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Econometrics III
Problem Set 1
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Exercise 1

(a) We want to prove:

$$\frac{1}{n-1}\sum_{i=1}^{n}(Y_i-\overline{Y})^2=\frac{1}{n-1}\sum_{i=1}^{n}Y_i^2-\frac{n}{n-1}\overline{Y}^2$$

Starting with the left-hand side:

$$\begin{aligned}\frac{1}{n-1}\sum_{i=1}^{n}(Y_i-\overline{Y})^2&=\frac{1}{n-1}\sum_{i=1}^{n}\left(Y_i^2-2Y_i\overline{Y}+\overline{Y}^2\right)\\&=\frac{1}{n-1}\left(\sum_{i=1}^{n}Y_i^2-2\overline{Y}\sum_{i=1}^{n}Y_i+n\overline{Y}^2\right)\\&=\frac{1}{n-1}\left(\sum_{i=1}^{n}Y_i^2-2n\overline{Y}^2+n\overline{Y}^2\right)\\&=\frac{1}{n-1}\left(\sum_{i=1}^{n}Y_i^2-n\overline{Y}^2\right)\end{aligned}$$

(b) We start from the variance identity:

$$V(Y)=\mathbb{E}(Y^2)-\mathbb{E}(Y)^2\quad\Rightarrow\quad\mathbb{E}(Y^2)=V(Y)+\mathbb{E}(Y)^2$$

Then:

$$\mathbb{E}\left(\sum_{i=1}^{n}Y_i^2\right)=\sum_{i=1}^{n}\mathbb{E}(Y_i^2)=\sum_{i=1}^{n}\left(V(Y)+\mathbb{E}(Y)^2\right)=nV(Y)+n\mathbb{E}(Y)^2$$

Also:

$$V(\overline{Y})=\frac{1}{n^2}V\left(\sum_{i=1}^{n}Y_i\right)=\frac{1}{n^2}\cdot n\cdot V(Y)=\frac{V(Y)}{n}$$

So:

$$\mathbb{E}(\overline{Y}^2)=V(\overline{Y})+\mathbb{E}(\overline{Y})^2=\frac{V(Y)}{n}+\mathbb{E}(Y)^2$$

Now plug into the expression from 1.1:

$$\begin{aligned}
\frac{1}{n-1}\sum_{i=1}^{n} Y_i^2 - \frac{n}{n-1}\mathbb{E}(\overline{Y}^2) &= \frac{1}{n-1}\left(\sum_{i=1}^{n}\mathbb{E}(Y_i^2) - n\cdot\mathbb{E}(\overline{Y}^2)\right) \\
&= \frac{1}{n-1}\left(\sum_{i=1}^{n}\left(V(Y)+\mathbb{E}(Y)^2\right) - n\left(\frac{V(Y)}{n}+\mathbb{E}(Y)^2\right)\right) \\
&= \frac{1}{n-1}\left(n\cdot\left(V(Y)+\mathbb{E}(Y)^2\right) - \left(V(Y)+n\cdot\mathbb{E}(Y)^2\right)\right) \\
&= \frac{1}{n-1}\left(V(Y)(n-1)\right) = V(Y)
\end{aligned}$$

Exercise 2

(a) We know:

$$\mathbb{E}[Y_i] = \frac{0+\theta}{2} = \frac{\theta}{2} \quad \Rightarrow \quad \theta = 2\cdot\mathbb{E}[Y_i]$$

So a natural unbiased estimator is:

$$\hat{\theta}_1 = 2\overline{Y}, \quad \text{where } \overline{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$$

(b) Using the first moment:

$$\mathbb{E}[Y] = \frac{\theta}{2} \quad \text{and} \quad \overline{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$$

Setting sample moment equal to population moment gives:

$$\hat{\theta}_{\text{MM}} = 2\overline{Y}$$

This matches the estimator in part (a).

(c) **Asymptotic Normality of $\hat{\theta}_1 = 2\overline{Y}$**

Since Y_i has finite mean and variance, by the Central Limit Theorem:

$$\sqrt{n}(\overline{Y} - \mathbb{E}[Y]) \xrightarrow{d} \mathcal{N}\left(0, \operatorname{Var}(Y)\right)$$

We have:

$$\mathbb{E}[Y] = \frac{\theta}{2}, \quad \operatorname{Var}(Y) = \frac{\theta^2}{12} \quad \Rightarrow \quad \sqrt{n}(\overline{Y} - \theta/2) \xrightarrow{d} \mathcal{N}\left(0, \frac{\theta^2}{12}\right)$$

Then:

$$\sqrt{n}(\hat{\theta}_1 - \theta) = \sqrt{n}(2\overline{Y} - \theta) = 2\sqrt{n}(\overline{Y} - \theta/2) \xrightarrow{d} \mathcal{N}\left(0, \frac{\theta^2}{3}\right)$$

Therefore, $\hat{\theta}_1$ is asymptotically normal with asymptotic variance $4\operatorname{Var}(Y) = \frac{\theta^2}{3}$.

(d) Because θ is the upper bound of the distribution, the maximum value of $Y_{(n)}$ in the sample is a natural estimator for θ.

(e) we have:

$$P(\hat{\theta}_{ML} \leq x) = \begin{cases} 0, & x < 0, \\ P\big(Y_1 \leq x, \dots, Y_n \leq x\big), & 0 \leq x \leq \theta, \\ 1, & x > \theta. \end{cases}$$

For $0 \leq x \leq \theta$, by independence,

$$P(\hat{\theta}_{ML} \leq x) = \prod_{i=1}^{n} P(Y_i \leq x) = \big(P(Y_1 \leq x)\big)^n.$$

Since $Y_1 \sim \text{Unif}[0, \theta]$, its CDF on $[0, \theta]$ is $F_Y(x) = x/\theta$. Hence,

$$P(\hat{\theta}_{ML} \leq x) = \left(\frac{x}{\theta}\right)^n, \qquad 0 \leq x \leq \theta.$$

When $x < 0$, $P(\hat{\theta}_{ML} \leq x) = 0$ since $\hat{\theta}_{ML} \geq 0$ almost surely. When $x > \theta$, $P(\hat{\theta}_{ML} \leq x) = 1$ since $\hat{\theta}_{ML} \leq \theta$ almost surely.

Putting the three regions together,

$$P(\hat{\theta}_{ML} \leq x) = \begin{cases} 0, & x < 0, \\ \left(\dfrac{x}{\theta}\right)^n, & 0 \leq x \leq \theta, \\ 1, & x > \theta. \end{cases}$$

(f) We define:

$$Z_n = n\left(\frac{\theta - \hat{\theta}_2}{\theta}\right)$$

Then use change of variable:

$$\mathbb{P}(Z_n \leq z) = \mathbb{P}\left(\hat{\theta}_2 \geq \theta\left(1 - \frac{z}{n}\right)\right) = 1 - \left(1 - \frac{z}{n}\right)^n \to 1 - e^{-z}$$

Hence:

$$n\left(\frac{\theta - \hat{\theta}_2}{\theta}\right) \xrightarrow{d} \text{Exp}(1)$$

(g) The MM estimator $\hat{\theta}_{MM} = 2\bar{Y}$ is unbiased with variance $\theta^2/(3n)$ and, by the CLT, satisfies $\sqrt{n}(\hat{\theta}_{MM} - \theta) \xrightarrow{d} \mathcal{N}(0, \theta^2/3)$; it is therefore $\sqrt{n}$-consistent and asymptotically normal. The ML estimator $\hat{\theta}_{ML} = \max_i Y_i$ is downward biased ($\mathbb{E}[\hat{\theta}_{ML}] = \frac{n}{n+1}\theta$) but consistent and converges faster: $n(\theta - \hat{\theta}_{ML})/\theta \xrightarrow{d} \text{Exp}(1)$, so its error is $O_p(1/n)$ with $\text{MSE} \sim 2\theta^2/n^2$. So it is not clear which estimator is better.

(h) See Stata code in Appendix.

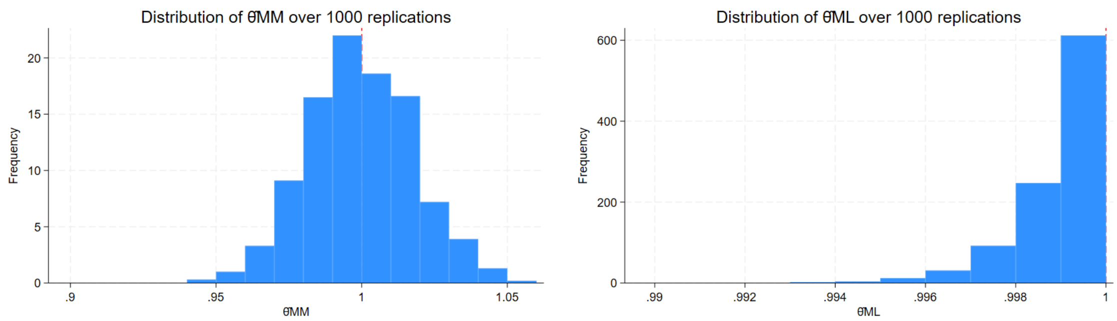

Figure 1: Histograms of $\hat{\theta}_{MM}$ (left) and $\hat{\theta}_{ML}$ (right) across 100 replications. The red vertical line marks the true parameter $\theta = 1$.

(i) From (f),

$$\frac{n(\theta - \hat{\theta}_{ML})}{\theta} \xrightarrow{d} U, \qquad U \sim \text{Exp}(1).$$

Since $\hat{\theta}_{ML} \xrightarrow{p} \theta$, hence also $\theta/\hat{\theta}_{ML} \xrightarrow{p} 1$. Therefore, by Slutsky's lemma,

$$\frac{n(\theta - \hat{\theta}_{ML})}{\hat{\theta}_{ML}} = \frac{n(\theta - \hat{\theta}_{ML})}{\theta} \cdot \frac{\theta}{\hat{\theta}_{ML}} \xrightarrow{d} U \cdot 1 \;=\; U \sim \text{Exp}(1).$$

Let $t_{1-\alpha}$ be the $(1-\alpha)$-quantile of $\text{Exp}(1)$ (so $t_{1-\alpha} = -\ln \alpha$). Then

$$P\left(\frac{n(\theta - \hat{\theta}_{ML})}{\hat{\theta}_{ML}} \leq t_{1-\alpha}\right) \;\to\; 1-\alpha,$$

which is equivalent to

$$P\left(\theta \leq \hat{\theta}_{ML} + \frac{\hat{\theta}_{ML}}{n}\, t_{1-\alpha}\right) \;\to\; 1-\alpha.$$

Since $\hat{\theta}_{ML} \leq \theta$ almost surely, we obtain the asymptotic $(1-\alpha)$ CI

$$IC(\alpha) = \left[\, \hat{\theta}_{ML}\, ,\; \hat{\theta}_{ML} + \frac{\hat{\theta}_{ML}}{n}\, t_{1-\alpha} \right].$$

Exercise 3

3.1

Define the continuous function:

$$f(u, v) = uv$$

Given:
$$U_n \xrightarrow{P} \ell, \quad V_n \xrightarrow{P} \ell'$$
and since f is continuous in $\mathbb{R}^2$, the Continuous Mapping Theorem implies:
$$U_n V_n = f(U_n, V_n) \xrightarrow{P} f(\ell, \ell') = \ell\ell'$$

3.2

We are given:
$$U_n \xrightarrow{P} \ell, \quad V_n \xrightarrow{P} \ell'$$
By the first part of Slutsky lemma, since convergence in probability implies convergence in distribution, we have:
$$U_n \xrightarrow{d} \ell, \quad V_n \xrightarrow{d} \ell'$$
Define the continuous function:
$$f(u, v) = uv$$
Similar to 3.1, by the Continuous Mapping Theorem, we have:
$$f(U_n, V_n) = U_n V_n \xrightarrow{d} f(\ell, \ell') = \ell\ell'$$
Since $\ell\ell'$ is a constant, by the second statement of Slutsky's lemma, convergence in distribution to a constant implies convergence in probability:
$$U_n V_n \xrightarrow{P} \ell\ell'$$