Counting

① a) unusual

Consider $\{u, \_, \_, \_, \_\}$ choose 4 spots to fill with $\{n, s, a, l\}$
Fix 1 u:
$= \binom{4}{4} = 1$

Consider $\{u, u, \_, \_, \_\}$ choose 3 spots to fill with $\{n, s, a, l\}$
Fix 2 u's: $\Rightarrow \{u, u, n, s, l\}$
$\{u, u, n, a, l\}$
$= \binom{4}{3} = 4$

Consider $\{u, u, u, \_, \_\}$ choose 2 spots to fill with $\{n, s, a, l\}$
Fix 3 u's:
$= \binom{4}{2} = 6$

$1 + 4 + 6 = \boxed{11}$ combinations of unique subsets

b) $\{u, n, s, a, l\}$ 1u
$5 \times 4 \times 3 \times 2 \times 1 = 5! = 120$

$\{u, u, n, a, l\}$ 2 u's $\Rightarrow$ double counting

$= \frac{5!}{2!} \times 4 = \frac{120}{2} \times 4 = 240$

possible strings

consider other sets

$\{u_1, u_2, u_3, n, a\}$ 3 u's $\Rightarrow$ triple counting

$= \frac{5!}{3!} \times 6 = \frac{120}{6} \times 6 = 120$

$120 + 240 + 120 = \boxed{480}$ different strings

(2) 5 card hand

2 pairs

5th card is whatever

out of 13 values, choose 2

$\underline{\heartsuit 2}$ $\underline{\diamondsuit 2}$ $\underline{\spadesuit K}$ $\underline{\clubsuit K}$ $\underline{\quad}$

$\underbrace{\quad\quad\quad\quad}_{A,2\ldots}$ $\underbrace{\quad\quad\quad\quad}_{K,3\ldots}$

$$\begin{array}{r} 15 \\ -8 \\ \hline 44 \end{array}$$ → b/c you cant use the rest of 2 or K

same for 2nd pair

$$\binom{13}{2} \times \binom{4}{2} \times \binom{4}{2} \times \binom{44}{1}$$

out of 4 suits, choose 2 for 1st pair

↳ out of 13 vals, want to choose 2 vals (ie. King, Ace)

$$= \frac{13!}{2!\,11!} \cdot \frac{4!}{2!\,2!} \cdot \frac{4!}{2!\,2!} \cdot \frac{44!}{43!\,1!}$$

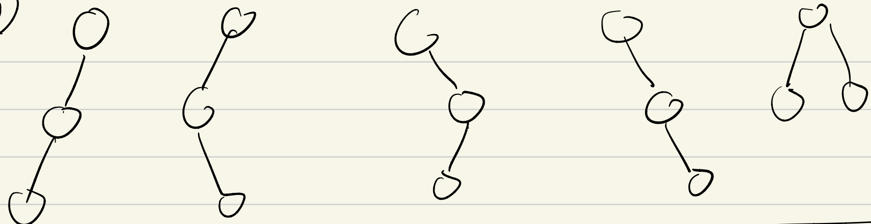
$$= 13! \cdot 3! \cdot 3 \cdot 44 = 4{,}931{,}800{,}473{,}600$$

(3) $\underline{16}$ $\underbrace{\_\ \_\ \_\ \_\ \_\ \_}_{15 \text{ choose } n,\ 6 \text{ boxes}}$

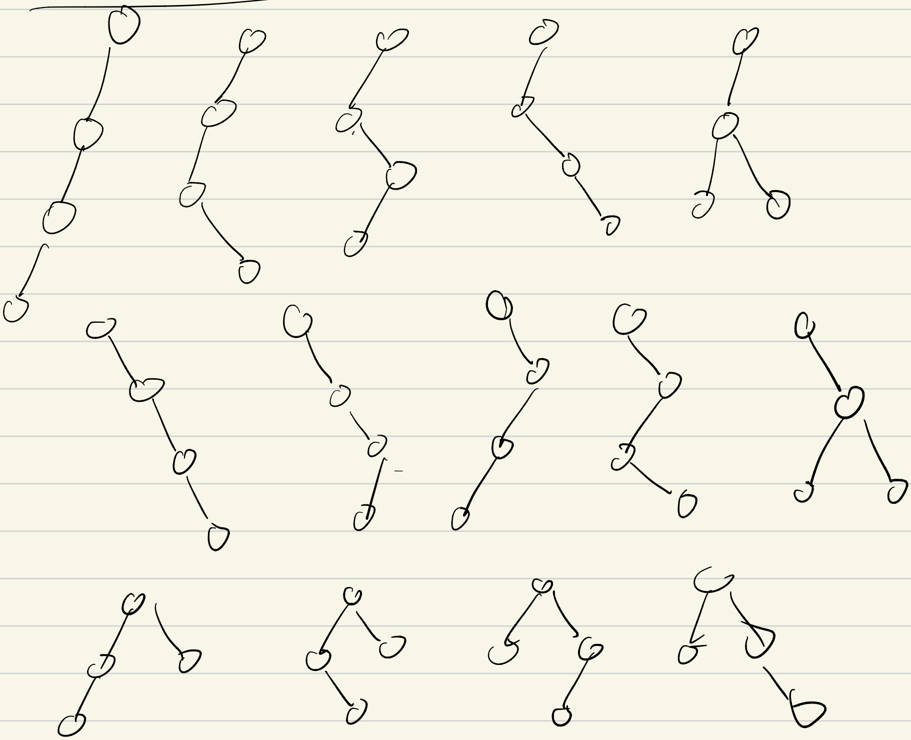
$$\binom{n-1+r}{r} = \binom{15-1+6}{6} = \binom{20}{6} = \frac{20!}{6!\,12!}$$

$$\Rightarrow 16\left(\frac{20!}{6!\,12!}\right) = 112{,}865{,}120$$

4



5 combinations



14 combinations

$5 \cdot 5 \cdot 14 = \boxed{350 \text{ combinations}}$

5 (7,1,1,1), (6,2,1,1) (5,2,2,1), (4,3,2,1)
(4,4,1,1), (5,3,1,1), (4,2,2,2), (3,3,3,1), (3,3,2,2)

(8,1,1), (7,2,1), (6,2,2), (6,3,1), (5,4,1), (5,3,2),
(4,4,2), (4,3,3)

9 + 8 = 17 combinations

Probability

① 15 students, 8 qs

P(no student has to answer twice)

$= \underset{\text{1st q}}{\frac{15}{15}} \cdot \frac{14}{15} \cdot \frac{13}{15} \cdot \frac{12}{15} \cdot \frac{11}{15} \cdot \frac{10}{15} \cdot \frac{9}{15} \cdot \frac{8}{15} = \frac{\frac{15!}{7!}}{15!} = \frac{1}{2}$

② $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$\{1, 3, 5, 7, 8\}$ $\{3, 5, 7, 9\}$ $\{0, 2, 4, 6, 8\}$

$\underset{\text{odd}}{\underline{5}} \times \underset{\text{odd}}{\underline{4}} \times \underline{7} \times \underline{6} \times \underset{\text{even}}{\underline{5}} = 4200$

$\{5, 7, 9, 2, 4, 6, 8\}$

same as A, just 1 # less

③ At least → consider 2 dice showing ≥ 4
3 dice showing ≥ 4

Event A

A = event at least 2 dice ≥ 4 $\{1, 2, 3, 4, 5, 6\}$

B = event all dice show same value

$P(A) \cdot P(B) = P(A \cap B)$ ⇒ if formula holds then is independent

$P(A) = P(\text{just 2 out of 3 dice show} \geq 4) + P(\text{all 3 dice show} \geq 4)$

A1 A2

Binomial distribution - out of n trials did you get k successes?

$= P(k) = \binom{n}{k} p^k (1-p)^{n-k}$

# of trials success

$P(A_1) = \binom{3}{2} \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^1 = \frac{3}{8}$

success fail

probability

$$P(A_2) = \binom{3}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^0 = \frac{1}{8}$$

$$\Rightarrow P(A) = \frac{3}{8} + \frac{1}{8} = \frac{1}{2}$$

$$P(B) = \frac{6}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$$

1st roll 2nd roll 3rd roll

$P(A \cap B) \Rightarrow P(B)$ overrides $P(A)$

$\{4,4,4\}$ $\{5,5,5\}$ $\{6,6,6\}$

$$= \frac{3}{6 \cdot 6 \cdot 6} = \frac{1}{72}$$

$$P(A) \cdot P(B) = P(A \cap B)$$

$$\Rightarrow \frac{1}{2} \cdot \frac{1}{36} = \frac{1}{72}$$

$$\frac{1}{72} = \frac{1}{72}$$

$\Rightarrow$ Yes, they are independent

④ 5 cards → find success (flush)

P(success) = P(get flush)

= p

$$p = \frac{\binom{4}{1}\binom{13}{5}}{\binom{52}{5}} = 0.0019807 9$$

$\binom{4}{1}\binom{13}{5}$ ← flush

$\binom{52}{5}$ ← all possible combos of 5 card hands from a deck of 52

S1 S S S

{D, H, C, S} → $\binom{4}{1}$

{1, 2, 3 ... 10, 11, 12, 13} → $\binom{13}{5}$

Choose 1 suit out of 4 × choose 5 card values out of 13 ⇒ flush

Expected value = average

$$P\left(\text{play 3 hands, get flush at 3rd hand}\right) = (1-p)\cdot(1-p)\cdot p = (1-p)^2 p$$

failure success

$$P\left(\text{play k hands, get flush at kth hand}\right) = (1-p)^{k-1} p$$

Geometric Distribution: $P(X=k) = (1-p)^{k-1} p$

E[X] ← geometric r.v.

$= \sum k \cdot p(k)$ ← k hands, P(getting a flush after k hand)

$$E[X] = \frac{1}{p}$$

$$= \frac{1}{.0019807 9}$$

$= 504.85$

5) Conditional Probability

E = event that team wins 4/5 games

F = event that superstar played.

F^c = event that superstar doesn't play

$P(F) = 75\% = \frac{3}{4}$

$P(F^c) = 1 - 75\% = 25\%$

trying to find

P(superstar played | team won $\frac{4}{5}$ games)

$= P(F|E)$

Baye's rule: $P(F|E) = \dfrac{P(E|F)\cdot \overset{0.75}{P(F)}}{P(E|F)\underset{0.75}{P(F)} + P(E|F^c)\underset{0.25}{P(F^c)}}$

$P(E|F) \rightarrow$ Binomial Distribution

$= P$(team won $\frac{4}{5}$ games | superstar play)

$\binom{n}{k} p^k (1-p)^{n-k}$

$X = P(E|F) = \binom{5}{4}(.70)^4(.30)^1 = \frac{5!}{4!1!}(.70)^4(.30) = 0.36015$

$P(E|F^c) = P$(team won $\frac{4}{5}$ games | superstar did not play)

$y = \binom{5}{4}(0.5)^4(0.5)^1 = 0.15625$

$P(F|E) = \dfrac{P(E|F)\cdot P(F)}{P(E|F)P(F) + P(E|F^c)P(F^c)}$

$= \dfrac{X \cdot 0.75}{X(0.75) + y(0.25)}$ = probability superstar played given team won $\frac{4}{5}$ games

$= \dfrac{0.36015 \cdot 0.75}{0.36015(0.75) + 0.15625(0.25)}$

$= 0.8736556966$