Problem Set 1

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Exercise 1

Question 1:

$$\max_{(c_t,k_{t+1})} \sum_{t=0}^{\infty} \beta^t \log(c_t) = \max_{(c_0,k_{t+1})} \Big\{ \log(c_0) + \sum_{t=1}^{\infty} \beta^t \log(c_t) \Big\}.$$

$$= \max_{(c_0,k_{t+1})} \log(c_0) + \beta \max_{(c_t,k_{t+1})} \sum_{t=0}^{\infty} \beta^t \log(c_{t+1}).$$

Therefore,

$$V(k_0) = \max_{(c_0,k_1)} \Big\{ \log(c_0) + \beta V(k_1) \Big\}.$$

The recursive form of the problem is:

$$V(k_t) = \max_{(c_t,k_{t+1})} \Big\{ \log(c_t) + \beta V(k_{t+1}) \Big\}.$$

State variable: k_t.
Control variables: c_t, k_{t+1}.
Feasible set: $\Gamma(k) = \{(c, k_{t+1}) : c_t + k_{t+1} \leq \theta k_t^\alpha,\ c_t, k_{t+1} \geq 0\}$.

Using the resource constraint $c_t = \theta k_t^\alpha - k_{t+1}$, we can rewrite the Bellman equation as:

$$V(k_t) = \max_{(c_t,k_{t+1})} \Big\{ \log(\theta k_t^\alpha - k_{t+1}) + \beta V(k_{t+1}) \Big\}.$$

Question 2: a. FOC wrt to k_{t+1}:

$$\frac{1}{\theta k_t^\alpha - k_{t+1}} = \beta V'(k_{t+1}),$$

so

$$\frac{1}{c_t} = \beta V'(k_{t+1}).$$

b. Envelope theorem: Consider $V'(k_t)$ at optimal value $k_{t+1} = g_k(k_t)$

$$V'(k_t) = \frac{1}{\theta k_t^\alpha - k_{t+1}} \cdot \alpha\theta k_t^{\alpha-1} - \frac{1}{\theta k_t^\alpha - k_{t+1}} \cdot g_k'(k_t) + \beta V'(g_k(k_t)) \cdot g_k'(k_t).$$

Thus

$$V'(k_t) = \frac{1}{\theta k_t^\alpha - k_{t+1}} \alpha\theta k_t^{\alpha-1} + g_k'(k_t)(\beta V'(k_{t+1}) - \frac{1}{\theta k_t^\alpha - k_{t+1}}).$$

Given the FOC, the second term is zero, so

$$V'(k_t) = \frac{1}{c_t}\alpha\theta k_t^{\alpha-1}.$$

c. Combining the FOC and the envelope condition, we have

$$\frac{1}{\theta k_t^\alpha - k_{t+1}} = \beta\frac{1}{c_{t+1}}\alpha\theta k_{t+1}^{\alpha-1},$$

or simply

$$u'(c_t) = \beta u'(c_{t+1})f'(k_{t+1}).$$

Question 3: a. Initial guess:

$$V_0(k) = 0.$$

Then

$$V_1(k) = \max_{k_{t+1}} \log(\theta k_t^\alpha - k_{t+1}).$$

Since there is no continuation value, the optimal choice is $k_{t+1} = 0$, hence

$$V_1(k) = \log(\theta k_t^\alpha).$$

b. We then compute

$$V_2(k) = \max_{k_{t+2}} \Big\{ \log(\theta k_{t+1}^\alpha - k_{t+2}) + \beta\log(\theta(k_{t+1})^\alpha) \Big\}.$$

FOC wrt to k_{t+2}:

$$-\frac{1}{\theta k_{t+1}^\alpha - k_{t+2}} + \frac{\beta\alpha}{k_{t+2}} = 0.$$

So,

$$\frac{1}{\theta k_{t+1}^\alpha - k_{t+2}} = \frac{\beta\alpha}{k_{t+2}}.$$

This implies

$$k_{t+2} = \beta\alpha\,\theta k_{t+1}^\alpha.$$

or in general,

$$k_{t+1} = \beta\alpha\,\theta k_t^\alpha.$$

Question 4: We guess the value function has the form

$$V(k) = a_1 + a_2\log k.$$

The Bellman equation is

$$V(k_t) = \max_{k_{t+1}} \Big\{ \log(\theta k_t^\alpha - k_{t+1}) + \beta V(k_{t+1}) \Big\}.$$

So

$$V(k_{t+1}) = a_1 + a_2 \log k_{t+1}$$

and thus

$$V(k_t) = \max_{k_{t+1}} \Big\{ \log(\theta k_t^\alpha - k_{t+1}) + \beta(a_1 + a_2 \log k_{t+1}) \Big\}.$$

FOC wrt to k_{t+1}:

$$-\frac{1}{\theta k_t^\alpha - k_{t+1}} + \frac{\beta a_2}{k_{t+1}} = 0.$$

Thus

$$\theta k_t^\alpha - k_{t+1} = \frac{k_{t+1}}{\beta a_2}.$$

Rearranging,

$$k_{t+1} = \frac{\beta a_2}{1 + \beta a_2} \theta k_t^\alpha.$$

also,

$$c_t = \theta k_t^\alpha - k_{t+1} = \frac{1}{1 + \beta a_2} \theta k_t^\alpha.$$

Back to the Bellman equation:

$$V(k) = \log\left(\frac{\theta k^\alpha}{1 + \beta a_2}\right) + \beta\Big(a_1 + a_2 \log\left(\frac{\beta a_2}{1 + \beta a_2} \theta k^\alpha\right)\Big).$$

$$V(k) = \log(\theta k^\alpha) - \log(1 + \beta a_2) + \beta a_1 + \beta a_2 \Big(\log\left(\frac{\beta a_2}{1 + \beta a_2}\right) + \log(\theta k^\alpha)\Big).$$

$$= log(\theta) + \alpha \log(k) - \log(1 + \beta a_2) + \beta a_1 + \beta a_2 \log\left(\frac{\beta a_2}{1 + \beta a_2}\right) + \beta a_2 \log(\theta) + \alpha \beta a_2 \log(k).$$

Looking for coefficients on $\log(k)$, we have

$$a_2 = \alpha + \beta a_2 \alpha,$$

which gives

$$a_2(1 - \alpha\beta) = \alpha,$$

so

$$a_2 = \frac{\alpha}{1 - \alpha\beta}.$$

similarly, looking for constant terms, we have

$$a_1 = \log(\theta) - \log(1 + \beta a_2) + \beta a_1 + \beta a_2 \left(\log\left(\frac{\beta a_2}{1 + \beta a_2}\right) + \log \theta \right).$$

Denote $C = \log(\theta) - \log(1 + \beta a_2) + \beta a_2 \left(\log\left(\frac{\beta a_2}{1+\beta a_2}\right) + \log \theta\right)$, then

$$a_1 = C + \beta a_1,$$

so

$$a_1 = \frac{C}{1-\beta}.$$

Recall that $a_2 = \frac{\alpha}{1-\alpha\beta}$ and noticing that

$$1 + \beta a_2 = \frac{1-\alpha\beta+\alpha\beta}{1-\alpha\beta} = \frac{1}{1-\alpha\beta},$$

we have

$$(1+\beta a_2)(1-\alpha\beta) = 1.$$

Thus

$$\beta a_2 - \beta^2 a_2 \alpha = \alpha\beta,$$

and then

$$\frac{\beta a_2}{1+\beta a_2} = \alpha\beta.$$

Plugging these into C, we have

$$C = \log(\theta(1-\alpha\beta)) + \beta\frac{\alpha}{1-\alpha\beta}\log(\alpha\beta\theta).$$

Question 5: Plug back into the function for k_{t+1}:

$$k_{t+1} = \frac{\beta a_2}{1+\beta a_2}\theta k_t^\alpha = \alpha\beta\theta k_t^\alpha.$$

similarly for c_t:

$$c_t = \frac{1}{1+\beta a_2}\theta k_t^\alpha = (1-\alpha\beta)\theta k_t^\alpha.$$

Question 6: Plug in the values of parameters:

$$c_0 = (1 - 0.5 * 0.9) * 2 * 1^{0.5} = 1.1,$$

$$k_1 = 0.5 * 0.9 * 2 * 1^{0.5} = 0.9,$$
$$c_1 = (1 - 0.5 * 0.9) * 2 * 0.9^{0.5} = 1.043,$$
$$k_2 = 0.5 * 0.9 * 2 * 0.9^{0.5} = 0.854,$$
$$c_2 = (1 - 0.5 * 0.9) * 2 * 0.854^{0.5} = 1.016.$$