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Econometrics II
Problem Set 4
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Part II — Theory

Problem 2

You have proven in class that the OLS estimator $\hat{\beta}$ is a consistent estimator of β in the following simple regression:

$$y = \alpha + \beta x + u$$

Given such an estimator, define an estimator of α by $\hat{\alpha}$ = $\bar{y} - \hat{\beta}\bar{x}$. Show that this estimator is consistent.
We first take the average of the regression equation:

$$\bar{y} = \alpha + \beta \bar{x} + \bar{u}$$

as α and β are constants, we can remove the bars from them.
Then we substitute $\bar{y}$ inton the estimator:

$$\begin{aligned}\hat{\alpha} &= \bar{y} - \hat{\beta}\bar{x} \\ &= \alpha + \beta\bar{x} + \bar{u} - \hat{\beta}\bar{x} \\ &= \alpha + (\beta - \hat{\beta})\bar{x} + \bar{u}\end{aligned}$$

As $\hat{\beta}$ is a consistent estimator of β, we have $\hat{\beta} \xrightarrow{p} \beta$, implying that $(\beta - \hat{\beta}) \xrightarrow{p} 0$.
Also, $\bar{u} \xrightarrow{p} 0$ as the error term is assumed to be mean zero.
Therefore, $\hat{\alpha} \xrightarrow{p} \alpha$ and $\hat{\alpha}$ is a consistent estimator of α.

Problem 3

Let's consider the following model $y_i = \beta_0 + \beta_1 x_{1i} + ...\beta_k x_{ki} + u_i$, for $i = 1...n$. We use OLS to estimate the model. We assume $\mathbb{E}[x_i u_i = 0]$ and that $\mathbb{E}[x_i x_i^T]$ is non-singular. Prove that $\hat{\sigma^2} = \frac{1}{n-k-1}\sum_{i=1}^{n} \hat{u_i}^2$ is a consistent estimator of $\sigma^2 = Var(u_i)$.
Hint: Start by proving that $\hat{u}_i = (u_i - \bar{u}) - \sum_{j=1}^{k}(\hat{\beta}_j - \beta_j)(x_{ij} - \bar{x}_j)$.
We use OLS to estimate the parameters and define the residuals as

$$\hat{u}_i = y_i - \hat{y}_i,$$

where the fitted values are

$$\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_{1i} + \cdots + \hat{\beta}_k x_{ki}.$$

Using the results from Problem 2, we can express the intercept as,

$$\hat{\beta}_0 = \bar{y} - \sum_{j=1}^{k} \hat{\beta}_j \bar{x}_j,$$

Thus, the fitted value can be rewritten as

$$\hat{y}_i = \bar{y} + \sum_{j=1}^{k} \hat{\beta}_j (x_{ij} - \bar{x}_j),$$

and the residual becomes

$$\hat{u}_i = y_i - \bar{y} - \sum_{j=1}^{k} \hat{\beta}_j (x_{ij} - \bar{x}_j).$$

Then we try to express $y_i - \bar{y}$ in terms of the true model.

$$y_i = \beta_0 + \beta_1 x_{1i} + \cdots + \beta_k x_{ki} + u_i,$$

the sample mean is

$$\bar{y} = \beta_0 + \beta_1 \bar{x}_1 + \cdots + \beta_k \bar{x}_k + \bar{u},$$

Subtracting $\bar{y}$ from y_i, we have

$$y_i - \bar{y} = \beta_1 (x_{1i} - \bar{x}_1) + \cdots + \beta_k (x_{ki} - \bar{x}_k) + (u_i - \bar{u}).$$

Substitute this expression into the residual:

$$\hat{u}_i = [\beta_1 (x_{1i} - \bar{x}_1) + \cdots + \beta_k (x_{ki} - \bar{x}_k) + (u_i - \bar{u})] - \sum_{j=1}^{k} \hat{\beta}_j (x_{ij} - \bar{x}_j).$$

By summing the terms with β

$$\hat{u}_i = (u_i - \bar{u}) + \sum_{j=1}^{k} \beta_j (x_{ij} - \bar{x}_j) - \sum_{j=1}^{k} \hat{\beta}_j (x_{ij} - \bar{x}_j).$$

Then we factor out $x_{ij} - \bar{x}_j$ to have

$$\hat{u}_i = (u_i - \bar{u}) - \sum_{j=1}^{k} (\hat{\beta}_j - \beta_j)(x_{ij} - \bar{x}_j)$$

We now turn to the estimator

$$\hat{\sigma}^2 = \frac{1}{n-k-1} \sum_{i=1}^{n} \hat{u}_i^2$$

we square both sides of the expression for $\hat{u}_i$:

$$\hat{u}_i^2 = (u_i - \bar{u})^2 - 2(u_i - \bar{u}) \sum_{j=1}^{k} (\hat{\beta}_j - \beta_j)(x_{ij} - \bar{x}_j) + \left[\sum_{j=1}^{k} (\hat{\beta}_j - \beta_j)(x_{ij} - \bar{x}_j) \right]^2.$$

Averaging over i gives

$$\frac{1}{n}\sum_{i=1}^{n}\hat{u}_i^2 = \frac{1}{n}\sum_{i=1}^{n}(u_i-\bar{u})^2 - \frac{2}{n}\sum_{i=1}^{n}(u_i-\bar{u})\sum_{j=1}^{k}(\hat{\beta}_j-\beta_j)(x_{ij}-\bar{x}_j) + \frac{1}{n}\sum_{i=1}^{n}\left[\sum_{j=1}^{k}(\hat{\beta}_j-\beta_j)(x_{ij}-\bar{x}_j)\right]^2.$$

1. **First Term:**

   $$\frac{1}{n}\sum_{i=1}^{n}(u_i-\bar{u})^2.$$

   By the Law of Large Numbers, this sample variance converges in probability to σ^2.

2. **Second Term:** Since $\mathbb{E}[x_i u_i = 0]$ and that $\mathbb{E}[x_i x_i^T]$ is non-singular, we have that $\hat{\beta}_j$ is a consistent estimator of β_j and $\hat{\beta}_j \xrightarrow{p} \beta_j$. Thus, we have

   $$\frac{2}{n}\sum_{i=1}^{n}(u_i-\bar{u})\sum_{j=1}^{k}(\hat{\beta}_j-\beta_j)(x_{ij}-\bar{x}_j) \xrightarrow{p} 0.$$

3. **Third Term:** Similarly, we have

   $$\frac{1}{n}\sum_{i=1}^{n}\left[\sum_{j=1}^{k}(\hat{\beta}_j-\beta_j)(x_{ij}-\bar{x}_j)\right]^2 \xrightarrow{p} 0.$$

Therefore, we have

$$\frac{1}{n}\sum_{i=1}^{n}\hat{u}_i^2 = \frac{1}{n}\sum_{i=1}^{n}(u_i-\bar{u})^2,$$

The right hand side is by definition the sample variance of the residuals, which converges in probability to σ^2. Thus, we conclude that

$$\frac{1}{n}\sum_{i=1}^{n}\hat{u}_i^2 \xrightarrow{p} \sigma^2.$$

Since

$$\frac{n}{n-k-1} \to 1 \quad \text{as} \quad n \to \infty,$$

dividing by $n-k-1$ rather than n does not affect the limit. Thus, we conclude that

$$\hat{\sigma}^2 = \frac{1}{n-k-1}\sum_{i=1}^{n}\hat{u}_i^2 \xrightarrow{p} \sigma^2,$$

which means that $\hat{\sigma}^2$ is a consistent estimator of σ^2.