Finite-Volume Second Order Solution to Steady-State Temperature Distribution of Uniformly Heated Disc

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Outline

Problem Introduction and Brief Background

- Finite-Volume methods form using Gauss-Divergence Theorem, and one part of this is information from nearby cell(s) to get a flux estimate.
- First-order approximations are limited to nearest-neighbor cells for flux adjustments, leading to over-diffusion.
- Neglected flux contributions from further neighbors ($k = 2, 3, 4, \ldots$).
- Goal: Use a second-order finite volume solver to improve accuracy.
- Method: Use uniformly heated disc as a comparison and benchmark.

Numerical Methods – Formulation

- PDE transformed to Finite-Volume form using Gauss-Divergence Theorem:

$$T_i = \frac{\sum_{j \in \mathcal{N}_k(i)} \frac{T_j}{d_{ij}} l_{ij} - \frac{Q_i A_i}{\lambda}}{\sum_{j \in \mathcal{N}_k(i)} \frac{l_{ij}}{d_{ij}}}$$

- Neighbor levels:
  - Level 1 ($k = 1$): Weight $= 1$
  - Level 2 ($k = 2$): Weight $= -\frac{4}{3}$
  - Level 3 ($k = 3$): Weight $= \frac{1}{3}$
- If Level 3 does not exist (I.e. near edge of circle), switch to 1st order methods for approximation
- For this disc, $Q = 1$, 2000 elements in a $R = 1$ radius
- Gauss SOR to resolve iterations

Numerical Methods – FV Elements (1st order)

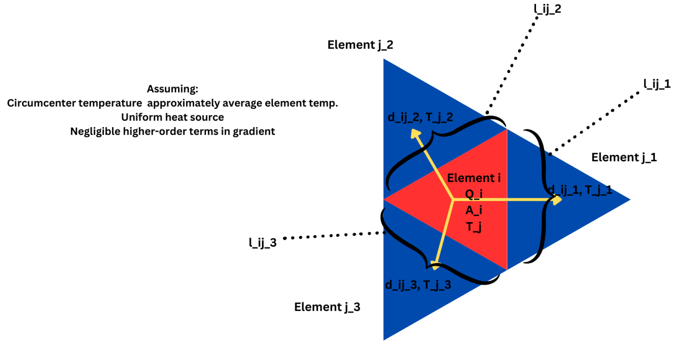

Figure: Illustration of FV elements, j-th neighbor cells up to $k = 1$ levels.

Numerical Methods – FV Elements (2nd order)

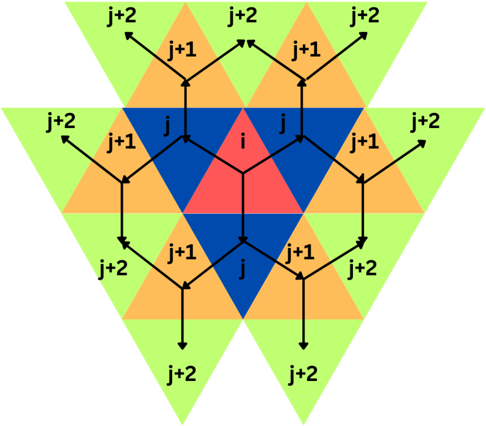

Figure: Illustration of FV elements, j-th neighbor cells up to $k = 3$ levels.

Numerical Methods – BC

- BC from the ghost cell – Dirichlet conditions:

$$T_{ghost} = -T_i$$

- 1st order for BC – k-neighbors is $k = 1$.

Numerical Methods – Benchmark and Validation

The analytical Laplace solution for a uniformly heated disc is:

$$T_i = \frac{Q}{4\lambda}\left(R - r_i^2\right)$$

This will be used for comparison and benchmarking in numerical results.

Results (Numerical vs Analytical) – Key Results

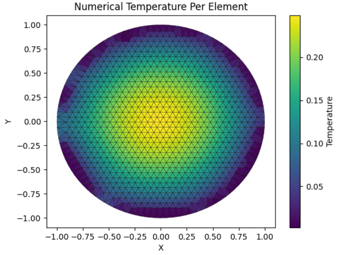

Figure: Numerical Solution (2nd order)

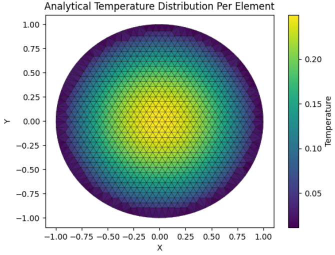

Figure: Analytical Solution

Both similar, but the edges are not satisfied properly
$k = 3$ levels influence from cells near edge.

Results (Comparison)

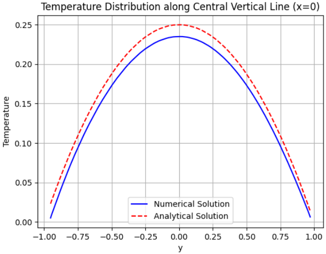

Figure: 1st order forward-difference

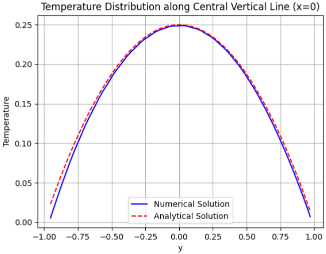

Figure: 2nd order forward-difference

Results (Comparison)

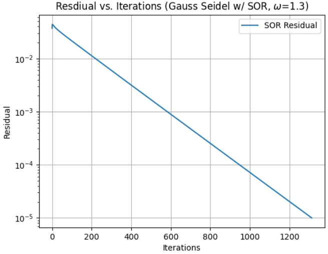

Figure: Error vs. Iterations (2nd order)

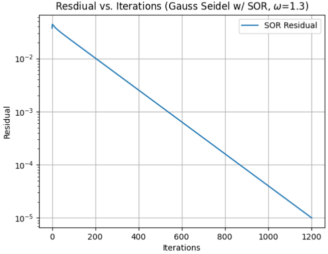

Figure: Error vs. Iterations (1st order)

There is not much difference between 1st and 2nd order iterations required for convergence (tol $= 1E - 5$)

Conclusions

- Improved accuracy with second-order finite volume solver.
- Reduction in over-diffusion and better alignment with the analytical solution.
- 2nd order closely matched 1st order iteration count, but did take more time.