1. If λ_n is the least eigenvalues of a Hermitian matrix H, show that $\lambda_n = \min_{x \neq 0} \frac{\langle Hx, x \rangle}{\langle x. x \rangle}$.

Answer: From Rayleigh's theorem we get:

$$\underset{\text{largest}}{\lambda_*} = \frac{\langle Hx, x \rangle}{\langle x. x \rangle}$$

n can be shown as a diagonal matrix D

$$U * D * u = H$$

$$\langle Hx, x \rangle = x^* Hx$$

$$= x^* u^* D u n$$

$$= (Ux)^* D u n$$

we know that

$\because$ U is unitary matrix.

$$(Un)^* (Ux) = x^* x$$

Hence, $\min D(x) = \min \frac{\langle Dx, x \rangle}{\langle x, x \rangle}$

$\because$ Diagonal matrices contains eigenvalues

$$\therefore \lambda n = \min_{n \neq 0} \frac{\langle Dx, x \rangle}{\langle x, x \rangle} = \min_{n \neq 0} \frac{\langle Hx, x \rangle}{\langle x, x \rangle}$$

2. Find all possible values of μ guaranteeing the positive-definiteness of

$$H = \begin{pmatrix} 1 & 2 & 3 \\ 2 & \mu & 4 \\ 3 & 4 & 5 \end{pmatrix}$$

Answer: we know that a positive definite matrix is a Symmetric matrix with all the eigenvalues (λ).

Given $H = \begin{bmatrix} 1 & 2 & 3 \\ 2 & \mu & 4 \\ 3 & 4 & 5 \end{bmatrix}$

we also know that eigenvalues can be computed as: $|H - \lambda I| = \begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & \mu-\lambda & 4 \\ 3 & 4 & 5-\lambda \end{vmatrix} = 0$

$= (1-\lambda)[(\mu-\lambda)(5-\lambda)-16] - 2[10-2\lambda-12] + 3[8-3\mu-3\lambda]$

$= \lambda^3 - \lambda^2(6+\mu) + \lambda(6\mu-2\mu) - (12-4\mu) = 0$

we know that all roots of the equation would be the

so $(6+\mu)>0$, $(6\mu-2\mu)>0$, $(12-4\mu)>0$

$\Rightarrow \mu > -6$, $\mu > 4$, $\mu < 3$

Hence these are no such μ values for which H can be a positive definite matrix

3. Show that $|x| = \max_k |x_k|$, denoted as $|x|_\infty$, and $|x| = \sum_k |x_k|$, denoted as $|x|_1$, are both norms. what are their associated matrix norms?

Ansver: Given $|x|_\infty = \max_k |x_k|$

and $|x|_1 = \sum_k |x_k|$

To show that $|x|_\infty$ & $|x|_1$ are norms, we need to show that it satisfies norm properties.

for $|x|_\infty = \max_k |x_k|$

① & $|x|_\infty = 0$ iff $x = 0$, as $\max |x_k| = 0$

if $x_k = 0$ $\forall k = 0$

& $|x|_\infty \geq 0$ as $\max_k |x_k| > 0$ as absolute function is always true.

② To show that $|\alpha x|_\infty = |\alpha| \cdot |x|_\infty$

$$
\begin{aligned}
|\alpha \cdot x|_\infty &= \max_k |\alpha x_k| \\
&= \max_k |\alpha| \cdot |x_k| \\
&= |\alpha| \cdot \max_k |\alpha| \cdot |x_k| \\
&= |\alpha| \cdot \max_k \cdot |x_k| \\
&= |\alpha| \cdot |x_k|_\infty \\
&= |\alpha| \cdot |x_k|_\infty \ \#
\end{aligned}
$$

③ To show that, $|x+y|_\infty \leq |x|_\infty + |y|_\infty$

$$
\begin{aligned}
|x+y|_\infty &= \max_k |x_k + y_k| \\
&\leq \max_k (|x_k| + |y_k|) \\
&\leq \max_k |x_k| + \max_k |y_k| \\
&\leq |x_k|_\infty + |y_k|_\infty \quad \#
\end{aligned}
$$

Hence $|x|_\infty$ is a norm

Now for $|x|_1 = \sum_k |x_k|$

① $|x|_1 = 0$, iff $\sum_k |x_k| = 0$, when $|x_k| = 0, \forall k$

$|n|_1 \geq 0$ as $\sum_k |n_k| \geq 0$ as absolute function is always tru

② To show that $|\alpha x_1| = |\alpha|\ |x_1|$

now $|\alpha . x|_1 = \sum_k |\alpha x_k|$

$$
\begin{aligned}
\text{Now } |\alpha . x|_1 &= \sum_k |\alpha x_k| \\
&= \sum_k |\alpha| \cdot \sum_k |x_k| \\
&= |\alpha| \cdot \sum_k |\cancel{x}_k| \\
&= |\alpha| \cdot |x|_1 \quad \#
\end{aligned}
$$

③ $|x+y|_1 = |x|_1 + |y|_1$

$$
\begin{aligned}
\text{Now } |x+y|_1 &= \sum_k |x_k + y_k| \\
&\leq \sum_k |x_k| + \sum_k |y_k| \\
&\leq |x|_1 + |y|_1 \quad \#
\end{aligned}
$$

Hence, $|x|_1$ is also a norm

Now, Associated matrix norm:

elements of A

let A be a matrix, $\|A\|_1 = \max_j \sum_{i=1}^{n} |a_{ij}|$

$\|A\|_n = \max \sum^{n} |n_{ii}|$

$\|A\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}|$