
Lecture 4 Solving Equations $dx/P = dy/Q = dz/R$

In the previous lecture, we have seen that the integral curves of the set of differential equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \quad (1)$$

form a two-parameter family of curves in three-dimensional space. If we can derive two relation of the form

$$u_1(x, y, z) = c_1, \quad u_2(x, y, z) = c_2, \quad (2)$$

then varying c_1 and c_2 we get a two-parameter family of curves satisfying (1). In this lecture, we shall describe methods for finding integral curves of the set of differential equations (1).

Method I: Along any tangential direction through a point (x, y, z) to $u_1(x, y, z) = c_1$ we have

$$\frac{\partial u_1}{\partial x} dx + \frac{\partial u_1}{\partial y} dy + \frac{\partial u_1}{\partial z} dz = 0. \quad (3)$$

If $u_1(x, y, z) = c$ is a suitable one-parameter system of surfaces, then the tangential direction to the integral curve through the point (x, y, z) is also a tangential direction to this surface. Hence

$$\begin{aligned} (P, Q, R) \cdot \nabla u_1 &= 0 \\ \Rightarrow P \frac{\partial u_1}{\partial x} + Q \frac{\partial u_1}{\partial y} + R \frac{\partial u_1}{\partial z} &= 0. \end{aligned}$$

To find u_1 , choose functions P_1 , Q_1 and R_1 such that

$$\begin{aligned} (P, Q, R) \cdot (P_1, Q_1, R_1) &= 0, \\ \Rightarrow PP_1 + QQ_1 + RR_1 &= 0. \end{aligned} \quad (4)$$

Thus, there exists a function u_1 such that

$$P_1 = \frac{\partial u_1}{\partial x}, \quad Q_1 = \frac{\partial u_1}{\partial y}, \quad R_1 = \frac{\partial u_1}{\partial z}.$$

and this leads to the equation

$$du_1 = P_1 dx + Q_1 dy + R_1 dz, \quad (5)$$

which is an exact differential.

REMARK 1. *The method described above for finding solutions of (1) is by inspection. A good deal of intuition is required to determine the forms of the functions P_1 , Q_1 and R_1 (cf. [10]).*

EXAMPLE 2. *Find the integral curves of the equations*

$$\frac{dx}{y(x+y)} = \frac{dy}{x(x+y)} = \frac{dz}{z(x+y)}. \quad (6)$$

Solution. Comparing with (1), we find that

$$P = y(x+y), \quad Q = x(x+y), \quad R = z(x+y).$$

If we choose

$$P_1 = \frac{1}{z}, \quad Q_1 = \frac{1}{z}, \quad R_1 = -\frac{x+y}{z^2}$$

then condition $PP_1 + QQ_1 + RR_1 = 0$ is satisfied. The function $u_1(x, y, z)$ is then determined as follows:

$$\begin{aligned} u_1(x, y, z) &= \int \frac{1}{z} dx + \int \frac{1}{z} dy + \int \left(-\frac{x+y}{z^2}\right) dz \\ &= \frac{x}{z} + \frac{y}{z} + \frac{x+y}{z} = c \\ \implies 2\frac{x+y}{z} &= c. \end{aligned}$$

Similarly, choose $P_1 = x$, $Q_1 = -y$ and $R_1 = 0$ and verify that the condition (4) is get satisfied. The function u_2 is then determined as

$$u_2(x, y, z) = \frac{1}{2}(x^2 - y^2).$$

Thus, the integral curves of the differential equations (6) are the member of the two-parameter family of curves

$$x + y = c_1 z, \quad x^2 - y^2 = c_2.$$

Method II: Suppose that we are able to find three functions P_1 , Q_1 and R_1 such that the ratio

$$\frac{P_1 dx + Q_1 dy + R_1 dz}{PP_1 + QQ_1 + RR_1} = dW_1, \quad (7)$$

an exact differential. Similarly, suppose we can find other three functions P_2 , Q_2 and R_2 such that

$$\frac{P_2 dx + Q_2 dy + R_2 dz}{PP_2 + QQ_2 + RR_2} = dW_2 \quad (8)$$

is also an exact differential. Since the ratios

$$\frac{P_1 dx + Q_1 dy + R_1 dz}{PP_1 + QQ_1 + RR_1} = \frac{P_2 dx + Q_2 dy + R_2 dz}{PP_2 + QQ_2 + RR_2} = \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R},$$

it now follows that

$$dW_1 = dW_2,$$

which yields the relation

$$W_1(x, y, z) = W_2(x, y, z) + c_1,$$

where c_1 denotes an arbitrary constant.

EXAMPLE 3. Find the integral curves of the equations

$$\frac{dx}{y-x} = \frac{dy}{x+y} = \frac{zdz}{x^2+y^2}. \quad (9)$$

Solution. Here $P = y - x$, $Q = x + y$ and $R = \frac{x^2+y^2}{z}$. Observe that $P + Q = 2y$. Now choose $P_1 = 1$, $Q_1 = 1$ and $R_1 = 0$ to obtain

$$\begin{aligned} \frac{dx + dy}{2y} &= \frac{dy}{x+y} \\ \implies (x+y)(dx + dy) &= 2y \\ \implies \frac{1}{2}d\{(x+y)^2\} &= 2y. \end{aligned}$$

It has solution of the form

$$u_1(x, y, z) = \frac{(x+y)^2}{2} - y^2 = c_1.$$

Similarly, with $P_2 = x$, $Q_2 = -y$ and $R_2 = z$, we find that

$$xdx - ydy + zdz = 0,$$

which has solution

$$u_2(x, y, z) = x^2 - y^2 + z^2 = c_2.$$

The equations

$$\frac{(x+y)^2}{2} - y^2 = c_1, \quad x^2 - y^2 + z^2 = c_2$$

constitute the integral curves of (9).

Method III: When one of the variables is absent from (1), we can derive the integral curves in a simple way.

For the sake of definiteness, let P and Q be functions of x and y only. Then the equation

$$\frac{dx}{P} = \frac{dy}{Q}$$

may be written as

$$\frac{dy}{dx} = f(x, y), \quad \text{where } f(x, y) = \frac{Q}{P}.$$

Let this equation has a solution of the form

$$\phi(x, y, c_1) = 0. \quad (10)$$

Solving (10) for x and substituting the value of x in the equation

$$\frac{dy}{Q} = \frac{dz}{R}$$

leads to an equation of the form

$$\frac{dy}{dz} = g(y, z, c_1). \quad (11)$$

Let the solution of (11) be expressed by

$$\psi(y, z, c_1, c_2) = 0. \quad (12)$$

EXAMPLE 4. Find the integral curves of the equations

$$\frac{dx}{x} = \frac{dy}{y + x^2} = \frac{dz}{y + z} \quad (13)$$

Solution. The first two equations may be expressed as

$$\begin{aligned} \frac{dy}{dx} - \frac{y}{x} &= x \\ \implies \frac{d}{dx} \left(\frac{y}{x} \right) &= 1, \end{aligned}$$

which has solution

$$y = c_1 x + x^2.$$

Using the first and third equations of (13), we note that

$$\begin{aligned} \frac{dz}{dx} &= \frac{y}{x} + \frac{z}{x} = c_1 + x + \frac{z}{x} \\ \implies \frac{d}{dx} \left(\frac{z}{x} \right) &= \frac{c_1}{x} + 1, \end{aligned}$$

which has solution

$$z = c_1 x \log x + c_2 x + x^2.$$

Hence, the integral curves of the differential equations (13) are given by the equations

$$y = c_1 x + x^2, \quad z = c_1 x \log x + c_2 x + x^2.$$

PRACTICE PROBLEMS 4

Find the integral curves of the following system of ODEs:

$$1. \frac{dx}{y-z} = \frac{dy}{z-x} = \frac{dz}{x-y}$$

$$2. \frac{dx}{z} = \frac{dy}{xz} = \frac{dz}{y}$$

$$3. \frac{dx}{xz-y} = \frac{dy}{yz-x} = \frac{dz}{xy-z}$$

$$4. \frac{dx}{y+3z} = \frac{dy}{z+5x} = \frac{dz}{x+7y}$$



INVOICE

Customer Details

Name: Digiversal Consultants Pvt. Ltd.
Bill No: EXINV-MAS-Apr-2023-000021
State: Uttar Pradesh
Date: 05 Apr 2023
State Code: 09
GST No: 09AAGCD0307EIZ5

Description (Brand: MAS)	Amount
Professional Fee for the period	INR 6500.00
(-) Penalty Charged	INR 600.00
Net payable	INR 5900.00
(-) TDS (20%)	INR 1180.00
To Pay	INR 4720.00

Amount in Words

Four thousand seven hundred and twenty rupees only

Self Details

Name: Jeremiah Karulo
Country: Kenya
PAN No: 1234
Bank Account details
Bank Name: Absa bank
Account No: 0704860350
IFSC Code: 0000

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INVOICE # 102

Issue-Date : 2023-08-01

Due-Date : 2023-08-01

Description	Amount
It	123,000/-
Sub-Total Amount:	123,000/-
Tax Rate:	16%
Tax Amount:	19,680/-
Total:	142,680/-

Payment Details

Account Name: Napoh Holdings

Bank Name: NCBA Bank

Account Number: 123456789

Routing Number: 987654321