Lecture 4 Solving Equations $dx/P = dy/Q = dz/R$

In the previous lecture, we have seen that the integral curves of the set of differential equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \tag{1}$$

form a two-parameter family of curves in three-dimensional space. If we can derive two relation of the form

$$u_1(x, y, z) = c_1, \quad u_2(x, y, z) = c_2, \tag{2}$$

then varying c_1 and c_2 we get a two-parameter family of curves satisfying (1). In this lecture, we shall describe methods for finding integral curves of the set of differential equations (1).

Method I: Along any tangential direction through a point (x, y, z) to $u_1(x, y, z) = c_1$ we have

$$\frac{\partial u_1}{\partial x}dx + \frac{\partial u_1}{\partial x}dy + \frac{\partial u_1}{\partial x}dz = 0. \tag{3}$$

If $u_1(x, y, z) = c$ is a suitable one-parameter system of surfaces, then the tangential direction to the integral curve through the point (x, y, z) is also a tangential direction to this surface. Hence

$$\begin{aligned} & (P, Q, R) \cdot \nabla u_1 = 0 \\ \Longrightarrow \quad & P\frac{\partial u_1}{\partial x} + Q\frac{\partial u_1}{\partial x} + R\frac{\partial u_1}{\partial x} = 0. \end{aligned}$$

To find u_1, choose functions P_1, Q_1 and R_1 such that

$$\begin{aligned} & (P, Q, R) \cdot (P_1, Q_1, R_1) = 0, \\ \Longrightarrow \quad & PP_1 + QQ_1 + RR_1 = 0. \end{aligned} \tag{4}$$

Thus, there exists a function u_1 such that

$$P_1 = \frac{\partial u_1}{\partial x}, \quad Q_1 = \frac{\partial u_1}{\partial y}, \quad R_1 = \frac{\partial u_1}{\partial z}.$$

and this leads to the equation

$$du_1 = P_1 dx + Q_1 dy + R_1 dz, \tag{5}$$

which is an exact differential.

Remark 1. *The method described above for finding solutions of (1) is by inspection. A good deal of intuition is required to determine the forms of the functions P_1, Q_1 and R_1 (cf. [10]).*

Example 2. *Find the integral curves of the equations*

$$\frac{dx}{y(x+y)} = \frac{dy}{x(x+y)} = \frac{dz}{z(x+y)}. \tag{6}$$

Solution. Comparing with (1), we find that

$$P = y(x+y), \quad Q = x(x+y), \quad R = z(x+y).$$

If we choose

$$P_1 = \frac{1}{z}, \quad Q_1 = \frac{1}{z}, \quad R_1 = -\frac{x+y}{z^2}$$

then condition $PP_1 + QQ_1 + RR_1 = 0$ is satisfied. The function $u_1(x, y, z)$ is then determined as follows:

$$\begin{aligned} u_1(x,y,z) &= \int \frac{1}{z}\,dx + \int \frac{1}{z}\,dy + \int (-\frac{x+y}{z^2})\,dz \\ &= \frac{x}{z} + \frac{y}{z} + \frac{x+y}{z} = c \\ &\implies 2\frac{x+y}{z} = c. \end{aligned}$$

Similarly, choose $P_1 = x$, $Q_1 = -y$ and $R_1 = 0$ and verify that the condition (4) is get satisfied. The function u_2 is then determined as

$$u_2(x,y,z) = \frac{1}{2}(x^2 - y^2).$$

Thus, the integral curves of the differential equations (6) are the member of the two-parameter family of curves

$$x + y = c_1 z, \quad x^2 - y^2 = c_2.$$

Method II: Suppose that we are able to find three functions P_1, Q_1 and R_1 such that the ratio

$$\frac{P_1dx + Q_1dy + R_1dz}{PP_1 + QQ_1 + RR_1} = dW_1, \tag{7}$$

an exact differential. Similarly, suppose we can find other three functions P_2, Q_2 and R_2 such that

$$\frac{P_2dx + Q_2dy + R_2dz}{PP_2 + QQ_2 + RR_2} = dW_2 \tag{8}$$

is also an exact differential. Since the ratios

$$\frac{P_1dx + Q_1dy + R_1dz}{PP_1 + QQ_1 + RR_1} = \frac{P_2dx + Q_2dy + R_2dz}{PP_2 + QQ_2 + RR_2} = \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R},$$

it now follows that

$$dW_1 = dW_2,$$

which yields the relation

$$W_1(x, y, z) = W_2(x, y, z) + c_1,$$

where c_1 denotes an arbitrary constant.

Example 3. *Find the integral curves of the equations*

$$\frac{dx}{y - x} = \frac{dy}{x + y} = \frac{zdz}{x^2 + y^2}. \tag{9}$$

Solution. Here $P = y - x$, $Q = x + y$ and $R = \frac{x^2+y^2}{z}$. Observe that $P + Q = 2y$. Now choose $P_1 = 1$, $Q_1 = 1$ and $R_1 = 0$ to obtain

$$\begin{aligned} & \frac{dx + dy}{2y} = \frac{dy}{x + y} \\ \Longrightarrow\quad & (x + y)(dx + dy) = 2y \\ \Longrightarrow\quad & \frac{1}{2}d\{(x + y)^2\} = 2y. \end{aligned}$$

It has solution of the form

$$u_1(x, y, z) = \frac{(x + y)^2}{2} - y^2 = c_1.$$

Similarly, with $P_2 = x$, $Q_2 = -y$ and $R_2 = z$, we find that

$$xdx - ydy + zdz = 0,$$

which has solution

$$u_2(x, y, z) = x^2 - y^2 + z^2 = c_2.$$

The equations

$$\frac{(x + y)^2}{2} - y^2 = c_1, \quad x^2 - y^2 + z^2 = c_2$$

constitute the integral curves of (9).

Method III: When one of the variables is absent from (1), we can derive the integral curves in a simple way.

For the sake of definiteness, let P and Q be functions of x and y only. Then the equation

$$\frac{dx}{P} = \frac{dy}{Q}$$

may be written as

$$\frac{dy}{dx} = f(x,y), \;\text{ where }\; f(x,y) = \frac{Q}{P}.$$

Let this equation has a solution of the form

$$\phi(x, y, c_1) = 0. \tag{10}$$

Solving (10) for x and substituting the value of x in the equation

$$\frac{dy}{Q} = \frac{dz}{R}$$

leads to an equation of the form

$$\frac{dy}{dz} = g(y, z, c_1). \tag{11}$$

Let the solution of (11) be expressed by

$$\psi(y, z, c_1, c_2) = 0. \tag{12}$$

Example 4. *Find the integral curves of the equations*

$$\frac{dx}{x} = \frac{dy}{y+x^2} = \frac{dz}{y+z} \tag{13}$$

Solution. The first two equations may be expressed as

$$\begin{aligned} & \frac{dy}{dx} - \frac{y}{x} = x \\ \implies \quad & \frac{d}{dx}\left(\frac{y}{x}\right) = 1, \end{aligned}$$

which has solution

$$y = c_1 x + x^2.$$

Using the first and third equations of (13), we note that

$$\begin{aligned} & \frac{dz}{dx} = \frac{y}{x} + \frac{z}{x} = c_1 + x + \frac{z}{x} \\ \implies \quad & \frac{d}{dx}\left(\frac{z}{x}\right) = \frac{c_1}{x} + 1, \end{aligned}$$

which has solution

$$z = c_1 x \log x + c_2 x + x^2.$$

Hence, the integral curves of the differential equations (13) are given by the equations

$$y = c_1 x + x^2, \quad z = c_1 x \log x + c_2 x + x^2.$$

Practice Problems 4

Find the integral curves of the following system of ODEs:

1. $\dfrac{dx}{y-z} = \dfrac{dy}{z-x} = \dfrac{dz}{x-y}$
2. $\dfrac{dx}{z} = \dfrac{dy}{xz} = \dfrac{dz}{y}$
3. $\dfrac{dx}{xz-y} = \dfrac{dy}{yz-x} = \dfrac{dz}{xy-z}$
4. $\dfrac{dx}{y+3z} = \dfrac{dy}{z+5x} = \dfrac{dz}{x+7y}$