PS 1, Econometrics 3

Exercise 1: an unbiased estimator of the variance of iid random variables

Let $(Y_i)_{1\leq i\leq n}$ be n iid random variables. Let $\overline{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$ denote the average of these variables. Let Y be a random variable with the same distribution as the Y_is. The goal of the exercise is to show that $\frac{1}{n-1}\sum_{i=1}^{n}\left(Y_i-\overline{Y}\right)^2$ is an unbiased estimator of $V(Y)$, the variance of the Y_is.

1) Show that $\frac{1}{n-1}\sum_{i=1}^{n}\left(Y_i-\overline{Y}\right)^2 = \frac{1}{n-1}\sum_{i=1}^{n} Y_i^2 - \frac{n}{n-1}\left(\overline{Y}\right)^2$.

2) Use the result in question 1) to prove that $E\left(\frac{1}{n-1}\sum_{i=1}^{n}\left(Y_i-\overline{Y}\right)^2\right) = V(Y)$.

Exercise 2: A super consistent estimator

Assume you observe an iid sample of n random variables $(Y_i)_{1\leq i\leq n}$ following the uniform distribution on $[0,\theta]$, where θ is an unknown strictly positive real number we would like to estimate. Let Y be a random variable with the same distribution as the Y_is.

a) Compute $E(Y)$. Write θ as a function of $E(Y)$.

b) Use question a) to propose an estimator $\widehat{\theta}_{MM}$ for θ using the method of moments (reminder: that method amounts to replacing expectations by sample means).

c) Show that $\widehat{\theta}_{MM}$ is an asymptotically normal estimator of θ, and show that its asymptotic variance is $4V(Y)$.

Consider the following alternative estimator for θ: $\widehat{\theta}_{ML} = \max\limits_{1\leq i\leq n}\{Y_i\}$.

d) Why does using $\widehat{\theta}_{ML}$ to estimate θ sounds like a natural idea?

e) Show that

$$P(\widehat{\theta}_{ML}\leq x) = \left\{\begin{array}{cc} 0 & \text{if } x<0 \\ \left(\frac{x}{\theta}\right)^n & \text{if } x\in[0,\theta] \\ 1 & \text{if } x>\theta \end{array}\right\}.$$

f) Use the result in question e) to show that $n\left(\frac{\theta-\widehat{\theta}_{ML}}{\theta}\right) \xrightarrow{d} U$, where U follows an exponential distribution with parameter 1. Hint: to prove this, you need to use the definition of convergence in distribution in your lecture notes. Also, use the fact that the cdf of an exponential distribution with parameter 1 is

$$F(x) = \left\{\begin{array}{cc} 0 & \text{if } x<0 \\ 1-\exp(-x) & \text{if } x\geq 0 \end{array}\right\}.$$

g) Which estimator is the best: $\widehat{\theta}_{MM}$, or $\widehat{\theta}_{ML}$?

h) Illustrate this through a Monte-Carlo study. Draw 1000 iid realizations of variables following a uniform distribution on $[0,1]$ in Stata (you need to use the "uniform()" command), compute $\widehat{\theta}_{MM}$ and $\widehat{\theta}_{ML}$. What is the value of θ in this example? Which estimator is the closest to θ?

i) For any $q\in(0,1)$, let t_q denote the q^{th} quantile of the $exp(1)$ distribution: $t_q = F^{-1}(q)$. Show that $IC(\alpha) = \left[\widehat{\theta}_{ML}, \widehat{\theta}_{ML} + \widehat{\theta}_{ML}\frac{t_{1-\alpha}}{n}\right]$ is a confidence interval for θ with asymptotic coverage $1-\alpha$. You should use the result from the previous question and the Slutsky lemma.

You can use without proving it the fact that $\widehat{\theta}_{ML} \xrightarrow{P} \theta$ (actually, that directly follows from the fact $\widehat{\theta}_{ML}$ is an n-consistent estimator of θ).

To conclude, just a few words as to why this exercise can actually be useful in the real world. Assume you are a military analyst working for the UK during World War 2. You are being asked by Churchill to answer the following question: "How many tanks does Germany produce each month?" Conventional espionage methods, such as aerial photographs, are dangerous, and the estimate they yield (1400 tanks per month) seems too high to you. A way to answer the question goes as follows. The allies have salvaged n German tanks on the battlefield. The serial numbers of the salvaged tanks go by Year/Month/monthly serial number. Let Y_i denote the last part of that serial number for salvaged tank i. It is realistic to assume that from an ex-ante perspective, before the tanks are salvaged, the Y_is all follow uniform distributions on $\{1, 2, ..., \theta\}$, where θ is the monthly production of tanks by Germany. It is also realistic to assume that the Y_is are almost independent. Therefore, the only difference with the modelling framework in the exercise is that here, the Y_is follow a discrete uniform distribution, while in the exercise they follow a continuous uniform distribution. Believe my word, the results we derived in the exercise would still apply almost as is if we had assumed the Y_is follow a discrete uniform distribution. Therefore, the allied (or some statistician working for them) solved the same exercise as you did, and used $\widehat{\theta}_{ML}$ to estimate the monthly production of tanks by Germany. They found $\widehat{\theta}_{ML} = 255$. After the war, German production records showed that the true production of tanks was 256 per month, which is very close to 255. So this statistical model did a very good job at inferring the true production of tanks. On the other hand, the conventional espionage method did a very poor job since it estimated that Germans produced 1400 tanks per month. Note that the same method can be used in many other contexts. For instance Samsung could use it to infer Apple's monthly production of IPhones, and conversely Apple could use it to infer Samsung's production.

Exercise 3

Assume you observe two sequences of random variables $(U_n)_{n\in\mathbb{N}}$ and $(V_n)_{n\in\mathbb{N}}$. Assume that $U_n \xrightarrow{P} l$ and $V_n \xrightarrow{P} l'$, where l and l' are two real numbers.

1) Use the continuous mapping theorem to prove that $U_n \times V_n \xrightarrow{P} l \times l'$.

2) Use the Slutsky lemma to prove that $U_n \times V_n \xrightarrow{P} l \times l'$. You need to use the two following facts:

1. If $X_n \xrightarrow{P} X$, then $X_n \xrightarrow{d} X$ (convergence in probability implies convergence in distribution)
2. If $X_n \xrightarrow{d} x$ and x is a real number, then $X_n \xrightarrow{P} x$ (convergence in distribution **towards a real number** implies convergence in probability towards that real number)