

# MA3111 AY1819 Sem 2 Solutions

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1.

$$(2i)^{5/4} = 2^{5/4}(e^{i\frac{\pi}{2}})^{5/4} = 2^{5/4}(e^{i\frac{\pi}{8}})^{1/4} = 2^{5/4}e^{i(\frac{\pi}{8} + \frac{\pi n}{2})}, \quad n = 0, 1, 2, 3$$

$$\begin{aligned} \text{Log}((2i)^{5/4}) &= \text{Log}(2^{5/4}e^{i(\frac{\pi}{8} + \frac{\pi n}{2})}) \\ &= \ln(2^{5/4}) + i \text{Arg}(e^{i(\frac{\pi}{8} + \frac{\pi n}{2})}) \\ &= \frac{5}{4} \ln 2 + i\left(\frac{\pi}{8} + \frac{\pi n}{2}\right) \end{aligned}$$

It could take on values  $\frac{5}{4} \ln 2 + i(\frac{\pi}{8} + \frac{\pi n}{2})$  for  $n = 0, 1, 2, 3$ .

$$(\frac{5}{4} \ln 2, \frac{\pi}{8}), (\frac{5}{4} \ln 2, \frac{\pi}{8} + \frac{\pi}{2}), (\frac{5}{4} \ln 2, \frac{\pi}{8} + \frac{2\pi}{2}), (\frac{5}{4} \ln 2, \frac{\pi}{8} + \frac{3\pi}{2})$$

2. (a) Note that

$$\frac{1}{z+c} = \frac{1}{c} - \frac{z}{c^2} + \frac{z^2}{c^3} - \dots$$

Hence,

$$\begin{aligned} \frac{z^3-1}{z^2+3z-4} &= (z^2+z+1) \left( \frac{1}{z+4} \right) \\ &= (z^2+z+1) \left( \frac{1}{4} - \frac{z}{4^2} + \frac{z^2}{4^3} - \dots \right) \\ &= \frac{1}{4} + \left( \frac{1}{4} - \frac{1}{16} \right) z + \left( \frac{1}{4} - \frac{1}{4^2} + \frac{1}{4^3} \right) z^2 + \dots \\ &= \frac{1}{4} + \frac{3}{16} z + \frac{13}{64} z^2 + \dots \end{aligned}$$

The first 3 terms of the Taylor series is  $\frac{1}{4}, \frac{3}{16}, \frac{13}{64}$ .

Note that the radius of convergence of  $\frac{1}{z+4}$  is 4, and  $(z^2+z+1)$  converges everywhere. Hence, the radius of convergence of  $\frac{z^3-1}{z^2+3z-4} = (z^2+z+1) \left( \frac{1}{z+4} \right)$  is 4.

(b)

$$\frac{1}{z+4} = \frac{1}{3} - \frac{z+1}{3^2} + \frac{(z+1)^2}{3^3} - \dots$$

$$\begin{aligned} \frac{z^3-1}{z^2+3z-4} &= ((z+1)^2 - (z+1) + 1) \left( \frac{1}{z+4} \right) \\ &= ((z+1)^2 - (z+1) + 1) \left( \frac{1}{3} - \frac{z+1}{3^2} + \frac{(z+1)^2}{3^3} - \dots \right) \\ &= \frac{1}{3} + \left( -\frac{1}{3^2} - \frac{1}{3} \right) (z+1) + \left( \frac{1}{3^3} + \frac{1}{3^2} + \frac{1}{3} \right) (z+1)^2 + \dots \end{aligned}$$

Hence,

$$\frac{z^3 - 1}{z^2 + 3z - 4} = \sum_{n=0}^{\infty} a_n (z + 1)^n$$

where

$$a_0 = \frac{1}{3}$$

$$a_1 = -\frac{4}{9}$$

$$a_n = (-1)^n \left( \frac{1}{3^{n+1}} + \frac{1}{3^n} + \frac{1}{3^{n-1}} \right), \quad \text{for } n \geq 2$$

3. (a)  $f(z)$  is not analytic when  $z = 0$  and when  $\sin(\pi z) = 0$ , ie when  $z \in \mathbb{Z}$ .

I claim  $f$  has pole of order 3 at 0, and of order 1 at  $\mathbb{Z} - \{0\}$ .

For  $z = 0$ , define a function  $\phi(z)$  in  $B(0, 0.1)$  such that  $\phi(z) = \frac{e^z z}{\sin(\pi z)}$  for  $z \neq 0$ , and  $\phi(0) = \frac{1}{\pi}$ .

Note that  $\phi$  is analytic and non-zero. And since  $f(z) = \frac{\phi(z)}{z^3}$ , hence,  $f$  has pole of order 3 at 0.

For at  $z = n$  for some  $n \in \mathbb{Z} - \{0\}$ , define a function  $\phi(z)$  in  $B(n, 0.1)$  such that  $\phi(z) = \frac{e^z (z - n)}{z^2 \sin(\pi z)}$  for  $z \neq n$  and  $\phi(n) = \frac{e^z}{z^2 \pi}$ . Note that  $\phi$  is analytic and non-zero. And since  $f(z) = \frac{\phi(z)}{z - n}$ , hence,  $f$  has pole of order 1 at  $n$ .

- (b) The singular points inside  $\gamma$  are  $-1, 0, 1$ . Hence, by Cauchy residue theorem,

$$\int_{\gamma} f(z) dz = 2\pi i (\text{Res}_{z=-1}(z) + \text{Res}_{z=0}(z) + \text{Res}_{z=1}(z))$$

Now, we need to find each residue.

$$\begin{aligned} \text{Res}_{z=-1} f(z) &= \lim_{z \rightarrow -1} (z + 1) f(z) \\ &= \lim_{z \rightarrow -1} \frac{e^z (z + 1)}{z^2 \sin(\pi z)} \\ &= \lim_{z \rightarrow -1} \frac{e^{-1}}{\pi \cos(\pi z)} \quad \text{by L'hospital} \\ &= -\frac{1}{e\pi} \end{aligned}$$

$$\begin{aligned} \text{Res}_{z=1} f(z) &= \lim_{z \rightarrow 1} (z - 1) f(z) \\ &= \lim_{z \rightarrow 1} \frac{e^z (z - 1)}{z^2 \sin(\pi z)} \\ &= \lim_{z \rightarrow 1} \frac{e^1}{\pi \cos(\pi z)} \\ &= -\frac{e}{\pi} \end{aligned}$$

And to find the residue at  $z = 0$ , we find the laurent series.

Note that the laurent series of  $\frac{1}{\sin \pi z}$  is  $\frac{1}{\pi z} + \sum_{n=1} a_n z^n$ , and the laurent series of  $e^z$  is  $1 + x + \frac{x^2}{2} + \dots$ .

$$\begin{aligned} f(z) &= \frac{e^z}{z^2 \sin(\pi z)} \\ &= \frac{1 + x + \frac{x^2}{2} + \dots}{z^2} \left( \frac{1}{\pi z} + \sum_{n=1} a_n z^n \right) \\ &= (\dots) + (\dots) \frac{1}{z} + (\dots) \frac{1}{z^2} + \left( \frac{1}{\pi} \right) \frac{1}{z^3} \end{aligned}$$

$$\therefore \text{Res}_{z=0} f(z) = \frac{1}{\pi}$$

Hence,

$$\begin{aligned} \int_{\gamma} f(z) dz &= 2\pi i (\text{Res}_{z=-1}(z) + \text{Res}_{z=0}(z) + \text{Res}_{z=1}(z)) \\ &= 2\pi i \left( -\frac{1}{e\pi} - \frac{e}{\pi} + \frac{1}{\pi} \right) \\ &= \left( -\frac{2}{e} - 2e + 2 \right) i \end{aligned}$$

4. WLOG, assume  $a > 0$ . For  $a < 0$ , the answer would be the negation.

Consider the function

$$f(z) = \frac{e^{iaz}}{z^4 + 16}$$

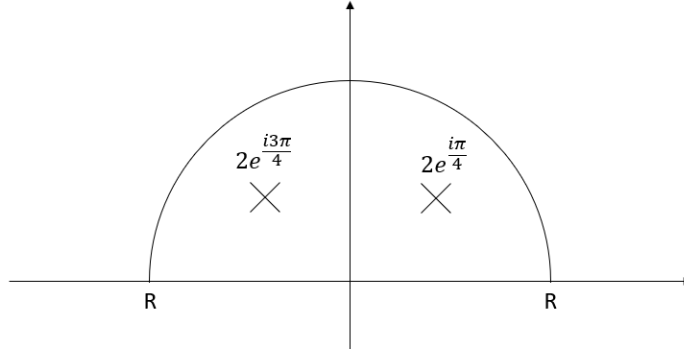
Note that

$$\frac{\cos(ax)}{x^4 + 16} = \text{Re}(f(z))$$

and

$$\int_0^R \frac{\cos(ax)}{x^4 + 16} dz = \frac{1}{2} \int_{-R}^R \frac{\cos(ax)}{x^4 + 16} dz$$

And for  $R > 10$ , let  $\gamma_R(t) = Re^{it}, 0 \leq t \leq \pi$ .



There are 2 residue points inside the semicircle. Hence, by Cauchy Residue Theorem,

$$\int_{-R}^R f(z) dz + \int_{\gamma_R} f(z) dz = 2\pi i (\text{Res}_{z=2e^{i\pi/4}} f(z) + \text{Res}_{z=2e^{i3\pi/4}} f(z))$$

Now we want to find the residue at those 2 points.

$$f(z) = \frac{e^{iaz}}{z^4 + 16} = \frac{e^{iaz}}{(z - 2e^{i\pi/4})(z - 2e^{i3\pi/4})(z - 2e^{-i\pi/4})(z - 2e^{-i3\pi/4})}$$

Hence,

$$\begin{aligned} \text{Res}_{z=2e^{i\pi/4}} f(z) &= \lim_{z \rightarrow 2e^{i\pi/4}} (z - 2e^{i\pi/4}) f(z) \\ &= \lim_{z \rightarrow 2e^{i\pi/4}} \frac{e^{iaz}}{(z - 2e^{i3\pi/4})(z - 2e^{-i\pi/4})(z - 2e^{-i3\pi/4})} \\ &= \frac{e^{ia(\sqrt{2}+i\sqrt{2})}}{(2\sqrt{2})^3(1+i)(i)} \\ &= \frac{e^{-a\sqrt{2}}e^{ia\sqrt{2}}}{16\sqrt{2}(i-1)} \end{aligned}$$

$$\begin{aligned}
\text{Res}_{z=2e^{\frac{i3\pi}{4}}} f(z) &= \lim_{z \rightarrow 2e^{\frac{i3\pi}{4}}} (z - 2e^{\frac{i\pi}{4}}) f(z) \\
&= \lim_{z \rightarrow 2e^{\frac{i3\pi}{4}}} \frac{e^{iaz}}{(z - 2e^{\frac{i\pi}{4}})(z - 2e^{-\frac{i\pi}{4}})(z - 2e^{-\frac{i3\pi}{4}})} \\
&= \frac{e^{ia(-\sqrt{2}+i\sqrt{2})}}{(2\sqrt{2})^3(-1)(-1+i)(i)} \\
&= \frac{e^{-a\sqrt{2}}e^{-ia\sqrt{2}}}{16\sqrt{2}(i+1)}
\end{aligned}$$

Hence,

$$\int_{-R}^R f(z) dz = 2\pi i \left( \frac{e^{-a\sqrt{2}}e^{ia\sqrt{2}}}{16\sqrt{2}(i-1)} + \frac{e^{-a\sqrt{2}}e^{-ia\sqrt{2}}}{16\sqrt{2}(i+1)} \right) - \int_{\gamma_R} f(z) dz$$

Hence,

$$\int_0^R \frac{\cos(ax)}{x^4 + 16} dz = \frac{1}{2} \text{Re} \left( 2\pi i \left( \frac{e^{-a\sqrt{2}}e^{ia\sqrt{2}}}{16\sqrt{2}(i-1)} + \frac{e^{-a\sqrt{2}}e^{-ia\sqrt{2}}}{16\sqrt{2}(i+1)} \right) - \int_{\gamma_R} f(z) dz \right)$$

Now, we want to solve  $\int_{\gamma_R} f(z) dz$ :

$$\left| \int_{\gamma_R} f(z) dz \right| \leq \pi R |f(z)| = \pi R \left| \frac{e^{iaz}}{z^4 + 16} \right| \leq \pi R \frac{e^{-ay}}{|z^4| - 16} \leq \pi R \frac{1}{R^4 - 16}$$

Which  $\rightarrow 0$  as  $R \rightarrow \infty$ . Hence,

$$\begin{aligned}
\int_0^\infty \frac{\cos(ax)}{x^4 + 16} dz &= \frac{1}{2} \text{Re} \left( 2\pi i \left( \frac{e^{-a\sqrt{2}}e^{ia\sqrt{2}}}{16\sqrt{2}(i-1)} + \frac{e^{-a\sqrt{2}}e^{-ia\sqrt{2}}}{16\sqrt{2}(i+1)} \right) \right) \\
&= -\pi \frac{e^{-a\sqrt{2}}}{16\sqrt{2}} \text{Im} \left( \frac{(-i-1)e^{ia\sqrt{2}}}{2} + \frac{(-i+1)e^{-ia\sqrt{2}}}{2} \right) \\
&= -\frac{\pi}{2} \frac{e^{-a\sqrt{2}}}{16\sqrt{2}} \text{Im} \left( (-i-1)e^{ia\sqrt{2}} + (-i+1)e^{-ia\sqrt{2}} \right) \\
&= -\frac{\pi}{2} \frac{e^{-a\sqrt{2}}}{16\sqrt{2}} \left( -\cos(a\sqrt{2}) - \sin(a\sqrt{2}) - \cos(-a\sqrt{2}) + \sin(-a\sqrt{2}) \right) \\
&= -\frac{\pi}{2} \frac{e^{-a\sqrt{2}}}{16\sqrt{2}} \left( -2\cos(a\sqrt{2}) - 2\sin(a\sqrt{2}) \right) \\
&= \frac{\pi e^{-a\sqrt{2}}(\cos(a\sqrt{2}) + \sin(a\sqrt{2}))}{16\sqrt{2}}
\end{aligned}$$

Hence, for  $a > 0$ ,

$$\int_0^\infty \frac{\cos(ax)}{x^4 + 16} dz = \frac{\pi e^{-a\sqrt{2}}(\cos(a\sqrt{2}) + \sin(a\sqrt{2}))}{16\sqrt{2}}$$

and for  $a < 0$ ,

$$\int_0^\infty \frac{\cos(ax)}{x^4 + 16} dz = -\frac{\pi e^{-|a|\sqrt{2}}(\cos(|a|\sqrt{2}) + \sin(|a|\sqrt{2}))}{16\sqrt{2}}$$

5. (a) The only singular point is  $z = 0$ .

$$\begin{aligned}
\text{Res}_{z=0} \frac{\exp(rz^n)}{z} &= \text{Res}_{z=0} \frac{1}{z} \left( 1 + rz^n + \frac{(rz^n)^2}{2} + \frac{(rz^n)^3}{3!} + \dots \right) \\
&= 1
\end{aligned}$$

Then, by Cauchy residue theorem,

$$\int_C \frac{\exp(rz^n)}{z} dz = 2\pi i$$

(b) Let the  $C$  be the equation  $e^{i\theta}$  for  $0 \leq \theta \leq 2\pi$ .

$$\begin{aligned} \int_C \frac{\exp(rz^n)}{z} dz &= \int_0^{2\pi} \frac{\exp(r(e^{i\theta})^n)}{e^{i\theta}} (ie^{i\theta}) d\theta \\ &= i \int_0^{2\pi} \exp(r(e^{i\theta})^n) d\theta \\ &= 2\pi i \end{aligned}$$

$$\therefore \int_0^{2\pi} \exp(r(e^{i\theta})^n) d\theta = 2\pi$$

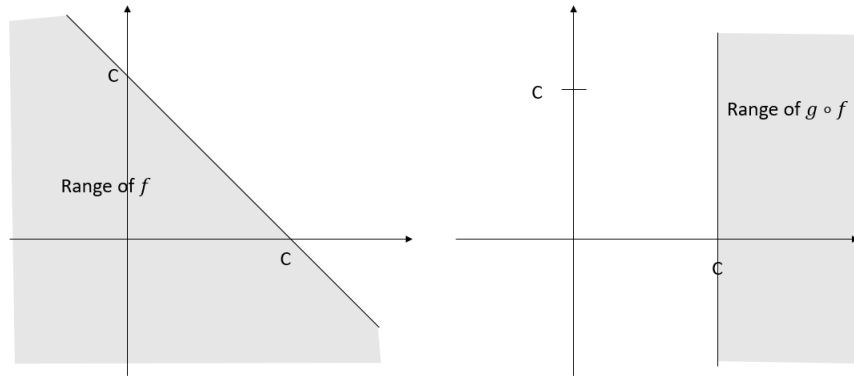
On the other hand,

$$\begin{aligned} \operatorname{Re} \left( \int_0^{2\pi} \exp(r(e^{i\theta})^n) d\theta \right) &= \operatorname{Re} \left( \int_0^{2\pi} \exp[\operatorname{Re}(r(e^{i\theta})^n) + i \operatorname{Im}(r(e^{i\theta})^n)] d\theta \right) \\ &= \operatorname{Re} \left( \int_0^{2\pi} \exp[r \cos(\theta n) + ir \sin(\theta n)] d\theta \right) \\ &= \int_0^{2\pi} \operatorname{Re} (\exp[r \cos(\theta n) + ir \sin(\theta n)]) d\theta \\ &= \int_0^{2\pi} \exp[r \cos(\theta n)] \operatorname{Re} (\exp[ir \sin(\theta n)]) d\theta \\ &= \int_0^{2\pi} \exp(r \cos(\theta n)) \cos(r \sin(\theta n)) d\theta \end{aligned}$$

Hence,

$$\int_0^{2\pi} \exp(r \cos(\theta n)) \cos(r \sin(\theta n)) d\theta = \operatorname{Re}(2\pi) = 2\pi$$

6. (a) Consider the function  $g(z) = (z - C)e^{i\frac{3\pi}{4}} + C$  (rotate  $\frac{3\pi}{4}$  anti-clockwise about  $C$ ).



Note that  $\operatorname{Re}(g(f(z))) \geq C$ .

Then, consider  $g_2(z) = \frac{1}{e^z}$ .

$$|g_2(g(f(z)))| = \frac{1}{|e^{g(f(z))}|} \leq \frac{1}{C}$$

And since all of  $f, g, g_2$  are analytic, and  $g_2(g(f(z)))$  is bounded, hence, by Liouville's Theorem,  $g_2(g(f(z)))$  is a constant, and hence,  $g(f(z))$  is a constant, and hence,  $f(z)$  is a constant.

(b) Consider the function

$$g(z) = \exp(f(z) - f(iz))$$

This function is analytic. We want to show it is bounded.

$$\begin{aligned} |g(z)| &= |\exp(f(z) - f(iz))| \\ &= \exp(\operatorname{Re}(f(z) - f(iz))) \\ &= \exp(u(x, y) - u(-y, x)) \\ &\leq \exp(C) \end{aligned}$$

Hence, by Liouville's Theorem,  $g(z) = \exp(f(z) - f(iz))$  is a constant. Hence,  $f(z) - f(iz) = k$  for some constant  $k \in \mathbb{C}$ .

Substitute  $z := 0$  into  $f(z) - f(iz) = k$  to get  $f(0) - f(0) = k$ , hence,  $k = 0$ . Hence,

$$f(z) = f(iz)$$