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We present a mathematical model for stock market volatility flocking. Our proposed model consists of geometric Brownian motions with time-varying volatilities coupled with Cucker–Smale (C–S) flocking and regime switching mechanisms. For all-to-all interactions, we assume that all assets' volatilities are coupled to each other with a constant interaction weight, and we show that the common volatility emerges asymptotically

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and discuss its financial applications. We also provide several numerical simulations and compare them to existing analytical results.

Keywords: Cucker–Smale model; geometric Brownian motion; flocking; regime switching; volatility; Wiener process.

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1. Introduction

Collective behaviors of complex systems such as aggregation, flocking, and herding are often observed in biological systems[5–8,18,19,21,23,43,44,47] and human society.[9,10,13,24–26,34,46] Herein, the term "*flocking*" is used to denote the collective phenomena in which self-propelled agents are organized from disordered states into an ordered state using only simple rules. The systematic study of mathematical flocking models began only two decades ago by statistical physicists Vicsek and his collaborators.[45] Vicsek studied the flocking modeling of planar self-propelled particles moving with a unit speed constraint. Later, Vicsek's planar model was generalized by Cucker and Smale.[19] Cucker and Smale introduced a simple second-order Newton-like model called the Cucker–Smale (C–S) flocking mechanism and provided several sufficient conditions to guarantee asymptotic flocking, which depends on the spatial decay rate of communication weights. Other flocking models have also been proposed[7,21,43,44] to model the swarming of fish and aggregation of bacteria.

Among the many conceivable applications of the C–S flocking mechanism, our interest is in its application to finance. In particular, we focus on the flocking pattern between stock returns' volatilities. For this, we first express stock price movement in geometric Brownian motion following the conventional model[36]:

$$dS = \mu S dt + v S dW,$$

where S is the stock price, μ is the expected rate of return, and v is the variance of the stock price change. A stock return's variance or volatility represents the risk in uncertain stock returns, which is compensated by a risk premium. Ross[40] has argued that volatility contains a noteworthy implication for understanding the rate of information flow between market participants. We find that stock return volatilities tend to flock to each other, i.e. a degree of correlation exists, and the degree of flocking corresponds to the probability of a stock market crash. When a piece of information is released (for example, a crash-related rumor), a market easily becomes chaotic and fragile, rendering it difficult to examine firm-specific news for determining a fair price of a stock; therefore, the current stock price becomes highly volatile. More importantly, in fact, many stocks follow the same pattern resulting in a high correlation between volatilities. Although a few papers[14,31,35] study the correlation between volatilities, to the best of our knowledge, this paper is the first attempt to find and model volatility flocking in association with the discontinuous information flow and communication between particles or agents in the C–S mechanism.

As a first step in modeling the flocking of volatilities, Bae and his collaborators[1] studied a generalized Heston model with stochastic volatility whose square process follows noisy C–S flocking dynamics. More precisely, for N stock returns, let S_i and $\sqrt{v_i}$ be the stock return and volatility of the ith stock return, respectively. The generalized Heston model[1] is as follows:

$$\begin{aligned} dS_i &= \mu S_i dt + \sqrt{v_i} S_i dW_i^s, \quad i = 1, \ldots, N, \ \ t > 0, \\ dv_i &= \frac{K}{N}\sum_{j=1}^{N} \psi_{ji}(v_j - v_i)dt + \gamma_i \sqrt{v_i} dW_i^v, \end{aligned} \tag{1.1}$$

where ψ_{ji} is a communication weight between the ith and jth volatilities, $W_i^s(t)$ and $W_i^v(t)$ are one-dimensional Brownian motions under the risk neutral measure with correlation $\rho_i \in [-1, 1]$, and each W_i^v, $i = 1, \ldots, N$, is an independent and identically distributed ($i.i.d.$) one-dimensional Brownian motion. It is well known that the sample paths of (1.1) are continuous[39]; therefore, it does not model the random discontinuous change of a volatility process as in Ref. 38 due to the discontinuous arrival of information.

To model such discontinuous flocking patterns, we employ a regime switching idea that is pioneered in Refs. 32 and 38. In particular, we rely on the fact that some volatilities tend to flock to others while other volatilities are free from flocking. By borrowing the model of Naik,[38] we assume that one volatility series follows a hidden Markov process, $\varepsilon_t, t \in [0, T]$; that is, in either a normal state, 0, or a flocking state, 1. A set of processes that change their deterministic structure at random points in time evolves by a two-state Markov chain. A formulation of Markov systems with a random switch within a certain number of deterministic states at random time appear in many research areas such as operation research, stochastic hybrid system, statistical physics, and financial mathematics. More detailed review and analysis can be found in Ref. 4.

A similar idea to the regime switching, the piecewise deterministic process, was introduced earlier in Ref. 20. As we did in this paper, especially, the two-state or dichotomic model was studied in Ref. 37. This clearly improves the authors' earlier modeling[1] on the stochastic flocking volatilities process in which the interactions between square roots of the volatility process are modeled via the C–S flocking mechanism only. In Ref. 4, they reported numerical results of their Fokker–Planck (FP) control applied to a linear-filtered process to study the sensitivity of the model with respect to the transition probability. Our sensitivity analysis on the C–S flocking mechanism according to the transition rule in (1.1) is provided in Sec. 5. The results demonstrate the noteworthy role of a transition parameter in a hidden Markov process in determining the degree of flocking.

In the absence of noise in the evolution of the volatility process, our model for the spot evolution of N assets' prices S^i and volatility $V^i(t, \varepsilon_t)$ is as follows:

$$dS^i = \mu^i(t)S^i dt + V^i(t,\varepsilon_t)S^i dW^i, \quad i = 1,\dots,N, \ t > 0,$$

$$\frac{d}{dt}V^i(t,\varepsilon_t) = \begin{cases} 0, & \varepsilon_t = 0, \\ \dfrac{K}{N}\displaystyle\sum_{j=1}^{N}\psi_{ji}(V^j(t,\varepsilon_t) - V^i(t,\varepsilon_t)), & \varepsilon_t = 1, \end{cases} \tag{1.2}$$

where for each asset $i = 1,\dots,N$, μ^i is the expected rate of return, dW^i is a Wiener process, and K is the coupling strength. Note that the first equation in (2.3) is the geometric Brownian motion of stock return S^i and the second is its volatility equation. The statistical estimates of (2.3) will be given in Sec. 3.

The movement of an individual volatility is determined by its communication with an environment where the amount of news arrives stochastically. This communication contains stochastic noise. We model this empirical regularity by introducing stochastic C–S dynamics governed by a multi-dimensional white noise process. By considering the communication weight ψ_{ij} in (2.2) as the decomposition of a deterministic and stochastic term, we provide a volatility model with noise given by (4.2). This is the second main model introduced in this paper. We analyze this model in Sec. 4, and a numerical proof for general Brownian motion is given in Sec. 5.

In summary, we provide a mathematical model for a stock return's volatility that incorporates three key concepts: flocking, regime switching, and geometric Brownian motion, and rigorous mathematical analysis for the large-time behavior of the proposed model.

The rest of this paper is organized as follows. In Sec. 2, we review a first-order C–S flocking model, discuss its relationship to phenomenological volatility flocking, and discuss our proposed model (2.3). In Sec. 3, we present the stochastic movements of stock returns' volatilities based on regime switching and the interacting geometric Brownian process. Moreover, we study the flocking dynamics of our model for a two-stock market in the absence of noise in the dynamics of volatilities (Theorem 3.1). In Sec. 4, we consider the effect of stochastic noise on the deterministic dynamics of volatilities in (2.3) and provide a sufficient condition for the strong stochastic flocking estimate (Theorem 4.1). In Sec. 5, we present several numerical experiments to validate analytic results. In Sec. 6, we present empirical implications of these numerical results. Finally, Sec. 7 is devoted to summarizing our main results. In Appendices A and B, we provide some elementary estimates used in Secs. 3 and 4.

2. Preliminaries

In this section, we first briefly review the C–S flocking mechanism introduced by Cucker and Smale[19] and discuss the phenomenology for the volatility flocking often observed in financial markets. Furthermore, we also recall a regime switching idea[38] for the volatility process. Finally, we provide a mathematical model for volatility movement.

2.1. *A first-order C–S flocking model*

In the C–S flocking mechanism, the rate of change of a test state is proportional to the weighted average of relative differences between neighboring states and a test state. The weight is assumed to be a function of states. This simple C–S flocking idea has been extensively employed by various models and studied for its application to collective behaviors of multi-agent systems.[2,17,26–30,34]

Consider a financial market consisting of N stocks. These stocks interact with each other via some communication network. In this situation, it is reasonable to think that the stocks are inter-correlated; under this assumption, the volatilities often affect each other. To model the weighted averaging effects on the dynamics of volatilities due to pairwise interactions, we employ linear relaxation dynamics weighted by their respective communication weights. One very important and intriguing question is how to decipher the communication weight from the observed collective behavior.[18] In this paper, we assume that this communication weight takes a given functional form *a priori*.

Let $\zeta_i = \zeta_i(t)$ be the quantitative measure of stock return i at time t. Then the C–S dynamics of ζ_i are as follows:

$$\frac{d\zeta_i}{dt} = \sum_{j=1}^{N} \psi_{ji}(\zeta_j - \zeta_i), \quad i = 1, \ldots, N, \ \ t > 0, \tag{2.1}$$

where ψ_{ji} is the communication weight between ζ_j and ζ_i. Although deciphering ψ_{ji} from empirical observed data is important, it is often dependent on the specific situation. This issue is beyond the scope of this paper. First-order models, similar to (2.1), that address this problem can be found in Refs. 3, 11–13, 15, 16, 33 and 45.

2.2. *Modeling of volatility flocking*

In this subsection, we provide the discussion on empirical evidence of volatility-flocking observed in financial markets.

In Fig. 1, the top graph shows the volatilities of three companies (Apple Inc., Exxon Mobile Corp., and Wal-Mart Stores Inc.) from January 1990–January 2014. In the middle panel, the co-movements of the implied volatilities are captured during recessionary periods that are published by the National Bureau of Economic Research (NBER) from April 2001–November 2001 and January 2008–June 2009. When the movements are drawn from the percent changes in stock (closing) prices (two bottom graphs in Fig. 1), volatilities tend to flock to each other more strongly during these periods. During these times, the correlation between volatilities increases; this strong co-movement is in contrast to the co-movement over the entire period. When a market becomes speculative by highly volatile returns,

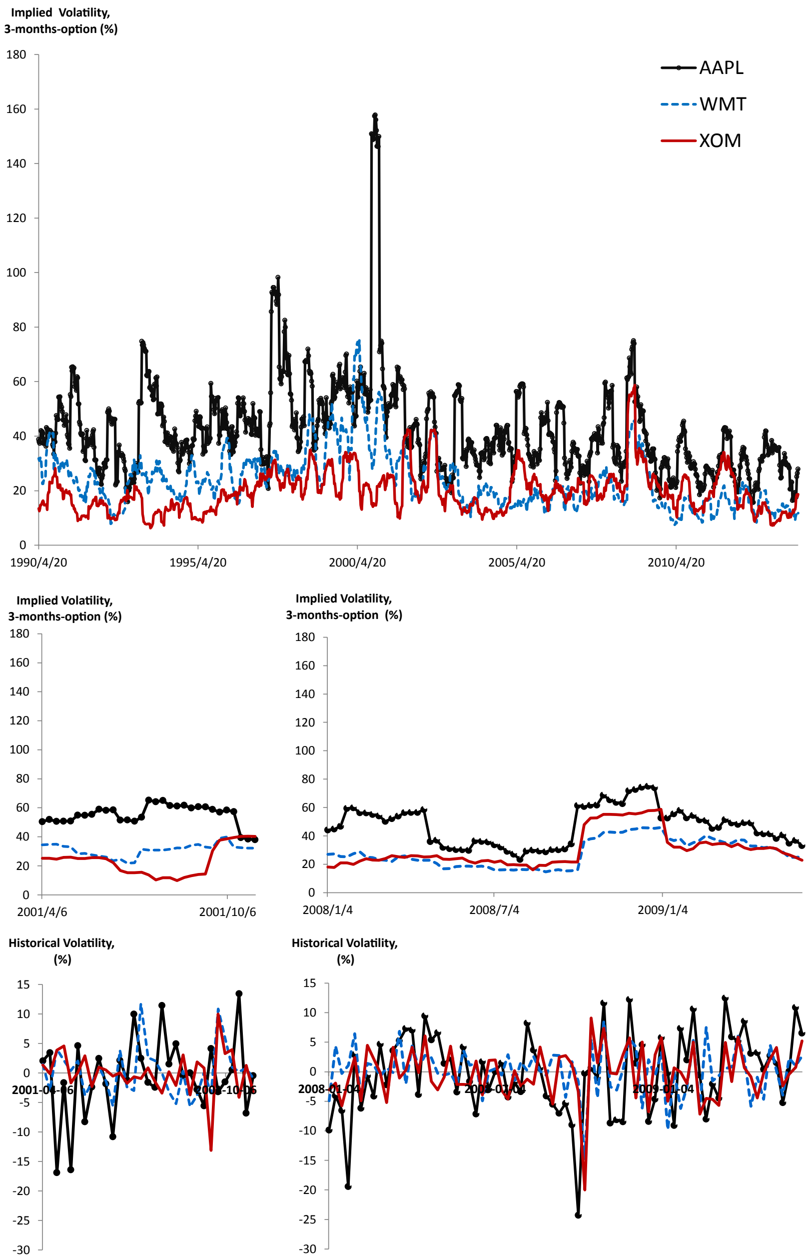


Fig. 1. Volatilities of Apple Inc., Exxon Mobile Corp., and Wal-Mart Stores Inc. Two types of volatilities are used: the implied volatility from a three-month-option price and the historical volatility obtained from the percentage change in stock prices (weekly close). Among the firms listed in the S&P 500, we chose the top three firms based on market capitalization size: Apple Inc. (AAPL with market cap size of $559,002 million), Exxon Mobile Corp. (XOM, $408,778 million), and Wal-Mart Stores Inc. (WMT, $208,358 million). Note that these values are current as of April 30, 2013. *Source*: Thompson Reuters.

or perhaps by a chaotic environment, there is intense interaction between volatilities (volatility flocking). In this flocking state, it becomes difficult to obtain a consensus of a stock price based solely on the fundamentals because traders either add momentum to price changes or cause prices to overshoot. Thus, the correlation between stock volatilities becomes higher even when those stocks are in heterogeneous sectors.

2.3. *Regime switching in volatility and mathematical modeling*

In this subsection, we discuss the regime switching concept,[4,20,32,37,38] and present a mathematical model for volatility flocking that incorporates a regime switching mechanism. To gain insight into regime switching for volatility modeling, we first consider Naik's remarks[38] on the spirit of random change of volatilities between low and high regimes:

> "*Stock price movements in our model fluctuate randomly between low and high volatility regimes. Discontinuous changes in the volatility of stock prices may be a result of significant changes in a firm's operating and financial structure, its competitive environment, or its corporate plan. Reorganizations caused by takeovers and mergers may also occasion such volatility shifts.*"

In this paper, we assume two market regimes represent the state of information in the investor community: a normal state and a flocking state. Based on the two-state regime switching idea,[38] we define a two-state continuous-time Markov chain $\varepsilon_t, t \in [0, T]$, with two states denoted by 0 and 1. We borrow Hamilton's volatility approach[32] and apply regime switching to explain the transition between discrete regimes. A sudden perturbation and its effect are described by a regime-switch, where the motion of a volatility is subject to a different interaction rule in a different regime. In the financial context, news on an unexpected event triggers the abrupt change for the structure of the deterministic flocking motion at random points in time by changing the information distribution in a financial market. We focus on the fact that it is difficult to detect this switching mechanism explicitly due to the hidden or variable property of the informational environment of each state. Subsequently, in our model, we develop a hidden Markov process denoted by ε_t, which follows a first-order Markov process with a homogeneous transition probability.

We now present a model based on two key modeling components: C–S flocking, with the relationship between volatility and information flow, and the regime switching mechanism. As shown in Ref. 40, the price volatility is directly related to the rate of flow of information in the market. Since the communication, or interaction, between news and information can be represented by a communication weight, we are able to model volatility dynamics using the results in Ref. 40 and the C–S flocking mechanism.

Consider N assets with volatilities V^i, $i = 1, \ldots, N$ with all-to-all communication weight $\psi_{ij} = 1$ between volatilities V^i and V^j. In the absence of regime switching, the dynamics of the volatilities is given by the following C–S type flocking model:

$$\frac{dV^i}{dt} = \sum_{j=1}^{N} (V^j - V^i), \quad t > 0, \tag{2.2}$$

where again, N is the number of assets. Here the volatilities are time-dependent functions converging to the same constant volatility asymptotically. We now consider a more general situation whether the volatilities are subject to flocking and regime switching mechanisms.

Recall the Itô process (2.3) for the spot evolution of N assets' prices S^i with corresponding volatilities $V^i(t, \varepsilon_t)$, where $i = 1, \ldots, N$:

$$\frac{dS^i(t)}{S^i(t)} = \mu^i(t)dt + V^i(t, \varepsilon_t)dW^i(t), \quad t > 0,$$

$$\frac{dV^i(t, \varepsilon_t)}{dt} = \begin{cases} 0, & \varepsilon_t = 0, \\ \dfrac{K}{N} \displaystyle\sum_{j=1}^{N} (V^j(t, \varepsilon_t) - V^i(t, \varepsilon_t)), & \varepsilon_t = 1. \end{cases}$$

Let Q be the generator matrix of the Markov chain defined by the intensities of the Poisson process; specifically, Q is defined by

$$Q = \begin{pmatrix} -p_{00} & p_{01} \\ p_{10} & -p_{11} \end{pmatrix}, \tag{2.3}$$

where p_{ij} is the transition intensity of a jump from state i to state j. If at time t, $\varepsilon_t = 0$, then it will transition to $\varepsilon_t = 1$ in the time interval $(t, t + \delta)$ with probability $p_{01}\delta$. The probability of remaining in the current state $\varepsilon_t = 0$ is given by $1 - p_{01}\delta$. Similarly, p_{10} is the transition intensity of a change from $\varepsilon_t = 1$ to $\varepsilon_t = 0$. The transition matrix of the state process in a small interval $(t, t + \delta)$ is given by $I_2 + Q\delta$, where I_2 is the 2×2 identity matrix (see Sec. 3.1 for more details).

By considering this regime switching concept, our flocking model has the ability to describe the long-run dynamics of volatilities' ensemble averages. Furthermore, the relationship between these macroscopic dynamics and an individual volatility is examined by the deviations in asymptotic ensemble averages. The suggested model successfully explains the pattern that more volatilities tend to flock to each other, even in situations where it is considerably uncertain to predict a company's stock return solely on a company's fundamental performance. To explain volatility movements in a flocking regime, we use the C–S flocking mechanism based on the previous work[19,26,27] to explain random switching across two states.

3. Volatility Dynamics: Noiseless Case

In this section, we first briefly discuss the idea of modeling for the volatility dynamics in (2.3) and study the asymptotic flocking dynamics of volatilities in a two-stock market in the absence of noises.

3.1. *Volatility dynamics*

Suppose there exist two states for the return volatilities of N assets: flocking or normal. In this case, a volatility transitions between two states — flocking state (noisy state) or non-flocking state (normal state) — according to the regime switching mechanism. A switch occurs when the random arrival of information strikes a market with intense surprise. Such a process can be described by the continuous random process $\varepsilon = (\varepsilon_t)_{t\geq 0}$, which takes the binary values for each t: $\varepsilon_t = 0$ in the normal state and $\varepsilon_t = 1$ in the flocking state. When $\varepsilon_t = 1$, interacting geometric Brownian motion is used to describe the flocking pattern. Furthermore, the volatility process $V^i(t, \varepsilon_t)$ follows a Markov chain process:

$$\frac{dV^i(t,\varepsilon_t)}{dt} = \begin{cases} 0, & \varepsilon_t = 0, \\ \dfrac{K}{N}\displaystyle\sum_{j=1}^{N}(V^j(t,\varepsilon_t) - V^i(t,\varepsilon_t)), & \varepsilon_t = 1, \end{cases} \tag{3.1}$$

where K is the positive coupling strength. A two-state continuous-time Markov chain $\varepsilon_t, t \in [0, T]$, is defined for states $\{0, 1\}$. The Markov chain ε_t is independent of the Brownian motion W_t and moves between the two states.

In Sec. 2, we have introduced the generating matrix Q of the Markov chain. The matrix Q is determined by Poisson process intensity parameters defined in (2.3). Recall that the transition matrix of the state process in an interval $(t, t + \delta)$, where δ is sufficiently small, is given by $I_2 + Q\delta$, where I_2 is the 2×2 identity matrix. The equation summarizes the transition between a normal state, $\varepsilon_t = 0$, and a flocking state, $\varepsilon_t = 1$. It is natural to assume that $p_{00} = p_{01}$ and $p_{11} = p_{10}$. In this case, the transition rate during the period $(t, t + \delta)$ for $t \geq 0$ is given by:

$$\mathbb{P}(\varepsilon_{t+\delta} = 0 \,|\, \varepsilon_t = 0) = 1 - p_{00}\delta, \quad \mathbb{P}(\varepsilon_{t+\delta} = 1 \,|\, \varepsilon_t = 0) = p_{00}\delta, \quad \delta > 0,$$
$$\mathbb{P}(\varepsilon_{t+\delta} = 0 \,|\, \varepsilon_t = 1) = p_{11}\delta, \quad \mathbb{P}(\varepsilon_{t+\delta} = 1 \,|\, \varepsilon_t = 1) = 1 - p_{11}\delta,$$

where $\mathbb{P}$ denotes the probability.

In a non-flocking regime, i.e. when $\varepsilon_t = 0$, all volatilities follow the market's long-run average. In this scenario, when a shock occurs in the market, it is difficult to identify credible information or measure the impact of the shocking information on the fair stock price. As an investor becomes sensitive to news in this uncertain market, equity prices fluctuate more than in the normal state. More interestingly, high volatility is found not only in one particular stock price, but also in many other prices because a stock's supply and demand are in association with other stock's equilibrium price when an investor generates a portfolio for multiple stocks.

We call this state a flocking regime. We also model the transition between two states/regimes using a Markov regime switching model. We borrow the notation given in Chap. 2 of Ref. 22, including the Markovian binary state denoted by ε_t with states $\{0,1\}$. We assume that for each $i = 1,\dots,N$, the volatility $V^i(t,\varepsilon_t)$ in (2.3) can be separated by the multiplication of the volatility $V^i(t)$ and the Markovian binary state ε_t; that is, $V^i(t,\varepsilon_t) = V^i(t)\varepsilon_t$.

For each $i = 1,\dots,N$, let $V_0^i(t)$ and $V_1^i(t)$ denote the volatilities of states $\varepsilon_t = 0$ and $\varepsilon_t = 1$, respectively. Moreover, let $(\varepsilon_t^0, \varepsilon_t^1)$ denote the value of the random state such that $\varepsilon_t^0 = 1$ and $\varepsilon_t^1 = 0$ when $\varepsilon_t = 0$, and $\varepsilon_t^0 = 0$ and $\varepsilon_t^1 = 1$ when $\varepsilon_t = 1$. In this case, the volatility $V^i(t,\varepsilon_t)$ is

$$V^i(t,\varepsilon_t) = V^i(t)\varepsilon_t := V_0^i(t)\varepsilon_t^0 + V_1^i(t)\varepsilon_t^1$$

and

$$dV^i(t,\varepsilon_t) = \frac{dV^i}{dt}dt + V^i d\varepsilon_t = \frac{dV^i}{dt}dt + V^i Q\varepsilon_t dt.$$

From (3.1), we have:

$$\begin{cases} \dfrac{dV_0^i(t)}{dt} = p_{10}V_1^i - p_{00}V_0^i & (\varepsilon_t = 0), \\ \dfrac{dV_1^i(t)}{dt} = \dfrac{K}{N}\displaystyle\sum_{j=1}^{N}(V_1^j(t) - V_1^i(t)) + p_{01}V_0^i - p_{11}V_1^i & (\varepsilon_t = 1). \end{cases} \tag{3.2}$$

The following definition is related to the flocking (herding) of volatilities whose dynamics are governed by (3.2).

Definition 3.1. (Deterministic flocking) Let $(V_0^i(t), V_1^i(t))$ be the solution to system (3.2). Then the volatility process $(V_0^i(t), V_1^i(t))$ exhibits asymptotic flocking if and only if the following estimates hold:

$$\lim_{t\to\infty} |V_0^i(t) - V_0^j(t)| = 0 \quad \text{and} \quad \lim_{t\to\infty} |V_1^i(t) - V_1^j(t)| = 0, \quad 1 \le i,\ j \le N.$$

3.2. *Two-stock market*

In this subsection, we consider the simple case when $N = 2$. In this case, system (3.2) leads to:

$$\begin{aligned} \frac{dV_0^1(t)}{dt} &= p_{10}V_1^1 - p_{00}V_0^1, \quad \frac{dV_0^2(t)}{dt} = p_{10}V_1^2 - p_{00}V_0^2, \\ \frac{dV_1^1(t)}{dt} &= \left(-\frac{K}{2} - p_{11}\right)V_1^1 + p_{01}V_0^1 + \frac{K}{2}V_1^2(t), \\ \frac{dV_1^2(t)}{dt} &= \left(-\frac{K}{2} - p_{11}\right)V_1^2 + p_{01}V_0^2 + \frac{K}{2}V_1^1(t). \end{aligned} \tag{3.3}$$

Equivalently, system (3.3) can be written in the following compact form:

$$\mathbf{V}_2' = \mathbf{A}_2\mathbf{V}_2, \quad \mathbf{V}_2 = (V_0^1, V_0^2, V_1^1, V_1^2)^\top, \tag{3.4}$$

where the coefficient matrix $\mathbf{A}_2$ is defined in Appendix A. The general solution of (3.4) is

$$\mathbf{V}_2(t) = c_0^+\mathbf{E}_0^+ e^{\lambda_0^+ t} + c_0^-\mathbf{E}_0^- e^{\lambda_0^- t} + c_1^+\mathbf{E}_1^+ e^{\lambda_1^+ t} + c_1^-\mathbf{E}_1^- e^{\lambda_1^- t}, \tag{3.5}$$

where $c_i^\pm$, $i = 0, 1$, are any constant numbers, and $\lambda_i^\pm$ and $\mathbf{E}_i^\pm$, $i = 0, 1$, are the eigenvalues and corresponding eigenvectors of matrix $\mathbf{A}_2$ given in (A.2) and (A.3), respectively. Note that if (3.5) is substituted into the first equation of (1.2), the stock prices can be obtained.

In the case that $p_{00} = p_{01}$ and $p_{11} = p_{10}$,

$$\begin{aligned}
\lambda_0^+ &= 0, \quad \lambda_0^- = -p_{00} - p_{11}, \\
\lambda_1^+ &= \frac{1}{2}[-p_{00} - p_{11} - K + \sqrt{(p_{00}+p_{11}+K)^2 - 4Kp_{00}}], \\
\lambda_1^- &= \frac{1}{2}[-p_{00} - p_{11} - K - \sqrt{(p_{00}+p_{11}+K)^2 - 4Kp_{00}}],
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{E}_0^j &= \begin{pmatrix} 1 & 1 & \dfrac{\lambda_0^j + p_{00}}{p_{11}} & \dfrac{\lambda_0^j + p_{00}}{p_{11}} \end{pmatrix}^\top, \\
\mathbf{E}_1^j &= \begin{pmatrix} 1 & -1 & \dfrac{\lambda_1^j + p_{00}}{p_{11}} & -\dfrac{\lambda_1^j + p_{00}}{p_{11}} \end{pmatrix}^\top.
\end{aligned} \tag{3.6}$$

Figure 2 presents the movements of volatilities in each state. In Figs. 2(c) and 2(d), as time passes, the volatilities in the normal state decrease. In contrast, volatilities in the other regime flock to each other quickly, and their levels increase. The increasing pattern of the ensemble average continues until the arrival of news that makes the regime jump to the other regime. In Figs. 2(a) and 2(b), for different parameter values of Q, the pattern is shown in an opposite way.

Theorem 3.1. (Deterministic flocking) *Let* $\mathbf{V}_2 = \mathbf{V}_2(t)$ *be a solution to* (3.2). *Then for any initial volatility vector* $V_2(0)$, *asymptotic flocking exists. More precisely*, *there exists a positive constant* C *such that*:

(i)

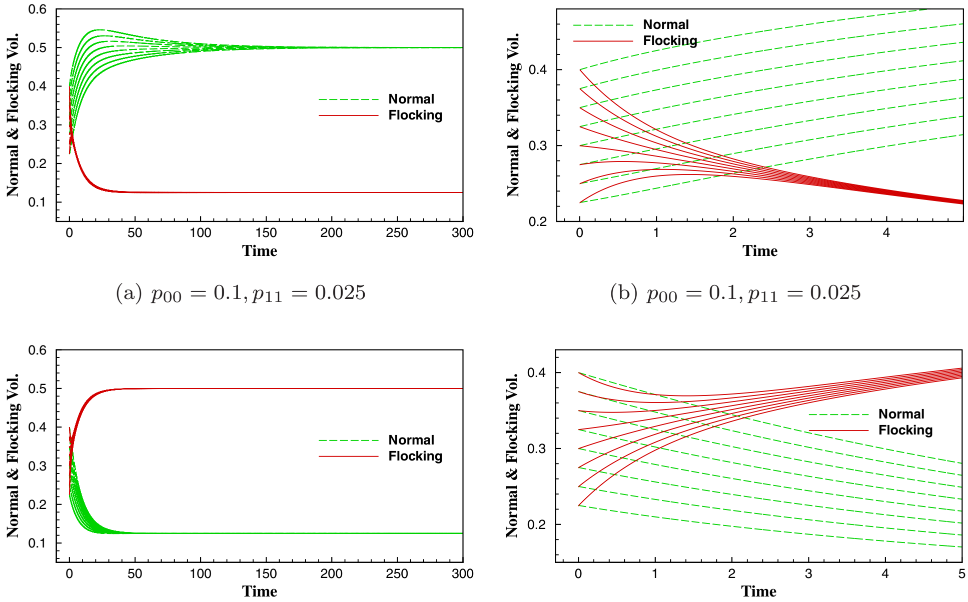
$$\frac{V_j^1(t) + V_j^2(t)}{2} = c_0^+\chi_j^+ e^{\lambda_0^+ t} + c_0^-\chi_j^- e^{\lambda_0^- t}, \quad j = 0, 1,$$

(ii)

$$|V_0^1 - V_0^2|^2 + |V_1^1 - V_1^2|^2 \le Ce^{2\lambda_1^+ t},$$

where $\chi_j^\pm$ *are constants defined by*

$$\chi_0^\pm = 1, \quad \chi_1^+ = \frac{\lambda_0^+ + p_{00}}{p_{11}}, \quad \chi_1^- = \frac{\lambda_0^- + p_{00}}{p_{11}}.$$



(a) $p_{00} = 0.1, p_{11} = 0.025$ (b) $p_{00} = 0.1, p_{11} = 0.025$

(c) $p_{00} = 0.025, p_{11} = 0.1$ (d) $p_{00} = 0.025, p_{11} = 0.1$

Fig. 2. Evolution of volatilities for eight stocks, where volatilities range from 22.5–40%. (a) and (c): Long-term evolutions of volatilities in the range $[0, 300]$. (b) and (d): Close-up of (a) and (c) on $[0, 5]$.

Proof. (i) It follows from (3.5) and (3.6) that:

$$V_0^1(t) + V_0^2(t) = 2[c_0^+ e^{\lambda_0^+ t} + c_0^- e^{\lambda_0^- t}],$$
$$V_1^1(t) + V_1^2(t) = 2\left[c_0^+ \left(\frac{\lambda_0^+ + p_{00}}{p_{11}}\right) e^{\lambda_0^+ t} + c_0^- \left(\frac{\lambda_0^- + p_{00}}{p_{11}}\right) e^{\lambda_0^- t}\right].$$

(ii) It also follows from (3.5) and (3.6) that:

$$|V_0^1 - V_0^2|^2 + |V_1^1 - V_1^2|^2 = 4(c_1^+ e^{\lambda_1^+ t} + c_1^- e^{\lambda_1^- t})^2 + \frac{4}{p_{11}^2}[c_1^+(\lambda_1^+ + p_{00})e^{\lambda_1^+ t} + c_1^-(\lambda_1^- + p_{00})e^{\lambda_1^- t}]^2.$$

Using the fact that $\lambda_1^- < \lambda_1^+ < 0$, the desired estimate is obtained. □

Remark 3.1. Note that the average of the flocking and normal states, $\frac{V_j^1 + V_j^2}{2}$ $(j = 0, 1)$, depends only on the eigenvalues $\lambda_0^\pm$ of the generating matrix Q; the average is not relevant to $\lambda_1^\pm$. When $p_{01} = p_{00}$ and $p_{10} = p_{11}$, the volatilities converge to specific values as time approaches infinity since $\lambda_0^+ = 0$ and $\lambda_0^- < 0$ (shown in Fig. 2). The second estimate of Theorem 3.1 implies asymptotic flocking in the sense of Definition 3.1.

4. Volatility Dynamics: Noisy Case

In this section, we first discuss a brief motivation for the noisy volatility model as a stochastic perturbation of constant all-to-all coupling strength in (3.3) and provide a rigorous definition of stochastic flocking. For the stochastic estimate of the noisy model, we decompose the volatility into macroscopic components (ensemble averages) and microscopic components (fluctuations), and then study the dynamics of macro and micro components separately.

4.1. *Motivation of the noisy volatility model*

As studied in Ref. 41, it is well known that volatility is governed by a mean-reverting arithmetic Ornstein–Uhlenbeck (or AR1) process. The movement of an individual volatility is also determined by its communication with an environment where the amount of news arrives stochastically. Therefore, subsequent communications contain stochastic noise. We model this empirical regularity by introducing stochastic C–S dynamics governed by a multi-dimensional white noise process. Now, the communication weight ψ_{ij} in (2.2) is decomposed into deterministic and stochastic terms:

$$\psi_{ij} = \bar{\psi}_{ij} + \text{white noise}, \tag{4.1}$$

where $\bar{\psi}_{ij} = \frac{K}{N}$ when $\varepsilon_t = 1$, and $\bar{\psi}_{ij} = 0$ when $\varepsilon_t = 0$.

Referring to Ref. 28, the system in (3.2) with a Wiener process explains the mutual interactions between a sole stochastic volatility (the particle system) and an environment. We employ multiplicative white noise for the stochastic forces acting on the ith agent:

$$Dg^i_j(V)dW^i_j(t), \quad V := (V^1_j, V^2_j, \ldots, V^N_j), \quad i = 1, \ldots, N, \;\; j = 0, 1,$$

where D denotes a non-negative constant that is proportional to the noise strength, g^i_j is a function of V, and $W^i_j(t)$ is the one-dimensional Wiener process. The white noise $dW^i_j(t)$ has mean zero and is characterized by its covariance relations:

$$\begin{aligned} &\langle dW^i_j(t)\rangle = 0, \quad \langle dW^i_j(t), dW^i_j(t_*)\rangle = \delta(t - t_*), \quad \text{for } i = 1, \ldots, N, \;\; j = 0, 1, \\ &\langle dW^i_j(t), dW^l_k(t_*)\rangle = 0, \quad \text{for } j \neq k \text{ or } i \neq l, \end{aligned}$$

where $\langle\cdot\rangle$ represents the ensemble average.

In this environment, the dynamics of volatility processes $V^i_j(t)$ are governed by the following stochastic dynamical system: for $1 \le i \le N$,

$$dV^i_{\varepsilon_t}(t) = \begin{cases} (p_{10}V^i_1 - p_{00}V^i_0)dt + Dg^i_0(V)dW^i_0(t) & (\varepsilon_t = 0), \\ \left(\dfrac{K}{N}\displaystyle\sum_{k=1}^N (V^k_1(t) - V^i_1(t)) + p_{01}V^i_0 - p_{11}V^i_1\right) dt & \\ \qquad + Dg^i_1(V)dW^i_1(t) & (\varepsilon_t = 1), \end{cases} \tag{4.2}$$

subject to deterministic initial data:

$$(V^1(0), V^2(0), \dots, V^N(0)) = (v^1, v^2, \dots, v^N).$$

In the sequel, we assume g_j^i takes the following values:

$$g_0^i(V) := V_0^i - V_0^e, \quad g_1^i(V) := V_1^i - V_1^e,$$

where $V_j^e,\ j = 0, 1$, is a constant state in $\mathbb{R}$. Thus, system (4.2) leads to:

$$\begin{cases} dV_0^i = (p_{10}V_1^i - p_{00}V_0^i)dt + D(V_0^i - V_0^e)dW_0^i & (\varepsilon_t = 0), \\ dV_1^i = \left(\dfrac{K}{N}\displaystyle\sum_{k=1}^{N}(V_1^k - V_1^i) + p_{01}V_0^i - p_{11}V_1^i\right)dt & \\ \qquad\quad + D(V_1^i - V_1^e)dW_1^i & (\varepsilon_t = 1). \end{cases} \tag{4.3}$$

Remark 4.1. Note that $V_j^i = V_j^e$, $(i = 1, \dots, N,\ j = 0, 1)$, is an equilibrium solution if $p_{10}V_1^e = p_{00}V_0^e$ and $p_{01}V_0^e = p_{11}V_1^e$.

Next, we introduce two quantities: macro observables (ensemble averages) and micro observables (fluctuations). For $1 \le i \le N$,

$$V_0^c := \frac{1}{N}\sum_{i=1}^{N} V_0^i, \quad V_1^c := \frac{1}{N}\sum_{i=1}^{N} V_1^i,$$
$$\hat{V}_0^i := V_0^i - V_0^c, \quad \hat{V}_1^i := V_1^i - V_1^c.$$

The macroscopic observables register the long-time dynamics of the volatility processes, while the microscopic observables monitor the transient fluctuations. For simplicity, we assume that all Wiener processes are equal. In Sec. 5, we relax this restrictive assumption in the numerical simulations.

Note that the ensemble averages and fluctuations satisfy:

$$\begin{aligned} dV_0^c &= (p_{10}V_1^c - p_{00}V_0^c)dt + D(V_0^c - V_0^e)dW, \\ dV_1^c &= (-p_{11}V_1^c + p_{01}V_0^c)dt + D(V_1^c - V_1^e)dW, \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} d\hat{V}_0^i &= (p_{10}\hat{V}_1^i - p_{00}\hat{V}_0^i)dt + D\hat{V}_0^i dW, \quad 1 \le i \le N, \\ d\hat{V}_1^i &= \left[\frac{K}{N}\sum_{j=1}^{N}(\hat{V}_1^j - \hat{V}_1^i) - p_{11}\hat{V}_1^i + p_{01}\hat{V}_0^i\right]dt + D\hat{V}_1^i dW, \end{aligned} \tag{4.5}$$

with zero sum constraints

$$\sum_{i=1}^{N} \hat{V}_0^i = 0, \quad \sum_{i=1}^{N} \hat{V}_1^i = 0, \quad t \ge 0.$$

Also note that the macro and micro dynamics in (4.4) and (4.5), respectively, are completely decoupled from each other. Moreover, the coupling strength K does not appear in (4.4); it is only present in (4.5). This is consistent with our intuition on the role of flocking, i.e. the definition of flocking in deterministic, and random settings only involve the differences $V_0^i(t) - V_0^j(t)$ and $V_1^i(t) - V_1^j(t)$.

Later, we consider the matrix:

$$Q - K\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -p_{00} & p_{01} \\ p_{10} & -p_{11} - K \end{pmatrix}.$$

Let $\bar{\lambda}_0(K)$ and $\bar{\lambda}_1(K)$ be the eigenvalues of the above matrix. Then, by direct calculation, we obtain:

$$\begin{aligned} \bar{\lambda}_0(K) &:= \frac{1}{2}[-p_{00} - p_{11} - K - \sqrt{(p_{00} + p_{11} + K)^2 - 4(p_{00}p_{11} - p_{01}p_{10}) - 4Kp_{00}}], \\ \bar{\lambda}_1(K) &:= \frac{1}{2}[-p_{00} - p_{11} - K + \sqrt{(p_{00} + p_{11} + K)^2 - 4(p_{00}p_{11} - p_{01}p_{10}) - 4Kp_{00}}]. \end{aligned} \tag{4.6}$$

Definition 4.1. (Stochastic flocking[42]) Let $(V_0^i(t), V_1^i(t))$ be the solution to system (4.3).

(1) The volatility process $(V_0^i(t), V_1^i(t))$ exhibits probabilistic asymptotic stochastic flocking if and only if for any $\epsilon > 0$,

$$\lim_{t\to\infty} \mathbb{P}(|V_0^i(t) - V_0^j(t)| > \epsilon) = 0 \quad \text{and}$$
$$\lim_{t\to\infty} \mathbb{P}(|V_1^i(t) - V_1^j(t)| > \epsilon) = 0, \quad \text{for all } i, j = 1, \ldots, N.$$

(2) The volatility process $(V_0^i(t), V_1^i(t))$ exhibits asymptotic stochastic flocking almost surely if and only if the relative volatilities $|V_0^i(t) - V_0^j(t)|$ and $|V_1^i(t) - V_1^j(t)|$ approach zero almost surely as $t \to \infty$, i.e.

$$\lim_{t\to\infty} |V_0^i(t) - V_0^j(t)| = 0, \quad \lim_{t\to\infty} |V_1^i(t) - V_1^j(t)| = 0, \quad i, j = 1, \ldots, N.$$

In the following subsections, we study the statistical estimates of macroscopic and microscopic observables.

4.2. *Macro dynamics*

Let V_j^c, $j = 0, 1$, be the macroscopic processes, determined by the Cauchy problem, for the coupled stochastic differential equations in (4.4):

$$\begin{aligned} dV_0^c &= (p_{10}V_1^c - p_{00}V_0^c)dt + D(V_0^c - V_0^e)dW, \quad t > 0, \\ dV_1^c &= (-p_{11}V_1^c + p_{01}V_0^c)dt + D(V_1^c - V_1^e)dW, \end{aligned} \tag{4.7}$$

subject to initial data:

$$V^c(0) = v^c := \frac{1}{N}\sum_{i=1}^N v^i. \tag{4.8}$$

Since flocking is defined as the zero convergence of relative volatilities $V_j^i - V_j^k$ in a suitable sense (see Definition 4.1), the macro dynamics in (4.7) are irrelevant to the flocking of the entire system (4.2), even though they govern the long-run dynamics of (4.2).

Proposition 4.1. (Statistical estimates) *Suppose the transition frequency p_{ij} satisfies*

$$(p_{00}+p_{11})^2 - 4(p_{00}p_{11} - p_{01}p_{10}) \neq 0. \tag{4.9}$$

Then the following estimates hold:

(1) *The solution to* (4.7) *and* (4.8) *is of the following form:*

$$V_0^c(t) = \frac{\bar\lambda_1(0)+p_{00}}{p_{01}(\bar\lambda_1(0)-\bar\lambda_0(0))}U_0^c(t) - \frac{\bar\lambda_0(0)+p_{00}}{p_{01}(\bar\lambda_1(0)-\bar\lambda_0(0))}U_1^c(t) + V_0^e, \quad t \geq 0, \tag{4.10}$$

$$V_1^c(t) = \frac{1}{\bar\lambda_0(0)-\bar\lambda_1(0)}U_0^c(t) - \frac{1}{\bar\lambda_0(0)-\bar\lambda_1(0)}U_1^c(t) + V_1^e, \tag{4.11}$$

where

$$\begin{aligned} U_j^c(t) := e^{\bar\lambda_j(0)t - \frac{1}{2}D^2 t + DW(t)} \bigg[&(p_{01}+p_{00}+\bar\lambda_j(0))v^c - p_{01}V_0^e - (\bar\lambda_j(0)+p_{00})V_1^e \\ &+ \bar\lambda_j(0)(p_{01}V_0^e + (\bar\lambda_j(0)+p_{00})V_1^e) \\ &\times \int_0^t e^{-\bar\lambda_j(0)s + \frac{1}{2}D^2 s - DW(s)}ds \bigg], \quad j = 0,1. \end{aligned} \tag{4.12}$$

(2) *The expectations $\mathbb{E}[V_0^c(t)]$ and $\mathbb{E}[V_1^c(t)]$ satisfy:*

$$\begin{aligned} \mathbb{E}[V_0^c(t)] &= \bigg[\frac{\bar\lambda_1(0)+p_{00}}{p_{01}(\bar\lambda_1(0)-\bar\lambda_0(0))}(p_{01}+p_{00}+\bar\lambda_0(0))e^{\bar\lambda_0(0)t} \\ &\quad - \frac{\bar\lambda_0(0)+p_{00}}{p_{01}(\bar\lambda_1(0)-\bar\lambda_0(0))}(p_{01}+p_{00}+\bar\lambda_1(0))e^{\bar\lambda_1(0)t} \bigg] v^c, \\ \mathbb{E}[V_1^c(t)] &= \bigg[\frac{1}{\bar\lambda_0(0)-\bar\lambda_1(0)}(p_{01}+p_{00}+\bar\lambda_0(0))e^{\bar\lambda_0(0)t} \\ &\quad - \frac{1}{\bar\lambda_0(0)-\bar\lambda_1(0)}(p_{01}+p_{00}+\bar\lambda_1(0))e^{\bar\lambda_1(0)t} \bigg] v^c. \end{aligned} \tag{4.13}$$

Remark 4.2. When the condition (4.9) is not satisfied, eigenvalues of two states are identical, that is, $\bar\lambda_0(0) = \bar\lambda_1(0)$. In other words, two states are degenerated and some singularities might occur as shown in (4.10).

Proof of Proposition 4.1. (i) By considering Eq. (4.4) together with the fact that $\bar{\lambda}_j(0)$ is the root of the characteristic equation of Q, i.e.

$$\bar{\lambda}_j(0)^2 + (p_{00} + p_{11})\bar{\lambda}_j(0) + p_{00}p_{11} - p_{01}p_{10} = 0,$$

the coefficient of $V_1^c dt$ can be simplified as follows:

$$p_{01}p_{10} - p_{11}(\bar{\lambda}_j(0) + p_{00}) = \bar{\lambda}_j(0)(\bar{\lambda}_j(0) + p_{00}).$$

In light of this fact,

$$\begin{aligned}
d(p_{01}V_0^c &+ (\bar{\lambda}_j(0) + p_{00})V_1^c) \\
&= [p_{01}\bar{\lambda}_j(0)V_0^c + (p_{01}p_{10} - p_{11}(\bar{\lambda}_j(0) + p_{00}))V_1^c]dt \\
&\quad + D(p_{01}V_0^c + (\bar{\lambda}_j(0) + p_{00})V_1^c - p_{01}V_0^e - (\bar{\lambda}_j(0) + p_{00})V_1^e)dW \\
&= \bar{\lambda}_j(0)(p_{01}V_0^c + (\bar{\lambda}_j(0) + p_{00})V_1^c)dt \\
&\quad + D(p_{01}V_0^c + (\bar{\lambda}_j(0) + p_{00})V_1^c - p_{01}V_0^e - (\bar{\lambda}_j(0) + p_{00})V_1^e)dW. \qquad (4.14)
\end{aligned}$$

By setting

$$U_j^c(t) := p_{01}V_0^c(t) + (\bar{\lambda}_j(0) + p_{00})V_1^c(t) - p_{01}V_0^e - (\bar{\lambda}_j(0) + p_{00})V_1^e, \qquad (4.15)$$

relation (4.14) leads to

$$dU_j^c = \bar{\lambda}_j(0)\{U_j^c + p_{01}V_0^e + (\bar{\lambda}_j(0) + p_{00})V_1^e\}dt + DU_j^c dW,$$

which is of the same form as (B.1) found in Appendix B. Thus, by applying Lemma B.1, we obtain U_j^c in (4.12). Having U_0^c and U_1^c, the desired representations for V_0^c and V_1^c are derived using (4.15).

(ii) For the second part of Proposition 4.1, it suffices to estimate $\mathbb{E}[U_j^c]$. We use the fact that

$$\mathbb{E}\left[e^{(\bar{\lambda}_j(0) - \frac{D^2}{2})t + DW}\right] = e^{\bar{\lambda}_j(0)t} \quad \text{and} \quad \mathbb{E}[e^{D[W(t) - W(s)]}] = e^{\frac{1}{2}D^2(t-s)}$$

in (4.12) to obtain

$$\mathbb{E}[U_j^c] = (p_{01} + p_{00} + \bar{\lambda}_j(0))v^c e^{\alpha_j t} - p_{01}V_0^e - (\bar{\lambda}_j(0) + p_{00})V_1^e. \qquad (4.16)$$

On the other hand, it follows from (4.15) that:

$$\begin{aligned}
\mathbb{E}[U_0^c(t)] &= p_{01}\mathbb{E}[V_0^c(t)] + (\bar{\lambda}_0(0) + p_{00})\mathbb{E}[V_1^c(t)] - p_{01}V_0^e - (\bar{\lambda}_j(0) + p_{00})V_1^e, \\
\mathbb{E}[U_1^c(t)] &= p_{01}\mathbb{E}[V_0^c(t)] + (\bar{\lambda}_1(0) + p_{00})\mathbb{E}[V_1^c(t)] - p_{01}V_0^e - (\bar{\lambda}_j(0) + p_{00})V_1^e.
\end{aligned} \qquad (4.17)$$

Combining (4.16) and (4.17) yields the desired estimates. □

Remark 4.3. If $p_{00}p_{11} > p_{01}p_{10}$, it is easy to see from (4.6) that

$$\bar{\lambda}_0(0) < \bar{\lambda}_1(0) < 0.$$

Thus, (4.13) implies

$$\lim_{t\to\infty} \mathbb{E}[V_0^c(t)] = 0 \quad \text{and} \quad \lim_{t\to\infty} \mathbb{E}[V_1^c(t)] = 0.$$

On the other hand, if $p_{00}p_{11} < p_{01}p_{10}$,

$$\bar{\lambda}_0(0) < 0 < \bar{\lambda}_1(0).$$

Thus,

$$\lim_{t\to\infty} |\mathbb{E}[V_0^c(t)]| = \infty \quad \text{and} \quad \lim_{t\to\infty} |\mathbb{E}[V_1^c(t)]| = \infty.$$

Now, set:

$$\begin{aligned} U_0^c(t) &= e^{-(p_{00}+p_{11})t - \frac{1}{2}D^2 t + DW(t)} \Bigg[(p_{00} - p_{11})v^c - p_{00}V_0^e + p_{11}V_1^e \\ &\quad - (p_{00}+p_{11})(p_{00}V_0^e - p_{11}V_1^e)\int_0^t e^{(p_{00}+p_{11})s + \frac{1}{2}D^2 s - DW(s)} ds\Bigg], \\ U_1^c(t) &= e^{-\frac{1}{2}D^2 t + DW(t)} p_{00}(2v^c - V_0^e - V_1^e). \end{aligned}$$

The following result is a corollary to Proposition 4.1.

Corollary 4.1. *Suppose the transition frequency* p_{ij} *satisfies*

$$p_{00} = p_{01} \quad \text{and} \quad p_{10} = p_{11}. \tag{4.18}$$

Then the following assertions hold:

(1) *The solution to* (4.4) *is of the following form*:

$$V_0^c = \frac{p_{00}U_0^c + p_{11}U_1^c}{p_{00}(p_{00}+p_{11})} + V_0^e, \quad V_1^c = \frac{U_1^c - U_0^c}{p_{00}+p_{11}} + V_1^e.$$

(2) *Expectations* $\mathbb{E}[V_0^c]$ *and* $\mathbb{E}[V_1^c]$ *are given by the following relations*:

$$\mathbb{E}[V_0^c] = [2p_{11} + (p_{00} - p_{11})e^{-(p_{00}+p_{11})t}]\frac{v^c}{p_{00}+p_{11}},$$

$$\mathbb{E}[V_1^c] = [2p_{00} - (p_{00} - p_{11})e^{-(p_{00}+p_{11})t}]\frac{v^c}{p_{00}+p_{11}}.$$

Proof. Assumption (4.18) implies:

$$(p_{00}+p_{11})^2 - 4(p_{00}p_{11} - p_{01}p_{10}) = (p_{00}+p_{11})^2 > 0,$$

$$\bar{\lambda}_0(0) = -(p_{00}+p_{11}) < 0, \quad \bar{\lambda}_1(0) = 0.$$

Applying all estimates in Proposition 4.1 yields the desired claim. □

Remark 4.4. (1) Note that the results of Corollary 4.1 imply

$$\lim_{t\to\infty} V_0^c(t) = V_0^e, \quad \lim_{t\to\infty} V_1^c(t) = V_1^e,$$

$$\lim_{t\to\infty}\mathbb{E}[V_0^c(t)] = \frac{2p_{11}v^c}{p_{00}+p_{11}}, \quad \lim_{t\to\infty}\mathbb{E}[V_1^c(t)] = \frac{2p_{00}v^c}{p_{00}+p_{11}}.$$

If V_0^e and V_1^e are chosen according to

$$V_0^e := \frac{2p_{11}v^c}{p_{00}+p_{11}}, \quad V_1^e := \frac{2p_{00}v^c}{p_{00}+p_{11}},$$

then,

$$\lim_{t\to\infty} V_0^c(t) = V_0^e, \quad \lim_{t\to\infty} V_1^c(t) = V_1^e, \quad \lim_{t\to\infty}\mathbb{E}[V_0^c(t)] = V_0^e, \quad \lim_{t\to\infty}\mathbb{E}[V_1^c(t)] = V_1^e.$$

Note that the larger p_{00} is, the larger $\mathbb{E}[V_1^c]$ is, and similarly, the larger p_{11} is, the larger $\mathbb{E}[V_0^c]$ is. This is consistent with our intuition: if the transition probability from one state to another increases, the expected value of the second state increases. Additionally, the ensemble volatility in each state does not increase without bound, but converges to a control term.

(2) In addition to (4.18), if $p_{00}V_0^e = p_{11}V_1^e$,

$$\begin{aligned} V_0^c &= V_0^e + \frac{1}{p_{00}+p_{11}}[(2v^c p_{11} - (p_{11}+p_{00})V_0^e) \\ &\quad + (p_{00}-p_{11})v^c e^{-(p_{00}+p_{11})t}]e^{-\frac{1}{2}D^2t+DW(t)}, \\ V_1^c &= V_1^e + \frac{1}{p_{00}+p_{11}}\left[p_{00}\left(2v^c - \left(1+\frac{p_{00}}{p_{11}}\right)V_0^e\right)\right. \\ &\quad \left. - (p_{00}-p_{11})v^c e^{-(p_{00}+p_{11})t}\right] e^{-\frac{1}{2}D^2t+DW(t)}. \end{aligned}$$

4.3. *Micro dynamics*

In this subsection, we investigate the flocking dynamics of system (4.2) via the micro dynamics in (4.5). To begin, we use the zero sum constraint to simplify the mean-field interaction term:

$$\frac{K}{N}\sum_{j=1}^{N}(\hat{V}_1^j - \hat{V}_1^i) = \frac{K}{N}\sum_{j=1}^{N}\hat{V}_1^j - K\hat{V}_1^i = -K\hat{V}_1^i.$$

Thus, micro dynamics (4.5) are reduced to:

$$\begin{aligned} d\hat{V}_0^i &= (p_{10}\hat{V}_1^i - p_{00}\hat{V}_0^i)dt + D\hat{V}_0^i dW, \\ d\hat{V}_1^i &= (-(K+p_{11})\hat{V}_1^i + p_{01}\hat{V}_0^i)dt + D\hat{V}_1^i dW. \end{aligned} \tag{4.19}$$

Lemma 4.1. *The solution to* (4.19) *is given by the following explicit formula*:

$$\begin{aligned} \hat{V}_0^i &= \frac{\bar{\lambda}_1(K)+p_{00}}{p_{01}(\bar{\lambda}_1(K)-\bar{\lambda}_0(K))}\hat{U}_0^i - \frac{\bar{\lambda}_0(K)+p_{00}}{p_{01}(\bar{\lambda}_1(K)-\bar{\lambda}_0(K))}\hat{U}_1^i, \\ \hat{V}_1^i &= \frac{-1}{(\bar{\lambda}_1(K)-\bar{\lambda}_0(K))}\hat{U}_0^i + \frac{1}{(\bar{\lambda}_1(K)-\bar{\lambda}_0(K))}\hat{U}_1^i, \end{aligned} \tag{4.20}$$

where $\hat{U}_j^i$ *is given by*

$$\begin{aligned}\hat{U}_j^i(t) &= (p_{01} + p_{00} + \bar{\lambda}_j(K))\hat{v}^i e^{(\bar{\lambda}_j(K) - \frac{1}{2}D^2)t + DW(t)} \\ &= \hat{U}_j^i(0) e^{(\bar{\lambda}_j(K) - \frac{1}{2}D^2)t + DW(t)}. \end{aligned} \tag{4.21}$$

Proof. It follows from (4.5) that

$$\begin{aligned}&d(p_{01}\hat{V}_0^i + (\bar{\lambda}_j(K) + p_{00})\hat{V}_1^i) \\ &\quad = \bar{\lambda}_j(K)(p_{01}\hat{V}_0^i + (\bar{\lambda}_j(K) + p_{00})\hat{V}_1^i)dt + D(p_{01}\hat{V}_0^i + (\bar{\lambda}_j(K) + p_{00})\hat{V}_1^i)dW.\end{aligned}$$

By setting

$$\hat{U}_j^i(t) := p_{01}\hat{V}_0^i(t) + (\bar{\lambda}_j(K) + p_{00})\hat{V}_1^i(t),$$

it follows that

$$d\hat{U}_j^i := \bar{\lambda}_j(K)\hat{U}_j^i dt + D\hat{U}_j^i dW.$$

Taken together, the above stochastic differential equation and Lemma B.1 yield the desired estimates. □

Next, we provide the flocking estimate for system (4.2).

Theorem 4.1. (Stochastic flocking estimate) *Suppose that the coupling strength and transition intensity* p_{ij} *satisfy*

$$K > 0, \quad p_{00} = p_{01} \quad \text{and} \quad p_{10} = p_{11}. \tag{4.22}$$

Then for any initial volatilities, the volatility process applied to system (4.2) *exhibits asymptotic stochastic flocking almost surely, i.e.*

$$\lim_{t\to\infty} |V_0^i(t) - V_0^j(t)| = 0 \quad \text{and} \quad \lim_{t\to\infty} |V_1^i(t) - V_1^j(t)| = 0, \quad \text{a.s. } i,j = 1,\ldots,N.$$

Proof. We only provide the estimate for $\lim_{t\to\infty} |V_0^i(t) - V_0^j(t)|) = 0$ since the other case can be treated similarly.

Note that assumptions (4.6) and (4.22) yield

$$\bar{\lambda}_0(K) < \bar{\lambda}_1(K) < 0. \tag{4.23}$$

On the other hand, it follows from (4.20) that:

$$\begin{aligned}|V_0^i(t) - V_0^j(t)| &= |\hat{V}_0^i(t) - \hat{V}_0^j(t)| \\ &\leq \left|\frac{\bar{\lambda}_1(K) + p_{00}}{p_{01}(\bar{\lambda}_1(K) - \bar{\lambda}_0(K))}\right| \cdot |\hat{U}_0^i - \hat{U}_0^j| \\ &\quad + \left|\frac{\bar{\lambda}_0(K) + p_{00}}{p_{01}(\bar{\lambda}_1(K) - \bar{\lambda}_0(K))}\right| \cdot |\hat{U}_1^i - \hat{U}_1^j|. \end{aligned} \tag{4.24}$$

We claim that

$$|\hat{U}_0^i - \hat{U}_0^j| \to 0, \quad |\hat{U}_1^i - \hat{U}_1^j| \to 0, \quad \text{a.s. as } t \to \infty. \tag{4.25}$$

Proof of claim: Owing to the defining relation of $\hat{U}_j^i$ in (4.21), it suffices to verify that

$$e^{(\bar{\lambda}_j(K) - \frac{1}{2}D^2)t + DW(t)} \to 0 \quad \text{as } t \to \infty.$$

Using an iterative law of logarithms on $W(t)$,

$$\limsup_{t \to \infty} \frac{|W(t)|}{\sqrt{2t \log \log t}} = 1, \quad \text{a.s.},$$

we see that

$$e^{(\bar{\lambda}_j(K) - \frac{1}{2}D^2)t + DW(t)} \le e^{\frac{(\bar{\lambda}_j(K) - \frac{1}{2}D^2)t}{2}}, \quad \text{a.s. } t \gg 1.$$

Using (4.23) and the above relation, we conclude

$$e^{(\bar{\lambda}_j(K) - \frac{1}{2}D^2)t + DW(t)} \to 0, \quad \text{a.s. as } t \to \infty.$$

Hence, the desired estimate (4.25) is obtained. Finally, combining (4.24) and (4.25) yields the desired estimate. □

Remark 4.5. Note that the stochastic flocking estimate for (4.3) occurs independently of D. This is due to the fact that multiplicative noise $D(V_0^i - V_0^e)dW_0^i$ and $D(V_1^i - V_1^e)dW_1^i$ is employed in (4.3), which enforces stochastic flocking. This effect is called stochastic resonance due to noise. However, if additive noise is employed instead, say DdW_0^i and DdW_1^i in (4.3), stochastic flocking as in Theorem 4.1 is not expected.

Theorem 4.2. *Let $\hat{V}_j^i$ be a fluctuation process given by the explicit representation* (4.20). *Then for $1 \le i \le N$ and $j = 0, 1$,*

(i)

$$\mathbb{E}[\hat{V}_0^i(t)] = \frac{(\bar{\lambda}_1(K) + p_{00})\hat{U}_0^i(0)}{p_{01}(\bar{\lambda}_1(K) - \bar{\lambda}_0(K))} e^{\bar{\lambda}_0(K)t} - \frac{(\bar{\lambda}_0(K) + p_{00})\hat{U}_1^i(0)}{p_{01}(\bar{\lambda}_1(K) - \bar{\lambda}_0(K))} e^{\bar{\lambda}_1(K)t},$$

$$\mathbb{E}[\hat{V}_1^i(t)] = \frac{-\hat{U}_0^i(0)}{(\bar{\lambda}_1(K) - \bar{\lambda}_0(K))} e^{\bar{\lambda}_0(K)t} + \frac{\hat{U}_1^i(0)}{(\bar{\lambda}_1(K) - \bar{\lambda}_0(K))} e^{\bar{\lambda}_1(K)t},$$

(ii)

$$\mathrm{Var}[\hat{V}_0^i(t)] \le \mathcal{O}(1) e^{2\bar{\lambda}_1(K)t}(e^{D^2 t} - 1),$$
$$\mathrm{Var}[\hat{V}_1^i(t)] \le \mathcal{O}(1) e^{2\bar{\lambda}_1(K)t}(e^{D^2 t} - 1), \quad \textit{as } t \to \infty.$$

Proof. (i) First, note that (4.21) and the fact that $\mathbb{E}\left[e^{(\bar{\lambda}_j(K) - \frac{D^2}{2})t + DW}\right] = e^{\bar{\lambda}_j(K)t}$ imply

$$\mathbb{E}[\hat{U}_j^i(t)] = \hat{U}_j^i(0)\mathbb{E}\left[e^{(\bar{\lambda}_j(K) - \frac{1}{2}D^2)t + DW_t}\right] = \hat{U}_j^i(0) e^{\bar{\lambda}_j(K)t}.$$

By substituting the above estimate into (4.20), the desired estimates are obtained.
(ii) Taken together, (4.26) and the fact that $\mathbb{E}[e^{(2\bar{\lambda}_j(K)-D^2)t+2DW}] = e^{(2\bar{\lambda}_j(K)+D^2)t}$ yield:

$$\begin{aligned}\mathrm{Var}[\hat{V}_0^i] &= \left[\left(\frac{(\bar{\lambda}_1(K)+p_{00})\hat{U}_0^i(0)}{p_{01}(\bar{\lambda}_1(K)-\bar{\lambda}_0(K))}\right)e^{\bar{\lambda}_0(K)t} - \left(\frac{(\bar{\lambda}_0(K)+p_{00})\hat{U}_1^i(0)}{p_{01}(\bar{\lambda}_1(K)-\bar{\lambda}_0(K))}\right)e^{\bar{\lambda}_1(K)t}\right]^2 \\ &\quad \times (e^{D^2t}-1), \\ \mathrm{Var}[\hat{V}_1^i] &= \left[\frac{\hat{U}_0^i(0)}{(\bar{\lambda}_1(K)-\bar{\lambda}_0(K))}e^{\bar{\lambda}_0(K)t} - \frac{\hat{U}_1^i(0)}{(\bar{\lambda}_1(K)-\bar{\lambda}_0(K))}e^{\bar{\lambda}_1(K)t}\right]^2(e^{D^2t}-1).\end{aligned}$$

Since $\bar{\lambda}_0(K) < \bar{\lambda}_1(K)$, the above estimates yield the desired estimates. □

Remark 4.6. If $p_{01} = p_{00} > 0$ and $p_{10} = p_{11}$,

$$\bar{\lambda}_0(K) < \bar{\lambda}_1(K) < 0, \quad \lim_{K\to\infty} \bar{\lambda}_1(K) = -p_{00}.$$

Thus, the result of Theorem 4.2 implies that expectations of the fluctuation process asymptotically converge to zero, i.e.

$$\lim_{t\to\infty} \mathbb{E}[\hat{V}_j^i(t)] = 0.$$

Furthermore, if D is sufficiently small compared to p_{00},

$$-p_{00} + \frac{1}{2}D^2 < 0.$$

Then for sufficiently large coupling strength K,

$$\bar{\lambda}_1(K) + \frac{1}{2}D^2 < 0.$$

Thus, the variances also vanish asymptotically; that is,

$$\lim_{t\to\infty} \mathrm{Var}[\hat{V}_j^i(t)] = 0.$$

5. Numerical Examples

In this section, we examine the results of numerical experiments. Within the more general configuration, these numerical results support the analytic results of the previous section.

Recall that we proved the volatility flocking phenomena for each state of (4.3) for a set of N-couple volatility processes in Sec. 4. In Sec. 4, we assumed identical Wiener processes; however, in this section, we remove this assumption. In an experiment, we demonstrate the flocking phenomena for an N-coupled system of *i.i.d.* volatility processes in (4.3). We will also show that the analytic results for average processes (Corollary 4.1(2), Theorem 4.1) are still valid.

For the numerical simulation, we adopted the Euler scheme to generate numerical data: $(V_0^{1,n}, V_1^{i,n}) \equiv (V_0^i(t_n), V_1^i(t_n))$:

$$\begin{aligned}
V_0^{i,n+1} &= V_0^{i,n} + (p_{10}V_1^{i,n} - p_{00}V_0^{i,n})\Delta t + D(V_0^{i,n} - V_0^e)\Delta W_0^{i,n},\\
V_1^{i,n+1} &= V_1^{i,n} + \left(\frac{K}{N}\sum_{k=1}^{N}(V_1^{k,n} - V_1^{i,n}) + p_{01}V_0^{i,n} - p_{11}V_1^{i,n}\right)\Delta t\\
&\qquad + D(V_1^{i,n} - V_1^e)\Delta W_1^{i,n}, \qquad (5.1)\\
V_0^{c,n} &:= \frac{1}{N}\sum_{k=1}^{N} V_0^{i,n}, \quad V_1^{c,n} := \frac{1}{N}\sum_{k=1}^{N} V_1^{i,n},\\
\hat{V}_0^{i,n} &:= V_0^{i,n} - V_0^{c,n}, \quad \hat{V}_1^{i,n} := V_1^{i,n} - V_1^{c,n}, \quad i = 1, \ldots, N,
\end{aligned}$$

subject to the following initial volatility configuration:

$$V_0^i(0) = v^i = V_1^i(0).$$

For the simulation, the values of $p_{ij}, (i, j = 0, 1)$, K, N, Δt, V_0^e, and V_1^e were:

$$p_{00} = p_{01} = 0.1, \quad p_{11} = p_{10} = 0.025, \quad K = 1.0, \quad N = 100, \quad \text{and} \quad \Delta t = 10^{-2},$$

$$V_0^e = \frac{2p_{11}}{p_{00} + p_{11}} v_0^c, \quad V_1^e = \frac{2p_{00}}{p_{00} + p_{11}} v_0^c.$$

The initial volatilities $v^i, (i = 1, \ldots, 100)$, were chosen in a uniformly random manner from the interval $(0.1, 0.65)$, as shown in Fig. 3. The random variables $\Delta W_0^{i,n}$ and $\Delta W_1^{i,n}$ were *i.i.d.* and followed the normal distribution $N(0, \sqrt{\Delta t})$. We simulated (5.1) in the time interval $[0, 300]$ with 10^4 paths according to the noise strength parameter $D = 0.1$, 0.2, and 0.4. Note that the choice of parameters follow the condition in Remark 4.6, i.e. $D^2 < -2\bar{\lambda}_1(K)$. We chose five sample initial volatilities among 100 initial V^is. Figure 4 depicts the evolutionary paths and the evolution of expectations, respectively. In this figure, five sample paths

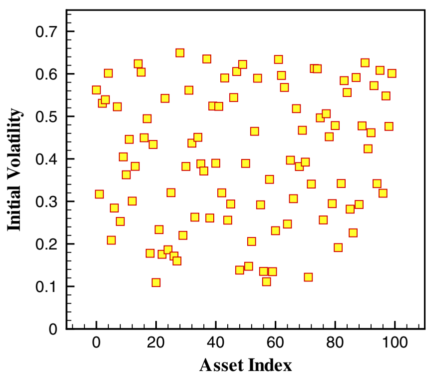
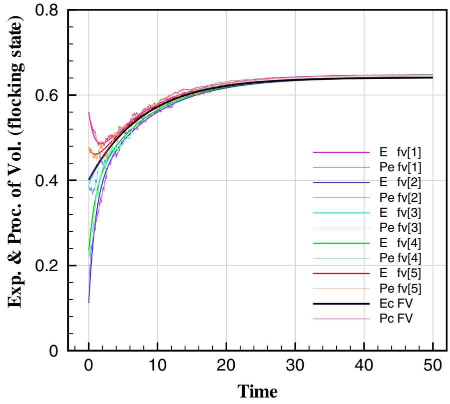
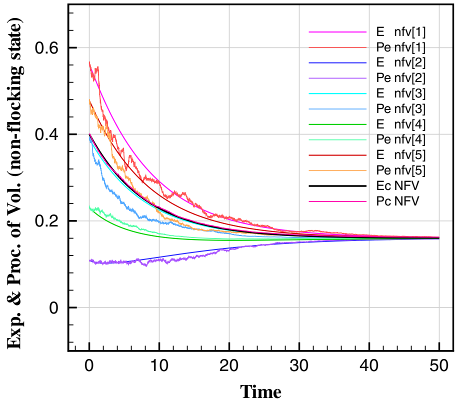


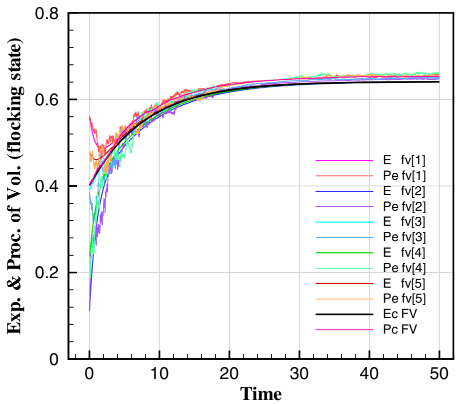
Fig. 3. The initial configuration of volatilities $v^i, (i = 1, \ldots, 100)$.



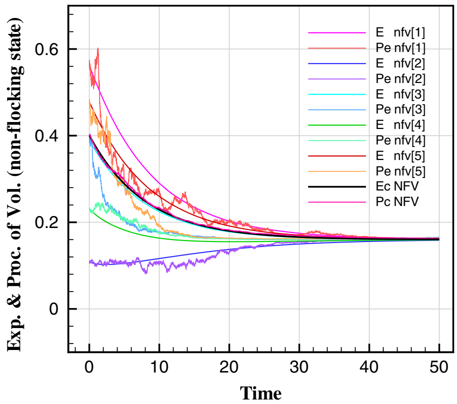
(a) $V_1^i,\ D = 0.1$



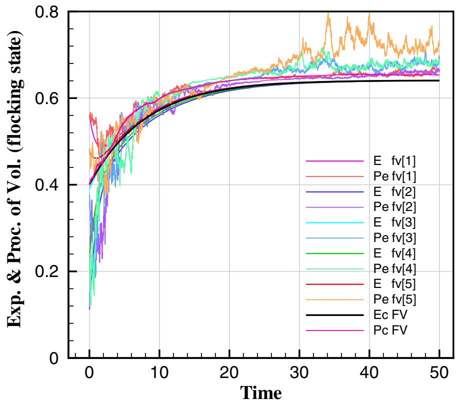
(b) $V_0^i,\ D = 0.1$



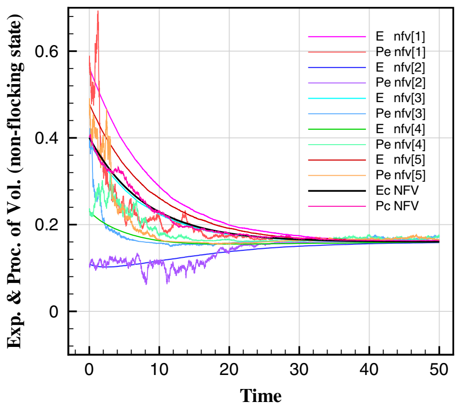
(c) $V_1^i,\ D = 0.2$



(d) $V_0^i,\ D = 0.2$



(e) $V_1^i,\ D = 0.4$

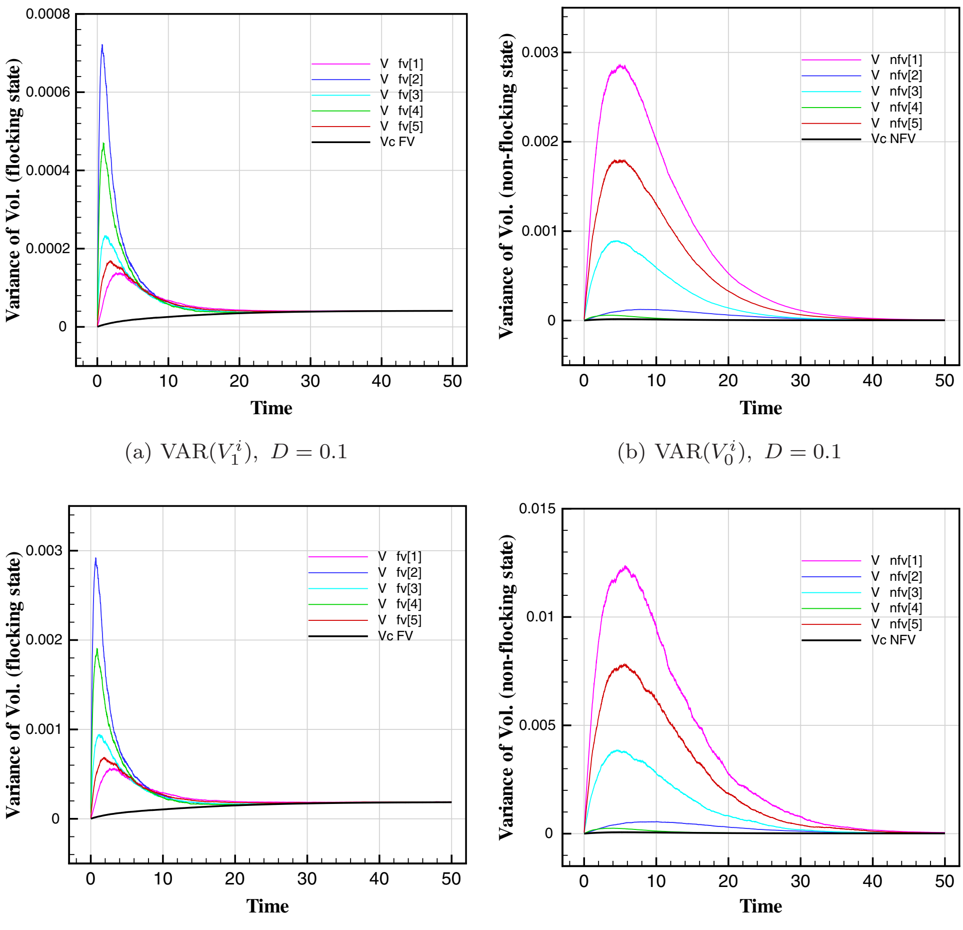


(f) $V_0^i,\ D = 0.4$

Fig. 4. Profile of V_0^i, V_1^i, the average process $\mathbb{E}[V_0^i], \mathbb{E}[V_i^i]$, and macro dynamics V_0^c and V_1^c.

$(P_e fv[i], P_e nfv[i])$ of $V_k^{i,n}, (k = 0, 1)$, are drawn in a serrated shape. Their expectations $(\mathbb{E}(fv[i]), \mathbb{E}(nfv[i]))$, however, are smooth. In fact, the expectation of each sample congregates to the expectation $(\mathbb{E}_c(FV), \mathbb{E}_c(NFV))$ of the average process $(P_c(FV), P_c(NFV))$.

It is worth mentioning that the size of the noise strength D is related to the strength of the attraction. Figure 5 depicts the variance of sample processes $\mathrm{Var}(V_k^{i,n}), (k = 0, 1)$. The variances of the sample are denoted by $(Vfv[i])$ and $(Vnfv[i])$. We denote the variances of the average processes by (V_cFV) and (V_cNFV). The macro dynamics are stable as the variances of V_0^c and V_1^c are uniformly bounded. Additionally, the flocking state process V_1^i generates more intensive flocking than a normal state process V_0^i. The results of process V_1^i are predicted for



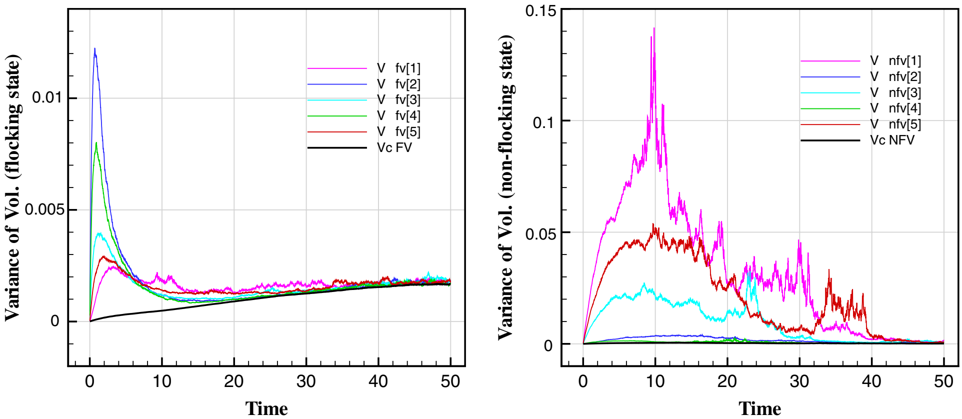
(a) $\mathrm{VAR}(V_1^i),\ D = 0.1$

(b) $\mathrm{VAR}(V_0^i),\ D = 0.1$

(c) $\mathrm{VAR}(V_1^i),\ D = 0.2$

(d) $\mathrm{VAR}(V_0^i),\ D = 0.2$

Fig. 5. Variance of the average process $\mathbb{E}[V_0^i], \mathbb{E}[V_i^i]$ and macro dynamics V_0^c and V_1^c.

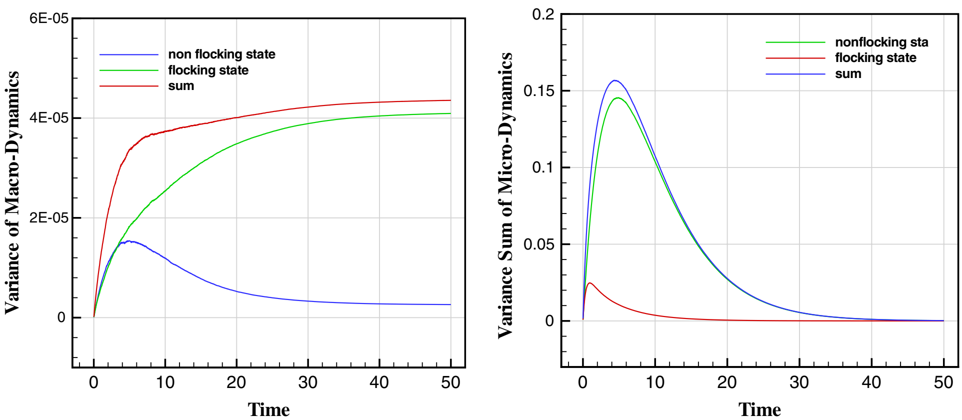


(e) $\mathrm{VAR}(V_1^i)$, $D = 0.4$ (f) $\mathrm{VAR}(V_0^i)$, $D = 0.4$

Fig. 5. (*Continued*)

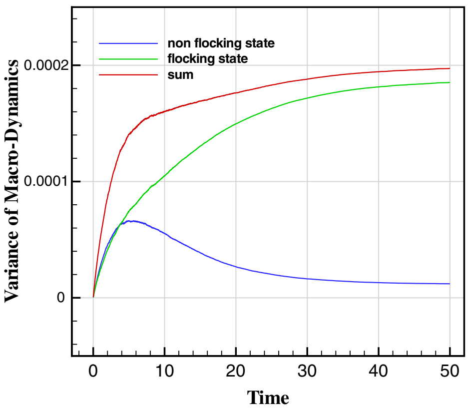
the volatility in (4.2). Consequently, a smaller noise strength D results in a stronger attraction among each state of volatilities.

Figure 6 presents the variance of macro dynamics $(\mathrm{Var}(V_k^c), k = 0, 1)$. Including the sum of the variance in the fluctuation process $\left(\sum_{i=1}^{100} \mathrm{Var}(\hat{V}_k^i), k = 0, 1\right)$, Fig. 6 leads to two conjectures: (1) the normal state process (V_0^i) results in a more stable average process than the flocking state process (V_1^i), and (2) the flocking state process (V_1^i) is realized with a smaller variance than the normal state process, which implies a stronger flocking phenomenon.

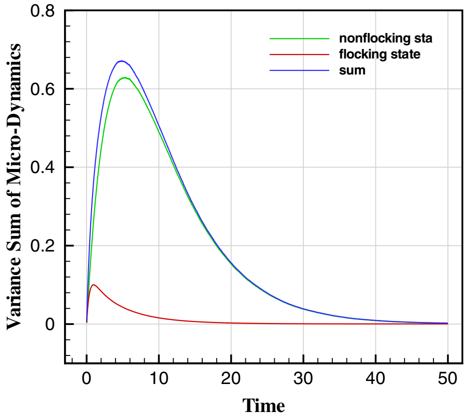


(a) $\mathrm{VAR}(V^c)$, $D = 0.1$ (b) $\sum_{i=1}^{100} \mathrm{VAR}(\hat{V}^i)$, $D = 0.1$

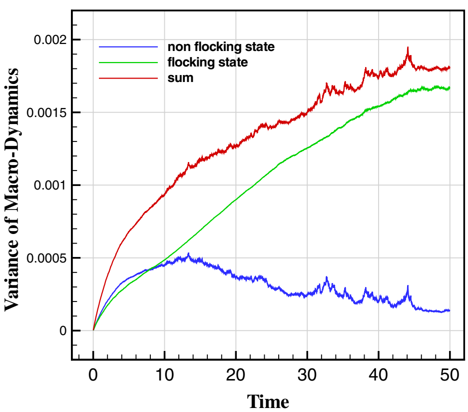
Fig. 6. Variance of macro dynamics V_0^c and V_1^c, and the sum of variance for micro dynamics $\hat{V}_0^i$ and $\hat{V}_1^i$.



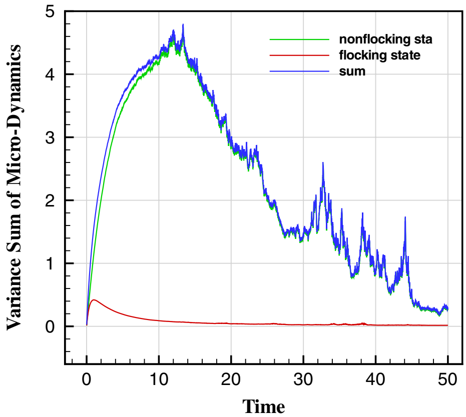
(c) $\mathrm{VAR}(V^c),\ D = 0.2$



(d) $\sum_{i=1}^{100} \mathrm{VAR}(\hat{V}^i),\ D = 0.2$



(e) $\mathrm{VAR}(V^c),\ D = 0.4$



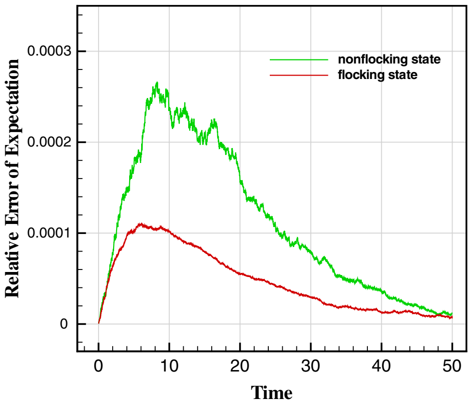
(f) $\sum_{i=1}^{100} \mathrm{VAR}(\hat{V}^i),\ D = 0.4$

Fig. 6. (*Continued*)

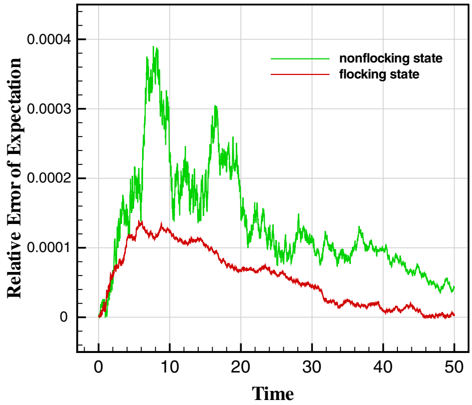
In Sec. 4, we derived the analytic formula (Corollary 4.1(2)) of the expected average process. Its behavior was drawn under the assumption that the Wiener process is equi-random. To check the robustness of numerical solutions, we calculated the relative errors between the numerical results and the analytic solutions. More precisely, when the numerical expectation of the average process is $(\mathbb{E}^h[V_k^{c,n}], k = 0, 1)$, and the expectation of the analytic form is $(\mathbb{E}[V_k^c(t_n)], k = 0, 1)$ as in Corollary 4.1(2), the relative errors, shown in Fig. 7, are

$$R\_Err_k(t_n) \equiv \left| \frac{\mathbb{E}^h[V_k^{c,n}] - \mathbb{E}[V_k^c(t_n)]}{\mathbb{E}[V_k^c(t_n)]} \right|, \quad k = 0, 1.$$

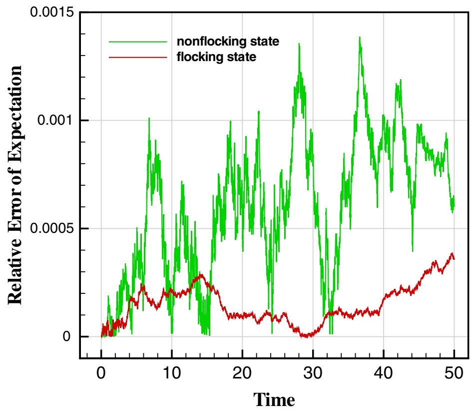
In this numerical analysis, the results from Corollary 4.1 are valid under the general configuration of the random variables.



(a) $D = 0.1$



(b) $D = 0.2$



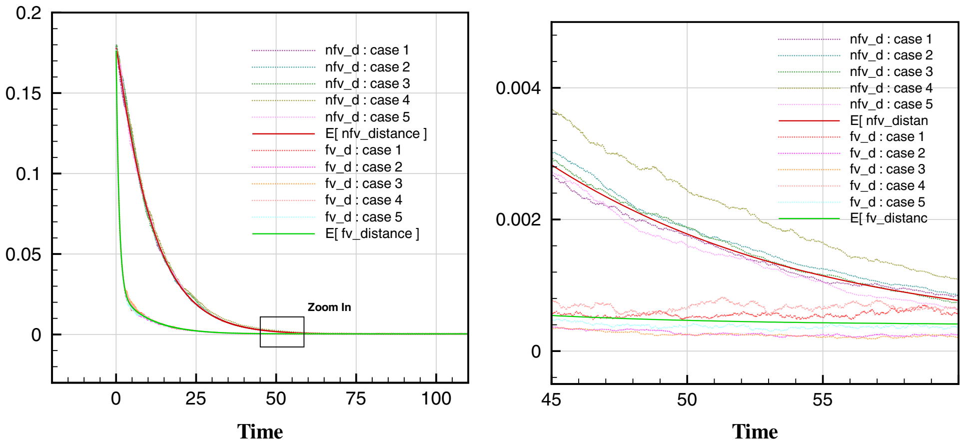
(c) $D = 0.4$

Fig. 7. Relative errors between numerical data $\mathbb{E}^h[V_k^{c,n}]$, and the analytic formula $\mathbb{E}[V_k^c(t)]$ in Corollary 4.1(2), $(k = 0, 1)$.

Figure 8 presents the samples and the expectations of the ensemble average of the mutual distances $\frac{\sum_{i=1}^n \sum_{j=i+1}^n \|V_k^i - V_k^j\|}{\frac{n(n+1)}{2}}$, $k = 0, 1$. Five sets of samples among 10^4 paths are indicated by the dotted lines, and their expectations are depicted using solid lines. These numerical results are in good agreement with the analytic results in Theorem 4.1.

6. Empirical Implication

This section presents the empirical implications of the numerical results. Figure 9 shows the dynamics of the weighted average pairwise correlation between 43 stocks' return volatilities from January 1995–January 2014. We selected 43 firms from the S&P 500 index; all selected firm's market capital size was substantial to cover



(a) $D = 0.1$, samples and expectations

(b) Zoomed figure

Fig. 8. Profile of $\frac{\sum_{i=1}^{n}\sum_{j=i+1}^{n}\|V_k^i - V_k^j\|}{\frac{n(n+1)}{2}}$, $k = 0, 1$ and $n = 100$.

almost 50% of the total market capitalization. A historical volatility (a dotted red line) is obtained by the standard deviation from the mean price over the period in contrast to the implied volatility (a dashed blue line) that is based on the prices of a three-month option. To investigate the trend in the volatility-correlation, we used a rolling pairwise correlation with a four-week window. Within a window, we first calculated the average correlation of one volatility to another. The aggregate correlation level was obtained by averaging the 1848 pairwise correlations. In the weighted average, the weight for each pairwise correlation was according to the firm's market capital size relative to the capital size of the S&P 500 index. Finally, the rolling means were smoothed with a Gaussian kernel, and a 95% confidence interval was chosen.

In Fig. 9, the degree of flocking intensifies when a market is in recessionary periods (a gray-shaded area highlights those years that are published by the National Bureau of Economic Research (NBER)). In particular, the weighted average correlation of volatilities, either historical or implied, is at its peak during the financial turmoil between the first quarter of 2008 and the second quarter of 2009. This period was known for high uncertainty, as indicated by the market's volatility index or VIX (represented by the solid black line). According to VIX, a considerable number of market participants expected that the market would remain highly volatile during this period. Its volatility was even greater than that of other recessions such as those in 1990 and 2001. In our model, a realized stock return becomes more volatile and flocks to another as a result of interaction with other volatilities. This state appears more intensively when market participants refer to another investor's opinion in portfolio selection by a lack of market consensus or firm-specific news. This is related to the degree of uncertainty in a stock market; the more heterogeneous

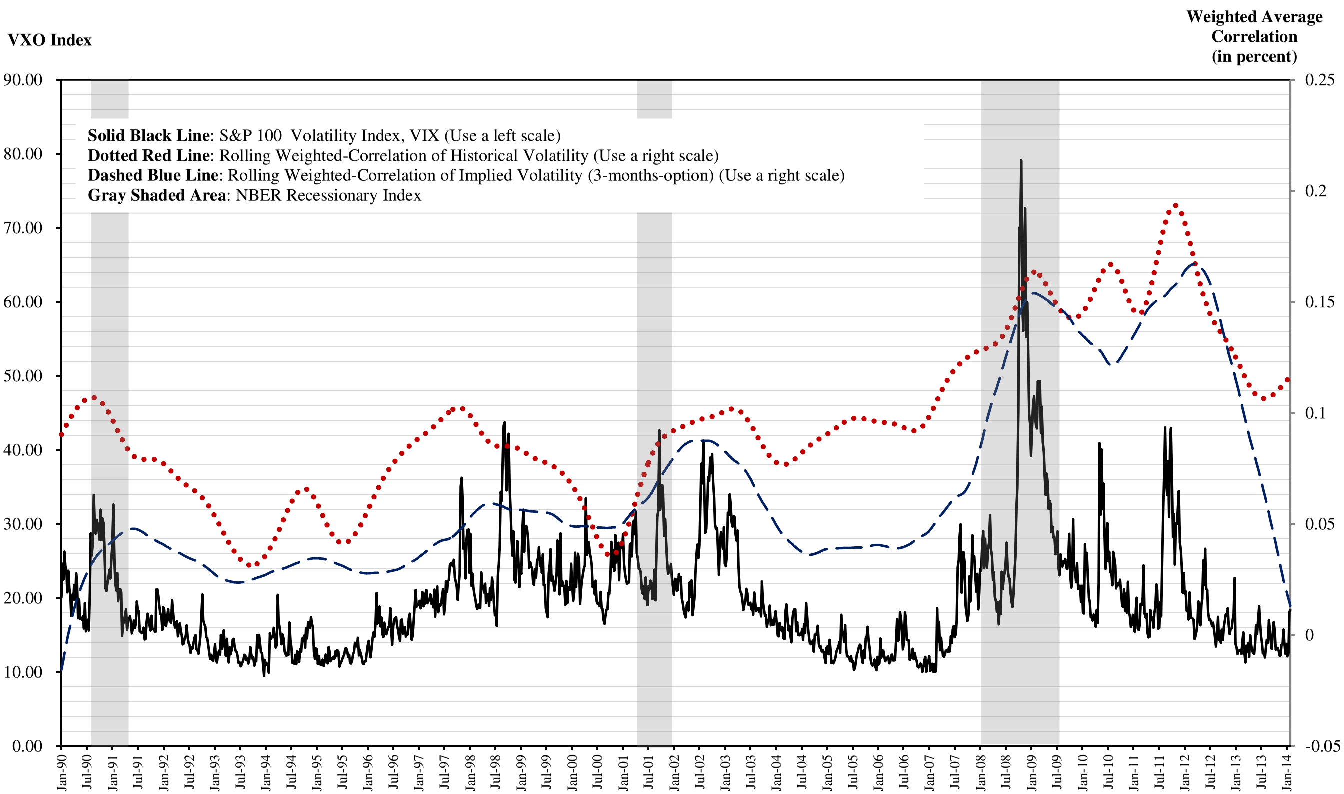


Fig. 9. (Color online) Rolling weighted correlation of historical volatilities and volatility index. Historical and implied (by a price of a three-month option) volatilities were used for the top 43 firms listed in the S&P 500. Firms were selected according to their size of market capitalization as of April 2013. In selection process, we tried to avoid sample bias from the dominant industry by choosing a limited number of firms in some industries that have more than five firms in "the top 50". The window-selection was robust when the correlation was compared to that from eight-week and 12-week windows. Results are available from the authors on request. *Source*: Thompson Reuters Eikon.

volatilities flock, the more uncertainties in a market where the information is not fairly distributed.

7. Conclusion

In this paper, we proposed an analytical model to capture the volatility-flocking phenomenon. In modeling the volatilities of N stock returns, we focused on the following three observations:

- Some volatilities tend to flock together.
- The flocking phenomenon is intensified when a shock brings considerable uncertainty to a market.
- As a market's information is absorbed to make a consensus, some volatilities move back to a normal state. Either being in a flocking state or remaining in a normal state is related to the degree of consensus on the market's environment: a moderate market, in which a price is cleared, or a chaotic market that has numerous noisy traders with respect to uncertainties. A volatility can be in either of these states and transition randomly between them, shifting either from a non-flocking state to a flocking state or vice versa.

In this work, we presented a mathematical model to capture these noteworthy patterns. With time-varying, geometric Brownian volatilities, coupled through the C–S flocking and regime switching mechanisms, we provided a realistic model for volatility flocking (herding). For the all-to-all interactions, where all assets' volatilities are coupled to each other with a constant interaction weight, we showed that the common volatility emerges asymptotically. Finally, we discussed its financial applications and the correlation between heterogeneous volatilities.

Appendix A. First-Order Linear ODE System

In this section, we present an explicit solution to the linear system (3.2).

A.1. *Two-stock market*

Consider the 2×2 linear system of ordinary differential equations:

$$\mathbf{V}_2' = \mathbf{A}_2 \mathbf{V}_2, \tag{A.1}$$

where

$$\mathbf{V}_2 = \begin{pmatrix} V_0^1 \\ V_0^2 \\ V_1^1 \\ V_1^2 \end{pmatrix}, \quad \mathbf{A}_2(t) = \begin{pmatrix} -p_{00} & 0 & p_{10} & 0 \\ 0 & -p_{00} & 0 & p_{10} \\ p_{01} & 0 & -\dfrac{K}{2} - p_{11} & \dfrac{K}{2} \\ 0 & p_{01} & \dfrac{K}{2} & -\dfrac{K}{2} - p_{11} \end{pmatrix}.$$

We first derive the characteristic equation of $\mathbf{A}_2$:

$$\begin{aligned} 0 &= \det(\mathbf{A}_2 - \lambda \mathbf{I}_2) \\ &= (\lambda^2 - (\mathrm{tr}(Q))\lambda + \det(Q))(\lambda^2 + (-\mathrm{tr}(Q) + K)\lambda + \det(Q) + Kp_{00}), \end{aligned}$$

where Q is the 2×2 matrix defined in (2.3) and

$$\det(Q) = p_{00}p_{11} - p_{01}p_{10}, \quad \mathrm{tr}(Q) = -p_{00} - p_{11}.$$

Four eigenvalues $\lambda_i^j, (i = 0, 1,\ j = +, -)$, of matrix $\mathbf{A}_2$ are obtained:

$$\begin{aligned} \lambda_0^{\pm} &= \frac{1}{2}[\mathrm{tr}(Q) \pm \sqrt{(\mathrm{tr}(Q))^2 - 4\det(Q)}], \\ \lambda_1^{\pm} &= \frac{1}{2}[\mathrm{tr}(Q) - K \pm \sqrt{(-\mathrm{tr}(Q) + K)^2 - 4\det(Q) - 4Kp_{00}}]. \end{aligned} \tag{A.2}$$

The eigenvectors $\mathbf{E}_0^j$ and $\mathbf{E}_1^j$, corresponding to eigenvalues λ_0^j, λ_1^j, $(j = +, -)$, are as follows:

$$\begin{aligned} \mathbf{E}_0^j &= \begin{pmatrix} 1 & 1 & \dfrac{\lambda_0^j + p_{00}}{p_{10}} & \dfrac{\lambda_0^j + p_{00}}{p_{10}} \end{pmatrix}^\top, \\ \mathbf{E}_1^j &= \begin{pmatrix} 1 & -1 & \dfrac{\lambda_1^j + p_{00}}{p_{10}} & -\dfrac{\lambda_1^j + p_{00}}{p_{10}} \end{pmatrix}^\top. \end{aligned} \tag{A.3}$$

The general solution to (A.1) is

$$\mathbf{V}_2(t) = c_0^+\mathbf{E}_0^+ e^{\lambda_0^+ t} + c_0^-\mathbf{E}_0^- e^{\lambda_0^- t} + c_1^+\mathbf{E}_1^+ e^{\lambda_1^+ t} + c_1^-\mathbf{E}_1^- e^{\lambda_1^- t}.$$

• **Example:** Consider an initial condition given by

$$\mathbf{V}_2(0) = (V_0^1(0), V_0^2(0), V_1^1(0), V_1^2(0)) = (a, b, a, b).$$

Then,

$$\begin{aligned} c_0^+ &= \left(\frac{a+b}{2}\right)\left(\frac{p_{11} - p_{00} - \lambda_0^-}{\lambda_0^+ - \lambda_0^-}\right), \quad & c_0^- &= -\left(\frac{a+b}{2}\right)\left(\frac{p_{11} - p_{00} - \lambda_0^+}{\lambda_0^+ - \lambda_0^-}\right), \\ c_1^+ &= \left(\frac{a-b}{2}\right)\left(\frac{p_{11} - p_{00} - \lambda_1^-}{\lambda_1^+ - \lambda_1^-}\right), \quad & c_1^- &= -\left(\frac{a-b}{2}\right)\left(\frac{p_{11} - p_{00} - \lambda_1^+}{\lambda_1^+ - \lambda_1^-}\right). \end{aligned}$$

A.2. *Multi-stock market*

In this subsection, we consider the general situation when $N \geq 3$. Recall the dynamics of $\mathbf{V}$:

$$\begin{aligned} \frac{dV_1^i(t)}{dt} &= \left(-\frac{K(1-N)}{N} - p_{00}\right)V_1^i + p_{10}V_2^i + \frac{K}{N}\sum_{j\neq i}^{N} V_1^j(t), \quad & \varepsilon_t = 0, \\ \frac{dV_2^i(t)}{dt} &= o_{01}V_1^i - p_{11}V_2^i, & \varepsilon_t = 1. \end{aligned}$$

The above equations can be written in a matrix form. To do this, we denote $\mathbf{V}_N$ and $\mathbf{A}_N(t)$ by

$$\mathbf{V}_N = (V_0^1, V_0^2, \ldots, V_0^N, V_1^1, V_1^2, \ldots, V_1^N)^T,$$

and

$$\mathbf{A}_N(t) = \begin{pmatrix} -p_{00} & 0 & \cdots & 0 & p_{10} & 0 & \cdots & 0 \\ 0 & -p_{00} & \cdots & 0 & 0 & p_{10} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdot & 0 & -p_{00} & 0 & \cdot & 0 & p_{10} \\ p_{01} & 0 & \cdots & 0 & \alpha & \dfrac{K}{N} & \cdots & \dfrac{K}{N} \\ 0 & p_{01} & \cdots & 0 & \dfrac{K}{N} & \alpha & \cdots & \dfrac{K}{N} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdot & 0 & p_{01} & \dfrac{K}{N} & \cdot & \dfrac{K}{N} & \alpha \end{pmatrix}, \quad \alpha = \frac{K(1-N)}{N} - p_{11}.$$

Thus, the first-order linear system can be written as

$$\mathbf{V}_N' = \mathbf{A}_N \mathbf{V}_N.$$

By direct calculation, the characteristic equation of $\mathbf{A}_N$ is

$$0 = (\lambda^2 - (\mathrm{tr}(Q))\lambda + \det(Q))(\lambda^2 + (-\mathrm{tr}(Q) + K)\lambda + \det(Q) + Kp_{00})^{2N-2}.$$

Four eigenvalues of $\mathbf{A}$ are obtained, λ_i, $(i = 1, 2, 3, 4)$, regardless of N (≥ 2):

$$\lambda_0^{\pm} = \frac{\mathrm{tr}(Q) \pm \sqrt{(\mathrm{tr}(Q))^2 - 4\det(Q)}}{2}$$

$$\lambda_1^{\pm} = \frac{\mathrm{tr}(Q) - K \pm \sqrt{(-\mathrm{tr}(Q) + K)^2 - 4\det(Q) - 4Kp_{00}}}{2},$$

and α_1^+, α_1^- are repeated eigenvalues. The eigenvectors $\mathbf{E}_1^{jm}$, $(m = 1, \ldots, N-1)$ corresponding to eigenvalues λ_1^j, $(j = +, -)$ are represented by the multiplication $\mathbf{E}_1^{jm} = \mathbf{E}_1^j I_N^k$ of $\mathbf{E}_1^j$ and the $(2N \times 4)$ matrix I_N^k, of which only the $(1,1)$, $(2,k)$, $(3,N)$ and $(4, N+k)$ entries are 1, and all other entries are zeros. The eigenvector $\mathbf{E}_0^{j1}$ corresponding to eigenvalues λ_0^j, $(j = +, -)$ is represented by the multiplication of $\mathbf{E}_0^j$ and the $(2N \times 4)$ matrix I_N^1; that is, $\mathbf{E}_0^{j1} = \mathbf{E}_0^j I_N^1$. The general solution is:

$$\begin{aligned}\mathbf{V}_N(t) = \sum_{j=+,-} & (c_1^{j1}\mathbf{E}_1^{j1} e^{\lambda_1^j t} + c_1^{j2}(\mathbf{E}_1^{j1} t e^{\lambda_1^j t} + \mathbf{E}_1^{j2} e^{\lambda_1^j t}) + \cdots + c_1^{jN-1} \\ & \times (\mathbf{E}_1^{j1} t^{N-2} e^{\lambda_1^j t} + \cdots + \mathbf{E}_1^{jN-1} e^{\lambda_1^j t})) + c_0^{+1}\mathbf{E}_0^{+1} e^{\lambda_0^+ t} + c_0^{-1}\mathbf{E}_0^{-1} e^{\lambda_0^- t}.\end{aligned}$$

Appendix B. General Solution of a Linear Stochastic System

In this section, we derive a solution formula for a linear stochastic system.

Consider an Itô process X_t whose dynamics are governed by the following stochastic differential equation:

$$dX_t = (a_1(t)X_t + a_2(t))dt + (b_1(t)X_t + b_2(t))dW_t, \quad t \in (0, T), \tag{B.1}$$

where $a_1(t), a_2(t), b_1(t), b_2(t) \in \mathbb{R}$ and $t \in [0, T)$.

Lemma B.1. *The general solution to* (B.1) *has the following explicit representation*:

$$X_t = \Phi_t \left(X_0 + \int_0^t (a_2(s) - b_1(s)b_2(s))\Phi_s^{-1} ds + \int_0^t b_2(s)\Phi_s^{-1} dW_s \right),$$

where Φ_t *is given by the following relation*:

$$\Phi_t := \exp\left[\int_0^t \left(a_1(s) - \frac{b_1^2(s)}{2} \right) ds + \int_0^t b_1(s) dW_s \right].$$

Proof. Applying the chain rule $X_t\Phi_t^{-1}$ yields:

$$\begin{aligned} d(X_t\Phi_t^{-1}) &= X_t d\Phi_t^{-1} + \Phi_t^{-1} dX_t + dX_t d\Phi_t^{-1} \\ &= X_t\Phi_t^{-1}(b_1^2(t)dt - a_1(t)dt - b_1(t)dW_t) \\ &\quad + \Phi^{-1}(X_t a_1(t)dt + a_2(t)dt + X_t b_1(t)dW_t + b_2(t)dW_t) \\ &\quad - X_t b_1^2(t)\Phi_t^{-1} dt - b_1(t)b_2(t)\Phi_t^{-1} dt. \end{aligned}$$

Rearranging the first term in the right-hand side of the above relation results in

$$d(X_t(\Phi_t^{-1})) = (a_2(t) - b_1(t)b_2(t))\Phi_t^{-1} dt + b_2(t)\Phi_t^{-1} dW_t.$$

This yields the explicit representation formula for X_t. □

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