Calculus 18/19 Sem 1 Suggested Answers

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Question 1

i) Increasing on $(5, \infty)$, decreasing on $(-\infty, 5)$.

$$f'(x) = \frac{6x - 30}{5(x - 6)^{4/5}}$$

When $f'(x) > 0, 6x - 30 > 0 \implies x > 5$. When $f'(x) < 0 \implies x < 5$. We also check the points where $f'(x)$ is undefined.

x	6^-	6^+
$f'(x)$	+ve	+ve

ii) Checking the point at $x = 5$ gives us that it is a local min.

x	5^-	5^+
$f'(x)$	-ve	+ve

Local min occurs at $(5, -5)$ and there is no local max.

iii) Concave up when $(-\infty, 6)$ and $(10, \infty)$ and concave down when $(6, 10)$.

$$f''(x) = \frac{6x - 60}{25(x - 6)^{1/5}}$$

When $f''(x) > 0$, one needs to have $6x - 60 > 0$ or $x - 6 > 0 \implies x > 10$ or $x > 6 \implies x > 10$. One could also have $6x - 60 < 0$ or $x - 6 < 0 \implies x < 10$ or $x < 6 \implies x < 6$.

When $f''(x) < 0$, one needs to have $6x - 60 < 0$ or $x - 6 > 0 \implies x < 10$ or $x > 6 \implies 6 < x < 10$. Alternatively, $6x - 60 > 0$ or $x - 6 < 0 \implies x > 10$ or $x < 6$, but that's impossible.

iv) Inflection points are $x = 6, 10$ as that is when the graph changes concavity.

Question 2

a) $\lim_{x\to 1^+} \frac{x+3}{x-1} = \infty$ iff $\forall M > 0, \exists \delta > 0$ such that $0 < x - 1 < \delta \implies f(x) > M$.

Rough work: $f(x) > M \implies \frac{x+3}{x-1} > M$. So whichever δ we pick must ensure $\frac{x+3}{x-1} > M$. Intuitively, what this means is that when $0 < x - 1 < \delta$, we must find a lower bound for $x + 3$ and an upper bound for $x - 1$ (which is δ) so that we can establish the following inequality:

$$\frac{x+3}{x-1} > \frac{\text{Lower bound for } (x+3)}{x-1} > \frac{\text{Lower bound for } (x+3)}{\delta} > M$$

Assume $0 < x < 1$ is sufficient, then we have $0 < x < 2 \implies 3 < x + 3 < 5$, so the lower bound for $x + 3$ is 3.

Proof: Picking $\delta = \min\{1, \frac{3}{M}\}$ is sufficient. If $M > 3$, pick $\delta = \frac{3}{M} < 1$. So one has:

$$\frac{x+3}{x-1} > \frac{3}{x-1} > \frac{3}{3/M} = M$$

If $M \leqslant 3$, pick $\delta = 1$.

$$\frac{x+3}{x-1} > \frac{3}{x-1} > \frac{3}{1} = 3 \geqslant M$$

b) 6

$$\lim_{x\to\infty} \left[\frac{1}{3} \left(3^{1/x} + 8^{1/x} + 9^{1/x} \right) \right]^x = \lim_{u\to 0} \left[\frac{1}{3} \left(3^u + 8^u + 9^u \right) \right]^{1/u} \qquad \text{(Let } u = 1/x\text{)}$$

$$= \exp\left(\lim_{u\to 0} \frac{\ln(\frac{1}{3}[3^u + 8^u + 9^u])}{u} \right) \quad \text{(Use L'hospital rule, top and bottom go to 0)}$$

$$= \exp\left(\lim_{u\to 0} \frac{3^u \ln 3 + 8^u \ln 8 + 9^u \ln 9}{3^u + 8^u + 9^u} \right)$$

$$= \exp\left(\frac{\ln 3 + \ln 8 + \ln 9}{3} \right)$$

$$= \exp\left(\ln 216^{1/3} \right)$$

$$= 6$$

c) $y = \left(\frac{1}{x} \right)^{\ln x} \implies \ln y = (\ln x)(\ln \frac{1}{x})$.

Implicitly differenciating:

$$\frac{1}{y} \frac{dy}{dx} = (\ln x)\left(\frac{1}{1/x}\right)\left(-\frac{1}{x^2}\right) + \ln\left(\frac{1}{x}\right)\left(\frac{1}{x}\right)$$

$$\frac{1}{y} \frac{dy}{dx} = -\frac{1}{x}\ln x + \frac{1}{x}\ln\left(\frac{1}{x}\right) = -\frac{2}{x}\ln x$$

When $x = e, y = 1/e$. We have:

$$\frac{1}{1/e} \frac{dy}{dx} = -\frac{2}{e}\ln e$$

$$e\frac{dy}{dx} = -\frac{2}{e}$$

$$\frac{dy}{dx} = -\frac{2}{e^2}$$

Equation of tangent:

$$\frac{y - \frac{1}{e}}{x - e} = \frac{-2}{e^2}$$

$$y = -\frac{2x}{e^2} + \frac{3}{e}$$

Question 3 $D = \sqrt{3}R, W = R$.

We know:

$$(\frac{W}{2})^2 + (\frac{D}{2})^2 = R^2$$
$$W^2 = 4R^2 - D^2$$

We model stiffness as:

$$S = kWD^3$$
$$S^2 = k^2W^2D^6$$
$$S^2 = k^2(4R^2 - D^2)D^6$$

Implicitly differentiating w.r.t D,

$$2S\frac{dS}{dD} = k^2(24R^2D^5 - 8D^7)$$

Setting $dS/dD = 0$, we have $D = 0$ or $D = \pm\sqrt{3}R$. Rejecting 0 and $-\sqrt{3}R$, we check for maxima:

D	$\sqrt{3}R^-$	$\sqrt{3}R^+$
dS/dD	+ve	-ve

When $D = \sqrt{3}R, W = R$.

Question 4

i) Let $y = f(x)$. Then $x = g(y)$. Using inverse differentiation: $g'(y) = \frac{1}{f'(x)} = [f'(x)]^{-1}$. Differentiating this with respect to x:

$$g''(y)f'(x) = -[f'(x)]^{-2}f''(x)$$
$$g''(y) = -\frac{f''(x)}{[f'(x)]^3}$$
$$= -\frac{f''(g(y))}{[f'(g(y))]^3}$$

ii) From part(i), we have: $g''(y) = -f''(x)[f'(x)]^{-3}$

Differentiating with respect to x:

$$g'''(y)f'(x) = -f'''(x)[f'(x)]^{-3} - (-3)(f''(x))[f'(x)]^{-4}(f''(x))$$
$$= [f'(x)]^{-4}[-f'''(x)f'(x) + 3[f''(x)]^2]$$
$$g'''(y) = \frac{3[f''(x)]^2 - f'''(x)f'(x)}{[f'(x)]^5}$$
$$= \frac{3[f''(g(y))]^2 - f'(g(y))f'''(g(y))}{[f'(g(y))]^5}$$

Question 5

a) (i) Using washer method:

$$\text{Volume} = \int_0^1 \pi(2)^2 dx - \int_0^1 \pi y^2 dx$$

$$= 4\pi x\big|_0^1 - \pi \int_0^1 \frac{1}{(1+x)^4} dx$$

$$= 4\pi - \pi\big[\frac{-1}{3(1+x)^3}\big|_0^1\big]$$

$$= 4\pi + \frac{\pi}{3}\big(\frac{1}{8} - \frac{1}{1}\big)$$

$$= 4\pi - \frac{7\pi}{24}$$

$$= \frac{89\pi}{24}$$

(ii) Using method of cylindrical shells:

$$\text{Volume} = \int_0^1 2\pi(2-x)(2)dx - \int_0^1 2\pi(2-x)\big(\frac{1}{1+x}\big)^2 dx$$

$$= 4\pi(2x - \frac{1}{2}x^2)\big|_0^1 - 2\pi \int_0^1 \frac{2-x}{x^2 + 2x + 1} dx$$

$$= 6\pi - 2\pi \int_0^1 -\frac{1}{2}[\frac{2x+2}{(1+x)^2}] + \frac{3}{(1+x)^2} dx$$

$$= 6\pi - 2\pi\big[-\ln(1+x) - \frac{3}{1+x}\big]\big|_0^1$$

$$= 6\pi - 2\pi\big[-\ln 2 - \frac{3}{2} + \ln 1 + \frac{3}{1}\big]$$

$$= 3\pi + 2\pi \ln 2$$

b) Arc length $= \int_0^b \sqrt{1 + (f'(x))^2} dx$.

$$\int_0^b \sqrt{1 + (f'(x))^2} dx = b + \frac{2}{3}b^3$$

Differentiating with respect to b:

$$\sqrt{1 + (f'(b))^2} = 1 + 2b^2$$

$$1 + (f'(b))^2 = 1 + 4b^2 + 4b^4$$

$$f'(b) = \pm\sqrt{4b^2 + 4b^4}$$

Note that in the case of $f'(b) = -\sqrt{4b^2 + 4b^4}$, $\lim_{b\to\infty} f'(b) = -\infty$.

This implies that $\lim_{b\to\infty} f(b) = -\infty$ which contradicts the given condition that f is a nonnegative function. Thus we reject the case of $f'(b) = -\sqrt{4b^2 + 4b^4}$.

$$f'(b) = 2b\sqrt{b^2 + 1}$$

$$f(b) = \int 2b\sqrt{b^2 + 1} db$$

$$= \int \sqrt{u + 1} du \quad (\text{Substitute } u = b^2, \frac{du}{db} = 2b)$$

$$= \frac{2}{3}(u+1)^{\frac{3}{2}} + C$$

$$= \frac{2}{3}(b^2 + 1)^{\frac{3}{2}} + C$$

$$f(0) = \frac{2}{3} \implies \frac{2}{3} = \frac{2}{3}(1)^{\frac{3}{2}} + C \implies C = 0$$

$$f(x) = \frac{2}{3}(x^2 + 1)^{\frac{3}{2}}$$

Question 6

a)

$$\int \frac{x}{\sqrt{9 + 8x^2 - x^4}} dx = \frac{1}{2} \int \frac{1}{\sqrt{9 + 8u - u^2}} du \quad \text{(Use substitution } u = x^2, \frac{du}{dx} = 2x)$$

$$= \frac{1}{2} \int \frac{1}{\sqrt{25 - (u^2 - 8u + 16)}} du$$

$$= \frac{1}{2} \int \frac{1}{\sqrt{25 - (u - 4)^2}} du$$

$$= \frac{1}{2} \int \frac{5 \cos v}{\sqrt{25 - 25 \sin^2 v}} dv \quad \text{(Use substitution } 5 \sin v = u - 4, \frac{du}{dv} = 5 \cos v)$$

$$= \frac{1}{2} \int \frac{5 \cos v}{5 \cos v} dv$$

$$= \frac{1}{2} \int 1 dv$$

$$= \frac{1}{2} v + C$$

$$= \frac{1}{2} \sin^{-1} \frac{u - 4}{5} + C$$

$$= \frac{1}{2} \sin^{-1} \frac{x^2 - 4}{5} + C$$

b) Consider 2 cases:

Case 1: $p = -1$.

$$\int_0^1 x^{-1} \ln x dx = (\ln x)^2 \big|_0^1 - \int_0^1 x^{-1} \ln x dx$$

$$2 \int_0^1 x^{-1} \ln x dx = (\ln x)^2 \big|_0^1$$

$$\int_0^1 x^{-1} \ln x dx = \frac{1}{2} (\ln x)^2 \big|_0^1$$

Clearly the integral does not converge as $\lim_{x \to 0} \ln x \to -\infty$.

Case 2: $p \neq -1$.

$$\int_0^1 x^p \ln x dx = \frac{x^{p+1}}{p+1} \ln x \big|_0^1 - \int_0^1 \frac{x^{p+1}}{p+1} (\frac{1}{x}) dx$$

$$= \frac{x^{p+1}}{p+1} \ln x \big|_0^1 - \int_0^1 \frac{x^p}{p+1} dx$$

$$= \frac{x^{p+1}}{p+1} \ln x \big|_0^1 - \frac{x^{p+1}}{(p+1)^2} \big|_0^1$$

$$= \frac{x^{p+1}}{p+1} \ln x \big|_0^1 - \frac{1}{(p+1)^2}$$

$$= 0 - \lim_{x \to 0} \frac{x^{p+1} \ln x}{p+1} - \frac{1}{(p+1)^2}$$

Note that $\frac{1}{(p+1)^2}$ which is a finite value as $p \neq -1$. For the integral to converge, $\lim_{x \to 0} \frac{x^{p+1} \ln x}{p+1}$ must also be some finite value. Now we claim that $\lim_{x \to 0} (x^{1+p} \ln x)$ converges iff $p > -1$.

Proof:

If $p < -1$, we would have $\lim_{x \to 0} x^{1+p} \to \infty$ and $\lim_{x \to 0} \ln x \to -\infty$, and thus their product cannot be finite.

But when $p > -1$:

$$\lim_{x \to 0} \frac{x^{p+1} \ln x}{p+1} = \frac{1}{p+1} \lim_{x \to 0} \frac{\ln x}{x^{-p-1}} [\frac{-\infty}{\infty}]$$

$$= \frac{1}{p+1} \lim_{x \to 0} \frac{\frac{1}{x}}{(-p-1)x^{-p-2}} \quad \text{(By L'Hopital's rule)}$$

$$= \frac{1}{p+1} \lim_{x \to 0} \frac{x^{p+1}}{(-p-1)}$$

$$= 0$$

Thus when $p > -1$, $\frac{x^{p+1}}{p+1} \ln x \big|_0^1 = 0$.

The integral converges when $p > -1$ and $\int_0^1 x^p \ln x \, dx = \frac{x^{p+1}}{p+1} \ln x \big|_0^1 - \frac{x^{p+1}}{(p+1)^2} \big|_0^1 = -\frac{1}{(p+1)^2}$.

Question 7

a) Find the integrating factor:

$$e^{\int \frac{2x+1}{x} dx} = e^{\int 2 + \frac{1}{x} dx}$$

$$= e^{2x + \ln x}$$

$$= xe^{2x}$$

$$xe^{2x}y = \int xe^{2x}(2x)dx$$

$$= \int 2x^2 e^{2x} dx$$

$$= x^2 e^{2x} - \int 2xe^{2x} dx$$

$$= x^2 e^{2x} - xe^{2x} + \int e^{2x} dx$$

$$= x^2 e^{2x} - xe^{2x} + \frac{1}{2}e^{2x} + C$$

$$y = x - 1 + \frac{1}{2x} + \frac{C}{xe^{2x}}$$

When $x = 1, y = 1$,

$$1 = 1 - 1 + \frac{1}{2} + \frac{C}{e^2}$$

$$C = \frac{e^2}{2}$$

$$y = x - 1 + \frac{1}{2x} + \frac{e^2}{2xe^{2x}}$$

b) (i)

$$\frac{dQ}{dt} = a - bQ$$

$$\frac{1}{a - bQ}\frac{dQ}{dt} = 1$$

$$\int \frac{1}{a - bQ}dQ = \int 1dt$$

$$-\frac{1}{b}\ln|a - bQ| = t + C$$

$$|a - bQ| = e^{-bt - bc}$$

$$a - bQ = Ae^{-bt} \quad (\text{where } A = \pm e^{-bc})$$

$$Q = \frac{a - Ae^{-bt}}{b}$$

As $t \to \infty, e^{-bt} \to 0, Q \to \frac{a}{b}$.

Limiting concentration $= \frac{a}{b}$. (ii) When $t = 0, Q = 0$.

$$0 = \frac{a - Ae^0}{b} \implies a - A = 0 \implies A = a$$

$$Q = \frac{a - ae^{-bt}}{b}$$

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When Q $= \frac{1}{2}\left(\frac{a}{b}\right)$:

$$\frac{a}{2b} = \frac{a - ae^{-bt}}{b}$$

$$\frac{a}{2} = a - ae^{-bt}$$

$$e^{-bt} = \frac{1}{2}$$

$$bt = -\ln\frac{1}{2}$$

$$t = \frac{\ln 2}{b}$$

$$\frac{a}{2b} = \frac{a - ae^{-bt}}{b}$$

$$\frac{a}{2} = a - ae^{-bt}$$

$$e^{-bt} = \frac{1}{2}$$

Question 8

a) By Rolle's, $\exists c$ such that $f'(c) = 0$. Take the gradient between 0 and c, one has by MVT the existence of point $p_1 \in (0, c)$ such that

$$f'(p_1) = \frac{f(c) - f(0)}{c - 0} = \frac{f(c)}{c} > 0$$

Likewise, taking gradient between c and 1, one has by MVT the existence of point $p_2 \in (c, 1)$ such that
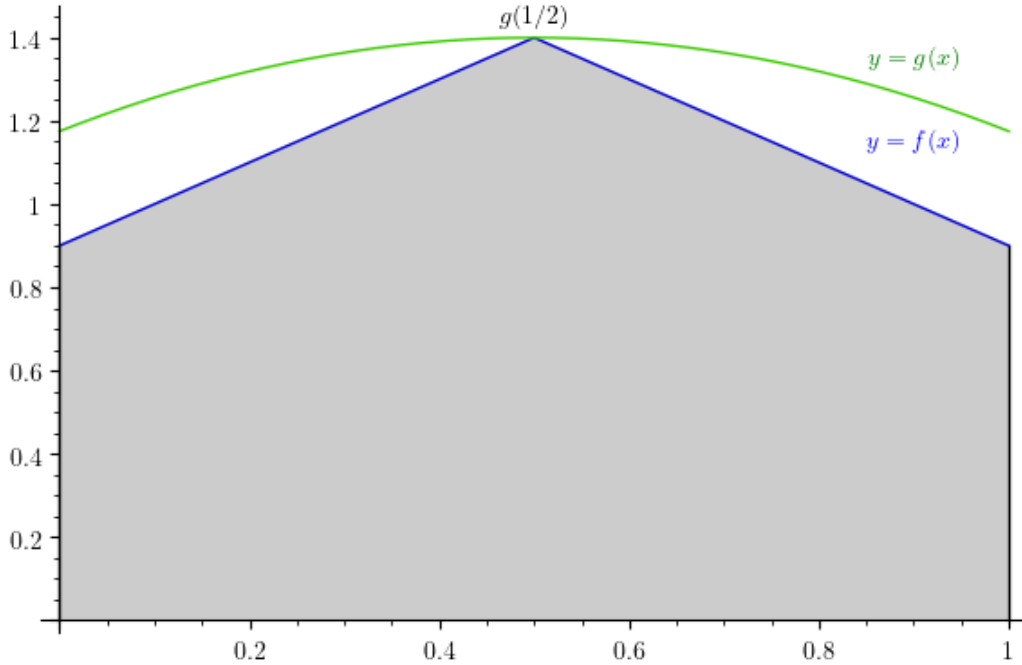
$$f'(p_2) = \frac{f(1) - f(c)}{1 - c} = \frac{-f(c)}{1 - c} < 0$$

By MVT theorem on the graph of $f'(x)$, there exists $p_3 \in (p_1, p_2)$ such that

$$f''(p_3) = \frac{f'(p_2) - f'(p_1)}{p_2 - p_1} < 0$$

Since $p_2 > p_1$. So there exists a point such that $f''(x) < 0$.

b) The intuitive idea behind the proof is to draw a triangle of gradient M from the point $(1/2, g(1/2))$. Then $g(x)$ must always lie above this triangle, else it would have point which has gradient $> M$.

The area of the shaded trapezium is $g(1/2) - M/4$ and the inequality follows trivially. However, to formalise the proof:

Consider the function defined by:

$$f(x) = \begin{cases} Mx - \frac{1}{2}M + g(1/2), & \text{if } 0 \geqslant x \leqslant 1/2 \\ -Mx + \frac{1}{2}M + g(1/2) & \text{if } 1/2 < x \leqslant 1 \end{cases}$$

$f(x)$ has a maximum point at $(1/2, g(1/2))$. We claim that for all x, $f(x) \leqslant g(x)$. Suppose there exists $c \neq 1/2$ such that $f(c) > g(c)$. WLOG, assume $c < 1/2$. We thus have $\frac{g(1/2) - g(c)}{1/2 - c} > M$. By MVT, this implies there exists $p \in (c, 1/2)$ such that $g'(p) > M$, which is a contradiction. Similarly, we can conclude that if $c > 1/2$, we have $\frac{g(c) - g(1/2)}{c - 1/2} < -M$.

Consider $\int_0^1 f(x)dx$. We have:

$$\int_0^1 f(x)dx = \int_0^{1/2} Mx - \frac{1}{2}M + g(1/2) \, dx + \int_{1/2}^1 -Mx + \frac{1}{2}M + g(1/2) \, dx = g(1/2) - M/4$$

Since $g(x) \geqslant f(x)$ for all x, one has $\int_0^1 g(x)dx \geqslant \int_0^1 f(x)dx = g(1/2) - M/4$, which gives the desired inequality. $\qquad \square$