Problem Set 1: Bellman equations

Sciences Po - Macroeconomics 3 - Fall 2025
Professor: Xavier Ragot
TA: Paloma Péligry

The deadline submission is 21st, September 23:59.

Literature: Wealth distribution

Present and comment on an **empirical** result or a descriptive statistics about wealth distribution of an article of your choice, typically a table or a graph. Please confirm the article with me by email (paloma.peligry@sciencespo.fr). Once it has been approved, register the paper on the Google Sheet.

Exercise 1: Recursive problem and Value Function Iteration

Consider an economy in which the representative consumer lives forever. There is a good in each period that can be consumed or saved as capital. The consumer's utility function is

$$\sum_{t=0}^{\infty} \beta^t log(c_t)$$

Where $\beta \in (0, 1)$. The consumer is also endowed with k_0 units of capital in the first period. The feasible allocations satisfy:

$$c_t + k_{t+1} \leq \theta k_t^{\alpha} \quad \forall t \geq 0$$

Where $0 < \alpha < 1$ and $\theta > 0$. Notice we also have the following constraints:

$$c_t, k_t \geq 0$$

1. Write the problem in recursive form. Specify the Bellman equation, the state variables(s), the control variable(s) and the feasible set(s).
2. **Euler equation**
   (a) Derive the F.O.C.
   (b) Derive the Envelope condition.
   (c) Find the Euler equation.
3. **Value Function Iteration:**
   (a) Let's make the initial guess that the value function has the following form: $V_0(x) = 0$ $\forall x \in \Gamma$, where Γ is the feasible set. Find the next guess V_1 such that $V_1 = TV_0$.
   Hint: here, to get V_1, you need to choose the value of k' that maximizes V_1 given V_0. The choice might not require to take the FOC!
   (b) Find the next guess V_2 such that $V_2 = TV_1$.
4. **Guess and verify**: Assume that the value function has the form $V(k) = a_1 + a_2 logk$. Solve for the analytical solution of the value function and recover the values of a_1 and a_2.
   Note: this question is independent of question 3.

5. Find the policy functions.
6. **Recover the sequence of consumption.** Assume that $k_0 = 1$ $\alpha = 0.5$, $\beta = 0.9$ and $\theta = 2$, find the values of c_0, c_1 and c_2.

Exercise 2: Housing problem

Houses are durable goods from which households derive some utility. To model the demand for houses, a simple shortcut consists in introducing houses in the utility function. The goal of this exercise is to be able to use data on house prices and interest rates to derive properties of the demand for houses.

Households thus derive utility from consumption and from having houses. The instantaneous utility function is $u(c_t, H_t)$ where H_t is the amount of housing. Households also have access to financial savings denoted b_t at period t, remunerated at a real rate r_t between period t and $t+1$. Houses depreciate at rate δ. The program of the households is

$$\max_{\{c_t, H_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t, H_t)$$

$$c_t + b_t + P_t H_t = y_t + (1 + r_{t-1}) b_{t-1} + P_t (1 - \delta) H_{t-1}$$

1. What are the underlying assumptions about the transactions in the housing market?
2. State the transversality conditions.
3. Write the **Bellman equation** of the problem, with the value function denoted as $V(b, H)$ (be careful about the timing notation). What are the control variable(s)? What are the state variable(s)?
4. Find the two **Euler equations** (step by step).
5. BONUS: Assume the utility function has the following form:

   $$u(c_t, H_t) = (c_t^{\rho} + H_t^{\rho})^{\frac{1}{\rho}}$$

   Explain what is the economic meaning of the ρ coefficient. Using the two Euler equations, express $\frac{c_t}{H_t}$ as a function of P_t, P_{t+1} and r_t. How can we get ρ from the data?

Exercise 3: Policy functions with VFI on Julia

We consider the same program as in Exercise 1, in its recusrive form, and consider the prices as given. The utility function and production function are the same.

$$V(k) = \max_{k'} u(f(k) - k') + \beta V(k')$$

1. Define the utility and production functions.
2. Build the 2D utility array.
3. Given the guess of the value function, find the fixed point.
4. Recover the policy functions.
5. Plot the results.
6. BONUS: Start with a new guess, how does it improve the performance? Assume that $a' = a$.