Impact of Nonstationarity on Estimating and Modeling Empirical Copulas of Daily Stock Returns

Marcel Wollschläger[*,a,b] and Rudi Schäfer[a,c]

[a] *Faculty of Physics, University of Duisburg-Essen,Lotharstraße 1, 47048 Duisburg, Germany*
[b] *Chair for Energy Trading and Finance, University of Duisburg-Essen, Universitätsstraße 12, 45141 Essen, Germany*
[c] *Research and Prototyping, Arago GmbH, Eschersheimer Landstraße 526–532, 60433 Frankfurt am Main, Germany*

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The R package `kcopula` and Mathematica code is provided by Kremer (2020a,b).

Abstract

All too often, measuring statistical dependencies between financial time series is reduced to a linear correlation coefficient. However, this may not capture all facets of reality. This paper studies empirical dependencies of daily stock returns by their pairwise copulas. We investigate in particular to which extent the nonstationarity of financial time series affects both the estimation and the modeling of empirical copulas. We estimate empirical copulas from the nonstationary, original return time series and stationary, locally normalized ones. Thereby, we are able to explore the empirical dependence structure on two different scales: globally and locally. Additionally, the asymmetry of the empirical copulas is emphasized as a fundamental characteristic. We compare our empirical findings with a single Gaussian copula, a correlation-weighted average of Gaussian copulas, the K-copula which directly addresses the nonstationarity of dependencies as a model parameter, and the skewed Student's t-copula. The K-copula covers the empirical dependence structure on the local scale most adequately, whereas the skewed Student's t-copula best captures the asymmetry of the empirical copula on the global scale.

Keywords: Copulas; Financial time series; Nonstationarity; Asymmetry; Multivariate mixture; K-copula

JEL Classification: C13; C46; C55; G12; G10

[*]Corresponding author. Email: marcel.wollschlaeger@uni-due.de; marcel.kremer@uni-due.de.

1 Introduction

The study of empirical dependencies of financial time series is not only important for risk management and portfolio optimization, but it is a prerequisite for a deeper understanding. It is common practice to reduce the question of statistical dependence to the study of the Pearson correlation coefficient; the most notable works in this direction include Markowitz (1952); Martens and Poon (2001); Pelletier (2006); Engle and Colacito (2006). The correlation coefficient, however, only measures the degree of linear dependence between two random variables. Non-linear dependencies or more complicated dependence structures are not captured appropriately. In this paper, we choose a copula approach to investigate the full statistical dependence. In contrast to multivariate distributions, which also contain the marginal distributions, copulas introduced by Sklar (1973) transform these marginal distributions to uniform distributions. This allows for a direct study of statistical dependencies. Copulas find application in various fields of research, such as civil engineering (Kilgore and Thompson, 2011), medicine (Onken et al., 2009), climate and weather research (Schölzel and Friederichs, 2008), and the generation of random vectors (Strelen, 2010). In finance, copulas are primarily used in risk and portfolio management (see, e.g., Brigo et al., 2010; Embrechts et al., 2002, 2003; Low et al., 2013; McNeil et al., 2005; Meucci, 2011; Rosenberg and Schuermann, 2006). A compact survey of the vastly existing and growing literature of copula models is provided by Patton (2012b,a). For a more general introduction to copulas, the reader is referred to Joe (1997); Nelsen (2006).

In all these contexts, easily tractable analytical copulas are used as fundamental building blocks for multivariate distributions. Empirical studies which do not assume an analytical dependence structure *a priori* are few and far between (see, e.g., Münnix et al., 2012; Ning et al., 2008). This is where our contribution fits in. We study empirical dependencies of daily returns of S&P 500 stocks. To achieve a good statistical significance for the estimation of a full copula, a very large amount of data is necessary. Therefore, we restrict ourselves to the bivariate case and average over many pairwise copulas. In addition, we consider a rather large time horizon. The consideration of a large time horizon confronts us with two problems: the nonstationarity of the individual time series and their dependencies. The former, i.e., time-varying trends and volatilities, has to be taken into account when we estimate empirical copulas. To this end, we apply the local normalization to the empirical return time series, which rids them of nonstationary trends and volatilities. The local normalization was introduced by Schäfer and Guhr (2010) and yields stationary, standard normally distributed time series while preserving dependencies between different time series. In particular, it allows us to study the empirical dependence structure on a local scale, whereas the copula of original returns reveals the dependence structure on a global scale.

We compare our empirical findings with different analytical copulas, addressing particularly the question how far the assumption of a Gaussian dependence structure may carry. Since we average over many stock pairs, it would be naive to assume that the resulting average pairwise copula could be described by a single Gaussian copula where only the average correlation is included. Indeed, we find rather poor agreement, especially on the global scale. A correlation-weighted average of Gaussian copulas, which takes into account the different pairwise correlations, provides only a slightly better description. We have to consider the time-varying dependencies as well. These can be addressed by an ensemble average over random correlations Schmitt et al. (2013). This ansatz yields good agreement for multivariate return distributions. In this paper, we introduce the bivariate K-copula resulting from this random matrix model and compare it with our empirical findings. Overall, we find rather good agreement, especially on the local scale. However, the empirical copulas show an asymmetry in the tail dependence which our model cannot account for. As a model that allows for an asymmetry in the dependence structure, the skewed Student's t-distribution introduced by Hansen (1994), and its corresponding copula introduced by Demarta and McNeil (2005), are steadily gaining popularity, in particular with regard to credit risk management (see, e.g., Hu and Kercheval, 2006; Dokov et al., 2008) and the study of asymmetric dependencies (see, e.g., Sun et al., 2008). On the global scale, where the asymmetry is more pronounced, the skewed Student's t-copula yields improved agreement with our empirical copulas.

The paper is structured as follows. In Section 2, we present our data set and the theoretical background for the local normalization, correlations and analytical copulas. In particular, we introduce the K-copula. We present our results in Section 3 and conclude in Section 4.

2 Data set and theoretical background

2.1 Prices and returns

In our empirical study, we consider the daily closing prices of all stocks in the S&P 500 stock index, which have been adjusted for splits and dividends. Our observation period ranges from March 1, 2006 to December 31, 2012. The data are obtained from `https://finance.yahoo.com`. From the price time series, we calculate returns – later referred to as original returns – as the percentage change,

$$r_k(t) = \frac{S_k(t+\Delta t) - S_k(t)}{S_k(t)} \ , \tag{1}$$

where $S_k(t)$ is the price of a stock k at time t and $\Delta t = 1$ trading day. We denote the return time series by $r_k = \{r_k(t)\}_{t=1,\dots,T}$. In total, we arrive at $T = 1760$ daily returns for $K = 460$ stocks which were continuously traded and listed in the S&P 500 index during the entire observation period. It is well known that original returns show strongly nonstationary behavior.

2.2 Local normalization

To correct for nonstationary trends and volatilities in the original returns, we employ a method called local normalization which has been introduced by Schäfer and Guhr (2010). The locally normalized return is defined by

$$\rho_k(t) = \frac{r_k(t) - \mu_k(t)}{\sigma_k(t)} \ , \tag{2}$$

with the local average $\mu_k(t)$ and local volatility $\sigma_k(t)$ of a stock k, which are both estimated on a time window over the preceding 13 trading days. As has been shown in Schäfer and Guhr (2010), the time interval of 13 trading days for the estimation of local average $\mu_k(t)$ and volatility $\sigma_k(t)$ yields the best approximation of stationary, standard normally distributed time series.

2.3 Correlations

To measure the statistical dependence between two continuous random variables X, Y, one often considers the Pearson correlation coefficient,

$$C_{X,Y} = \frac{\mathrm{Cov}(X,Y)}{\sigma_X \, \sigma_Y} \ , \tag{3}$$

where $\mathrm{Cov}(X,Y)$ is the covariance between X, Y, and σ_X, σ_Y are the respective standard deviations. We are interested in the correlation between returns of different stocks. Nonstationarity of the individual time series is a fundamental problem for the estimation of correlation coefficients. For return time series, the time-varying volatilities lead to estimation errors in Equation (3) (see, e.g, Schäfer and Guhr, 2010; Münnix et al., 2012). This can be taken into account by the local normalization introduced in Section 2.2. As mentioned in Section 1, the correlation coefficient only measures the linear dependence between two random variables. Therefore, it is not well suited to measure arbitrary dependencies. For this purpose, we consider copulas in the following.

2.4 Definition of copulas

The complete information of the statistical dependence between random variables is contained in their joint distribution. However, it is difficult to compare joint distributions if the marginal distributions are not the same for all random variables. To achieve comparability, we can transform each marginal distribution to a uniform distribution. Then, the joint distribution of such transformed, uniformly distributed random variables is called a copula.

In our short introduction to copulas, we shall confine ourselves to the bivariate case since we only study empirical pairwise copulas. We consider two continuous random variables X, Y with joint probability density function $f_{X,Y}(x, y)$. The joint cumulative distribution function is given by

$$F_{X,Y}(x,y) = \int\limits_{-\infty}^{x} \mathrm{d}x' \int\limits_{-\infty}^{y} \mathrm{d}y' \, f_{X,Y}(x',y') \ . \tag{4}$$

We denote the marginal cumulative distribution functions of X, Y by $F_X(x), F_Y(y)$, and the corresponding marginal probability density functions by $f_X(x) = \frac{\mathrm{d}}{\mathrm{d}x}F_X(x)$, $f_Y(y) = \frac{\mathrm{d}}{\mathrm{d}y}F_Y(y)$. According to Sklar (1973), we define the copula by

$$\begin{aligned} & F_{X,Y}(x,y) = \mathrm{Cop}_{X,Y}(F_X(x), F_Y(y)) \\ \Leftrightarrow \quad & \mathrm{Cop}_{X,Y}(u,v) = F_{X,Y}(F_X^{-1}(u), F_Y^{-1}(v)) \ , \end{aligned} \tag{5}$$

where we used the transformation $u = F_X(x)$, $v = F_Y(y)$, and F_X^{-1}, F_Y^{-1} indicate the inverse marginal distribution functions, also referred to as quantile functions. Thus, a joint cumulative distribution function may be expressed by the marginal distribution functions and a copula. According to the usual definition of a probability density function, the copula density is defined as the derivative of the copula,

$$\begin{aligned} \mathrm{cop}_{X,Y}(u,v) &= \frac{\partial^2 \mathrm{Cop}_{X,Y}(u,v)}{\partial u \partial v} \\ &= \left. \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y} \right|_{\substack{x=F_X^{-1}(u) \\ y=F_Y^{-1}(v)}} \frac{\partial F_X^{-1}(u)}{\partial u} \frac{\partial F_Y^{-1}(v)}{\partial v} \\ &= \left. \frac{f_{X,Y}(x,y)}{f_X(x) f_Y(y)} \right|_{\substack{x=F_X^{-1}(u) \\ y=F_Y^{-1}(v)}} \ , \end{aligned} \tag{6}$$

where we used the inverse function rule in the last step.

2.5 Empirical pairwise copula

In order to calculate the empirical pairwise copula of two return time series r_k, r_l, we have to transform their marginal distributions to uniformity. We then refer to the transformed time series as $u = u_k$, $v = u_l$. To obtain uniformly distributed time series, we utilize the empirical distribution function,

$$u_k(t) = F_k(r_k(t)) = \frac{1}{T} \sum_{\tau=1}^{T} \mathbf{1}\{r_k(\tau) \leq r_k(t)\} - \frac{1}{2T} \ , \tag{7}$$

where $\mathbf{1}$ denotes the indicator function. The term $-\frac{1}{2}$ in Equation (7) ensures that all values $u_k(t)$ lie strictly in the interval (0,1). We arrive at the empirical pairwise copula density as a two-dimensional histogram of data pairs $(u_k(t), u_l(t))$, $t = 1, \ldots, T$.

2.6 Bivariate Gaussian copula

The Gaussian copula is the dependence structure that arises from a normal distribution. We consider two random variables X, Y, which follow a joint normal distribution with correlation coefficient c. The bivariate Gaussian copula is given by

$$\mathrm{Cop}_c^{\mathrm{G}}(u,v) = \Phi_c(\Phi^{-1}(u), \Phi^{-1}(v)) \ , \tag{8}$$

where Φ_c is the bivariate standard normal distribution function, and Φ^{-1} the inverse marginal standard normal distribution function,

$$\Phi_c(x,y) = \int\limits_{-\infty}^{x} \mathrm{d}x' \int\limits_{-\infty}^{y} \mathrm{d}y' \, \frac{1}{2\pi\sqrt{1-c^2}} \exp\left(-\frac{(x')^2 - 2cx'y' + (y')^2}{2(1-c^2)}\right) , \tag{9}$$

$$\Phi(x) = \int\limits_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x')^2}{2}\right) \mathrm{d}x' \ . \tag{10}$$

Partial differentiation yields the bivariate Gaussian copula density,

$$\mathrm{cop}_c^{\mathrm{G}}(u,v) = \frac{1}{\sqrt{1-c^2}} \times \exp\left(-\frac{c^2\Phi^{-1}(u)^2 - 2c\,\Phi^{-1}(u)\Phi^{-1}(v) + c^2\Phi^{-1}(v)^2}{2(1-c^2)}\right) \ . \tag{11}$$

In Figure 1, we show the Gaussian copula density $\mathrm{cop}_c^{\mathrm{G}}(u,v)$ for a positive and a negative correlation c. We can identify the inherently symmetric character as a fundamental property. Gaussian copula densities with positive correlation describe strong dependencies between events in the same tail of each marginal distribution, while negative correlations describe strong dependencies between events in the opposite tails of the marginal distributions.

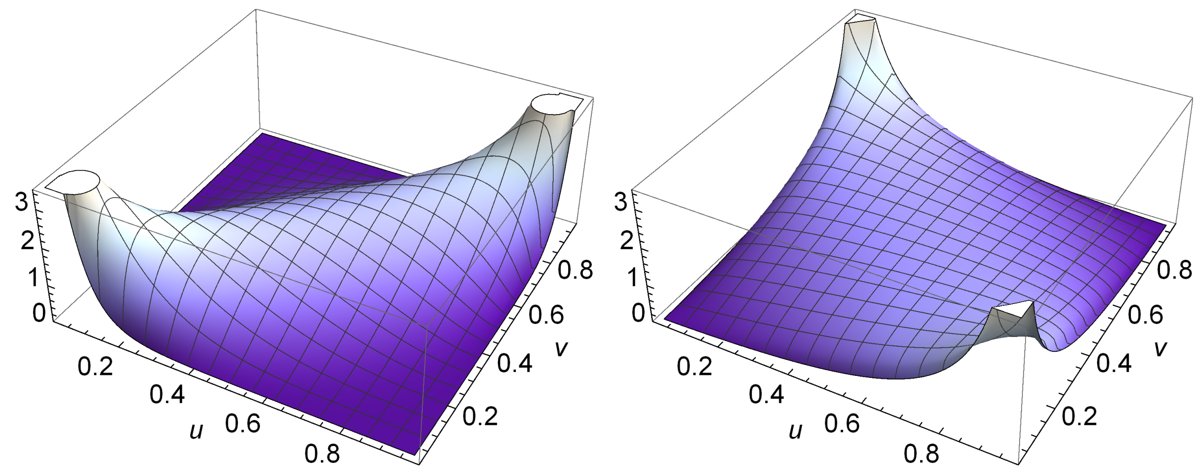

Figure 1: Gaussian copula density $\mathrm{cop}_c^{\mathrm{G}}(u,v)$ with correlation $c = 0.8$ (left) and $c = -0.5$ (right).

2.7 Bivariate K-copula

Correlations between financial time series are time-dependent (see, e.g., Engle, 2002; Tse and Tsui, 2002; Cappiello et al., 2006; Schäfer and Guhr, 2010; Münnix et al., 2012). To take this empirical fact into account, we consider the random matrix model introduced by Schmitt et al. (2013). The random matrix model assumes, for each return vector $\mathbf{r}(t) = (r_1(t), \ldots, r_K(t))$ at time $t = 1, \ldots, T$, a multivariate normal distribution,

$$g(\mathbf{r}(t), \Sigma_t) = \frac{1}{\sqrt{\det(2\pi\Sigma_t)}} \exp\left(-\frac{1}{2}\mathbf{r}^\dagger(t)\Sigma_t^{-1}\mathbf{r}(t)\right) , \tag{12}$$

with time-dependent covariance matrix Σ_t. Each covariance matrix Σ_t at time $t = 1, \ldots, T$, is then modeled by a random matrix $\Sigma_t \to AA^\dagger/N$ drawn from a Wishart distribution (Wishart, 1928). To this end, we choose a multivariate normal distribution,

$$w(A, \Sigma) = \frac{1}{\det^{N/2}(2\pi\Sigma)} \exp\left(-\frac{1}{2}\mathrm{tr}A^\dagger\Sigma^{-1}A\right) , \tag{13}$$

for the matrix elements of A, which fluctuate around the empirical average covariance matrix $\Sigma = \langle\Sigma_t\rangle_{t=1,\ldots,T}$. The distribution of a sample of return vectors $\mathbf{r}$ is then given by a multivariate normal–Wishart mixture: we average a multivariate normal distribution over an ensemble of Wishart-distributed covariances, which finally results in a K-distribution,

$$\begin{aligned}
\langle g\rangle(\mathbf{r}, \Sigma, N) &= \int \mathrm{d}[A]\, g\left(\mathbf{r}, \frac{1}{N}AA^\dagger\right) w(A, \Sigma) \\
&= \frac{1}{2^{N/2+1}\Gamma(N/2)\sqrt{\det(2\pi\Sigma/N)}} \\
&\qquad \frac{\mathcal{K}_{(K-N)/2}\left(\sqrt{N\mathbf{r}^\dagger\Sigma^{-1}\mathbf{r}}\right)}{\sqrt{N\mathbf{r}^\dagger\Sigma^{-1}\mathbf{r}}^{(K-N)/2}} \\
&= \frac{1}{(2\pi)^K\Gamma(N/2)\sqrt{\det\Sigma}} \\
&\qquad \int\limits_0^\infty \mathrm{d}z\, z^{\frac{N}{2}-1}\mathrm{e}^{-z}\sqrt{\frac{\pi N}{z}}^K \exp\left(-\frac{N}{4z}\mathbf{r}^\dagger\Sigma^{-1}\mathbf{r}\right) ,
\end{aligned} \tag{14}$$

where $\int \mathrm{d}[A]$ denotes the integral over all matrix elements of A, and $\mathcal{K}_\lambda$ is the modified Bessel function of the second kind of order λ. The K-distribution (14) contains only two parameters: the empirical average covariance matrix Σ and a free parameter N, which characterizes the fluctuations of covariances around the empirical average Σ. In this manner, the empirically observed nonstationarity of covariances enters directly into the random matrix model. Note that the K-distribution (14) is a special case of the multivariate generalized hyperbolic distribution (see, e.g., McNeil et al., 2005; Schmidt et al., 2006; Aas and Haff, 2006; Necula, 2009; Socgnia and Wilcox, 2014; Vilca et al., 2014; Browne and McNicholas, 2015).

By considering the K-distribution (14) for the bivariate case, we can derive the bivariate K-copula density via Equation (6). In our case, the probability density function is the bivariate K-distribution, $f_{X,Y}(x, y) = \langle g\rangle(\mathbf{r}, \Sigma, N)$ with $K = 2$. Since the copula density is independent of the marginal distributions, we can choose the standard deviations $\sigma_X = \sigma_Y = 1$. This leads to the covariance matrix,

$$\Sigma = \begin{pmatrix} \sigma_X^2 & \sigma_X\sigma_Y c \\ \sigma_X\sigma_Y c & \sigma_Y^2 \end{pmatrix} = \begin{pmatrix} 1 & c \\ c & 1 \end{pmatrix} , \tag{15}$$

which is merely a correlation matrix with the empirical average correlation coefficient c. Thus, the parameter N characterizes the fluctuation strength of the correlations around their mean value c: the smaller N, the larger the fluctuations. In the limit $N \to \infty$, the fluctuations vanish and we arrive at a normal distribution and a Gaussian copula, respectively. The marginal probability density functions of the K-distribution (14) are identical, $f_X(\cdot) = f_Y(\cdot)$, with

$$
\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} \mathrm{d}y\, f_{X,Y}(x,y) \\
&= \frac{1}{\Gamma(N/2)} \int_0^{\infty} \mathrm{d}z\, z^{\frac{N}{2}-1} \mathrm{e}^{-z} \sqrt{\frac{N}{4\pi z}} \exp\left(-\frac{N}{4z}x^2\right) .
\end{aligned} \tag{16}
$$

The marginal cumulative distribution functions, $F_X(\cdot) = F_Y(\cdot)$, with

$$
\begin{aligned}
F_X(x) &= \int_{-\infty}^{x} \mathrm{d}\xi\, f_X(\xi) \\
&= \frac{1}{\Gamma(N/2)} \int_0^{\infty} \mathrm{d}z\, z^{\frac{N}{2}-1} \mathrm{e}^{-z} \int_{-\infty}^{x} \mathrm{d}\xi \sqrt{\frac{N}{4\pi z}} \exp\left(-\frac{N}{4z}\xi^2\right) ,
\end{aligned} \tag{17}
$$

have to be calculated and inverted numerically. Insertion into equation (6) then yields the K-copula density $\mathrm{cop}^{\mathrm{K}}_{c,N}(u,v)$.

For illustration, Figure 2 shows the K-copula density for an average correlation $c = 0$ and a fluctuation strength $N = 5$, and for $c = 0.2$, $N = 4$. We observe that the K-copula density contains both positive and negative correlations in contrast to the Gaussian copula density. In particular, negative correlations are covered even if the average correlation coefficient c is positive but the fluctuations around it are large enough. However, the K-copula density is still symmetric with respect to the facing corners.

Kremer (2020b) develops the R package `kcopula` and Kremer (2020a) provides Mathematica code for the bivariate K-copula.

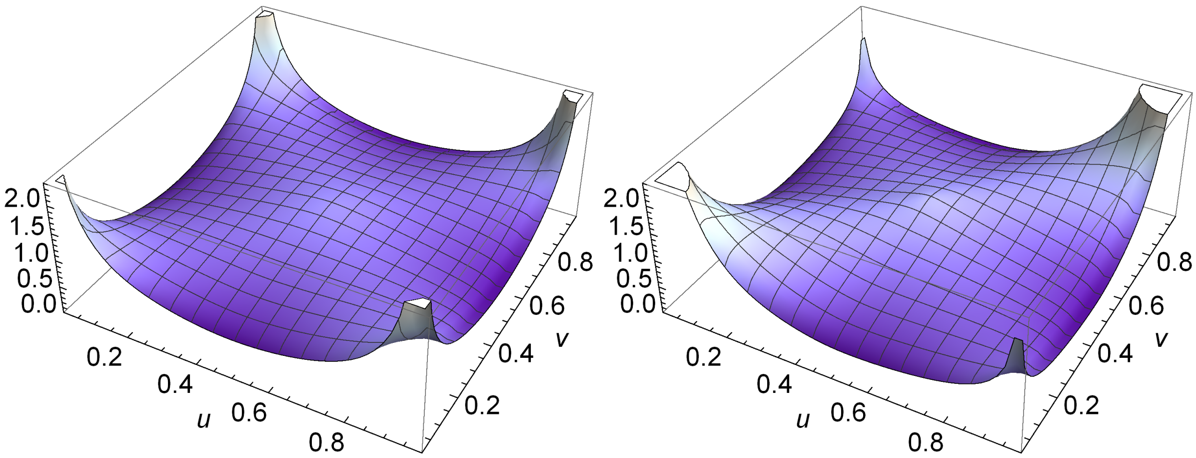

Figure 2: K-copula density $\mathrm{cop}^{\mathrm{K}}_{c,N}(u,v)$ with average correlation $c = 0$ and parameter $N = 5$ (left) and $c = 0.2$, $N = 4$ (right).

2.8 Bivariate skewed Student's t-copula

We consider a bivariate skewed Student's t-distributed random variable $\mathbf{Z} = (X, Y)$ represented by the normal mean–variance mixture,

$$
\mathbf{Z} = \boldsymbol{\mu} + \boldsymbol{\gamma} W + \mathbf{Q}\sqrt{W} , \tag{18}
$$

where $\boldsymbol{\mu} = (\mu_1, \mu_2)$ is a location parameter vector, $W \sim IG(\nu/2, \nu/2)$ follows an inverse gamma distribution with ν degrees of freedom, $\mathbf{Q} \sim N(0, \Sigma)$ is independent of W and drawn from a bivariate normal distribution with zero mean and covariance matrix Σ, and $\boldsymbol{\gamma} = (\gamma_1, \gamma_2)$ is the skewness parameter vector.

Since we aim at deriving a skewed Student's t-copula density from its corresponding distribution following Equation (6), we are able to drop the location parameter vector $\boldsymbol{\mu}$ in the stochastic representation (18), i.e. $\boldsymbol{\mu} = \mathbf{0}$, and set the standard deviations to 1 such that Σ conforms to the correlation matrix (15). In this case, the bivariate skewed Student's t probability density function reads

$$f_{\mathbf{Z}}(\mathbf{z}) = \frac{1}{2^{\nu/2}\Gamma(\nu/2)\pi\nu\sqrt{\det\Sigma}} \frac{\exp(\mathbf{z}^\dagger\Sigma^{-1}\boldsymbol{\gamma})}{(1+\frac{\mathbf{z}^\dagger\Sigma^{-1}\mathbf{z}}{\nu})^{\nu/2+1}} \frac{\mathcal{K}_{\nu/2+1}(\sqrt{(\nu+\mathbf{z}^\dagger\Sigma^{-1}\mathbf{z})\boldsymbol{\gamma}^\dagger\Sigma^{-1}\boldsymbol{\gamma}})}{(\sqrt{(\nu+\mathbf{z}^\dagger\Sigma^{-1}\mathbf{z})\boldsymbol{\gamma}^\dagger\Sigma^{-1}\boldsymbol{\gamma}})^{-(\nu/2+1)}} \ , \tag{19}$$

where $\mathcal{K}_\lambda$ is the modified Bessel function of the second kind of order λ. The marginal probability density functions are identical, $f_X(\cdot) = f_Y(\cdot)$, with

$$f_X(x) = \frac{1}{2^{(1-\nu)/2}\Gamma(\nu/2)\sqrt{\pi\nu}\sqrt{\det\Sigma}} \frac{\exp(x\gamma_1)}{(1+\frac{x^2}{\nu})^{(\nu+1)/2}} \frac{\mathcal{K}_{(\nu+1)/2}(\sqrt{(\nu+x^2)\gamma_1^2})}{(\sqrt{(\nu+x^2)\gamma_1^2})^{-(\nu+1)/2}} \ . \tag{20}$$

The marginal cumulative distribution functions, $F_X(\cdot) = F_Y(\cdot)$, with

$$F_X(x) = \int_{-\infty}^{x} \mathrm{d}\xi \, f_X(\xi) \ , \tag{21}$$

have to be calculated and inverted numerically. Bringing it all together corresponding to Equation (6) with $f_{X,Y}(x,y) = f_{\mathbf{Z}}(\mathbf{z})$ then yields the bivariate skewed Student's t-copula density $\mathrm{cop}^t_{c,\nu,\gamma}(u,v)$. In the limit $\nu \to \infty$, we receive the Gaussian copula density.

3 Empirical results and model comparison

In Section 3.1, we present the empirical copula densities for original returns and locally normalized returns. In Section 3.2, we examine the asymmetry of the tail dependence of the empirical copulas in more detail. We compare the empirical copulas to a single Gaussian copula in Section 3.3, to a correlation-weighted Gaussian copula in Section 3.4, to the K-copula in Section 3.5, and to the skewed Student's t-copula in Section 3.6.

3.1 Empirical copula densities

We consider empirical pairwise copula densities averaged over all $K(K-1)/2$ stock pairs,

$$\mathrm{cop}(u,v) = \frac{2}{K(K-1)} \sum_{k,l=1,\ l>k}^{K} \mathrm{cop}_{k,l}(u,v) \ , \tag{22}$$

where $\mathrm{cop}_{k,l}(u,v)$ is calculated as a two-dimensional histogram of the data pairs $(u_k(t), u_l(t))$, $t = 1, \ldots, T$, according to Equation (7). For the bin size of these histograms, we choose $\Delta u = \Delta v = 1/20$. In the following, we denote the empirical copula density for original returns by $\mathrm{cop}^{(\mathrm{glob})}(u,v)$, and for locally normalized returns by $\mathrm{cop}^{(\mathrm{loc})}(u,v)$. As mentioned above, the main difference between these two cases is the consideration of nonstationarity: the original returns exhibit time-varying trends and volatilities; thus, one might argue the return time series are nonstationary. The locally normalized returns, on the other hand, show stationary behavior. We compare the results for both cases because each has its merits. The copula for the original returns provides the statistical dependence over the full time horizon, i.e., on a global scale, while the copula for the locally normalized returns reveals the statistical dependence on a local scale.

In Figure 3, we compare the empirical copula densities. At first sight, the dependence structure seems very similar for both cases. Deviations exist mainly in the corners. Overall, the statistical dependence is preserved rather well when we strip the time series from time-varying trends and volatilities. For the local normalization, Schäfer and Guhr (2010) showed that correlations are preserved under this procedure. Apparently, this also holds for the copula to some degree. For the original returns, the two peaks in the corners are higher. This can be explained as follows: the returns of periods with high volatility are more likely to end up in the lowest or highest quantile, i.e., $u_k(t)$ and $u_l(t)$ are close to 0 or 1. Therefore, periods with high volatility contribute strongly to the corners of this copula. Since high volatility typically coincides with strong correlations in the market, the corners exhibit a stronger dependence.

Qualitatively, the empirical results resemble Gaussian copula densities except for the corners, where an asymmetry is observed. This empirical asymmetry implies that large negative returns of two stocks show stronger dependence than large positive ones. Although the asymmetry can be captured neither by the Gaussian copula or a Gaussian mixture nor by the K-copula, we investigate how well these analytical copulas can approximate the overall dependence structure of empirical stock returns. In addition, we compare our empirical copulas to the skewed Student's t-copula which is able to model asymmetries in the dependence structure. First, however, we take a closer look at the empirical asymmetry of the tail dependence.

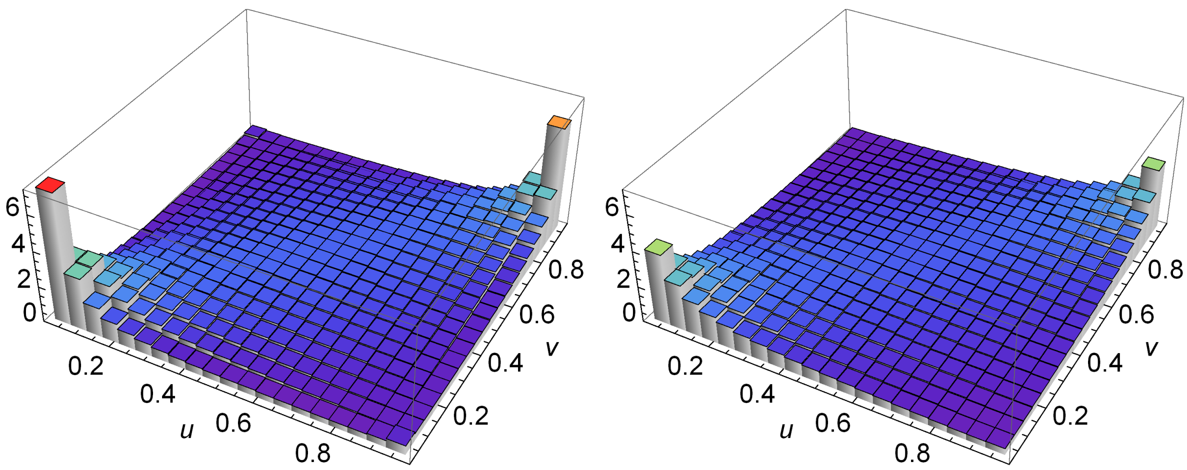

Figure 3: Empirical copula density for original returns, $\mathrm{cop}^{(\mathrm{glob})}(u,v)$, (left) and locally normalized returns, $\mathrm{cop}^{(\mathrm{loc})}(u,v)$, (right).

3.2 Asymmetry of the tail dependence

We elaborate on the features of the empirical copula densities for original returns $\mathrm{cop}^{(\mathrm{glob})}(u,v)$ and locally normalized returns $\mathrm{cop}^{(\mathrm{loc})}(u,v)$. Their essential characteristic is the asymmetry between the lower–lower and the upper–upper corner: the dependence between contemporaneous large negative returns is always stronger than the dependence between contemporaneous large positive returns. Consequently, we observe a stronger dependence between coincident downside movements than between coincident upside movements. In other words, we distinguish an asymmetry between bearish and bullish markets. This asymmetry is particularly evident for original returns. Ang and Chen (2002), studying the dependence between US stocks and the US market, observe the same asymmetry based on correlations.

To quantify the empirical asymmetry, we estimate the tail dependence by integrating over the 0.2×0.2 area in all four corners for each of the $K(K-1)/2$ empirical pairwise copula densities.

We receive the tail-dependence asymmetries by subtracting the integrated areas in the opposite corners of the empirical copula densities:

$$p_{k,l} = \int_{0.8}^{1} \mathrm{d}u \int_{0.8}^{1} \mathrm{d}v \, \mathrm{cop}_{k,l}(u,v) - \int_{0}^{0.2} \mathrm{d}u \int_{0}^{0.2} \mathrm{d}v \, \mathrm{cop}_{k,l}(u,v) \ , \tag{23}$$

$$q_{k,l} = \int_{0}^{0.2} \mathrm{d}u \int_{0.8}^{1} \mathrm{d}v \, \mathrm{cop}_{k,l}(u,v) - \int_{0.8}^{1} \mathrm{d}u \int_{0}^{0.2} \mathrm{d}v \, \mathrm{cop}_{k,l}(u,v) \ . \tag{24}$$

We call $p_{k,l}$ the positive tail-dependence asymmetry, and $q_{k,l}$ the negative tail-dependence asymmetry. The histograms of these tail-dependence asymmetry values, $f(p_{k,l})$ and $f(q_{k,l})$, are shown in Figure 4 for original and locally normalized returns.

While the negative tail-dependence asymmetries $q_{k,l}$ are centered around zero for both original and locally normalized returns, we observe a distinct negative offset for the positive tail-dependence asymmetries $p_{k,l}$. Hence, on average, there is no asymmetry in the negative tail dependence, i.e., between concurrent large positive and negative movements, but there is a significant asymmetry in the positive tail dependence, i.e., between concurrent large negative movements, for one thing, and concurrent large positive movements, for another thing.

Additionally, we notice that the locally normalized returns show a much weaker asymmetry in the positive tail dependence than the original returns. This empirical finding can be interpreted as follows: for the original return time series, events in the tails of the distribution reflect periods of high volatility; since high volatility often occurs simultaneously for most stocks, the tail dependence also reflects these periods in particular. In contrast, the locally normalized returns are stationary with a constant volatility of 1. Therefore, the tail dependence in this case reflects the average behavior over the entire time period. Thus, our empirical results imply that the asymmetry in the tail dependence is not stationary, but it is particularly strong in periods with high volatility. In fact, periods with high volatility coincide with both strong correlations and strong asymmetry in the dependence structure.

3.3 Comparison to the Gaussian copula

At first sight, our empirical copula densities seem to roughly resemble the Gaussian copula. How good is this agreement quantitatively? In Figure 5, we compare the empirical copula densities with the analytical Gaussian copula density $\mathrm{cop}_c^{\mathrm{G}}(u,v)$. Here, we set the correlation coefficient in each Gaussian copula to the empirical average correlation: $\overline{c}^{(\mathrm{glob})} = 0.44$ for original returns and $\overline{c}^{(\mathrm{loc})} = 0.39$ for locally normalized returns.

In Figure 5, we observe clear deviations from the respective Gaussian copula densities. This is to be expected since the empirical copula densities are in fact an average over $K(K-1)/2$ pairwise copula densities with different correlations. Hence, the comparison to a single Gaussian copula with average correlation cannot yield suitable results. Indeed, for original returns, we find considerable deviations not only in the corners but over the entire dependence structure. For locally normalized returns, the corners are rather well described. Nonetheless, deviations over the whole dependence structure are clearly visible.

Table 1 summarizes the deviation between empirical and analytical copula densities in terms of the sum of squared residuals. For the original returns, we find a sum of squared residuals of 27.52, while the locally normalized returns yield a smaller 5.26.

3.4 Comparison to the correlation-weighted Gaussian copula

Aiming for a more suitable description of the empirical copulas, we consider a weighted average of Gaussian copulas for different correlation coefficients. This takes into account the empirical average over different stock pairs in Equation (22). Figure 6 shows the histogram of empirical

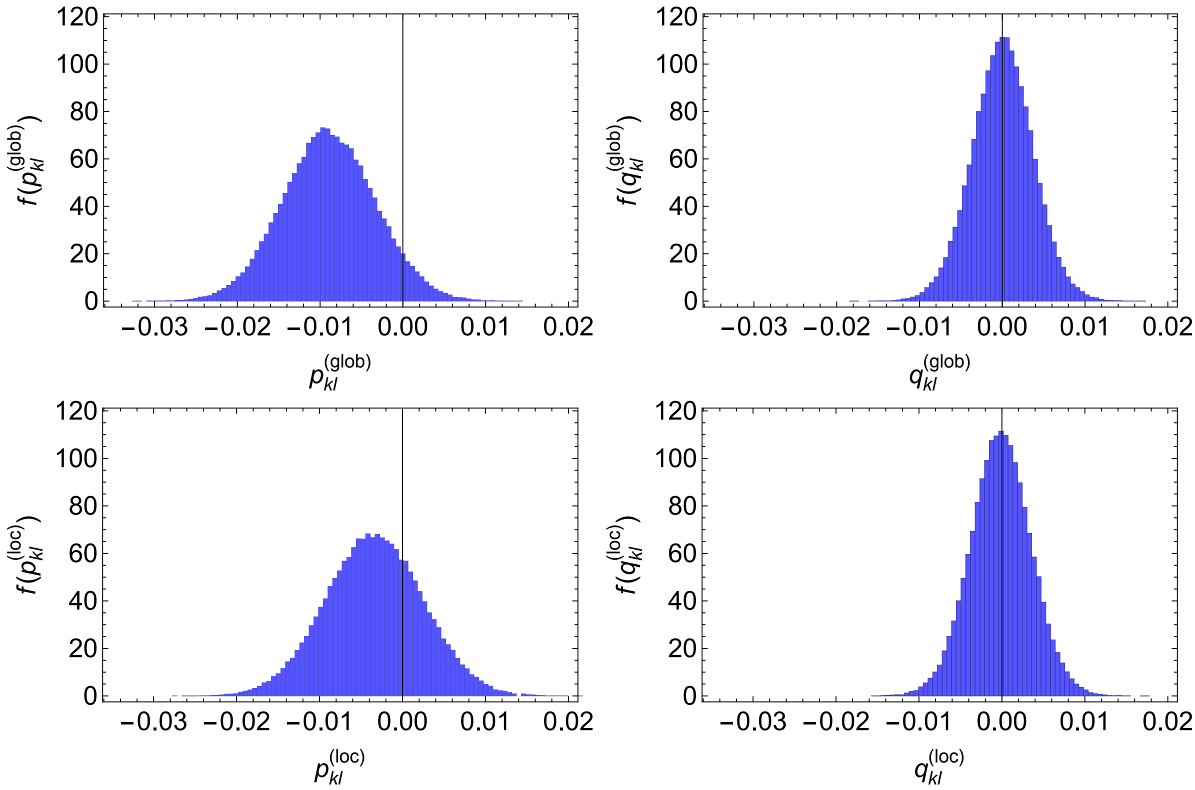

Figure 4: Histogram of positive tail-dependence asymmetries $p_{k,l}$ (left) and negative tail-dependence asymmetries $q_{k,l}$ (right); top: for original returns; bottom: for locally normalized returns.

correlation coefficients for original and locally normalized returns estimated over the entire time horizon, $f(C_{k,l}^{(\mathrm{glob})})$ and $f(C_{k,l}^{(\mathrm{loc})})$. The bin size is $\Delta c = 0.02$. We observe only positive correlations for both original and locally normalized returns.

We obtain a Gaussian mixture by weighting each Gaussian copula density with the probability of occurrence of the respective correlation, $h(C_{k,l}) = f(C_{k,l})\,\Delta c$. This yields the correlation-weighted Gaussian copula density,

$$\mathrm{cop}^{\mathrm{CWG}}(u,v) = \sum_{C_{k,l}=-1}^{1} h(C_{k,l})\,\mathrm{cop}^{\mathrm{G}}_{C_{k,l}}(u,v)\ , \tag{25}$$

which is compared to the empirical copula densities in Figure 7. For both return time series, we find only slight improvement over the single Gaussian copula. The sum of squared errors between the empirical and analytical copulas are only slightly smaller as well, see Table 1.

This can be attributed to the fact that correlations also vary in time (see, e.g., Schäfer and Guhr, 2010; Münnix et al., 2012). Thus, it is not sufficient to take the average over different correlation coefficients into account. For a better approximation of the empirical copula densities, the correlations would have to be estimated on shorter time intervals. This, however, leads to increasing estimation errors for shorter estimation intervals. Consequently, a reliable attainment of time-dependent correlations is problematic. This is where the K-copula introduced in Section 2.7 comes into play.

3.5 Comparison to the K-copula

We consider the K-copula density $\mathrm{cop}^{\mathrm{K}}_{c,N}(u,v)$, which takes inhomogeneous and time-varying correlations into account. Here, the fluctuations of correlations around their mean value c are

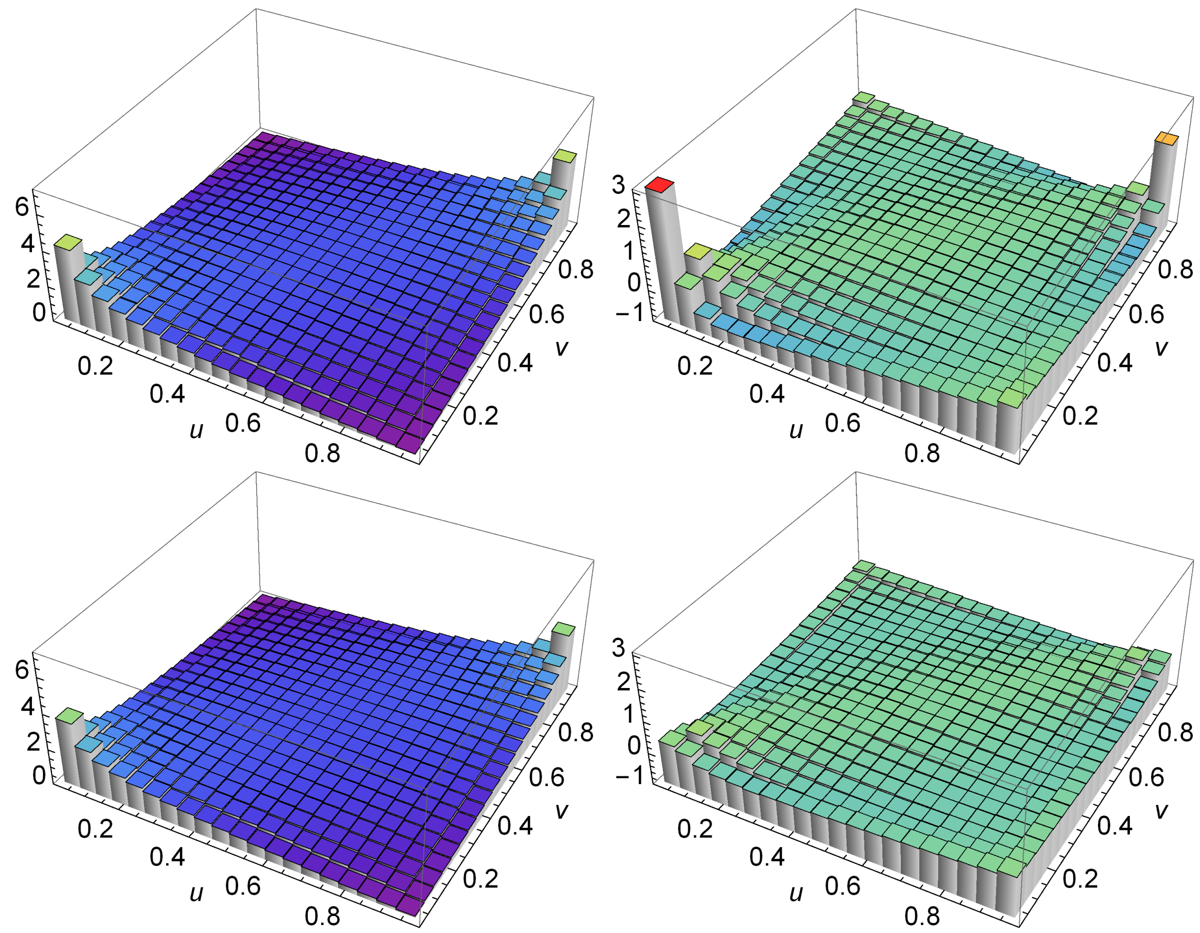

Figure 5: Gaussian copula density $\mathrm{cop}_c^{\mathrm{G}}(u, v)$ (left) and difference between the empirical copula and the Gaussian copula density, $\mathrm{cop}(u, v) - \mathrm{cop}_c^{\mathrm{G}}(u, v)$, (right); top: for original returns with $\bar{c}^{(\mathrm{glob})} = 0.44$; bottom: for locally normalized returns with $\bar{c}^{(\mathrm{loc})} = 0.39$.

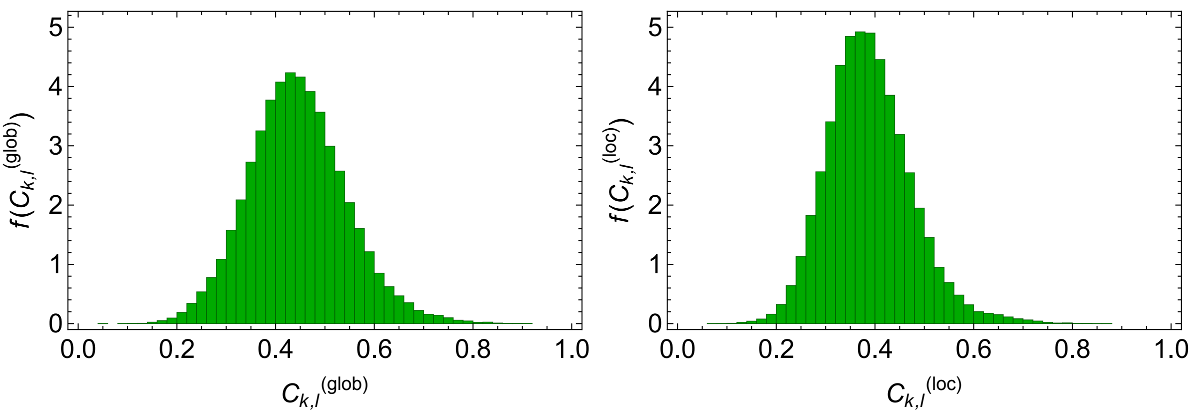

Figure 6: Histogram of correlation coefficients $C_{k,l}$ for original returns (left) and locally normalized returns (right).

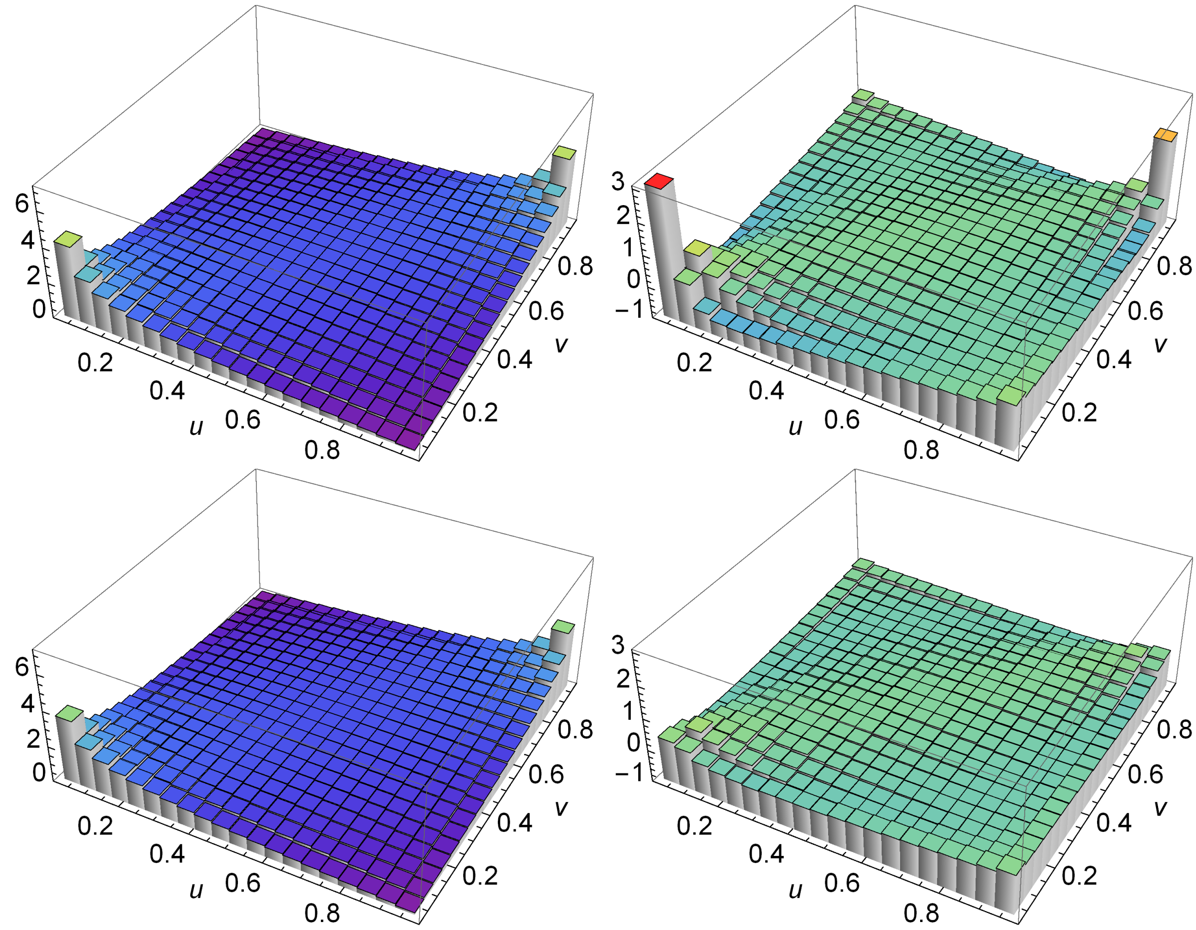

Figure 7: Correlation-weighted Gaussian copula density $\mathrm{cop}^{\mathrm{CWG}}(u, v)$ (left) and difference between the empirical copula and the correlation-weighted Gaussian copula density, $\mathrm{cop}(u, v) - \mathrm{cop}^{\mathrm{CWG}}(u, v)$, (right); top: for original returns; bottom: for locally normalized returns.

characterized by the free parameter N. We calculate the mean correlation coefficient for original and locally normalized returns. As noted above, we find $\bar{c}^{(\mathrm{glob})} = 0.44$ and $\bar{c}^{(\mathrm{loc})} = 0.39$. The parameter N is fitted to the empirical copula densities by the method of least squares. We obtain $N^{(\mathrm{glob})} = 3.2$ for original returns and $N^{(\mathrm{loc})} = 7.8$ for locally normalized returns. The parameter values for $\bar{c}$ and N are summarized in Table 2.

The lower value of N for the original returns reflects the fact that the locally normalized returns have a constant volatility of 1. Hence, there are smaller fluctuations in the covariances, which are in this case merely the correlations. In the K-copula, smaller fluctuations of the covariances or correlations are described by larger values of N.

In Figure 8, the resulting K-copula densities are compared to the empirical copula densities. In both cases, we find improved agreement with the empirical results. This is also reflected in much lower values for the sum of squared residuals in Table 1. However, for original returns, a large deviation between empirical and analytical results remains. The overall structure of the empirical dependence is not captured very well by the K-copula. For locally normalized returns, in contrast, the K-copula yields a very good description of the empirical dependence. Only slight deviations persist in the lower–lower and upper–upper corner. The asymmetry of the empirical copula densities cannot be captured by the K-copula density due to its symmetric character. This asymmetry is only weakly present in the case of locally normalized returns, which leads to a better agreement between K-copula and empirical copula.

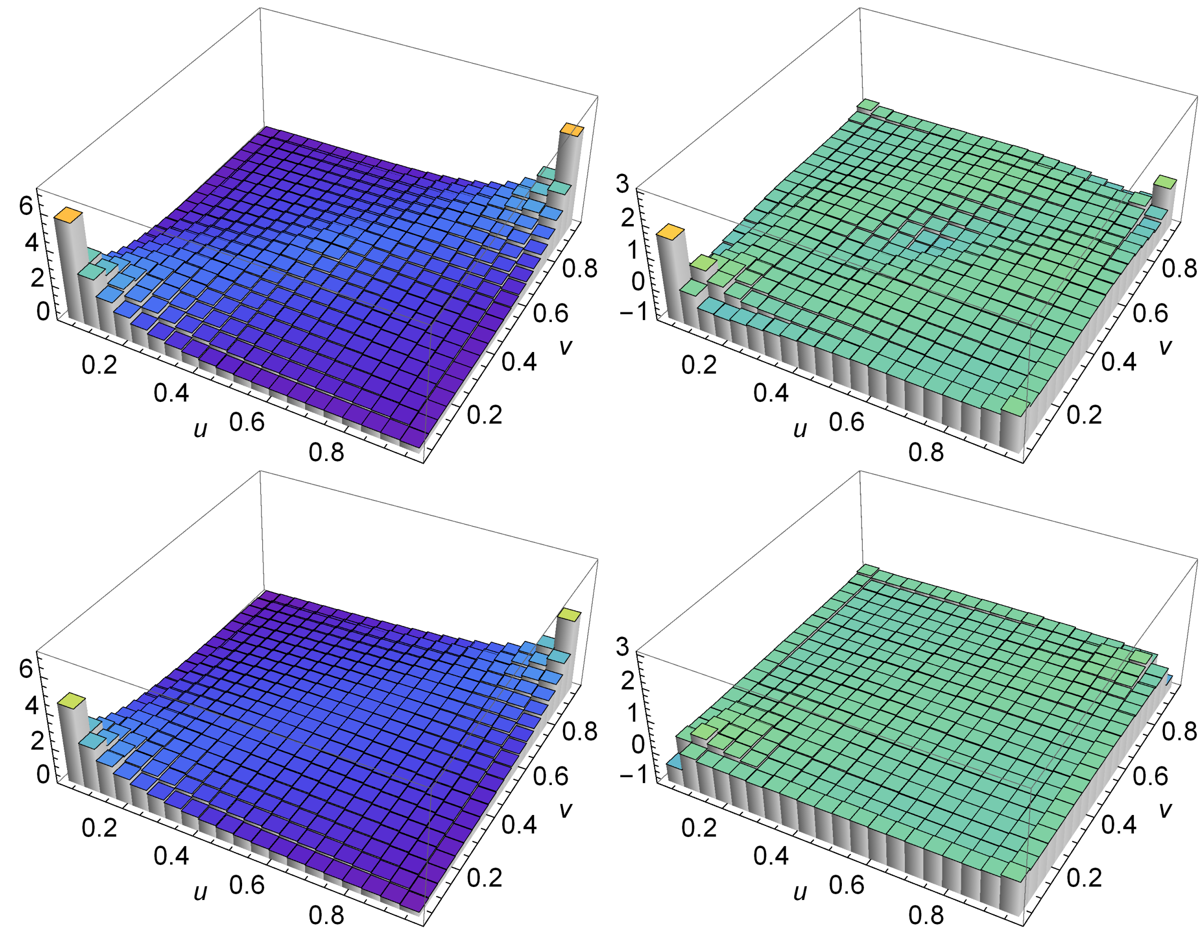

Figure 8: K-copula density $\mathrm{cop}^{\mathrm{K}}_{c,N}(u,v)$ (left) and difference between the empirical copula and the K-copula density, $\mathrm{cop}(u,v) - \mathrm{cop}^{\mathrm{K}}_{c,N}(u,v)$, (right); top: for original returns with $\bar{c}^{(\mathrm{glob})} = 0.44$, $N^{(\mathrm{glob})} = 3.2$; bottom: for locally normalized returns with $\bar{c}^{(\mathrm{loc})} = 0.39$, $N^{(\mathrm{loc})} = 7.8$.

3.6 Comparison to the skewed Student's t-copula

The skewed Student's t-copula allows for an asymmetry in the dependence structure. Hence, it is a natural candidate to compare our empirical findings with. Since we observe no asymmetry in the negative tail dependence of the empirical copula densities on average, see histograms in Figure 4 on the right, we choose the same value for both components of the skewness parameter vector $\boldsymbol{\gamma}$, that is, $\gamma_1 = \gamma_2 = \gamma$. This leaves only an asymmetry in the positive tail dependence of the skewed Student's t-copula and reduces the number of parameters. The parameter values for ν and γ are fitted to the empirical copula densities by least squares, whereas the mean correlation c is empirically determined by $\overline{c}^{(\mathrm{glob})} = 0.44$ and $\overline{c}^{(\mathrm{loc})} = 0.39$. We find $\nu^{(\mathrm{glob})} = 3.3$, $\gamma^{(\mathrm{glob})} = 0.06$ for original returns and $\nu^{(\mathrm{loc})} = 8.0$, $\gamma^{(\mathrm{loc})} = 0.04$ for locally normalized returns. The parameter values are summarized in Table 3.

Figure 9 illustrates the resulting skewed Student's t-copula densities and their difference to the empirical copula densities. For original returns, i.e., the global scale, we find a remarkable agreement. The skewed Student's t-copula is able to capture the overall dependence structure very well, including the asymmetry in the positive tail dependence. There are only small deviations in the corners, in particular a mild overestimation for the negative tail dependence of extreme events. For locally normalized returns, i.e., the local scale, the skewed Student's t-copula provides a slightly worse fit of the empirical data than the K-copula does. There is not much asymmetry to capture in the empirical copula density. Hence, the skewness parameter plays only a minor role in this case. Furthermore, the positive tail dependence is more pronounced for the skewed Student's t-copula than for the K-copula. On a local scale, however, this is not reflected in the empirical dependence structure.

Analytical density	original returns	loc. normalized returns
Gaussian copula	27.52	5.26
Correlation-weighted Gaussian copula	26.42	5.11
K-copula	6.26	2.27
Skewed Student's t-copula	0.65	2.87

Table 1: Sum of squared errors between analytical and empirical copula densities for original and locally normalized returns.

Parameter	original returns	loc. normalized returns
$\overline{c}$	0.44	0.39
N	3.2	7.8

Table 2: Parameter values for the K-copula density.

Parameter	original returns	loc. normalized returns
$\overline{c}$	0.44	0.39
ν	3.3	8.0
γ	0.06	0.04

Table 3: Parameter values for the skewed Student's t-copula density.

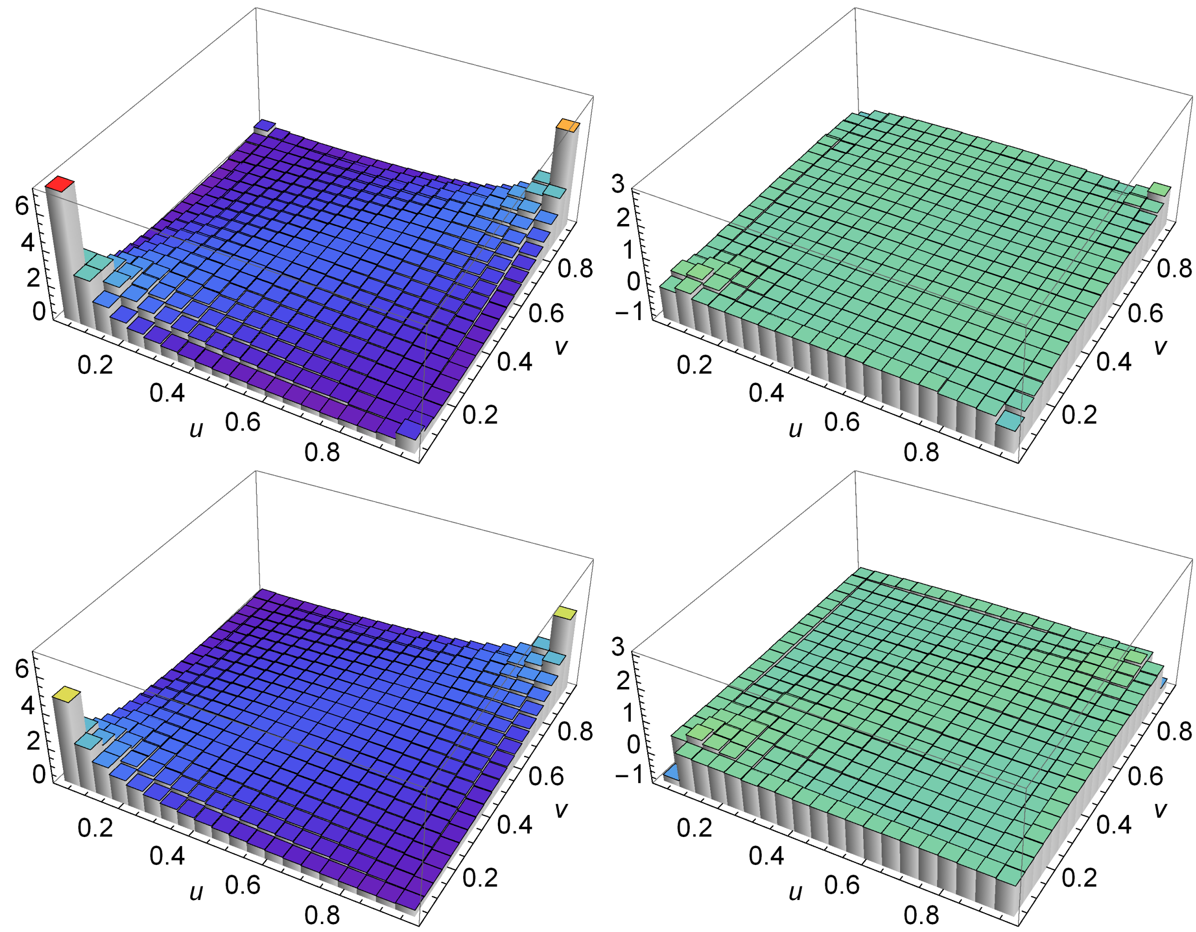

Figure 9: Skewed Student's t-copula densities $\mathrm{cop}^{\mathrm{t}}_{c,\nu,\gamma}(u,v)$ (left) and difference between the empirical copula and the skewed Student's t-copula density, $\mathrm{cop}(u,v) - \mathrm{cop}^{\mathrm{t}}_{c,\nu,\gamma}(u,v)$, (right); top: for original returns with $\bar{c}^{(\mathrm{glob})} = 0.44$, $\nu^{(\mathrm{glob})} = 3.3$, $\gamma^{(\mathrm{glob})} = 0.06$; bottom: for locally normalized returns with $\bar{c}^{(\mathrm{loc})} = 0.39$, $\nu^{(\mathrm{loc})} = 8.0$, $\gamma^{(\mathrm{loc})} = 0.04$.

4 Conclusion

We present an empirical study on the statistical dependencies of daily stock returns. To this end, we estimate empirical pairwise copulas for original returns and for locally normalized returns. Considering the former allows us to study the dependence structure on a global scale. In the latter case, the nonstationary characteristics of return time series, i.e., time-varying trends and volatilities, are removed. This provides us with the dependence structure on a local scale.

How far does the concept of Gaussian dependence carry in the light of our empirical findings? To answer this question, we compare the empirical results not only with a single Gaussian copula, but also with a correlation-weighted average of Gaussian copulas and with the K-copula. The latter arises from a random matrix approach which models time-varying covariances by a Wishart distribution. This yields a K-distribution which adequately describes multivariate returns. We derive the resulting K-copula for the bivariate case and found very good agreement with empirical pairwise dependencies of locally normalized returns.

Thus, we arrive at a consistent picture within the random matrix model: the K-distribution is able to describe the tail behavior of the marginal return distributions, and the K-copula captures the overall empirical dependence structure. This implies that Gaussian statistics, and thus also a Gaussian dependence structure, provide a good description on a local scale. However, on a global scale, we find rather significant deviations from the K-copula. In particular, we observe a pronounced asymmetry in the positive tail dependence.

Therefore, we also compare our empirical findings with a model that explicitly allows for such an asymmetry: the skewed Student's t-copula. Indeed, we find a rather compelling agreement with the empirical dependence structure of original returns. On the local scale, however, the empirical copula exhibits only a mild asymmetry and is overall better described by the K-copula. How can we understand this? For original returns, the tail dependence reflects periods with high volatility, while for locally normalized returns, all periods contribute equally to the tail dependence. Thus, our results imply that the asymmetry in the tail dependence is not stationary, and it is stronger in periods with high volatility.

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