

describe the effect of parents' height on a child's adult height. Regression analysis also is widely applied in actuarial science; as evidence, it is a required educational component of the two main actuarial bodies in the U.S. and Canada, the Society of Actuaries and the Casualty Actuarial Society.

Although widely applicable, regression analysis is limited by the basic setup that requires the analyst to identify one dimension of the outcome as the primary measure of interest (the dependent variable) and other dimensions as supporting or "explaining" this variable (the independent variables). This article examines problems in which this relationship is not of primary interest; hence, we focus on the more basic problem of understanding the distribution of several outcomes, a *multivariate* distribution. For an example in actuarial science, when two lives are subject to failure, such as under a joint life insurance or annuity policy, we are concerned with joint distribution of lifetimes. As another example, when we simulate the distribution of a scenario that arises out of a financial security system, we need to understand the distribution of several variables interacting simultaneously, not in isolation of one another.

The normal distribution has long dominated the study of multivariate distributions. For example, leading references on multivariate analysis, such as Anderson (1958) and Johnson and Wichern (1988), focus exclusively on the multivariate normal and related distributions that can be derived from normal distributions, including multivariate extensions of Student's *t*- and Fisher's *F*-distributions. Multivariate normal distributions are appealing because the marginal distributions are also normal. For example, in the Galton dataset, the distribution of adult children's height and the distribution of parents' height are each approximately normal, in isolation of the other. Multivariate normal distributions are also appealing because the association between any two random outcomes can be fully described knowing only (1) the marginal distributions and (2) one additional parameter, the correlation coefficient.

More recent texts on multivariate analysis, such as Krzanowski (1988), have begun to recognize the need for examining alternatives to the normal distribution setup. This is certainly true for actuarial science applications such as for lifetime random variables (Bowers et al. 1997, Chap. 3) and long-tailed claims variables (Hogg and Klugman 1984), where the normal distribution does not provide an adequate approximation to many datasets. An extensive literature in statistics deals with nonnormal

multivariate distributions; see, for example, Johnson and Kotz (1973) and Johnson, Kotz and Balakrishnan (1997). However, historically many multivariate distributions have been developed as immediate extensions of univariate distributions, examples being the bivariate Pareto, bivariate gamma, and so on. The drawbacks of these types of distributions are that (1) a different family is needed for each marginal distribution, (2) extensions to more than just the bivariate case are not clear, and (3) measures of association often appear in the marginal distributions.

A construction of multivariate distributions that does not suffer from these drawbacks is based on the *copula* function. To define a copula, begin as you might in a simulation study by considering p uniform (on the unit interval) random variables, u_1, u_2, \dots, u_p . Here, p is the number of outcomes that you wish to understand. Unlike many simulation applications, we do not assume that u_1, u_2, \dots, u_p are independent; yet they may be related. This relationship is described through their joint distribution function

$$C(u_1, u_2, \dots, u_p) \\ = \text{Prob}(U_1 \leq u_1, U_2 \leq u_2, \dots, U_p \leq u_p).$$

Here, we call the function C a *copula*. Further, U is a (ex-ante) uniform random variable, whereas u is the corresponding (ex-post) realization. To complete the construction, we select arbitrary marginal distribution functions $F_1(x_1), F_2(x_2), \dots, F_p(x_p)$. Then, the function

$$C[F_1(x_1), F_2(x_2), \dots, F_p(x_p)] \\ = F(x_1, x_2, \dots, x_p) \quad (1.1)$$

defines a multivariate distribution function, evaluated at x_1, x_2, \dots, x_p , with marginal distributions F_1, F_2, \dots, F_p .

With copula construction in Equation (1.1), we select different marginals for each outcome. For example, suppose we are considering modeling male and female lifetimes for a joint-life annuity product. Then, with $p=2$, we might choose the Gompertz distribution to represent mortality at the older ages, yet with different parameters to reflect gender differences in mortality. As another example, in Section 4, we consider a bivariate outcome associated with the loss and the expense associated with administering a property and casualty claim. There, we could elect to use a lognormal distribution for expenses and a longer tail distribution, such as Pareto, for losses associated with

the claim. The copula construction does not constrain the choice of marginal distributions.

In Section 2 we see that the copula method for understanding multivariate distributions has a relatively short history in the statistics literature; most of the statistical applications have arisen in the last ten years. However, copulas have been studied in the probability literature for about 40 years (Schweizer 1991), and thus several desirable properties of copulas are now widely known. To begin, it is easy to check from the construction in Equation (1.1) that F is a multivariate distribution function. Sklar (1959) established the converse. He showed that any multivariate distribution function F can be written in the form of Equation (1.1), that is, using a copula representation. Sklar also showed that if the marginal distributions are continuous, then there is a unique copula representation. In this sense copulas provide a unifying theme for our study of multivariate distributions. Sections 3 and 5 describe other desirable properties of copulas.

Given that copulas are fundamental building blocks for studying multivariate distributions, we now turn to the question of how to build a copula function for a problem at hand. Despite Sklar's result that a copula function always exists, Example 1.1 shows that it is *not* always convenient to identify the copula. Example 1.2 illustrates a useful way of building a copula, using the method of *compounding*. We describe this method of constructing copulas in detail in Section 3.1.

Example 1.1 Marshall-Olkin (1967) Exponential Shock Model

Suppose that we wish to model $p=2$ lifetimes that we suspect are subject to some common disaster, or "shock," that may induce a dependency between the lives. For simplicity, let us assume that Y_1 and Y_2 are two independent (underlying) lifetimes with distribution functions H_1 and H_2 . We further assume there exists an independent exponential random variable Z with parameter λ that represents the time until common disaster. Both lives are subject to the same disaster, so that actual ages-at-death are represented by $X_1 = \min(Y_1, Z)$ and $X_2 = \min(Y_2, Z)$. Thus, the marginal distributions are

$$\begin{aligned} \text{Prob}(X_j \leq x_j) &= F_j(x_j) \\ &= 1 - \exp(-\lambda x_j)(1 - H_j(x_j)), \text{ for } j = 1, 2. \end{aligned}$$

Basic calculations show that the joint distribution is¹

$$\begin{aligned} F(x_1, x_2) &= F_1(x_1) + F_2(x_2) - 1 \\ &\quad + \exp(\lambda \max(x_1, x_2))(1 - F_1(x_1))(1 - F_2(x_2)). \end{aligned}$$

This expression, although intuitively appealing, is not in the form of the copula construction (1.1) because the joint distribution function F is not a function of the marginals $F_1(x_1)$ and $F_2(x_2)$. For further discussions in the actuarial literature of this bivariate distribution, see Frees (1996) and Bowers et al. (1997, Sec. 9.6).

Example 1.2 Bivariate Pareto Model

Consider a claims random variable X that, given a risk classification parameter γ , can be modeled as an exponential distribution; that is,

$$\text{Prob}(X \leq x | \gamma) = 1 - e^{-\gamma x}.$$

As is well known in credibility theory (see, for example, Klugman et al. 1997), if γ has a gamma distribution, then the marginal distribution (over all risk classes) of X is Pareto. That is, if γ is gamma(α, λ), then

$$\begin{aligned} F(x) &= \text{Prob}(X \leq x) \\ &= \int \text{Prob}(X \leq x | \gamma) \frac{\alpha^\lambda}{\Gamma(\lambda)} \gamma^{\alpha-1} e^{-\lambda \gamma} d\gamma \\ &= 1 - \int e^{-\gamma x} \frac{\alpha^\lambda}{\Gamma(\lambda)} \gamma^{\alpha-1} e^{-\lambda \gamma} d\gamma \\ &= 1 - (1 + x/\lambda)^{-\alpha}, \end{aligned} \tag{1.2}$$

a Pareto distribution.

Suppose, conditional on the risk class γ , that X_1 and X_2 are independent and identically distributed. Assuming that they come from the same risk class γ induces a dependency. The joint distribution is²

¹ $F(x_1, x_2) = \text{Prob}(X_1 \leq x_1, X_2 \leq x_2) = 1 - \text{Prob}(X_1 > x_1) - \text{Prob}(X_2 > x_2) + \text{Prob}(X_1 > x_1, X_2 > x_2)$
 $= 1 - \exp(-\lambda x_1)(1 - H_1(x_1)) - \exp(-\lambda x_2)(1 - H_2(x_2)) + \exp(-\lambda \min(x_1, x_2))(1 - H_1(x_1))(1 - H_2(x_2))$
 $= 1 - (1 - F_1(x_1)) - (1 - F_2(x_2)) + \exp(-\lambda \min(x_1, x_2)) \exp(\lambda(x_1 + x_2))(1 - F_1(x_1))(1 - F_2(x_2)).$

$$F(x_1, x_2) = F_1(x_1) + F_2(x_2) - 1 + \left[(1 - F_1(x_1))^{-1/\alpha} + (1 - F_2(x_2))^{-1/\alpha} - 1 \right]^{-\alpha}.$$

This yields the copula function

$$C(u_1, u_2) = u_1 + u_2 - 1 + \left[(1 - u_1)^{-1/\alpha} + (1 - u_2)^{-1/\alpha} - 1 \right]^{-\alpha}. \quad (1.3)$$

With this function, we can express the bivariate distribution function as $H(x_1, x_2) = C(F_1(x_1), F_2(x_2))$.

Alternatively, we can consider the copula

$$C_*(u_1, u_2) = C(1 - u_1, 1 - u_2) = (u_1^{-1/\alpha} + u_2^{-1/\alpha} - 1)^{-\alpha} - 1$$

and express the joint survival function as $\text{Prob}(X_1 > x_1, X_2 > x_2) = C_*(S_1(x_1), S_2(x_2))$, where $S = 1 - F$. Because our motivating examples in Section 2 concern lifetime (positive) random variables, we often find it intuitively appealing to work with survival in lieu of distribution functions.

Several methods are available for constructing multivariate distributions; see Hougaard (1987) and Hutchinson and Lai (1990) for detailed reviews. Example 1.1 illustrates the so-called “variables-in-common” technique in which a common element serves to induce dependencies among several random variables. This article focuses on the compounding method illustrated in Example 1.2 for two reasons. First, there is a long history of using compound distributions for risk classification in the actuarial science literature, particularly within the credibility framework. Second, Marshall and Olkin (1988) showed that compounding can be used to generate several important families of copulas. Additional discussion of this point appears in Sections 2 and 3.

Examples 1.1 and 1.2 each describe bivariate distributions through probabilistic interpretations of random quantities. It is also useful to explore (in Section

3) a class of functions called “Archimedean copulas,” which arise from the mathematical theory of associativity. An important special case of this class, due to Frank (1979), is

$$C(u, v) = \frac{1}{\alpha} \ln \left(1 + \frac{(e^{\alpha u} - 1)(e^{\alpha v} - 1)}{e^{\alpha} - 1} \right). \quad (1.4)$$

Although Frank’s copula does not appear to have a natural probabilistic interpretation, its other desirable properties make it well suited for empirical applications (Nelson 1986 and Genest 1987).

The purpose of this paper is to introduce copulas, their characteristics and properties, and their applicability to specific situations. Section 2 reviews empirical applications of copulas in analyzing survival of multiple lives and competing risks. Both are familiar topics to actuaries. Section 3 discusses properties and characteristics of copulas. In particular, we show (1) how to specify a copula, (2) how the association structure of copulas can be summarized in terms of familiar measures of dependence, and (3) how simulation of multivariate outcomes can be easily accomplished when the distribution is expressed as a copula. Section 4 provides an illustration of fitting a copula to insurance company losses and expenses. Section 5 reviews additional applications of copulas. We conclude in Section 6.

2. EMPIRICAL APPLICATIONS

Copulas are useful for examining the dependence structure of multivariate random vectors. In this section, we describe two biological science subject areas that are related to actuarial science and that have used copulas to understand empirical relationships among multivariate observations.

2.1 Survival of Multiple Lives

In epidemiological and actuarial studies, it is often of interest to examine the joint mortality pattern of groups of more than a single individual. This group

² $F(x_1, x_2) = 1 - \text{Prob}(X_1 > x_1) - \text{Prob}(X_2 > x_2) + \text{Prob}(X_1 > x_1, X_2 > x_2)$

$$\begin{aligned} &= 1 - \left(1 + \frac{x_1}{\lambda}\right)^{-\alpha} - \left(1 + \frac{x_2}{\lambda}\right)^{-\alpha} + \int \text{Prob}(X_1 > x_1 | \gamma) \text{Prob}(X_2 > x_2 | \gamma) \frac{\alpha^\lambda}{\Gamma(\lambda)} \gamma^{\alpha-1} e^{-\lambda\gamma} d\gamma \\ &= 1 - \left(1 + \frac{x_1}{\lambda}\right)^{-\alpha} - \left(1 + \frac{x_2}{\lambda}\right)^{-\alpha} + \int e^{-\gamma x_1} e^{-\gamma x_2} \frac{\alpha^\lambda}{\Gamma(\lambda)} \gamma^{\alpha-1} e^{-\lambda\gamma} d\gamma \\ &= 1 - \left(1 + \frac{x_1}{\lambda}\right)^{-\alpha} - \left(1 + \frac{x_2}{\lambda}\right)^{-\alpha} + \left[1 + \frac{x_1 + x_2}{\lambda}\right]^{-\alpha}. \end{aligned}$$

could be, for example, a husband and wife, a family with children, or twins (identical or nonidentical). There is strong empirical evidence that supports the dependence of mortality on pairs of individuals. For example, statistical analyses of mortality patterns of married couples are frequently made to test the “broken heart” syndrome. Using a dataset consisting of 4,486 55-year-old widowers, Parkes et al. (1969) showed that there is a 40% increase in mortality among the widowers during the first few months after the death of their wives; see also Ward (1976). Intuitively, pairs of individuals exhibit dependence in mortality because they share common risk factors. These factors may be purely genetic, as in the case of twins, or environmental, as in the case of a married couple.

The first application of copulas in joint-life models arose indirectly through the work of Clayton (1978) in his study of bivariate life tables of fathers and sons. Clayton developed the bivariate distribution function given in Equation (1.3) as the solution of a second-order partial differential equation. Clayton also pointed out the random effects interpretation of the model that was subsequently developed by Oakes (1982). See also Cook and Johnson (1981).

Random effects models are important in biological and epidemiological studies because they provide a method of modeling heterogeneity. A random effects model particularly suited for multivariate survival analysis is the *frailty* model, due to Vaupel, Manton and Stallard (1979) and Hougaard (1984). To describe frailty models, we first introduce some notation. In survival analysis, it is customary to consider the complement of the distribution function, the survival function, and the negative derivative of its logarithmic transform, the hazard function. Thus, for a continuous random survival time T , we define

$$S(t) = \text{Prob}(T > t) = 1 - F(t)$$

and

$$h(t) = -\frac{\partial \ln S(t)}{\partial t} = \frac{f(t)}{S(t)}.$$

Actuaries know the hazard function $h(t)$ as the force of mortality (see, for example, Bowers et al. 1997, Chap. 3).

To understand explanatory variables Z in survival analysis, we can use the Cox (1972) proportional hazards model, which represents the hazard function as

$$h(t, Z) = e^{\beta Z} b(t),$$

where $b(t)$ is the so-called “baseline” hazard function and β is a vector of regression parameters. It is

proportional in the sense that all the information contained in the explanatory variables is in the multiplicative factor $\gamma = e^{\beta Z}$. Integrating and exponentiating the negative hazard, we can also express Cox’s proportional hazard model as

$$S(t|\gamma) = \exp\left(-\int_0^t h(s, Z) ds\right) = B(t)^\gamma.$$

Here,

$$B(t) = \exp\left(-\int_0^t b(s) ds\right)$$

is the survival function corresponding to the baseline hazard. Frailty models arise when Z , and hence γ , is unobserved. The factor γ is called a *frailty* because larger values of γ imply a smaller survival function, $S(t|\gamma)$, indicating poorer survival. As demonstrated in Example 1.2, the marginal distribution for a single life T is obtained by taking expectations over the potential values of γ ; that is, $S(t) = E_\gamma S(t|\gamma)$.

Oakes (1989, 1994) described how frailties can be used to model the dependencies among multiple lives. Other studies, such as Jagger and Sutton (1991), used a Cox regression model with *known* explanatory variables Z to account for the dependencies among multiple lives. Multivariate frailty models are obtained when the investigator does not wish to attribute, or does not have knowledge of, specific characteristics that may induce dependencies. For multivariate frailty models, we assume that “ p ” lives T_1, T_2, \dots, T_p are independent given the frailty γ . That is,

$$\begin{aligned} \text{Prob}(T_1 > t_1, \dots, T_p > t_p|\gamma) \\ &= \text{Prob}(T_1 > t_1|\gamma) \cdots \text{Prob}(T_p > t_p|\gamma) \\ &= S_1(t_1|\gamma) \cdots S_p(t_p|\gamma) \\ &= B_1(t_1)^\gamma \cdots B_p(t_p)^\gamma. \end{aligned}$$

The joint multivariate survival function is defined as

$$\begin{aligned} \text{Prob}(T_1 > t_1, \dots, T_p > t_p) \\ &= E_\gamma \left\{ B_1(t_1) \cdots B_p(t_p) \right\}^\gamma. \quad (2.1) \end{aligned}$$

Example 2.1 Hougaard’s Copula Family

To illustrate, an important frailty model was given by Hougaard (1986), who assumed that the distribution of γ could be modeled as a positive “stable distribution” with Laplace transform $E_\gamma e^{-s\gamma} = \exp(-s^\alpha)$

and parameter α . Recall that the Laplace transform of a positive random variable γ is defined by

$$\tau(s) = E_{\gamma} e^{-s\gamma} = \int e^{-st} dG_{\gamma}(t),$$

where G_{γ} is the distribution function of γ . This is also the moment generating function evaluated at $-s$; thus, knowledge of $\tau(s)$ determines the distribution.

With a positive stable distribution for γ , using Equation (2.1) we have

$$\begin{aligned} \text{Prob}(T_1 > t_1, \dots, T_p > t_p) \\ &= E_{\gamma} \exp\left(\gamma \ln \{B_1(t_1) \cdots B_p(t_p)\}\right) \\ &= \exp\left(-\left\{-\ln B_1(t_1) - \dots - \ln B_p(t_p)\right\}^{\alpha}\right). \end{aligned}$$

Because

$$S_i(t_i) = \exp\left(-\left\{-\ln B_i(t_i)\right\}^{\alpha}\right),$$

we can write the joint survival function as

$$\begin{aligned} \text{Prob}(T_1 > t_1, \dots, T_p > t_p) \\ &= \exp\left(-\left[\left\{-\ln S_1(t_1)\right\}^{1/\alpha} + \dots + \left\{-\ln S_p(t_p)\right\}^{1/\alpha}\right]^{\alpha}\right), \end{aligned} \quad (2.2)$$

a copula expression. In particular, for bivariate lifetimes with $p=2$, Hougaard proposed examining Weibull marginals so that $B_i(t) = \exp(-a_i t^{b_i})$ and $S_i(t|\gamma) = \exp(-a_i \gamma t^{b_i})$. This yields the bivariate survivor function

$$\begin{aligned} \text{Prob}(T_1 > t_1, T_2 > t_2) \\ &= \exp\left(-\left[a_1 t_1^{b_1} + a_2 t_2^{b_2}\right]^{\alpha}\right). \end{aligned} \quad (2.3)$$

This is desirable in the sense that both the conditional and marginal distributions are Weibull.

Equations (1.2) of Example 1.2 and (2.2) of Example 2.1 show that these frailty models can be written as copulas. Marshall and Olkin (1988) showed that these are special cases of a more general result; they demonstrated that all frailty models of the form in Equation (2.1) can be easily written as copulas. Further, the copula form is a special type called an Archimedean copula, which we will introduce in Section 3.

In addition to the Clayton and Oakes studies, other works have investigated the use of copula models in studying behavior of multiple lives. Hougaard et al. (1992) analyzed the joint survival of Danish twins born

between 1881 and 1930. They use the frailty model arising from a positive stable distribution as well as Cox's proportional hazard model. Frees et al. (1995) investigated mortality of annuitants in joint- and last-survivor annuity contracts using Frank's copula (Equation 1.3). They found that accounting for dependence in mortality produced approximately a 3% to 5% reduction in annuity values when compared to standard models that assume independence.

2.2 Competing Risks—Multiple Decrement Theory

The subject of competing risks deals with the study of the lifetime distribution of a system subject to several competing causes; this subject is called multiple decrement theory in actuarial science (see, for example, Bowers et al. 1997, Chap. 10 and 11). The problem of competing risks arises in survival analysis, systems reliability, and medical studies as well as in actuarial science. For example, a person dies because of one of several possible causes: cancer, heart disease, accident, and so on. As yet another example, a mechanical device fails because a component fails. For mathematical convenience, the general framework begins with an unobserved multivariate lifetime vector (T_1, T_2, \dots, T_p) ; each element in the vector denotes the lifetime due to one of p competing causes. The quantities typically observed are $T = \min(T_1, T_2, \dots, T_p)$ and the cause of failure J . To illustrate, in life insurance, T usually denotes the lifetime of the insured individual and J denotes the cause of death such as cancer or accidental death. Several texts lay the foundation of the theory of competing risks. For example, see Bowers et al. (1997), Cox and Oakes (1984), David and Moeschberger (1978), and Elandt-Johnson and Johnson (1980).

When formulating the competing risk model, it is often assumed that the component lifetimes T_i are statistically independent. With independence, the model is easily tractable and avoids the problem of identifiability encountered in inference. However, many authors, practitioners, and academicians recognize that this assumption is not practical, realistic, or reasonable; see Carriere (1994), Makeham (1874), and Seal (1977).

To account for dependence in competing risk models, one general approach is to apply copulas. In particular, the frailty model seems well suited for handling competing risks. Assuming that causes of death are independent given a frailty γ , we have

$$\begin{aligned}
\text{Prob}(T > t|\gamma) &= \text{Prob}(\min(T_1, \dots, T_p) > t) \\
&= \text{Prob}(T_1 > t|\gamma) \cdots \text{Prob}(T_p > t|\gamma) \\
&= B_1(t)^\gamma \cdots B_p(t)^\gamma.
\end{aligned}$$

Thus, similar to Equation (2.1), the overall survival function is

$$\text{Prob}(T > t) = E_\gamma \{ B_1(t) \cdots B_p(t) \}^\gamma. \quad (2.4)$$

Example 2.1 (Continued)

For a positive stable distribution for γ , the survival function is

$$\begin{aligned}
\text{Prob}(T > t) &= \exp\left(-\left[\left\{-\ln S_1(t)\right\}^{1/\alpha} \right. \right. \\
&\quad \left. \left. + \cdots + \left\{-\ln S_p(t)\right\}^{1/\alpha} \right]^\alpha\right),
\end{aligned}$$

similar to Equation (2.2). For bivariate lifetimes with Weibull marginals, we have

$$\text{Prob}(T > t) = \exp\left(-[a_1 t^{b_1} + a_2 t^{b_2}]^\alpha\right).$$

There have been several applications of frailty models for studying competing risk situations. Oakes (1989) discussed the number of cycles of two chemotherapy regimes tolerated by 109 cancer patients. Shih and Louis (1995) analyzed HIV-infected patients by using Clayton's family, positive stable frailties, as well as Frank's copula. Zheng and Klein (1995) considered data from a clinical trial of patients with non-Hodgkin's lymphoma using gamma copula (as in Clayton's family). In a nonbiological context, Hougaard (1987) described how dependent competing risk models using positive stable copulas can be used to assess machine failure.

3. PROPERTIES OF COPULAS

This section discusses several properties and characteristics of copulas, specifically (1) how to generate copulas, (2) how copulas can summarize association between random variables, and (3) how to simulate copula distributions.

3.1 Specifying Copulas: Archimedean and Compounding Approaches

Copulas provide a general structure for modeling multivariate distributions. The two main methods for specifying a family of copulas are the Archimedean approach and the compounding approach, the latter illustrated in Example 1.2.

The Archimedean representation allows us to reduce the study of a multivariate copula to a single univariate function. For simplicity, we first consider bivariate copulas so that $p=2$. Assume that ϕ is a convex, decreasing function with domain $(0, 1]$ and range $[0, \infty)$ such that $\phi(1)=0$. Use ϕ^{-1} for the inverse function of ϕ . Then the function

$$C_\phi(u, v) = \phi^{-1}(\phi(u) + \phi(v)) \text{ for } u, v \in (0, 1]$$

is said to be an Archimedean copula. We call ϕ a *generator* of the copula C_ϕ . Genest and McKay (1986a, 1986b) give proofs of several basic properties of C_ϕ , including the fact that it is a distribution function. As seen in Table 1, different choices of generator yield several important families of copulas. A generator uniquely determines (up to a scalar multiple) an Archimedean copula. Thus, this representation helps identify the copula form. This point is further developed in Section 3.2.

Examples 1.2 and 2.1 show that compound distributions can be used to generate copulas of interest.

TABLE 1
ARCHIMEDEAN COPULAS AND THEIR GENERATORS

Family	Generator $\phi(t)$	Dependence Parameter (α) Space	Bivariate Copula $C_\phi(u, v)$
Independence	$-\ln t$	Not applicable	uv
Clayton (1978), Cook-Johnson (1981), Oakes (1982)	$t^{-\alpha} - 1$	$\alpha > 1$	$(u^{-\alpha} + v^{-\alpha} - 1)^{-1/\alpha}$
Gumbel (1960), Hougaard (1986)	$(-\ln t)^\alpha$	$\alpha \geq 1$	$\exp\left\{-\left[(-\ln u)^\alpha + (-\ln v)^\alpha\right]^{1/\alpha}\right\}$
Frank (1979)	$\ln \frac{e^{\alpha t} - 1}{e^\alpha - 1}$	$-\infty < \alpha < \infty$	$\frac{1}{\alpha} \ln \left(1 + \frac{(e^{\alpha u} - 1)(e^{\alpha v} - 1)}{e^\alpha - 1}\right)$

These examples are special cases of a general method for constructing copulas due to Marshall and Olkin (1988). To describe this method, suppose that X_i is a random variable whose conditional, given a positive latent variable γ_i , distribution function is specified by $H_i(x|\gamma_i) = H_i(x)^{\gamma_i}$, where $H_i(\cdot)$ is some baseline distribution function, for $i=1, \dots, p$. Marshall and Olkin considered multivariate distribution functions of the form

$$F(x_1, x_2, \dots, x_p) = EK(H_1(x_1)^{\gamma_1}, \dots, H_p(x_p)^{\gamma_p}).$$

Here, K is a distribution function with uniform marginals and the expectation is over $\gamma_1, \gamma_2, \dots, \gamma_p$. As a special case, take all latent variables equal to one another so that $\gamma_1 = \gamma_2 = \dots = \gamma_p = \gamma$ and use the distribution function corresponding to independent marginals. Marshall and Olkin (1988) showed that

$$\begin{aligned} F(x_1, x_2, \dots, x_p) &= E_{\gamma}(H_1(x_1)^{\gamma} \cdots H_p(x_p)^{\gamma}) \\ &= \tau(\tau^{-1}\{F_1(x_1)\} + \cdots + \tau^{-1}\{F_p(x_p)\}) \quad (3.2) \end{aligned}$$

where F_i is the i -th marginal distribution of F and $\tau(\cdot)$ is the Laplace transform of γ , defined by $\tau(s) = E_{\gamma}e^{-s\gamma}$.

Laplace transforms have well-defined inverses. Thus, from Equation (3.2), we see that the inverse function τ^{-1} serves as the generator for an Archimedean copula. In this sense, Equation (3.2) provides a probabilistic interpretation of generators. To illustrate, Table 2 provides the inverse Laplace transform for the generators listed in Table 1. Here, we see how well-known distributions can be used to generate compound distributions. Because generators are defined uniquely only up to scalar multiple, any positive constant in the family

of Laplace transforms determines the same class of generators. (Indeed, this methodology suggests new copula families.) Thus, the inverse of a Laplace transform represents an important type of generator for Archimedean copulas.

To summarize, assume that X_1, X_2, \dots, X_p are conditionally, given γ , independent with distribution functions $H_i(x)^{\gamma}$. Then, the multivariate distribution is given by the copula form with the generator being the inverse of the Laplace transform of the latent variable γ . Because of the form of the conditional distribution, we follow Joe (1997) and call this a *mixture of powers* distribution. We remark that

$$\tau[-\ln H_i(x)] = E_{\gamma} \exp\{-[-\ln H_i(x)]\gamma\} = F_i(x)$$

so that

$$H_i(x) = \exp\{-\tau^{-1}[F_i(x)]\}.$$

This provides a way of specifying the baseline function given the marginal distribution and the distribution of the latent variable.

For the applications involving lifetimes in Section 2, we found it natural to discuss distributions in terms of survival functions. Marshall and Olkin (1988) pointed out that the construction as in Equation (3.2) could also be used for survivor functions. That is, with the frailty model $\text{Prob}(T_i > t|\gamma) = B_i(t)^{\gamma}$ and conditional independence of T_1, T_2, \dots, T_p from Equation (2.1), we have

$$\begin{aligned} \text{Prob}(T_1 > t_1, \dots, T_p > t_p) &= E_{\gamma}\{B_1(t_1) \cdots B_p(t_p)\}^{\gamma} \\ &= \tau(\tau^{-1}\{S_1(x_1)\} \\ &\quad + \cdots + \tau^{-1}\{S_p(x_p)\}) \quad (3.3) \end{aligned}$$

TABLE 2
ARCHIMEDEAN GENERATORS AND THEIR INVERSES

Family	Generator $\phi(t)$	Inverse Generator (Laplace Transform) $\tau(s) = \phi^{-1}(s)$	Laplace Transform Distribution
Independence	$-\ln t$	$\exp(-s)$	Degenerate
Clayton (1978), Cook-Johnson (1981), Oakes (1982)	$t^{-\alpha} - 1$	$(1+s)^{-1/\alpha}$	Gamma
Gumbel (1960), Hougaard (1986)	$(-\ln t)^{\alpha}$	$\exp(-s^{1/\alpha})$	Positive stable
Frank (1979)	$\ln \frac{e^{\alpha t} - 1}{e^{\alpha} - 1}$	$\alpha^{-1} \ln [1 + e^s(e^{\alpha} - 1)]$	Logarithmic series distribution on the positive integers

As before,

$$\tau[-\ln B_i(t)] = S_i(t) = 1 - F_i(t)$$

and

$$B_i(t) = \exp\{-\tau^{-1}[S_i(t)]\}.$$

If the mixing distribution remains the same, the Laplace transform and hence the generator remain the same. However, the two constructions yield different multivariate distributions because

$$\text{Prob}(T \leq t|\gamma) = 1 - B(t)^\gamma \neq (1 - F(t))^\gamma.$$

We follow Marshall and Olkin and first present the copula construction in Equation (3.2) by using distribution functions because it is useful for all random variables. However, for positive lifetime random variables, the concept of frailty models is intuitively appealing; thus, using survival functions in the construction is preferred for these applications. We finally remark, unlike the gamma and positive stable families, that Frank's copula $C_\phi(u, v)$ is symmetric about the point $(1/2, 1/2)$ [for example, $C_\phi(u, v) = C_\phi(1/2 - u, 1/2 - v)$]. Thus, it is invariant to the choice of F or $S = 1 - F$ in the construction (Genest 1987).

3.2 Measures of Association

Recall the copula representation of a distribution function in Equation (1.1),

$$F(x_1, x_2, \dots, x_p) = C(F_1(x_1), F_2(x_2), \dots, F_p(x_p)).$$

With this expression, we see that F is a function of its marginals and the copula. As pointed out by Genest and Rivest (1993), this suggests that a natural way of specifying the distribution function is to examine the copula and marginals separately. Moreover, the case of independence is a special form of the copula $C(u_1, u_2, \dots, u_p) = u_1 \cdot u_2 \cdots u_p$ (regardless of the marginals), and this suggests that we examine the copula function to understand the association among random variables. Because we are concerned with correlation measures, we restrict our consideration to $p=2$.

Schweizer and Wolff (1981) established that the copula accounts for all the dependence between two random variables, X_1 and X_2 , in the following sense. Consider g_1 and g_2 , strictly increasing (but otherwise arbitrary) functions over the range of X_1 and X_2 . Then, Schweizer and Wolff showed that the transformed variables $g_1(X_1)$ and $g_2(X_2)$ have the same copula as X_1 and X_2 . Thus, the manner in which X_1 and X_2 “move

together” is captured by the copula, regardless of the scale in which each variable is measured.

Schweizer and Wolff also showed that two standard nonparametric correlation measures could be expressed solely in terms of the copula function. These are Spearman's correlation coefficient, defined by

$$\begin{aligned} \rho(X_1, X_2) &= 12E\{(F_1(x_1) - 1/2)(F_2(x_2) - 1/2)\} \\ &= 12 \iint \{C(u, v) - uv\} dudv \end{aligned}$$

and Kendall's correlation coefficient, defined by

$$\begin{aligned} \tau(X_1, X_2) &= \text{Prob}\{(X_1 - X_1^*)(X_2 - X_2^*) > 0\} \\ &\quad - \text{Prob}\{(X_1 - X_1^*)(X_2 - X_2^*) < 0\} \\ &= 4 \iint C(u, v) dC(u, v) - 1. \end{aligned}$$

For these expressions, we assume that X_1 and X_2 have a jointly continuous distribution function. Further, the definition of Kendall's τ uses an independent copy of (X_1, X_2) , (X_1^*, X_2^*) to define the measure of “concordance.” See Section 5 for more details. Also, the widely used Pearson correlation coefficient, $\text{Cov}(X_1, X_2)/(\text{Var}X_1 \text{Var}X_2)^{1/2}$, depends not only on the copula but also on the marginal distributions. Thus, this measure is affected by (nonlinear) changes of scale.

Table 3 illustrates the calculation of these correlation measures. The correlations from Frank's copula rely on the so-called “Debye” functions, defined as

$$D_k(x) = \frac{k}{x^k} \int_0^x \frac{t^k}{e^t - 1} dt,$$

for $k=1,2$. To evaluate negative arguments of the Debye function D_k , basic calculus shows that

$$D_k(-x) = D_k(x) + \frac{kx}{k+1}.$$

An important point of this table is that there is a one-to-one relationship between each correlation measure and the association parameter α . Further, Table 3 allows us to see a drawback of the Clayton/Cook-Johnson/Oakes and Gumbel-Hougaard copula families. Because of the limited dependence parameter space as shown in Table 1, these families permit only non-negative correlations, a consequence of the latent variable model. However, Frank's family permits negative as well as positive dependence.

Correlation measures summarize information in the copula concerning the dependence, or association, between random variables. Following a procedure due to Genest and Rivest (1993), we can also

TABLE 3
ARCHIMEDEAN COPULAS AND THEIR MEASURES OF DEPENDENCE

Family	Bivariate Copula $C_\phi(u, v)$	Kendall's τ	Spearman's ρ
Independence	uv	0	0
Clayton (1978), Cook-Johnson (1981), Oakes (1982)	$(u^{-\alpha} + v^{-\alpha} - 1)^{-1/\alpha}$	$\frac{\alpha}{\alpha + 2}$	Complicated form
Gumbel (1960), Hougaard (1986)	$\exp \left\{ - \left[(-\ln u)^\alpha + (-\ln v)^\alpha \right]^{1/\alpha} \right\}$	$1 - \alpha^{-1}$	No closed form
Frank (1979)	$\frac{1}{\alpha} \ln \left(1 + \frac{(e^{\alpha u} - 1)(e^{\alpha v} - 1)}{e^\alpha - 1} \right)$	$1 - \frac{4}{\alpha} \{D_1(-\alpha) - 1\}$	$1 - \frac{12}{\alpha} \{D_2(-\alpha) - D_1(-\alpha)\}$

use the dependence measure to specify a copula form in empirical applications, as follows.

Genest and Rivest's procedure for identifying a copula begins by assuming that we have available a random sample of bivariate observations, $(X_{11}, X_{21}), \dots, (X_{1n}, X_{2n})$. Assume that the distribution function F has associated Archimedean copula C_ϕ ; we wish to identify the form of ϕ . We work with an intermediate (unobserved) random variable $Z_i = F(X_{1i}, X_{2i})$ that has distribution function $K(z) = \text{Prob}(Z_i \leq z)$. Genest and Rivest showed that this distribution function is related to the generator of an Archimedean copula through the expression

$$K(z) = z - \frac{\phi(z)}{\phi'(z)}.$$

To identify ϕ , we:

1. Estimate Kendall's correlation coefficient using the usual (nonparametric or distribution-free) estimate

$$\tau_n = \binom{n}{2}^{-1} \sum_{i < j} \text{sign}[(X_{1i} - X_{1j})(X_{2i} - X_{2j})].$$

2. Construct a nonparametric estimate of K , as follows:
 - a. First, define the pseudo-observations $Z_i = \{\text{number of } (X_{1j}, X_{2j}) \text{ such that } X_{1j} < X_{1i} \text{ and } X_{2j} < X_{2i}\} / (n - 1)$ for $i = 1, \dots, n$.
 - b. Second, construct the estimate of K as $K_n(z) = \text{proportion of } Z_i's \leq z$.
3. Now construct a parametric estimate of K using the relationship

$$K_\phi(z) = z - \frac{\phi(z)}{\phi'(z)}.$$

For example, refer to Table 1 for various choices of ϕ and use the estimate τ_n to calculate an

estimate of α , say α_n . Use α_n to estimate $\phi(x)$, say $\phi_n(x)$. Finally, use $\phi_n(x)$ to estimate $K_\phi(z)$, say $K_{\phi_n}(z)$.

Repeat step 3 for several choices of ϕ . Then, compare each parametric estimate to the nonparametric estimate constructed in Step 2. Select the choice of ϕ so that the parametric estimate $K_{\phi_n}(z)$ most closely resembles the nonparametric estimate $K_n(z)$. Measuring "closeness" can be done by minimizing a distance such as $\int [K_{\phi_n}(z) - K_n(z)]^2 dK_n(z)$ or graphically. Graphical representations include (1) plots of $K_n(z)$ and $K_{\phi_n}(z)$ versus z , and (2) the corresponding quantile plots. See Section 4 for an example.

3.3 Simulation

Actuaries routinely deal with complex nonlinear functions, such as present values, of random variables. Simulation is a widely used tool for summarizing the distribution of stochastic outcomes and for communicating the results of complex models. The copula construction allows us to simulate outcomes from a multivariate distribution easily.

The two primary simulation strategies are the Archimedean and compounding methods for constructing copulas; each has relative advantages and disadvantages.

We begin with the Archimedean construction. Our goal is to construct an algorithm to generate X_1, X_2, \dots, X_p having known distribution function $F(x_1, x_2, \dots, x_p) = C(F_1(x_1), F_2(x_2), \dots, F_p(x_p))$, where the copula function is

$$C(u_1, u_2, \dots, u_p) = \phi^{-1}(\phi(u_1) + \dots + \phi(u_p)).$$

For this construction, Genest and Rivest (1986b) and Genest (1987) introduced the idea of simulating

the full distribution of (X_1, X_2, \dots, X_p) by recursively simulating the conditional distribution of X_k given X_1, \dots, X_{k-1} , for $k=2, \dots, p$. This idea, subsequently developed by Lee (1993), is as follows. For simplicity, we assume that the joint probability density function of X_1, X_2, \dots, X_p exists. Using the copula construction, the joint probability density function of X_1, \dots, X_k is

$$\begin{aligned} f_k(x_1, \dots, x_k) &= \frac{\partial^k}{\partial x_1 \dots \partial x_k} \phi^{-1} \\ &\quad \left\{ \phi[F_1(x_1)] + \dots + \phi[F_k(x_k)] \right\} \\ &= \phi^{-1(k)} \left\{ \phi[F_1(x_1)] + \dots + \phi[F_k(x_k)] \right\} \\ &\quad \prod_{j=1}^k \phi^{(1)} [F_j(x_j)] F_j^{(1)}(x_j). \end{aligned}$$

Here, the superscript notation (j) means the j -th mixed partial derivative. Thus, the conditional density of X_k given X_1, \dots, X_{k-1} is

$$\begin{aligned} f_k(x_k | x_1, \dots, x_{k-1}) &= \frac{f_k(x_1, \dots, x_k)}{f_{k-1}(x_1, \dots, x_{k-1})} \\ &= \phi^{(1)} [F_k(x_k)] F^{(1)}(x_k) \\ &\quad \frac{\phi^{-1(k-1)} \left\{ \phi[F_1(x_1)] + \dots + \phi[F_k(x_k)] \right\}}{\phi^{-1(k-1)} \left\{ \phi[F_1(x_1)] + \dots + \phi[F_{k-1}(x_{k-1})] \right\}}. \end{aligned}$$

Further, the conditional distribution function of X_k given X_1, \dots, X_{k-1} is

$$\begin{aligned} F_k(x_k | x_1, \dots, x_{k-1}) &= \int_{-\infty}^{x_k} f_k(x | x_1, \dots, x_{k-1}) dx \\ &= \frac{\phi^{-1(k-1)} \left\{ \phi[F_1(x_1)] + \dots + \phi[F_k(x_k)] \right\}}{\phi^{-1(k-1)} \left\{ \phi[F_1(x_1)] + \dots + \phi[F_{k-1}(x_{k-1})] \right\}} \\ &= \frac{\phi^{-1(k-1)} \left\{ c_{k-1} + \phi[F_k(x_k)] \right\}}{\phi^{-1(k-1)} (c_{k-1})}, \end{aligned}$$

where $c_k = \phi[F_1(x_1)] + \dots + \phi[F_k(x_k)]$. With this distribution function, we can now use the usual procedure of solving for the inverse distribution function and evaluating this at a uniform random number; that is, use $F^{-1}(U) = X_k$.

To summarize, the algorithm is (Lee 1993):

Algorithm 3.1 Generating Multivariate Outcomes from an Archimedean Copula

1. Generate U_1, U_2, \dots, U_p independent uniform (0,1) random numbers.
2. Set $X_1 = F_1^{-1}(U_1)$ and $c_0 = 0$.
3. For $k=2, \dots, p$, recursively calculate X_k as the solution of

$$\begin{aligned} U_k &= F_k(X_k | x_1, \dots, x_{k-1}) \\ &= \frac{\phi^{-1(k-1)} \left\{ c_{k-1} + \phi[F_k(x_k)] \right\}}{\phi^{-1(k-1)} (c_{k-1})}. \end{aligned} \quad (3.4)$$

This algorithm was initially introduced in the context of Frank's copula for $p=2$. Here, pleasant calculations show that the algorithm reduces to:

Algorithm 3.2 Generating Bivariate Outcomes from Frank's Copula

1. Generate U_1, U_2 independent uniform (0,1) random numbers.
2. Set $X_1 = F_1^{-1}(U_1)$.
3. Calculate X_2 as the solution of

$$U_2 = e^{-\alpha U_1} \left(\frac{e^{-\alpha} - e^{-\alpha F_2(X_2)}}{e^{-\alpha F_2(X_2)} - 1} + 1 \right)^{-1}.$$

That is, calculate $X_2 = F_2^{-1}(U_{*2})$ where

$$U_{*2} = \frac{U_2 e^{-\alpha} - e^{-\alpha U_1} (1 - U_2)}{U_2 + e^{-\alpha U_1} (1 - U_2)}.$$

Genest (1987) gave this algorithm.

The algorithm can also be readily used to simulate distributions from Clayton's family. From Table 2, we have $\phi^{-1}(s) = (1+s)^{-1/\alpha}$ so that

$$\phi^{-1(1)}(s) = -\alpha^{-1} (1+s)^{-(1/\alpha)-1}.$$

This expression with $p=2$ and Equation (3.4) show that

$$U_2 = \frac{-\alpha^{-1} \left((1 + F_2(X_2))^{-1/\alpha} - 1 + U_1^{-1/\alpha} - 1 \right)^{-(1/\alpha)-1}}{-\alpha^{-1} \left(1 + U_1^{-1/\alpha} - 1 \right)^{1(1/\alpha)-1}}.$$

That is, calculate $X_2 = F_2^{-1}(U_{*2})$ where

$$U_{*2} = \left(1 + U_1^{-1/\alpha} (U_{*2}^{-\alpha/(\alpha+1)} - 1) \right)^{-\alpha}.$$

For the Gumbel-Hougaard copula, determining X_2 in Equation (3.4) requires an iterative solution. Although straightforward, this is computationally

expensive because many applications require large numbers of simulated values. This drawback leads us to introducing an alternative algorithm suggested by Marshall and Olkin (1988) for compound constructions of copulas.

To generate X_1, X_2, \dots, X_p having a mixture of powers distribution specified in Equation (3.2), the algorithm is:

Algorithm 3.3 Generating Multivariate Outcomes from a Compound Copula

1. Generate a (latent) random variable γ having Laplace transform τ .
2. Independently of step 1, generate U_1, U_2, \dots, U_p independent uniform (0, 1) random numbers.
3. For $k=1, \dots, p$, calculate $X_k = F_k^{-1}(U_{*k})$ where

$$U_{*k} = \tau(-\gamma^{-1} \ln U_k). \quad (3.5)$$

Recall that the marginal distribution function can be calculated from the baseline distribution function using $F_k(x) = \tau(-\ln H_k(x))$. To illustrate for the Gumbel-Hougaard copula, from Equation (3.5), we have

$$U_{*k} = \exp\left[-\left(-\gamma^{-1} \ln U_k\right)^{-1/\alpha}\right].$$

The algorithm is straightforward for most copulas of interest that are generated by the compounding method. Like the conditional distribution approach, it can be easily implemented for more than two dimensions ($p > 2$). It is computationally more straightforward than the conditional distribution approach. A disadvantage is that it requires the generation of an additional variable, γ . For bivariate applications, this means generating 50% more uniform random variates; this can be expensive in applications.

4. INSURANCE COMPANY LOSS AND EXPENSE APPLICATION

This section illustrates methods of fitting copulas to insurance company indemnity claims. The data comprise 1,500 general liability claims randomly chosen from late settlement lags and were provided by Insurance Services Office, Inc. Each claim consists of an indemnity payment (the loss, X_1) and an allocated loss adjustment expense (ALAE, X_2). Here, ALAE are types of insurance company expenses that are specifically attributable to the settlement of individual claims such as lawyers' fees and claims investigation expenses; see, for example, Hogg and Klugman (1984). Our objective is to describe the joint distribution of losses and expenses.

Estimation of the joint distribution of losses and expenses is complicated by the presence of censoring, a common feature of loss data (Hogg and Klugman 1984). Specifically, in addition to loss and expense information, for each claim we have a record of the policy limit, the maximal claim amount. With the presence of the policy limit, the loss variable is censored because the amount of claim cannot exceed the stated policy limit. For some claims, the policy limit was unknown, and for these policies, we assumed there was no policy limit.

Table 4 summarizes the data. Here, only 34 of 1,500 policies have claims that equaled the policy limit and thus are considered censored. However, the censored losses cannot be ignored; for example, the mean loss of censored claims is much higher than the corresponding mean for uncensored claims. Table 4 also shows that our sample comprises only claims with positive losses and expenses. Separate models would be required for claims with positive losses but no expenses or for claims with zero losses but positive expenses.

Figure 1 is a scatterplot of loss versus ALAE. The corresponding correlation coefficient is 0.41. This

TABLE 4
SUMMARY STATISTICS OF LOSSES AND EXPENSES

	ALAE	Loss	Policy Limit	Loss (Uncensored)	Loss (Censored)
Number	1,500	1,500	1,352	1,466	34
Mean	12,588	41,208	559,098	37,110	217,491
Median	5,471	12,000	500,000	11,048	100,000
Standard Deviation	28,146	102,748	418,649	92,513	258,205
Minimum	15	10	5,000	10	5,000
Maximum	501,863	2,173,595	7,500,000	2,173,595	1,000,000
25th Percentile	2,333	4,000	300,000	3,750	50,000
75th Percentile	12,577	35,000	1,000,000	32,000	300,000

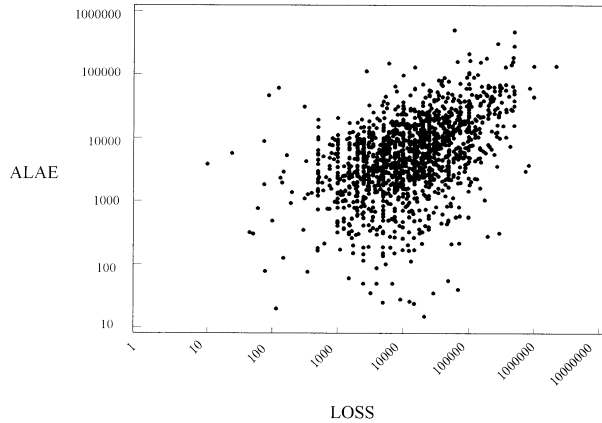
FIGURE 1

PLOT OF ALAE VERSUS LOSS.

BOTH VARIABLES ARE ON A LOGARITHMIC SCALE.

THIRTY-FOUR LOSS OBSERVATIONS ARE CENSORED.

THIS PLOT DEMONSTRATES A STRONG RELATIONSHIP BETWEEN ALAE AND LOSS.



statistic, together with the plot, suggests a strong relationship between losses and expenses.

4.1 Fitting Marginal Distributions

The initial step in our model fitting is to determine the appropriate marginals. Determining appropriate parametric distributions for univariate data is well described in Hogg and Klugman (1984). For these data, this step was examined in detail earlier by Klugman

and Parsa (1995). Thus, for simplicity, we present here only the fit of the univariate marginals using a Pareto distribution. With parameters λ and θ , the distribution function is

$$F(x) = 1 - \left(\frac{\lambda}{\lambda + x} \right)^\theta.$$

The quality of the fit of the marginal distributions can be examined with a graphical comparison of the fitted distribution function against their empirical versions, as displayed in Figures 2 and 3. Because of censoring, we used the Kaplan-Meier empirical distribution function for the loss variable.

4.2 Fitting a Copula to the Bivariate Data

To fit the copula, we first identify the form of the copula in Section 4.2.1 and then estimate it using maximum likelihood in Section 4.2.2.

4.2.1 Identifying a Copula

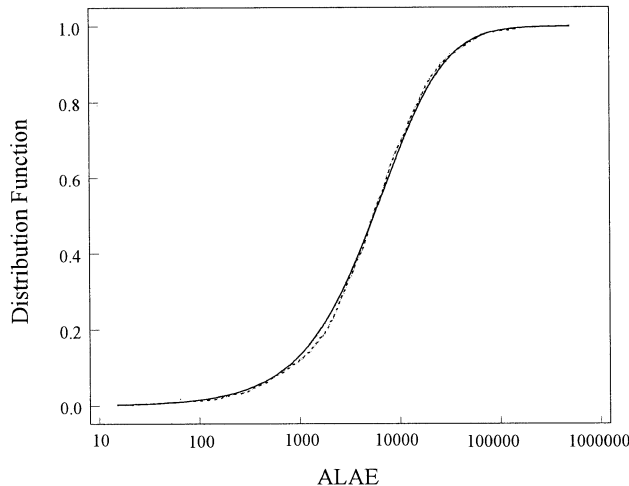
We use the procedure developed by Genest and Rivest (1993) for identifying an appropriate copula, outlined in Section 3.2. According to the procedure, we examine the degree of closeness of the parametric and non-parametric versions of the distribution function $K(z)$. This procedure is based on estimates of $K(z)$, the distribution function of pseudo-observations $Z = F(X_1, X_2)$. The idea behind the procedure is to compare a

FIGURE 2

FITTED DISTRIBUTION FUNCTIONS OF ALAE.

THE DOTTED CURVE IS THE EMPIRICAL DISTRIBUTION FUNCTION.

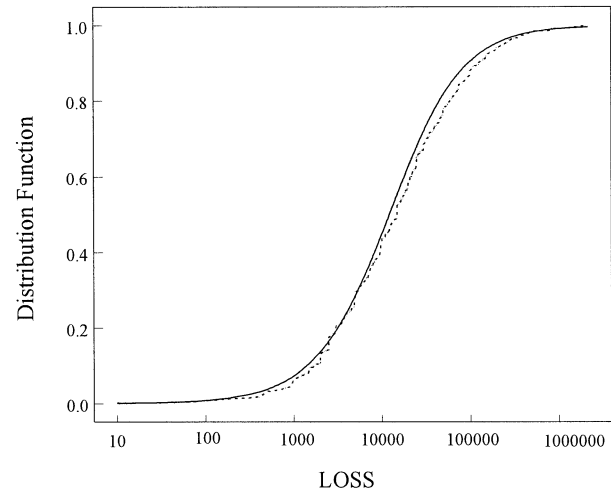
THE SMOOTH CURVE IS A FIT USING THE PARETO DISTRIBUTION.

**FIGURE 3**

FITTED DISTRIBUTION FUNCTIONS OF LOSS.

THE DOTTED CURVE IS A KAPLAN-MEIER EMPIRICAL

DISTRIBUTION FUNCTION. THE SMOOTH CURVE IS A FIT USING THE PARETO DISTRIBUTION.



nonparametric estimate of $K(z)$ to those based on a specific form of the copula. To compare the two estimates of $K(z)$, we examine quantiles from each estimated distribution. Scatterplots of the two sets of quantiles are widely known in statistics as quantile-quantile, or q - q , plots. For identification purposes, we ignore the mild censoring in the loss variable although we do accommodate it in the more formal maximum likelihood fitting in Section 4.2.2.

We now discuss identification for three widely used copulas, namely, the Gumbel-Hougaard, Frank, and Cook-Johnson copulas. The form of the copula for each of these families appears in Table 3. The comparison of the resulting q - q plots is displayed in Figure 4. Because of the close agreement between nonparametric and parametric quantiles, the procedure suggests the use of the Gumbel-Hougaard copula. The quantiles based on Frank's copula are also close to the nonparametric quantiles, although there is greater disparity at the higher quantiles, corresponding to high losses and expenses. We therefore identify both Frank's and Gumbel-Hougaard's as copulas that we fit more formally in Section 4.2.2.

4.2.2 Fitting a Copula Using Maximum Likelihood

Recall that our data consist of losses (X_1) and expenses (X_2) and that we also have available an indicator for censoring (δ), so that $\delta=1$ indicates the claim is censored. Parameters were estimated using maximum likelihood procedures that were programmed using the SAS procedure IML. In the development of the likelihood equation, we use the following partial derivatives:

$$F_1(x_1, x_2) = \frac{\partial F(x_1, x_2)}{\partial x_1},$$

$$F_2(x_1, x_2) = \frac{\partial F(x_1, x_2)}{\partial x_2},$$

and

$$f(x_1, x_2) = \frac{\partial^2 F(x_1, x_2)}{\partial x_1 \partial x_2}.$$

Similarly, the first partial derivatives for the copula will be denoted by C_1 and C_2 ; the second mixed partial derivative by C_{12} .

Using a one-parameter copula and Pareto marginals, we have a total of five parameters to estimate: two each for the marginals and one for the dependence parameter. To develop the likelihood, we distinguish between the censored and uncensored cases. If the loss variable is not censored, then $\delta=0$ and the contribution to the likelihood function is

$$f(x_1, x_2) = f_1(x_1) f_2(x_2) C_{12} [F_1(x_1), F_2(x_2)]. \quad (4.1)$$

If the loss variable is censored, then $\delta=1$ and the joint probability is given by

$$\text{Prob}(X_1 \geq x_1, X_2 \leq x_2) = F_2(x_2) - F(x_1, x_2).$$

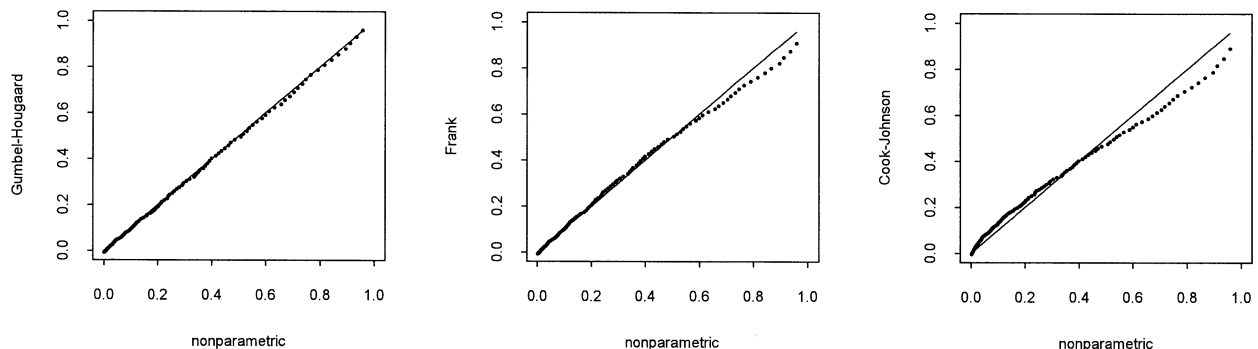
Thus, the contribution to the likelihood when the observation is censored is

$$\begin{aligned} f_2(x_2) - F_2(x_1, x_2) \\ = f_2(x_2) \{1 - C_2[F_1(x_1), F_2(x_2)]\}. \end{aligned} \quad (4.2)$$

Combining Equations (4.1) and (4.2), for a single observation, the logarithm of the likelihood function is

FIGURE 4

QUANTILE-QUANTILE (Q-Q) PLOTS, CORRESPONDING TO THE PARAMETRIC AND NONPARAMETRIC DISTRIBUTION ESTIMATES OF THE PSEUDO-OBSERVATIONS DEFINED IN SECTION 3.2. THE DOTTED LINES CORRESPOND TO THE QUANTILES OF NONPARAMETRIC AND PARAMETRIC ESTIMATES OF THE ARCHIMEDEAN GENERATOR ϕ . THE SMOOTHED LINES CORRESPOND TO THE CASE WHERE THE QUANTILES ARE EQUAL.



$$\log L(x_1, x_2, \delta) = (1 - \delta) \log f(x_1, x_2) + \delta \left\{ \log f_2(x_2) + \log \left[1 - C_2(F_1(x_1), F_2(x_2)) \right] \right\}. \quad (4.3)$$

The parameter estimates are then determined by maximizing the likelihood for the entire dataset:

$$\sum_{i=1}^n \log L(x_{1i}, x_{2i}, \delta).$$

Results of the maximum likelihood estimation fitting the Gumbel-Hougaard copula, whose partial derivatives are derived in Appendix A, are summarized in Table 5. Here, we see that the parameter estimates of the marginal distributions are largely unchanged when we compare the univariate to the bivariate estimation. Standard errors are smaller in the bivariate fit, indicating greater precision of the parameter estimates. The estimate of the dependence parameter is significantly different from 1; the estimate of α is approximately 13 standard errors above 1. This provides strong statistical evidence that losses and expenses are not independent. Using Table 3 for the Gumbel-Hougaard copula, we can convert the dependence parameter into a more familiar measure of association. Thus, the parameter value of $\hat{\alpha} = 1.453$ corresponds to an approximate Spearman's correlation measure of 31%. A

95% confidence interval for α is therefore given by $\hat{\alpha} \pm 1.96 \cdot se(\hat{\alpha}) = 1.453 \pm 1.96(0.034) = (1.386, 1.520)$. This corresponds to a 95% confidence interval of (28%, 34%) for the Spearman's correlation.

Results of the maximum likelihood estimation fitting the Frank copula are summarized in Table 6. Here, the behavior of parameter estimates and standard errors is similar to the fit using the Gumbel-Hougaard copula. For our parameterization of Frank's copula, the case of independence corresponds to $\alpha = 0$. In our case, the parameter estimate of $\hat{\alpha} = -3.158$ corresponds to an approximate Spearman's correlation measure of 32%, which is close to the estimate using the Gumbel-Hougaard copula.

It is difficult to compare the fit of the two copulas directly because they are non-nested models. However, we did compute Akaike's Information Criteria (AIC) for each model, given by $AIC = [-2 \ln(\text{maximized likelihood}) + 2(5)]/1500$. The results are 15.02 and 15.06 for the Gumbel-Hougaard and Frank copula models, respectively. The smaller AIC value for the Gumbel-Hougaard model indicates that this model is preferred.

4.3 Uses of the Bivariate Fit

This section describes two applications using the estimated bivariate distribution of losses and expenses.

TABLE 5
BIVARIATE DATA PARAMETER ESTIMATES USING GUMBEL-HOUGAARD'S COPULA WITH PARETO MARGINALS

	Parameter	Bivariate Distribution		Univariate Distribution	
		Estimate	Standard Error	Estimate	Standard Error
Loss (X_1)	λ_1	14,036	1,298	14,453	1,397
	θ_1	1.122	0.062	1.135	0.066
ALAE (X_2)	λ_2	14,219	1,426	15,133	1,633
	θ_2	2.118	0.153	2.223	0.175
Dependence	α	1.453	0.034	Not applicable	Not applicable

TABLE 6
BIVARIATE DATA PARAMETER ESTIMATES USING FRANK'S COPULA WITH PARETO MARGINALS

	Parameter	Bivariate Distribution		Univariate Distribution	
		Estimate	Standard Error	Estimate	Standard Error
Loss (X_1)	λ_1	14,558	1,390	14,453	1,397
	θ_1	1.115	0.065	1.135	0.066
ALAE (X_2)	λ_2	16,678	1,824	15,133	1,633
	θ_2	2.309	0.187	2.223	0.175
Dependence	α	-3.158	0.174	Not applicable	Not applicable

4.3.1 Calculating Reinsurance Premiums

After having identified the joint distribution of (X_1, X_2) , we can examine the distribution of any known function of X_1 and X_2 , say, $g(X_1, X_2)$. To illustrate, consider a reinsurer's expected payment on a policy with limit L and insurer's retention R . Then, assuming a pro-rata sharing of expenses, we have

$$g(X_1, X_2) = \begin{cases} 0 & \text{if } X_1 < R \\ X_1 - R + \frac{X_1 - R}{X_1} X_2 & \text{if } R \leq X_1 < L \\ L - R + \frac{L - R}{L} X_2 & \text{if } X_1 \geq L \end{cases}$$

The expected payment $Eg(X_1, X_2)$ could be calculated using numerical integration when the joint density of losses and expenses is available. However, simulation is a simpler, numerical evaluation tool. The procedure for simulation is described in Section 3.3.

The idea with simulation is to generate a sequence of bivariate data (x_{1i}, x_{2i}) from the bivariate distribution model. The procedure for the Gumbel-Hougaard copula is summarized in Algorithm 3.3, using Equation (3.6) for Equation (3.5). In the procedure, the inverse of the marginal distribution functions is needed. In the Pareto case, it is not difficult to verify that $F^{-1}(x) = \lambda[(1-x)^{-1/\theta} - 1]$.

The simulation steps outlined in Algorithm 3.3, using the Gumbel-Hougaard copula, are repeated a large number of times. Let $nsim$ be the number of simulations performed so that we generate the sequence of sample (x_{1i}, x_{2i}) , $i=1, \dots, nsim$. Thus, the estimated value for the reinsurer's expected payment is given by

$$\hat{g}^*(L, R) = \frac{1}{nsim} \sum_{i=1}^{nsim} g(x_{1i}, x_{2i}),$$

with standard error

$$se(\hat{g}^*(L, R)) = \sqrt{\frac{\frac{1}{nsim} \sum_{i=1}^{nsim} g(x_{1i}, x_{2i})^2 - \hat{g}^*(L, R)^2}{nsim}}.$$

We performed a simulation study of size $nsim=100,000$; the results are summarized in Table 7.

The results in Table 7 provide the adjusted premiums the reinsurer would have assessed to cover costs of losses and expenses according to various policy limits and ratios of insurer's retention to policy limit. On a crude basis, these results appear to make sense. For example, from the summary statistics in Table 4, the average policy limit is 559,098 with an average of losses plus expenses of 53,796. Without any reinsurance, our results indicate an adjusted premium of 49,367, with a standard error of 733, for a policy limit of 500,000. Furthermore, the results are intuitively appealing because as expected we observe (1) higher premium for larger policy limits and (2) lower premium when the ratio R/L is higher, that is, insurer retains larger amount of losses.

Because it is common practice to assume independence, we provide Table 8, which gives ratios of dependence to independence reinsurance premiums. The dependence assumption is based on the Gumbel-Hougaard estimation results, while the independence assumption is based on the estimation results when the joint distribution is assumed to be the product of the marginals. In previous sections, we argue that the estimation results using dependence are statistically significant. Table 8 shows that substantial mispricing could result if the unrealistic assumption of independence between losses and expenses is made. A ratio below 1.0 from the table suggests an overvalued reinsurance premium; a ratio above 1.0 suggests undervalued premiums. According to the

TABLE 7
SIMULATION-BASED REINSURANCE PREMIUMS (SIMULATION STANDARD ERRORS ARE IN PARENTHESIS)

Policy Limit (L)	Ratio of Insurer's Retention to Policy Limit (R/L)				
	0.00	0.25	0.50	0.75	0.95
10,000	15,636 (640)	11,232 (480)	7,220 (320)	3,498 (160)	684 (32)
100,000	34,264 (655)	17,965 (493)	10,003 (328)	4,425 (164)	819 (33)
500,000	49,367 (733)	17,457 (544)	9,234 (359)	4,007 (179)	739 (36)
1,000,000	55,683 (818)	16,762 (597)	8,740 (390)	3,716 (193)	672 (38)

TABLE 8
SIMULATION-BASED RATIOS OF DEPENDENCE TO INDEPENDENCE
REINSURANCE PREMIUMS

Policy Limit (L)	Ratio of Insurer's Retention to Policy Limit (R/L)				
	0.00	0.25	0.50	0.75	0.95
10,000	0.80	0.95	1.02	1.07	1.10
100,000	0.89	1.24	1.36	1.44	1.50
500,000	0.92	1.31	1.40	1.47	1.52
1,000,000	0.93	1.31	1.39	1.47	1.53

table, undervalued premiums result from higher retention by the reinsured. This undervaluation is more important for higher policy limits. The greatest overvalued premiums are for large retention levels and policy limits. This is intuitively plausible; pricing in the tail of the distribution is very sensitive to model misspecification.

4.3.2 Estimating Regression Functions

As described in Section 1, the regression function is the most widely used tool for describing multivariate relationships. To illustrate in the context of losses and claims, we examine situations in which it is useful to estimate the expected amount of expenses for a given level of loss. Copulas can help us understand the full joint distribution and thus be used to address some important applications described in Sections 2, 4.3.1, and 5. We can also use copulas to define regression functions, as follows.

To be specific, let us assume an Archimedean form of the copula as in Section 3.3 so that the conditional distribution of X_k given X_1, \dots, X_{k-1} is

$$F_k(x_k | x_1, \dots, x_{k-1}) = \frac{\phi^{-1(k-1)} \{c_{k-1} + \phi[F_k(x_k)]\}}{\phi^{-1(k-1)} (c_{k-1})},$$

where $c_k = \phi\{F_1(x_1)\} + \dots + \phi\{F_k(x_k)\}$. Basic results from mathematical statistics show that the regression function can be expressed as

$$\begin{aligned} E(X_k | x_1, \dots, x_{k-1}) &= \int_0^\infty [1 - F_k(x | x_1, \dots, x_{k-1})] dx \\ &\quad + \int_{-\infty}^0 F_k(x | x_1, \dots, x_{k-1}) dx. \end{aligned}$$

In certain situations, this expression is convenient to evaluate. For example, assuming $k=2$ and using uniform marginals and Frank's copula, Genest (1987) gave the regression function

$$E(X_2 | X_1 = x) = \frac{(1 - e^{-\alpha})xe^{-\alpha x} + e^{-\alpha}(e^{-\alpha x} - 1)}{(e^{-\alpha x} - 1)(e^{-\alpha} - e^{-\alpha x})}.$$

In general, however, the calculation of the regression function is tedious. As an alternative, copulas are well-suited to the concept of "quantile" regression. Here, in lieu of examining the mean of a conditional distribution, one looks at the median or some other percentile (quantile) of the distribution. Specifically, define the p -th quantile to be the solution x_p of the equation:

$$p = F_k(x_p | x_1, \dots, x_{k-1}).$$

For the case $k=2$, we have

$$p = F_2(x_p | X_1 = x_1) = C_1[F_1(x_1), F_2(x_p)]. \quad (4.5)$$

The first partial derivative for the case of the Gumbel-Hougaard copula is derived in Appendix A. For the case of Frank copula, the first partial derivative is given by

$$C_1(u, v) = \frac{e^{\alpha v}(e^{\alpha v} - 1)}{e^{\alpha} - 1 + (e^{\alpha u} - 1)(e^{\alpha v} - 1)}. \quad (4.6)$$

For a complete derivation of partial derivatives of the Frank's copula, see also Frees et al. (1996).

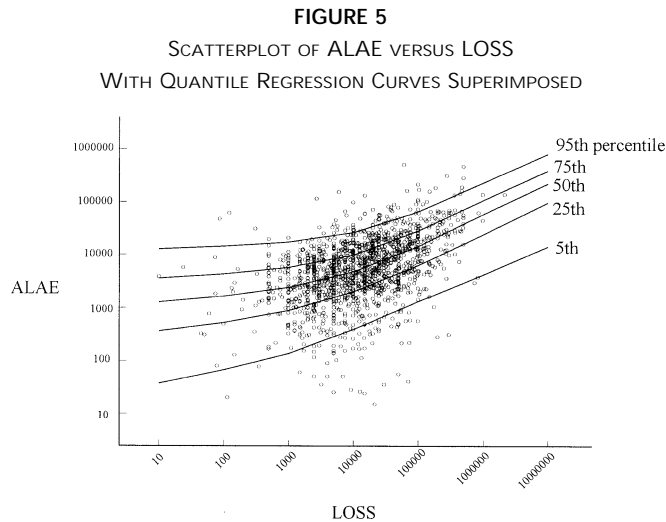
Thus, for a specified proportion p and amount of loss x_1 , we can find the percentile of the corresponding expenses by solving Equation (4.5). For the case of the Gumbel-Hougaard copula, we use Equation (A.2) to get the p -th percentile of the expense level, which is given by

$$x_p = F_2^{-1}(v_p),$$

where v_p is the solution to the following equation:

$$\left(\frac{\ln F_1(x_1)}{\ln C(F_1(x_1), v_p)} \right)^{\alpha-1} \frac{C(F_1(x_1), v_p)}{F_1(x_1)} = p.$$

For various percentiles, Figure 5 graphically displays the result of regression curves that provide estimates of expenses conditional on the amount of loss using the Gumbel-Hougaard copula. We superimposed these regression curves on the scatterplot of losses and expenses. This plot allows a manager to estimate expenses for a prespecified loss amount. By providing several percentiles, the manager can choose the degree of conservatism that is appropriate for the business decision at hand.



5. ADDITIONAL APPLICATIONS OF COPULAS

We now discuss three different subject areas useful to the actuary in which copulas have been applied: stochastic ordering, fuzzy logic, and insurance pricing. Stochastic ordering refers to relationships among distribution functions of random variables. Fuzzy logic is an approach for dealing with uncertainty, analogous to probability theory. All actuaries are familiar with insurance pricing, which involves premium calculation.

5.1 Stochastic Ordering

In actuarial science and the economics of decision-making, the comparison of the attractiveness of various risks is usually of interest; therefore the subject of stochastic orderings is of prime importance. To illustrate, we say the random variable X_1 *stochastically dominates* the random variable X_2 with respect to a class of functions, say U , if for any function $u \in U$, we have

$$Eu(X_1) \geq Eu(X_2).$$

Often, u is the utility-of-wealth function that defines risk preferences of decision-makers. As special cases, we consider the well-known classes of first (FSD) and second (SSD) stochastic dominance. In FSD, the class U is taken to be the class of increasing utility functions. In SSD, U is the smaller class of increasing, concave utility functions, applicable to risk-averse decision-makers; see Kaas et al. (1994), Heilman and Schroter (1991), and Gooavaerts et al. (1982).

Copulas are used in the analysis for the demand of insurance coverage studied by Tibiletti (1995). The

problem is to find the optimal insurance coverage for a decision-maker when a proportion of the wealth is uninsurable. Suppose that X_1 is the amount of uninsurable asset and X_2 is the insurable loss with $0 \leq x_2 \leq m$; that is, m is the maximum value of the insurable asset. In a two-period model, define the final wealth to be

$$Z = X_1 + m - X_2 + \delta(X_2 - p). \quad (5.1)$$

Here, δ is the coinsurance rate and p is the premium rate. In this insurance setup, the questions typically explored are: (1) what coinsurance rate δ maximizes $E(Z)$ and (2) when do certain “beneficial” changes that increase expected utility affect optimal coverage? Tibiletti explored beneficial changes such as (1) a shift in the distribution of either X_1 or X_2 , (2) a change in the dependence between X_1 and X_2 , and (3) a change in both (1) and (2). Because dependence comes into play, copulas provide a natural tool in this situation. Most of the theorems described and proved use conditions that involve a certain dependence ordering called *more concordance*. If (X_1, X_2) and (Y_1, Y_2) are pairs with associated copulas C_x and C_y , then the pair (X_1, X_2) is said to be *more concordant* than (Y_1, Y_2) if $C_x(a, b) \geq C_y(a, b)$ for all $(a, b) \in [0, 1]^2$.

The *more concordant* ordering is just one of several types of dependence ordering used to order multivariate distributions. See Tchen (1980), Kimeldorf and Sampson (1987, 1989), and Metry and Sampson (1991) for alternative orderings. For example, another type is the so-called *more regression dependent* ordering. Here, the random vector (X_1, X_2) is said to be *more regression dependent* than (Y_1, Y_2) if for any $x_1' > x_1$, we have

$$\frac{\Pr(X_2 \leq x_2 | X_1 = x_1')}{\Pr(Y_2 \leq y_2 | Y_1 = x_1')} \geq 1$$

whenever we have

$$\frac{\Pr(X_2 \leq x_2 | X_1 = x_1)}{\Pr(Y_2 \leq y_2 | Y_1 = x_1)} \geq 1.$$

See Bilodeau (1989). Dependence ordering is particularly useful for determining the range of possible dependence in multivariate random variables. Other areas in which stochastic orderings are useful are probability theory, reliability, and operations research; see Mosler and Scarsini (1991).

5.2 Fuzzy Sets

Like probability theory, fuzzy set theory deals with uncertainty. To start, let U be a non-empty set whose

subsets are of primary interest; in fuzzy set theory, it is called the universe of discourse. A fuzzy set is defined to be a pair (E, μ_E) where $E \subset U$ and $\mu_E: U \rightarrow [0, 1]$. For any x in U , $\mu_E(x)$ denotes the degree to which x belongs to the fuzzy set, almost like a probability.

Operations, such as unions and intersections, on fuzzy sets are common. To illustrate, if we have two fuzzy sets (A, μ_A) and (B, μ_B) , then their union is the fuzzy set (C, μ_C) where $\mu_C(x) = \max[\mu_A(x), \mu_B(x)]$ and their intersection is the fuzzy set (D, μ_D) with $\mu_D(x) = \min[\mu_A(x), \mu_B(x)]$. Other operations on fuzzy sets are performed using triangular norms. Triangular norms were considered originally in the context of probabilistic metric spaces and provide the link to copulas.

Following Ostaszewski (1993) and Klement (1982a, 1982b), a triangular norm (t-norm for short), T is a mapping $T: [0, 1]^2 \rightarrow [0, 1]$ with the following properties: (1) boundary condition: $T(x, 1) = x$; (2) monotonicity: $T(x, y) \leq T(u, v)$ whenever $x \leq u, y \leq v$; (3) commutativity: $T(x, y) = T(y, x)$; and (4) associativity: $T(T(x, y), z) = T(x, T(y, z))$. In Schweizer and Sklar (1983), several results are given that relate triangular norms and copulas. Many copulas can serve as triangular norms, as the following examples illustrate.

Example 5.1 Frechet Bounds

$$T(x, y) = \min(x, y) \text{ and } T(x, y) = \max(x + y - 1, 0).$$

Example 5.2 Independence

$$T(x, y) = xy.$$

Example 5.3 Frank

$$T(x, y) = \frac{\ln 1 + \frac{(\eta^x - 1)(\eta^y - 1)}{\eta - 1}}{\ln \eta}$$

where $\eta > 0, \eta \neq 1$.

Therefore, by considering familiar copulas, as well as newly constructed copulas, we can define new operations that can be performed on fuzzy sets. Fuzzy set theory has been applied in specific areas of interest to many actuaries such as risk economics, interest theory, and underwriting/classification of risks. See Ostaszewski (1993), LeMaire (1990), and Young (1993) for descriptions of these applications.

5.3 Insurance Pricing

The calculation of a premium to be assigned to an insurance risk X is a fundamental job of the actuary. Recently, Wang (1996) developed a principle that exhibits several desirable properties of a premium principle. Wang (1997) used the concept of a distortion function and extended this, using copulas, to a portfolio of dependent risks.

Let X be an insurance risk with survival function $S(x) = \Pr(X > x)$. A class of premium principles can be defined by $\pi(X) = \int_0^\infty g[S(z)] dz$, where g , called the distortion function, is increasing with $g(0) = 0$ and $g(1) = 1$. In the special case of the function $g(t) = t^{1/\eta}$ with $\eta \geq 0$, we have the class of *proportional hazard (PH) transforms*.

For the bivariate case, consider a random vector (X_1, X_2) with distribution function $F(x_1, x_2) = C[F_1(x_1), F_2(x_2)]$; then the function $\tilde{F}(x_1, x_2) = g[F(x_1, x_2)]$ is another joint distribution function with marginals $g(F_1)$ and $g(F_2)$. The associated copula is therefore $\tilde{C}(u_1, u_2) = g[C(g^{-1}(u_1), g^{-1}(u_2))]$. Some illustrations of distortion functions follow.

Example 5.4 Power Distortion Function

Here, we have $g(t) = t^{1/\eta}$, where $\eta \geq 0$. The independence structure is preserved under this distortion function. To see this, note that if $C(u_1, u_2) = u_1 u_2$, then $\tilde{C}(u_1, u_2) = g[C(g^{-1}(u_1), g^{-1}(u_2))]$.

Example 5.5 Exponential Distortion Function

Here, we have

$$g(t) = \frac{1 - e^{-\eta t}}{1 - e^{-\eta}},$$

where $\eta > 0$. Again, beginning with the independence copula $C(u_1, u_2) = u_1 u_2$, it is straightforward to show that we can generate Frank's family of copulas

$$\tilde{C}(u_1, u_2) = \ln \frac{1 + \frac{(\eta^{u_1} - 1)(\eta^{u_2} - 1)}{\eta - 1}}{\ln \eta}.$$

Using the concept of copulas, Wang (1997) extends the PH measure to a portfolio of risks. In many situations, for a portfolio of risks, the proposed measure is useful because individual risks are considered dependent. Wang justified dependence of individual risks by stating that they are "influenced by the same underlying market environment."

6. SUMMARY AND CONCLUSIONS

In analyzing the impact of future contingent events, actuaries are faced with problems involving multivariate outcomes. In this paper, we review the problems of (1) estimating distributions of joint lifetimes of paired individuals, useful in the analysis of survivorship insurance protection, and (2) investigating mortality experience, for the actuary who needs to distinguish among causes of death. We introduced, and provided a solution for, the problem of dependence between an insurer's losses and expenses. Failures of ignoring dependencies can lead to mispricing. Thus, it is important for actuaries to be able to adequately model multivariate outcomes.

The tool used to study multivariate outcomes is the copula function; it couples univariate marginals to the full multivariate distribution. The biological frailty models and the mathematical Archimedean models can motivate copulas. A statistical mixture of powers model serves as a bridge between these two sets of families.

Because copulas are parametric families, standard techniques such as maximum likelihood can be used for estimation. Other statistical tools have been recently developed to help fit copulas. We described a graphical tool to identify the form of the copula. We discussed how copulas could be used to simulate multivariate outcomes, an important tool for actuaries. We also developed the connection between copulas and the regression function, a widely used tool for summarizing what we expect based on conditional distributions.

This article has focused on the connection between copulas and statistics, the theory of data. Yet, much of the development of copulas has historically arisen from probability theory. To recognize this connection, we briefly reviewed topics in applied probability theory pertaining to copulas that are of the greatest interest to actuaries: stochastic ordering, fuzzy set theory, and insurance pricing.

Our knowledge of copulas has been rapidly developing recently. Many of the articles cited in this review paper were written in the 1990s. In another recent survey paper, Kotz (1997) cites three conferences in the last five years that were largely devoted to copulas. Copulas offer analysts an intuitively appealing structure, first for investigating univariate distributions and second for specifying a dependence structure. Copulas offer a flexible structure that can be applied in many situations. We hope that this

article encourages actuaries to seek new applications for this promising tool.

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- This paper studies bivariate survival data where the risks may be dependent but do not preclude one another. Each of the two risks is subject to censoring by an independent mechanism. The dependency between risks is modeled using a one-parameter copula function. The marginal survival functions are estimated (i) nonparametrically, using Kaplan-Meier estimators and (ii) parametrically. A two-stage estimation procedure is proposed; both finite and asymptotic properties are established. An AIDS dataset illustrates the procedures.
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- The author is one of the pioneers of copulas. In this paper, he proves elementary results that relate copulas to distribution functions and random variables. In particular, given a copula function C and an n -tuple distribution function (F_1, \dots, F_n) , he establishes the existence of a probability space and a set of random variables X_1, \dots, X_n defined over that space with C as the associated copula. In addition, the paper describes structural properties of a special type of copula called associative copulas.
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- This paper provides mechanisms for extracting the dependence information from parameterized copulas. The author defines various dependence functions to capture dependence information. Approximations to copulas using Taylor's expansion are discussed. Most of the derivations relate to two-dimensional copulas although a brief attempt to extend to three-dimensional copulas is made.
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- For bivariate survival data, it is well-known that without assumptions about the relationship between the competing survival times, the marginal survival functions are unidentifiable. This article assumes a known copula and introduces nonparametric estimators of the marginal survival functions. The estimates reduce to the Kaplan-Meier product limit estimator in the case of independent survival times. Further, bounds on the survival functions are provided when the copula is known only approximately.

APPENDIX A

GUMBEL-HOUGAARD COPULA AND ITS PARTIAL DERIVATIVES

In this appendix, we derive the formulas needed to evaluate the likelihood in Equation (4.3) in the case in which we apply the Gumbel-Hougaard copula as given in Table 1. First, we re-express the Gumbel-Hougaard copula as follows:

$$[-\ln C(u, v)]^\alpha = (-\ln u)^\alpha + (-\ln v)^\alpha.$$

For simplicity, we denote $C(u, v)$ by C . Now, take the partial derivative with respect to u of both sides of the equation to get:

$$\frac{(-\ln C)^{\alpha-1}}{C} \frac{\partial C}{\partial u} = \frac{(-\ln u)^{\alpha-1}}{u}. \quad (\text{A.1})$$

Solving for $\partial C / \partial u$, we have:

$$\frac{\partial C}{\partial u} = \left(\frac{\ln u}{\ln C} \right)^{\alpha-1} \frac{C}{u}. \quad (\text{A.2})$$

Applying symmetry, we get:

$$\frac{\partial C}{\partial v} = \left(\frac{\ln v}{\ln C} \right)^{\alpha-1} \frac{C}{v}. \quad (\text{A.3})$$

From Equation (A.1), we can take the partial derivative with respect to v to get:

$$\frac{(-\ln C)^{\alpha-1}}{C} \frac{\partial^2 C}{\partial u \partial v} - \frac{\partial C}{\partial u} \frac{\partial C}{\partial v} \frac{(-\ln C)^{\alpha-1}}{C^2} [(\alpha-1)(-\ln C)^{-1} + 1] = 0. \quad \text{Equations (A.2), (A.3), and (A.4) are used to evaluate the log-likelihood given in Equation (4.3).}$$

$$\frac{\partial^2 C}{\partial u \partial v} = \frac{1}{C} \frac{\partial C}{\partial u} \frac{\partial C}{\partial v} [(\alpha-1)(-\ln C)^{-1} + 1]. \quad (\text{A.4})$$

Rearranging terms yields the second partial derivative of the Gumbel-Hougaard copula:

APPENDIX B

TABLE OF ONE-PARAMETER FAMILY OF COPULAS

Family	General Form of Copula	Parameter Constraint	Kendall's Tau	Spearman's Rho
Ali-Mikhail-Haq	$uv[1 - \alpha(1-u)(1-v)]^{-1}$	$-1 \leq \alpha \leq 1$	$\left(\frac{3\alpha-2}{\alpha}\right) - \frac{2}{3}\left(1 - \frac{1}{\alpha}\right)^2 \ln(1-\alpha)$	Complicated form
Cook-Johnson	$[u^{-\alpha} + v^{-\alpha} - 1]^{-1/\alpha}$	$\alpha \geq 0$	$\frac{\alpha}{\alpha+2}$	Complicated form
Farlie-Gumbel-Morgenstern	$uv[1 + \alpha(1-u)(1-v)]$	$-1 \leq \alpha \leq 1$	$\frac{2}{9}\alpha$	$\frac{1}{3}\alpha$
Frank	$\frac{1}{\alpha} \ln \left[1 + \frac{(e^{\alpha u} - 1)(e^{\alpha v} - 1)}{(e^{\alpha} - 1)} \right]$	$\alpha \neq 0$	$1 - \frac{4}{\alpha} [D_1(-\alpha) - 1]$	$1 - \frac{12}{\alpha} [D_2(-\alpha) - D_1(-\alpha)]$
Gumbel-Hougaard	$\exp\left\{-[(-\ln u)^{\alpha} + (-\ln v)^{\alpha}]^{1/\alpha}\right\}$	$\alpha \geq 1$	$1 - \alpha^{-1}$	No closed form
Normal (Bivariate)	$H(\Phi^{-1}(u), \Phi^{-1}(v))$ where H is the bivariate normal distribution function with correlation coefficient α and Φ^{-1} is the inverse of a univariate normal distribution function.	$-1 \leq \alpha \leq 1$	$\frac{2}{\pi} \arcsin(\alpha)$	$\frac{6}{\pi} \arcsin\left(\frac{\alpha}{2}\right)$
Plackett	$\frac{1}{2}(\alpha-1)^{-1} \left\{ 1 + (\alpha-1)(u+v) - \left[(1 + (\alpha-1)(u+v))^2 + 4\alpha(1-\alpha) \right]^{1/2} \right\}$	$\alpha \geq 0$	No closed form	$\frac{(\alpha+1)}{(\alpha-1)} - \frac{2\alpha \ln \alpha}{(\alpha-1)^2}$