Proposal – Finite-Volume Second Order Solution to Steady-State Temperature Distribution of Uniformly Heated Disc

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Problem Introduction and Brief Background

Throughout the discussion of a first-order approximation of temperature gradients for the integral, the approximation was limited to the first nearest-neighbors of the approximation. At maximum, there are up to 3 such cells derived for a triangle whose adjacent cells share faces, say in temrs of k-hopes at $k = 1$ triangle units away. However, the original derivation neglected the neighboring cells of these adjacent j-th cells, leading to over-diffusion in problems with low resolution. The end result is an underprediction for temperature at the center of each mesh tested. Typically, diffusivity issues like this are caused when the flux from adjacent j-th cells to further points (i.e., $k = 2, 3, 4...$ triangles hops away) are not considered, leading to an underprediction of flux.

However, by considering the first-order derivative with further neighboring adjacent cells (e.g., $k = 1, 2, 3...$ levels of neighbors), over-diffusion may be corrected with respect to the original cell. This assignment is to use a second-order finite difference approximation for a uniform triangular mesh to improve this accuracy. The goal of this project is to provide public access to a 2nd order finite volume solver alongside exploring the increased numerical accuracy of using such a solver over the 1st order.

Analysis Brief Overview

The following steps were applied to transform the PDE into a FV (Finite-Volume) with Gauss-Divergence Theorem. T_i represents the temperature of the element at the circumcenter, and T_j is the temperature of neighboring elements at the circumcenter, defined by the average of their respective triangular nodes. d_{ij} represents the circumcenter distance from $i \to j$, and l_{ij} represents the elemental face length between circumcenter $i \to j$ in this representation. An illustration of these variables with element $i \to j$ is shown in Figure 1.

$$T_i = \left(\sum_{j \epsilon \mathcal{N}_k(i)} \frac{T_j}{d_{ij}} l_{ij} - \frac{Q_i A_i}{\lambda} \right) / \sum_{j \epsilon \mathcal{N}_k(i)} \frac{l_{ij}}{d_{ij}} \tag{1}$$

$\mathcal{N}_k(i) =$ Set of neighbors that are k hops away from triangle i

Level 1: Weighting $= 1$ (direct neighbors).
Level 2: Weighting $= -\frac{4}{3}$ (for neighbors of level 1).
Level 3: Weighting $= \frac{1}{3}$ (for neighbors of level 2).

For each outward neighbor, a reduced weighting is applied to each level. To correct for this additional layering of cells, the following weightings are considered. A flux weighting of 1 is applied for each j cell at $k = 1$. For each $k = 2$ neighboring cell (up to 6 additional cells), a weighting of $-\frac{4}{3}$ is applied to account for flux loss assignments. For the third level, a weighting of $\frac{1}{3}$ is imposed on $k = 3$ neighbors. The weighting is following a 2nd order accurate forward-difference scheme, relative to the j cell neighbor.

The analytical solution is as follows, which will be compared and bench-marked in the numerical results.

$$T_i = \frac{Q}{4\lambda}\left(R - r_i^2\right)$$

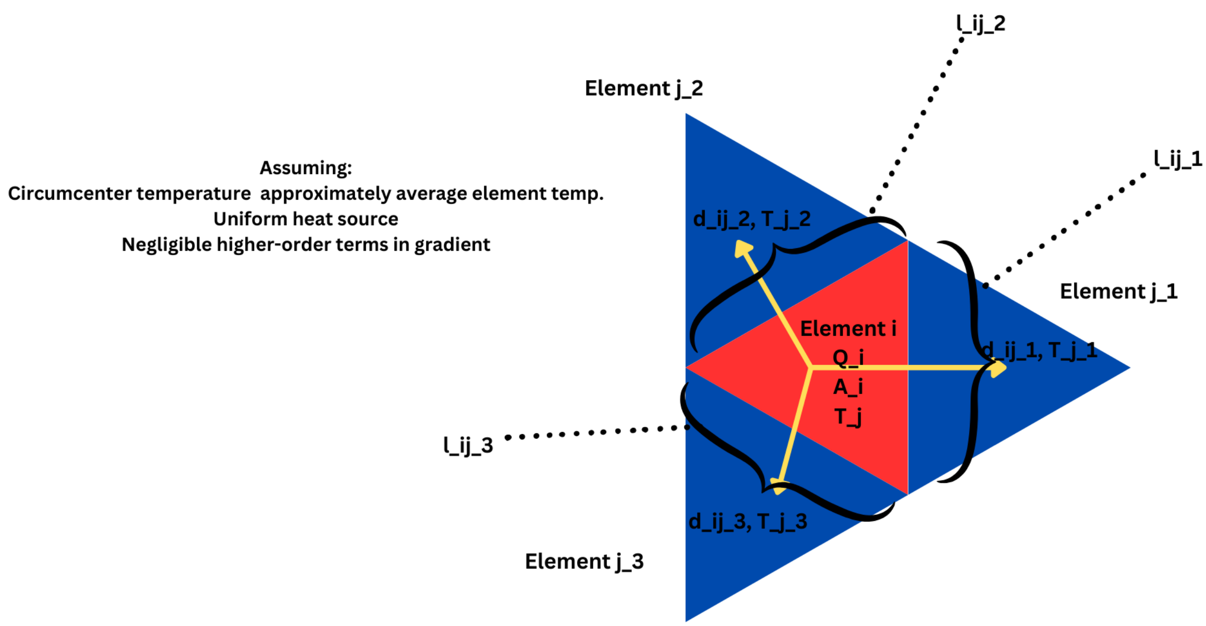

Figure 1: Illustration of FV elements proposed, j-th neighbor cell (note 2nd order will include up to $j + 2$ levels

Preliminary results

The following are preliminary results gained from the code adjustment. Note that the improved 2nd order is able to capture the analytical solution better than the prior 1st order numerical solution.

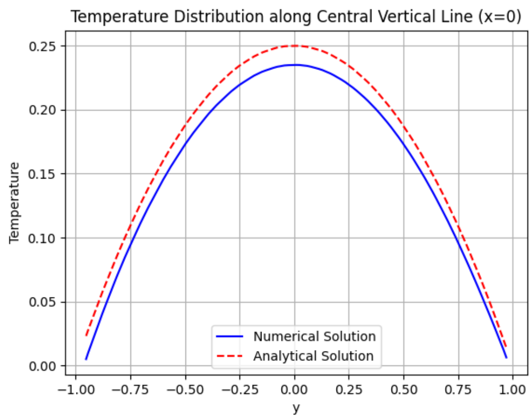

Figure 2: 1st order forward-difference

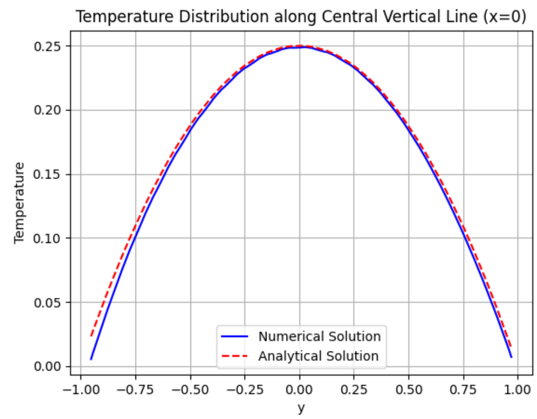

Figure 3: 2nd order forward-difference

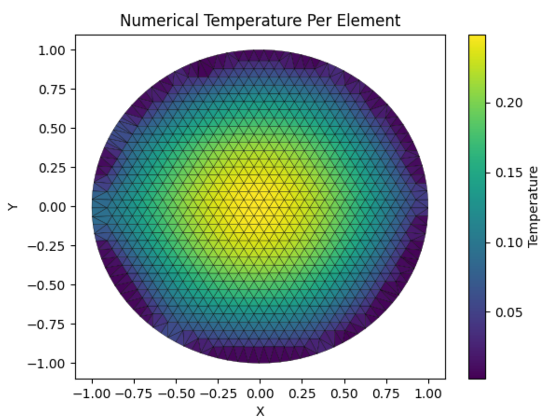

Figure 4: 2nd order forward-difference

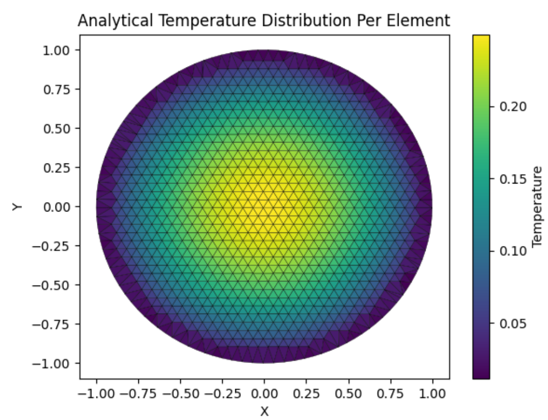

Figure 5: Analytical solution