Calculus 19/20 Sem 1 Suggested Answers

NUS LaTeXify Proj Team

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Done by: Yip Jung Hon and Pan Jing Bin

Question 1

a) True. First note that $f(x)$ is continuous on $(0, \pi/2)$. Then $f'(x) = 1 + \frac{\cos x}{\sin x} = 1 + \cot x$. On $(0, \pi/2)$, $\cot x > 0 \implies f'(x) > 0$. Thus, on $(0, \pi/2), f$ is continuous and strictly increasing and so f has an inverse.

b) False. Consider $f(x) = x^3$. Then $f'(x) = 3x^2$. f is an increasing function, but $f'(0) = 0$. Thus the statement is false.

c) False. Consider $f(x) = x^4$. Then $f'(x) = 4x^3$ and $f''(x) = 12x^2$. Note that f has a local minimum at $x = 0$ but $f''(0) = 0$ thus the statement is false.

d) False. f can be a strictly increasing function, in which case $f'(c) \neq 0\ \forall x \in (0, 1)$.

e) True. First note that since f is continuous on $[-1, 1]$, $f(x)$ is defined $\forall x \in [-1, 1]$. Thus $\int_0^1 f(x)dx$ and $\int_{-1}^0 f(x)dx$ exists.

Since f is odd, then $f(x) = -f(-x)$. Then one has:

$$\begin{aligned}
\int_{-1}^1 f(x)dx &= \int_0^1 f(x)dx + \int_{-1}^0 f(x)dx \\
&= \int_0^1 f(x)dx + \int_{-1}^0 -f(-x)dx \\
&= \int_0^1 f(x)dx + \int_1^0 f(u)du \quad (\text{Sub } u = -x,\ \frac{du}{dx} = -1) \\
&= \int_0^1 f(x)dx - \int_0^1 f(u)du \\
&= 0
\end{aligned}$$

Question 2

a) $\sin(x^2+1) + 2x^2\cos(x^2+1)$

b) $D = \int_0^{\pi/2} |\cos(2t)|dt = 1.$

c) $\int_0^1 f(x)dx \approx 0.25 \times 1 + 0.25 \times 0 + 0.25 \times 2 + 0.25 \times 0 = 0.75.$

d) $f(0.0001) \approx f(0) + 0.0001f'(0) = 1.0002.$

e) $e^{2a}\tan(a)$

$$\begin{aligned}
\int_0^a e^{2x}(+1)^2\ dx &= \int_0^a e^{2x}\tan^2 x\ dx + \int_0^a e^{2x} 2\tan x\ dx + \int_0^a e^{2x}\ dx \\
&= \int_0^a e^{2x}\sec^2 x\ dx + \int_0^a 2e^{2x}\tan x\ dx \\
&= \int_0^a e^{2x}\sec^2 x\ dx + e^{2x}\tan x\Big|_0^a - \int_0^a e^{2x}\sec^2 x\ dx \quad \text{(By IBP on } 2e^{2x}\tan x\text{)} \\
&= e^{2a}\tan(a)
\end{aligned}$$

Question 3

i) Define $h(x) = f(x) - g(x)$. Then $h(x) = -3x^2 - 12x + 1 - x^3/3 - \sin(2x)$.
$h(0) = 1, h(1) = -\frac{43}{3} - \sin(2) < 0$.
Since $h(0) = 1 > 0$ and $h(1) < 0$ and h is continuous on $\mathbb{R}$, by IVT, $\exists\ c \in (0,1)$ such that $h(c) = 0 \implies f(c) = g(c)$.

ii) We first suppose there 2 roots. Assume that $\exists\ c_1, c_2 \in \mathbb{R}$ such that $h(c_1) = h(c_2) = 0$. WLOG, let. $c_2 > c_1$.

$$\begin{aligned}
h'(x) &= -6x - 12 - x^2 - 2\cos(2x) \\
&= -(x+3)^2 - 2\cos(2x) - 3
\end{aligned}$$

Note that $-(x+3)^2 \leqslant 0$ and $-2\cos(2x) - 3 < 0$. Thus $h'(x) < 0\ \forall x \in \mathbb{R}$.
Since h is continuous and differentiable on $\mathbb{R}$, by the MVT, $\exists d \in (c_1, c_2)$ such that $h'(d) = 0$. This is a contradiction as we have just shown $h'(x) < 0$.

Question 4

$f(x) = (x-6)^{2/3}(5-x)^{2/3}$.

$$f'(x) = \frac{16-3x}{3(x-6)^{1/3}(5-x)^{2/3}} \tag{1}$$

Note that $(5-x)^{2/3}$ is always positive.

(i) When $f'(x) > 0$, you need $16-3x > 0$ and $x-6 > 0 \implies x < 16/3$ and $x > 6$. You could also have $16-3x < 0$ and $x-6 < 0 \implies x > 16/3$ and $x < 6$.

For $f'(x) < 0$, you need $16-3x < 0$ and $x-6 > 0 \implies x > 16/3$ and $x > 6$. You could also have $16-3x > 0$ and $x-6 < 0 \implies x < 16/3$ and $x < 6$.

We also have to check $f'(x)$ at the points near where $f'(x)$ is undefined:

x	5^-	5^+	6^-	6^+
$f'(x)$	-ve	-ve	+ve	-ve

Thus, we have that $f(x)$ is increasing on the open interval $(16/3, 6)$ and $f(x)$ is decreasing on the open interval $(-\infty, 16/3)$ and $(6, \infty)$.

(ii) When $f'(x) = 0, x = 16/3$.

x	$16/3^-$	$16/3^+$
$f'(x)$	-ve	+ve

From the table in part (i), we also note that 6 is a local maximum point. We have:

Local min: $(16/3, -\frac{1}{3}(2^{2/3}))$
Local max: $(6, 0)$

Question 5

a) Let $\epsilon > 0$. Choose $\delta = \min\{\epsilon/7, 1\}$. Then whenever $0 < |x-1| < \delta$,

$$\begin{aligned}
\left|\frac{1}{x^3+1} - \frac{1}{2}\right| &= \left|\frac{2 - x^3 - 1}{x^3+1}\right| \\
&= \left|\frac{1-x^3}{x^3+1}\right| \\
&= \left|\frac{x^3-1}{x^3+1}\right| \\
&= \left|\frac{(x-1)(x^2+x+1)}{x^3+1}\right| \\
&= \left|\frac{(x-1)[(x-1)^2+3(x-1)+3]}{x^3+1}\right| \\
&\leqslant \left|\frac{|x-1|(|x-1|^2+3|x-1|+3)}{x^3+1}\right| \\
&< \frac{\delta(\delta^2+3\delta+3)}{1} \\
&\leqslant 7\delta \\
&\leqslant \epsilon
\end{aligned}$$

b) Define $f(t) = \sqrt{t^2+t+1} - \sqrt{t^2-t}$

$$\lim_{x\to\infty} \frac{\int_1^x (\sqrt{t^2+t+1} - \sqrt{t^2-t})(x-t)dt}{x^2+x+1} = \lim_{x\to\infty} \frac{x\int_1^x f(t)dt - \int_1^x tf(t)dt}{x^2+x+1}$$

Since the top and bottom goes to infinity, we can apply L'Hopital's Rule. The term $x\int_1^x f(t)dt$ can be differenciated using the product rule.

$$\begin{aligned}
&= \lim_{x\to\infty} \frac{xf(x) + \int_1^x f(t)dt - xf(x)}{2x+1} \\
&= \lim_{x\to\infty} \frac{\int_1^x f(t)dt}{2x+1}
\end{aligned}$$

The top and bottom goes to infinity, and we apply L'Hopital's Rule again.

$$\begin{aligned}
&= \lim_{x\to\infty} \frac{f(x)}{2} \\
&= \lim_{x\to\infty} \frac{\sqrt{x^2+x+1} - \sqrt{x^2-x}}{2}
\end{aligned}$$

The trick to evaluating these kinds of limits is to rationalise the surd and mutiply $1/x$ to both the top and bottom.

$$\begin{aligned}
&= \lim_{x\to\infty} \frac{\sqrt{x^2+x+1} - \sqrt{x^2-x}}{2} . \frac{\sqrt{x^2+x+1} + \sqrt{x^2+x}}{\sqrt{x^2+x+1} + \sqrt{x^2+x}} \\
&= \lim_{x\to\infty} \frac{2x+1}{2\left[\sqrt{x^2+x+1} + \sqrt{x^2-x}\right]} \\
&= \lim_{x\to\infty} \frac{2+1/x}{2\left[\sqrt{1+1/x+1/x^2} + \sqrt{1-1/x}\right]} \\
&= \frac{2}{2(1+1)} = \frac{1}{2}
\end{aligned}$$

Question 6

a) We first implicitly differenciate the curve. We obtain:

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}\frac{dy}{dx}y^{-1/3} = 0$$
$$x^{-1/3} + \frac{dy}{dx}y^{-1/3} = 0$$
$$\frac{dy}{dx} = -\frac{-y^{1/3}}{x^{1/3}}$$

So one has:

$$1 + (\frac{dy}{dx})^2 = \frac{y^{2/3}}{x^{2/3}} + \frac{x^{2/3}}{x^{2/3}} = \frac{1}{x^{2/3}}$$

We first calculate the length of the curve, L, for just the first quadrant.

$$\begin{aligned}\frac{1}{4}L &= \int_0^1 \sqrt{1 + f'(x)}\ dx \\ &= \int_0^1 \sqrt{\frac{1}{x^{2/3}}}\ dx \\ &= \int_0^1 \frac{1}{x^{1/3}}\ dx \\ &= \frac{3}{2}x^{2/3}\Big|_0^1 \\ &= \frac{3}{2}\end{aligned}$$

So $L = 6$.

b) Using Surface Area $= \int 2\pi f(x)\sqrt{1 + f'(x)}\ dx$.

$$\text{Surface Area} = \int_{-\pi/2}^{\pi/2} 2\pi \cos x\sqrt{1 + \sin^2 x}\ dx$$

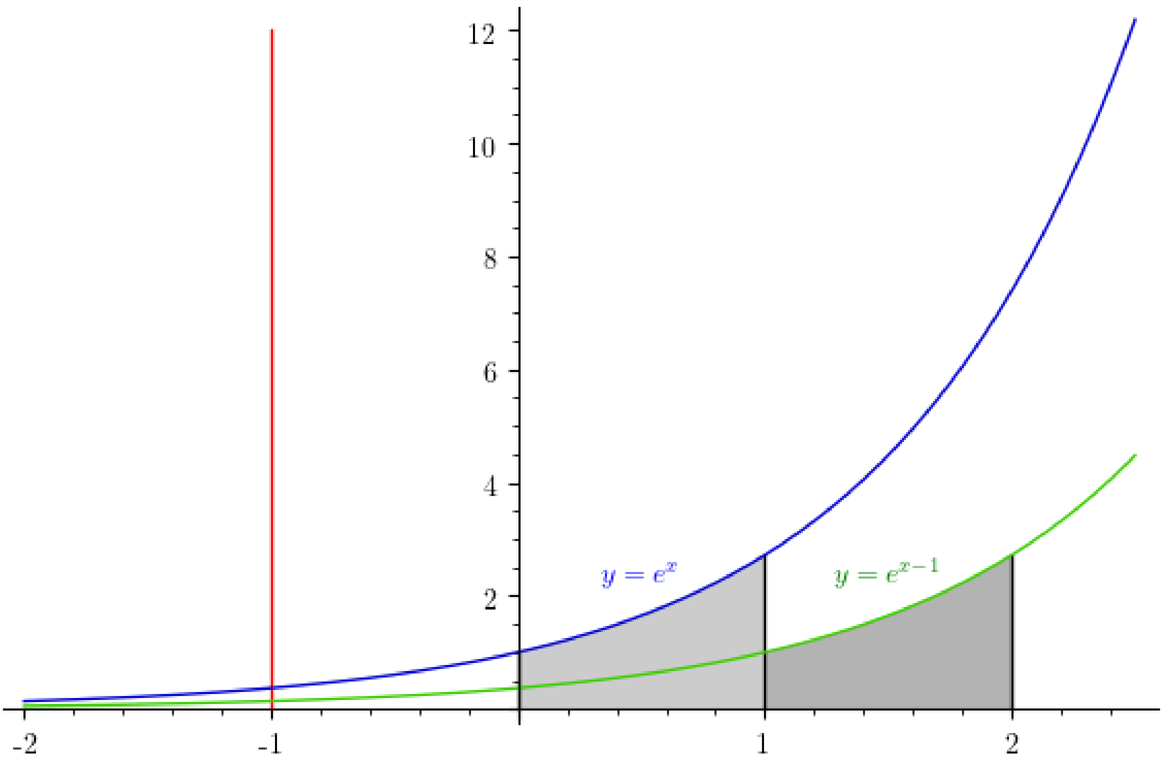
Let $u = \sin x, du/dx = \cos x$,

$$\begin{aligned}&= \int_{-1}^{1} 2\pi \cos x\sqrt{1 + u^2}\ \frac{du}{\cos x} \\ &= \int_{-1}^{1} 2\pi\sqrt{1 + u^2}\ du\end{aligned}$$

Let $u = \tan x, du/dx = \sec^2 x$.

$$\begin{aligned}&= \int_{-\pi/4}^{\pi/4} 2\pi \sec x \sec^2 x\ dx \\ &= 2\pi \int_{-\pi/4}^{\pi/4} \sec^3 x\ dx \\ &= 2\pi \left[\frac{1}{2}\sec\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{4}\right) + \frac{1}{2}\ln\left(\sec\left(\frac{\pi}{4}\right) + \tan\left(\frac{\pi}{4}\right)\right)\right] - \\ &\quad 2\pi \left[\frac{1}{2}\sec\left(-\frac{\pi}{4}\right)\tan\left(-\frac{\pi}{4}\right) + \frac{1}{2}\ln\left(\sec\left(-\frac{\pi}{4}\right) + \tan\left(-\frac{\pi}{4}\right)\right)\right] \\ &= \pi\left[\sqrt{2} + \ln(\sqrt{2}+1) - (\sqrt{2} + \ln(\sqrt{2}-1)\right] \\ &= \pi\left[2\sqrt{2} + \ln\frac{\sqrt{2}+1}{\sqrt{2}-1}\right]\end{aligned}$$

c) Note that this is equivalent to finding the volume of $y = e^{x-1}$ revolving about $x = 0$ from $x = 1$ to $x = 2$.



By method of cylindrical shells,

$$\text{Volume} = \int_1^2 2\pi x e^{x-1}\ dx$$

Let $u = x - 1$,

$$\begin{aligned}
&= \int_0^1 2\pi(u+1)e^u\ du \\
&= 2\pi\left[\int_0^1 ue^u\ du + \int_0^1 e^u\ du\right] \\
&= 2\pi\left[ue^u\Big|_0^1 - \int_0^1 e^u\ du + \int_0^1 e^u\ du\right] \quad \text{(By IBP on } ue^u\text{)} \\
&= 2\pi e
\end{aligned}$$

Question 7

a) This is a homogeneous DE.

$$\frac{dy}{dx} = \frac{x^2 - y^2}{2xy} = \frac{1}{2}\left(\frac{x}{y}\right) - \frac{1}{2}\left(\frac{y}{x}\right)$$

Substitute $z = y/x$, then $\frac{dy}{dx} = z + x\frac{dz}{dx}$.

$$\begin{aligned}
&\frac{dz}{dx}x + z = \frac{1}{2z} - \frac{1}{2}z \\
&\frac{dz}{dx}x = \frac{1}{2}\left(\frac{1-3z^2}{z}\right) \\
&\frac{z}{1-3z^2}\frac{dz}{dx} = \frac{1}{2x} \\
&\int \frac{z}{1-3z^2}dz = \frac{1}{2}\int \frac{1}{x}dx \\
&-\frac{1}{3}\ln|1-3z^2| = \ln x + C \quad (x > 0) \\
&\ln\left|1 - 3\left(\frac{y}{x}\right)^2\right| = -3\ln x - 3C \\
&\left|1 - 3\left(\frac{y}{x}\right)^2\right| = e^{-3\ln x - 3C} \\
&1 - 3(\frac{y}{x})^2 = Ae^{-3\ln x} \quad (\text{A}=\pm e^{-3C}) \\
&1 - 3\left(\frac{y}{x}\right)^2 = \frac{A}{x^3} \\
&3\left(\frac{y}{x}\right)^2 = 1 - \frac{A}{x^3} \\
&y^2 = \frac{1}{3}x^2 - \frac{A}{3x}
\end{aligned}$$

When $x = 1, y = 1, A = -2$.

$$\begin{aligned}
&y^2 = \frac{1}{3}x^2 + \frac{2}{3x} \\
&y = \pm\sqrt{\frac{1}{3}x^2 + \frac{2}{3x}} \\
&y = \sqrt{\frac{1}{3}x^2 + \frac{2}{3x}} \quad (\text{Rej -ve, } y > 0)
\end{aligned}$$

b) Let x be the fertiliser in the tank at time t. Let V be the volume of water in the tank at time t.
Rate of adding fertiliser $= 0.1 * 4 = 0.4$.
Rate of fertiliser pumped out $= 12x/V$.
Note that $V = 400 - 8t$.

$$\begin{aligned}
&\frac{dx}{dt} = 0.4 - \frac{12x}{400-8t} \\
&\frac{dx}{dt} + \frac{12x}{400-8t} = 0.4
\end{aligned}$$

Find integrating factor:

$$\begin{aligned}
e^{\int \frac{12}{400-8t}dt} &= e^{-\frac{3}{2}\ln|400-8t|} \\
&= (400-8t)^{-\frac{3}{2}} \quad (\text{Note that } 400 - 8t \geqslant 0).
\end{aligned}$$

Mutiplying $(400-8t)^{-\frac{3}{2}}$ to both sides and integrating, we get:

$$\begin{aligned}
(400-8t)^{-\frac{3}{2}}x &= \int 0.4(400-8t)^{-\frac{3}{2}}dt \\
&= \frac{1}{10}(400-8t)^{-\frac{1}{2}} + C \\
x &= \frac{1}{10}(400-8t) + C(400-8t)^{\frac{3}{2}}.
\end{aligned}$$

When $t = 0, x = 0$.

$$0 = 40 + C(400)^{\frac{3}{2}} \implies C = -\frac{1}{200}$$

.

So one has that,

$$x = 40 - \frac{4}{5}t - \frac{(400-8t)^{\frac{3}{2}}}{200}$$

At maximum fertiliser, $\frac{dx}{dt} = 0$.

$$\begin{aligned}
&0.4 = \frac{12x}{400-8t} \\
&x = \frac{400-8t}{30} \\
&\frac{400-8t}{30} = 40 - \frac{4}{5}t - \frac{(400-8t)^{\frac{3}{2}}}{200} \\
&(400-8t)^{\frac{3}{2}} = \frac{40}{3}(400-8t)
\end{aligned}$$

Moving all the terms to one side and factorising,

$$\begin{aligned}
&400 - 8t = 0 \text{ or } (400-8t)^{\frac{1}{2}} = \frac{40}{3} \\
&t = 50 \text{ or } 400 - 8t = \frac{1600}{9} \\
&t = \frac{250}{9}
\end{aligned}$$

Note that when $t \geqslant 50, x = 0$ as all the water has been drained away.

At $t = 250/9, x = 160/27$.

Thus the maximum amount of fertiliser in the tank is $x = 160/27$ and the time required to reach it is $t = 250/9$.

Question 8

Note the following identities:

$$\sin^2(x/2) = \frac{1-\cos x}{2} \qquad \cos^2(x/2) = \frac{1+\cos x}{2}$$

Also note that $1 + a\cos x = A(1+\cos x) + B(1-\cos x)$ where $A + B = 1, A - B = a$.

Solving, we have $A = \frac{1}{2}(1+a), B = \frac{1}{2}(1-a)$.

So we can write:

$$\begin{aligned}
\frac{1}{1+a\cos x} &= \frac{1}{\frac{1}{2}(1+a)(1+\cos x) + \frac{1}{2}(1-a)(1-\cos x)} \\
&= \frac{1}{(1+a)\cos^2(x/2) + (1-a)\sin^2(x/2)} \\
&= \frac{1}{1+a}\frac{\sec^2(x/2)}{1+\frac{1-a}{1+a}\tan^2(x/2)}
\end{aligned}$$

If $a = 1$, then we have:

$$\int \frac{1}{1+a}\sec^2(x/2)\ dx = \int \frac{1}{2}\sec^2(x/2)\ dx = \tan(x/2) + c$$

If $a < 1$, sub $t = \sqrt{\frac{1-a}{1+a}}\tan(x/2)$, then $\frac{dx}{dt} = \sqrt{\frac{1+a}{1-a}}\frac{2}{\sec^2(x/2)}$.

$$\begin{aligned}
\int \frac{1}{1+a}\frac{\sec^2(x/2)}{1+\frac{1-a}{1+a}\tan^2(x/2)}\ dx &= \int \frac{1}{1+a}\frac{\sec^2(x/2)}{1+t^2}\sqrt{\frac{1+a}{1-a}}\frac{2}{\sec^2(x/2)}\ dt \\
&= \int \frac{1}{\sqrt{(1+a)(1-a)}}\frac{2}{1+t^2}\ dt \\
&= \int \frac{1}{\sqrt{1-a^2}}\frac{2}{1+t^2}\ dt \\
&= \frac{2\tan^{-1}t}{\sqrt{1-a^2}} + c \\
&= \frac{2\tan^{-1}\left[\sqrt{\frac{1-a}{1+a}}\tan(\frac{x}{2})\right]}{\sqrt{1-a^2}} + c
\end{aligned}$$

Else, if $a > 1$, sub $t = \sqrt{\frac{a-1}{1+a}}\tan(x/2)$, then $\frac{dx}{dt} = \sqrt{\frac{1+a}{a-1}}\frac{2}{\sec^2(x/2)}$.

$$\begin{aligned}
\int \frac{1}{1+a}\frac{\sec^2(x/2)}{1+\frac{1-a}{1+a}\tan^2(x/2)}\ dx &= \int \frac{1}{1+a}\frac{\sec^2(x/2)}{1-t^2}\sqrt{\frac{1+a}{a-1}}\frac{2}{\sec^2(x/2)}\ dt \\
&= \int \frac{1}{\sqrt{(1+a)(a-1)}}\frac{2}{1-t^2}\ dt \\
&= \int \frac{1}{\sqrt{a^2-1}}\frac{2}{1-t^2}\ dt \\
&= \frac{1}{\sqrt{a^2-1}}\ln\left|\frac{t+1}{t-1}\right| + c \\
&= \frac{1}{\sqrt{a^2-1}}\ln\left|\frac{\sqrt{\frac{a-1}{1+a}}\tan(\frac{x}{2})+1}{\sqrt{\frac{a-1}{1+a}}\tan(\frac{x}{2})-1}\right| + c
\end{aligned}$$