1 回旋平均

By gyro-averaging over the perturbed potential in the drift equation, the gyrokinetic equation is obtained as

$$\frac{d\delta g}{dt}=\frac{iZe}{M}\frac{\partial F}{\partial \epsilon}\left(\omega-n\omega_{*h}'\right)\mathbf{v}_d\cdot\nabla\left\langle U\right\rangle \tag{1}$$

with

$$\mathbf{v}_d\cdot\nabla U=\frac{\epsilon}{\omega_{c0}}\left(2b-\Lambda\right)\hat{\mathbf{b}}\times\vec{\kappa}\cdot\nabla U$$

thus,

$$\frac{d\delta g}{dt}=\frac{iZe}{M}\frac{\partial F}{\partial \epsilon}\left(\omega-n\omega_{*h}'\right)\frac{\epsilon}{\omega_{c0}}G \tag{2}$$

$$G=\left(2b-\Lambda\right)\hat{\mathbf{b}}\times\vec{\kappa}\cdot\nabla\left\langle U\right\rangle \tag{3}$$

Since

$$\begin{aligned}G'&=\left(2b-\Lambda\right)\hat{\mathbf{b}}\times\vec{\kappa}\cdot\nabla U\\&=\sum G_m\left(r,\Lambda,\theta\right)e^{-im\theta}e^{in\phi-i\omega t}\end{aligned} \tag{4}$$

and

$$U=\sum_m U_m e^{-i\omega t-im\theta+in\phi}$$

$$U_m=\sum_l A_m\left(l\right)h_l\left(r\right)$$

$$\vec{B}_0=B_\phi R\nabla\phi+\nabla\phi\times\nabla\psi$$

$$\vec{\kappa}=\kappa_r\nabla r+\kappa_\theta\nabla\theta$$

then

$$\begin{aligned}&\hat{\mathbf{b}}\times\vec{\kappa}\cdot\nabla U\\&=\frac{B_\phi R}{JB}\left(\kappa_r\frac{\partial U}{\partial\theta}-\kappa_\theta\frac{\partial U}{\partial r}\right)+O\left(r^2/R^2\right)\\&=\sum_m\sum_l A_m\left(l\right)\frac{B_\phi R}{JB}\left(-im\kappa_r h_l\left(r\right)-\kappa_\theta\frac{\partial h_l\left(r\right)}{\partial r}\right)e^{-im\theta+in\phi-i\omega t}\end{aligned}$$

finally

$$
\begin{aligned}
G_m(\Lambda, r, \theta) &= \frac{B_\phi R}{JB}(2b-\Lambda)\left(-im\kappa_r U_m - \kappa_\theta \frac{\partial U_m}{\partial r}\right) \\
&= \sum_l A_m(l)\frac{B_\phi R}{JB}(2b-\Lambda)\left(-im\kappa_r h_l(r) - \kappa_\theta \frac{\partial h_l(r)}{\partial r}\right)
\end{aligned} \tag{5}
$$

$$
\begin{aligned}
&\sum_m G_m(\Lambda, r', \theta')\exp(in\phi' - im\theta' - i\omega t') \\
&= \sum_m \sum_l A_m(l)\frac{B_\phi R}{JB}(2b-\Lambda) \\
&\cdot \left(-im\kappa_r h_l(r) - \kappa_\theta \frac{\partial h_l(r)}{\partial r}\right)_{r=r',\theta=\theta'} \exp(in\phi' - im\theta' - i\omega t')
\end{aligned}
$$

with

$$
G_m(\Lambda, r, \theta) = \sum_l A_m(l)\frac{B_\phi R}{JB}(2b-\Lambda)\left(-im\kappa_r h_l(r) - \kappa_\theta \frac{\partial h_l(r)}{\partial r}\right) \clubsuit
$$

$$
\begin{aligned}
G_m(\Lambda, r', \theta')e^{in\tilde{\phi}'} &= \left[(2b-\Lambda)\hat{\mathbf{b}}_0 \times \vec{\kappa}\cdot\nabla\hat{U}\right]e^{in\tilde{\phi}'} \\
&= \sum_l A_m(l)\frac{B_\phi R}{JB}(2b-\Lambda)\left(-im\kappa_r h_l(r') - \kappa_\theta \frac{\partial h_l(r')}{\partial r}\right)e^{-im\theta'}e^{in\tilde{\phi}'} \\
&= \sum_l A_m(l)Y_m^l(\Lambda, r', \theta')e^{in\tilde{\phi}'} \\
&= \sum_{l,p} A_m(l)\hat{Y}_{m,p}^l(\Lambda, \bar{r})\exp(ip\omega_\theta t'(\theta))
\end{aligned} \tag{6}
$$

with

$$
Y_m^l(\Lambda, r', \theta') = (2b-\Lambda)\frac{B_\phi R}{JB}\left(-im\kappa_r h_l(r') - \kappa_\theta \frac{\partial h_l(r')}{\partial r}\right)e^{-im\theta'} \tag{7}
$$

$$
Y_m^l(\Lambda, r', \theta')e^{in\tilde{\phi}'} = \sum_p \hat{Y}_{m,p}^l(\Lambda, \bar{r})\exp(ip\omega_\theta t') \tag{8}
$$

where

$$
\begin{aligned}
\hat{Y}_{m,p}^l(\Lambda, \bar{r}) &= \frac{1}{\tau_\theta}\oint dt' Y_m^l(\Lambda, r', \theta')\exp\left(in\tilde{\phi}'\right)\exp(-ip\omega_\theta t') \\
&= \frac{1}{\tau_\theta}\oint dt'(2b(t')-\Lambda)\frac{B_\phi R}{J(t')B(t')}\left(-im\kappa_r h_l(r(t')) - \kappa_\theta \frac{\partial h_l(r(t'))}{\partial r}\right)e^{-im\theta(t')+in\tilde{\phi}(t')}e^{-ip\omega_\theta t'}
\end{aligned} \tag{9}
$$

类似地，相关 $G_k^*(\Lambda, r, \theta)$ 项的结果，如下：

$$
\begin{aligned}
G_k^*(\Lambda, r, \theta) &= \vec{v}_d \cdot \nabla \hat{V}^* \\
&= \frac{E}{\omega_{c0}}(2b - \Lambda)\hat{\mathbf{b}}_0 \times \vec{\kappa} \cdot \nabla \hat{V}^* \\
&= \frac{E}{\omega_{c0}}(2b - \Lambda)\frac{B_\phi R}{JB}\left(ik\kappa_r h_i(r') - \kappa_\theta \frac{\partial h_i(r')}{\partial r}\right)e^{ik\theta'} \\
&= \frac{E}{\omega_{c0}} Y_k^{i*}(\Lambda, r', \theta')
\end{aligned} \tag{10}
$$

$$
Y_k^{i*}(\Lambda, r', \theta')e^{-in\tilde{\phi}'} = \sum_{p'} \hat{Y}_{k,p'}^{*i}(\Lambda, \bar{r}) \exp(-ip'\omega_\theta t') \tag{11}
$$

where

$$
\begin{aligned}
\hat{Y}_{k,p'}^{*i}(E, \Lambda, \bar{r}) &= \frac{1}{\tau_\theta}\oint dt' Y_k^{*i}(E, \Lambda, r', \theta')\exp\left(-in\tilde{\phi}'\right)\exp(ip'\omega_\theta t') \\
= \frac{1}{\tau_\theta}\oint dt'(2b - \Lambda)\frac{B_\phi R}{JB}&\left(ik\kappa_r h_i(r') - \kappa_\theta \frac{\partial h_i(r')}{\partial r}\right)e^{ik\theta' - in\tilde{\phi}'}e^{ip'\omega_\theta t'}
\end{aligned} \tag{12}
$$

To normalize,

$$
\begin{aligned}
\bar{\hat{Y}}_{k,p'}^{*i}(\Lambda, \bar{r}) &= \frac{1}{\bar{\tau}_\theta}\oint dt' Y_k^{*i}(E, \Lambda, r', \theta')\exp\left(-in\tilde{\phi}'\right)\exp(ip'\omega_\theta t') \\
= \frac{1}{\bar{\tau}_\theta}\oint dt'(2b - \Lambda)\frac{1}{\bar{J}\bar{B}}&\left(ik\bar{\kappa}_r h_i(x')e^{ik\theta'} - \bar{\kappa}_\theta \frac{\partial h_i(x')}{\partial x}e^{ik\theta'}\right)e^{-in\tilde{\phi}'}e^{ip'\omega_\theta t'}
\end{aligned} \tag{13}
$$

在非正交坐标系 (r, θ, φ) 中做回旋平均，方法类似于 GTC，GMEC。首先建立径向网格 $n_r = nog$，极向网格 n_θ 和环向网格 n_φ 以及径向步长为 h，极向步长为 $d_\theta = 2\pi/(n_\theta - 1)$，环向步长为 $d_\varphi = 2\pi/(n_\varphi - 1)$.（但是在本征值程序中，径向用有限元离散是有网格的，极向并没有离散用 Fourier 分解。其中，径向网格大小为 $1.0/nog$，极向在本征值程序中是 fourier 分解没有相应的离散的网格，怎么取？）其次，得到粒子导心运动的位置 $(r_i, \theta_i, \varphi_i)$，进而得到大柱坐标系下的位置 (R_i, Z_i, φ_i). 然后，找到 2 点分别对应于只变径向位置，即在导心位置的基础上增加 h，$(r_1, \theta_1, \varphi_1) = (r_i + h, \theta_i, \varphi_i)$，和只变极向位置，即在导心位置的基础上增加 d_θ，$(r_2, \theta_2, \varphi_2) = (r_i, \theta_i + d_\theta, \varphi_i)$。这 2 点到导心的距离为 $d_{1,2}$：

$$
d_1 = \sqrt{(R_1 - R_i)^2 + (Z_1 - Z_i)^2} \tag{14}
$$

$$R_1 = R(r_i + h, \theta_i) \tag{15}$$

$$Z_1 = Z(r_i + h, \theta_i) \tag{16}$$

$$d_2 = \sqrt{(R_2 - R_i)^2 + (Z_2 - Z_i)^2} \tag{17}$$

$$R_2 = R(r_i, \theta_i + d_\theta) \tag{18}$$

$$Z_2 = Z(r_i, \theta_i + d_\theta) \tag{19}$$

$$R_i = R(r_i, \theta_i) \tag{20}$$

$$Z_i = Z(r_i, \theta_i) \tag{21}$$

最后得到极向平面内 2 个方向的小量:

$$\delta r = (\rho_c/d_1)h \tag{22}$$

$$\delta\theta = (\rho_c/d_2)d_\theta \tag{23}$$

where $\rho_c = \frac{m_i v_\perp}{ZeB}$.

$$\Lambda = \frac{\mu B\left(0\right)}{E} \tag{24}$$

$$\frac{\Lambda}{b} = \frac{v_\perp^2}{v^2}$$

$$E_i = \frac{1}{2}Mv^2$$

$$\rho_c = \frac{v_\perp}{\omega_{c0}}b = \frac{\sqrt{\frac{\Lambda}{b}\frac{2E_i}{M}}}{\omega_{c0}}b$$

$$\rho_h = \frac{\sqrt{\frac{T_h(\text{ekev})}{M}}}{\omega_{c0}}$$

$$\frac{\bar{\rho}_c}{\bar{\rho}_h} = \sqrt{2\Lambda b}\sqrt{\frac{E_i}{\text{ekev}}}$$

with $\bar{\rho}_h = 1.02 \times 10^2\sqrt{\mu}/Z\sqrt{\text{ekev} \times 10^3}/(\text{bkg} \times 10^3)/R_0$. 因此，多点平均的位置为 $(r_i + \delta r_j, \theta_i + \delta\theta_j, \varphi_i)$, 其中

$$\delta r_j = \sin\left(\zeta_j + \frac{\alpha}{2}\right)\frac{\delta r}{\sin\alpha} \tag{25}$$

$$\delta\theta_j = \sin\left(\zeta_j - \frac{\alpha}{2}\right)\frac{\delta\theta}{\sin\alpha} \tag{26}$$

$$\zeta_j = \frac{2\pi j}{N}, j = 1, 2, \cdots, N \tag{27}$$

$$\cos\alpha = \frac{g^{r\theta}}{\sqrt{g^{rr}g^{\theta\theta}}} \tag{28}$$

To normalize and gyro-average,

$$\left\langle \bar{\hat{Y}}^l_{m,p}\left(E,\Lambda,\bar{x}\right)\right\rangle = \frac{1}{\bar{\tau}_\theta}\oint dt'\left\langle \bar{Y}^l_m\left(\Lambda,x',\theta'\right)\right\rangle \exp\left(in\tilde{\phi}'\right)\exp\left(-ip\omega_\theta t'\right)$$
$$= \frac{1}{\bar{\tau}_\theta}\oint dt'\left(2b-\Lambda\right)\frac{1}{\bar{J}\bar{B}}\left(-im\bar{\kappa}_r\left\langle h_l\left(x'\right)e^{-im\theta'}\right\rangle - \bar{\kappa}_\theta\left\langle \frac{\partial h_l\left(r'\right)}{\partial x}e^{-im\theta'}\right\rangle\right)e^{in\tilde{\phi}'}e^{-ip\omega_\theta t'} \tag{29}$$

note that the quantites of equlirium do not make gyro-averaging. Specially,

$$\left\langle h_l\left(x'\right)e^{-im\theta'}\right\rangle = \frac{1}{N}\sum_j h_l\left(x_i+\delta r_j\right)e^{-im\left(\theta_i+\delta\theta_j\right)} \tag{30}$$

$$\left\langle \frac{\partial h_l\left(r'\right)}{\partial x}e^{-im\theta'}\right\rangle = \frac{1}{N}\sum_j \frac{\partial h_l\left(x_i+\delta r_j\right)}{\partial x}e^{-im\left(\theta_i+\delta\theta_j\right)} \tag{31}$$

对于上面的问题，傅老师给出的建议不一定用 GMEC 的方法，想办法对粒子位置进行回旋半径的 Taylor 展开就可以了。

根据文献 Yingfeng Xu, Computer Physics Communications 244 (2019) 40–48, 磁面坐标系下回旋平均表示为

$$\left\langle F\left(r,\theta\right)\right\rangle = \frac{1}{N}\sum_{j=1}^{N} F\left(r'_j,\theta'_j\right) \tag{32}$$

$$r'_j = r(\vec{X}+\rho_j) \approx r + \vec{\rho}_j\cdot\nabla r \tag{33}$$

$$\theta'_j = \theta(\vec{X}+\rho_j) \approx \theta + \vec{\rho}_j\cdot\nabla\theta \tag{34}$$

where $\vec{\rho}_j = \vec{\rho}_j(r,\theta,\xi_j,\Lambda)$ and $\xi_j = \frac{(j-1)2\pi}{N}$.

$$\vec{\rho} = \rho\left(\cos\xi\vec{e}_1 - \sin\xi\vec{e}_2\right) \tag{35}$$

where $\vec{e}_1 = \nabla r$ and $\vec{e}_2 = \vec{b}\times\vec{e}_1$ with $\vec{B} = B_\phi R\nabla\phi + \nabla\phi\times\nabla\psi$, $\rho = \frac{m_i v_\perp}{ZeB}$. Thus,

$$\vec{\rho} = \rho\left(\cos\xi\nabla r - \sin\xi\frac{B_\phi R}{B_0}\nabla\phi\times\nabla r\right) \tag{36}$$

傅老师指出了单位矢量的定义问题，建议优化方案把计算好的 x_j, θ_j 都保存为数组。这样在 Y_p 的计算中不用重复计算这些量了。

$$\vec{\rho} = \rho \left(\cos\xi \vec{e}_1 - \sin\xi \vec{e}_2\right) \tag{37}$$

where $\vec{e}_1 = \frac{\nabla r}{|\nabla r|}$ and $\vec{e}_2 = R\nabla\phi \times \frac{\nabla r}{|\nabla r|}$, $\rho = \frac{m_i v_\perp}{ZeB}$. Thus,

$$\vec{\rho} = \rho \left(\cos\xi \frac{\nabla r}{|\nabla r|} - \sin\xi R\nabla\phi \times \frac{\nabla r}{|\nabla r|}\right) \tag{38}$$

$$r'_j = r(\vec{X} + \rho_j) \approx r + \rho\cos\xi_j \sqrt{g^{rr}} \tag{39}$$

$$\theta'_j = \theta(\vec{X} + \rho_j) \approx \theta + \rho\left(\cos\xi_j \frac{g^{r\theta}}{\sqrt{g^{rr}}} - \sin\xi_j \frac{R}{J\sqrt{g^{rr}}}\right) \tag{40}$$

Furthermore,

$$x'_j \approx x + \frac{\bar{\rho}}{\epsilon}\cos\xi_j \sqrt{\bar{g}^{rr}} \tag{41}$$

$$\begin{aligned} \theta'_j &\approx \theta + \frac{\bar{\rho}}{\epsilon}\left(\cos\xi_j \frac{\bar{g}^{r\theta}}{\sqrt{\bar{g}^{rr}}} - \sin\xi_j \frac{\bar{R}}{\bar{J}\sqrt{\bar{g}^{rr}}}\right) \\ &= \theta + \frac{\bar{\rho}}{\epsilon\sqrt{\bar{g}^{rr}}}\left(\cos\xi_j \bar{g}^{r\theta} - \sin\xi_j \frac{1}{\bar{R}(\bar{J}/\bar{R}^2)}\right) \end{aligned} \tag{42}$$

$$\frac{\bar{\rho}_c}{\bar{\rho}_h} = \sqrt{2\Lambda b}\sqrt{\frac{E_i}{\text{ekev}}}$$