Calculus 18/19 Sem 1 Suggested Answers

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Question 1

i) Increasing on $(5, \infty)$, decreasing on $(-\infty, 5)$.

$$f'(x) = \frac{6x - 30}{5(x-6)^{4/5}}$$

When $f'(x) > 0, 6x - 30 > 0 \implies x > 5$. When $f'(x) < 0 \implies x < 5$. We also check the points where $f'(x)$ is undefined.

x	6^-	6^+
$f'(x)$	+ve	+ve

ii) Checking the point at $x = 5$ gives us that it is a local min.

x	5^-	5^+
$f'(x)$	-ve	+ve

Local min occurs at $(5, -5)$ and there is no local max.

iii) Concave up when $(-\infty, 6)$ and $(10, \infty)$ and concave down when $(6, 10)$.

$$f''(x) = \frac{6x - 60}{25(x-6)^{1/5}}$$

When $f''(x) > 0$, one needs to have $6x - 60 > 0$ or $x - 6 > 0 \implies x > 10$ or $x > 6 \implies x > 10$. One could also have $6x - 60 < 0$ or $x - 6 < 0 \implies x < 10$ or $x < 6 \implies x < 6$.

When $f''(x) < 0$, one needs to have $6x - 60 < 0$ or $x - 6 > 0 \implies x < 10$ or $x > 6 \implies 6 < x < 10$. Alternatively, $6x - 60 > 0$ or $x - 6 < 0 \implies x > 10$ or $x < 6$, but that's impossible.

iv) Inflection points are $x = 6, 10$ as that is when the graph changes concavity.

Question 2

a) $\lim_{x\to 1^+} \frac{x+3}{x-1} = \infty$ iff $\forall M > 0, \exists \delta > 0$ such that $0 < x-1 < \delta \implies f(x) > M$.

Rough work: $f(x) > M \implies \frac{x+3}{x-1} > M$. So whichever δ we pick must ensure $\frac{x+3}{x-1} > M$. Intuitively, what this means is that when $0 < x-1 < \delta$, we must find a lower bound for $x+3$ and an upper bound for $x-1$ (which is δ) so that we can establish the following inequality:

$$\frac{x+3}{x-1} > \frac{\text{Lower bound for }(x{+}3)}{x-1} > \frac{\text{Lower bound for }(x{+}3)}{\delta} > M$$

Assume $0 < x < 1$ is sufficient, then we have $0 < x < 2 \implies 3 < x+3 < 5$, so the lower bound for $x+3$ is 3.

Proof: Picking $\delta = \min\{1, \frac{3}{M}\}$ is sufficient. If $M > 3$, pick $\delta = \frac{3}{M} < 1$. So one has:

$$\frac{x+3}{x-1} > \frac{3}{x-1} > \frac{3}{3/M} = M$$

If $M \leqslant 3$, pick $\delta = 1$.

$$\frac{x+3}{x-1} > \frac{3}{x-1} > \frac{3}{1} = 3 \geqslant M$$

b) 6

$$\begin{aligned}
\lim_{x\to\infty}\left[\frac{1}{3}\left(3^{1/x}+8^{1/x}+9^{1/x}\right)\right]^x &= \lim_{u\to 0}\left[\frac{1}{3}\left(3^u+8^u+9^u\right)\right]^{1/u} && \text{(Let } u = 1/x\text{)}\\
&= \exp\left(\lim_{u\to 0}\frac{\ln(\frac{1}{3}[3^u+8^u+9^u])}{u}\right) && \text{(Use L'hospital rule, top and bottom go to 0)}\\
&= \exp\left(\lim_{u\to 0}\frac{3^u\ln 3+8^u\ln 8+9^u\ln 9}{3^u+8^u+9^u}\right)\\
&= \exp\left(\frac{\ln 3+\ln 8+\ln 9}{3}\right)\\
&= \exp\left(\ln 216^{1/3}\right)\\
&= 6
\end{aligned}$$

c) $y = (\frac{1}{x})^{\ln x} \implies \ln y = (\ln x)(\ln \frac{1}{x})$.

Implicitly differenciating:

$$\begin{aligned}
\frac{1}{y}\frac{dy}{dx} &= (\ln x)(\frac{1}{1/x})(-\frac{1}{x^2}) + \ln(\frac{1}{x})(\frac{1}{x})\\
\frac{1}{y}\frac{dy}{dx} &= -\frac{1}{x}\ln x + \frac{1}{x}\ln(\frac{1}{x}) = -\frac{2}{x}\ln x
\end{aligned}$$

When $x = e, y = 1/e$. We have:

$$\begin{aligned}
\frac{1}{1/e}\frac{dy}{dx} &= -\frac{2}{e}\ln e\\
e\frac{dy}{dx} &= -\frac{2}{e}\\
\frac{dy}{dx} &= -\frac{2}{e^2}
\end{aligned}$$

Equation of tangent:

$$\begin{aligned}
\frac{y-\frac{1}{e}}{x-e} &= \frac{-2}{e^2}\\
y &= -\frac{2x}{e^2} + \frac{3}{e}
\end{aligned}$$

Question 3 $D = \sqrt{3}R, W = R$.

We know:

$$
\begin{aligned}
&(\frac{W}{2})^2 + (\frac{D}{2})^2 = R^2 \\
&W^2 = 4R^2 - D^2
\end{aligned}
$$

We model stiffness as:

$$
\begin{aligned}
&S = kWD^3 \\
&S^2 = k^2W^2D^6 \\
&S^2 = k^2(4R^2 - D^2)D^6
\end{aligned}
$$

Implicitly differentiating w.r.t D,

$$2S\frac{dS}{dD} = k^2(24R^2D^5 - 8D^7)$$

Setting $dS/dD = 0$, we have $D = 0$ or $D = \pm\sqrt{3}R$. Rejecting 0 and $-\sqrt{3}R$, we check for maxima:

D	$\sqrt{3}R^-$	$\sqrt{3}R^+$
dS/dD	+ve	-ve

When $D = \sqrt{3}R, W = R$.

Question 4

i) Let $y = f(x)$. Then $x = g(y)$. Using inverse differentiation: $g'(y) = \frac{1}{f'(x)} = [f'(x)]^{-1}$. Differentiating this with respect to x:

$$
\begin{aligned}
g''(y)f'(x) &= -[f'(x)]^{-2}f''(x) \\
g''(y) &= -\frac{f''(x)}{[f'(x)]^3} \\
&= -\frac{f''(g(y))}{[f'(g(y))]^3}
\end{aligned}
$$

ii) From part(i), we have: $g''(y) = -f''(x)[f'(x)]^{-3}$

Differentiating with respect to x:

$$
\begin{aligned}
g'''(y)f'(x) &= -f'''(x)[f'(x)]^{-3} - (-3)(f''(x))[f'(x)]^{-4}(f''(x)) \\
&= [f'(x)]^{-4}[-f'''(x)f'(x) + 3[f''(x)]^2] \\
g'''(y) &= \frac{3[f''(x)]^2 - f'''(x)f'(x)}{[f'(x)]^5} \\
&= \frac{3[f''(g(y))]^2 - f'(g(y))f'''(g(y))}{[f'(g(y))]^5}
\end{aligned}
$$

Question 5

a) (i) Using washer method:

$$\begin{aligned}\text{Volume} &= \int_0^1 \pi(2)^2 dx - \int_0^1 \pi y^2 dx \\ &= 4\pi x\Big|_0^1 - \pi\int_0^1 \frac{1}{(1+x)^4}dx \\ &= 4\pi - \pi[\frac{-1}{3(1+x)^3}\Big|_0^1] \\ &= 4\pi + \frac{\pi}{3}(\frac{1}{8}-\frac{1}{1}) \\ &= 4\pi - \frac{7\pi}{24} \\ &= \frac{89\pi}{24}\end{aligned}$$

(ii) Using method of cylindrical shells:

$$\begin{aligned}\text{Volume} &= \int_0^1 2\pi(2-x)(2)dx - \int_0^1 2\pi(2-x)(\frac{1}{1+x})^2 dx \\ &= 4\pi(2x-\frac{1}{2}x^2)\Big|_0^1 - 2\pi\int_0^1 \frac{2-x}{x^2+2x+1}dx \\ &= 6\pi - 2\pi\int_0^1 -\frac{1}{2}[\frac{2x+2}{(1+x)^2}] + \frac{3}{(1+x)^2}dx \\ &= 6\pi - 2\pi[-\ln(1+x) - \frac{3}{1+x}]\Big|_0^1 \\ &= 6\pi - 2\pi[-\ln 2 - \frac{3}{2} + \ln 1 + \frac{3}{1}] \\ &= 3\pi + 2\pi\ln 2\end{aligned}$$

b) Arc length $= \int_0^b \sqrt{1+(f'(x))^2}dx$.

$$\int_0^b \sqrt{1+(f'(x))^2}dx = b + \frac{2}{3}b^3$$

Differentiating with respect to b:

$$\begin{aligned}\sqrt{1+(f'(b))^2} &= 1+2b^2 \\ 1+(f'(b))^2 &= 1+4b^2+4b^4 \\ f'(b) &= \pm\sqrt{4b^2+4b^4}\end{aligned}$$

Note that in the case of $f'(b) = -\sqrt{4b^2+4b^4}$, $\lim_{b\to\infty} f'(b) = -\infty$.
This implies that $\lim_{b\to\infty} f(b) = -\infty$ which contradicts the given condition that f is a nonnegative function. Thus we reject the case of $f'(b) = -\sqrt{4b^2+4b^4}$.

$$\begin{aligned}f'(b) &= 2b\sqrt{b^2+1} \\ f(b) &= \int 2b\sqrt{b^2+1}db \\ &= \int \sqrt{u+1}du \quad (\text{Substitute } u = b^2, \frac{du}{db} = 2b) \\ &= \frac{2}{3}(u+1)^{\frac{3}{2}} + C \\ &= \frac{2}{3}(b^2+1)^{\frac{3}{2}} + C \\ f(0) &= \frac{2}{3} \implies \frac{2}{3} = \frac{2}{3}(1)^{\frac{3}{2}} + C \implies C = 0 \\ f(x) &= \frac{2}{3}(x^2+1)^{\frac{3}{2}}\end{aligned}$$

Question 6

a)

$$
\begin{aligned}
\int \frac{x}{\sqrt{9+8x^2-x^4}}dx &= \frac{1}{2}\int \frac{1}{\sqrt{9+8u-u^2}}du \quad \text{(Use substitution } u = x^2, \frac{du}{dx} = 2x\text{)} \\
&= \frac{1}{2}\int \frac{1}{\sqrt{25-(u^2-8u+16)}}du \\
&= \frac{1}{2}\int \frac{1}{\sqrt{25-(u-4)^2}}du \\
&= \frac{1}{2}\int \frac{5\cos v}{\sqrt{25-25\sin^2 v}}dv \quad \text{(Use substitution } 5\sin v = u-4, \frac{du}{dv} = 5\cos v\text{)} \\
&= \frac{1}{2}\int \frac{5\cos v}{5\cos v}dv \\
&= \frac{1}{2}\int 1 dv \\
&= \frac{1}{2}v + C \\
&= \frac{1}{2}\sin^{-1}\frac{u-4}{5} + C \\
&= \frac{1}{2}\sin^{-1}\frac{x^2-4}{5} + C
\end{aligned}
$$

b) Consider 2 cases:

Case 1: $p = -1$.

$$
\begin{aligned}
\int_0^1 x^{-1}\ln x dx &= (\ln x)^2\Big|_0^1 - \int_0^1 x^{-1}\ln x dx \\
2\int_0^1 x^{-1}\ln x dx &= (\ln x)^2\Big|_0^1 \\
\int_0^1 x^{-1}\ln x dx &= \frac{1}{2}(\ln x)^2\Big|_0^1
\end{aligned}
$$

Clearly the integral does not converge as $\lim_{x\to 0}\ln x \to -\infty$.

Case 2: $p \neq -1$.

$$
\begin{aligned}
\int_0^1 x^p \ln x dx &= \frac{x^{p+1}}{p+1}\ln x\Big|_0^1 - \int_0^1 \frac{x^{p+1}}{p+1}(\frac{1}{x})dx \\
&= \frac{x^{p+1}}{p+1}\ln x\Big|_0^1 - \int_0^1 \frac{x^p}{p+1}dx \\
&= \frac{x^{p+1}}{p+1}\ln x\Big|_0^1 - \frac{x^{p+1}}{(p+1)^2}\Big|_0^1 \\
&= \frac{x^{p+1}}{p+1}\ln x\Big|_0^1 - \frac{1}{(p+1)^2} \\
&= 0 - \lim_{x\to 0}\frac{x^{p+1}\ln x}{p+1} - \frac{1}{(p+1)^2}
\end{aligned}
$$

Note that $\frac{1}{(p+1)^2}$ which is a finite value as $p \neq -1$. For the integral to converge, $\lim_{x\to 0}\frac{x^{p+1}\ln x}{p+1}$ must also be some finite value. We note that as $x \to 0$, $\ln x \to -\infty$, so intuitively, for the limit to exist, x^{1+p} should go to 0. Now we claim that $\lim_{x\to 0}(x^{1+p}\ln x)$ converges iff $p > -1$.

Proof:

If $p < -1$, we would have $\lim_{x\to 0} x^{1+p} \to \infty$ and $\lim_{x\to 0}\ln x \to -\infty$, and thus their product cannot be finite.

But when $p > -1$:

$$\begin{aligned}
\lim_{x\to 0} \frac{x^{p+1}\ln x}{p+1} &= \frac{1}{p+1} \lim_{x\to 0} \frac{\ln x}{x^{-p-1}} [\frac{-\infty}{\infty}] \\
&= \frac{1}{p+1} \lim_{x\to 0} \frac{\frac{1}{x}}{(-p-1)x^{-p-2}} \quad \text{(By L'Hopital's rule)} \\
&= \frac{1}{p+1} \lim_{x\to 0} \frac{x^{p+1}}{(-p-1)} \\
&= 0
\end{aligned}$$

Thus when $p > -1$, $\frac{x^{p+1}}{p+1} \ln x \Big|_0^1 = 0$.

The integral converges when $p > -1$ and $\int_0^1 x^p \ln x dx = \frac{x^{p+1}}{p+1} \ln x \Big|_0^1 - \frac{x^{p+1}}{(p+1)^2} \Big|_0^1 = -\frac{1}{(p+1)^2}$.

Question 7

a) Find the integrating factor:

$$\begin{aligned}
e^{\int \frac{2x+1}{x}dx} &= e^{\int 2+\frac{1}{x}dx} \\
&= e^{2x+\ln x} \\
&= xe^{2x} \\
xe^{2x}y &= \int xe^{2x}(2x)dx \\
&= \int 2x^2e^{2x}dx \\
&= x^2e^{2x} - \int 2xe^{2x}dx \\
&= x^2e^{2x} - xe^{2x} + \int e^{2x}dx \\
&= x^2e^{2x} - xe^{2x} + \frac{1}{2}e^{2x} + C \\
y &= x - 1 + \frac{1}{2x} + \frac{C}{xe^{2x}}
\end{aligned}$$

When $x = 1, y = 1$,

$$\begin{aligned}
1 &= 1 - 1 + \frac{1}{2} + \frac{C}{e^2} \\
C &= \frac{e^2}{2} \\
y &= x - 1 + \frac{1}{2x} + \frac{e^2}{2xe^{2x}}
\end{aligned}$$

b) (i)

$$\begin{aligned}
&\frac{dQ}{dt} = a - bQ \\
&\frac{1}{a-bQ}\frac{dQ}{dt} = 1 \\
&\int \frac{1}{a-bQ}dQ = \int 1dt \\
&-\frac{1}{b}\ln|a-bQ| = t + C \\
&|a-bQ| = e^{-bt-bc} \\
&a - bQ = Ae^{-bt} \quad (\text{where } A = \pm e^{-bc}) \\
&Q = \frac{a - Ae^{-bt}}{b}
\end{aligned}$$

As $t \to \infty, e^{-bt} \to 0, Q \to \frac{a}{b}$.
Limiting concentration $= \frac{a}{b}$. (ii) When $t = 0, Q = 0$.

$$\begin{aligned}
0 &= \frac{a - Ae^0}{b} \implies a - A = 0 \implies A = a \\
Q &= \frac{a - ae^{-bt}}{b}
\end{aligned}$$

When Q $=\frac{1}{2}(\frac{a}{b})$:

$$\frac{a}{2b} = \frac{a - ae^{-bt}}{b}$$
$$\frac{a}{2} = a - ae^{-bt}$$
$$e^{-bt} = \frac{1}{2}$$
$$bt = -\ln\frac{1}{2}$$
$$t = \frac{\ln 2}{b}$$

Question 8

a) By Rolle's, $\exists c$ such that $f'(c) = 0$. Take the gradient between 0 and c, one has by MVT the existence of point $p_1 \in (0, c)$ such that

$$f'(p_1) = \frac{f(c) - f(0)}{c - 0} = \frac{f(c)}{c} > 0$$

Likewise, taking gradient between c and 1, one has by MVT the existence of point $p_2 \in (c, 1)$ such that

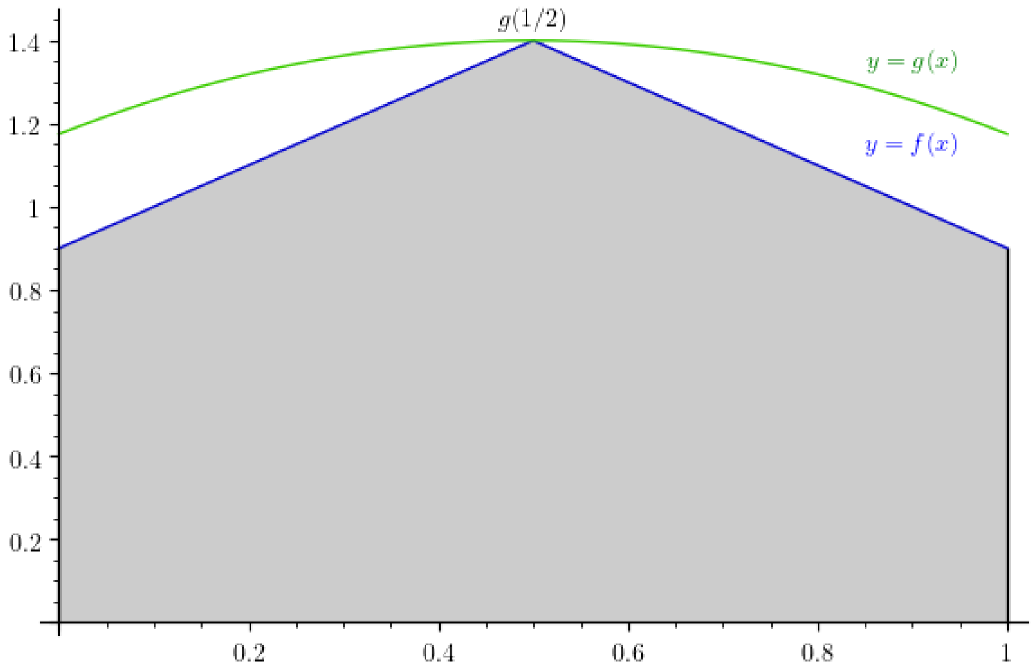
$$f'(p_2) = \frac{f(1) - f(c)}{1 - c} = \frac{-f(c)}{1 - c} < 0$$

By MVT theorem on the graph of $f'(x)$, there exists $p_3 \in (p_1, p_2)$ such that

$$f''(p_3) = \frac{f'(p_2) - f'(p_1)}{p_2 - p_1} < 0$$

Since $p_2 > p_1$. So there exists a point such that $f''(x) < 0$.

b) The intuitive idea behind the proof is to draw a triangle of gradient M from the point $(1/2, g(1/2))$. Then $g(x)$ must always lie above this triangle, else it would have point which has gradient $> M$.



The area of the shaded trapezium is $g(1/2) - M/4$ and the inequality follows trivially. However, to formalise the proof:

Consider the function defined by:

$$f(x) = \begin{cases} Mx - \frac{1}{2}M + g(1/2), & \text{if } 0 \geqslant x \leqslant 1/2 \\ -Mx + \frac{1}{2}M + g(1/2) & \text{if } 1/2 < x \leqslant 1 \end{cases}$$

$f(x)$ has a maximum point at $(1/2, g(1/2))$. We claim that for all x, $f(x) \leqslant g(x)$. Suppose there exists $c \neq 1/2$ such that $f(c) > g(c)$. WLOG, assume $c < 1/2$. We thus have $\frac{g(1/2)-g(c)}{1/2-c} > M$. By MVT, this implies there exists $p \in (c, 1/2)$ such that $g'(p) > M$, which is a contradiction. Similarly, we can conclude that if $c > 1/2$, we have $\frac{g(c)-g(1/2)}{c-1/2} < -M$.

Consider $\int_0^1 f(x)dx$. We have:

$$\int_0^1 f(x)dx = \int_0^{1/2} Mx - \frac{1}{2}M + g(1/2)\ dx + \int_{1/2}^1 -Mx + \frac{1}{2}M + g(1/2)\ dx = g(1/2) - M/4$$

Since $g(x) \geqslant f(x)$ for all x, one has $\int_0^1 g(x)dx \geqslant \int_0^1 f(x)dx = g(1/2) - M/4$, which gives the desired inequality. $\square$