

## Exercise 1

(a) We want to prove:

$$\frac{1}{n-1} \sum_{i=1}^n (Y_i - \bar{Y})^2 = \frac{1}{n-1} \sum_{i=1}^n Y_i^2 - \frac{n}{n-1} \bar{Y}^2$$

Starting with the left-hand side:

$$\begin{aligned} \frac{1}{n-1} \sum_{i=1}^n (Y_i - \bar{Y})^2 &= \frac{1}{n-1} \sum_{i=1}^n (Y_i^2 - 2Y_i\bar{Y} + \bar{Y}^2) \\ &= \frac{1}{n-1} \left( \sum_{i=1}^n Y_i^2 - 2\bar{Y} \sum_{i=1}^n Y_i + n\bar{Y}^2 \right) \\ &= \frac{1}{n-1} \left( \sum_{i=1}^n Y_i^2 - 2n\bar{Y}^2 + n\bar{Y}^2 \right) \\ &= \frac{1}{n-1} \left( \sum_{i=1}^n Y_i^2 - n\bar{Y}^2 \right) \end{aligned}$$

(b) We start from the variance identity:

$$V(Y) = \mathbb{E}(Y^2) - \mathbb{E}(Y)^2 \quad \Rightarrow \quad \mathbb{E}(Y^2) = V(Y) + \mathbb{E}(Y)^2$$

Then:

$$\mathbb{E} \left( \sum_{i=1}^n Y_i^2 \right) = \sum_{i=1}^n \mathbb{E}(Y_i^2) = \sum_{i=1}^n (V(Y) + \mathbb{E}(Y)^2) = nV(Y) + n\mathbb{E}(Y)^2$$

Also:

$$V(\bar{Y}) = \frac{1}{n^2} V \left( \sum_{i=1}^n Y_i \right) = \frac{1}{n^2} \cdot n \cdot V(Y) = \frac{V(Y)}{n}$$

So:

$$\mathbb{E}(\bar{Y}^2) = V(\bar{Y}) + \mathbb{E}(\bar{Y})^2 = \frac{V(Y)}{n} + \mathbb{E}(Y)^2$$

Now plug into the expression from 1.1:

$$\begin{aligned}
\frac{1}{n-1} \sum_{i=1}^n Y_i^2 - \frac{n}{n-1} \mathbb{E}(\bar{Y}^2) &= \frac{1}{n-1} \left( \sum_{i=1}^n \mathbb{E}(Y_i^2) - n \cdot \mathbb{E}(\bar{Y}^2) \right) \\
&= \frac{1}{n-1} \left( \sum_{i=1}^n (V(Y) + \mathbb{E}(Y)^2) - n \left( \frac{V(Y)}{n} + \mathbb{E}(Y)^2 \right) \right) \\
&= \frac{1}{n-1} (n \cdot (V(Y) + \mathbb{E}(Y)^2) - (V(Y) + n \cdot \mathbb{E}(Y)^2)) \\
&= \frac{1}{n-1} (V(Y)(n-1)) = V(Y)
\end{aligned}$$

## Exercise 2

(a) We know:

$$\mathbb{E}[Y_i] = \frac{0 + \theta}{2} = \frac{\theta}{2} \quad \Rightarrow \quad \theta = 2 \cdot \mathbb{E}[Y_i]$$

So a natural unbiased estimator is:

$$\hat{\theta}_1 = 2\bar{Y}, \quad \text{where } \bar{Y} = \frac{1}{n} \sum_{i=1}^n Y_i$$

(b) Using the first moment:

$$\mathbb{E}[Y] = \frac{\theta}{2} \quad \text{and} \quad \bar{Y} = \frac{1}{n} \sum_{i=1}^n Y_i$$

Setting sample moment equal to population moment gives:

$$\hat{\theta}_{\text{MM}} = 2\bar{Y}$$

This matches the estimator in part (a).

(c) Since  $Y_i$  has finite mean and variance, by the Central Limit Theorem:

$$\sqrt{n}(\bar{Y} - \mathbb{E}[Y]) \xrightarrow{d} \mathcal{N}(0, \text{Var}(Y))$$

We have:

$$\mathbb{E}[Y] = \frac{\theta}{2}, \quad \text{Var}(Y) = \frac{\theta^2}{12} \quad \Rightarrow \quad \sqrt{n}(\bar{Y} - \theta/2) \xrightarrow{d} \mathcal{N}\left(0, \frac{\theta^2}{12}\right)$$

Then:

$$\sqrt{n}(\hat{\theta}_1 - \theta) = \sqrt{n}(2\bar{Y} - \theta) = 2\sqrt{n}(\bar{Y} - \theta/2) \xrightarrow{d} \mathcal{N}\left(0, \frac{\theta^2}{3}\right)$$

Therefore,  $\hat{\theta}_1$  is asymptotically normal with asymptotic variance  $4 \text{Var}(Y) = \frac{\theta^2}{3}$ .

- (d) Because  $\theta$  is the upper bound of the distribution, the maximum value of  $Y_{(n)}$  in the sample is a natural estimator for  $\theta$ .
- (e) we have:

$$P(\hat{\theta}_{ML} \leq x) = \begin{cases} 0, & x < 0, \\ P(Y_1 \leq x, \dots, Y_n \leq x), & 0 \leq x \leq \theta, \\ 1, & x > \theta. \end{cases}$$

For  $0 \leq x \leq \theta$ , by independence,

$$P(\hat{\theta}_{ML} \leq x) = \prod_{i=1}^n P(Y_i \leq x) = (P(Y_1 \leq x))^n.$$

Since  $Y_1 \sim \text{Unif}[0, \theta]$ , its CDF on  $[0, \theta]$  is  $F_Y(x) = x/\theta$ . Hence,

$$P(\hat{\theta}_{ML} \leq x) = \left(\frac{x}{\theta}\right)^n, \quad 0 \leq x \leq \theta.$$

When  $x < 0$ ,  $P(\hat{\theta}_{ML} \leq x) = 0$  since  $\hat{\theta}_{ML} \geq 0$  almost surely. When  $x > \theta$ ,  $P(\hat{\theta}_{ML} \leq x) = 1$  since  $\hat{\theta}_{ML} \leq \theta$  almost surely.

Putting the three regions together,

$$P(\hat{\theta}_{ML} \leq x) = \begin{cases} 0, & x < 0, \\ \left(\frac{x}{\theta}\right)^n, & 0 \leq x \leq \theta, \\ 1, & x > \theta. \end{cases}$$

- (f) We define:

$$Z_n = n \left( \frac{\theta - \hat{\theta}_2}{\theta} \right)$$

Then use change of variable:

$$\mathbb{P}(Z_n \leq z) = \mathbb{P}\left(\hat{\theta}_2 \geq \theta \left(1 - \frac{z}{n}\right)\right) = 1 - \left(1 - \frac{z}{n}\right)^n \rightarrow 1 - e^{-z}$$

Hence:

$$n \left( \frac{\theta - \hat{\theta}_2}{\theta} \right) \xrightarrow{d} \text{Exp}(1)$$

- (g) The MM estimator  $\hat{\theta}_{MM} = 2\bar{Y}$  is unbiased with variance  $\theta^2/(3n)$  and, by the CLT, satisfies  $\sqrt{n}(\hat{\theta}_{MM} - \theta) \xrightarrow{d} \mathcal{N}(0, \theta^2/3)$ ; it is therefore  $\sqrt{n}$ -consistent and asymptotically normal. The ML estimator  $\hat{\theta}_{ML} = \max_i Y_i$  is downward biased ( $\mathbb{E}[\hat{\theta}_{ML}] = \frac{n}{n+1}\theta$ ) but consistent and converges faster:  $n(\theta - \hat{\theta}_{ML})/\theta \xrightarrow{d} \text{Exp}(1)$ , so its error is  $O_p(1/n)$  with  $\text{MSE} \sim 2\theta^2/n^2$ . So it is not clear which estimator is better.

- (h) See Stata code in Appendix.

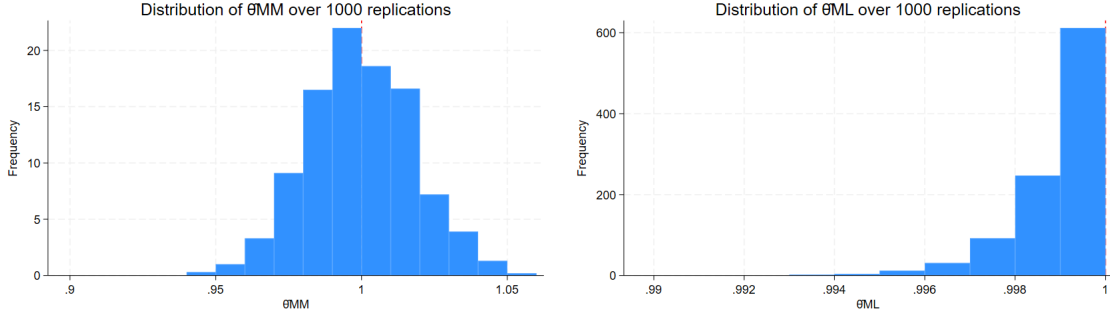


Figure 1: Histograms of  $\hat{\theta}_{MM}$  (left) and  $\hat{\theta}_{ML}$  (right) across 100 replications. The red vertical line marks the true parameter  $\theta = 1$ .

(i) From (f),

$$\frac{n(\theta - \hat{\theta}_{ML})}{\theta} \xrightarrow{d} U, \quad U \sim \text{Exp}(1).$$

Since  $\hat{\theta}_{ML} \xrightarrow{p} \theta$ , hence also  $\theta/\hat{\theta}_{ML} \xrightarrow{p} 1$ . Therefore, by Slutsky's lemma,

$$\frac{n(\theta - \hat{\theta}_{ML})}{\hat{\theta}_{ML}} = \frac{n(\theta - \hat{\theta}_{ML})}{\theta} \cdot \frac{\theta}{\hat{\theta}_{ML}} \xrightarrow{d} U \cdot 1 = U \sim \text{Exp}(1).$$

Let  $t_{1-\alpha}$  be the  $(1 - \alpha)$ -quantile of  $\text{Exp}(1)$  (so  $t_{1-\alpha} = -\ln \alpha$ ). Then

$$P\left(\frac{n(\theta - \hat{\theta}_{ML})}{\hat{\theta}_{ML}} \leq t_{1-\alpha}\right) \rightarrow 1 - \alpha,$$

which is equivalent to

$$P\left(\theta \leq \hat{\theta}_{ML} + \frac{\hat{\theta}_{ML}}{n} t_{1-\alpha}\right) \rightarrow 1 - \alpha.$$

Since  $\hat{\theta}_{ML} \leq \theta$  almost surely, we obtain the asymptotic  $(1 - \alpha)$  CI

$$IC(\alpha) = \left[ \hat{\theta}_{ML}, \hat{\theta}_{ML} + \frac{\hat{\theta}_{ML}}{n} t_{1-\alpha} \right].$$

## Exercise 3

### 3.1

Define the continuous function:

$$f(u, v) = uv$$

Given:

$$U_n \xrightarrow{P} \ell, \quad V_n \xrightarrow{P} \ell'$$

and since  $f$  is continuous in  $\mathbb{R}^2$ , the Continuous Mapping Theorem implies:

$$U_n V_n = f(U_n, V_n) \xrightarrow{P} f(\ell, \ell') = \ell \ell'$$

## 3.2

We are given:

$$U_n \xrightarrow{P} \ell, \quad V_n \xrightarrow{P} \ell'$$

By the first part of Slutsky lemma, since convergence in probability implies convergence in distribution, we have:

$$U_n \xrightarrow{d} \ell, \quad V_n \xrightarrow{d} \ell'$$

Define the continuous function:

$$f(u, v) = uv$$

Similar to 3.1, by the Continuous Mapping Theorem, we have:

$$f(U_n, V_n) = U_n V_n \xrightarrow{d} f(\ell, \ell') = \ell \ell'$$

Since  $\ell \ell'$  is a constant, by the second statement of Slutsky's lemma, convergence in distribution to a constant implies convergence in probability:

$$U_n V_n \xrightarrow{P} \ell \ell'$$