① Consider the Matrix

$$A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}_{3\times 3}$$

Definition: The set of solutions to $A\vec{x} = \vec{0}$ is called null space (kernel) of A, often denoted as null(A)

The homogeneous system is:

$$X_1 + 4X_2 + 7X_3 = 0$$
$$2X_1 + 5X_2 + 8X_3 = 0$$
$$3X_1 + 6X_2 + 9X_3 = 0$$

The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 4 & 7 & 0 \\ 2 & 5 & 8 & 0 \\ 3 & 6 & 9 & 0 \end{array}\right]$$

Its row-echelon form is

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore, X_3 is free variable $\Rightarrow$ $\text{null}(A) = \left\{ \left(X_3, \frac{X_3}{2}, X_3\right) \mid X_3 \in R \right\}$

A basis for the null space is $(1, \frac{1}{2}, 1)$.

The dimension of the null space : nullity(A) = 1

The rank of A is the dimension of the row space of A (same as the dimension of the column space of A, and same as the number of leading 1 in the row-echelon form of A)

$$\text{rank}(A) = 2$$

If A is $m\times n$ matrix, $\text{rank}(A) + \text{nullity}(A) = n$

Since the given A is 3×3 matrix, $3 = \text{rank}(A) + \text{nullity}(A) = 2 + 1$ (agreed)

② For any pair of $n \times n$ matrices A B, show that

$$\det(AB) = \det(BA) = \det(A)\det(B)$$

First note the identity

$$\begin{bmatrix} A & 0 \\ I & B \end{bmatrix} = \begin{bmatrix} I & A \\ 0 & I \end{bmatrix} \begin{bmatrix} -AB & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ B & I \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$$

1^{st} 2^{nd} 3^{rd} 4^{th}

The first and third matrices add multiples of rows and columns of $\begin{bmatrix} -AB & 0 \\ 0 & I \end{bmatrix}$ to other rows and columns of $\begin{bmatrix} -AB & 0 \\ 0 & I \end{bmatrix}$. These operations do not affect the determinant of $\begin{bmatrix} -AB & 0 \\ 0 & I \end{bmatrix}$, as already proved. Also, the fourth Matrix Interchanges n pairs of columns of $\begin{bmatrix} 0 & A \\ B & I \end{bmatrix}$, which affects the determinant of $\begin{bmatrix} 0 & A \\ B & I \end{bmatrix}$ by a factor of $(-1)^n$.

Hence,

$$\det(A)\det(B) = \det\begin{bmatrix} A & 0 \\ I & B \end{bmatrix} \quad \text{(proved in HW\#1 for triangular matrix)}$$

$$= (-1)^n \det\begin{bmatrix} -AB & 0 \\ 0 & I \end{bmatrix}$$

$$= (-1)^n \det(-AB)\det(I)$$

$$= (-1)^n \det(-AB)\cdot 1$$

$$= (-1)^n(-1)^n \det(AB) \quad \left(\det \alpha A \triangleq \alpha^n \det(A), \text{ where } A \text{ is } n\times n \text{ matrix}\right)$$

$$= (-1)^{2n}\det(AB)$$

$$= \det(AB) \quad \text{(proved)}$$

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Give some examples to show that $AB \neq BA$

1st example:

Suppose $A = (a_{ij})_{2\times2} = \begin{pmatrix} 3 & 1 \\ 4 & 1 \end{pmatrix}_{2\times2}$

$B = (b_{ij})_{2\times2} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}_{2\times2}$

$$AB = \begin{bmatrix} 3\times1+1\times3 & 3\times2+1\times4 \\ 4\times1+1\times3 & 4\times2+1\times4 \end{bmatrix}_{2\times2} = \begin{pmatrix} 6 & 10 \\ 7 & 12 \end{pmatrix}_{2\times2}$$

$$BA = \begin{bmatrix} 1\times3+4\times2 & 1\times1+2\times1 \\ 3\times3+4\times4 & 3\times1+4\times1 \end{bmatrix}_{2\times2} = \begin{pmatrix} 12 & 3 \\ 23 & 7 \end{pmatrix}_{2\times2} \neq AB$$

2nd example:

Suppose $A = (a_{ij})_{3\times3} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}_{3\times3}$, $B = (b_{ij})_{3\times3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{3\times3}$

$$AB = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}_{3\times3}$$

$$BA = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{3\times3} \cdot \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}_{3\times3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}_{3\times3} \neq AB$$

(4) Given:

$$X_1 + 2X_2 + 3X_3 + 4X_4 = 0$$
$$2X_1 + 4X_2 + \lambda_1 X_3 + \lambda_2 X_4 = 0$$

The augmented matrix is:

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 2 & 4 & \lambda_1 & \lambda_2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 0 & 0 & \lambda_1-6 & \lambda_2-8 & 0 \end{array}\right]$$

Trivial solution means $X_1 = X_2 = X_3 = X_4 = 0$

Non-zero solutions means at least one of X_i's will not be zero, where $i = 1,2,3,4$

And we know there are $4-2=2$ free variables

Let X_2 & X_4 be free variables, Case I

$$(\lambda_1 - 6)X_3 = -(\lambda_2 - 4)X_4$$

$$X_3 = \frac{-(\lambda_2-4)}{\lambda_1 - 6} X_4$$

$$X_1 + 2X_2 + 3X_3 + 4X_4 = 0$$

$$X_1 + 2X_2 - \frac{3(\lambda_2-4)}{\lambda_1-6}X_4 + 4X_4 = 0$$

$$X_1 = -2X_2 + X_4\left[\frac{3(\lambda_2-4)}{\lambda_1-6} - 4\right] = -2X_2 + X_4\left[\frac{3\lambda_2 - 12 - 4\lambda_1 + 24}{\lambda_1 - 6}\right]$$

$$= -2X_2 + X_4\left[\frac{3\lambda_2 - 4\lambda_1 + 12}{\lambda_1 - 6}\right]$$

$\therefore$ The solution set is $\left\langle -2X_2 + X_4\left(\frac{3\lambda_2 - 4\lambda_1 + 12}{\lambda_1 - 6}\right),\ X_2,\ \frac{-(\lambda_2-4)}{\lambda_1-6}X_4,\ X_4 \right\rangle$

Case II:

When $\lambda_1 = 6$ and $\lambda_2 = 8$, we will have

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Hence, we'll have 3 free variables.

Let x_2, x_3 and x_4 be free variables

$$x_1 + 2x_2 + 3x_3 + 4x_4 = 0$$

$$x_1 = -2x_2 - 3x_3 - 4x_4$$

$\therefore$ The solution set is $\langle -2x_2 - 3x_3 - 4x_4,\ x_2,\ x_3,\ x_4 \rangle$

Therefore, λ_1 & λ_2 could be anything and there will be 3 free variables only when $\lambda_1 = 6$ and $\lambda_2 = 8$; there will be 2 free variables, otherwise.