Liming Lin
Professor
Econometrics III
Problem Set 1
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Exercise 1

1.1

We want to prove:

$$\frac{1}{n-1}\sum_{i=1}^{n}(Y_i-\overline{Y})^2=\frac{1}{n-1}\sum_{i=1}^{n}Y_i^2-\frac{n}{n-1}\overline{Y}^2$$

Starting with the left-hand side:

$$\begin{aligned}\frac{1}{n-1}\sum_{i=1}^{n}(Y_i-\overline{Y})^2&=\frac{1}{n-1}\sum_{i=1}^{n}\left(Y_i^2-2Y_i\overline{Y}+\overline{Y}^2\right)\\&=\frac{1}{n-1}\left(\sum_{i=1}^{n}Y_i^2-2\overline{Y}\sum_{i=1}^{n}Y_i+n\overline{Y}^2\right)\\&=\frac{1}{n-1}\left(\sum_{i=1}^{n}Y_i^2-2n\overline{Y}^2+n\overline{Y}^2\right)\\&=\frac{1}{n-1}\left(\sum_{i=1}^{n}Y_i^2-n\overline{Y}^2\right)\end{aligned}$$

1.2

We start from the variance identity:

$$V(Y)=\mathbb{E}(Y^2)-\mathbb{E}(Y)^2\quad\Rightarrow\quad\mathbb{E}(Y^2)=V(Y)+\mathbb{E}(Y)^2$$

Then:

$$\mathbb{E}\left(\sum_{i=1}^{n}Y_i^2\right)=\sum_{i=1}^{n}\mathbb{E}(Y_i^2)=\sum_{i=1}^{n}\left(V(Y)+\mathbb{E}(Y)^2\right)=nV(Y)+n\mathbb{E}(Y)^2$$

Also:

$$V(\overline{Y})=\frac{1}{n^2}V\left(\sum_{i=1}^{n}Y_i\right)=\frac{1}{n^2}\cdot n\cdot V(Y)=\frac{V(Y)}{n}$$

So:

$$\mathbb{E}(\overline{Y}^2)=V(\overline{Y})+\mathbb{E}(\overline{Y})^2=\frac{V(Y)}{n}+\mathbb{E}(Y)^2$$

Now plug into the expression from 1.1:

$$\begin{aligned}\frac{1}{n-1}\sum_{i=1}^{n} Y_i^2 - \frac{n}{n-1}\mathbb{E}(\overline{Y}^2) &= \frac{1}{n-1}\left(\sum_{i=1}^{n}\mathbb{E}(Y_i^2) - n\cdot\mathbb{E}(\overline{Y}^2)\right)\\ &= \frac{1}{n-1}\left(\sum_{i=1}^{n}\left(V(Y)+\mathbb{E}(Y)^2\right) - n\left(\frac{V(Y)}{n}+\mathbb{E}(Y)^2\right)\right)\\ &= \frac{1}{n-1}\left(n\cdot\left(V(Y)+\mathbb{E}(Y)^2\right) - \left(V(Y)+n\cdot\mathbb{E}(Y)^2\right)\right)\\ &= \frac{1}{n-1}\left(V(Y)(n-1)\right) = V(Y)\end{aligned}$$

Exercise 3

3.1

Define the continuous function:

$$f(u,v) = uv$$

Given:

$$U_n \xrightarrow{P} \ell, \quad V_n \xrightarrow{P} \ell'$$

and since f is continuous in $\mathbb{R}^2$, the Continuous Mapping Theorem implies:

$$U_nV_n = f(U_n, V_n) \xrightarrow{P} f(\ell,\ell') = \ell\ell'$$

3.2

We are given:

$$U_n \xrightarrow{P} \ell, \quad V_n \xrightarrow{P} \ell'$$

By the first part of Slutsky lemma, since convergence in probability implies convergence in distribution, we have:

$$U_n \xrightarrow{d} \ell, \quad V_n \xrightarrow{d} \ell'$$

Define the continuous function:

$$f(u,v) = uv$$

Similar to 3.1, by the Continuous Mapping Theorem, we have:

$$f(U_n, V_n) = U_nV_n \xrightarrow{d} f(\ell,\ell') = \ell\ell'$$

Since $\ell\ell'$ is a constant, by the second statement of Slutsky's lemma, convergence in distribution to a constant implies convergence in probability:

$$U_nV_n \xrightarrow{P} \ell\ell'$$