Midterm Solution

November 10, 2022

Problem 1

1. True. Let $Ax = \lambda x$ with $\lambda \in \mathbb{C}$ and non-zero $x \in \mathbb{C}^n$. Because $A^2 = A$, $Ax = A^2x = A\lambda x = \lambda^2 x = \lambda x$, which implies that $(\lambda^2 - \lambda)x = 0$. Since x contains at least one non-zero element, $\lambda^2 - \lambda = 0$ must hold. It follows that $\lambda = 0$ or 1.

2. True. Note that the rank of a zero matrix is 0 and its dimension of kernel (null) space is n by the rank-nullity theorem. Besides, $\mathrm{Ker}(A) = \mathrm{Ker}(A^TA)$ (Since if $Ax = 0$, $A^TAx = 0$. Conversely, if $A^TAx = 0$, $Ax = 0$ follows from $x^TA^TAx = 0$). Therefore, by the rank-nullity theorem, $\mathrm{rank}A = n - \dim \mathrm{Ker}(A) = n - \dim \mathrm{Ker}(A^TA) = 0$, which implies that A is zero matrix. (An alternative proof is that $\mathrm{tr}(A^TA) = 0$ implies $\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^2 = 0$. Thus $a_{ij} = 0$ where a_{ij} denotes the element in the ith row and jth column of A)

3. True. Since $A-B$ is real symmetric, there exist an orthogonal matrix O and a real diagonal matrix D with diagonal elements $\{d_1, d_2, ..., d_n\}$ such that $A - B = O^TDO$. From $x^T(A-B)x = 0$, $x^TO^TDOx = 0$. We can construct a series of vectors x_i such that $Ox_i = e_i$ where e_i is the standard unit vector (only i th element is 1 and other elements are 0 in this vector). Since $d_i = e_i^TDe_i = 0$ for $i = 1, 2, ..., n$, we conclude that $D = 0$ and it follows that $A - B = 0$ and $A = B$.

Problem 2

1. Let A have distinct eigenvalues $\lambda_1, ..., \lambda_n$ with corresponding eigenvectors $x_1, ..., x_n$. If follows that $Ax_i = \lambda_i x_i$ for $i = 1, ...n$. Since A commutes with B, $ABx_i = BAx_i = B\lambda_i x_i = \lambda_i Bx_i$ which implies that Bx_i is also an eigenvector of A with respect to eigenvalue λ_i (when $Bx_i = 0$, the following analysis still holds). Therefore, Bx_i lies in the eigenspace spanned by x_i so that there exists μ_i such that $Bx_i = \mu_i x_i$. Hence (μ_i, x_i) is an eigenvalue and eigenvector pair for B, for each $i = 1, ...n$. Let $O = [x_1, x_2, ..., x_n]$ and D be a diagonal matrix with diagonal elements $\{\mu_1, \mu_2, ..., \mu_n\}$, then $BO = OD$ and $B = ODO^{-1}$. Therefore, B is diagonalizable.

2. Let Λ be a diagonal matrix with diagonal elements $\lambda_1, ..., \lambda_n$, then $A = O\Lambda O^{-1}$. Since $B = ODO^{-1}$, it is sufficient to prove that there exist coefficients $a_0, a_1, ..., a_{n-1}$ such that $D = a_{n-1}\Lambda^{n-1} + a_{n-2}\Lambda^{n-2} + ... + a_1\Lambda + a_0 I$. Since D and Λ are diagonal matrices, $\mu_i = a_{n-1}\lambda_i^{n-1} + a_{n-2}\lambda_i^{n-2} + ... + a_1\lambda_i + a_0$ should hold for $i = 1, ..., n$. Equivalently, there should exist a solution for the linear equation

$$\begin{bmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_n & \cdots & \lambda_n^{n-1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{bmatrix}$$

where the leftmost square matrix denoted as V is a Vandermonde matrix and

$$\det(V) = \prod_{1 \le i < j \le n} (\lambda_j - \lambda_i)$$

is non-zero since A has distinct eigenvalues. Thus V is non-singular and there exists a unique solution for the linear equation, which completes the proof.

Problem 3

1. Characteristic polynomial of A: $A - \lambda I = -\lambda^3 + 2\lambda^2 - \lambda = -\lambda(\lambda - 1)^2$.
   Therefore, eigenvalues of A: 0 or 1.
2. Algebraic multiplicity for $\lambda = 0$ is 1, and for $\lambda = 1$ is 2.
3. Geometric multiplicity for $\lambda = 0$ is 1, and for $\lambda = 1$ is 1.
4. A matrix is diagonalizable if and only if the algebraic multiplicity equals the geometric multiplicity of each eigenvalue. Therefore, matrix A is *not diagonalizable*
5. For $Av_1 = 0$, $v_1 = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$, For $(A - I)v_2 = 0$, $v_2 = \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix}$, For $(A - I)v_3 = v_2$, $v_3 = \begin{pmatrix} 0 \\ 3 \\ -5 \end{pmatrix}$
6. $P = \begin{pmatrix} 0 & 1 & 0 \\ -1 & -1 & 3 \\ 2 & 5 & -5 \end{pmatrix}$ and rank of matrix P is 3. $J = P^{-1}AP = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$