1. How can you solve a linear system using row operations? Give an example and solve it.

2. What is a pivot position and a pivot column of a given matrix? Give an example and find them.

3. How do we define the multiplication of a matrix with a vector? and when are we able to define such a multiplication in terms of the number of rows or columns?

4. What is a linear combination of a set of vectors? Why do we say “a” linear combination, not “the” linear combination?

5. What is the span of a set of vectors? Why do we say “the” span, not “a” span?

6. Explain the relation between

$$\mathbf{b} \text{ is a linear combination of columns of } A. \quad \Longleftrightarrow \quad A\mathbf{x} = \mathbf{b} \text{ is consistent.}$$

7. Explain the relation between

$$\text{Every vector } \mathbf{b} \text{ in } \mathbb{R}^m \text{ is a linear combination of columns of } A. \quad \Longleftrightarrow \quad \text{The columns of } A \text{ span } \mathbb{R}^m.$$

8. Why is the existence of at least one free variable for an equation equivalent to that $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution?

9. What are the concepts linearly INdependent and linearly dependent?

10. Why does every set containing the zero vector is always linearly dependent?

11. Explain the relation between

$$\text{The columns of } A \text{ form a linearly INdependent set.} \quad \Longleftrightarrow \quad A\mathbf{x} = \mathbf{0} \text{ has only the trivial solution.}$$

12. Every elementary row operation corresponds to a matrix. Explain what this means. (Note. Such a matrix is called an *elementary matrix*.)

13. How can you find such a matrix given an elementary row operation?

14. Such a matrix is invertible because every elementary row operation is reversible. Explain this.

15. Explain why every elementary row operation preserves "linear relations among column vectors". (You might use elementary matrices for this problem.)

16. What is the meaning of a map f being a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$?

17. Why is every matrix transformation a linear transformation?

18. Why do you only need images of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to describe a linear transformation T from $\mathbb{R}^2$ to $\mathbb{R}^n$?

19. Explain the concept of a map being one-to-one (injective)? Once your map turns out to be a linear transformation, how can you change the condition for the map begin one-to-one?

20. Explain the concept of a map being onto (surjective)?

21. Explain the relation between

$$\text{A matrix transformation is one-to-one.} \iff \text{The columns form a linearly INdependent set.}$$

22. Explain the relation between

$$\text{A matrix transformation is onto } \mathbb{R}^m. \iff \text{The columns span } \mathbb{R}^m.$$

23. In terms of the multiplication of a matrix with a vector, explain how to multiply two matrices. What would be the condition for defining matrix multiplication?

24. What is the transpose of a matrix A? (Note. By hand, you can prove that $(AB)^T = B^T A^T$. Let's just take the formula for granted.)

25. What is the inverse matrix of a matrix A? What is the meaning of a matrix A being invertible? (Note. We only define the inverse matrix for a SQUARE matrix A.)

26. Why do we say "the" inverse matrix, not "a" inverse matrix?

27. Why is this true?

$$(AB)^{-1} = B^{-1}A^{-1}$$

28. Once you have the inverse matrix of A, how can you use the inverse matrix A^{-1} to find the solution for $A\mathbf{x} = \mathbf{b}$?

29. Why is the RREF of an invertible matrix A the identity matrix? (Hint. Explain this in terms of "linear relations of columns".)

30. Explain an algorithm for finding A^{-1} using row operations. Why does this hold?

31. What is a subspace of $\mathbb{R}^n$? (Note. A subspace satisfies "linear combination condition.")

32. What is Col A? Explain why Col A is a subspace of $\mathbb{R}^m$.

33. What is Nul A? Explain why Nul A is a subspace of $\mathbb{R}^n$.

34. What is a basis (for a subspace of $\mathbb{R}^n$)?

35. Why do we say "a" basis, not "the" basis?

36. How can you find a basis for Col A given the RREF of A?

37. What is the dimension (of a subspace of $\mathbb{R}^n$)?

38. In terms of ‘the number of pivots’, what can you say about the dimension of Col A and Nul A?

39. We can say that the dimension is the number of ‘maximal linearly independent vectors’. What does it mean?

40. What is the rank (of a matrix A)?

41. Why do we say “the” dimension and “the” rank, not “a“ of them?

42. Explain the change-of-basis matrix for two bases $\mathcal{B}$ and $\mathcal{C}$ of V. Specifically, explain how to get $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ from the equation (= the definition)

$$[\mathbf{v}]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[\mathbf{v}]_{\mathcal{B}}.$$

43. Given a linear transformation $T : V \to W$, explain what *the matrix for* T *relative to* $\mathcal{B}$ *and* $\mathcal{C}$ is. (Let's use the notation $[T]^{\mathcal{C}}_{\mathcal{B}}$.) The equation (= the defintion) for this matrix is given by

$$[T(\mathbf{v})]_{\mathcal{C}} = [T]^{\mathcal{C}}_{\mathcal{B}}[\mathbf{v}]_{\mathcal{B}}.$$