MA3236 AY1819 Sem 1 Answers

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Question 1

By KKT first order necessary condition, we have:

$$\nabla f(\bar{x}) + \lambda \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} + \mu^T \begin{pmatrix} -1 \\ \vdots \\ -1 \end{pmatrix} = 0$$

Where μ is a vector such that $\mu \geq 0$, and $\mu_i = 0$ if $\bar{x}_i \neq 0$ ($x_i \geq 0$ is not active).

Hence, we consider the j-th coordinate:

$$\delta_j + \lambda - \mu_j = 0$$

Hence, we can choose the scalar $\kappa = -\lambda$, then

$$\delta_j + \lambda - \mu_j = 0 \implies \delta_j + \lambda \geq 0 \implies \delta_j \geq \kappa$$

$$\mu_i = 0 \text{ if } \bar{x}_i \neq 0 \implies (\mu_i)\bar{x}_i = 0 \implies (\delta_j + \lambda)\bar{x}_i = 0$$

Hence, κ satisfies the conditions.

Question 2

WLOG, we can assume $a = 0$. This is because we can consider another function $\phi_2(x) = \phi(x + a)$ and $f_2(x) = f(x - a)$, so that $\phi_2 = f_2(x) + ||x||_2^2$.

WLOG, we can also assume $f(0) = 0$, because we can consider another function $\phi_2(x) = \phi(x) - f(0)$.

Hence, it now suffices to prove that $\phi(x) = f(x) + ||x||_2^2$ is coercive given an extra condition that $f(0) = 0$.

Lemma 1. *For x with $||x|| \geq 1$, $f(x) \geq ||x|| f(\frac{x}{||x||})$*

Proof. Since f is convex, hence,

$$\frac{1}{||x||} f(x) + \frac{||x|| - 1}{||x||} f(0) \geq f(\frac{x}{||x||})$$

And since $f(0) = 0$,

$$\therefore f(x) \geq ||x|| f(\frac{x}{||x||})$$

□

Hence, we can let $y := \arg\min_{||x||=1}(f(x))$. Then, by the above lemma, for all x with $||x|| > 1$, we have $f(x) \geq ||x||f(\frac{x}{||x||})$, and since y minimizes $f(\frac{x}{||x||})$, hence, we have $f(x) \geq ||x||f(\frac{x}{||x||}) \geq ||x||f(y)$. Hence,

$$\phi(x) = f(x) + ||x||_2^2 \geq ||x||f(y) + ||x||_2^2$$

Which is quadratic in terms of $||x||$. Hence, as $||x|| \to \infty$, $\phi(x) \to \infty$.

Question 3

(a) Yes. The function is continuous. Since $(x_1 - 1)^2 + x_2^2 = 0$, hence $|x_1| \leq 100$, $|x_2| \leq 100$, which is bounded. The constraints are all closed as well. Hence, The feasible set is closed and bounded, which means the optimal solution exists.

(b) Yes.

$$\nabla g_1(x) = \begin{pmatrix} 2(x_1 - 1) \\ 2x_2 \\ 0 \end{pmatrix}$$

$$\nabla g_2(x) = \begin{pmatrix} 0 \\ 1 \\ -3x_3^2 \end{pmatrix}$$

Case 1: both g_1 and g_2 are non-zero

Then, they have to be not linearly independent. Hence, there must exist some scalar such that $\nabla g_1(x) = k\nabla g_2(x)$

$$\begin{pmatrix} 2(x_1 - 1) \\ 2x_2 \\ 0 \end{pmatrix} = k \begin{pmatrix} 0 \\ 1 \\ -3x_3^2 \end{pmatrix}$$

Hence, $k = 2x_2, x_1 = 1, x_3 = 0$.

$x_3 = 0$ implies that $x_2 = 0$ (because of g_2). But $(1, 0, 0)$ is not feasible (because of g_1). Hence, all feasible points are regular for this case.

Case 2: $\nabla g_1(x) = 0$

Then, $x_1 = 1, x_2 = 0$. But this is not feasible (because of g_1). Hence, all feasible points are regular for this case.

(c)

$$\nabla f(x) = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$$

KKT first order necessary condition:

$$\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2(x_1 - 1) \\ 2x_2 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ -3x_3^2 \end{pmatrix} = 0$$

Hence, $\lambda_2 = 0$.

$$\begin{pmatrix} 2 \\ -1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2(x_1 - 1) \\ 2x_2 \end{pmatrix} = 0$$

Hence, $(x_1 - 1) = -2x_2$. Substitute this into g_1:

$$(-2x_2)^2 + x_2^2 - 4 = 0 \implies x_2 = \pm\sqrt{\frac{4}{5}}$$

$$x_1 = -2x_2 + 1 = \mp 2\sqrt{\frac{4}{5}} + 1$$

The KKT points are $(\sqrt{\frac{4}{5}}, -2\sqrt{\frac{4}{5}} + 1, \sqrt[3]{-2\sqrt{\frac{4}{5}} + 1})$ and $(-\sqrt{\frac{4}{5}}, 2\sqrt{\frac{4}{5}} + 1, \sqrt[3]{2\sqrt{\frac{4}{5}} + 1})$.

(d) Sub $(x_1, x_2) = (\sqrt{\frac{4}{5}}, -2\sqrt{\frac{4}{5}} + 1)$ into f:

$$f(x) = 4\sqrt{\frac{4}{5}} - 1$$

Sub $(x_1, x_2) = (-\sqrt{\frac{4}{5}}, 2\sqrt{\frac{4}{5}} + 1)$ into f:

$$f(x) = -4\sqrt{\frac{4}{5}} - 1$$

The optimal is $-4\sqrt{\frac{4}{5}} - 1$.

Question 4

If $d = 0$, then $\phi(t) = 0$, a constant, which is clearly monotone increasing. Hence, WLOG, we can now assume that $d \neq 0$.

WLOG, we can assume $x = 0$. This is because we can define another $f_2(y) = f(y + x)$ which is also convex, and $\phi(t) = \frac{f_2(td) - f_2(0)}{t}$.

WLOG, we can also assume $f(0) = 0$. This is because we can define another $f_2(y) = f(y) - f(0)$ which is also convex, and $\phi(t) = \frac{f_2(td)}{t}$.

Hence, it now suffices to show that

$$\phi(t) = \frac{f(td)}{t}$$

is monotone increasing.

Let $t_1 > t_2 > 0$ be numbers in $\mathbb{R}_+$. Consider the two points at $t_1 d$ and 0. By convexity of f, we know that

$$\frac{t_2}{t_1} f(t_1 d) + \frac{t_1 - t_2}{t_1} f(0) \geq f(t_2 d)$$

Since $t_1 > t_2$, hence,

$$f(t_1 d) \geq \frac{t_2}{t_1} f(t_1 d) \geq f(t_2 d)$$

Hence,

$$\phi(t_1) \geq \phi(t_2)$$

Hence, ϕ is monotone increasing.

Question 5

$$\nabla f(x) = \begin{pmatrix} 2x_1 - x_2 - 2 \\ 2x_2 - x_1 - 3 \end{pmatrix}$$

First iteration:

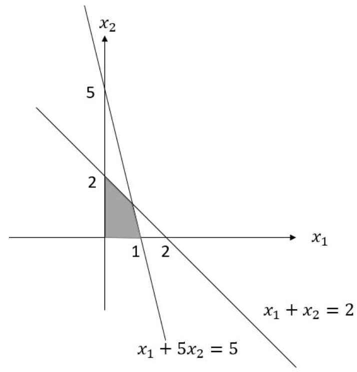

Figure 1: Plot of feasible area (shaded)

Want to solve for $\min z(x) = f(\begin{pmatrix}5/4\\3/4\end{pmatrix}) + \nabla f(\begin{pmatrix}5/4\\3/4\end{pmatrix})^T(x - \begin{pmatrix}5/4\\3/4\end{pmatrix})$.

$$\begin{aligned} z(x) &= -\frac{57}{16} + \begin{pmatrix}-1/4\\-11/4\end{pmatrix}^T (x - \begin{pmatrix}5/4\\3/4\end{pmatrix}) \\ &= -\frac{19}{16} - \begin{pmatrix}1/4\\11/4\end{pmatrix}^T (x) \end{aligned}$$

It suffices to check the four points $(0,0), (2,0), (0,1), (\frac{5}{4}, \frac{3}{4})$, and we can find that $z(x)$ is minimum at $(0,1)$.
Then, direction $d = (0,1) - (\frac{5}{4}, \frac{3}{4}) = (-\frac{5}{4}, \frac{1}{4})$.
Now, we use line search to find $\min\{\psi(t) = f((\frac{5}{4}, \frac{3}{4}) + td) : 0 \le t \le 1\}$

$$x_1 = \frac{5}{4} - \frac{5}{4}t$$

$$x_2 = \frac{3}{4} + \frac{1}{4}t$$

$$f((x_1,x_2)) = (\frac{5}{4} - \frac{5}{4}t)^2 + (\frac{3}{4} + \frac{1}{4}t)^2 - (\frac{5}{4} - \frac{5}{4}t)(\frac{3}{4} + \frac{1}{4}t) - 2(\frac{5}{4} - \frac{5}{4}t) - 3(\frac{3}{4} + \frac{1}{4}t)$$

$$f((x_1,x_2)) = \frac{1}{16}[(5-5t)^2 + (3+t)^2 - (5-5t)(3+t) - 8(5-5t) - 3(3+t)]$$

$$\psi'(t) = \frac{1}{16}[-5(5-5t) + (3+t) + 5(3+t) - (5-5t) + 40 - 3]$$

$$\psi'(t) = \frac{1}{16}[36t + 25]$$

Hence, $\psi(t)$ is minimum at $t = -\frac{25}{36}$. The point is updated from $(\frac{5}{4}, \frac{3}{4})$ to $(\frac{5}{4}, \frac{3}{4}) - \frac{25}{36}(-\frac{5}{4}, \frac{1}{4}) = (\frac{305}{144}, \frac{83}{144})$.
Second iteration:
(i dunno how to continue, the numbers are very ugly)

Question 6

(i)

$$\begin{aligned} \theta(\lambda) &= \inf_{x \in X}\{f(x) + \lambda g(x)\} \\ &= \inf_{x \in X}\{2x_1 + 3x_2 + \lambda(x_1 + 3x_2 - 3)\} \end{aligned}$$

Dual problem:

$$\max_{\lambda\in\mathbb{R}}\{\theta(\lambda)\}$$

(ii) To find $\inf_{x\in X}\{2x_1+3x_2+\lambda(x_1+3x_2-3)\}$, since the function is linear, it suffices to check the "corners" of the set X, which are the points $(0,0),(1,0),(0,2)$.

$$\begin{aligned}\inf_{x\in X}\{2x_1+3x_2+\lambda(x_1+3x_2-3)\} &= \inf_{x\in X}\{(2+\lambda)x_1+(3+3\lambda)x_2\}-3\lambda\\ &= \inf_{x\in\{(0,0),(1,0),(0,2)\}}\{(2+\lambda)x_1+(3+3\lambda)x_2\}-3\lambda\\ &= \inf\{0,2+\lambda,6+6\lambda\}-3\lambda\end{aligned}$$

$$\therefore\theta(\lambda)=\begin{cases}6+3\lambda & \text{if } \lambda\le -1\\ -3\lambda & \text{otherwise}\end{cases}$$

We calculate the max of each case above:

$$\max\{6+3\lambda|\lambda\le -1\}=3$$

$$\max\{-3\lambda|\lambda\ge -1\}=3$$

Hence,

$$\max_{\lambda\in\mathbb{R}}\theta(\lambda)=3$$

(iii) Optimal solution is at $x^*=(0,1)$.

Sub $\lambda=-1$ into $\inf_{x\in X}\{2x_1+3x_2+\lambda(x_1+3x_2-3)\}$, and we get

$$\inf_{x\in X}\{x_1+3\}$$

And we can see that the equation is minimum in the set X when $x_1=0$. Substitute this into g to get the optimal solution is $(0,1)$.

We can then also verify that $f((0,1))=3$.

(iv)

$$\partial\theta(\lambda)=\begin{cases}\{3\} & \text{if } \lambda<-1\\ [-3,3] & \text{if } \lambda=-1\\ \{-3\} & \text{if } \lambda>-1\end{cases}$$

(v) For $\lambda<-1$, the steepest direction is 1.

For $\lambda>-1$, the steepest direction is -1.

For $\lambda=-1$, there is no ascent direction.