Network Games with Incomplete Information*

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Abstract

We consider a network game with strategic complementarities where the individual reward or the strength of interactions is only partially known by the agents. Players receive different correlated signals and they make inferences about other players' information. We demonstrate that there exists a unique Bayesian-Nash equilibrium. We characterize the equilibrium by disentangling the information effects from the network effects and show that the equilibrium effort of each agent is a weighted combinations of different Katz-Bonacich centralities.

Keywords: social networks, strategic complementarities, Bayesian games.
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1 Introduction

Social networks are important in numerous facets of our lives. For example, the decision of an agent to buy a new product, attend a meeting, commit a crime, find a job is often influenced by the choices of his or her friends and acquaintances. The emerging empirical evidence on these issues motivates the theoretical study of network effects. For example, job offers can be obtained from direct and indirect acquaintances through word-of-mouth communication. Also, risk-sharing devices and cooperation usually rely on family and friendship ties. Spread of diseases, such as AIDS infection, also strongly depends on the geometry of social contacts. If the web of connections is dense, we can expect higher infection rates.

Network analysis is a growing field within economics[1] because it can analyze the situations described above and provides interesting predictions in terms of equilibrium behavior. A recent branch of the network literature has focused on how network structure influences individual behaviors. This is modeled by what are sometimes referred to as "games on networks"or "network games".[2] The theory of "games on networks" considers a game with n agents (that can be individuals, firms, regions, countries, etc.) who are embedded in a network. Agents choose actions (e.g., buying products, choosing levels of education, engaging in criminal activities, investing in R&D, etc.) to maximize their payoffs, given how they expect others in their network to behave. Thus, agents implicitly take into account interdependencies generated by the social network structure. An important paper in this literature is that of Ballester et al. (2006). They compute the Nash equilibrium of a network game with *strategic complementarities* when agents choose their efforts simultaneously. In their setup, restricted to linear-quadratic utility functions, they establish that, for any possible network, the peer effects game has a unique Nash equilibrium where each agent effort's is proportional to her Katz-Bonacich centrality measure. This is a measure introduced by Katz (1953) and Bonacich (1987), which counts all paths starting from an agent but gives a smaller value to connection that are farther away.

While settings with a fixed network are widely applicable,[3] there are also many applications where players choose actions *without fully knowing* with whom they will interact. For example, learning a language, investing in education, investing in a software program, and so forth. These can be better modeled using the machinery of incomplete information games. This is what we do in this paper.

[1]For overviews on the network literature, see Goyal (2007), Jackson (2008, 2011), Ioannides (2012), Jackson et al. (2015) and Zenou (2015).

[2]For a recent overview of this literature, see Jackson and Zenou (2015).

[3]See e.g. Belhaj and Deroïan (2013), König et al. (2014) and Zhou and Chen (2015).

To be more precise, we consider a model similar to that of Ballester et al. (2006) but where the individual reward or the strength of interactions is *partially known*. In other words, this is a model where the state of world (i.e. the marginal return of effort or the synergy parameter) is common to all agents but only partially known by them. We assume that there is no communication between the players and that the network does not affect the possible channels of communication between them.

We start with a simple model with imperfect information on the marginal return of effort and where there are two states of the world. All individuals share a common prior and each individual receives a private signal, which is partially informative. Using the same condition as in the perfect information case, we show that there exists a unique Bayesian-Nash equilibrium. We can also characterize the Nash equilibrium of this game for each agent and for each signal received by disentangling the network effects from the information effects by showing that each effort is a weighted combination of two Katz-Bonacich centralities where the decay factors are the eigenvalues of the information matrix times the synergy parameter while the weights involve conditional probabilities, which include beliefs about the states of the world given the signals received by all agents. We then extend our model to any number of the states of the world and any signal. We demonstrate that there also exists a unique Bayesian-Nash equilibrium and give a complete characterization of equilibrium efforts as a function of weighted Katz-Bonacich centralities and information aspects. We also derive similar results for the case when the strength of interactions is partially known.

The paper unfolds as follows. In the next section, we relate our paper to the network literature with incomplete information. In Section 3, we characterize the equilibrium in the model with perfect information and show under which condition there exists a unique Nash equilibrium. Section 4 deals with a simple model with only two states of the world and two signals when the marginal return of effort is partially known. In Section 5, we analyze the general model when there is a finite number of states of the world and signals for the case when the marginal return of effort is unknown. In Section 6, we discuss the case when the information matrix is not diagonalizable. Finally, Section 7 concludes. In Appendix A.1, we discuss the implications of some important assumptions of the model. In Appendix A.2, we analyze the general model when there is a finite number of states of the world and signals for the case when the strength of interactions is unknown. Appendix A.3 deals with case when the information matrix is not diagonalizable and where we resort to the Jordan decomposition. The proofs of all lemmas and propositions in the main text can be found in Appendix A.4.

2 Related literature

Our paper is a contribution to the literature on "games on networks" or "network games" (Jackson and Zenou, 2015). We consider a game with *strategic complements* where an increase in the actions of other players leads a given player's higher actions to have relatively higher payoffs compared to that player's lower actions. In this framework, we consider a game with imperfect information on either the marginal payoff of effort or the strength of interaction. There is a relatively small literature which looks at the issue of imperfect information in this class of games.

Galeotti et al. (2010) and Jackson and Yariv (2007) are related to our paper but they study a very different dimension of uncertainty—i.e., uncertainty about the network structure. They show that an incomplete information setting can actually simplify the analysis of games on networks. In particular, results can be derived showing how agents' actions vary with their degree.[4]

There is also an interesting literature on learning in networks. Bala and Goyal (1998) were among the first to study this issue and show that each agent in a connected network will obtain the same long-run utility and that, if the network is large enough and there are enough agents who are optimistic about each action spread throughout the network, then the probability that the society will converge to the best overall action can be made arbitrarily close to 1. More recently, Acemoglu et al. (2011) study a model where the state of the world is unknown and affects the action and the utility function of each agent. Each agent forms beliefs about this state from a private signal and from her observation of the actions of other agents. As in our model, agents can update their beliefs in a Bayesian way. They show that when private beliefs are unbounded (meaning that the implied likelihood ratios are unbounded), there will be asymptotic learning as long as there is some minimal amount of "expansion in observations".[5]

Compared to the literature of incomplete information in networks, our paper is the first to consider a model with a common unknown state of the world (i.e. the marginal return of effort or the synergy parameter), which is partially known by the agents and where there is neither communication nor learning.[6] We first show that there exists a unique Bayesian-Nash equilibrium. We are also able to completely characterize this unique equilibrium. This char-

[4] See Jackson and Yariv (2011) for an overview of this literature.

[5] For overviews on these issues, see Jackson (2008, 2011) and Goyal (2011).

[6] Bergemann and Morris (2013) propose an interesting paper on these issues but without an explicit network analysis. Blume et al. (2015) develop a network model with incomplete information but mainly focus on identification issues.

acterization is such that each equilibrium effort is a combination of different Katz-Bonacich centralities, where the decay factors are the corresponding eigenvalues of the information matrix while the weights are the elements of matrices that have eigenvectors as columns. We are able to do so because we could diagonalize both the adjacency matrix of the network, which lead to the Katz-Bonacich centralities, and the information matrix.

3 The complete information case

3.1 The model

The network Let $\mathcal{I} := \{1, \ldots, n\}$ denote the set of players, where $n > 1$, connected by a network $\mathbf{g}$. We keep track of social connections in this network by its symmetric *adjacency matrix* $\mathbf{G} = [g_{ij}]$, where $g_{ij} = g_{ji} = 1$ if i and j are linked to each other, and $g_{ij} = 0$, otherwise. We also set $g_{ii} = 0$. The *neighborhood* of individual i is the set of i's neighbors given by $N_i = \{j \neq i \mid g_{ij} = 1\}$. The cardinality of the set N_i is $g_i = \sum_{j=1}^{n} g_{ij}$, which is known as the *degree* of i in graph theory.

Payoffs Each agent takes action $x_i \in [0, +\infty)$ that maximizes the following quadratic utility function:

$$u_i\left(x_i, x_{-i}; \mathbf{G}\right) = \alpha x_i - \frac{1}{2} x_i^2 + \beta \sum_{j=1}^{n} g_{ij} x_i x_j \tag{1}$$

where α is the *marginal return of effort* and β is the *strength of strategic interactions* (*synergy* parameter).

The first two terms of the utility function correspond to a standard cost-benefit analysis without the influence of others. In other words, if individual i was isolated (not connected in a network), she will choose the optimal action $x_i^* = \alpha$, independent of what the other agents choose. The last term in (1) reflects the network effects, i.e. the impact of the agents' links *aggregate* effort levels on i's utility. As agents may have different locations in a network and their friends may choose different effort levels, the term $\sum_{j=1}^{n} g_{ij} x_i x_j$ is heterogeneous in i. The coefficient β captures the *local-aggregate* endogenous peer effect. More precisely, bilateral influences for individual i, j $(i \neq j)$ are captured by the following cross derivatives

$$\frac{\partial^2 u_i\left(x_i, x_{-i}; \mathbf{G}\right)}{\partial x_i \partial x_j} = \beta g_{ij} \tag{2}$$

As we assume $\beta > 0$, if i and j are linked, the cross derivative is positive and reflects *strategic complementarity* in efforts. That is, if j increases her effort, then the utility of i

will be higher if i also increases her effort. Furthermore, the utility of i increases with the number of friends.

In equilibrium, each agent maximizes her utility (1). From the first-order condition, we obtain the following best-reply function for individual i

$$x_i^* = \alpha + \beta \sum_{j=1}^{n} g_{ij} x_j^* \tag{3}$$

The Katz-Bonacich network centrality measure Let $\mathbf{G}^k$ be the kth power of $\mathbf{G}$, with coefficients $g_{ij}^{[k]}$, where k is some nonnegative integer. The matrix $\mathbf{G}^k$ keeps track of the indirect connections in the network: $g_{ij}^{[k]} \geq 0$ measures the number of walks of length $k \geq 1$ in g from i to j.[7] In particular, $\mathbf{G}^0 = \mathbf{I}_n$, where $\mathbf{I}_n$ is the $n \times n$ identity matrix. Denote by $\lambda_{\max}(\mathbf{G})$ the largest eigenvalue of $\mathbf{G}$, which means that, for every eigenvalue $\lambda_i(\mathbf{G})$ of a non-negative matrix $\mathbf{G}$, $|\lambda_i(\mathbf{G})| \leq \lambda_{\max}(\mathbf{G})$ for all i (Perron-Frobenius Theorem). From this fact, it is straightforward to conclude that $\mathbf{G}^k$ converges as k goes to infinity (i.e., $\beta |\lambda_i(\mathbf{G})| < 1$ for all i) if and only if $\beta\lambda_{\max}(\mathbf{G}) < 1$. We have the following definition:

Definition 1 *Consider a network* $\mathbf{g}$ *with adjacency* $n-$*square matrix* $\mathbf{G}$ *and a scalar* $\beta > 0$ *such that* $\beta\lambda_{\max}(\mathbf{G}) < 1$.

(i) *Given a vector* $\mathbf{u}_n \in \mathbb{R}_+^n$, *the vector of* $\mathbf{u}_n$*-weighted Katz-Bonacich centralities of parameter* β *in* $\mathbf{g}$ *is:*

$$\mathbf{b}_{\mathbf{u}_n}(\beta, \mathbf{G}) := \sum_{k=0}^{+\infty} \beta^k \mathbf{G}^k \mathbf{u}_n = (\mathbf{I}_n - \beta\mathbf{G})^{-1} \mathbf{u}_n \tag{4}$$

(ii) *If* $\mathbf{u}_n = \mathbf{1}_n$, *where* $\mathbf{1}_n$ *is the* $n-$*dimensional vector of ones, then the* ***unweighted*** *Katz-Bonacich centrality of parameter* β *in* $\mathbf{g}$ *is:*[8]

$$\mathbf{b}(\beta, \mathbf{G}) := \sum_{k=0}^{+\infty} \beta^k \mathbf{G}^k \mathbf{1}_n = (\mathbf{I}_n - \beta\mathbf{G})^{-1} \mathbf{1}_n \tag{5}$$

[7] A *walk* of length k from i to j is a sequence $\langle i_0, ..., i_k \rangle$ of players such that $i_0 = i$, $i_k = j$, $i_p \neq i_{p+1}$, and $g_{i_p i_{p+1}} > 0$, for all $0 \leq p \leq k-1$, that is, players i_p and i_{p+1} are directly linked in $\mathbf{g}$. In fact, $g_{ij}^{[k]}$ accounts for the total weight of all walks of length k from i to j. When the network is un-weighted, that is, $\mathbf{G}$ is a $(0,1)-$matrix, $g_{ij}^{[k]}$ is simply the number of walks of length k from i to j.

[8] To avoid cumbersome notations, when $\mathbf{u}_n = \mathbf{1}_n$, the *unweighted* Katz-Bonacich centrality vector is denoted by $\mathbf{b}(\beta, \mathbf{G})$ and not $\mathbf{b}_{\mathbf{1}}(\beta, \mathbf{G})$ and the individual one by $b_i(\beta, \mathbf{G})$ and not $b_{i,1}(\beta, \mathbf{G})$. For any other *weighted* Katz-Bonacich centralities, we will use the notations $\mathbf{b}_{\mathbf{u}}(\beta, \mathbf{G})$ and $b_{i,u}(\beta, \mathbf{G})$.

If we consider the *unweighted* Katz-Bonacich centrality of node i (defined by (5)), i.e. $b_i(\beta, \mathbf{G})$, it counts the *total* number of walks in $\mathbf{g}$ starting from i and discounted by distance. By definition, $\mathbf{b}(\beta, \mathbf{G}) \geq 1$, with equality when $\beta = 0$. The $\mathbf{u}_n$-weighted Katz-Bonacich centrality of node i (defined by (4)), i.e. $b_{i,u}(\beta, \mathbf{G})$, has a similar interpretation with the additional fact that the walks have to be weighted by the vector $\mathbf{u}_n$.

We have a first result due to Ballester et al. (2006) and Calvó-Armengol et al. (2009).[9]

Proposition 1 *If $\alpha > 0$ and $0 < \beta < 1/\lambda_{\max}(\mathbf{G})$, then the network game with payoffs (1) has a unique interior Nash equilibrium in pure strategies given by*

$$x_i^* = \alpha\, b_i(\beta, \mathbf{G}) \tag{6}$$

The equilibrium Katz-Bonacich centrality measure $\mathbf{b}(\beta, \mathbf{G})$ is thus the relevant network characteristic that shapes equilibrium behavior. This measure of centrality reflects both the direct and the indirect network links stemming from each agent.

To understand why the above characterization in terms of the largest eigenvalue $\lambda_{\max}(\mathbf{G})$ works, and to connect the analysis to what follows below, we provide now a characterization of the solution using the fact that the adjacency matrix $\mathbf{G}$ is diagonalizable. The system that characterizes the equilibrium of the game (6) is:

$$\mathbf{x}^* = \alpha\,(\mathbf{I}_n - \beta\mathbf{G})^{-1}\,\mathbf{1}_n$$

To resolve this system we are going to diagonalize $\mathbf{G}$, which is assumed to be symmetric and thus diagonalizable. We have that $\mathbf{G} = \mathbf{C}\mathbf{D}_{\mathbf{G}}\mathbf{C}^{-1}$, where $\mathbf{D}_{\mathbf{G}}$ is a $n \times n$ diagonal matrix with entries equal to the eigenvalues of the matrix $\mathbf{G}$, i.e.

$$\mathbf{D}_{\mathbf{G}} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $\mathbf{G}$. The Neuman series $\sum_{k\geq 0} \beta^k (\mathbf{D}_{\mathbf{G}})^k$ converges if and only if $\beta < 1/\lambda_{\max}(\mathbf{G})$. If it converges, then $\sum_{k=0}^{+\infty} \beta^k (\mathbf{D}_{\mathbf{G}})^k = (\mathbf{I}_n - \beta\mathbf{D}_{\mathbf{G}})^{-1}$. In such

[9] It is well-known that for any symmetric adjacency matrix $\mathbf{G}$, the maximum eigenvalue has a bound:

$$d_{avg}(\mathbf{G}) \leq \lambda_{\max}(\mathbf{G}) \leq d_{\max}(\mathbf{G})$$

where $d_{avg}(\mathbf{G})$ denotes the average degree of network $\mathbf{G}$ and $d_{\max}(\mathbf{G})$ denotes the maximum degree of network $\mathbf{G}$. Hence, a necessary condition for the existence of an equilibrium is given by $\beta d_{avg}(\mathbf{G}) < 1$.

cases, we have that

$$(\mathbf{I}_n - \beta\mathbf{G})^{-1} = \sum_{k=0}^{+\infty} \beta^k \mathbf{G}^k = \mathbf{C}\left[\sum_{k=0}^{+\infty} \beta^k \left(\mathbf{D_G}\right)^k\right]\mathbf{C}^{-1}$$

Observe that the "if and only if condition" is due to the fact that the diagonal entries of $\sum_{k\geq 0} \beta^k \left(\mathbf{D_G}\right)^k$ are power series of rates equal to $\beta\lambda_i(\mathbf{G})$, and all these power series converge if and only if $\beta\lambda_{\max}(\mathbf{G}) < 1$, which is equivalent to the condition written in Proposition 1. These terms are obviously very easy to compute. Indeed, they are equal to $\frac{1}{1-\beta\lambda_i(\mathbf{G})}$ for each $i \in \{1, \ldots, n\}$. The off-diagonal elements of $\sum_{k=0}^{+\infty} \beta^k \left(\mathbf{D_G}\right)^k$ are all equal to 0.

4 The incomplete information case: A simple model when α is unknown

We develop a simple model with common values and private information where there are only two states of the world and two signals.

4.1 The model

Assume that the marginal return of effort α in the payoff function (1) is *common* to all agents but only *partially known* by the agents. Agents know, however, the exact value of the synergy parameter β.[10]

Information We assume that there are *two states of the world*, so that the parameter α can only take two values: $\alpha_l < \alpha_h$. All individuals share a common prior:

$$\mathbb{P}\left(\{\alpha = \alpha_h\}\right) = p \in (0, 1)$$

Each individual i receives a private signal, $s_i \in \{h, l\}$, such that

$$\mathbb{P}\left(\{s_i = h\} \mid \{\alpha = \alpha_h\}\right) = \mathbb{P}\left(\{s_i = l\} \mid \{\alpha = \alpha_l\}\right) = q \geq 1/2$$

where $\{s_i = h\}$ and $\{s_i = l\}$ denote, respectively, the event that agent i has received a signal h and l. Assume that there is no communication between the players and that the network does not affect the possible channels of communication between them.

[10] We consider the case of unknown β in Appendix A.2.

The Bayesian Game Given that there is incomplete information about the state of the world α and about others' information, this is a Bayesian game. Agent i has to choose an action $x_i(s_i) \geq 0$ for each signal $s_i \in \{l, h\}$. The expected utility of agent i can be written as:

$$\mathbb{E}[u_i|s_i] = \mathbb{E}[\alpha|s_i]\, x_i(s_i) - \frac{1}{2}[x_i(s_i)]^2 + \beta x_i(s_i) \sum_{j=1}^{n} g_{ij} \mathbb{E}[x_j|s_i]$$

4.2 Equilibrium Analysis

The first-order conditions are given by

$$\forall i \in \mathcal{I}, \; \frac{\partial \mathbb{E}[u_i|s_i]}{\partial x_i} = \mathbb{E}[\alpha|s_i] - x_i^*(s_i) + \beta \sum_{j=1}^{n} g_{ij} \mathbb{E}\left[x_j^*|s_i\right] = 0$$

Hence, the best reply of agent i is given by

$$x_i^*(s_i) = \mathbb{E}[\alpha|s_i] + \beta \sum_{j=1}^{n} g_{ij} \mathbb{E}\left[x_j^*|s_i\right] \tag{7}$$

Denote: $\widehat{\alpha}_l := \mathbb{E}[\alpha|\{s_i = l\}]$, $\widehat{\alpha}_h := \mathbb{E}_i[\alpha|\{s_i = h\}]$, $\gamma_l := \mathbb{P}(\{s_j = l\} \,|\, \{s_i = l\})$ and $\gamma_h := \mathbb{P}(\{s_j = h\} \,|\, \{s_i = h\})$. Denote also: $\underline{x}_i := x_i(\{s_i = l\}) := x_i(l)$, the action taken by agent i when receiving signal l and $\overline{x}_i := x_i(\{s_i = h\}) := x_i(h)$, the action taken by agent i when receiving signal h. Then, the optimal actions can be written as:

$$\underline{x}_i^* = \widehat{\alpha}_l + \beta \sum_{j=1}^{n} g_{ij} \left[\gamma_l \, \underline{x}_j^* + (1-\gamma_l)\, \bar{x}_j^*\right] \tag{8}$$

and

$$\bar{x}_i^* = \widehat{\alpha}_h + \beta \sum_{j=1}^{n} g_{ij} \left[(1-\gamma_h)\, \underline{x}_j^* + \gamma_h \, \bar{x}_j^*\right] \tag{9}$$

where $\widehat{\alpha}_l$, $\widehat{\alpha}_h$, γ_h and γ_l are given in Lemma 7 in Appendix A.4. Let us introduce the following notations: $\underline{\mathbf{x}} := (\underline{x}_1, ..., \underline{x}_n)^\mathsf{T}$ and $\overline{\mathbf{x}} := (\overline{x}_1, ..., \overline{x}_n)^\mathsf{T}$ are $n-$dimensional vectors, $\mathbf{x} := \begin{pmatrix} \underline{\mathbf{x}} \\ \overline{\mathbf{x}} \end{pmatrix}$ and $\widehat{\boldsymbol{\alpha}} := \begin{pmatrix} \widehat{\alpha}_l \mathbf{1}_n \\ \widehat{\alpha}_h \mathbf{1}_n \end{pmatrix}$ are $2n-$dimensional vectors, and

$$\mathbf{\Omega} := \begin{pmatrix} \gamma_l \mathbf{G} & (1-\gamma_l)\,\mathbf{G} \\ (1-\gamma_h)\,\mathbf{G} & \gamma_h \mathbf{G} \end{pmatrix}$$

is a $2n \times 2n$ matrix. Then the $2n$ equations of the best-reply functions (8) and (9) can be written in matrix form as follows:

$$\mathbf{x}^* = \widehat{\boldsymbol{\alpha}} + \beta\,\boldsymbol{\Omega}\,\mathbf{x}^*$$

If $\mathbf{I}_{2n} - \beta\,\boldsymbol{\Omega}$ is invertible, then we obtain

$$\begin{pmatrix} \underline{\mathbf{x}}^* \\ \overline{\mathbf{x}}^* \end{pmatrix} = \left[\mathbf{I}_{2n} - \beta\,\boldsymbol{\Gamma} \otimes \mathbf{G}\right]^{-1} \begin{pmatrix} \hat{\alpha}_l \mathbf{1}_n \\ \hat{\alpha}_h \mathbf{1}_n \end{pmatrix} \tag{10}$$

where

$$\boldsymbol{\Gamma} := \begin{pmatrix} \gamma_l & 1-\gamma_l \\ 1-\gamma_h & \gamma_h \end{pmatrix}$$

is a stochastic matrix and $\boldsymbol{\Gamma} \otimes \mathbf{G}$ is the Kronecker product of $\boldsymbol{\Gamma}$ and $\mathbf{G}$. $\boldsymbol{\Gamma}$ is called the *information matrix* since it keeps track of all the information received by the agent about the states of the world while $\mathbf{G}$, the adjacency matrix, is the "network" matrix since it keeps track of the position of each individual in the network. Our main result in this section can be stated as follows:

Proposition 2 *Consider the network game with payoffs (1) and unknown parameter α that can only take two values: $0 < \alpha_l < \alpha_h$. Then, if $0 < \beta < 1/\lambda_{\max}(\mathbf{G})$, there exists a unique interior Bayesian-Nash equilibrium in pure strategies given by*

$$\underline{\mathbf{x}}^* = \widehat{\alpha}\,\mathbf{b}(\beta, \mathbf{G}) - \frac{(1-\gamma_l)}{(2-\gamma_h-\gamma_l)}(\widehat{\alpha}_h - \widehat{\alpha}_l)\,\mathbf{b}((\gamma_h+\gamma_l-1)\,\beta, \mathbf{G}) \tag{11}$$

$$\overline{\mathbf{x}}^* = \widehat{\alpha}\,\mathbf{b}(\beta, \mathbf{G}) + \frac{(1-\gamma_h)}{(2-\gamma_h-\gamma_l)}(\widehat{\alpha}_h - \widehat{\alpha}_l)\,\mathbf{b}((\gamma_h+\gamma_l-1)\,\beta, \mathbf{G}) \tag{12}$$

where

$$\widehat{\alpha} \equiv \frac{(1-\gamma_h)\,\widehat{\alpha}_l + (1-\gamma_l)\,\widehat{\alpha}_h}{(2-\gamma_h-\gamma_l)}, \tag{13}$$

γ_l and γ_h are given by (47) and $\widehat{\alpha}_l$ and $\widehat{\alpha}_h$ by (48) and (49).

The following comments are in order. First, the condition for existence and uniqueness of a Bayesian-Nash equilibrium (i.e. $0 < \beta < 1/\lambda_{\max}(\mathbf{G})$) is exactly the same as the condition for the complete information case (see Proposition 1). This is due to the fact that the information matrix $\boldsymbol{\Gamma}$ is a stochastic matrix and its largest eigenvalue is thus equal to 1.

Second, we characterize the Nash equilibrium of this game for each agent and for each signal received by disentangling the network effects (captured by $\mathbf{G}$) from the information effects (captured by $\mathbf{\Gamma}$). We are able to do so because $\mathbf{G}$ is symmetric and $\mathbf{\Gamma}$ is of order 2 (i.e., it is a 2×2 matrix) and thus both are diagonalizable. We show that each effort is a combination of two Katz-Bonacich centralities where the decay factors are the eigenvalues of the information matrix $\mathbf{\Gamma}$ times the synergy parameter β while the weights are the conditional probabilities, which include beliefs about the states of the world given the signals received by all agents. To understand this result, observe that the diagonalization of $\mathbf{G}$ leads to the Katz-Bonacich centrality while the diagonalization of $\mathbf{\Gamma}$ to a matrix $\mathbf{A}$ with eigenvectors as columns. The different eigenvalues of $\mathbf{\Gamma}$ determine the number of the different Katz-Bonacich centrality vectors (two here) and the discount (or decay) factor in each of them ($1 \times \beta$ for the first Katz-Bonacich centrality and $(\gamma_h + \gamma_l - 1) \times \beta$ for the second Katz-Bonacich centrality, where 1 and $\gamma_h + \gamma_l - 1$ are the two eigenvalues of $\mathbf{\Gamma}$) and $\mathbf{A}$ and $\mathbf{A}^{-1}$ characterize the weights (i.e. $\widehat{\alpha}$ and $\frac{(1-\gamma_l)}{2-\gamma_h-\gamma_l}(\widehat{\alpha}_h - \widehat{\alpha}_l)$) of the different Katz-Bonacich centrality vectors in equilibrium strategies.

Third, observe that γ_h and γ_l, which are measures of the informativeness of private signals, enter both in $\mathbf{\Gamma}$ and therefore in the Kronecker product of $\mathbf{\Gamma} \otimes \mathbf{G}$ and in the vector $\widehat{\boldsymbol{\alpha}}$. So, when γ_h is close to 1 *and* γ_l is close to 1, which means that the signals are very informative, the gap between both eigenvalues[11] (which is a measure of the entanglement of actions in both states) tends to 0. More generally, we should expect this to be also true in the case of M different possible states of the world (we will show it formally in Section 5.3 below), bearing resemblance with the analysis in Golub and Jackson (2010, 2012), where they show that the second largest eigenvalue measures the speed of convergence of the DeGroot naive learning process, which at the same time relates to the speed of convergence of the Markov process. In our case, if the powers of $\mathbf{\Gamma}$ stabilize very fast, we can approximate very well equilibrium actions in different states with equilibrium actions in the complete information game. Finally, note that if, for example, $\widehat{\alpha}_l - \widehat{\alpha}_h \to 0$, meaning that both levels of αs (i.e. states of the world) are very similar, then $\underline{\mathbf{x}}^* \to \widehat{\alpha}_l \mathbf{b}(\lambda; \mathbf{G})$ and $\overline{\mathbf{x}}^* \to \widehat{\alpha}_l \mathbf{b}(\lambda; \mathbf{G})$. In other words, we end up with an equilibrium similar to the one obtained in the perfect information case.

[11] The largest eigenvalue of $\mathbf{\Gamma}$ is always 1 while the other eigenvalue is $\gamma_h + \gamma_l - 1$, so the gap between these two eigenvalues is $\gamma_h + \gamma_l - 2$.

5 The incomplete information case: A general model with a finite number of states and types

5.1 The model

The model of Section 4 with unknown α and two states of the world and two possible signals provides a good understanding on how the model works. Let us now consider a more general model when there is a finite number of states of the world and signals. We study a family of Bayesian games that share similar features and where there is incomplete information on either α or β. Hence we analyze Bayesian games with *common values* and *private information* (the level of direct reward of own activity, denoted by α, and the level of pairwise strategic complementarities, denoted by β).

As above, let $\mathcal{I} := \{1, \ldots, n\}$ denote the set of players, where $n > 1$. For all $i \in \mathcal{I}$, let s_i denote player i's signal, where $s_i \colon \Omega \to \mathcal{S} \subset \mathbb{R}$ is a random variable defined on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Assume that $\mathcal{S}$ is finite with $L := |\mathcal{S}| > 1$. Assume without loss of generality that $\mathcal{S} = \{1, \ldots, L\}$.[12] Let $(s_1, \ldots, s_n)^\mathsf{T}$ denote the random n-vector of the players' signals.

If $s_1, \ldots, s_n$ have the same distribution, then $s_1, \ldots, s_n$ are called *identically distributed*. Similarly, for all $2 \leq m \leq n$, for all $\{i_k\}_{k=1}^m \subset \mathcal{I}$, and for all $\{j_k\}_{k=1}^m \subset \mathcal{I}$, if $(s_{i_1}, \ldots, s_{i_m})^\mathsf{T}$ and $(s_{j_1}, \ldots, s_{j_m})^\mathsf{T}$ have the same (multivariate) distribution, then $(s_{i_1}, \ldots, s_{i_m})^\mathsf{T}$ and $(s_{j_1}, \ldots, s_{j_m})^\mathsf{T}$ are called *identically distributed*.

A permutation π of $\mathcal{I}$ is a bijection $\pi \colon \mathcal{I} \to \mathcal{I}$. Any permutation π of $\mathcal{I}$ can be uniquely represented by a non-singular $n \times n$ matrix $\mathbf{P}_\pi$, the so-called permutation matrix of π.

Definition 2 *The (multivariate) distribution of $(s_1, \ldots, s_n)^\mathsf{T}$, or equivalently, the joint distribution of $s_1, \ldots, s_n$, is called **permutation invariant** if for all permutations π of $\mathcal{I}$, $\mathbf{P}_\pi(s_1, \ldots, s_n)^\mathsf{T} = (s_{\pi(1)}, \ldots, s_{\pi(n)})^\mathsf{T}$ and $(s_1, \ldots, s_n)^\mathsf{T}$ are identically distributed.*

If the distribution of $(s_1, \ldots, s_n)^\mathsf{T}$ is permutation invariant, permuting the components of $(s_1, \ldots, s_n)^\mathsf{T}$ does not change its distribution. For example, if $n = 3$ and the (trivariate) distribution of $(s_1, s_2, s_3)^\mathsf{T}$ is permutation invariant, then $(s_1, s_2, s_3)^\mathsf{T}$, $(s_1, s_3, s_2)^\mathsf{T}$, $(s_2, s_1, s_3)^\mathsf{T}$, $(s_2, s_3, s_1)^\mathsf{T}$, $(s_3, s_1, s_2)^\mathsf{T}$, and $(s_3, s_2, s_1)^\mathsf{T}$ are identically distributed.

From now on, we assume that the two following assumptions hold throughout the paper:

Assumption 1: *For all $i \in \mathcal{I}$ and for all $\tau \in S$, $\mathbb{P}(\{s_i = \tau\}) > 0$.*

[12] This assumption is crucial for the definition of the players' information matrix (see Definition 3 and Remark 1).

Assumption 1 ensures that conditional probabilities of the form $\mathbb{P}(\{s_j = t\} \mid \{s_i = \tau\})$ are defined.

Assumption 2: *The distribution of* $(s_1, \ldots, s_n)^\intercal$ *is permutation invariant.*

In Appendix A.1, sections A.1.1 and A.1.2, we derive some results showing the importance of each of these two assumptions. We show that the information matrix $\mathbf{\Gamma}$ is well-defined if Assumptions 1 and 3a (defined in Appendix A.1) are satisfied. A sufficient condition for Assumption 3a to hold true is that the distribution of the players' signals is permutation invariant (Assumption 2). Observe that, in Proposition 3 in Appendix A.1, we show that Assumptions 1 and 2 guarantee that the eigenvalues of matrix $\mathbf{\Gamma}$ are all real. Observe also that Assumption 2 does not imply that $\mathbf{\Gamma}$ is symmetric. It just says that the identity of the player does not matter when calculating conditional probabilities. Below we give an example where Assumption 2 is satisfied and the matrix $\mathbf{\Gamma}$ is *not* symmetric.

Let us now go back to the model and let $\theta \in \{\alpha, \beta\}$ be the unknown common value. This parameter can take M different values (i.e. states of the world), $\theta \in \Theta = \{\theta_1, \ldots, \theta_M\}$. Agents can be of T different types, that we denote by $\mathcal{S} = \{1, \ldots, T\}$. These types can be interpreted as private informative signals of the value of θ. Next, we define the notation of the players' information matrix.

Definition 3 *The players'* information matrix, *denoted by* $\mathbf{\Gamma} = (\gamma_{t\tau})_{(t,\tau)\in\mathcal{S}^2}$, *is a square matrix of order* $T = L = |\mathcal{S}|$ *that is given by*

$$\forall (t,\tau) \in \mathcal{S}^2 \quad \gamma_{t\tau} = \mathbb{P}(\{s_i = \tau\} \mid \{s_j = t\}) = \frac{\mathbb{P}(\{s_i = \tau\} \cap \{s_j = t\})}{\mathbb{P}(\{s_j = t\})}, \tag{14}$$

where $(i,j) \in \mathcal{I}^2$ *with* $i \neq j$ *is arbitrary.*

Building on this definition, we can derive the *information* matrix $\mathbf{\Gamma} = (\gamma_{t\tau})_{(t,\tau)\in\mathcal{S}^2}$ where $\gamma_{t\tau}$ is defined by

$$\begin{aligned}
\gamma_{t\tau} &= \mathbb{P}(\{s_i = \tau\} \,|\, \{s_j = t\}) = \sum_{m=1}^{M} \mathbb{P}(\{\theta = \theta_m\} \cap \{s_i = \tau\} \,|\, \{s_j = t\}) \\
&= \frac{\sum_{m=1}^{M} \mathbb{P}(\{s_j = t\} \,|\, \{\theta = \theta_m\} \cap \{s_i = \tau\}) \mathbb{P}(\{\theta = \theta_m\} \cap \{s_i = \tau\})}{\mathbb{P}(\{s_j = t\})} \\
&= \frac{\sum_{m=1}^{M} \mathbb{P}(\{s_j = t\} \,|\, \{\theta = \theta_m\} \cap \{s_i = \tau\}) \mathbb{P}(\{s_i = \tau\} \,|\, \{\theta = \theta_m\}) \mathbb{P}(\theta_m)}{\mathbb{P}(\{s_j = t\})}
\end{aligned}$$

i.e. $\gamma_{t\tau}$ is the conditional probability of the event $\{s_i = \tau\}$ (that is, an agent i receives the signal τ) given the even $\{s_j = t\}$ (that is, another agent j receives the signal t). We obtain the following $T \times T$ matrix:

$$\underset{(T,T)}{\mathbf{\Gamma}} = \begin{pmatrix} \gamma_{11} & \cdots & \gamma_{1T} \\ \vdots & \ddots & \vdots \\ \gamma_{T1} & \cdots & \gamma_{TT} \end{pmatrix}$$

Agents know their own type but not the type of other agents. The strategy of each agent i is the function

$$x_i : \mathcal{S} \longrightarrow [0, \infty)$$

and the utility of each agent i by (1).

Let us now give an example where the distribution of $(s_1, \ldots, s_n)^\mathsf{T}$ is *permutation invariant* (Assumption 2) but the matrix $\mathbf{\Gamma}$ is *not* symmetric. It readily follows from Assumption 2 that $\mathbb{P}(\{s_i = \tau\} \cap \{s_j = t\}) = \mathbb{P}(\{s_i = t\} \cap \{s_j = \tau\})$. This implies that the probability mass function $\mathbb{P}(\{s_i = \tau\} \cap \{s_j = t\})$ of the (joint) distribution of (s_i, s_j) can be represented by a *symmetric* matrix as shown in the example below with three states of the world l, m and h and three signals:

t/τ	l	m	h	$\mathbb{P}(s_i = \tau)$
l	0.10	0.10	0.05	0.25
m	0.10	0.10	0.15	0.35
h	0.05	0.15	0.20	0.40
$\mathbb{P}(s_j = t)$	0.25	0.35	0.40	1

Notice that the marginal distributions for each s_i are the same. Assumption 2 therefore implies that $\mathbb{P}(\{s_i = \tau\}) = \mathbb{P}(\{s_j = \tau\})$. Indeed

$$\mathbb{P}(\{s_i = \tau\}) = \sum_t \mathbb{P}(\{s_i = \tau\} \cap \{s_j = t\}) = \sum_t \mathbb{P}(\{s_i = t\} \cap \{s_j = \tau\}) = \mathbb{P}(\{s_j = \tau\})$$

Observe, however, that this *does not* imply that the matrix of conditional probabilities $\mathbf{\Gamma}$, or the *information matrix*, will be symmetric. Indeed, Using Definition 3, it is straightforward to derive $\mathbf{\Gamma}$ for the above example

$$\gamma_{ll} = \mathbb{P}(\{s_i = l\} | \{s_j = l\}) = \frac{\mathbb{P}(\{s_i = l\} \cap \{s_j = l\})}{\mathbb{P}(\{s_j = l\})} = \frac{0.10}{0.25} = 0.4$$
$$\gamma_{lm} = \mathbb{P}(\{s_i = m\} | \{s_j = l\}) = \frac{\mathbb{P}(\{s_i = m\} \cap \{s_j = l\})}{\mathbb{P}(\{s_j = l\})} = \frac{0.10}{0.25} = 0.4$$
$$\gamma_{ml} = \mathbb{P}(\{s_i = l\} | \{s_j = m\}) = \frac{\mathbb{P}(\{s_i = l\} \cap \{s_j = m\})}{\mathbb{P}(\{s_j = m\})} = \frac{0.10}{0.35} = 0.286$$

It can be thus seen that $\gamma_{t\tau} \neq \gamma_{\tau t}$. Hence matrix $\mathbf{\Gamma}$ will be non-symmetric in general. In our example, it is given by:

$$\mathbf{\Gamma} = \begin{bmatrix} \gamma_{ll} & \gamma_{lm} & \gamma_{lh} \\ \gamma_{ml} & \gamma_{mm} & \gamma_{mh} \\ \gamma_{hl} & \gamma_{hm} & \gamma_{hh} \end{bmatrix} = \begin{bmatrix} 0.400 & 0.400 & 0.200 \\ 0.286 & 0.286 & 0.429 \\ 0.125 & 0.375 & 0.500 \end{bmatrix}$$

Observe that Assumptions 1 and 2 (see Proposition 3) guarantee that the eigenvalues are all real for matrix $\mathbf{\Gamma}$. In the present example, it can be shown that the eigenvalues of $\mathbf{\Gamma}$ are equal to: $\{1, 0.286, -0.1\} \subset \mathbb{R}$.

5.2 Example

To illustrate our information structure, consider the following example where there are $M = T$ states of the world (i.e., there are as many signals as possible θ's) but where the information structure is as follows. The priors are such that for all $m \in \{1, ..., T\}$, $\mathbb{P}(\theta_m) = 1/T$. Let us introduce the following $T \times M$ matrix $\mathbf{P} = (p_{tm})_{t,m}$. Given the type $t \in \mathcal{S}$ and a state $\theta_m \in \Theta$, we denote $p_{tm} := \mathbb{P}(\{s_i = t\} \mid \{\theta = \theta_m\})$, $t = 1, ..., T$, $m = 1, ..., M$. In this example, given the state realization, the private signals are conditionally independent and identically distributed. In that case, the matrix matrix $\mathbf{P} = (p_{tm})_{t,m}$ can be determined as:

$$p_{tm} = \mathbb{P}(\{s_i = t\} \mid \{\theta = \theta_m\}) = \begin{cases} p & \text{if} \quad t = m \\ (1-p)/(T-1) & \text{if} \quad t \neq m \end{cases}$$

where $p > 1/T$. Then, if agent i observes the signal t, she assigns the probability $p > 1/T$ of being in state t and probability $(1-p)/(T-1)$ of being in each other state. The $T \times T$ matrix $\mathbf{P}$ is then given by:[13]

$$\mathbf{P} = \begin{pmatrix} p & \frac{1-p}{T-1} & \cdots & \frac{1-p}{T-1} \\ \frac{1-p}{T-1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{1-p}{T-1} \\ \frac{1-p}{T-1} & \cdots & \frac{1-p}{T-1} & p \end{pmatrix} = p\mathbf{I}_n + \left(\frac{1-p}{T-1}\right)\left(\mathbf{1}_n\mathbf{1}_n^{\mathsf{T}} - \mathbf{I}_n\right) \tag{15}$$

where $\mathbf{1}_n$ is a $n-$dimensional vector of ones and $\mathbf{I}_n$ is the $n-$dimensional identity matrix. It is easily verified that $\mathbf{P}$ is symmetric.

[13]Observe that $\mathbf{P}$ is only introduced for this example and, in general, $\mathbf{P}$ is not sufficient to derive the information matrix $\mathbf{\Gamma}$ unless the private signals are conditionally independent.

Let us now determine the $T \times T$ information matrix $\mathbf{\Gamma}$. In this example, each element of information matrix $\mathbf{\Gamma}$ is easily computed by:

$$\gamma_{t\tau} = \sum_{m=1}^{M} \mathbb{P}(\{s_j = \tau\} \,|\, \{\theta = \theta_m\})\mathbb{P}(\{\theta = \theta_m\} \,|\, \{s_i = t\})$$

from the conditional independence assumption. Hence, if $t = \tau$, we obtain:

$$\gamma_{t\tau} = p^2 + (T-1)\left(\frac{1-p}{T-1}\right)^2 = p^2 + \frac{(1-p)^2}{T-1} \tag{16}$$

while, if $t \neq \tau$, we get:

$$\begin{aligned} \gamma_{t\tau} &= 2p\left(\frac{1-p}{T-1}\right) + (T-2)\left(\frac{1-p}{T-1}\right)^2 \\ &= \frac{(1-p)\,(Tp+T-2)}{(T-1)^2} \end{aligned} \tag{17}$$

The information matrix is thus given by:

$$\begin{aligned} \mathbf{\Gamma} &= \begin{pmatrix} p^2 + \frac{(1-p)^2}{T-1} & \frac{(1-p)(Tp+T-2)}{(T-1)^2} & \cdots & \frac{(1-p)(Tp+T-2)}{(T-1)^2} \\ \frac{(1-p)(Tp+T-2)}{(T-1)^2} & p^2 + \frac{(1-p)^2}{T-1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{(1-p)(Tp+T-2)}{(T-1)^2} \\ \frac{(1-p)(Tp+T-2)}{(T-1)^2} & \cdots & \frac{(1-p)(Tp+T-2)}{(T-1)^2} & p^2 + \frac{(1-p)^2}{T-1} \end{pmatrix} \\ &= \left[p^2 + \frac{(1-p)^2}{T-1}\right]\mathbf{I}_n + \frac{(1-p)\,(Tp+T-2)}{(T-1)^2}\left(\mathbf{1}_n\mathbf{1}_n^{\intercal} - \mathbf{I}_n\right) \end{aligned} \tag{18}$$

Evidently, $\mathbf{\Gamma}$ is symmetric.

5.3 The model with unknown α

Let us now solve the model with unknown α when there is a finite number of states of the world (M) and signals (T).

5.3.1 Equilibrium

Assume that the $T \times T$ information matrix $\mathbf{\Gamma}$ is diagonalizable. Then,

$$\mathbf{\Gamma} = \mathbf{A}\,\mathbf{D}_{\mathbf{\Gamma}}\,\mathbf{A}^{-1}$$

where

$$\mathbf{D}_{\mathbf{\Gamma}} = \begin{pmatrix} \lambda_1(\mathbf{\Gamma}) & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_T(\mathbf{\Gamma}) \end{pmatrix}$$

and $\lambda_1(\mathbf{\Gamma}), ..., \lambda_T(\mathbf{\Gamma})$ are the eigenvalues of $\mathbf{\Gamma}$, with $\lambda_{\max}(\mathbf{\Gamma}) := \lambda_1(\mathbf{\Gamma}) \geq \lambda_2(\mathbf{\Gamma}) \geq ... \geq \lambda_T(\mathbf{\Gamma})$. In this formulation, $\mathbf{A}$ is a $T \times T$ matrix where each ith column is formed by the eigenvector corresponding to the ith eigenvalue. Let us have the following notations:

$$\mathbf{A} = \begin{pmatrix} a_{11} & \cdots & a_{1T} \\ \vdots & \ddots & \vdots \\ a_{T1} & \cdots & a_{TT} \end{pmatrix} \text{ and } \mathbf{A}^{-1} = \begin{pmatrix} a_{11}^{(-1)} & \cdots & a_{1T}^{(-1)} \\ \vdots & \ddots & \vdots \\ a_{T1}^{(-1)} & \cdots & a_{TT}^{(-1)} \end{pmatrix}$$

where $a_{ij}^{(-1)}$ is the (i, j) cell of the matrix $\mathbf{A}^{-1}$.

The utility function of individual i receiving signal τ can be written as:

$$\mathbb{E}\left[u_i \middle| \{s_i = \tau\}\right] = \mathbb{E}\left[\alpha \middle| \{s_i = \tau\}\right] x_i(\tau) - \frac{1}{2}\left[x_i(\tau)\right]^2 + \beta x_i(\tau) \sum_{j=1}^{n} g_{ij} \mathbb{E}\left[x_j \middle| \{s_i = \tau\}\right]$$

The first order conditions are given by

$$\begin{aligned} \frac{\partial \mathbb{E}\left[u_i \middle| \{s_i = \tau\}\right]}{\partial x_i} &= \mathbb{E}\left[\alpha \middle| \{s_i = \tau\}\right] - x_i^*(\tau) + \beta \sum_{j=1}^{n} g_{ij} \mathbb{E}\left[x_j^* \middle| \{s_i = \tau\}\right] \\ &= \mathbb{E}\left[\alpha \middle| \{s_i = \tau\}\right] - x_i^*(\tau) + \beta \sum_{j=1}^{n} \sum_{t=1}^{T} g_{ij} \mathbb{P}\left[\{s_j = t\} \mid \{s_i = \tau\}\right] x_j^*(t) \\ &= \mathbb{E}\left[\alpha \middle| \{s_i = \tau\}\right] - x_i^*(\tau) + \beta \sum_{j=1}^{n} \sum_{t=1}^{T} g_{ij} \gamma_{\tau t} x_j^*(t) \end{aligned}$$

Define

$$\forall \tau \in \{1, ..., T\} \qquad \widehat{\alpha}_\tau := \mathbb{E}\left[\alpha \middle| \{s_i = \tau\}\right] = \sum_{m=1}^{M} \alpha_m \mathbb{P}\left(\{\alpha = \alpha_m\} \mid \{s_i = \tau\}\right)$$

We have the following result.

Theorem 1 *Consider the case when the marginal return of effort α is unknown. Assume that $\mathbf{\Gamma}$ is diagonalizable and that assumptions 1 and 2 hold. Let $\lambda_1(\mathbf{\Gamma}) \geq \lambda_2(\mathbf{\Gamma}) \geq ... \geq \lambda_T(\mathbf{\Gamma})$ be the eigenvalues of the information matrix $\mathbf{\Gamma}$. Then, if $\{\alpha_\tau\}_{\tau=1}^T \subset \mathbb{R}_{++}$ and $0 < \beta < 1/\lambda_{\max}(\mathbf{G})$, there exists a unique Bayesian-Nash equilibrium. In that case, if the signal received is $s = \tau$, then the equilibrium efforts are given by:*

$$\mathbf{x}^*(\{s = \tau\}) = \widehat{\alpha}_1 \sum_{t=1}^{T} a_{\tau t} a_{t1}^{(-1)} \mathbf{b}\left(\lambda_t\left(\mathbf{\Gamma}\right)\beta, \mathbf{G}\right) + ... + \widehat{\alpha}_T \sum_{t=1}^{T} a_{\tau t} a_{tT}^{(-1)} \mathbf{b}\left(\lambda_t\left(\mathbf{\Gamma}\right)\beta, \mathbf{G}\right) \tag{19}$$

for $\tau = 1, ..., T$.

Theorem 1 generalizes Proposition 2 when there are M states of the world, $\theta \in \Theta = \{\theta_1, \ldots, \theta_M\}$, and T different signals or types, $\mathcal{S} = \{1, \ldots, T\}$. Interestingly, the condition for existence and uniqueness of a Bayesian-Nash equilibrium (i.e. $0 < \beta < 1/\lambda_{\max}(\mathbf{G})$) is still the same because $\mathbf{\Gamma}$ is still a stochastic matrix whose largest eigenvalue is 1. Observe that the *Bayesian potential approach* (Monderer and Shapley 1996; van Heumen et al., 1996; Ui, 2000) is an alternative route one could take to prove the existence and uniqueness of a Bayesian equilibrium.[14] Lemma 6 in Ui (2000) considers a Bayesian game with quadratic payoff functions and shows the existence and uniqueness of a Bayesian equilibrium. Our uniqueness result is, however, stronger than what one would get via Bayesian potential approach because the latter requires the matrix $\mathbf{G}$ to be symmetric while our approach does not. Indeed, we only need $\mathbf{G}$ to be symmetric for the characterization of the Bayesian-Nash equilibrium but not for the existence and uniqueness result. Moreover, in Theorem 3 in Appendix A.2, where we deal with the case when β is unknown, we obtain a weaker condition than $\beta < 1/\lambda_{\max}(\mathbf{G})$ for the existence and uniqueness of the Bayesian Nash equilibrium, which is given by $\beta < 1/\left(\lambda_{\max}(\mathbf{G})\,\lambda_{\max}\left(\widetilde{\mathbf{\Gamma}}\right)\right)$ (see (25)). Indeed, when β is unknown, $\lambda_{\max}\left(\widetilde{\Gamma}\right)$ is not anymore equal to one as in the case when α is unknown because the matrix $\widetilde{\Gamma}$ is not row-normalized anymore (see (24)). To understand the difference in the information structure between the cases when α is unknow (which requires the same condition as in the Bayesian potential approach) and β is unknown (which requires a weaker condition), we

[14] Indeed, the (Bayesian) game considered in this paper is a (Bayesian) potential game with potential function

$$\Pi(\mathbf{x}, \alpha, \beta) = -\mathbf{x}^{\mathrm{T}}(\mathbf{I} - \beta\mathbf{G})\,\mathbf{x} + 2\alpha\mathbf{x}^{\mathrm{T}}\mathbf{1}$$

The potential function $\Pi(\mathbf{x}, \alpha, \beta)$ is strictly concave in $\mathbf{x}$ if $\mathbf{I} - \beta\mathbf{G}$ is positive semidefinite. Because $\mathbf{G}$ is symmetric, the eigenvalue decomposition reveals that $\mathbf{I} - \beta\mathbf{G}$ is positive semidefinite if and only if $1 - \beta\lambda_i(\mathbf{G})$ for all i. A sufficient condition is thus $\beta < 1/\lambda_{\max}(\mathbf{G})$, which is the condition given in Theorem 1.

compare the matrices $\mathbf{\Gamma}$ and $\widetilde{\mathbf{\Gamma}}$ for the model in Section 5.2 (see Section A.2.2 in Appendix A.2 for the case when β is unknown). We can see how the matrices Γ and $\widetilde{\Gamma}$ differ by comparing (16) and (17) with (27) and (28). When α is unknown, the $\gamma_{\tau t}$s only depend on p, the precision of the signal, and T, the number of signals or types, whereas, when β is unknown, the $\widetilde{\gamma}_{\tau t}$s depend on p and T but also on $\beta_{\max}$, $\widehat{\beta}$, β_τ and β_t (the maximum value of β, the expected value of β, the value of β when the signal is τ and when it is t).

Observe also that the proof of this Theorem 1 is relatively similar to that of Proposition 2 where we diagonalize the two matrices $\mathbf{G}$ and $\mathbf{\Gamma}$ to obtain a nice characterization of the equilibrium conditions. Assumptions 1 and 2 guarantee that the information matrix $\mathbf{\Gamma}$ is well-defined and that its eigenvalues are real (Proposition 3). The characterization obtained in Theorem 1 is such that each equilibrium effort (or action) is a combination of the T different Katz-Bonacich centralities, where the decay factors are the corresponding eigenvalues of the information matrix $\mathbf{\Gamma}$ multiplied by the synergy parameter β, while the weights are the elements of the $\mathbf{A}$ and $\mathbf{A}^{-1}$. This is because the diagonalization of $\mathbf{G}$ leads to the Katz-Bonacich centralities while the diagonalization of $\mathbf{\Gamma}$ leads to a matrix $\mathbf{A}$, with eigenvectors as columns. This implies that the number of the different eigenvalues of $\mathbf{\Gamma}$ determines the number of the different Katz-Bonacich centrality vectors and the discount factor in each of them, while the elements of $\mathbf{A}$ and $\mathbf{A}^{-1}$ characterize the weights of the different Katz-Bonacich vectors in equilibrium strategies.

More generally, in this characterization (19), $\mathbf{\Gamma}$ interacts in a fairly complicated way with $\mathbf{G}$ because different (positive and negative) eigenvalues of $\mathbf{\Gamma}$ are coefficients of the Katz-Bonacich centralities. This means that the Katz-Bonacich centralities can have both positive and negative decay factors, which, in fact, is also considered in the original article of Bonacich (1987). Indeed, Bonacich (1987) discusses the interpretation of his centrality measure when the decay factor alternates between negative and positive values, which means, in his case, that even powers of $\mathbf{G}$ are weighted negatively and odd powers positively. This implies that having many direct ties contributes to centrality (or power), but, if one's connections themselves have many connections, so that there are many paths of length two, centrality is reduced. This can be interpreted as a bargaining network because those one is in contact with have no options or because their other optional trading partners themselves also have many other options. A similar interpretation can be given here, where the lowest eigenvalues can have negative values while the highest ones positive values. For example, consider the case of 2 states and 2 signals of Section 4. Proposition 2 showed that the two eigenvalues of $\mathbf{\Gamma}$ are given by 1 and $\gamma_h + \gamma_l - 1$. For appropriate values of p and q, it is possible that $\gamma_h + \gamma_l < 1$ and therefore the equilibrium actions are characterized by a combination of two

Katz-Bonacich centralities where one puts *positive weights* on all the powers of matrix $\mathbf{G}$ for the first one and *negative weights* on all the powers of matrix $\mathbf{G}$ for the second one (see (11) and (12)).

Finally, observe that, in Theorem 1, we assume that $\mathbf{\Gamma}$ is diagonalizable. The case of nondiagonalizable $\mathbf{\Gamma}$ is *nongeneric* (Meyer, 2001). However, this does not mean that such matrices could not occur in practice. We consider the case of a nondiagonalizable $\mathbf{\Gamma}$ in Section 6 below.

5.3.2 Example

Consider the example of the previous section (Section 5.2) with $M = T$ and where the $T \times T$ information matrix $\mathbf{\Gamma}$ is given by (18). Assume that $p = 0.6$ and $T = 3$. This means that α can take three values α_l, α_w, α_h and that each agent receives a signal, which is either equal to l, w or h. In that case,

$$\mathbf{\Gamma} = \begin{pmatrix} 0.44 & 0.28 & 0.28 \\ 0.28 & 0.44 & 0.28 \\ 0.28 & 0.28 & 0.44 \end{pmatrix} \tag{20}$$

This matrix $\mathbf{\Gamma}$ has two distinct eigenvalues: $\lambda_1(\mathbf{\Gamma}) = 1$ and $\lambda_2(\mathbf{\Gamma}) = 0.16$. We can thus diagonalize $\mathbf{\Gamma}$ as follows:

$$\mathbf{\Gamma} = \mathbf{A} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.16 & 0 \\ 0 & 0 & 0.16 \end{pmatrix} \mathbf{A}^{-1}$$

where

$$\mathbf{A} = \begin{pmatrix} 0.577 & -0.765 & 0.286 \\ 0.577 & 0.630 & 0.520 \\ 0.577 & 0.135 & -0.805 \end{pmatrix}, \mathbf{A}^{-1} = \begin{pmatrix} 0.577 & 0.577 & 0.577 \\ -0.765 & 0.630 & 0.135 \\ 0.286 & 0.520 & -0.805 \end{pmatrix} \tag{21}$$

Assume that $\beta = 0.2$, which means that $\lambda_1(\mathbf{\Gamma})\beta = 0.2$ and $\lambda_2(\mathbf{\Gamma})\beta = 0.032$. Therefore, applying Theorem 1, if each agent i receives the signal $s_i = l$, then her equilibrium effort is equal to:

$$\begin{aligned} \mathbf{x}^*(l) &= \widehat{\alpha}_l \left[0.333\, \mathbf{b}(0.2, \mathbf{G}) + 0.667\, \mathbf{b}(0.032, \mathbf{G})\right] \\ &\quad + \widehat{\alpha}_w \left[0.333\, \mathbf{b}(0.2, \mathbf{G}) - 0.333\, \mathbf{b}(0.032, \mathbf{G})\right] \\ &\quad + \widehat{\alpha}_h \left[0.333\, \mathbf{b}(0.2, \mathbf{G}) - 0.333\, \mathbf{b}(0.032, \mathbf{G})\right] \end{aligned}$$

Similar calculations can be done when each agent i receives the signals $s_i = w$ and $s_i = h$.

As discussed above, in Appendix A.2, we derive the same results for the case when β is unknown. In particular, Theorem 3 gives the conditions for existence and uniqueness of a Bayesian-Nash equilibrium and its characterization.

6 Diagonalizable versus nondiagonalizable information matrix Γ

In Theorem 1, we assumed that the information matrix Γ was diagonalizable, which is generically true. First, let us give some *sufficient condition* on the primitives of the model (i.e. on the joint distribution of the signals) that guarantees that Γ is symmetric and thus diagonalizable. We have the following assumption:

Assumption 3: *The signal s_i of each player i has the discrete uniform distribution on S.*

In Appendix A.1, Section A.1.3, we show in Proposition 4 that, if Assumption 3 holds, then the information matrix $\mathbf{\Gamma}$ is symmetric, and therefore diagonalizable. In some sense, Assumption 3 is relatively similar to what is assumed in the literature with linear-quadratic utility functions and a continuum of signals (and states of the world) where the signals are assumed to follow a Normal distribution (see e.g. Calvó-Armengol and de Martí, 2009, and Bergemann and Morris, 2013).[15] Note that there are other, potentially less restrictive conditions that would ensure the diagonalizability of Γ. For example, it is sufficient to assume that Γ is strictly sign-regular. Then, it can be shown that all its eigenvalues will be real, distinct, and thus simple and the corresponding eigenbasis will consist of real vectors (see Ando, 1987, Theorem 6.2). The advantage of Assumption 3 is that it provides some sufficient conditions in terms of more primitive assumptions on the joint distribution of the signals.

Second, when Γ is nondiagonalizable, we can still characterize our unique Bayesian-Nash equilibrium using the Jordan decomposition and without assuming Assumption 3. We have the following result, whose proof is given in Appendix A.3:

[15] Indeed, the uniform distribution defined on a finite set is the maximum entropy distribution among all discrete distributions supported on this set. Similarly, the Normal distribution $N\left(\mu, \sigma^2\right)$ has maximum entropy among all real-valued distributions with specified mean μ and standard deviation σ.

Theorem 2 *Consider the case when the marginal return of effort is unknown and assume that assumptions 1 and 2 hold. Let* $\lambda_1(\mathbf{\Gamma}) \geq \lambda_2(\mathbf{\Gamma}) \geq ... \geq \lambda_T(\mathbf{\Gamma})$ *be the eigenvalues of the information matrix* $\mathbf{\Gamma}$*. Assume that the Jordan form of* $\mathbf{\Gamma}$ *is made up of* Q *Jordan blocks* $\mathbf{J}(\widehat{\lambda}_q)$*, where* $\widehat{\lambda}_q$ *is the eigenvalue associated with the* q*-th Jordan block of* $\mathbf{\Gamma}$*. Let* $\widehat{\lambda}_1(\mathbf{\Gamma}) \geq \widehat{\lambda}_2(\mathbf{\Gamma}) \geq ... \geq \widehat{\lambda}_Q(\mathbf{\Gamma})$*. Let* d_q *be the dimension of the* q*-th Jordan block of* $\mathbf{\Gamma}$*, and define* $D_q := \sum_{i=1}^{q} d_i$ *and* $\mathbf{u}_{n,h}(\widehat{\lambda}_q) := (\mathbf{I}_n - \widehat{\lambda}_q \beta \mathbf{G})^{-k} \beta^k \mathbf{G}^{\mathbf{k}} \mathbf{1}_n$*. Then, if* $\{\widehat{\alpha}_\tau\}_{\tau=1}^{T} \subset \mathbb{R}_{++}$ *and* $0 < \beta < 1/\lambda_{max}(\mathbf{G})$*, there exists a unique Bayesian-Nash equilibrium. In that case, if the signal received is* $s = \tau$*, then the equilibrium efforts are given by:*

$$\begin{aligned}\mathbf{x}^*(\{s=\tau\}) =& \widehat{\alpha}_1 \sum_{q=1}^{Q} \sum_{\substack{h= \\ D_{q-1}+1}}^{D_{q-1}+d_q} \sum_{\substack{\nu= \\ D_{q-1}+1}}^{h} a_{\tau\nu} a_{h1}^{(-1)} \mathbf{b}_{\mathbf{u}_{h-\nu}}(\widehat{\lambda}_q \beta, \mathbf{G}) + \dots \\ & \dots + \widehat{\alpha}_T \sum_{q=1}^{Q} \sum_{\substack{h= \\ D_{q-1}+1}}^{D_{q-1}+d_j} \sum_{\substack{\nu= \\ D_{q-1}+1}}^{h} a_{\tau\nu} a_{hT}^{(-1)} \mathbf{b}_{\mathbf{u}_{h-\nu}}(\widehat{\lambda}_q \beta, \mathbf{G})\end{aligned}$$

for $\tau = 1, \dots, T$*, or, more compactly*

$$\mathbf{x}^*(\{s=\tau\}) = \sum_{t=1}^{T} \widehat{\alpha}_t \sum_{q=1}^{Q} \sum_{\substack{h= \\ D_{q-1}+1}}^{D_{q-1}+d_q} \sum_{\substack{\nu= \\ D_{q-1}+1}}^{h} a_{\tau\nu} a_{ht}^{(-1)} \mathbf{b}_{\mathbf{u}_{h-\nu}}(\widehat{\lambda}_q \beta, \mathbf{G}) \tag{22}$$

where $\mathbf{b}_{\mathbf{u}_{h-\nu}}(\widehat{\lambda}_q \beta, \mathbf{G})$ *denotes the* $\mathbf{u}_{n,h-v}(\widehat{\lambda}_q)$*-weighted Katz-Bonacich centrality.*

We can see that the structure of the equilibrium characterization (22) is similar to that of Theorem 1, given by (19). It contains, however, additional terms, which are weighted Katz-Bonacich centralities $\mathbf{b}_{\mathbf{u}_{h-\nu}}(\widehat{\lambda}_q \beta, \mathbf{G})$, and is more complicated to calculate. The main advantage of this result is that it does not hinge on the diagonalizability of the information matrix Γ. Observe that the number and the weights of the Katz-Bonacich centralities given in (22) depend on the deficiency of the information matrix Γ. This implies that, when Γ is diagonalizable so that its eigenvalues are either simple or semi-simple, then the equilibrium characterization of efforts given by (22) collapses to (19), which is given by Theorem 1.

7 Conclusion

We analyze a family of tractable network games with incomplete information on relevant payoff parameters. We show under which condition there exists a unique Bayesian-Nash equilibrium. We are also able to explicitly characterize this Bayesian-Nash equilibrium by showing how it depends in a very precise way on both the network geometry (Katz-Bonacich centrality) and the informational structure.

There are many potential extensions and applications of the work described here. First, we have assumed that the network structure is common knowledge and known by everybody. This is clearly a restrictive assumption. For example, in financial networks (Acemoglu et al., 2015; Cohen-Cole et al., 2011; Denbee et al., 2014; Elliott et al., 2014), the balance sheet conditions or the strength of financial connections is heterogeneous across banks, and that information is partially known by each bank. Second, we have developed a model where uncertainty exists only for the common value component (α or β). If we consider, for example, criminal networks (Ballester et al., 2010; Calvó-Armengol and Zenou, 2004; Liu et al., 2013) or R&D networks (Goyal and Moraga-Gonzalez, 2001; König et al., 2014), the individual gain/loss from these activities may be private value (e.g., α_i rather than common α) or the synergy effects may be link-specific or idiosyncratic (e.g., g_{ij} or β_i is stochastic rather than common parameter β).[16] Finally, in our framework, the signal structure is symmetric (i.e. the probability assessment of the other player's signal realization is independent of the players' identities). In reality, some agents may have superior information to other agents because of the accumulation of experiences. Also, in many situations, agents may know a substantial amount about their neighbors but less about the neighbors' neighbors. We believe that our model captures some interesting aspects of networks and is one of the first dealing with uncertainty on private returns and synergy in networks. As stated above, many extensions should be considered to make it more realistic, especially with respect to real-world networks. We leave that to future research.

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[16] In fact, if one considers a model where the α_i are private value and assumes that the α_is are (pairwise) independent so that $\mathbb{E}[x_j(\alpha_j)|\alpha_i] = \mathbb{E}[x_j(\alpha_j)]$, then the equilibrium solution is relatively simple and less interesting. In the present paper, the α is a common component, but each player has a signal about it, which makes the inference/updating nontrivial and the results more interesting.

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A Appendix

A.1 Main assumptions of the model and their implications

A.1.1 Some useful results

Lemma 1 *If the distribution of* $(s_1, \ldots, s_n)^\intercal$ *is permutation invariant, then* $s_1, \ldots, s_n$ *are identically distributed.*

Proof. Assume that the distribution of $(s_1, \ldots, s_n)^\intercal$ is permutation invariant. Let $(i, j) \in \mathcal{I}^2$ with $i \neq j$, and let π be a permutation of $\mathcal{I}$ with $\pi(i) = j$. Let $B_i \subset \mathbb{R}$ be a Borel set, for example, $B_i = \{\tau\}$ for some $\tau \in \mathcal{S}$, and for all $k \in \mathcal{I} \setminus \{i\}$, let $B_k = \mathbb{R}$. We find

$$\mathbb{P}(\{s_i \in B_i\}) = \mathbb{P}\left(\bigcap_{k=1}^{n}\{s_k \in B_k\}\right) = \mathbb{P}\left(\bigcap_{k=1}^{n}\{s_{\pi(k)} \in B_k\}\right) = \mathbb{P}(\{s_j \in B_i\}).$$

The first equality follows from the fact that

$$\{s_i \in B_i\} = \{s_i \in B_i\} \cap \Omega = \{s_i \in B_i\} \cap \bigcap_{k=1, k\neq i}^{n} \{s_k \in B_k\} = \bigcap_{k=1}^{n}\{s_k \in B_k\}.$$

The second equality follows from the assumption that the distribution of $(s_1, \ldots, s_n)^\intercal$ is permutation invariant. The third equality follows from the fact that

$$\begin{aligned}\bigcap_{k=1}^{n}\{s_{\pi(k)} \in B_k\} = \{s_{\pi(i)} \in B_i\} \cap \bigcap_{k=1, k\neq i}^{n} \{s_{\pi(k)} \in B_k\}& \\ = \{s_{\pi(i)} \in B_i\} \cap \Omega = \{s_{\pi(i)} \in B_i\} = \{s_j \in B_i\}.&\end{aligned}$$

We conclude that $s_1, \ldots, s_n$ are identically distributed. ■

Lemma 2 *If the distribution of* $(s_1, \ldots, s_n)^\intercal$ *is permutation invariant, then for all* $(i, j) \in \mathcal{I}^2$ *with* $i \neq j$ *and for all* $(k, l) \in \mathcal{I}^2$ *with* $k \neq l$, $(s_i, s_j)^\intercal$ *and* $(s_k, s_l)^\intercal$ *are identically distributed.*

Proof. Assume that the distribution of $(s_1, \ldots, s_n)^\intercal$ is permutation invariant. Let $(i, j) \in \mathcal{I}^2$ with $i \neq j$, and let $(k, l) \in \mathcal{I}^2$ with $k \neq l$. Let π be a permutation of $\mathcal{I}$ with $\pi(i) = k$ and

$\pi(j) = l$. Let $B_i \subset \mathbb{R}$ and $B_j \subset \mathbb{R}$ be two Borel sets, for example, $B_i = \{\tau\}$ for some $\tau \in \mathcal{S}$ and $B_j = \{t\}$ for some $t \in \mathcal{S}$, and for all $k \in \mathcal{I} \setminus \{i, j\}$, let $B_k = \mathbb{R}$. We find

$$\begin{aligned}
\mathbb{P}\big((s_i, s_j)^\mathsf{T} \in B_i \times B_j\big) &= \mathbb{P}(\{s_i \in B_i\} \cap \{s_j \in B_j\}) \\
&= \mathbb{P}\left(\bigcap_{k=1}^{n} \{s_k \in B_k\}\right) \\
&= \mathbb{P}\left(\bigcap_{k=1}^{n} \{s_{\pi(k)} \in B_k\}\right) \\
&= \mathbb{P}(\{s_{\pi(i)} \in B_i\} \cap \{s_{\pi(j)} \in B_j\}) \\
&= \mathbb{P}(\{s_k \in B_i\} \cap \{s_l \in B_j\}) \\
&= \mathbb{P}\big((s_k, s_l)^\mathsf{T} \in B_i \times B_j\big).
\end{aligned}$$

The third equality follows from the assumption that the distribution of $(s_1, \ldots, s_n)^\mathsf{T}$ is permutation invariant. The other equalities are obvious. We conclude that $(s_i, s_j)^\mathsf{T}$ and $(s_k, s_l)^\mathsf{T}$ are identically distributed. ■

A.1.2 Main results using Assumptions 1 and 2

We introduce the following assumption concerning the distribution of the players' signals.

Assumption 3a: *For all $(i, j) \in \mathcal{I}^2$ with $i \neq j$, for all $(k, l) \in \mathcal{I}^2$ with $k \neq l$, and for all $(t, \tau) \in \mathcal{S}^2$, $\mathbb{P}(\{s_k = \tau\})\mathbb{P}(\{s_j = t\} \cap \{s_i = \tau\}) = \mathbb{P}(\{s_i = \tau\})\mathbb{P}(\{s_l = t\} \cap \{s_k = \tau\})$.*

Suppose Assumption 1 (defined in the text) is satisfied. Then, Assumption 3a states that for all pairs of signal values $(t, \tau) \in \mathcal{S}^2$, the conditional probability $\mathbb{P}(\{s_j = t\} \mid \{s_i = \tau\})$ is (functionally) independent of $(i, j) \in \mathcal{I}^2$ or, equivalently, $\mathbb{P}(\{s_j = t\} \mid \{s_i = \tau\})$ is only a function of (t, τ) but not of (i, j), where $i \neq j$.

The following lemma gives a sufficient condition for Assumption 3a to be satisfied.

Lemma 3 *If the distribution of $(s_1, \ldots, s_n)^\mathsf{T}$ is permutation invariant (Assumption 2), then Assumption 3a is satisfied.*

Proof. Assume that the distribution of $(s_1, \ldots, s_n)^\mathsf{T}$ is permutation invariant. Let $(i, j) \in \mathcal{I}^2$ with $i \neq j$, $(k, l) \in \mathcal{I}^2$ with $k \neq l$, and $(t, \tau) \in \mathcal{S}^2$. We find

$$\mathbb{P}(\{s_k = \tau\})\mathbb{P}(\{s_j = t\} \cap \{s_i = \tau\}) = \mathbb{P}(\{s_i = \tau\})\mathbb{P}(\{s_l = t\} \cap \{s_k = \tau\})$$

because $\mathbb{P}(\{s_k = \tau\}) = \mathbb{P}(\{s_i = \tau\})$ according to Lemma 1 and $\mathbb{P}(\{s_j = t\} \cap \{s_i = \tau\}) = \mathbb{P}(\{s_l = t\} \cap \{s_k = \tau\})$ according to Lemma 2. ■

Remark 1 *Note that $\mathcal{S} = \{1, \dots, L\}$. If $\mathcal{S} \neq \{1, \dots, L\}$, then we could not directly use $\mathcal{S}$ as an index set to define the components of $\mathbf{\Gamma}$. Clearly, $\mathbf{\Gamma}$ can still be defined in a reasonable way if $\mathcal{S} \neq \{1, \dots, L\}$. To see this, suppose $\mathcal{S} \neq \{1, \dots, L\}$. There exists a unique order isomorphism $h \colon \mathcal{S} \to \{1, \dots, L\}$.*[17] *Using h, we can restate Definition 3 as follows: Suppose Assumptions 1 and 3a are satisfied. The players' information matrix, denoted by $\mathbf{\Gamma} = (\gamma_{rs})_{(r,s)\in\{1,\dots,L\}^2}$, is a square matrix of order $L = |\mathcal{S}|$ that is given by*

$$\forall (r,s) \in \{1, \dots, L\}^2 \quad \gamma_{rs} = \mathbb{P}(\{s_j = h^{-1}(s)\} \mid \{s_i = h^{-1}(r)\}) \\ = \frac{\mathbb{P}(\{s_j = h^{-1}(s)\} \cap \{s_i = h^{-1}(r)\})}{\mathbb{P}(\{s_i = h^{-1}(r)\})},$$

where $(i,j) \in \mathcal{I}^2$ with $i \neq j$ is arbitrary.

We conclude this Appendix with a statement about the spectrum of $\mathbf{\Gamma}$.

Proposition 3 *Suppose Assumptions 1 and 2 are satisfied. Then the eigenvalues of $\mathbf{\Gamma}$ are real.*

Proof. Suppose Assumption 1 is satisfied and assume that the distribution of $(s_1, \dots, s_n)^\mathsf{T}$ is permutation invariant. Let $(i,j) \in \mathcal{I}^2$ with $i \neq j$. Let $\mathbf{\Lambda} = (\lambda_{\tau,t})_{(\tau,t)\in\mathcal{S}^2}$ be the diagonal matrix of order L given by

$$\forall \tau \in \mathcal{S} \quad \lambda_{\tau,\tau} = \mathbb{P}(\{s_i = \tau\}).$$

Note that, according to Assumption 1, $\mathbf{\Lambda}$ is positive definite (and therefore non-singular). We write $\mathbf{\Lambda}^{-1} = (\lambda_{\tau,t}^{(-1)})_{(\tau,t)\in\mathcal{S}^2}$. Let $\mathbf{\Sigma} = (\sigma_{\tau,t})_{(\tau,t)\in\mathcal{S}^2}$ be the square matrix of order L given by

$$\forall (\tau,t) \in \mathcal{S}^2 \quad \sigma_{\tau,t} = \mathbb{P}(\{s_j = t\} \cap \{s_i = \tau\}).$$

Note that $\mathbf{\Sigma}$ is symmetric because the distribution of $(s_1, \dots, s_n)^\mathsf{T}$ is permutation invariant. We have $\mathbf{\Lambda}^{-1}\mathbf{\Sigma} = \mathbf{\Gamma}$. Indeed, for all $(\tau,t) \in \mathcal{S}^2$,

$$\sum_{k=1}^{L} \lambda_{\tau,k}^{(-1)} \sigma_{k,t} = \lambda_{\tau,\tau}^{(-1)} \sigma_{\tau,t} \\ = \frac{1}{\mathbb{P}(\{s_i = \tau\})} \mathbb{P}(\{s_j = t\} \cap \{s_i = \tau\}) \\ = \mathbb{P}(\{s_j = t\} \mid \{s_i = \tau\}) \\ = \gamma_{\tau,t}.$$

[17] An order isomorphism is an order-preserving bijection.

Since $\mathbf{\Lambda}$ is symmetric and positive definite, it has a unique square root $\mathbf{\Lambda}^{1/2}$, which is symmetric and positive definite (and therefore non-singular). Let $\mathbf{\Lambda}^{-1/2}$ denote the inverse of $\mathbf{\Lambda}^{1/2}$. We have

$$\mathbf{\Lambda}^{1/2}\mathbf{\Gamma}\mathbf{\Lambda}^{-1/2} = \mathbf{\Lambda}^{1/2}(\mathbf{\Lambda}^{-1}\mathbf{\Sigma})\mathbf{\Lambda}^{-1/2} = \mathbf{\Lambda}^{-1/2}\mathbf{\Sigma}\,\mathbf{\Lambda}^{-1/2},$$

that is, $\mathbf{\Gamma}$ is similar to the symmetric matrix $\mathbf{\Lambda}^{-1/2}\mathbf{\Sigma}\,\mathbf{\Lambda}^{-1/2}$. Note that the spectrum of $\mathbf{\Lambda}^{-1/2}\mathbf{\Sigma}\,\mathbf{\Lambda}^{-1/2}$ is real because it is symmetric. We conclude that the spectrum of $\mathbf{\Gamma}$ is real because similar matrices have the same spectrum. ■

A.1.3 Main result using Assumption 3

Proposition 4 *If the distribution of $(s_1, \ldots, s_n)^\top$ is permutation invariant and s_1 has the discrete uniform distribution on $\mathcal{S}$, then $\mathbf{\Gamma}^\top = \mathbf{\Gamma}$, that is, $\mathbf{\Gamma}$ is symmetric.*

Proof. Assume that the distribution of $(s_1, \ldots, s_n)^\top$ is permutation invariant and s_1 has the discrete uniform distribution on $\mathcal{S}$. It follows that $s_1, \ldots, s_n$ are identically distributed with the discrete uniform distribution on $\mathcal{S}$ (Lemma 1). Let $(t, \tau) \in \mathcal{S}^2$. We need to show that $\gamma_{\tau,t} = \gamma_{t,\tau}$. We find

$$\gamma_{\tau,t} = \frac{\mathbb{P}(\{s_j = t\} \cap \{s_i = \tau\})}{\mathbb{P}(\{s_i = \tau\})} = \frac{\mathbb{P}(\{s_j = \tau\} \cap \{s_i = t\})}{\mathbb{P}(\{s_i = t\})} = \gamma_{t,\tau}.$$

The first and the third equality are according to (14). The second equality follows from the assumption that the distribution of $(s_1, \ldots, s_n)^\top$ is permutation invariant and s_1 has the discrete uniform distribution on $\mathcal{S}$. Indeed, $\mathbb{P}(\{s_j = t\} \cap \{s_i = \tau\}) = \mathbb{P}(\{s_i = t\} \cap \{s_j = \tau\}) = \mathbb{P}(\{s_j = \tau\} \cap \{s_i = t\})$ because the $(s_j, s_i)^\top$ and $(s_i, s_j)^\top$ are identically distributed (Lemma 2) and $\mathbb{P}(\{s_i = \tau\}) = \mathbb{P}(\{s_i = t\})$ because s_i has the discrete uniform distribution of $\mathcal{S}$. ■

A.2 The model with unknown β

Assume that the uncertainty is on the synergy parameter β, which is an unknown common value for all agents. As in the case of unknown α, there are M different states of the world so that β can take M different values: $\beta \in \{\beta_1, \ldots, \beta_M\}$. There are T different values for a signal, so that agents can be of T different types, which we denote by $\mathcal{S} = \{1, \ldots, T\}$.

A.2.1 Equilibrium

When agent i receives the signal $s_i = \tau$, individual i computes the following conditional expected utility:

$$\begin{aligned}
\mathbb{E}\left[u_i|\left\{s_i=\tau\right\}\right] &= \alpha\mathbb{E}\left[x_i|\left\{s_i=\tau\right\}\right]-\frac{1}{2}\mathbb{E}\left[x_i^2|\left\{s_i=\tau\right\}\right]+\sum_{j=1}^n g_{ij}x_i\mathbb{E}\left[\beta x_i x_j|\left\{s_i=\tau\right\}\right] \\
&= \alpha\, x_i\left(\tau\right)-\frac{1}{2}x_i\left(\tau\right)^2+\sum_{j=1}^n g_{ij}x_i\left(\tau\right)\mathbb{E}\left[\beta x_j|\left\{s_i=\tau\right\}\right]
\end{aligned}$$

The first-order conditions are given by:

$$\forall i=1,...,n \quad \frac{\partial\mathbb{E}\left[u_i|\left\{s_i=\tau\right\}\right]}{\partial x_i\left(\tau\right)}=\alpha-x_i^*\left(\tau\right)+\sum_{j=1}^n g_{ij}\mathbb{E}\left[\beta x_j^*|\left\{s_i=\tau\right\}\right]=0$$

When agent i receives the signal $s_i = \tau$, for each possible j, we have that

$$\begin{aligned}
\mathbb{E}\left[\beta x_j|\left\{s_i=\tau\right\}\right] &= \sum_{t=1}^T\sum_{m=1}^M \beta_m\, x_j(t)\mathbb{P}\left(\left\{\beta=\beta_m\right\}\cap\left\{s_j=t\right\}|\left\{s_i=\tau\right\}\right) \\
&= \sum_{t=1}^T\sum_{m=1}^M \mathbb{P}\left(\left\{\beta=\beta_m\right\}\cap\left\{s_j=t\right\}|\left\{s_i=\tau\right\}\right)\beta_m\, x_j\left(t\right) \\
&= \sum_{t=1}^T\left(\sum_{m=1}^M \mathbb{P}\left(\left\{\beta=\beta_m\right\}\cap\left\{s_j=t\right\}|\left\{s_i=\tau\right\}\right)\beta_m\right)x_j\left(t\right) \\
&= \beta_{\max}\sum_{t=1}^T\left(\sum_{m=1}^M \mathbb{P}\left(\left\{\beta=\beta_m\right\}\cap\left\{s_j=t\right\}|\left\{s_i=\tau\right\}\right)\frac{\beta_m}{\beta_{\max}}\right)x_j\left(t\right)
\end{aligned}$$

where $\beta_{\max} := \max\left\{\beta_1,\ldots,\beta_M\right\}$. We define a $T\times T$ matrix $\widetilde{\mathbf{\Gamma}}$ with entries equal to $\left(\widetilde{\gamma}_{\tau t}\right)_{(\tau,t)\in\{1,\ldots,T\}^2}$ where $\widetilde{\gamma}_{\tau t}$ is defined by (individual i receives signal τ while individual j receives signal t):

$$\widetilde{\gamma}_{\tau t}=\sum_{m=1}^M \mathbb{P}\left(\left\{\beta=\beta_m\right\}\cap\left\{s_j=t\right\}|\left\{s_i=\tau\right\}\right)\frac{\beta_m}{\beta_{\max}} \tag{23}$$

Observe that, in the case of incomplete information on β instead of α, for all $m \in \{1, ..., M\}$, $\frac{\beta_m}{\beta_{\max}} \leq 1$. Therefore, $\widetilde{\mathbf{\Gamma}}$ is non-stochastic because

$$\begin{aligned} & \sum_{t=1}^{T} \widetilde{\gamma}_{\tau t} \\ = & \sum_{t=1}^{T} \sum_{m=1}^{M} \mathbb{P}\left(\{\beta = \beta_m\} \cap \{s_j = t\} \mid \{s_i = \tau\}\right) \frac{\beta_m}{\beta_{\max}} \\ < & \sum_{t=1}^{T} \sum_{m=1}^{M} \mathbb{P}\left(\{\beta = \beta_m\} \cap \{s_j = t\} \mid \{s_i = \tau\}\right) = 1 \end{aligned} \tag{24}$$

Altogether, this means that we can write the first order conditions as follows:

$$\alpha - x_i^* (\tau) + \beta_{\max} \sum_{j=1}^{n} g_{ij} \sum_{t=1}^{T} \widetilde{\gamma}_{\tau t} \; x_j (t) = 0$$

Therefore the system of the best-replies is now given by:

$$\begin{pmatrix} \mathbf{x}(1) \\ \vdots \\ \mathbf{x}(T) \end{pmatrix} = \left(\mathbf{I}_{Tn} - \beta_{\max} \underbrace{\widetilde{\mathbf{\Gamma}}}_{\text{information}} \otimes \underbrace{\mathbf{G}}_{\text{network}} \right)^{-1} \begin{pmatrix} \alpha \mathbf{1} \\ \vdots \\ \alpha \mathbf{1} \end{pmatrix}$$

To characterize the equilibrium, we can use the same techniques as for the case when α was unknown. We have the following result.

Theorem 3 *Consider the case when the strength of interactions is unknown. Assume that $\widetilde{\mathbf{\Gamma}}$ is is diagonalizable and that assumptions 1 and 2 hold. Let $\lambda_1 \left(\widetilde{\mathbf{\Gamma}}\right) \geq \cdots \geq \lambda_T \left(\widetilde{\mathbf{\Gamma}}\right)$ be the eigenvalues of the information matrix $\widetilde{\mathbf{\Gamma}}$ and $\lambda_1 (\mathbf{G}) \geq \cdots \geq \lambda_n (\mathbf{G})$ be the eigenvalues of the adjacency matrix $\mathbf{G}$ where $\lambda_{\max} \left(\widetilde{\mathbf{\Gamma}}\right) = \max_t \left\{ \left| \lambda_t \left(\widetilde{\mathbf{\Gamma}}\right) \right| \right\}$ and $\lambda_{\max} (\mathbf{G}) = \max_i \{ |\lambda_i (\mathbf{G})| \}$. Then, there exists a unique Bayesian-Nash equilibrium if $\alpha > 0$, $\beta_{\min} > 0$ and*

$$\beta_{\max} < \frac{1}{\lambda_{\max} (\mathbf{G}) \, \lambda_{\max} \left(\widetilde{\mathbf{\Gamma}}\right)} \tag{25}$$

If the signal received is $s_i = \tau$, the equilibrium efforts are given by:

$$\mathbf{x}^*(\tau) = \alpha \left[\sum_{t=1}^{T} a_{\tau t} a_{t1}^{(-1)} \mathbf{b} \left(\lambda_t \left(\widetilde{\mathbf{\Gamma}}\right) \beta_{\max}, \mathbf{G} \right) ... + \sum_{t=1}^{T} a_{\tau t} a_{tT}^{(-1)} \mathbf{b} \left(\lambda_t \left(\widetilde{\mathbf{\Gamma}}\right) \beta_{\max}, \mathbf{G} \right) \right] \tag{26}$$

for $j = 1, ..., T$.

Proof: See Appendix A.4.

The results are relatively similar to the case when α was unknown. One of the main difference with Theorem 1 is that the condition (25) is weaker since it imposes a larger upper bound on $\beta_{\max}$ compared to $\beta_{\max} < 1/\lambda_{\max}(\mathbf{G})$ because $\widetilde{\mathbf{\Gamma}}$ is not stochastic and its largest eigenvalue $\lambda_{\max}\left(\widetilde{\mathbf{\Gamma}}\right)$ is not 1. We have the following remark that shows, however, that the condition $\beta_{\max} < 1/\lambda_{\max}(\mathbf{G})$ is still a sufficient condition.

Remark 2 *A sufficient condition for existence and uniqueness of a Bayesian-Nash equilibrium when β is unknown is that $\beta_{\max} < 1/\lambda_{\max}(\mathbf{G})$.*

Proof: See Appendix A.4.

Theorem 3 gives a complete characterization of equilibrium efforts as a function of weighted Katz-Bonacich centralities when β is unknown.

A.2.2 Example

Let us now consider the special model of Section 5.2 with $M = T$ and where the $T \times T$ matrix $\mathbf{P}$ is given by (15) and the $T \times T$ information matrix $\mathbf{\Gamma}$ by (18). We want to compute the matrix $\widetilde{\mathbf{\Gamma}}$ and then get a closed-form expression for the Bayesian-Nash equilibrium when β (and not α) is unknown. We have seen that

$$\begin{aligned}
&\mathbb{P}(\{s_j = \tau\} \cap \{s_i = t\} \mid \{\theta = \theta_m\}) \\
&= \begin{cases} \left(\frac{1-p}{T-1}\right)^2 & \text{if } \tau \neq m \text{ and } t \neq m \\ p\left(\frac{1-p}{T-1}\right) & \text{if either } (\tau \neq m \text{ and } t = m) \text{ or } (\tau = m \text{ and } t \neq m) \\ p^2 & \text{if } \tau = t = m \end{cases}
\end{aligned}$$

and that, if $t = \tau$, we obtain:

$$\gamma_{\tau t} = p^2 + \frac{(1-p)^2}{T-1}$$

while, if $t \neq \tau$, we get:

$$\gamma_{\tau t} = \frac{(1-p)(Tp + T - 2)}{(T-1)^2}$$

It is easily verified that, if $t = \tau$, then

$$\begin{aligned}
\widetilde{\gamma}_{\tau t} &= \frac{1}{\beta_{\max}}\left(p^2\beta_\tau + \left(\frac{1-p}{T-1}\right)^2 \sum_{m \neq \tau} \beta_m\right) && (27)\\
&= \frac{1}{\beta_{\max}}\left[\left(p^2 - \left(\frac{1-p}{T-1}\right)^2\right)\beta_\tau + \left(\frac{1-p}{T-1}\right)^2 T\widehat{\beta}\right]
\end{aligned}$$

where $\widehat{\beta}$ is the expected value of β with respect to the prior distribution, i.e. $\widehat{\beta} = \frac{1}{M}\sum_m \beta_m = \frac{1}{T}\sum_m \beta_m$. If, on the contrary, $t \neq \tau$, then

$$\begin{aligned}
\widetilde{\gamma}_{\tau t} &= \sum_m \mathbb{P}(\beta_m, s(t)|s(\tau))\frac{\beta_m}{\beta_{\max}} = \sum_m \mathbb{P}(s(t), s(\tau)|\beta_m)\frac{\beta_m}{\beta_{\max}} \\
&= \frac{1}{\beta_{\max}}\left(p\left(\frac{1-p}{T-1}\right)(\beta_\tau + \beta_t) + \left(\frac{1-p}{T-1}\right)^2 \sum_{\substack{m\neq\tau \\ m\neq t}} \beta_m\right) \qquad (28) \\
&= \frac{1}{\beta_{\max}}\left(\frac{1-p}{T-1}\right)\left[\left(\frac{Tp-1}{T-1}\right)(\beta_\tau + \beta_t) + \left(1 - \frac{Tp-1}{T-1}\right)\widehat{\beta}\right]
\end{aligned}$$

Clearly the matrix $\widetilde{\mathbf{\Gamma}}$ is symmetric and thus diagonalizable.

As in Section 5.3.2, assume that $p = 0.6$ and $T = 3$. This means that β can take three values $\beta_l = 0.2$, $\beta_w = 0.3$, $\beta_h = \beta_{\max} = 0.4$ so that $\widehat{\beta} = 0.3$ and that each agent i receives three signals: l, w or h. In that case,

$$\widetilde{\mathbf{\Gamma}} = \begin{pmatrix} 0.25 & 0.19 & 0.21 \\ 0.19 & 0.33 & 0.23 \\ 0.21 & 0.23 & 0.41 \end{pmatrix}$$

and the three eigenvalues are: $\lambda_1\left(\widetilde{\mathbf{\Gamma}}\right) = 0.76$, $\lambda_2\left(\widetilde{\mathbf{\Gamma}}\right) = 0.138$ and $\lambda_3\left(\widetilde{\mathbf{\Gamma}}\right) = 0.091$. Then:

$$\mathbf{A} = \begin{pmatrix} 0.485 & 0.166 & 0.859 \\ 0.569 & 0.686 & -0.454 \\ 0.664 & -0.709 & -0.238 \end{pmatrix} \quad , \quad \mathbf{A}^{-1} = \begin{pmatrix} 0.485 & 0.569 & 0.664 \\ 0.166 & 0.686 & -0.709 \\ 0.859 & -0.454 & -0.238 \end{pmatrix}$$

We can now use Theorem 3 and state that, if $\lambda_{\max}(\mathbf{G}) < \frac{1}{0.4\times 0.76} \approx 3.289$, then if each individual i receives the signal $s_i = l$, she provides a unique effort given by:

$$\begin{aligned}
\mathbf{x}^*(\{s_i = l\}) &= \alpha \begin{bmatrix} a_{ll}a_{ll}^{(-1)}\mathbf{b}(0.304, \mathbf{G}) + a_{lw}a_{wl}^{(-1)}\mathbf{b}(0.055, \mathbf{G}) + a_{lh}a_{hl}^{(-1)}\mathbf{b}(0.037, \mathbf{G}) \\ +a_{ll}a_{lw}^{(-1)}\mathbf{b}(0.304, \mathbf{G}) + a_{lw}a_{ww}^{(-1)}\mathbf{b}(0.055, \mathbf{G}) + a_{lh}a_{hw}^{(-1)}\mathbf{b}(0.037, \mathbf{G}) \\ +a_{ll}a_{lh}^{(-1)}\mathbf{b}(0.304, \mathbf{G}) + a_{lw}a_{wh}^{(-1)}\mathbf{b}(0.055, \mathbf{G}) + a_{lh}a_{hh}^{(-1)}\mathbf{b}(0.037, \mathbf{G}) \end{bmatrix} \\
&= \alpha \begin{bmatrix} 0.235\ \mathbf{b}(0.304, \mathbf{G}) + 0.028\ \mathbf{b}(0.055, \mathbf{G}) + 0.737\ \mathbf{b}(0.037, \mathbf{G}) \\ +0.276\ \mathbf{b}(0.304, \mathbf{G}) + 0.114\ \mathbf{b}(0.055, \mathbf{G}) - 0.39\ \mathbf{b}(0.037, \mathbf{G}) \\ +0.322\ \mathbf{b}(0.304, \mathbf{G}) - 0.118\ \mathbf{b}(0.055, \mathbf{G}) - 0.204\ \mathbf{b}(0.037, \mathbf{G}) \end{bmatrix}
\end{aligned}$$

Similar calculations can be done when each agent i receives the signals $s_i = w$ and $s_i = h$.

A.3 Jordan decomposition

A.3.1 Proof of Theorem 2

A general approach Recall that in the $n-$player game, the equilibrium efforts of the agents as a function of the signal they receive are given by

$$\begin{pmatrix} \mathbf{x}^*(\{s=1\}) \\ \vdots \\ \mathbf{x}^*(\{s=T\}) \end{pmatrix} = [\mathbf{I}_{Tn} - \beta(\mathbf{\Gamma} \otimes \mathbf{G})]^{-1} \begin{pmatrix} \widehat{\alpha}_1 \mathbf{1}_n \\ \vdots \\ \widehat{\alpha}_T \mathbf{1}_n \end{pmatrix} \tag{29}$$

Let us rewrite the proof of Theorem 1 using the Jordan decomposition of matrix $\mathbf{\Gamma}$ instead of its diagonal eigenvalue decomposition. We have:

$$\mathbf{\Gamma} = \mathbf{A} \ \mathcal{J}_{\mathbf{\Gamma}} \ \mathbf{A}^{-1}$$

where $\mathbf{A}$ is a non-singular $T \times T$ matrix and $\mathcal{J}_{\mathbf{\Gamma}}$ is the Jordan form of matrix $\mathbf{\Gamma}$. Recall that since $\mathbf{G}$ is a real symmetric matrix, it is diagonalizable with

$$\mathbf{G} = \mathbf{C} \ \mathbf{D_G} \ \mathbf{C}^{-1}$$

Let $\lambda_{\max}(\mathbf{G})$ denote the spectral radius of matrix $\mathbf{G}$, that is, $\lambda_{\max}(\mathbf{G}) := \max_{\lambda \in \sigma(\mathbf{G})} |\lambda|$. Then, assuming that $\beta\lambda_{\max}(\mathbf{\Gamma} \otimes \mathbf{G}) = \beta\lambda_{\max}(\mathbf{\Gamma})\lambda_{\max}(\mathbf{G}) < 1$, it follows that

$$\begin{aligned} [\mathbf{I}_{Tn} - \beta(\mathbf{\Gamma} \otimes \mathbf{G})]^{-1} &= \left[\mathbf{I}_{Tn} - \beta(\mathbf{A} \otimes \mathbf{C})(\mathcal{J}_{\mathbf{\Gamma}} \otimes \mathbf{D_G})(\mathbf{A}^{-1} \otimes \mathbf{C}^{-1})\right]^{-1} \\ &= \sum_{k=0}^{+\infty} (\mathbf{A} \otimes \mathbf{C})\beta^k(\mathcal{J}_{\mathbf{\Gamma}} \otimes \mathbf{D_G})^k(\mathbf{A}^{-1} \otimes \mathbf{C}^{-1}) \\ &= (\mathbf{A} \otimes \mathbf{C}) \sum_{k=0}^{+\infty} (\beta^k {\mathcal{J}_{\mathbf{\Gamma}}}^k \otimes {\mathbf{D_G}}^k)(\mathbf{A}^{-1} \otimes \mathbf{C}^{-1}) \end{aligned} \tag{30}$$

The above expression differs from the respective expression in the paper in the term ${\mathcal{J}_{\mathbf{\Gamma}}}^{\mathbf{k}}$. $\mathcal{J}_{\mathbf{\Gamma}}$ is a block diagonal matrix, consisting of Jordan blocks and zero matrices. Thus

$${\mathcal{J}_{\mathbf{\Gamma}}}^{\mathbf{k}} = \begin{bmatrix} \mathbf{J_1^k} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{J_2^k} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \dots & \mathbf{0} & \mathbf{J_T^k} \end{bmatrix}$$

An additional complication stems from calculating the terms $\mathbf{J_q^k}$. For Jordan blocks of diagonalizable matrices, or Jordan blocks of deficient matrices associated with semi-simple eigenvalues, it is easy to calculate these terms since $\mathbf{J_q}^{\mathbf{k}}$ will be a degenerate 1×1 matrix given by:

$$\mathbf{J_q}^{\mathbf{k}} = [\lambda_q]^k = [\lambda_q^k]$$

If, however, $\mathbf{\Gamma}$ is not diagonalizable, its Jordan form will contain at least one non-diagonal Jordan block. In that case, letting $f(x) = x^k$, we have:

$$\mathbf{J_q}^{\mathbf{k}} = \begin{bmatrix} \lambda_i^k & k\lambda_i^{k-1} & \frac{(k-1)k\lambda_i^{k-2}}{2} & \cdots & \frac{(k-(d_q-2))...(k-1)k\lambda_i^{k-(d_q-1)}}{(d_q-1)!} \\ & \lambda_i^k & k\lambda_i^{k-1} & \ddots & \vdots \\ & & \ddots & \ddots & \frac{(k-1)k\lambda_i^{k-2}}{2} \\ & \mathbf{0} & & \lambda_i^k & k\lambda_i^{k-1} \\ & & & & \lambda_i^k \end{bmatrix} \tag{31}$$

or more compactly

$$\mathbf{J_q}^{\mathbf{k}} = \begin{bmatrix} \lambda_i^k & \binom{k}{1}\lambda_i^{k-1} & \binom{k}{2}\lambda_i^{k-2} & \cdots & \binom{k}{d_q-1}\lambda_i^{k-(d_q-1)} \\ & \lambda_i^k & \binom{k}{1}\lambda_i^{k-1} & \ddots & \vdots \\ & & \ddots & \ddots & \binom{k}{2}\lambda_i^{k-2} \\ & \mathbf{0} & & \lambda_i^k & \binom{k}{1}\lambda_i^{k-1} \\ & & & & \lambda_i^k \end{bmatrix}$$

It follows then that $\mathbf{J_q}^{\mathbf{k}}$, and thus $\mathcal{J}_{\mathbf{\Gamma}}{}^{\mathbf{k}}$, will not be diagonal matrices as $\mathbf{D_\Gamma}^{\mathbf{k}}$ is. As a result, the breakdown of the vector of equilibrium efforts $\mathbf{x}^*$ into Katz-Bonacich measures is complicated and to obtain the expression (19), we will first consider the case of a 3×3 information matrix Γ (3 states of the world and 3 signals).

The case of a 3×3 information matrix Γ In order to see how expression (19) changes when the information matrix Γ is non-diagonalizable, we first start with an example. Suppose that there are three possible values for the signal, leading to a 3×3 information matrix Γ. Assume that Γ possesses a simple eigenvalue, λ_1, and a defective double eigenvalue, $\lambda_2 = \lambda_3$. Hence Γ will not be diagonalizable. Yet, as discussed above, a diagonal Jordan decomposition will still exist and given by:

$$\mathcal{J}_{\mathbf{\Gamma}} = \begin{pmatrix} \mathbf{J_1}_{\{\mathbf{1x1}\}} & \mathbf{0}_{\{1x2\}} \\ \mathbf{0}_{\{2x1\}} & \mathbf{J_2}_{\{\mathbf{2x2}\}} \end{pmatrix} = \left(\begin{array}{c|cc} \lambda_1 & 0 & 0 \\ \hline 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_3 \end{array}\right)$$

For notational convenience, it will useful to relabel the eigenvalues of $\mathbf{\Gamma}$ so that $\widehat{\lambda}_q$ denotes the eigenvalue associated with the $q-$th Jordan block, that is $\widehat{\lambda}_1 := \lambda_1 = \lambda_2$ and $\widehat{\lambda}_2 := \lambda_3$. Thus,

$$\mathcal{J}_{\mathbf{\Gamma}} = \left(\begin{array}{c|cc} \widehat{\lambda}_1 & 0 & 0 \\ \hline 0 & \widehat{\lambda}_2 & 1 \\ 0 & 0 & \widehat{\lambda}_3 \end{array}\right)$$

Using (31), we have:

$$\mathcal{J}_{\Gamma}^k = \left(\begin{array}{c|cc} \widehat{\lambda}_1^k & 0 & 0 \\ \hline 0 & \widehat{\lambda}_2^k & k\widehat{\lambda}_2^{k-1} \\ 0 & 0 & \widehat{\lambda}_2^k \end{array}\right)$$

Then, by using standard matrix algebra, it is easy to show that

$$\sum_{k=0}^{+\infty}(\beta^k \mathcal{J}_{\Gamma}^k \otimes \mathbf{D}_{\mathbf{G}}^{\mathbf{k}}) = \begin{pmatrix} \sum_k \beta^k \widehat{\lambda}_1^k \mathbf{D}_{\mathbf{G}}^{\mathbf{k}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \sum_k \beta^k \widehat{\lambda}_2^k \mathbf{D}_{\mathbf{G}}^{\mathbf{k}} & \sum_k k\beta^k \widehat{\lambda}_2^{k-1} \mathbf{D}_{\mathbf{G}}^{\mathbf{k}} \\ \mathbf{0} & \mathbf{0} & \sum_k \beta^k \widehat{\lambda}_2^k \mathbf{D}_{\mathbf{G}}^{\mathbf{k}} \end{pmatrix} \tag{32}$$

Recall that if $\mathbf{\Gamma}$ is diagonalizable, then matrix $\sum_{k=0}^{+\infty}(\beta^k \mathbf{D}_{\mathbf{\Gamma}}^{\mathbf{k}} \otimes \mathbf{D}_{\mathbf{G}}^{\mathbf{k}})$ will be diagonal, as shown in the proof of Theorem 1. Here, in our example, this will not be the case since the term $\sum_k k\beta^k \widehat{\lambda}_2^{k-1} \mathbf{D}_{\mathbf{G}}{}^k$ appears in the $(2,3)$ block of this matrix. This term is a source of potential concern since, apart from complicating the algebra, it is not straightforward to interpret this expression as some measure of Katz-Bonacich centrality. It will, however, turn out to be the case. Indeed, taking into account (32), expression (30) can be written as:

$$[\mathbf{I}_{Tn} - \beta(\mathbf{\Gamma} \otimes \mathbf{G})]^{-1} =$$

$$\begin{pmatrix} a_{11}\mathbf{C} & a_{12}\mathbf{C} & a_{13}\mathbf{C} \\ a_{21}\mathbf{C} & a_{22}\mathbf{C} & a_{23}\mathbf{C} \\ a_{31}\mathbf{C} & a_{32}\mathbf{C} & a_{33}\mathbf{C} \end{pmatrix} \begin{pmatrix} \sum_k \beta^k \widehat{\lambda}_1^k \mathbf{D}_{\mathbf{G}}^{\mathbf{k}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \sum_k \beta^k \widehat{\lambda}_2^k \mathbf{D}_{\mathbf{G}}^{\mathbf{k}} & \sum_k k\beta^k \widehat{\lambda}_2^{k-1} \mathbf{D}_{\mathbf{G}}^{\mathbf{k}} \\ \mathbf{0} & \mathbf{0} & \sum_k \beta^k \widehat{\lambda}_2^k \mathbf{D}_{\mathbf{G}}^{\mathbf{k}} \end{pmatrix}$$

$$\times \begin{pmatrix} a_{11}^{(-1)}\mathbf{C}^{-1} & a_{12}^{(-1)}\mathbf{C}^{-1} & a_{13}^{(-1)}\mathbf{C}^{-1} \\ a_{21}^{(-1)}\mathbf{C}^{-1} & a_{22}^{(-1)}\mathbf{C}^{-1} & a_{23}^{(-1)}\mathbf{C}^{-1} \\ a_{31}^{(-1)}\mathbf{C}^{-1} & a_{32}^{(-1)}\mathbf{C}^{-1} & a_{33}^{(-1)}\mathbf{C}^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}\mathbf{C}\sum_k \beta^k \widehat{\lambda}_1^k \mathbf{D}_{\mathbf{G}}{}^k & a_{12}\mathbf{C}\sum_k \beta^k \widehat{\lambda}_2^k \mathbf{D}_{\mathbf{G}}{}^k & a_{12}\mathbf{C}\sum_k k\beta^k \widehat{\lambda}_2^{k-1} \mathbf{D}_{\mathbf{G}}{}^k + a_{13}\mathbf{C}\sum_k \beta^k \widehat{\lambda}_2^k \mathbf{D}_{\mathbf{G}}{}^k \\ a_{21}\mathbf{C}\sum_k \beta^k \widehat{\lambda}_1^k \mathbf{D}_{\mathbf{G}}{}^k & a_{22}\mathbf{C}\sum_k \beta^k \widehat{\lambda}_2^k \mathbf{D}_{\mathbf{G}}{}^k & a_{22}\mathbf{C}\sum_k k\beta^k \widehat{\lambda}_2^{k-1} \mathbf{D}_{\mathbf{G}}{}^k + a_{23}\mathbf{C}\sum_k \beta^k \widehat{\lambda}_2^k \mathbf{D}_{\mathbf{G}}{}^k \\ a_{31}\mathbf{C}\sum_k \beta^k \widehat{\lambda}_1^k \mathbf{D}_{\mathbf{G}}{}^k & a_{32}\mathbf{C}\sum_k \beta^k \widehat{\lambda}_2^k \mathbf{D}_{\mathbf{G}}{}^k & a_{32}\mathbf{C}\sum_k k\beta^k \widehat{\lambda}_2^{k-1} \mathbf{D}_{\mathbf{G}}{}^k + a_{33}\mathbf{C}\sum_k \beta^k \widehat{\lambda}_2^k \mathbf{D}_{\mathbf{G}}{}^k \end{pmatrix}$$

$$\times \begin{pmatrix} a_{11}^{(-1)}\mathbf{C}^{-1} & a_{12}^{(-1)}\mathbf{C}^{-1} & a_{13}^{(-1)}\mathbf{C}^{-1} \\ a_{21}^{(-1)}\mathbf{C}^{-1} & a_{22}^{(-1)}\mathbf{C}^{-1} & a_{23}^{(-1)}\mathbf{C}^{-1} \\ a_{31}^{(-1)}\mathbf{C}^{-1} & a_{32}^{(-1)}\mathbf{C}^{-1} & a_{33}^{(-1)}\mathbf{C}^{-1} \end{pmatrix}$$

$$= \left(\begin{matrix} a_{11}a_{11}^{(-1)}\mathbf{M_0}\left(\widehat{\lambda}_1\beta,\mathbf{G}\right) + a_{12}a_{21}^{(-1)}\mathbf{M_0}\left(\widehat{\lambda}_2\beta,\mathbf{G}\right) + a_{12}a_{31}^{(-1)}\mathbf{M_1}\left(\widehat{\lambda}_2\beta,\mathbf{G}\right) + a_{13}a_{31}^{(-1)}\mathbf{M_0}\left(\widehat{\lambda}_2\beta,\mathbf{G}\right) \dots \\ a_{21}a_{11}^{(-1)}\mathbf{M_0}\left(\widehat{\lambda}_1\beta,\mathbf{G}\right) + a_{22}a_{21}^{(-1)}\mathbf{M_0}\left(\widehat{\lambda}_2\beta,\mathbf{G}\right) + a_{22}a_{31}^{(-1)}\mathbf{M_1}\left(\widehat{\lambda}_2\beta,\mathbf{G}\right) + a_{23}a_{31}^{(-1)}\mathbf{M_0}\left(\widehat{\lambda}_2\beta,\mathbf{G}\right) \dots \\ a_{31}a_{11}^{(-1)}\mathbf{M_0}\left(\widehat{\lambda}_1\beta,\mathbf{G}\right) + a_{32}a_{21}^{(-1)}\mathbf{M_0}\left(\widehat{\lambda}_2\beta,\mathbf{G}\right) + a_{32}a_{31}^{(-1)}\mathbf{M_1}\left(\widehat{\lambda}_2\beta,\mathbf{G}\right) + a_{33}a_{31}^{(-1)}\mathbf{M_0}\left(\widehat{\lambda}_2\beta,\mathbf{G}\right) \dots \end{matrix}\right.$$

$$\left. \begin{matrix} a_{11}a_{13}^{(-1)}\mathbf{M_0}\left(\widehat{\lambda}_1\beta,\mathbf{G}\right) + a_{12}a_{23}^{(-1)}\mathbf{M_0}\left(\widehat{\lambda}_2\beta,\mathbf{G}\right) + a_{12}a_{33}^{(-1)}\mathbf{M_1}\left(\widehat{\lambda}_2\beta,\mathbf{G}\right) + a_{13}a_{33}^{(-1)}\mathbf{M_0}\left(\widehat{\lambda}_2\beta,\mathbf{G}\right) \\ a_{21}a_{13}^{(-1)}\mathbf{M_0}\left(\widehat{\lambda}_1\beta,\mathbf{G}\right) + a_{22}a_{23}^{(-1)}\mathbf{M_0}\left(\widehat{\lambda}_2\beta,\mathbf{G}\right) + a_{22}a_{33}^{(-1)}\mathbf{M_1}\left(\widehat{\lambda}_2\beta,\mathbf{G}\right) + a_{23}a_{33}^{(-1)}\mathbf{M_0}\left(\widehat{\lambda}_2\beta,\mathbf{G}\right) \\ a_{31}a_{13}^{(-1)}\mathbf{M_0}\left(\widehat{\lambda}_1\beta,\mathbf{G}\right) + a_{32}a_{23}^{(-1)}\mathbf{M_0}\left(\widehat{\lambda}_2\beta,\mathbf{G}\right) + a_{32}a_{33}^{(-1)}\mathbf{M_1}\left(\widehat{\lambda}_2\beta,\mathbf{G}\right) + a_{33}a_{33}^{(-1)}\mathbf{M_0}\left(\widehat{\lambda}_2\beta,\mathbf{G}\right) \end{matrix} \right)$$

$$\begin{pmatrix} \sum_{q=1}^{2}\sum_{h=D_{q-1}+1}^{D_{q-1}+d_q}\sum_{v=D_{q-1}+1}^{h} a_{1v}a_{h1}^{(-1)}\mathbf{M}_{h-v}\left(\widehat{\lambda}_q\beta,\mathbf{G}\right) & \dots & \sum_{q=1}^{2}\sum_{h=D_{q-1}+1}^{D_{q-1}+d_q}\sum_{v=D_{q-1}+1}^{h} a_{1v}a_{h3}^{(-1)}\mathbf{M}_{h-v}\left(\widehat{\lambda}_q\beta,\mathbf{G}\right) \\ \sum_{q=1}^{2}\sum_{h=D_{q-1}+1}^{D_{q-1}+d_q}\sum_{v=D_{q-1}+1}^{h} a_{2v}a_{h1}^{(-1)}\mathbf{M}_{h-v}\left(\widehat{\lambda}_q\beta,\mathbf{G}\right) & \dots & \sum_{q=1}^{2}\sum_{h=D_{q-1}+1}^{D_{q-1}+d_q}\sum_{v=D_{q-1}+1}^{h} a_{2v}a_{h3}^{(-1)}\mathbf{M}_{h-v}\left(\widehat{\lambda}_q\beta,\mathbf{G}\right) \\ \sum_{q=1}^{2}\sum_{h=D_{q-1}+1}^{D_{q-1}+d_q}\sum_{v=D_{q-1}+1}^{h} a_{3v}a_{h1}^{(-1)}\mathbf{M}_{h-v}\left(\widehat{\lambda}_q\beta,\mathbf{G}\right) & \dots & \sum_{q=1}^{2}\sum_{h=D_{q-1}+1}^{D_{q-1}+d_q}\sum_{v=D_{q-1}+1}^{h} a_{3v}a_{h3}^{(-1)}\mathbf{M}_{h-v}\left(\widehat{\lambda}_q\beta,\mathbf{G}\right) \end{pmatrix} \tag{33}$$

where

$$\mathbf{M_0}(\widehat{\lambda}_q\beta,\mathbf{G}) := \sum_{k=0}^{+\infty}\beta^k\widehat{\lambda}_q\mathbf{G}^k = \mathbf{\Lambda}(\widehat{\lambda}_q\beta,\mathbf{G})$$

$$\mathbf{M_1}(\widehat{\lambda}_q\beta,\mathbf{G}) := \sum_{k=0}^{+\infty}k\beta^k\widehat{\lambda}_q^{k-1}\mathbf{G}^k$$

$$D_q := \sum_{i=1}^{q} d_i$$

where d_i denotes the size of the ith Jordan block. It can be observed that

$$\mathbf{M_0}(\widehat{\lambda}_q\beta,\mathbf{G})\mathbf{1}_n = \mathbf{\Lambda}(\widehat{\lambda}_q\beta,\mathbf{G})\mathbf{1}_n = \mathbf{b}(\widehat{\lambda}_q\beta,\mathbf{G}) \tag{34}$$

The interpretation of $\mathbf{M_1}(\widehat{\lambda}_q\beta,G)$ is not as straightforward. Yet it can be shown that $\mathbf{M_1}(\widehat{\lambda}_q\beta,G)$ leads to weighted Katz-Bonacich centrality. Indeed, we have the following result.

Lemma 4 *Let* $u_{n,1}(\widehat{\lambda}_q) := (\mathbf{I}_n - \widehat{\lambda}_q \beta G)^{-1} \beta G \mathbf{1}_n$, *and denote the* $u_{n,1}(\widehat{\lambda}_q)$*-weighted Katz-Bonacich centrality by* $b_{\mathbf{u}_1}$. *Then,*

$$\mathbf{M_1}(\widehat{\lambda}_q \beta, \mathbf{G})\mathbf{1}_n = \mathbf{b}_{\mathbf{u}_1}(\widehat{\lambda}_q \beta, \mathbf{G}) \tag{35}$$

Hence $\mathbf{M_1}(\widehat{\lambda}_q \beta, \mathbf{G})$, *the derivative of the unweighted Katz-Bonacich centrality with respect to* $\widehat{\lambda}_q$, *is the* $u_{n,1}(\widehat{\lambda}_q)$*-weighted Katz-Bonacich centrality.*

Proof: Recall that, by definition,

$$\mathbf{b}(\widehat{\lambda}_q \beta, \mathbf{G}) := \sum_{k=0}^{+\infty} (\widehat{\lambda}_q \beta)^k \mathbf{G}^k \mathbf{1}_n \tag{36}$$

If $\beta \widehat{\lambda}_q < 1/\lambda_{\max}(\mathbf{G})$, then

$$\mathbf{b}(\widehat{\lambda}_q \beta, \mathbf{G}) = (\mathbf{I}_n - \widehat{\lambda}_q \beta \mathbf{G})^{-1} \mathbf{1}_n \tag{37}$$

Hence, using the definition of Katz-Bonacich centrality in (36), we have:

$$\frac{\partial \mathbf{b}(\widehat{\lambda}_q \beta, \mathbf{G})}{\partial \widehat{\lambda}_q} = \frac{\partial}{\partial \widehat{\lambda}_q} \left[\sum_{k=0}^{+\infty} (\widehat{\lambda}_q \beta)^k \mathbf{G}^k \mathbf{1}_n \right] = \sum_{k=0}^{+\infty} k \beta^k \widehat{\lambda}_q^{k-1} \mathbf{G}^k \mathbf{1}_n = \mathbf{M_1}(\widehat{\lambda}_q \beta, \mathbf{G}) \mathbf{1}_n \tag{38}$$

Similarly, by the alternative expression for Katz-Bonacich given in (37), we have:

$$\begin{aligned}
\frac{\partial \mathbf{b}(\widehat{\lambda}_q \beta, \mathbf{G})}{\partial \widehat{\lambda}_q} &= \frac{\partial}{\partial \widehat{\lambda}_q} \left[(\mathbf{I}_n - \widehat{\lambda}_q \beta \mathbf{G})^{-1} \mathbf{1}_n \right] \\
&= -(\mathbf{I}_n - \widehat{\lambda}_q \beta \mathbf{G})^{-1} \frac{\partial (\mathbf{I}_n - \widehat{\lambda}_q \beta \mathbf{G})}{\partial \widehat{\lambda}_q} (\mathbf{I}_n - \widehat{\lambda}_q \beta \mathbf{G})^{-1} \mathbf{1}_n \\
&= -(\mathbf{I}_n - \widehat{\lambda}_q \beta \mathbf{G})^{-1} (-\beta \mathbf{G}) (\mathbf{I}_n - \widehat{\lambda}_q \beta \mathbf{G})_n^{-1} \mathbf{1} \\
&= (\mathbf{I}_n - \widehat{\lambda}_q \beta \mathbf{G})^{-1} \beta \mathbf{G} (\mathbf{I}_n - \widehat{\lambda}_q \beta \mathbf{G})_n^{-1} \mathbf{1} \\
&= (\mathbf{I}_n - \widehat{\lambda}_q \beta \mathbf{G})^{-1} (\mathbf{I}_n - \widehat{\lambda}_q \beta \mathbf{G})^{-1} \beta \mathbf{G} \ \mathbf{1}_n \\
&= (\mathbf{I}_n - \widehat{\lambda}_q \beta \mathbf{G})^{-1} \mathbf{u}_{n,1} \\
&= \mathbf{b}_{\mathbf{u}_1}(\widehat{\lambda}_q \beta, \mathbf{G})
\end{aligned} \tag{39}$$

where $\mathbf{u}_{n,1}(\widehat{\lambda}_q) := (\mathbf{I}_n - \widehat{\lambda}_q \beta \mathbf{G})^{-1} \beta \mathbf{G} \mathbf{1}_n$, and the fifth equality follows from the fact that $(\mathbf{I}_n - \widehat{\lambda}_q \beta \mathbf{G})^{-1}$ and $\mathbf{G}$ commute. Let us show that, indeed, matrices $\mathbf{G}$ and $(\mathbf{I}_n - \widehat{\lambda}_q \beta \mathbf{G})^{-1}$ are commutative by the following lemma.

Lemma 5 *Let* $\mathbf{A}$ *be a nonsingular matrix and let* $\mathbf{B}$ *be a conformable matrix that commutes with* $\mathbf{A}$. *The* $\mathbf{B}$ *also commutes with* $\mathbf{A}^{-h}$, *for* $h \in \mathbb{N}$.

Proof: Let us start by showing that $\mathbf{A}^{-1}$ and $\mathbf{B}$ commute. By using the assumption that $\mathbf{A}$ and $\mathbf{B}$ commute, and pre- and post-multiplying by $\mathbf{A}^{-1}$, we obtain:

$$\begin{aligned}\mathbf{A}\ \mathbf{B} &= \mathbf{B}\ \mathbf{A} \\ \mathbf{A}^{-1}\left(\mathbf{A}\ \mathbf{B}\right)\mathbf{A}^{-1} &= \mathbf{A}^{-1}\left(\mathbf{B}\ \mathbf{A}\right)\mathbf{A}^{-1} \\ \mathbf{B}\ \mathbf{A}^{-1} &= \mathbf{A}^{-1}\ \mathbf{B}\end{aligned} \tag{40}$$

It is now straightforward to show that matrices $\mathbf{A}^{-h}$ and $\mathbf{B}$ are commutative. Indeed

$$\begin{aligned}\mathbf{A}^{-h}\mathbf{B} &= \mathbf{A}^{-h+1}\mathbf{A}^{-1}\mathbf{B} \\ &= \mathbf{A}^{-h+1}\mathbf{B}\ \mathbf{A}^{-1} \\ &= \mathbf{A}^{-h+2}\mathbf{A}^{-1}\mathbf{B}\ \mathbf{A}^{-1} \\ &= \mathbf{A}^{-h+2}\mathbf{B}\ \mathbf{A}^{-2} \\ &= \ldots \\ &= \mathbf{A}^{-1}\mathbf{B}\ \mathbf{A}^{-h+1} \\ &= \mathbf{B}\ \mathbf{A}^{-h}\end{aligned}$$

where we have used (40). ∎

Let us go back to the proof of Lemma 4. We have shown that

$$\frac{\partial \mathbf{b}(\widehat{\lambda}_q\beta,\mathbf{G})}{\partial \widehat{\lambda}_q} = \mathbf{b}_{\mathbf{u}_1}(\widehat{\lambda}_q\beta,\mathbf{G})$$

Therefore, equalities (38) and (39) imply that:

$$\mathbf{M}_{\mathbf{1}}(\widehat{\lambda}_q\beta,\mathbf{G})\mathbf{1}_n = \mathbf{b}_{\mathbf{u}_1}(\widehat{\lambda}_q\beta,\mathbf{G})$$

which is the statement of the lemma. ∎

We have thus showed that $\mathbf{M}_{\mathbf{1}}(\widehat{\lambda}_q\beta,\mathbf{G})\mathbf{1}_n = \mathbf{b}_{\mathbf{u}_1}(\widehat{\lambda}_q\beta,\mathbf{G})$. Substituting (33) into (29), and taking into account (34), the vector of (stacked) equilibrium efforts can be written as:

$$\begin{pmatrix}\mathbf{x}^*(\{s=1\}) \\ \mathbf{x}^*(\{s=2\}) \\ \mathbf{x}^*(\{s=3\})\end{pmatrix} = \begin{pmatrix}\sum_{t=1}^{3}\widehat{\alpha}_t\sum_{q=1}^{2}\sum_{h=D_{q-1}+1}^{D_{q-1}+d_q}\sum_{\nu=D_{q-1}+1}^{h}a_{1\nu}a_{ht}^{(-1)}\mathbf{M}_{\mathbf{h}-\boldsymbol{\nu}}(\widehat{\lambda}_q\beta,\mathbf{G}) \\ \sum_{t=1}^{3}\widehat{\alpha}_t\sum_{q=1}^{2}\sum_{h=D_{q-1}+1}^{D_{q-1}+d_q}\sum_{\nu=D_{q-1}+1}^{h}a_{2\nu}a_{ht}^{(-1)}\mathbf{M}_{\mathbf{h}-\boldsymbol{\nu}}(\widehat{\lambda}_q\beta,\mathbf{G}) \\ \sum_{t=1}^{3}\widehat{\alpha}_t\sum_{q=1}^{2}\sum_{h=D_{q-1}+1}^{D_{q-1}+d_q}\sum_{\nu=D_{q-1}+1}^{h}a_{3\nu}a_{ht}^{(-1)}\mathbf{M}_{\mathbf{h}-\boldsymbol{\nu}}(\widehat{\lambda}_q\beta,\mathbf{G})\end{pmatrix} \tag{41}$$

It can thus be seen that the equilibrium strategies are a linear combination of unweighted and weighted Katz-Bonacich centralities.

Generalization to an arbitrary information matrix It can be shown that above conclusion carries over to the more general case of a $T \times T$ matrix $\mathbf{\Gamma}$ with any number of defective eigenvalues of arbitrary deficiency (Theorem 2). Indeed, assume that the Jordan form of Γ consists of Q Jordan blocks, $J_q\left(\widehat{\lambda}_q\right)$, $q \in \{1, \ldots, Q\}$. Using the same technique as above, it can be seen that the $(\tau, t)-$block of matrix $[\mathbf{I}_{Tn} - \beta(\mathbf{\Gamma} \otimes \mathbf{G})]^{-1}$ can be written as:

$$[\mathbf{I}_{Tn} - \beta(\mathbf{\Gamma} \otimes \mathbf{G})]^{-1}_{(\tau,t)} = \sum_{q=1}^{Q} \sum_{\substack{h= \\ D_{q-1}+1}}^{D_{q-1}+d_q} \sum_{\substack{\nu= \\ D_{q-1}+1}}^{h} a_{\tau\nu} a_{ht}^{(-1)} \mathbf{M}_{h-\nu}(\widehat{\lambda}_q \beta, \mathbf{G}) \tag{42}$$

where

$$\mathbf{M}_{h-\nu}(\widehat{\lambda}_q \beta, \mathbf{G}) := \sum_{k=0}^{+\infty} \binom{k}{h-v} \widehat{\lambda}_q^{k-(h-v)} \beta^k \mathbf{G}^k \tag{43}$$

and let D_Q, d_q and $\widehat{\lambda}_q$ defined as above. The following result is useful in obtaining the desired characterization of the equilibrium efforts.[18]

Lemma 6 *For $h \in N$, the matrix $\mathbf{M}_h(\widehat{\lambda}_q \beta, G)$ can be mapped into a vector of $\mathbf{u}_{n,h}(\widehat{\lambda}_q)$-weighted Katz-Bonacich centralities, with $\mathbf{u}_{n,h}(\widehat{\lambda}_q) := (\mathbf{I}_n - \beta G)^{-h} \beta^h \mathbf{G}^h \mathbf{1}_n$, as follows:*

$$\mathbf{M}_{\mathbf{h}}(\widehat{\lambda}_q \beta, \mathbf{G}) \mathbf{1}_n = \mathbf{b}_{\mathbf{u}_h}(\widehat{\lambda}_q \beta, \mathbf{G}) \tag{44}$$

Hence, the h-th order derivative of the unweighted Katz-Bonacich centrality measure $\mathbf{b}(\widehat{\lambda}_q \beta, \mathbf{G})$ with respect to $\widehat{\lambda}_q$ is still a weighted Katz-Bonacich centrality. More generally, for $m, \nu \in N$, the m-th order derivative of the weighted Katz-Bonacich centrality $\mathbf{b}_{\mathbf{u}_\nu}(\widehat{\lambda}_q \beta, \mathbf{G})$ with respect to $\widehat{\lambda}_q$ is still a weighted Katz-Bonacich centrality, albeit with a different weight.

[18] Observe that in order to keep notation as simple as possible and since there is no risk of confusion, similarly to Lemma 4, we define

$$\mathbf{b}_{\mathbf{u}_{h-\nu}}(\widehat{\lambda}_q \beta, \mathbf{G}) := \mathbf{b}_{\mathbf{u}_{n,h-\nu}(\widehat{\lambda}_q)}(\widehat{\lambda}_q \beta, \mathbf{G})$$

Proof: Using (36), we can calculate the second derivative of the Katz-Bonacich centrality measure as follows:

$$\begin{aligned}
\frac{\partial^2 \mathbf{b}(\widehat{\lambda}_q \beta, \mathbf{G})}{\partial \widehat{\lambda}_q^2} &= \frac{\partial \mathbf{b}_{\mathbf{u}_1}(\widehat{\lambda}_q \beta, \mathbf{G})}{\partial \widehat{\lambda}_q} \\
&= \frac{\partial}{\partial \widehat{\lambda}_q} \left[\sum_{k=0}^{+\infty} k \beta^k \widehat{\lambda}_q^{k-1} \mathbf{G}^k \mathbf{1}_n \right] \\
&= \sum_{k=0}^{+\infty} (k-1) k \widehat{\lambda}_q^{k-2} \beta^k \mathbf{G}^k \mathbf{1}_n \\
&= 2\ \mathbf{M_2}(\widehat{\lambda}_q \beta, \mathbf{G}) \mathbf{1}_n
\end{aligned}$$

Similarly, using (37), we get

$$\begin{aligned}
\frac{\partial^2 \mathbf{b}(\widehat{\lambda}_q \beta, \mathbf{G})}{\partial \widehat{\lambda}_q^2} &= \frac{\partial\ \mathbf{b}_{\mathbf{u}_1}(\widehat{\lambda}_q \beta, \mathbf{G})}{\partial \widehat{\lambda}_q} \\
&= \frac{\partial}{\partial \widehat{\lambda}_q} \left[(\mathbf{I}_n - \widehat{\lambda}_q \beta \mathbf{G})^{-1} (\mathbf{I}_n - \widehat{\lambda}_q \beta \mathbf{G})^{-1} \beta \mathbf{G} \mathbf{1}_n \right] \\
&= \left[\frac{\partial (\mathbf{I}_n - \widehat{\lambda}_q \beta \mathbf{G})^{-1}}{\partial \widehat{\lambda}_q} (\mathbf{I}_n - \widehat{\lambda}_q \beta \mathbf{G})^{-1} + (\mathbf{I}_n - \widehat{\lambda}_q \beta \mathbf{G})^{-1} \frac{\partial (\mathbf{I}_n - \widehat{\lambda}_q \beta \mathbf{G})^{-1}}{\partial \widehat{\lambda}_q} \right] \beta \mathbf{G} \mathbf{1}_n \\
&= \left[(\mathbf{I}_n - \widehat{\lambda}_q \beta \mathbf{G})^{-2} \beta \mathbf{G}\ (\mathbf{I}_n - \widehat{\lambda}_q \beta \mathbf{G})^{-1} + (\mathbf{I}_n - \widehat{\lambda}_q \beta \mathbf{G})^{-1} (\mathbf{I}_n - \widehat{\lambda}_q \beta \mathbf{G})^{-2} \beta \mathbf{G} \right] \beta \mathbf{G} \mathbf{1}_n \\
&= 2 \left[(\mathbf{I}_n - \widehat{\lambda}_q \beta \mathbf{G})^{-1} (\mathbf{I}_n - \widehat{\lambda}_q \beta \mathbf{G})^{-2} \beta \mathbf{G} \right] \beta \mathbf{G} \mathbf{1}_n \\
&= 2 (\mathbf{I}_n - \widehat{\lambda}_q \beta \mathbf{G})^{-1} (\mathbf{I}_n - \widehat{\lambda}_q \beta \mathbf{G})^{-2} \beta^2 \mathbf{G}^2 \mathbf{1}_n \\
&= 2 (\mathbf{I}_n - \widehat{\lambda}_q \beta \mathbf{G})^{-1} \mathbf{u}_2 \\
&= 2\ \mathbf{b}_{\mathbf{u}_2}(\widehat{\lambda}_q \beta, \mathbf{G})
\end{aligned}$$

where $\mathbf{u}_2 := (\mathbf{I}_n - \widehat{\lambda}_q \beta \mathbf{G})^{-2} \beta^2 \mathbf{G}^{\mathbf{2}} \mathbf{1}_n$, and the fifth equality follows from the commutation of $(\mathbf{I}_n - \widehat{\lambda}_q \beta \mathbf{G})^{-h}$ and $\mathbf{G}$ for $h \in \mathbb{N}$ (see Lemma 5).

The general pattern for the h-th order derivative of each expression of the Katz-Bonacich centrality $\mathbf{b}(\widehat{\lambda}_q\beta, \mathbf{G})$ starts now to emerge. Using (36), we obtain:

$$\begin{aligned}
\frac{\partial^h \mathbf{b}(\widehat{\lambda}_q\beta, \mathbf{G})}{\partial \widehat{\lambda}_q^h} &= \sum_{k=0}^{+\infty}\left[\prod_{i=0}^{h-1}(k-i)\right]\widehat{\lambda}_q^{k-h}\beta^k \mathbf{G}^k \mathbf{1}_n \\
&= \sum_{k=0}^{+\infty}(h!)\ \frac{\prod_{i=0}^{h-1}(k-i)}{h!}\widehat{\lambda}_q^{k-h}\beta^k \mathbf{G}_n^k \mathbf{1}_n \\
&= h!\sum_{k=0}^{+\infty}\binom{k}{h}\widehat{\lambda}_q^{k-h}\beta^k \mathbf{G}_n^k \mathbf{1}_n \\
&= (h!)\mathbf{M_h}(\widehat{\lambda}_q\beta, \mathbf{G})\mathbf{1}_n
\end{aligned} \tag{45}$$

where the last equality follows from the definition of $\mathbf{M_h}$ given in (43). Similarly, starting from (37) and taking into account that $\mathbf{G}$ and $(\mathbf{I}_n - \widehat{\lambda}_q\beta\mathbf{G})^{-h}$ commute, it can be shown that

$$\begin{aligned}
\frac{\partial^h \mathbf{b}(\widehat{\lambda}_q\beta, \mathbf{G})}{\partial \widehat{\lambda}_q^h} &= (\mathbf{I}_n - \widehat{\lambda}_q\beta\mathbf{G})^{-h-1}(h!)\ \beta^h\mathbf{G}^h\mathbf{1}_n \\
&= (h!)\ (\mathbf{I}_n - \widehat{\lambda}_q\beta\mathbf{G})^{-1}(\mathbf{I}_n - \widehat{\lambda}_q\beta\mathbf{G})^{-h}\beta^h\mathbf{G}^h\mathbf{1}_n \\
&= (h!)\ (\mathbf{I}_n - \widehat{\lambda}_q\beta\mathbf{G})^{-1}\mathbf{u}_h \\
&= (h!)\ \mathbf{b}_{\mathbf{u}_h}(\widehat{\lambda}_q\beta, \mathbf{G})
\end{aligned} \tag{46}$$

where $\mathbf{u}_h := (\mathbf{I}_n - \widehat{\lambda}_q\beta\mathbf{G})^{-h}\beta^h\mathbf{G}^h\mathbf{1}_n$. Now notice that (45) and (46) imply that

$$\mathbf{M_h}(\widehat{\lambda}_q\beta, \mathbf{G})\mathbf{1}_n = \mathbf{b}_{\mathbf{u}_h}(\widehat{\lambda}_q\beta, \mathbf{G})$$

which is precisely equation (44). ∎

We have shown that $\mathbf{M_h}(\widehat{\lambda}_q\beta, \mathbf{G})\mathbf{1}_n = \mathbf{b}_{\mathbf{u}_h}(\lambda_t\beta, \mathbf{G})$, that is, $\mathbf{M_h}(\widehat{\lambda}_q\beta, \mathbf{G})$ can be mapped into a vector of $\mathbf{u}_h$-weighted Katz-Bonacich centralities, with $\mathbf{u}_h := (\mathbf{I}_n - \widehat{\lambda}_q\beta\mathbf{G})^{-h}\beta^h\mathbf{G}^h\mathbf{1}_n$.

Then, substituting (42) into (29) and applying Lemma 6 yields

$$\begin{pmatrix} \vdots \\ \mathbf{x}^*(\{s=\tau\}) \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots & \vdots & \vdots \\ \cdots & \sum_{q=1}^{Q}\sum_{h=D_{q-1}+1}^{D_{q-1}+d_q}\sum_{\nu=D_{q-1}+1}^{h} a_{\tau\nu}a_{ht}^{(-1)}\mathbf{M}_{h-\nu}(\widehat{\lambda}_q\beta, \mathbf{G}) & \cdots \\ \vdots & \vdots & \vdots \end{pmatrix}\begin{pmatrix} \vdots \\ \widehat{\alpha}_\tau\mathbf{1}_n \\ \vdots \end{pmatrix}$$

$$= \begin{pmatrix} \vdots \\ \sum_{t=1}^{T} \widehat{\alpha}_t \left[\sum_{q=1}^{Q} \sum_{h=D_{q-1}+1}^{D_{q-1}+d_q} \sum_{\nu=D_{q-1}+1}^{h} a_{\tau\nu} a_{ht}^{(-1)} \mathbf{M}_{h-\nu}(\widehat{\lambda}_q \beta, \mathbf{G}) \right] \mathbf{1}_n \\ \vdots \end{pmatrix}$$

$$= \begin{pmatrix} \vdots \\ \sum_{t=1}^{T} \widehat{\alpha}_t \sum_{q=1}^{Q} \sum_{h=D_{q-1}+1}^{D_{q-1}+d_q} \sum_{\nu=D_{q-1}+1}^{h} a_{\tau\nu} a_{ht}^{(-1)} \mathbf{b}_{h-\nu}(\widehat{\lambda}_q \beta, \mathbf{G}) \\ \vdots \end{pmatrix}$$

and thus expression (22) in Theorem 2 is obtained.

Hence a generalized version of Theorem 1 (given by Theorem 2), applicable to any matrix $\mathbf{\Gamma}$, provides an expression for the equilibrium efforts that is a linear combination of *weighted* Katz-Bonacich centralities. In the light of the above discussion, deriving explicitly such expression seems straightforward, albeit quite tedious in terms of algebra and notation, since it must take into account the deficiency of the eigenvalues of $\mathbf{\Gamma}$.

A.4 Lemma 7 and Proofs

Lemma 7 *We have:*

$$\gamma_l = \frac{(1-p)\,q^2 + p\,(1-q)^2}{q\,(1-p) + (1-q)\,p} \text{ and } \gamma_h = \frac{(1-p)\,(1-q)^2 + pq^2}{qp + (1-q)\,(1-p)} \tag{47}$$

$$\widehat{\alpha}_l = \frac{q\,(1-p)}{q\,(1-p) + (1-q)\,p}\alpha_l + \frac{(1-q)\,p}{q\,(1-p) + (1-q)\,p}\alpha_h \tag{48}$$

and

$$\widehat{\alpha}_h = \frac{(1-q)\,(1-p)}{(1-q)\,(1-p) + qp}\alpha_l + \frac{qp}{(1-q)\,(1-p) + qp}\alpha_h \tag{49}$$

Proof of Lemma 7: We have:

$$\begin{aligned} & \mathbb{P}(\{s_i = l\}) \\ = \; & \mathbb{P}(\{s_i = l\} \mid \{\alpha = \alpha_l\})\, \mathbb{P}(\{\alpha = \alpha_l\}) + \mathbb{P}(\{s_i = l\} \mid \{\alpha = \alpha_h\})\, \mathbb{P}(\{\alpha = \alpha_h\}) \\ = \; & q\,(1-p) + (1-q)\,p \end{aligned}$$

and

$$\begin{aligned} & \mathbb{P}(\{s_i = h\}) \\ = \; & \mathbb{P}(\{s_i = h\} \mid \{\alpha = \alpha_l\})\, \mathbb{P}(\{\alpha = \alpha_l\}) + \mathbb{P}(\{s_i = h\} \mid \{\alpha = \alpha_h\})\, \mathbb{P}(\{\alpha = \alpha_h\}) \\ = \; & (1-q)\,(1-p) + qp \end{aligned}$$

We have:

$$\begin{aligned}
& \mathbb{P}\left(\{\alpha=\alpha_l\} \cap \{s_j=l\} \,|\, \{s_i=l\}\right) \\
= \; & \frac{\mathbb{P}\left(\{s_j=l\} \cap \{s_i=l\} \cap \{\alpha=\alpha_l\}\right)}{\mathbb{P}\left(\{s_i=l\}\right)} \\
= \; & \frac{\mathbb{P}\left(\{s_j=l\} \,|\, \{s_i=l\} \cap \{\alpha=\alpha_l\}\right) \mathbb{P}\left(\{s_i=l\} \,|\, \{\alpha=\alpha_l\}\right) \mathbb{P}\left(\{\alpha=\alpha_l\}\right)}{\mathbb{P}\left(\{s_i=l\}\right)} \\
= \; & \frac{\mathbb{P}\left(\{s_j=l\} \,|\, \{\alpha=\alpha_l\}\right) \mathbb{P}\left(\{s_i=l\} \,|\, \{\alpha=\alpha_l\}\right) \mathbb{P}\left(\{\alpha=\alpha_l\}\right)}{\mathbb{P}\left(\{s_i=l\}\right)} \\
= \; & \frac{q^2\left(1-p\right)}{q\left(1-p\right)+\left(1-q\right)p}
\end{aligned}$$

and thus

$$\begin{aligned}
\mathbb{P}\left(\{\alpha=\alpha_l\} \cap \{s_j=h\} \,|\, \{s_i=l\}\right) &= \frac{q\left(1-p\right)\left(1-q\right)}{q\left(1-p\right)+\left(1-q\right)p} \\
\mathbb{P}\left(\{\alpha=\alpha_h\} \cap \{s_j=l\} \,|\, \{s_i=l\}\right) &= \frac{p\left(1-q\right)^2}{q\left(1-p\right)+\left(1-q\right)p} \\
\mathbb{P}\left(\{\alpha=\alpha_h\} \cap \{s_j=h\} \,|\, \{s_i=l\}\right) &= \frac{\left(1-q\right)pq}{q\left(1-p\right)+\left(1-q\right)p}
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathbb{P}\left(\{\alpha=\alpha_l\} \cap \{s_j=l\} \,|\, \{s_i=h\}\right) &= \frac{\left(1-q\right)\left(1-p\right)q}{qp+\left(1-q\right)\left(1-p\right)} \\
\mathbb{P}\left(\{\alpha=\alpha_l\} \cap \{s_j=h\} \,|\, \{s_i=h\}\right) &= \frac{\left(1-p\right)\left(1-q\right)^2}{qp+\left(1-q\right)\left(1-p\right)} \\
\mathbb{P}\left(\{\alpha=\alpha_h\} \cap \{s_j=l\} \,|\, \{s_i=h\}\right) &= \frac{qp\left(1-q\right)}{qp+\left(1-q\right)\left(1-p\right)} \\
\mathbb{P}\left(\{\alpha=\alpha_h\} \cap \{s_j=h\} \,|\, \{s_i=h\}\right) &= \frac{pq^2}{qp+\left(1-q\right)\left(1-p\right)}
\end{aligned}$$

As a result,

$$\begin{aligned}
\gamma_l &= \mathbb{P}\left(\{s_j=l\} \,|\, \{s_i=l\}\right) \\
&= \mathbb{P}\left(\{\alpha=\alpha_l\} \cap \{s_j=l\} \,|\, \{s_i=l\}\right) + \mathbb{P}\left(\{\alpha=\alpha_h\} \cap \{s_j=l\} \,|\, \{s_i=l\}\right) \\
&= \frac{\left(1-p\right)q^2+p\left(1-q\right)^2}{q\left(1-p\right)+\left(1-q\right)p}
\end{aligned}$$

$$1-\gamma_l = \mathbb{P}\left(\{s_j=h\} \,|\, \{s_i=l\}\right) = \frac{q\left(1-q\right)}{q\left(1-p\right)+\left(1-q\right)p}$$

$$
\begin{aligned}
1-\gamma_h &= \mathbb{P}\left(\{s_j = l\} \mid \{s_i = h\}\right) \\
&= \mathbb{P}\left(\{\alpha = \alpha_l\} \cap \{s_j = l\} \mid \{s_i = h\}\right) + \mathbb{P}\left(\{\alpha = \alpha_h\} \cap \{s_j = l\} \mid \{s_i = h\}\right) \\
&= \frac{q\,(1-q)}{qp + (1-q)\,(1-p)}
\end{aligned}
$$

$$
\gamma_h = \mathbb{P}\left(\{s_j = h\} \mid \{s_i = h\}\right) = \frac{(1-p)\,(1-q)^2 + pq^2}{qp + (1-q)\,(1-p)}
$$

Similarly,

$$
\begin{aligned}
\widehat{\alpha}_l &:= \mathbb{E}\left[\alpha \mid \{s_i = l\}\right] = \mathbb{P}\left(\{\alpha = \alpha_l\} \mid \{s_i = l\}\right)\alpha_l + \mathbb{P}\left(\{\alpha = \alpha_h\} \mid \{s_i = l\}\right)\alpha_h \\
&= \frac{q\,(1-p)}{q\,(1-p) + (1-q)\,p}\alpha_l + \frac{(1-q)\,p}{q\,(1-p) + (1-q)\,p}\alpha_h
\end{aligned}
$$

and

$$
\begin{aligned}
\widehat{\alpha}_h &:= \mathbb{E}_i\left[\alpha \mid \{s_i = h\}\right] = \mathbb{P}\left(\{\alpha = \alpha_l\} \mid \{s_i = h\}\right)\alpha_l + \mathbb{P}\left(\{\alpha = \alpha_h\} \mid \{s_i = h\}\right)\alpha_h \\
&= \frac{(1-q)\,(1-p)}{(1-q)\,(1-p) + qp}\alpha_l + \frac{qp}{(1-q)\,(1-p) + qp}\alpha_h
\end{aligned}
$$

This proves the results. ∎

Proof of Proposition 2: The first-order conditions are given by (10), i.e.

$$
\mathbf{x} = \left[\mathbf{I}_{2n} - \beta\,\mathbf{\Gamma} \otimes \mathbf{G}\right]^{-1} \widehat{\boldsymbol{\alpha}}
$$

where $\mathbf{x} = \begin{pmatrix} \underline{\mathbf{x}} \\ \overline{\mathbf{x}} \end{pmatrix}$ and $\widehat{\boldsymbol{\alpha}} = \begin{pmatrix} \widehat{\alpha}_l \mathbf{1}_n \\ \widehat{\alpha}_h \mathbf{1}_n \end{pmatrix}$.

First, let us show that $\mathbf{I}_{2n} - \beta\,\mathbf{\Gamma} \otimes \mathbf{G}$ is non-singular. This is true if $0 < \beta < 1/\lambda_{\max}(\mathbf{G})$ holds true. Indeed, since $\mathbf{\Gamma}$ is a stochastic matrix and thus $\lambda_{\max}(\mathbf{\Gamma}) = 1$, then $\lambda_{\max}(\mathbf{\Gamma} \otimes \mathbf{G}) = \lambda_{\max}(\mathbf{\Gamma})\,\lambda_{\max}(\mathbf{G}) = \lambda_{\max}(\mathbf{G})$. Therefore, if $0 < \beta < 1/\lambda_{\max}(\mathbf{G})$ holds true, $\mathbf{I}_{2n} - \beta\,\mathbf{\Gamma} \otimes \mathbf{G}$ is non-singular. This shows that the system above has a unique solution and thus there exists a unique Nash-Bayesian equilibrium. This solution is interior since we have assumed that $0 < \alpha_l < \alpha_h$, which implies that $\widehat{\alpha}_l > 0$ and $\widehat{\alpha}_h > 0$.

Second, to show that the equilibrium action of each agent i is a linear function of the Katz-Bonacich centrality measures $b_i(\beta, \mathbf{G})$ and $b_i((\gamma_h + \gamma_l - 1)\,\beta, \mathbf{G})$, let us diagonalize the two main matrices $\mathbf{\Gamma}$ and $\mathbf{G}$.

Since $\mathbf{\Gamma}$ is a 2×2 stochastic matrix, it can be diagonalized as follows:

$$
\mathbf{\Gamma} = \mathbf{A}\,\mathbf{D}_{\mathbf{\Gamma}}\,\mathbf{A}^{-1}
$$

where

$$\mathbf{D}_{\mathbf{\Gamma}} = \begin{pmatrix} \lambda_1(\mathbf{\Gamma}) & 0 \\ 0 & \lambda_2(\mathbf{\Gamma}) \end{pmatrix}$$

and where $\lambda_1(\mathbf{\Gamma}) \geq \lambda_2(\mathbf{\Gamma})$ are the eigenvalues of $\mathbf{\Gamma}$. Observe that $\mathbf{\Gamma}$ is given by

$$\mathbf{\Gamma} = \begin{pmatrix} \gamma_l & 1-\gamma_l \\ 1-\gamma_h & \gamma_h \end{pmatrix}$$

It is easily verified that the two eigenvalues are $\lambda_1(\mathbf{\Gamma}) \equiv \lambda_{\max}(\mathbf{\Gamma}) = 1$ and $\lambda_2(\mathbf{\Gamma}) = \gamma_h + \gamma_l - 1$ and that

$$\mathbf{A} = \begin{pmatrix} 1 & -(1-\gamma_l)/(1-\gamma_h) \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{A}^{-1} = \left(\frac{1-\gamma_h}{2-\gamma_l-\gamma_h}\right) \begin{pmatrix} 1 & (1-\gamma_l)/(1-\gamma_h) \\ -1 & 1 \end{pmatrix}$$

Thus

$$\mathbf{D}_{\mathbf{\Gamma}} = \begin{pmatrix} 1 & 0 \\ 0 & \gamma_h + \gamma_l - 1 \end{pmatrix} \tag{50}$$

Moreover, the $n \times n$ network adjacency matrix $\mathbf{G}$ is symmetric and therefore can be diagonalized as follows:

$$\mathbf{G} = \mathbf{C}\mathbf{D}_{\mathbf{G}}\mathbf{C}^{-1}$$

where

$$\mathbf{D}_{\mathbf{G}} = \begin{pmatrix} \lambda_1(\mathbf{G}) & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n(\mathbf{G}) \end{pmatrix} \tag{51}$$

and where $\lambda_{\max}(\mathbf{G}) := \lambda_1(\mathbf{G}) \geq \lambda_2(\mathbf{G}) \geq ... \geq \lambda_n(\mathbf{G})$ are the eigenvalues of $\mathbf{G}$.

In this context, the equilibrium system (10) can be written as:

$$\begin{aligned} \mathbf{x}^* &= \left[\mathbf{I}_{2n} - \beta\,\mathbf{\Gamma} \otimes \mathbf{G}\right]^{-1} \widehat{\boldsymbol{\alpha}} \\ &= \left[\mathbf{I}_{2n} - \beta\left(\mathbf{A}\,\mathbf{D}_{\mathbf{\Gamma}}\,\mathbf{A}^{-1}\right) \otimes \left(\mathbf{C}\mathbf{D}_{\mathbf{G}}\mathbf{C}^{-1}\right)\right]^{-1} \widehat{\boldsymbol{\alpha}} \\ &= \left[\mathbf{I}_{2n} - \beta\left(\mathbf{A} \otimes \mathbf{C}\right)\left(\mathbf{D}_{\mathbf{\Gamma}} \otimes \mathbf{D}_{\boldsymbol{G}}\right)\left(\mathbf{A}^{-1} \otimes \mathbf{C}^{-1}\right)\right]^{-1} \widehat{\boldsymbol{\alpha}} \end{aligned}$$

We have:

$$\begin{aligned} &\left[\mathbf{I}_{2n} - \beta\left(\mathbf{A} \otimes \mathbf{C}\right)\left(\mathbf{D}_{\mathbf{\Gamma}} \otimes \mathbf{D}_{\boldsymbol{G}}\right)\left(\mathbf{A}^{-1} \otimes \mathbf{C}^{-1}\right)\right]^{-1} \\ =\; &\left(\mathbf{A} \otimes \mathbf{C}\right) \sum_{k=1}^{+\infty} \beta^k \left(\mathbf{D}_{\mathbf{\Gamma}} \otimes \mathbf{D}_{\boldsymbol{G}}\right)^k \left(\mathbf{A}^{-1} \otimes \mathbf{C}^{-1}\right) \end{aligned}$$

where we have used the properties of the Kronecker product.

Using (50), we have:

$$\mathbf{D_\Gamma} \otimes \mathbf{D_G} = \begin{pmatrix} \mathbf{D_G} & \mathbf{0} \\ \mathbf{0} & (\gamma_h + \gamma_l - 1)\,\mathbf{D_G} \end{pmatrix}$$

where $\mathbf{D_G}$ is defined by (51). Thus,

$$(\mathbf{D_\Gamma} \otimes \mathbf{D_G})^k = \begin{pmatrix} \mathbf{D_G}^k & \mathbf{0} \\ \mathbf{0} & (\gamma_h + \gamma_l - 1)^k\,\mathbf{D_G}^k \end{pmatrix}$$

Let us have the following notations:

$$\mathbf{\Lambda_1} := \sum_{k=1}^{+\infty} \beta^k \mathbf{D_G}^k \text{ and } \mathbf{\Lambda_2} := \sum_{k=1}^{+\infty} \beta^k (\gamma_h + \gamma_l - 1)^k\,\mathbf{D_G}^k$$

Then,

$$\begin{aligned}
& (\mathbf{A} \otimes \mathbf{C}) \sum_{k=1}^{+\infty} \beta^k (\mathbf{D_\Gamma} \otimes \mathbf{D}_{\boldsymbol{G}})^k \left(\mathbf{A}^{-1} \otimes \mathbf{C}^{-1}\right) \\
= \;& (\mathbf{A} \otimes \mathbf{C}) \begin{pmatrix} \mathbf{\Lambda_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Lambda_2} \end{pmatrix} \left(\mathbf{A}^{-1} \otimes \mathbf{C}^{-1}\right) \\
= \;& \left(\frac{1-\gamma_h}{2-\gamma_l-\gamma_h}\right) \begin{pmatrix} \mathbf{C} & -\left(\frac{1-\gamma_l}{1-\gamma_h}\right)\mathbf{C} \\ \mathbf{C} & \mathbf{C} \end{pmatrix} \begin{pmatrix} \mathbf{\Lambda_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Lambda_2} \end{pmatrix} \begin{pmatrix} \mathbf{C}^{-1} & \left(\frac{1-\gamma_l}{1-\gamma_h}\right)\mathbf{C}^{-1} \\ -\mathbf{C}^{-1} & \mathbf{C}^{-1} \end{pmatrix} \\
= \;& \left(\frac{1-\gamma_h}{2-\gamma_l-\gamma_h}\right) \begin{pmatrix} \mathbf{C} & -\left(\frac{1-\gamma_l}{1-\gamma_h}\right)\mathbf{C} \\ \mathbf{C} & \mathbf{C} \end{pmatrix} \begin{pmatrix} \mathbf{\Lambda_1}\mathbf{C}^{-1} & \left(\frac{1-\gamma_l}{1-\gamma_h}\right)\mathbf{\Lambda_1}\mathbf{C}^{-1} \\ -\mathbf{\Lambda_2}\mathbf{C}^{-1} & \mathbf{\Lambda_2}\mathbf{C}^{-1} \end{pmatrix} \\
= \;& \left(\frac{1-\gamma_h}{2-\gamma_l-\gamma_h}\right) \begin{pmatrix} \mathbf{C}\mathbf{\Lambda_1}\mathbf{C}^{-1} + \left(\frac{1-\gamma_l}{1-\gamma_h}\right)\mathbf{C}\mathbf{\Lambda_2}\mathbf{C}^{-1} & \left(\frac{1-\gamma_l}{1-\gamma_h}\right)\mathbf{C}\mathbf{\Lambda_1}\mathbf{C}^{-1} - \left(\frac{1-\gamma_l}{1-\gamma_h}\right)\mathbf{C}\mathbf{\Lambda_2}\mathbf{C}^{-1} \\ \mathbf{C}\mathbf{\Lambda_1}\mathbf{C}^{-1} - \mathbf{C}\mathbf{\Lambda_2}\mathbf{C}^{-1} & \left(\frac{1-\gamma_l}{1-\gamma_h}\right)\mathbf{C}\mathbf{\Lambda_1}\mathbf{C}^{-1} + \mathbf{C}\mathbf{\Lambda_2}\mathbf{C}^{-1} \end{pmatrix}
\end{aligned}$$

This implies that:

$$\begin{aligned}
\mathbf{x}^* &= \left[\mathbf{I}_{2n} - \beta\,(\mathbf{A} \otimes \mathbf{C})\,(\mathbf{D_\Gamma} \otimes \mathbf{D}_{\boldsymbol{G}})\left(\mathbf{A}^{-1} \otimes \mathbf{C}^{-1}\right)\right]^{-1} \widehat{\boldsymbol{\alpha}} \\
&= (\mathbf{A} \otimes \mathbf{C}) \sum_{k \geq 0} \beta^k (\mathbf{D_\Gamma} \otimes \mathbf{D}_{\boldsymbol{G}})^k (\mathbf{A} \otimes \mathbf{C})^{-1}\,\widehat{\boldsymbol{\alpha}}
\end{aligned}$$

$$= \left(\frac{1-\gamma_h}{2-\gamma_l-\gamma_h}\right) \begin{pmatrix} \mathbf{C}\mathbf{\Lambda_1}\mathbf{C}^{-1} + \left(\frac{1-\gamma_l}{1-\gamma_h}\right)\mathbf{C}\mathbf{\Lambda_2}\mathbf{C}^{-1} & \left(\frac{1-\gamma_l}{1-\gamma_h}\right)\mathbf{C}\mathbf{\Lambda_1}\mathbf{C}^{-1} - \left(\frac{1-\gamma_l}{1-\gamma_h}\right)\mathbf{C}\mathbf{\Lambda_2}\mathbf{C}^{-1} \\ \mathbf{C}\mathbf{\Lambda_1}\mathbf{C}^{-1} - \mathbf{C}\mathbf{\Lambda_2}\mathbf{C}^{-1} & \left(\frac{1-\gamma_l}{1-\gamma_h}\right)\mathbf{C}\mathbf{\Lambda_1}\mathbf{C}^{-1} + \mathbf{C}\mathbf{\Lambda_2}\mathbf{C}^{-1} \end{pmatrix} \begin{pmatrix} \widehat{\alpha}_l \mathbf{1}_n \\ \widehat{\alpha}_h \mathbf{1}_n \end{pmatrix}$$

$$= \left(\frac{1-\gamma_h}{2-\gamma_l-\gamma_h}\right)$$

$$\times \begin{pmatrix} \widehat{\alpha}_l \left[\mathbf{b}(\beta,\mathbf{G}) + \left(\frac{1-\gamma_l}{1-\gamma_h}\right)\mathbf{b}((\gamma_h+\gamma_l-1)\beta,\mathbf{G})\right] + \left(\frac{1-\gamma_l}{1-\gamma_h}\right)\widehat{\alpha}_h\left[\mathbf{b}(\beta,\mathbf{G}) - \mathbf{b}((\gamma_h+\gamma_l-1)\beta,\mathbf{G})\right] \\ \widehat{\alpha}_l\left[\mathbf{b}(\beta,\mathbf{G}) - \mathbf{b}((\gamma_h+\gamma_l-1)\beta,\mathbf{G})\right] + \left(\frac{1-\gamma_l}{1-\gamma_h}\right)\widehat{\alpha}_h\mathbf{b}(\beta,\mathbf{G}) + \widehat{\alpha}_h\mathbf{b}((\gamma_h+\gamma_l-1)\beta,\mathbf{G}) \end{pmatrix}$$

The last equality is obtained by observing that

$$\begin{aligned} \mathbf{b}(\beta,\mathbf{G}) &= \left(\sum_{k=0}^{+\infty}\beta^k\mathbf{G}^k\right)\mathbf{1}_n \\ &= \left(\sum_{k=0}^{+\infty}\beta^k\left(\mathbf{C}\mathbf{D}_{\mathbf{G}}\mathbf{C}^{-1}\right)^k\right)\mathbf{1}_n \\ &= \mathbf{C}\left(\sum_{k=0}^{+\infty}\beta^k\mathbf{D}_{\mathbf{G}}^k\right)\mathbf{C}^{-1}\mathbf{1}_n \\ &= \mathbf{C}\mathbf{\Lambda_1}\mathbf{C}^{-1}\mathbf{1}_n \end{aligned}$$

and

$$\begin{aligned} \mathbf{b}((\gamma_h+\gamma_l-1)\beta,\mathbf{G}) &= \left(\sum_{k=0}^{+\infty}((\gamma_h+\gamma_l-1)\beta)^k\mathbf{G}^k\right)\mathbf{1}_n \\ &= \left(\sum_{k=0}^{+\infty}((\gamma_h+\gamma_l-1)\beta)^k\left(\mathbf{C}\mathbf{D}_{\mathbf{G}}\mathbf{C}^{-1}\right)^k\right)\mathbf{1}_n \\ &= \mathbf{C}\left(\sum_{k=0}^{+\infty}((\gamma_h+\gamma_l-1)\beta)^k\mathbf{D}_{\mathbf{G}}^k\right)\mathbf{C}^{-\mathbf{1}}\mathbf{1}_n \\ &= \mathbf{C}\mathbf{\Lambda_2}\mathbf{C}^{-1}\mathbf{1}_n \end{aligned}$$

By rearranging the terms, we obtain the equilibrium values given in the Proposition. ∎

Proof of Theorem 1: The proof is relatively similar to that of Proposition 2. Indeed, $\mathbf{\Gamma}$ can be diagonalized as:

$$\mathbf{\Gamma} = \mathbf{A}\mathbf{D}_{\mathbf{\Gamma}}\mathbf{A}^{-1}, \text{ where } \mathbf{D}_{\mathbf{\Gamma}} = \begin{pmatrix} \lambda_1(\mathbf{\Gamma}) & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_T(\mathbf{\Gamma}) \end{pmatrix}$$

In this formulation, $\mathbf{A}$ is a $T\times T$ matrix where each ith column is formed by the eigenvector corresponding to the ith eigenvalue. Let us have the following notations:

$$\mathbf{A}=\begin{pmatrix} a_{11} & \cdots & a_{1T} \\ \vdots & \ddots & \vdots \\ a_{T1} & \cdots & a_{TT}\end{pmatrix} \text{ and } \mathbf{A}^{-1}=\begin{pmatrix} a_{11}^{(-1)} & \cdots & a_{1T}^{(-1)} \\ \vdots & \ddots & \vdots \\ a_{T1}^{(-1)} & \cdots & a_{TT}^{(-1)}\end{pmatrix}$$

where $a_{ij}^{(-1)}$ is the (i,j) cell of the matrix $\mathbf{A}^{-1}$.

The network adjacency matrix $\mathbf{G}$ is symmetric and, therefore, it can also be diagonalized as:

$$\mathbf{G}=\mathbf{C}\mathbf{D}_{\mathbf{G}}\mathbf{C}^{-1}\text{, where } \mathbf{D}_{\mathbf{G}}=\begin{pmatrix} \lambda_1(\mathbf{G}) & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n(\mathbf{G})\end{pmatrix}.$$

We can make use of the Kronecker product to rewrite the equilibrium system of the Bayesian game as:

$$\begin{pmatrix}\mathbf{x}(1)\\ \vdots \\ \mathbf{x}(T)\end{pmatrix}=\left(\mathbf{I}_{Tn}-\beta\mathbf{\Gamma}\otimes\mathbf{G}\right)^{-1}\begin{pmatrix}\widehat{\alpha}_1\mathbf{1}_n\\ \vdots \\ \widehat{\alpha}_T\mathbf{1}_n\end{pmatrix}$$

Applying the properties of the Kronecker product, this system becomes

$$\begin{pmatrix}\mathbf{x}(1)\\ \vdots \\ \mathbf{x}(T)\end{pmatrix}=\left[\mathbf{I}_{Tn}-\beta\left(\mathbf{A}\otimes\mathbf{C}\right)\left(\mathbf{D}_{\mathbf{\Gamma}}\otimes\mathbf{D}_{\mathbf{G}}\right)\left(\mathbf{A}^{-1}\otimes\mathbf{C}^{-1}\right)\right]^{-1}\begin{pmatrix}\widehat{\alpha}_1\mathbf{1}_n\\ \vdots \\ \widehat{\alpha}_T\mathbf{1}_n\end{pmatrix}$$

First, let us show that $\mathbf{I}_{Tn}-\beta\mathbf{\Gamma}\otimes\mathbf{G}$ is non-singular. This is true if $\beta\lambda_{\max}(\mathbf{\Gamma}\otimes\mathbf{G})=\beta\lambda_{\max}(\mathbf{\Gamma})\lambda_{\max}(\mathbf{G})<1$. Since $\mathbf{\Gamma}$ is stochastic we have that $\lambda_{\max}(\mathbf{\Gamma})=1$ and this condition boils down to $0<\beta<1/\lambda_{\max}(\mathbf{G})$. This shows that the system above has a unique solution and thus there exists a unique Nash-Bayesian equilibrium. This solution is interior since we have assumed that $\alpha_1>0$, ..., $\alpha_M>0$, which implies that $\widehat{\alpha}_1>0,...,\widehat{\alpha}_T>0$.

Second, let us show that the equilibrium effort of agent i is a linear function of the Katz-Bonacich centrality measures. We have

$$\begin{aligned}&\left[\mathbf{I}_{Tn}-\beta\left(\mathbf{A}\otimes\mathbf{C}\right)\left(\mathbf{D}_{\mathbf{\Gamma}}\otimes\mathbf{D}_{\mathbf{G}}\right)\left(\mathbf{A}^{-1}\otimes\mathbf{C}^{-1}\right)\right]^{-1}\\ =\;&\left(\mathbf{A}\otimes\mathbf{C}\right)\left(\sum_{k=0}^{+\infty}\beta^k\left(\mathbf{D}_{\mathbf{\Gamma}}\otimes\mathbf{D}_{\mathbf{G}}\right)^k\right)\left(\mathbf{A}^{-1}\otimes\mathbf{C}^{-1}\right)\end{aligned}$$

We also have that $(\mathbf{A}\otimes\mathbf{C})$, $(\mathbf{D}_{\mathbf{\Gamma}}\otimes\mathbf{D}_{\mathbf{G}})^k$ and $(\mathbf{A}^{-1}\otimes\mathbf{C}^{-1})$ are each a $Tn\times Tn$ matrix with

$$\mathbf{A}\otimes\mathbf{C}=\begin{pmatrix} a_{11}\mathbf{C} & \cdots & a_{1T}\mathbf{C}\\ \vdots & \ddots & \vdots\\ a_{T1}\mathbf{C} & \cdots & a_{TT}\mathbf{C}\end{pmatrix}$$

$$\sum_{k\geq 0}\beta^k\left(\mathbf{D}_{\mathbf{\Gamma}}\otimes\mathbf{D}_{\mathbf{G}}\right)^k=\begin{pmatrix} \sum_{k\geq 0}\beta^k\lambda_1^k\left(\mathbf{\Gamma}\right)\mathbf{D}_{\mathbf{G}}^k & 0 & \cdots & 0\\ 0 & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & 0\\ 0 & \cdots & 0 & \sum_{k\geq 0}\beta^k\lambda_T^k\left(\mathbf{\Gamma}\right)\mathbf{D}_{\mathbf{G}}^k\end{pmatrix}$$

and

$$\mathbf{A}^{-1}\otimes\mathbf{C}^{-1}=\begin{pmatrix} a_{11}^{(-1)}\mathbf{C}^{-1} & \cdots & a_{1T}^{(-1)}\mathbf{C}^{-1}\\ \vdots & \ddots & \vdots\\ a_{T1}^{(-1)}\mathbf{C}^{-1} & \cdots & a_{TT}^{(-1)}\mathbf{C}^{-1}\end{pmatrix}$$

It is easily verified that

$$\sum_{k\geq 0}\left(\mathbf{A}\otimes\mathbf{C}\right)\beta^k\left(\mathbf{D}_{\mathbf{\Gamma}}\otimes\mathbf{D}_{\mathbf{G}}\right)^k\left(\mathbf{A}^{-1}\otimes\mathbf{C}^{-1}\right)$$

$$=\begin{pmatrix} \sum_{t=1}^T a_{1t}a_{t1}^{(-1)}\mathbf{\Lambda}\left(\lambda_t\left(\mathbf{\Gamma}\right)\beta,\mathbf{G}\right) & \cdots & \cdots & \cdots & \sum_{t=1}^T a_{1t}a_{tT}^{(-1)}\mathbf{\Lambda}\left(\lambda_t\left(\mathbf{\Gamma}\right)\beta,\mathbf{G}\right)\\ \vdots & \cdots & \ddots & \cdots & \vdots\\ \vdots & \cdots & \sum_{t=1}^T a_{it}a_{tj}^{(-1)}\mathbf{\Lambda}\left(\lambda_t\left(\mathbf{\Gamma}\right)\beta,\mathbf{G}\right) & \cdots & \vdots\\ \vdots & \cdots & \cdots & \ddots & \vdots\\ \sum_{t=1}^T a_{Tt}a_{t1}^{(-1)}\mathbf{\Lambda}\left(\lambda_t\left(\mathbf{\Gamma}\right)\beta,\mathbf{G}\right) & \cdots & \cdots & \cdots & \sum_{t=1}^T a_{Tt}a_{tT}^{(-1)}\mathbf{\Lambda}\left(\lambda_t\left(\mathbf{\Gamma}\right)\beta,\mathbf{G}\right)\end{pmatrix}$$

where

$$\mathbf{\Lambda}\left(\lambda_t\left(\mathbf{\Gamma}\right)\beta,\mathbf{G}\right)=\mathbf{C}\sum_{k\geq 0}\beta^k\lambda_t^k\left(\mathbf{\Gamma}\right)\mathbf{D}_{\mathbf{G}}^k\mathbf{C}^{-1}$$

is an $n\times n$ matrix. Observe that

$$\mathbf{\Lambda}\left(\lambda_t\left(\mathbf{\Gamma}\right)\beta,\mathbf{G}\right)\mathbf{1}_n=\mathbf{b}\left(\lambda_t\left(\mathbf{\Gamma}\right)\beta,\mathbf{G}\right)$$

is a $n\times 1$ vector of Katz-Bonacich centrality measures. For $t=1,...,T$, define:

$$\widehat{\alpha}_t=\mathbb{E}_i\left[\alpha|\left\{s_i=t\right\}\right]=\sum_{m=1}^M\mathbb{P}\left(\left\{\alpha=\alpha_m\right\}|\left\{s_i=t\right\}\right)\alpha_m$$

Then, the result of the product

$$\left[\mathbf{I}_{Tn} - \beta\left(\mathbf{A}\otimes\mathbf{C}\right)\left(\mathbf{D}_{\mathbf{\Gamma}}\otimes\mathbf{D}_{\mathbf{G}}\right)\left(\mathbf{A}^{-1}\otimes\mathbf{C}^{-1}\right)\right]^{-1}\begin{pmatrix}\widehat{\alpha}_1\mathbf{1}_n\\ \vdots\\ \widehat{\alpha}_T\mathbf{1}_n\end{pmatrix}$$

depends on products of the form

$$\begin{aligned}\widehat{\alpha}_t\left(\sum_{t=1}^{T} a_{it}a_{tj}^{(-1)}\mathbf{C}\left(\lambda_t\left(\mathbf{\Gamma}\right)\beta\right)^k\mathbf{D}_{\mathbf{G}}^k\mathbf{C}^{-1}\right)\mathbf{1}_{Tn} &= \widehat{\alpha}_t\sum_{t=1}^{T} a_{it}a_{tj}^{(-1)}\left[\mathbf{C}\left(\lambda_t\left(\mathbf{\Gamma}\right)\beta\right)^k\mathbf{D}_{\mathbf{G}}^k\mathbf{C}^{-1}\mathbf{1}_{Tn}\right]\\ &= \widehat{\alpha}_t\sum_{t=1}^{t} a_{it}a_{tj}^{(-1)}\mathbf{b}\left(\lambda_t\left(\mathbf{\Gamma}\right)\beta,\mathbf{G}\right),\end{aligned}$$

which is a $Tn\times 1$ vector with entries equal to the combinations of the Katz-Bonacich centrality measures $b_i\left(\lambda_t\left(\mathbf{\Gamma}\right)\beta,\mathbf{G}\right)$, $t=1,\ldots,T$. In other words, the equilibrium effort of each agent is:

$$\begin{pmatrix}\mathbf{x}^*(1)\\ \cdots\\ \mathbf{x}^*(\tau)\\ \cdots\\ \mathbf{x}^*(T)\end{pmatrix} = \begin{pmatrix}\widehat{\alpha}_1\sum_{t=1}^{T} a_{1t}a_{t1}^{(-1)}\mathbf{b}\left(\lambda_t\left(\mathbf{\Gamma}\right)\beta,\mathbf{G}\right)+\ldots+\widehat{\alpha}_T\sum_{t=1}^{T} a_{1t}a_{tT}^{(-1)}\mathbf{b}\left(\lambda_t\left(\mathbf{\Gamma}\right)\beta,\mathbf{G}\right)\\ \vdots\\ \widehat{\alpha}_1\sum_{t=1}^{T} a_{\tau t}a_{t1}^{(-1)}\mathbf{b}\left(\lambda_t\left(\mathbf{\Gamma}\right)\beta,\mathbf{G}\right)+\ldots+\widehat{\alpha}_T\sum_{t=1}^{T} a_{\tau t}a_{tT}^{(-1)}\mathbf{b}\left(\lambda_t\left(\mathbf{\Gamma}\right)\beta,\mathbf{G}\right)\\ \vdots\\ \widehat{\alpha}_1\sum_{t=1}^{T} a_{Tt}a_{t1}^{(-1)}\mathbf{b}\left(\lambda_t\left(\mathbf{\Gamma}\right)\beta,\mathbf{G}\right)+\ldots+\widehat{\alpha}_T\sum_{t=1}^{T} a_{Tt}a_{tT}^{(-1)}\mathbf{b}\left(\lambda_t\left(\mathbf{\Gamma}\right)\beta,\mathbf{G}\right)\end{pmatrix}$$

This means, in particular, that if individual i receives a signal $s_i(\tau)$, then her effort is given by:

$$x_i^*(\tau) = \widehat{\alpha}_1\sum_{t=1}^{T} a_{\tau t}a_{t1}^{(-1)}b_i\left(\lambda_t\left(\mathbf{\Gamma}\right)\beta,\mathbf{G}\right)+\ldots+\widehat{\alpha}_T\sum_{t=1}^{T} a_{\tau t}a_{tT}^{(-1)}b_i\left(\lambda_t\left(\mathbf{\Gamma}\right)\beta,\mathbf{G}\right) \tag{52}$$

Hence, we obtain the final expressions of equilibrium efforts as linear functions of own Katz-Bonacich centrality measures. ▮

Proof of Theorem 3:

First, let us prove that this system has a unique solution and therefore there exists a unique Bayesian Nash equilibrium of this game. The system has one and only one well defined and non-negative solution if and only if $\beta_{\max} < 1/\lambda_{\max}\left(\widetilde{\mathbf{\Gamma}} \otimes \mathbf{G}\right)$. Since $\lambda_{\max}\left(\widetilde{\mathbf{\Gamma}} \otimes \mathbf{G}\right) = \lambda_{\max}\left(\widetilde{\mathbf{\Gamma}}\right)\lambda_{\max}(\mathbf{G})$, the result follows.

Let us now characterize the equilibrium. Assumptions 1 and 4 guarantee that $\widetilde{\mathbf{\Gamma}}$ is well-defined, symmetric and thus diagonalizable. As a result, the proof is analogous to the case with incomplete information on α (see proof of Theorem 1). ▮

Proof of Remark 2: Define a new matrix $\bar{\mathbf{\Gamma}} = (\bar{\gamma}_{t\tau})_{t,\tau}$ as follows:

$$\bar{\gamma}_{t\tau} = \sum_{m=1}^{M} \mathbb{P}(\{\beta = \beta_m\} \cap \{s_i = \tau\} \mid \{s_j = t\})$$

By definition we have that $\tilde{\gamma}_{t\tau} < \bar{\gamma}_{t\tau}$ for all t, τ and $\sum_{\tau=1}^{T} \bar{\gamma}_{t\tau} = 1$ for all t. This means that $\mathbf{0} \leq \tilde{\mathbf{\Gamma}} \leq \bar{\mathbf{\Gamma}}$ (elementwise inequalities). This implies that $\lambda_{\max}\left(\tilde{\mathbf{\Gamma}}\right) \leq \lambda_{\max}\left(\bar{\mathbf{\Gamma}}\right) = 1$. Thus, a sufficient condition for $\mathbf{I}_{Tn} - \beta_{\max}\bar{\mathbf{\Gamma}} \otimes \mathbf{G}$ to be non-singular is $\beta_{\max} < 1/\lambda_{\max}(\mathbf{G})$, and the result follows. ▮