MA3111 AY1819 Sem 2 Solutions

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1.
$$(2i)^{5/4} = 2^{5/4}(e^{i\frac{\pi}{2}})^{5/4} = 2^{5/4}(e^{i\frac{\pi}{2}})^{1/4} = 2^{5/4}e^{i(\frac{\pi}{8}+\frac{\pi n}{2})}, \quad n = 0,1,2,3$$

$$\begin{aligned}
\mathrm{Log}((2i)^{5/4}) &= \mathrm{Log}(2^{5/4}e^{i(\frac{\pi}{8}+\frac{\pi n}{2})}) \\
&= \ln(2^{5/4}) + i\,\mathrm{Arg}(e^{i(\frac{\pi}{8}+\frac{\pi n}{2})}) \\
&= \frac{5}{4}\ln 2 + i\,\mathrm{Arg}(e^{i(\frac{\pi}{8}+\frac{\pi n}{2})})
\end{aligned}$$

It could take on values $\frac{5}{4}\ln 2 + i\,\mathrm{Arg}(e^{i(\frac{\pi}{8}+\frac{\pi n}{2})})$ for $n = 0,1,2,3$.
$(\frac{5}{4}\ln 2, \frac{\pi}{8})$, $(\frac{5}{4}\ln 2, \frac{5\pi}{8})$, $(\frac{5}{4}\ln 2, -\frac{7\pi}{8})$, $(\frac{5}{4}\ln 2, -\frac{3\pi}{8})$

2. (a) Note that
$$\frac{1}{z+c} = \frac{1}{c} - \frac{z}{c^2} + \frac{z^2}{c^3} - \cdots$$

Hence,
$$\begin{aligned}
\frac{z^3-1}{z^2+3z-4} &= (z^2+z+1)\left(\frac{1}{z+4}\right) \\
&= (z^2+z+1)\left(\frac{1}{4} - \frac{z}{4^2} + \frac{z^2}{4^3} - \cdots\right) \\
&= \frac{1}{4} + (\frac{1}{4} - \frac{1}{16})z + (\frac{1}{4} - \frac{1}{4^2} + \frac{1}{4^3})z^2 + \cdots \\
&= \frac{1}{4} + \frac{3}{16}z + \frac{13}{64}z^2 + \cdots
\end{aligned}$$

The first 3 terms of the taylor series is $\frac{1}{4}, \frac{3}{16}, \frac{13}{64}$.
Note that the radius of convergence of $\frac{1}{z+4}$ is 4, and (z^2+z+1) converges everywhere. Hence, the radius of convergence of $\frac{z^3-1}{z^2+3z-4} = (z^2+z+1)\left(\frac{1}{z+4}\right)$ is 4.

(b)
$$\frac{1}{z+4} = \frac{1}{3} - \frac{z+1}{3^2} + \frac{(z+1)^2}{3^3} - \cdots$$

$$\begin{aligned}
\frac{z^3-1}{z^2+3z-4} &= ((z+1)^2 - (z+1) + 1)\left(\frac{1}{z+4}\right) \\
&= ((z+1)^2 - (z+1) + 1)\left(\frac{1}{3} - \frac{z+1}{3^2} + \frac{(z+1)^2}{3^3} - \cdots\right) \\
&= \frac{1}{3} + \left(-\frac{1}{3^2} - \frac{1}{3}\right)(z+1) + \left(\frac{1}{3^3} + \frac{1}{3^2} + \frac{1}{3}\right)(z+1)^2 + \cdots
\end{aligned}$$

Hence,

$$\frac{z^3-1}{z^2+3z-4} = \sum_{n=0}^{\infty} a_n (z+1)^n$$

where

$$a_0 = \frac{1}{3}$$

$$a_1 = -\frac{4}{9}$$

$$a_n = (-1)^n \left(\frac{1}{3^{n+1}} + \frac{1}{3^n} + \frac{1}{3^{n-1}} \right), \quad \text{for } n \geq 2$$

3. (a) $f(z)$ is not analytic when $z = 0$ and when $\sin(\pi z) = 0$, ie when $z \in \mathbb{Z}$.
I claim f has pole of order 3 at 0, and of order 1 at $\mathbb{Z} - \{0\}$.
For $z = 0$, define a function $\phi(z)$ in $B(0, 0.1)$ such that $\phi(z) = \frac{e^z z}{\sin(\pi z)}$ for $z \neq 0$, and $\phi(0) = \frac{1}{\pi}$.
Note that ϕ is analytic and non-zero. And since $f(z) = \frac{\phi(z)}{z^3}$, hence, f has pole of order 3 at 0.
For at $z = n$ for some $n \in \mathbb{Z} - \{0\}$, define a function $\phi(z)$ in $B(n, 0.1)$ such that $\phi(z) = \frac{e^z (z-n)}{z^2 \sin(\pi z)}$ for $z \neq n$ and $\phi(n) = \frac{e^z}{z^2 \pi}$. Note that ϕ is analytic and non-zero. And since $f(z) = \frac{\phi(z)}{z-n}$, hence, f has pole of order 1 at n.

(b) The singular points inside γ are $-1, 0, 1$. Hence, by Cauchy residue theorem,

$$\int_\gamma f(z)\ dz = 2\pi i (\mathrm{Res}_{z=-1}(z) + \mathrm{Res}_{z=0}(z) + \mathrm{Res}_{z=1}(z))$$

Now, we need to find each residue.

$$\begin{aligned}
\mathrm{Res}_{z=-1} f(z) &= \lim_{z \to -1} (z+1) f(z) \\
&= \lim_{z \to -1} \frac{e^z (z+1)}{z^2 \sin(\pi z)} \\
&= \lim_{z \to -1} \frac{e^{-1}}{\pi \cos(\pi z)} \quad \text{by L'hospital} \\
&= -\frac{1}{e\pi}
\end{aligned}$$

$$\begin{aligned}
\mathrm{Res}_{z=1} f(z) &= \lim_{z \to 1} (z-1) f(z) \\
&= \lim_{z \to 1} \frac{e^z (z-1)}{z^2 \sin(\pi z)} \\
&= \lim_{z \to 1} \frac{e^1}{\pi \cos(\pi z)} \\
&= -\frac{e}{\pi}
\end{aligned}$$

And to find the residue at $z = 0$, we find the laurent series.

$$\begin{aligned}
\frac{e^z}{z^2 \sin(\pi z)} &= \frac{1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \cdots}{z^2((\pi z) - \frac{(\pi z)^3}{3!} + \frac{(\pi z)^5}{5!} - \cdots)} \\
&= \frac{1}{z^3} \left(\frac{1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \cdots}{(\pi) - \frac{\pi^3 z^2}{3!} + \frac{\pi^5 z^4}{5!} - \cdots} \right) \\
&= \frac{1}{z^3} \left(\frac{1}{\pi} + \frac{1}{\pi} z + (\frac{1}{2\pi} + \frac{\pi}{6}) z^2 + \cdots \right)
\end{aligned}$$

$$\therefore \mathrm{Res}_{z=0} = \frac{1}{2\pi} + \frac{\pi}{6}$$

Hence,

$$\begin{aligned}
\int_\gamma f(z)\ dz &= 2\pi i(\mathrm{Res}_{z=-1}(z) + \mathrm{Res}_{z=0}(z) + \mathrm{Res}_{z=1}(z)) \\
&= 2\pi i\left(-\frac{1}{e\pi} - \frac{e}{\pi} + \frac{1}{2\pi} + \frac{\pi}{6}\right) \\
&= \left(-\frac{2}{e} - 2e + 1 + \frac{\pi^2}{3}\right) i
\end{aligned}$$

4. WLOG, assume $a > 0$. For $a < 0$, the answer would be the negation.

   Consider the function

$$f(z) = \frac{e^{iaz}}{z^4 + 16}$$

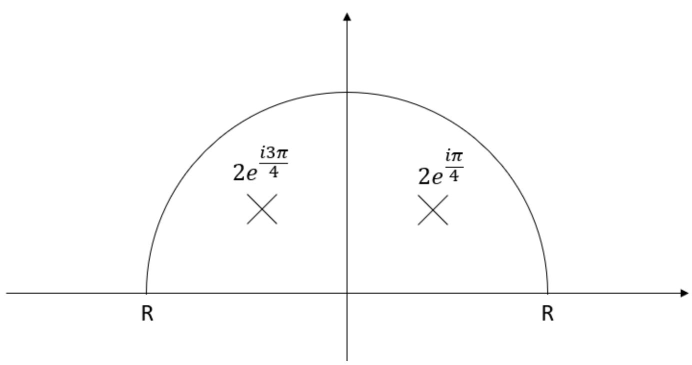
   Note that

$$\frac{\cos(ax)}{x^4 + 16} = \mathrm{Re}(f(z))$$

   and

$$\int_0^R \frac{\cos(ax)}{x^4 + 16}\ dz = \frac{1}{2}\int_{-R}^R \frac{\cos(ax)}{x^4 + 16}\ dz$$

   And for $R > 10$, let $\gamma_R(t) = Re^{it}, 0 \le t \le \pi$.



   There are 2 residue points inside the semicircle. Hence, by Cauchy Residue Theorem,

$$\int_{-R}^R f(z)\ dz + \int_{\gamma_R} f(z)\ dz = 2\pi i(\mathrm{Res}_{z=2e^{\frac{i\pi}{4}}} f(z) + \mathrm{Res}_{z=2e^{\frac{i3\pi}{4}}} f(z))$$

   Now we want to find the residue at those 2 points.

$$f(z) = \frac{e^{iaz}}{z^4 + 16} = \frac{e^{iaz}}{(z - 2e^{\frac{i\pi}{4}})(z - 2e^{\frac{i3\pi}{4}})(z - 2e^{-\frac{i\pi}{4}})(z - 2e^{-\frac{i3\pi}{4}})}$$

   Hence,

$$\begin{aligned}
\mathrm{Res}_{z=2e^{\frac{i\pi}{4}}} f(z) &= \lim_{z \to 2e^{\frac{i\pi}{4}}} (z - 2e^{\frac{i\pi}{4}}) f(z) \\
&= \lim_{z \to 2e^{\frac{i\pi}{4}}} \frac{e^{iaz}}{(z - 2e^{\frac{i3\pi}{4}})(z - 2e^{-\frac{i\pi}{4}})(z - 2e^{-\frac{i3\pi}{4}})} \\
&= \frac{e^{ia(\sqrt{2}+i\sqrt{2})}}{(2\sqrt{2})^3(1)(1+i)(i)} \\
&= \frac{e^{-a\sqrt{2}}e^{ia\sqrt{2}}}{16\sqrt{2}(i-1)}
\end{aligned}$$

$$
\begin{aligned}
\operatorname{Res}_{z=2e^{\frac{i3\pi}{4}}} f(z) &= \lim_{z\to 2e^{\frac{i3\pi}{4}}} (z-2e^{\frac{i\pi}{4}})f(z) \\
&= \lim_{z\to 2e^{\frac{i3\pi}{4}}} \frac{e^{iaz}}{(z-2e^{\frac{i\pi}{4}})(z-2e^{-\frac{i\pi}{4}})(z-2e^{-\frac{i3\pi}{4}})} \\
&= \frac{e^{ia(-\sqrt{2}+i\sqrt{2})}}{(2\sqrt{2})^3(-1)(-1+i)(i)} \\
&= \frac{e^{-a\sqrt{2}}e^{-ia\sqrt{2}}}{16\sqrt{2}(i+1)}
\end{aligned}
$$

Hence,

$$
\int_{-R}^{R} f(z)\ dz = 2\pi i\left(\frac{e^{-a\sqrt{2}}e^{ia\sqrt{2}}}{16\sqrt{2}(i-1)} + \frac{e^{-a\sqrt{2}}e^{-ia\sqrt{2}}}{16\sqrt{2}(i+1)}\right) - \int_{\gamma_R} f(z)\ dz
$$

Hence,

$$
\int_0^R \frac{\cos(ax)}{x^4+16}\ dz = \frac{1}{2}\operatorname{Re}\left(2\pi i\left(\frac{e^{-a\sqrt{2}}e^{ia\sqrt{2}}}{16\sqrt{2}(i-1)} + \frac{e^{-a\sqrt{2}}e^{-ia\sqrt{2}}}{16\sqrt{2}(i+1)}\right) - \int_{\gamma_R} f(z)\ dz\right)
$$

Now, we want to solve $\int_{\gamma_R} f(z)\ dz$:

$$
\left|\int_{\gamma_R} f(z)\ dz\right| \le \pi R|f(z)| = \pi R\left|\frac{e^{iaz}}{z^4+16}\right| \le \pi R\frac{e^{-ay}}{|z^4|-16} \le \pi R\frac{1}{R^4-16}
$$

Which $\to 0$ as $R\to\infty$. Hence,

$$
\begin{aligned}
\int_0^\infty \frac{\cos(ax)}{x^4+16}\ dz &= \frac{1}{2}\operatorname{Re}\left(2\pi i\left(\frac{e^{-a\sqrt{2}}e^{ia\sqrt{2}}}{16\sqrt{2}(i-1)} + \frac{e^{-a\sqrt{2}}e^{-ia\sqrt{2}}}{16\sqrt{2}(i+1)}\right)\right) \\
&= -\pi\frac{e^{-a\sqrt{2}}}{16\sqrt{2}}\operatorname{Im}\left(\frac{(-i-1)e^{ia\sqrt{2}}}{2} + \frac{(-i+1)e^{-ia\sqrt{2}}}{2}\right) \\
&= -\frac{\pi}{2}\frac{e^{-a\sqrt{2}}}{16\sqrt{2}}\operatorname{Im}\left((-i-1)e^{ia\sqrt{2}} + (-i+1)e^{-ia\sqrt{2}}\right) \\
&= -\frac{\pi}{2}\frac{e^{-a\sqrt{2}}}{16\sqrt{2}}\left(-\cos(a\sqrt{2}) - \sin(a\sqrt{2}) - \cos(-a\sqrt{2}) + \sin(-a\sqrt{2})\right) \\
&= -\frac{\pi}{2}\frac{e^{-a\sqrt{2}}}{16\sqrt{2}}\left(-2\cos(a\sqrt{2}) - 2\sin(a\sqrt{2})\right) \\
&= \frac{\pi e^{-a\sqrt{2}}(\cos(a\sqrt{2}+\sin(a\sqrt{2})))}{16\sqrt{2}}
\end{aligned}
$$

Hence, for $a>0$,

$$
\int_0^\infty \frac{\cos(ax)}{x^4+16}\ dz = \frac{\pi e^{-a\sqrt{2}}(\cos(a\sqrt{2}+\sin(a\sqrt{2})))}{16\sqrt{2}}
$$

and for $a<0$,

$$
\int_0^\infty \frac{\cos(ax)}{x^4+16}\ dz = -\frac{\pi e^{-|a|\sqrt{2}}(\cos(|a|\sqrt{2}+\sin(|a|\sqrt{2})))}{16\sqrt{2}}
$$

5. (a) The only singular point is $z=0$.

$$
\begin{aligned}
\operatorname{Res}_{z=0}\frac{\exp(rz^n)}{z} &= \operatorname{Res}_{z=0}\frac{1}{z}(1+rz^n+\frac{(rz^n)^2}{2}+\frac{(rz^n)^3}{3!}+\cdots) \\
&= 1
\end{aligned}
$$

Then, by Cauchy residue theorem,

$$\int_C \frac{\exp(rz^n)}{z}\ dz = 2\pi i$$

(b) Let the C be the equation $e^{i\theta}$ for $0 \le \theta \le 2\pi$.

$$\begin{aligned}\int_C \frac{\exp(rz^n)}{z}\ dz &= \int_0^{2\pi} \frac{\exp(r(e^{i\theta})^n)}{e^{i\theta}}(ie^{i\theta})\ d\theta \\ &= i\int_0^{2\pi} \exp(r(e^{i\theta})^n)\ d\theta \\ &= 2\pi i\end{aligned}$$

$$\therefore \int_0^{2\pi} \exp(r(e^{i\theta})^n)\ d\theta = 2\pi$$

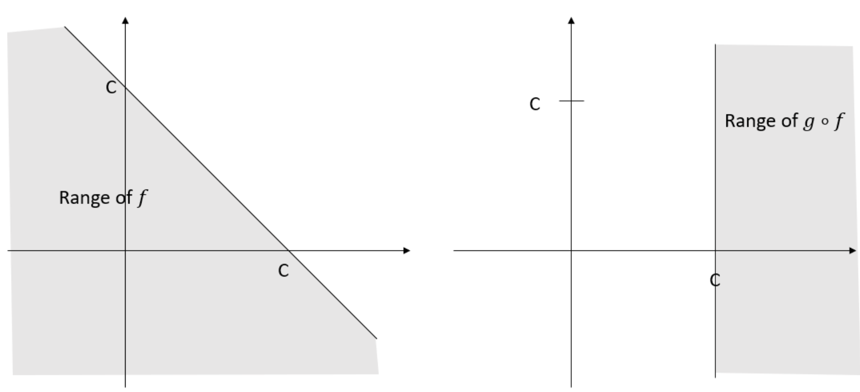
On the other hand,

$$\begin{aligned}\mathrm{Re}\left(\int_0^{2\pi} \exp(r(e^{i\theta})^n)\ d\theta\right) &= \mathrm{Re}\left(\int_0^{2\pi} \exp[\mathrm{Re}(r(e^{i\theta})^n) + i\,\mathrm{Im}(r(e^{i\theta})^n)]\ d\theta\right) \\ &= \mathrm{Re}\left(\int_0^{2\pi} \exp[r\cos(\theta n) + ir\sin(\theta n)]\ d\theta\right) \\ &= \int_0^{2\pi} \mathrm{Re}\left(\exp[r\cos(\theta n) + ir\sin(\theta n)]\right)\ d\theta \\ &= \int_0^{2\pi} \exp[r\cos(\theta n)]\,\mathrm{Re}\left(\exp[ir\sin(\theta n)]\right)\ d\theta \\ &= \int_0^{2\pi} \exp(r\cos(\theta n))\cos(r\sin(\theta n))\ d\theta\end{aligned}$$

Hence,

$$\int_0^{2\pi} \exp(r\cos(\theta n))\cos(r\sin(\theta n))\ d\theta = \mathrm{Re}(2\pi) = 2\pi$$

6. (a) Consider the function $g(z) = (z - C)e^{i\frac{3\pi}{4}} + C$ (rotate $\frac{3\pi}{4}$ anti-clockwise about C.



Note that $\mathrm{Re}(g(f(z)) \ge C$.
Then, consider $g_2(z) = \frac{1}{e^z}$.

$$|g_2(g(f(z)))| = \frac{1}{|e^{g(f(z))}|} \le \frac{1}{C}$$

And since all of f, g, g_2 are analytic, and $g_2(g(f(z)))$ is bounded, hence, by Liouville's Theorem, $g_2(g(f(z)))$ is a constant, and hence, $g(f(z))$ is a constant, and hence, $f(z)$ is a constant.

(b) Consider the function

$$g(z) = \exp(f(z) - f(iz))$$

This function is analytic. We want to show it is bounded.

$$\begin{aligned}
|g(z)| &= |\exp(f(z) - f(iz))| \\
&= \exp(\mathrm{Re}(f(z) - f(iz))) \\
&= \exp(u(x, y) - u(-y, x)) \\
&\leq \exp(C)
\end{aligned}$$

Hence, by Liouville's Theorem, $g(z) = \exp(f(z) - f(iz))$ is a constant. Hence, $f(z) - f(iz) = k$ for some constant $k \in \mathbb{C}$.

Substitute $z := 0$ into $f(z) - f(iz) = k$ to get $f(0) - f(0) = k$, hence, $k = 0$. Hence,

$$f(z) = f(iz)$$