Calculus 18/19 Sem 1 Suggested Answers

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Question 1

i) Let $\epsilon > 0$. Choose $\delta = \min\{1/2, \epsilon/6\}$.
Then whenever $0 < |x-2| < \delta$:

$$\begin{aligned}
\left|\frac{2x^2+3x+1}{x^2-1} - 5\right| &= \left|\frac{2x^2+3x+1-5x^2+5}{x^2-1}\right| \\
&= \left|\frac{-3x^2+3x+6}{x^2-1}\right| \\
&= 3\left|\frac{(x-2)(x+1)}{(x-1)(x+1)}\right| \\
&= 3\left|\frac{x-2}{x-1}\right| \\
&< 3\left|\frac{\delta}{\frac{1}{2}}\right| \\
&= 6\delta \\
&< \epsilon
\end{aligned}$$

ii) Consider 2 cases:
Case 1: If $f(0) = 0$ or $f(1) = 1$, then it definitely cuts the line $y = x$ at either $x = 0$ or $x = 1$.
Case 2: $f(0) \neq 0$ and $f(1) \neq 1$. We define $g(x) = f(x) - x$. Then g is continuous on $(0, 1)$.

$$f(0) \neq 0 \implies f(0) > 0 \implies g(0) > 0$$
$$f(1) \neq 1 \implies f(1) < 1 \implies g(1) < 0$$

Since $g(0) > 0$ and $g(1) < 0$ and g is continuous on $[0, 1]$, by the IVT, $\exists c \in (0, 1)$ such that $g(c) = 0 \implies f(c) = c$.

Question 2

i) Given $f(x+y) = f(x) + f(y) + x^2y + xy^2$.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x) + f(h) + x^2h + xh^2 - f(x)}{h} \\ &= \lim_{h\to 0} \frac{f(h)}{h} + x^2 + xh \\ &= x^2 + 1 \end{aligned} \tag{1}$$

Taking the derivative at 0, we have $f'(0) = 1$, however,

$$\begin{aligned} f'(0) = 1 = &= \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} \\ &= \lim_{x\to 0} \frac{f(x)}{x} - \frac{f(0)}{x} \\ &= 1 - \lim_{x\to 0} \frac{f(0)}{x} \end{aligned}$$

If $f(0)$ was not 0, then one would have $\lim_{x\to 0} \frac{f(0)}{x}$ going to infinity, which is a contradiction. Thus $f(0) = 0$. Integrating (1), $f(x) = \frac{1}{3}x^3 + x + c$. Subbing $f(0) = 0$, we have that $c = 0$, and hence $f(x) = \frac{1}{3}x^3 + x$.

ii) $\lim_{x\to 0} \frac{(ax+b)^{1/3}-2}{x} = 5/12$. Note that this implies that numerator must go to 0, as else if it approached some value greater than 0, it would mean the integral would go off to infinity. Solving for $\lim_{x\to 0}(ax + b)^{1/3} - 2 = 0$, we find that $\lim_{x\to 0} ax + b = 8 \implies b = 8$.

Since both the numerator and denominator go to 0, we can apply L'Hopital's Rule:

$$\begin{aligned} \lim_{x\to 0} \frac{(ax+b)^{1/3} - 2}{x} &= \lim_{x\to 0} \frac{a}{3}(ax+8)^{-2/3} \\ &= \frac{a}{3}(8^{-2/3}) = 5/12 \end{aligned}$$

Solving, $a = 5, b = 8$.

iii) If $f(x) = \frac{x^27 + x^26 + 2}{1+x}$, evaluate $f^{27}(x)$. By polynomial long division, we have $f(x) = x^{26} + \frac{2}{x+1}$. Differenciating f 27 times will make the x^{26} term disappear, so we only need to worry about the $\frac{2}{x+1}$ term. Note that if $g(x) = \frac{2}{x+1}$, then $g^1(x) = -\frac{2}{(x+1)^2}, g^2(x) = \frac{2\times 2}{(x+1)^3}, g^3(x) = -\frac{2\times 2\times 3}{(x+1)^4}$. In general, we realise that:

$$\begin{aligned} g^n(x) &= (-1)^n \frac{2n!}{(x+1)^n + 1} \\ g^{27}(3) &= -\frac{2\times 27!}{4^{28}} \end{aligned}$$

Question 3

i) Applying implicit differentiation:

$$\begin{aligned}2x + y + x\frac{dy}{dx} + 2y\frac{dy}{dx} &= 0\\ \frac{dy}{dx}(x+2y) &= -2x - y\\ \frac{dy}{dx} &= \frac{-2x-y}{x+2y}\end{aligned}$$

dy/dx is undefined at $x = -2y$. $dy/dx = 0 \implies x = -y/2$. At $x = -2y$:

$$\begin{aligned}(-2y)^2 + (-2y)y + y^2 &= 12\\ 4y^2 - 2y^2 + y^2 &= 12\\ 3y^2 &= 12\\ y &= \pm 2\end{aligned}$$

We also need to check the point where by dy/dx is undefined. At $x = -y/2$:

$$\begin{aligned}(-\frac{y}{2})^2 + (-\frac{y}{2})y + y^2 &= 12\\ \frac{y^2}{4} - \frac{y^2}{2} + y^2 &= 12\\ y^2 &= 16\\ y &= \pm 4\end{aligned}$$

We have 4 critical points: $(-4,2), (4,-2), (-2,4), (2,-4)$.

Highest point: $(-2,4)$. Lowest point: $(2,-4)$.

ii) The curve $y = cx^3 + e^x$ is continuous on $\mathbb{R}$.

$$\begin{aligned}y &= cx^3 + e^x\\ \frac{dy}{dx} &= 3cx^2 + e^x\\ \frac{d^2y}{dx^2} &= 6cx + e^x\end{aligned}$$

Note that $\forall c \in \mathbb{R}$:

$$\lim_{x\to+\infty} \frac{d^2y}{dx^2} = +\infty$$

Thus the curve have inflection points if $\exists d \in \mathbb{R}$ such that $6c(d) + e^d < 0$. (If such d exists, the intermediate value theorem will ensure that there exist a point where $\frac{d^2y}{dx^2} = 0$).

Consider 3 cases:

Case 1: $c > 0$.

$$\lim_{x\to-\infty} \frac{d^2y}{dx^2} = -\infty$$

Thus if $c > 0$,the curve will have inflection points.

Case 2: $c < 0$.

$$\begin{aligned}\frac{d^3y}{dx^3} &= 6c + e^x\\ \frac{d^3y}{dx^3} &= 0 \implies 6c = -e^x \implies x = \ln -6c\end{aligned}$$

$\frac{d^2y}{dx^2}$ has a local minimum at $x = \ln -6c$,

$$\begin{aligned}
&6c(\ln -6c) + e^{\ln -6c} < 0 \\
&6c(\ln -6c) < 6c \\
&\ln -6c > 1 \quad \text{(Since } 6c < 0\text{)} \\
&-6c > e \\
&c < -\frac{e}{6}
\end{aligned}$$

Case 3: $c = 0$.

$e^x > 0 \implies \frac{d^2y}{dx^2} > 0$ thus the curve does not have inflection points.

Combining the 3 cases, we have $c > 0$ or $c < -\frac{e}{6}$.

iii) There are two ways of doing this. One way is to simply recall the definition of e^a as:

$$e^a = \lim_{x\to\infty} \left(1 + \frac{a}{x}\right)^x$$

Then we simplify the given expression to see:

$$\lim_{x\to\infty} \left(\frac{x+a}{x-a}\right)^x = \lim_{x\to\infty} \left(1 + \frac{2a}{x-a}\right)^x = e^{2a}$$

Another way is to calculate limits by exponentiation:

$$\begin{aligned}
\lim_{x\to\infty} \left(\frac{x+a}{x-a}\right)^x &= \lim_{x\to\infty} e^{x \ln \frac{x+a}{x-a}} \\
\lim_{x\to\infty} x \ln \frac{x+a}{x-a} &= \lim_{x\to\infty} \frac{\ln \frac{x+a}{x-a}}{x^{-1}}
\end{aligned}$$

Since the top and bottom goes to 0, we can use L'Hopital's Rule.

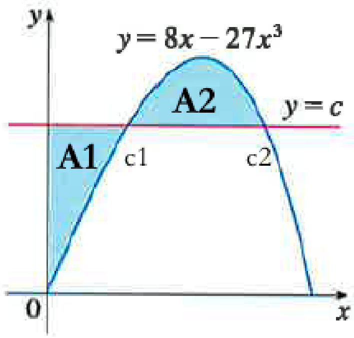
$$\begin{aligned}
&= \lim_{x\to\infty} \frac{\frac{(x-a)^{-1} + (x+a)(-1)(x-a)^{-2}}{\frac{x+a}{x-a}}}{-x^{-2}} \\
&= \lim_{x\to\infty} \frac{\frac{(x-a)-(x+a)}{(x+a)(x-a)}}{-x^{-2}} \\
&= \lim_{x\to\infty} \frac{2ax^2}{(x^2 - a^2)} \\
&= \lim_{x\to\infty} \frac{2a}{1 - \frac{a^2}{x^2}} \\
&= 2a
\end{aligned}$$

Comparing the powers, we see that $a = 1/2$.

$$e^{2a} = e \implies a = \frac{1}{2}$$

Question 4

a) Let the 2 points which the curve cuts c be c_1 and c_2, where $c_2 > c_1$.



$$A_1 = cc_1 - \int_0^{c_1} 8x - 27x^3 dx \qquad A_2 = \int_{c_1}^{c_2} 8x - 27x^3 dx - c[c_2 - c_1]$$

Equating:

$$\begin{aligned}
cc_1 - \int_0^{c_1} 8x - 27x^3 dx &= \int_{c_1}^{c_2} 8x - 27x^3 dx - cc_2 + cc_1 \\
cc_2 &= \int_0^{c_2} 8x - 27x^3 dx \\
cc_2 &= 4x^2 - \frac{27}{4}x^4 \Big|_0^{c_2} \\
cc_2 &= 4c_2^2 - \frac{27}{4}c_2^4 \\
c &= 4c_2 - \frac{27}{4}c_2^3
\end{aligned}$$

Since c_2 is also a solution to $y = 8x - 27x^3, c = 8c_2 - 27c_2^3$.

$$\begin{aligned}
8c_2 - 27c_2^3 &= 4c_2 - \frac{27}{4}c_2^3 \\
4 &= \frac{81}{4}c_2^2 \\
c_2 &= \frac{4}{9} \quad \text{(Reject -ve)}
\end{aligned}$$

We have $c = 32/27$.

b) We need to evaluate the integral $\int_0^1 2\pi(4 - x^2)\sqrt{1 + 4x^2}\ dx$.
For this question, we apply the integral reduction formula for $\sec^n x$.

$$I_n = \int \sec^n(x)dx = \frac{1}{n-1}(\sec^{n-2} x \tan x) + \frac{n-2}{n-1}I_{n-2} + C$$

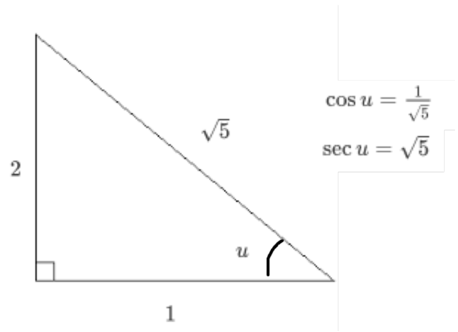
Letting $x = \tan u/2$, then $dx/du = \sec^2 u/2$.

$$\begin{aligned}
\int_0^1 2\pi(4 - x^2)\sqrt{1 + 4x^2}\ dx &= \int_0^{\tan^{-1} 2} 2\pi(4 - \frac{1}{4}\tan^2 u\sqrt{1 + \tan^2 u}\frac{1}{2}\sec^2 u du \\
&= \frac{1}{4}\pi \int_0^{\tan^{-1} 2} (16 - \tan^2 u)(\sec^3 u)du \\
&= \frac{\pi}{4} \int_0^{\tan^{-1} 2} (17\sec^3 u - \sec^5 u)du
\end{aligned}$$

Apply integral reduction on $\sec^5 u$, one has:

$$\int_0^{\tan^{-1} 2} \sec^5 u du = \frac{\tan u \sec^3 u}{4}\Big|_0^{\tan^{-1} 2} + \frac{3}{4}\int_0^{\tan^{-1} 2} \sec^3 u du$$

Considering a triangle:



$$\frac{\tan[\tan^{-1}(2)]5\sqrt{5}}{4} + \frac{3}{4}\int_0^{\tan^{-1} 2} \sec^3 u du = \frac{5\sqrt{5}}{2} + \frac{3}{4}\int_0^{\tan^{-1} 2} \sec^3 u du \tag{2}$$

Due to (2), the integral reduces to:

$$\frac{\pi}{4}\int_0^{\tan^{-1} 2} (17 \sec^3 u - \sec^5 u) du = -\frac{\pi}{4}\frac{5\sqrt{5}}{2} + \frac{\pi}{4}\int_0^{\tan^{-1} 2} \left(\frac{65}{4}\sec^3 u\right) du \tag{3}$$

Apply integral reduction on $\sec^3 u$, one has:

$$\begin{aligned}
\int_0^{\tan^{-1} 2} \sec^3 u du &= \frac{\tan u \sec u}{2}\Big|_0^{\tan^{-1} 2} + \frac{1}{2}\int_0^{\tan^{-1} 2} \sec u du \\
&= \sqrt{5} + \frac{1}{2}\ln(\sec x + \tan x)\Big|_0^{\tan^{-1} 2} \\
&= \sqrt{5} + \frac{1}{2}(\ln(\sqrt{5}+2) - \ln(1)) \\
&= \sqrt{5} + \frac{1}{2}(\ln(\sqrt{5}+2))
\end{aligned}$$

Going back to (3), we have:

$$\begin{aligned}
-\frac{\pi}{4}\frac{5\sqrt{5}}{2} + \frac{\pi}{4}\int_0^{\tan^{-1} 2} \left(\frac{65}{4}\sec^3 u\right) du &= -\frac{\pi}{4}\frac{5\sqrt{5}}{2} + \frac{65\pi}{16}(\sqrt{5} + \frac{1}{2}(\ln(\sqrt{5}+2))) \\
&\approx 33.4
\end{aligned}$$

Question 5

i) We have to split and evaluate the integral first.

$$\begin{aligned}
f(x) &= \int_0^\pi (\cos t)\cos(x-t)dt \\
&= \int_0^\pi (\cos t)(\cos x\cos t + \sin x \sin t)dt \\
&= \cos x\int_0^\pi \cos^2 t dt + \sin x\int_0^\pi \sin t\cos t dt \\
&= \cos x\int_0^\pi \frac{1}{2}\cos 2t + \frac{1}{2}dt + \sin x\int_0^\pi \frac{1}{2}\sin 2t dt \\
&= \cos x\left[\frac{1}{4}\sin 2t + \frac{t}{2}\right]_0^\pi - \sin x\left[\frac{1}{4}\cos 2t\right]_0^\pi \\
&= \cos x(\frac{\pi}{2}) - \sin x(0) \\
&= \frac{\pi}{2}\cos x \\
f'(x) &= -\frac{\pi}{2}\sin x
\end{aligned}$$

$f'(x) = 0 \implies \sin x = 0 \implies x = 0, x = \pi, x = 2\pi$.
$f(0) = \frac{\pi}{2}$, $f(\pi) = -\frac{\pi}{2}$, $f(2\pi) = \frac{\pi}{2}$.
Thus the minimum of $f(x)$ on $[0, 2\pi]$ is $(\pi, -\frac{\pi}{2})$.

ii) We prove this by strong induction. Let $P(n)$ be the predicate: $\int_0^1 (\ln x)^n dx = (-1)^n \cdot n!$

Base case($n = 0$ and $n = 1$):

$$\begin{aligned}
\int_0^1 (\ln x)^0 dx &= \int_0^1 1dx = 1 \\
(-1)^0 \cdot (0!) = 1 &= \int_0^1 (\ln x)^0 dx
\end{aligned}$$

Thus $P(0)$ is true.

$$\begin{aligned}
\int_0^1 \ln x dx &= [x\ln x - x]\Big|_0^1 = -1 \\
(-1)^1 \cdot (1!) = -1 &= \int_0^1 \ln x dx
\end{aligned}$$

Thus $P(1)$ is true.

Inductive step: Assume $P(n), P(n-1)\ldots P(0)$ is true, to show $P(n+1)$ is true.

$$\begin{aligned}
\int_0^1 (\ln x)^{n+1}dx &= \int_0^1 (\ln x)(\ln x)^n dx \\
&= [(x\ln x - x)(\ln x)^n]\Big|_0^1 - \int_0^1 (x\ln x - x)(n)(\ln x)^{n-1}(\frac{1}{x})dx \\
&= [0-0] - n\int_0^1 (\ln x)^n - (\ln x)^{n-1}dx \\
&= n\int_0^1 (\ln x)^{n-1}dx - n\int_0^1 (\ln x)^n dx \\
&= n(-1)^{n-1}\cdot(n-1)! - n(-1)^n \cdot n! \\
&= (-1)^{n+1}\cdot n! + n(-1)^{n+1}\cdot n! \\
&= (-1)^{n+1}\cdot(n+1)!
\end{aligned}$$

Thus $P(n+1)$ is true. By mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$. ☐

Question 6

a) e^{-2}

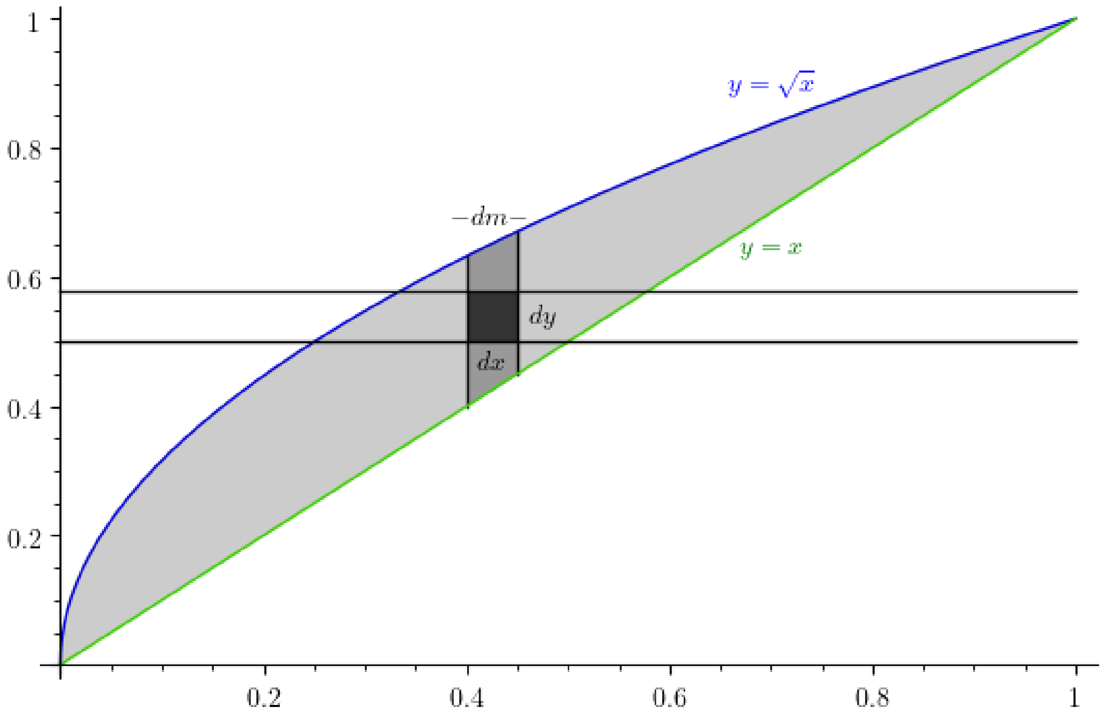
$$
\begin{aligned}
\lim_{x\to 0}\frac{\int_0^x [1-\tan(2t)]^{1/t}}{x} &= \lim_{x\to 0}(1-\tan(2x))^{1/x}\\
&= \exp\left(\lim_{x\to 0}\frac{1}{x}\ln(1-\tan(2x))\right)\\
&= \exp\left(\lim_{x\to 0}\frac{-2\sec^2(2x)}{1-\tan(2x)}\right)\\
&= \exp\left(\lim_{x\to 0}\frac{2\sec^2 2x}{\tan 2x-1}\right)\\
&= \exp\left(\frac{2}{-1}\right) = e^{-2}
\end{aligned}
$$

b) The formula for the x coordinate of the COM is given by:

$$x_{cm} = \lim_{x\to 0}\sum\frac{1}{M}x_i \bigtriangleup m_1 = \frac{1}{M}\int x dm \tag{4}$$

It can be intepreted as a weighted average of the mass of the shape. Given a shape (ie. the one below), one can divide the object into thin strips, with each of the little strips having little mass $\bigtriangleup m$. To find the x coordinate of the COM, one takes the weighted sum $\frac{1}{M}x_1 \bigtriangleup m_1 + x_2 \bigtriangleup m_2 + x_3 \bigtriangleup m_3 \ldots$. If the width of dm approaches 0, we get the formula of the COM given in (4).

The shape requested in the question is shown below. For the rest of the question, let the thickness of the shape be T.



Consider a small section of the shape (highlighted in black), with volume $= dxdyT$. It has small mass $ddm = dxdyT\delta(x,y) = dxdyT(1+y)$. To find the mass of thin strip dm, one must sum up all the masses of black squares from the top curve to the bottom curve, that is:

$$
\begin{aligned}
dm &= \int_x^{\sqrt{x}}(1+y)dydxT\\
&= Tdx(y+\frac{1}{2}y^2|_x^{\sqrt{x}}\\
&= T(\sqrt{x}-\frac{1}{2}x-\frac{1}{2}x^2)dx
\end{aligned} \tag{5}
$$

We can now find x_{cm} in terms of M and T.

$$\begin{aligned}
x_{cm} &= \frac{1}{M}\int_0^1 x dm \\
&= \frac{T}{M}\int_0^1 x\sqrt{x} - \frac{1}{2}x^2 - \frac{1}{2}x^3 dx \\
&= \frac{13}{120}\frac{T}{M}
\end{aligned}$$

To get rid of the T/M term, note that M is simply the total mass of the object. If we integrated dm from 0 to 1, we would get M. Using (5):

$$\begin{aligned}
M &= \int_0^1 dm \\
&= \int_0^1 T(\sqrt{x} - \frac{1}{2}x - \frac{1}{2}x^2)dx \\
&= T\frac{1}{4}
\end{aligned}$$

We have that $M/T = 1/4$ and hence $T/M = 4$. So $x_{cm} = 13/120 \times 4 = 13/30$.

y_{cm} is a little easier as the density does not vary in a strip. (ie. the density in the piece enclosed by the 2 horizontal lines in the picture above is the same throughout the piece). Dividing the piece into horizontal strips dm, to find the mass of a small strip dm, one must integrate ddm from y to y^2.

$$\begin{aligned}
dm &= \int_{y^2}^{y} (1+y)dydxT \\
&= T(y - y^3)dy
\end{aligned}$$

$$\begin{aligned}
y_{cm} &= \frac{1}{M}\int_0^1 y dm \\
&= \frac{T}{M}\int_0^1 y^2 - y^4 dy \\
&= \frac{2}{15}\frac{T}{M} \quad \text{(From 5)} \\
&= \frac{8}{15}
\end{aligned}$$

Thus the coordinates of the COM are $(x_{cm}, y_{cm}) = (13/30, 8/15)$.