For Question 1 and Question 2, only final answers are required. No working steps are required. Write your answer in **answer books.**

Question 1

[10 marks]

Indicate if the following statements are **true** or **false**.

(a) The function $f(x) = x + \ln(\sin x)$ has an inverse when x is restricted to $\left(0, \frac{\pi}{2}\right)$.

(b) If f is an increasing function and f is differentiable, then $f'(x) > 0$ for all x in the domain of f.

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$f'(x) = 4x^3$
$f''(x) = 12x^2.$

(c) If f is twice differentiable on $\mathbb{R}$ and attains its local min at $x = c$, then $f'(c) = 0$ and $f''(c) > 0$.

(d) If f is continuous on $[0, 1]$ and differentiable on $(0, 1)$, then $f'(c) = 0$ for some $c \in (0, 1)$.

(e) If f is a continuous, odd function on $[-1, 1]$, then $\displaystyle\int_{-1}^{1} f(x)\,dx = 0$.

Question 2

[10 marks]

(a) Let $f(x) = x\sin(x^2 + 1)$. Write down a formula for $f'(x)$.

(b) A particle is travelling in a straight line. It's velocity at time t is $v(t) = \cos(2t)$. Find the **total distance** travelled by the particle between $t = 0$ and $t = \frac{\pi}{2}$.

(c) Let f be a continuous function on $[0, 1]$ such that $f(0.1) = 1$, $f(0.3) = 0$, $f(0.6) = 2$, and $f(0.9) = 0$. Find an approximate value of $\displaystyle\int_0^1 f(x)\,dx$ using the Riemann sum with four intervals of equal length.

(d) Suppose that $f(0) = 1$ and $f'(0) = 2$. Find an approximation of $f(0.0001)$ using linear approximation at $x = 0$.

$\sec^2 x + 2\tan x$

(e) Suppose a is a positive real number. Evaluate $\displaystyle\int_0^a e^{2x}(\tan x + 1)^2\,dx$. Give your answer in terms of a.

$\tan(\pi + a) = \tan(a).$

let $x = \pi + u$

$[e^{2x} \tan x]_0^a - \int_0^a 2e^{2x} \tan x\,dx$

$\int_{-\pi}^{a-\pi} e^{\pi + u} (\tan u + 1)^2\,du$

$u =$

$= \int e^{\pi} e^{u} (\tan u + 1)^2\,du$

$\int$

$\frac{1}{2} e^{2x}(\tan x + 1)^2 - \int \frac{1}{2} e^{2x} (2)(\tan x + 1)(\sec^2 x)$

$\tan$

$\frac{1}{2} e^{2x}(\tan x + 1)^2 - \int e^{2x}$

For **Question 3 – Question 8, working steps are required.** Write your answer in **answer books.**

Question 3 [6 marks]

Let $f(x) = -3x^2 - 12x + 1$ and $g(x) = \dfrac{x^3}{3} + \sin(2x)$.

(i) Show that the graph of f and the graph of g intersect **at least once**.

(ii) Show that the graph of f and the graph of g intersect **at most once**.

Question 4 [6 marks]

For the function $f(x) = (x-6)^{2/3}(5-x)^{1/3}$,

(i) Find the open intervals on which it is increasing and decreasing;

(ii) Find the coordinates of all its local maximum and minimum points.

Question 5 [17 marks]

(a) Prove the following limit using only the precise definition:

$$\lim_{x\to 1} \frac{1}{x^3+1} = \frac{1}{2}.$$

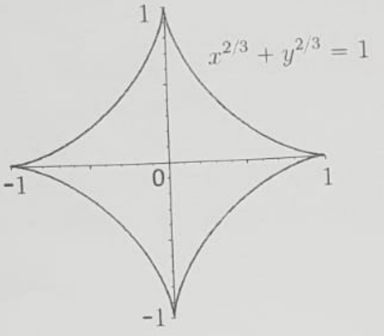
(b) Find the limit of

$$\lim_{x\to\infty} \frac{\int_1^x \left(\sqrt{t^2+t+1} - \sqrt{t^2-t}\right)(x-t)\,dt}{x^2+x+1}.$$

Question 6 [18 marks]

(a) Find the **length of the curve**

$$x^{2/3} + y^{2/3} = 1, \quad -1 \leqslant x \leqslant 1, -1 \leqslant y \leqslant 1.$$



(b) Find the **area of the surface** obtained by rotating the curve

$$y = \cos(x), \quad -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2},$$

about the x-axis.

You may use the following formulas without proof.

$$\int \sec x \, dx = \ln \left| \frac{1 + \tan(x/2)}{1 - \tan(x/2)} \right| + C, \quad \int \sec^3 x \, dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x| + C.$$

(c) Let R be the region enclosed by the curve

$$y = e^x, \quad 0 \leqslant x \leqslant 1,$$

the x-axis, the y-axis and the line $x = 1$. Find the **volume of the solid** generated by revolving R about the line $x = -1$.

[18 marks]

Question 7

(a) Solve the following initial value problem. **Give your answer in terms of** $y = f(x)$.

$$\frac{dy}{dx} = \frac{x^2 - y^2}{2xy}, \quad x > 0,\ y > 0,\ y(1) = 1.$$

(b) A tank contains 400 L of fresh water. A solution containing 0.1 kg/L of soluble lawn fertilizer runs into the tank at the rate of 4 L/min, and the mixture is pumped out of the tank at the rate of 12 L/min.

(i) Find the expression of the fertilizer in the tank at time t min.

(ii) Find the maximum amount of fertilizer in the tank and the time required to reach the maximum.

[15 marks]

Question 8

Suppose $a > 0$ is a real number. Find the indefinite integral

$$\int \frac{dx}{1 + a\cos(x)}.$$

(Hint: Consider 3 cases: $a > 1$, $0 < a < 1$, and $a = 1$.)

— **END OF PAPER** —