1 Problem 1

Consider a linear time invariant system $\dot{x} = Ax$, $x(0) = [0,1,1]^T \in \mathbb{R}^3$ with

$$A = \begin{bmatrix} -1 & 2 & -1 \\ 0 & -1 & 0 \\ 0 & 1 & -2 \end{bmatrix}.$$

1. Find the Jordan canonical form of A and give the transformation matrix.

Answer: The rank of the matrix is 3

step 1: computing Eigenvalues

$A - \lambda I = \begin{bmatrix} -1-\lambda & 2 & -1 \\ 0 & -1-\lambda & 0 \\ 0 & 1 & -2-\lambda \end{bmatrix}$ $\det(A-\lambda I) = (-1-\lambda)\cdot[(-1-\lambda)\cdot(-2-\lambda)-(0\cdot 1)]$

$\Rightarrow \lambda_1 = -2, \quad \lambda_2 = -1$

Step 2: find Eigenvectors

$\lambda_1 = -2 \quad A - \lambda I = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow[\text{from row3}]{\text{subtract row2}} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[\text{from row 1}]{\text{subtract 2(row2)}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

so we have $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

same Procedure for $\lambda_2 = -1$, we have Eigenvector $V_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

and we need to find the generized vector, $\Rightarrow V = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

so our matrix in basis $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$

Jordan form is compute by $A = S \cdot J \cdot S^{-1}$, we have $S = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

and $S^{-1} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix}$ $\quad J = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

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2. Find the solution of the differential equation under given initial conditions.

Answer:

To solve the initial problem, we have to find general solution:

$$x(t) = C_1 V_1 e^{\lambda_1 t} + C_2 V_2 e^{\lambda_2 t}$$ ~~from question 1~~

From question 1, we know $\lambda_1 = -1$, $\lambda_2 = -2$

when $\lambda_1 = -1$ $V_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

When $\lambda_2 = -2$ $V_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Substitude the value to the $x(t)$

$$x(t) = C_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{-t} + C_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{-2t}$$

$$x(0) = [0, 1, 1]^T = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$x(0) = C_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \begin{cases} C_1 + C_2 = 0 \\ C_2 = 1 \end{cases} \Rightarrow \begin{cases} C_2 = 1 \\ C_1 = -1 \end{cases}$$

So we have $x(t) = -\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{-t} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{-2t}$

3. Let A be perturbed by a constant $a \in \mathbb{R}$ and

$$A = \begin{bmatrix} -1 & 2 & -1 \\ 0 & -1 & a \\ 0 & 1 & -2 \end{bmatrix}.$$

Find the possible values of a such that the system $\dot{x} = Ax$ is asymptotically stable.

Answer: This equilibrium is said to be lyapunov stable. If, for every $\epsilon > 0$, there exists a $\delta > 0$, such that if $\|x(0) - x_e\| < \delta$, then for every $t \geq 0$ we have $\|x(t) - x_e\| < \epsilon$.

let $\det(IS - A) = 0$

$$IS - A = \begin{bmatrix} s+1 & -2 & 1 \\ 0 & s+1 & -a \\ 0 & -1 & s+2 \end{bmatrix}$$

$$\det(IS - A) = (s+1)[(s^2+3s+2) - a] = 0 \Rightarrow \begin{cases} s_1 = -1 \\ s^2+3s+2 = a \end{cases}$$

we can get $s_2 = -\dfrac{3 - \sqrt{1+4a}}{2}$ $\quad s_3 = \dfrac{-3 + \sqrt{1+4a}}{2}$

so $-3 + \sqrt{1+4a} < 0$ and $1 + 4a \geq 0$

$\Rightarrow -\frac{1}{4} \leq a < 2$

so when $Re\{\lambda i\} < 0$ for all λi then we called A is Hurwitz

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we can see the real part of each eigenvalue is strictly negative

if $-\frac{1}{4} \leq a < 2$, the system is asymptotically stable

2 Problem 2

Are the following statements true or false? If true, prove the statement. If false, give a counterexample.

1. Assume $X, Y \in \mathbb{R}^{n\times n}$ satisfying $XY = YX$ and x is a generalized eigenvector of X. Then, Yx is also a generalized eigenvector of X.

Answer: True

let assume $X, Y \in \mathbb{R}^{n\times n}$ satisfying $XY = YX$

we know that $(X-\lambda I)^{P} y_x = 0$ for some P and eigenvalue λ.

If X is an $n\times n$ matrix, a generilized eigenvector of a corresponding to the eigenvalue λ is nonzero vector. y_x satisfying $(X-\lambda I)^{P} y_x = 0$ for some positive integer P. Equivalently, it is a nonzero element of the null space of $(X-\lambda I)^{P}$.

now $(X-\lambda I)^{P} \cancel{X} y_x = y \cancel{X} (X-\lambda I)^{k} x = 0$

Since Y_x is also generalized Eigenvector of X.

Hence proved!

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2. Let $A \in \mathbb{R}^{m \times n}$. Then, $\text{rank}(AA^T) = \text{rank}(A)$.

Answer : True

The null space $N(A^T)$ of A^T contains vectors x which satisfies the equation $A^T \cdot x = 0$. These vectors ~~can~~ then will also satisfy the equation $AA^T \cdot x = 0$ which implies that $N(A^T) \subset N(AA^T)$. To prove that $N(A^T) = N(AA^T)$. we now need to show that $N(AA^T) \subset N(A^T)$. In order to do that, Let's pick any vector $x \in N(AA^T)$. So,

$AA^T \cdot x = 0 \Rightarrow x^T \cdot AA^T \cdot x = 0 \Rightarrow \|A^T \cdot x\|^2 = 0 \Rightarrow A^T \cdot x = 0$
$\Rightarrow x \in N(A)$,

So from the above implication we can say that $N(AA^T) \subset N(A^T)$. So $N(AA^T) = N(A^T)$

and $\dim(N(AA^T)) = \dim(N(A))$

$\Rightarrow \text{rank}(AA^T) = \text{rank}(A)$

3. Let $A \in \mathbb{R}^{4\times 6}$, and its null space is 3-dimensional. Then, for any $b \in \mathbb{R}^4$, the equation $Ax = b$ has a solution.

Answer: False

Since A has 6 columns and the nulity of A is 3 the rank of A is 6-3=3. Thus the dimension of the column space of A is 3.
if $Ax=b$ has solution, dim $-b \leq 3$

counter Example:

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

we can see that $Ax=b$ doesn't have solution

3 Problem 3

Suppose there are three birds flying in the sky and assume the birds' speeds are $v_1(k)$, $v_2(k)$ and $v_3(k)$ at moment k, respectively. Each bird will adjust its speed based on the average of the other two birds' speeds, i.e. at moment $k+1$, the birds' speeds are

$$v_1(k+1) = \frac{v_2(k)+v_3(k)}{2}, \quad v_2(k+1) = \frac{v_1(k)+v_3(k)}{2} \quad \text{and} \quad v_3(k+1) = \frac{v_1(k)+v_2(k)}{2},$$

respectively. Assume the initial speeds are $v_1(0) = 8\text{m/s}$, $v_2(0) = 9$ m/s and $v_3(0) = 13$ m/s.

1. What are the speeds of the birds at moment $k = 1$?

Answer:

$$V_1(1) = \frac{V_2(0)+V_3(0)}{2} = \frac{9+13}{2} = 11\text{m/s}$$

$$V_2(1) = \frac{V_1(0)+V_3(0)}{2} = \frac{8+13}{2} = 10.5\text{ m/s}$$

$$V_3(1) = \frac{V_1(0)+V_2(0)}{2} = \frac{8+9}{2} = 8.5\text{ m/s}$$

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2. Assume we lump the speeds into a vector as $x(k) = [v_1(k), v_2(k), v_3(k)]^T \in \mathbb{R}^3$, then the relation between $x(k+1)$ and $x(k)$ can be described as $x(k+1) = Ax(k)$ with $A \in \mathbb{R}^{3\times 3}$. Find the A matrix.

Answer: $X(K) = [V_1(K), V_2(K), V_3(K)]^T = \begin{bmatrix} V_1(K) \\ V_2(K) \\ V_3(K) \end{bmatrix}$

$$X(K+1) = \begin{bmatrix} \frac{V_2(k)+V_3(k)}{2} \\ \frac{V_1(k)+V_3(k)}{2} \\ \frac{V_1(k)+V_2(k)}{2} \end{bmatrix} \quad \text{assume } A = \begin{bmatrix} A_1 & A_2 & A_3 \\ A_4 & A_5 & A_6 \\ A_7 & A_8 & A_9 \end{bmatrix}$$

So we have

$$\begin{bmatrix} \frac{V_2(K)+V_3(K)}{2} \\ \frac{V_1(K)+V_3(K)}{2} \\ \frac{V_1(K)+V_2(K)}{2} \end{bmatrix} = \begin{bmatrix} A_1 & A_2 & A_3 \\ A_4 & A_5 & A_6 \\ A_7 & A_8 & A_9 \end{bmatrix} \begin{bmatrix} V_1(K) \\ V_2(K) \\ V_3(K) \end{bmatrix} \Rightarrow \begin{cases} A_1V_1(K)+A_2V_2(K)+A_3V_3(K) = \frac{V_2(K)+V_3K}{2} \quad ① \\ \end{cases}$$

$$A_4V_1(K)+A_5V_2(K)+A_6V_3(K) = \frac{V_1(K)+V_3(K)}{2} \quad ②$$

$$\cancel{A_7(V_1} \; A_7V_1(K)+A_8V_2(K)+A_9V_3(K) = \frac{V_1(K)+V_2(K)}{2} \quad ③$$

Combine ①, ②, ③, we get $A_1=0$, $A_2=\frac{1}{2}$ $A_3=\frac{1}{2}$, $A_4=\frac{1}{2}$ $A_5=0$
$A_6=\frac{1}{2}$ $A_7=\frac{1}{2}$ $A_8=\frac{1}{2}$ $A_9=0$

so $A = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$ #

3. Find a canonical diagonal matrix similar to A, and give the orthogonal transformation matrix.

Answer:

step1: find eigenvalue

$$A=\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \quad A-\lambda I=\begin{bmatrix} -\lambda, & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\lambda & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\lambda \end{bmatrix}$$

$$\det(A-\lambda I)=0=-\lambda\left[(-\lambda\cdot-\lambda)-(\tfrac{1}{2}\cdot\tfrac{1}{2})\right]-\tfrac{1}{2}\left[(\tfrac{1}{2}\cdot-\lambda)-(\tfrac{1}{2}\cdot\tfrac{1}{2})\right]+\tfrac{1}{2}\left[\tfrac{1}{4}-(\tfrac{1}{2}\cdot-\lambda)\right]$$

$$=\frac{3\lambda}{4}-\lambda^3+\frac{1}{4} \Rightarrow \lambda_1=1 \quad \lambda_2=-\frac{1}{2}$$

step 2: find eigenvectors

$\lambda_1=1$ Eigenvectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\lambda_2=\frac{1}{2}$ Eigenvectors $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

So $P=\begin{bmatrix} 1 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad P^{-1}=\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$

orthogonal transformation matrix

$$O=P^{-1}AP=\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}P=\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{6} & -\frac{1}{3} \\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \end{bmatrix}\cdot\begin{bmatrix} 1 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} \#$$

4. Find the speeds of the birds when $k \to \infty$, i.e. $\lim_{k\to\infty} x(k)$.

Answer:

when $k \to \infty$ $\lim_{k\to\infty} x(k)$

from 3.3, we have eigenvalue $\lambda_1 = 1$

and the Correspondant Vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

the initial speed is $V_1(0) + V_2(0) + V_3(0) = 8 + 9 + 13 = 30$.

So ~~the~~ it will be $\begin{bmatrix} 10 \\ 10 \\ 10 \end{bmatrix}$

Thus $\lim_{k\to\infty} x(k) = \begin{bmatrix} 10 \\ 10 \\ 10 \end{bmatrix}$ #