All of them are diagonalizable over $\mathbb{C}$.

4) If A and B are matrices such that $AB = 0$, we can safely conclude that

(a) $\operatorname{Nul} A$ contains $\operatorname{Nul} B$
(b) $BA = 0$
(c) $\operatorname{Nul} A$ contains $\operatorname{Col} B$
(d) $\operatorname{Col} A$ contains $\operatorname{Nul} B$

5) Which subspace of $\mathbb{R}^4$ is the orthogonal complement of the subspace defined by the conditions

$$\{[x_1, x_2, x_3, x_4]^T : x_1 + x_3 = 0 \text{ and } x_1 - x_2 + x_3 - x_4 = 0\}?$$

(a) $\operatorname{Span}([1,0,1,0]^T, [1,-1,1,-1]^T)$
(b) $\operatorname{Span}([1,2,-1,-2]^T, [1,1,1,1]^T)$
(c) $\operatorname{Span}([1,2,-1,2]^T, [1,1,-1,-1]^T)$
(d) $\operatorname{Span}([0,1,0,1]^T, [1,1,1,1]^T)$

6) Which linear transformation T has the image not of dimension 2?

(a) $T : \mathbb{R}^3 \to \mathbb{R}^4$ sending $[x,y,z]^T$ to $[x,y,0,0]^T$
(b) $T : \mathbb{P}_2 \to \mathbb{P}_2$ sending $at^2 + bt + c$ to $2at + b - 2c$
(c) $T : \mathbb{P}_3 \to \mathbb{P}_3$ sending $f(t)$ to $f''(t)$
(d) The orthogonal projection map from $\mathbb{R}^3$ to the plane defined by $x - 2y + 3z = 0$
(e) All of them have 2-dimensional images.

3. Consider the 2×2 matrix
$$M_a = \begin{bmatrix} a & 2-a \\ 2+a & -a \end{bmatrix}.$$

a) Find all real values of a such that M_a is invertible.

b) Find all real values of a such that M_a is diagonalizable.

c) Find all eigenvectors of M_1.

a) M_a is invertible if and only if $\det M_a \neq 0$ and $\det M_a = a\cdot(-a) - (2-a)(2+a)$
$= -a^2 - 4 + a^2$
$= -4 \neq 0$ for any a.

So, M_a is always invertible, that is a can be any real numbers.

b) $\chi_M(\lambda) = (\lambda - a)(\lambda + a) - (2-a)(2+a) = \lambda^2 - 4$.

Hence, a does not affect to the eigenvalues and $\lambda_1 = -2$, $\lambda_2 = 2$ are distinct. This implies that the corresponding eigenvectors are linearly independent so that they form a basis. So, M_a is diagonalizable for any real number a.

c) $M_1 = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$. $\lambda = 2$ and -2.

$\lambda_1 = 2 \Rightarrow \mathrm{Nul}\begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix} \ni \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. So, $E_{\lambda_1} = \mathrm{Span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$

$\lambda_2 = -2 \Rightarrow \mathrm{Nul}\begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix} \ni \begin{bmatrix} 1 \\ -3 \end{bmatrix}$. $E_{\lambda_2} = \mathrm{Span}\left\{\begin{bmatrix} 1 \\ -3 \end{bmatrix}\right\}$.

4. Given a linear second-order equation
$$y''(t) + ay'(t) + by(t) = f(t),$$
only information you have is a set of three solutions to the equation. They are
$$t + e^t \cos 2t + e^{2t} \sin t,\ t + e^{2t} \sin t,\ t + e^t \sin 2t + e^{2t} \sin t.$$
Find a, b, and $f(t)$.

By superposition principle, you know that the difference of any two will give you homogeneous case solutions. $1^{st} - 2^{nd} = e^t \cos 2t$. $2^{nd} - 3^{rd} = -e^t \sin 2t$. So, they correspond to $1 \pm 2i$. This implies that $(r - (1+2i))(r - (1-2i))$ is the auxiliary equation. It is $r^2 - 2r + 5$.

So, $a = -2$, $b = 5$. For $f(t)$, you can plug in the second function $y_p(t) = t + e^{2t}\sin t$.

$y_p'(t) = 1 + 2e^{2t}\sin t + e^{2t}\cos t$, $y_p''(t) = 3e^{2t}\sin t + 4e^{2t}\cos t$.

Hence, $f(t) = 5t - 2 + 2e^{2t}\cos t + 4e^{2t}\sin t$.

5. Solve the following initial value problems:

   a) $y'' + y' = t^2$ with $y(0) = 0$ and $y'(0) = 0$.

   b) $y'' + y = \sec t$ with $y(0) = 0$ and $y'(0) = 0$.

   c) $\mathbf{x}(t)' = A\mathbf{x}(t) + \mathbf{f}(t)$ where $A = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$ and $\mathbf{f}(t) = \begin{bmatrix} 0 \\ 4e^t \end{bmatrix}$ with $\mathbf{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

a) Aux. eq. is $r^2+r=0$. $r=-1$ and 0 are roots.
t^2 is $t^2 \cdot e^{0\cdot t}$. So, (using Undetermined Coefficients Method)

$y_p(t) = (at^2+bt+c)\cdot t$
$y_p'(t) = 3at^2+2bt+c$
$y_p''(t) = 6at+2b$.

$$\Rightarrow y_p''+y_p' = \underbrace{3a}_{=1}t^2 + \underbrace{(6a+2b)}_{=0}t + \underbrace{(2b+c)}_{=0}$$

$a = 1/3$, $b=-1$, $c=2$.

So, $y(t) = \frac{1}{3}t^3 - t^2 + 2t - 2 + 2e^{-t}$.

Hom case: $\underbrace{e^{0\cdot t}}_{=1}$ and $\underbrace{e^{-1\cdot t}}_{=e^{-t}}$.

General: $y(t) = (\frac{1}{3}t^2 - t + 2)t + C_1 + C_2 e^{-t}$.

$y'(t) = t^2 - 2t + 2 - C_2 e^{-t}$

Plug in $t=0 \Rightarrow C_1 + C_2 = 0$ and $2 - C_2 = 0$

b) Variation of Parameters.
Hom case solutions: Aux eq: $r^2+1=0$. $y_1(t) = \cos t$, $y_2(t) = \sin t$. $W[y_1,y_2](t) = \cos^2 t + \sin^2 t = 1$.

$v_1(t) = \int \frac{-y_2 g}{W} = \int \frac{-\sin t \cdot \sec t}{1} dt = \int -\frac{\sin t}{\cos t} dt = -\ln(\cos t)$

use $\frac{f'}{f} = (\ln f)'$.

$v_2(t) = \int \frac{y_1 g}{W} = \int \frac{\cos t \cdot \sec t}{1} dt = \int \frac{1}{1} dt = t$.

So, $y_p(t) = \ln(\cos t)\cdot \cos t + t\cdot \sin t$.
General $\Rightarrow y(t) = \ln(\cos t)\cdot\cos t + t\sin t + C_1 \cos t + C_2 \sin t$.
Use the initial values to conclude that $C_1 = C_2 = 0$. Answer: $y(t) = \ln(\cos t)\cos t + t\sin t$.

c) Method of Undetermined Coefficients.
$\chi_A(\lambda) = (\lambda-2)(\lambda+2)+3 = \lambda^2 - 1 = (\lambda-1)(\lambda+1)$. $\lambda = 1$ and -1 are eigenvalues.
But, in $f(t)$, we have e^t which corresponds to $\lambda=1$. So, we need to try $(ut+v)e^t$ as our $x_p(t)$. (LHS) we get $(ut+u+v)e^t$ (RHS) we get $Aute^t + Ave^t + \begin{bmatrix}0\\4\end{bmatrix}e^t$.
Terms w/ $t\cdot e^t$: $u = Au$. Terms w/ e^t: $u+v = Av + \begin{bmatrix}0\\4\end{bmatrix}$. So, $\underbrace{(I-A)}_{=\begin{bmatrix}-1 & 1\\-3 & 3\end{bmatrix}}u = 0$ and $u = \begin{bmatrix}c\\c\end{bmatrix}$ for some $c \in \mathbb{R}$. Now, the second equation gives
$(I-A)v = \begin{bmatrix}0\\4\end{bmatrix} - \begin{bmatrix}c\\c\end{bmatrix} = \begin{bmatrix}-c\\4-c\end{bmatrix}$. Here, you need to find c s.t. the system is consistent.
this is in $\text{Span}\{\begin{bmatrix}1\\3\end{bmatrix}\}$. So, $\begin{bmatrix}-c\\4-c\end{bmatrix}$ should be a multiple of $\begin{bmatrix}1\\3\end{bmatrix} \Rightarrow 4-c = 3\cdot(-c)$. So, $c=-2$.
Now, $(I-A)\cdot v = \begin{bmatrix}2\\6\end{bmatrix}$ gives a solution $v = \begin{bmatrix}0\\2\end{bmatrix}$.

Hom. case: eigenvectors are
$\lambda = 1 \Rightarrow v = \begin{bmatrix}1\\1\end{bmatrix}$, $\lambda=-1 \Rightarrow v = \begin{bmatrix}1\\3\end{bmatrix}$.
$x_1(t) = e^t \cdot \begin{bmatrix}1\\1\end{bmatrix}$, $x_2(t) = e^{-t}\cdot\begin{bmatrix}1\\3\end{bmatrix}$.

General: $\begin{bmatrix}-2\\-2\end{bmatrix}te^t + \begin{bmatrix}0\\2\end{bmatrix}e^t + C_1\begin{bmatrix}1\\1\end{bmatrix}e^t + C_2\begin{bmatrix}1\\3\end{bmatrix}e^{-t}$.
Use initial conditions to conclude that $C_1 = 1, C_2 = -1$.
Answer: $\begin{bmatrix}-2\\-2\end{bmatrix}te^t + \begin{bmatrix}1\\3\end{bmatrix}e^t - \begin{bmatrix}1\\3\end{bmatrix}e^{-t}$.

6. Consider the following differential equation in normal form:

$$\mathbf{x}(t)' = A\mathbf{x}(t) + \begin{bmatrix}1\\2\end{bmatrix} t^{-1}e^{3t} \text{ where } A = \begin{bmatrix}-1 & 2\\ -6 & 6\end{bmatrix}.$$

a) Find a fundamental matrix of the corresponding homogeneous equation.
b) Compute e^{At} using (a).
c) Find a particular solution using *eigenvector method.*[1]

a) Since A is 2×2, we need to find two linearly independent solutions of the corresponding homogeneous equation $X'(t) = AX(t)$.

$\chi_A(\lambda) = (\lambda+1)(\lambda-6)+12 = \lambda^2-5\lambda+6 = (\lambda-2)(\lambda-3)$. $\lambda=2 \Rightarrow \operatorname{Nul}\begin{bmatrix}-3 & 2\\ -6 & 4\end{bmatrix} \ni \begin{bmatrix}2\\3\end{bmatrix}$.

Let $X_1(t)$ be $e^{2t}\cdot\begin{bmatrix}2\\3\end{bmatrix}$ and $X_2(t)$ be $e^{3t}\begin{bmatrix}1\\2\end{bmatrix}$. $\lambda=3 \Rightarrow \operatorname{Nul}\begin{bmatrix}-4 & 2\\ -6 & 3\end{bmatrix} \ni \begin{bmatrix}1\\2\end{bmatrix}$.

Then, $X_1(t)$ and $X_2(t)$ are linearly independent solutions. (Note that $2\neq 3 \Rightarrow$ e.vectors are lin. indep.)

So, $X(t) = \begin{bmatrix} X_1(t) & X_2(t)\end{bmatrix} = \boxed{\begin{bmatrix}2e^{2t} & e^{3t}\\ 3e^{2t} & 2e^{3t}\end{bmatrix}}$ is a fundamental matrix.

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b) We know that e^{At} is a fundamental matrix b/c 1) $(e^{At}\cdot e_i)' = A\cdot e^{At}\cdot e_i$ ($i=1$ or 2) and 2) e^{At} is invertible.

So, e^{At} should be $X(t)\times M$ for some 2×2 invertible matrix M.

How to find M?

[I said it is $M\cdot X(t)$ in the discussion, but it is wrong. It should be $X(t)\cdot M$. The order matters b/c they are matrices.]

Because $e^{At} = X(t)\times M$, this is still true when $t=0$ especially. (LHS) is $e^{A\cdot 0} = e^{O_2} = I_2$. (RHS) is $X(0)\times M$. So, M should be the inverse matrix of $X(0)$.

From a), we have $X(0) = \begin{bmatrix}2 & 1\\ 3 & 2\end{bmatrix}$. So, $M = \frac{1}{2\cdot 2-1\cdot 3}\begin{bmatrix}2 & -1\\ -3 & 2\end{bmatrix} = \begin{bmatrix}2 & -1\\ -3 & 2\end{bmatrix}$.

Therefore $e^{At} = X(t)\times M = \begin{bmatrix}2e^{2t} & e^{3t}\\ 3e^{2t} & 2e^{3t}\end{bmatrix}\times\begin{bmatrix}2 & -1\\ -3 & 2\end{bmatrix} = \boxed{\begin{bmatrix}4e^{2t}-3e^{3t} & -2e^{2t}+2e^{3t}\\ 6e^{2t}-6e^{3t} & -3e^{2t}+4e^{3t}\end{bmatrix}}$

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c) From a), we have $\lambda_1=2, v_1=\begin{bmatrix}2\\3\end{bmatrix}$ and $\lambda_2=3, v_2=\begin{bmatrix}1\\2\end{bmatrix}$. Let $X_p(t) = \xi_1(t)v_1 + \xi_2(t)v_2$.

(LHS): $\xi_1(t)'v_1 + \xi_2'(t)v_2$

(RHS): $\xi_1(t)Av_1 + \xi_2(t)Av_2 + \begin{bmatrix}1\\2\end{bmatrix}t^{-1}e^{3t} = 2\xi_1(t)v_1 + 3\xi_2(t)v_2 + 0\cdot v_1 + t^{-1}e^{3t}\cdot v_2$

$f_1(t)v_1 + f_2(t)v_2 = \begin{bmatrix}1\\2\end{bmatrix}t^{-1}e^{3t}$

$\quad\|$

$\begin{bmatrix}v_1 & v_2\end{bmatrix}\begin{bmatrix}f_1(t)\\f_2(t)\end{bmatrix}$

$\Rightarrow \begin{bmatrix}f_1(t)\\f_2(t)\end{bmatrix} = \begin{bmatrix}2 & -1\\ -3 & 2\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}t^{-1}e^{3t} = \begin{bmatrix}0\\1\end{bmatrix}t^{-1}e^{3t}$

So, we get $\xi_1(t)' - 2\xi_1(t) = 0 \quad \cdots (1)$

$\xi_2(t)' - 3\xi_2(t) = t^{-1}e^{3t} \quad \cdots (2)$

$(1) \Rightarrow \xi_1(t) = e^{2t}$. $(2) \Rightarrow$ Multiply the integrating factor e^{-3t} and then $\xi_2(t)\cdot e^{-3t} = \ln t$,

so $\xi_2(t) = \ln t\cdot e^{3t}$.

Therefore, $\boxed{X_p(t) = \begin{bmatrix}2\\3\end{bmatrix}e^{2t} + \begin{bmatrix}1\\2\end{bmatrix}\ln t\cdot e^{3t}}$.

[1] That is, for v_1 and v_2 eigenvectors composing a basis, set $\mathbf{x}_p(t) = \xi_1(t)v_1 + \xi_2(t)v_2$ and solve for $\xi_1(t)$ and $\xi_2(t)$.

7. (Extra) Let A be a 2×2 matrix such that[2]

$$A \begin{bmatrix} -1 \\ 4 \end{bmatrix} = -4 \begin{bmatrix} -1 \\ 4 \end{bmatrix}, \; A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -4 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 4 \end{bmatrix}.$$

Find the functions $x(t)$ and $y(t)$ with initial values $x(0) = -2$, $y(0) = 11$ that satisfy the system of differential equations

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}' = A \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}.$$

$A \cdot \begin{bmatrix} -1 \\ 4 \end{bmatrix} = -4 \begin{bmatrix} -1 \\ 4 \end{bmatrix}$ and $A \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -4 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 4 \end{bmatrix}$ imply $A \cdot \begin{bmatrix} -1 & 0 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 4 & 1 \end{bmatrix} \cdot \begin{bmatrix} -4 & 1 \\ 0 & -4 \end{bmatrix}$.

so $A = PDP^{-1}$, $P = \begin{bmatrix} -1 & 0 \\ 4 & 1 \end{bmatrix}$, $D = \begin{bmatrix} -4 & 1 \\ 0 & -4 \end{bmatrix}$ (note that it's not diagonal.)

Then, the equation becomes $\mathbb{X}'(t) = PDP^{-1}\mathbb{X}(t)$.
So, it is $P^{-1}\mathbb{X}'(t) = DP^{-1}\mathbb{X}(t)$. If we let $\mathbb{Y}(t) = P^{-1}\mathbb{X}(t)$,
It becomes $\mathbb{Y}'(t) = D \cdot \mathbb{Y}(t)$. Initial conditions become $\mathbb{Y}(0) = P^{-1}\begin{bmatrix} -2 \\ 11 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 11 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.
[P^{-1} happens to be the same as P.]
D is $\begin{bmatrix} -4 & 1 \\ 0 & -4 \end{bmatrix}$ so -4 is the only eigenvalue, but the dimension of the eigenspace is $1 < 2$. Hence, we have $\mathbb{Y}_1(t) = e^{-4t}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ as one solution.

Now, we try $\mathbb{Y}_2(t) = t \cdot e^{-4t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + u \cdot e^{-4t}$. (This is the method you use for nondiagonalizable matrices.)

$\mathbb{Y}_2'(t) = e^{-4t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} - 4 \cdot t \cdot e^{-4t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} - 4u \cdot e^{-4t}$

$D\mathbb{Y}_2(t) = t \cdot e^{-4t} \cdot D\begin{bmatrix} 1 \\ 0 \end{bmatrix} + e^{-4t} \cdot Du$.

Terms w/ $t \cdot e^{-4t}$ match.
w/ e^{-4t}: $\begin{bmatrix} 1 \\ 0 \end{bmatrix} - 4u = Du$.
So, $(4I + D)u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.
$4I + D = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow$ choose $u = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

$\therefore \mathbb{Y}_2(t) = t \cdot e^{-4t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + e^{-4t}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

General: $\mathbb{Y}(t) = C_1\mathbb{Y}_1(t) + C_2\mathbb{Y}_2(t)$ satisfies $\mathbb{Y}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

So, $C_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + C_2\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \Rightarrow C_1 = 2$, $C_2 = 3$.

$$\therefore \mathbb{X}(t) = P \cdot \mathbb{Y}(t) = P \cdot (2\mathbb{Y}_1(t) + 3\mathbb{Y}_2(t)) = \begin{bmatrix} -1 & 0 \\ 4 & 1 \end{bmatrix} \cdot \left(3t \cdot e^{-4t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + e^{-4t}\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right)$$

$$= t \cdot e^{-4t}\begin{bmatrix} -3 \\ 12 \end{bmatrix} + e^{-4t}\begin{bmatrix} -2 \\ 11 \end{bmatrix}.$$

Finally, $x(t) = -3t \cdot e^{-4t} - 2e^{-4t}$ & $y(t) = 12t \cdot e^{-4t} + 11 \cdot e^{-4t}$.

[2] Hint. This tells you that $A = PDP^{-1}$ where $P = \begin{bmatrix} -1 & 0 \\ 4 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} -4 & 1 \\ 0 & -4 \end{bmatrix}$.