1. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} X = \begin{bmatrix} 10 & 13 \\ 11 & 14 \\ 12 & 15 \end{bmatrix}$.

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, $X = [X_1 \ X_2]$, $b_1 = \begin{bmatrix} 10 \\ 11 \\ 12 \end{bmatrix}$, $b_2 = \begin{bmatrix} 13 \\ 14 \\ 15 \end{bmatrix}$

$\Rightarrow$ Solve $AX_1 = b_1$, $AX_2 = b_2$

1° $AX_1 = b_1$, let $B_1 = [A \ b_1]$

$\text{rank}(A) = 2$, $\text{rank}(B_1) = 2 = \text{rank}(A)$

There are infinite number of solutions.

$X_1 = X_{p_1} + X_{h_1}$ where X_{p_1} is the particular solution
X_{h_1} is the homogeneous solution

$X_{p_1} = \begin{bmatrix} 1 \\ -11 \\ 31/3 \end{bmatrix}$,

Find X_{h_1},

$\Rightarrow$ Solve $\begin{cases} x_{11} + 2x_{12} + 3x_{13} = 0 & -① \\ 4x_{11} + 5x_{12} + 6x_{13} = 0 & -② \\ 7x_{11} + 8x_{12} + 9x_{13} = 0 & -③ \end{cases}$

①×4 − ②, $3x_{12} + 6x_{13} = 0$

$x_{12} = -2x_{13}$

Substitute it into ③, $7x_{11} + (-16x_{13}) + 9x_{13} = 0$

$x_{11} = x_{13}$

$\therefore x_{h_1} = \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \end{bmatrix} = \begin{bmatrix} t_1 \\ -2t_1 \\ t_1 \end{bmatrix} = t_1 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, $t_1 \in \mathbb{R}$

2° Find the solution to $A x_2 = b_2$. , let $B_2 = [A \ b_2]$

$\text{rank}(A) = 2$, $\text{rank}(B_2) = 2 = \text{rank}(A)$

~~There are infinite number of solutions.~~

$$x_2 = x_{p_2} + x_{h_2}$$

$$x_{p_2} = \begin{bmatrix} 1 \\ -14 \\ 40/3 \end{bmatrix}, \quad x_{h_2} = t_2 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \quad t_2 \in \mathbb{R}$$

(x_{h_2} is the solution to $Ax_2 = 0$)

$$\therefore X = [x_1 \ x_2] = [x_{p_1} + x_{h_1} \quad x_{p_2} + x_{h_2}],$$

~~where~~ $x_{p_1} = \begin{bmatrix} 1 \\ -11 \\ 31/3 \end{bmatrix}$, $x_{h_1} = t_1 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, $t_1 \in \mathbb{R}$; $x_{p_2} = \begin{bmatrix} 1 \\ -14 \\ 40/3 \end{bmatrix}$, $x_{h_2} = t_2 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, $t_2 \in \mathbb{R}$.

2.

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} x = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \qquad \text{let } x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\Rightarrow \quad x_1 + 3x_2 = 1 \quad — ①$$

$$2x_1 + 6x_2 = 3 \quad — ②$$

① × 2 − ② , $0 = -1$ conflict !

Hence, there is no solution.

Verified from the theory :

let $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $B = [A \ b] = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 6 & 3 \end{bmatrix}$

$\text{rank}(A) = 1$, $\text{rank}(B) = 2 \neq \text{rank}(A)$

$\Rightarrow$ There is no solution.

#

3. $b = x_0 + x_1 a^1 + x_2 a^2 = [1 \ a^1 \ a^2] \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix}$,

Find a least - squares fit

$\Rightarrow$ Equivalently, Find the solutions to $A^T A x = A^T b$

where $A = [1 \ a^1 \ a^2] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 3 & 0 \\ 1 & 4 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 3 & 0 \\ 1 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 10 & 1 \\ 10 & 30 & 4 \\ 1 & 4 & 1 \end{bmatrix}$$

$$(A^T A)^{-1} = \frac{1}{6} \begin{bmatrix} 14 & -6 & 10 \\ -6 & 3 & -6 \\ 10 & -6 & 20 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$x = (A^T A)^{-1} A^T b = \frac{1}{6} \begin{bmatrix} 14 & -6 & 10 \\ -6 & 3 & -6 \\ 10 & -6 & 20 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4/3 \\ -1/2 \\ 2/3 \end{bmatrix}$$

best linear fit :

$$b \approx 4/3 - 1/2 \, a^1 + 2/3 \, a^2$$

#

Solution to Problem 4

The eigenvalues of a matrix

$$A = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}$$

are the roots of its characteristic polynomial

$$\begin{aligned} p(\lambda) =& \det(\lambda I - A) \\ =& (\lambda - 1)^2 + 6 \end{aligned}$$

We get $\lambda_1 = 1 + \sqrt{6}j$ and $\lambda_2 = 1 - \sqrt{6}j$.
Their corresponding eigenvectors satisfy

$$\begin{aligned} (\lambda_1 I - A)v_1 =& 0 \\ (\lambda_2 I - A)v_2 =& 0 \end{aligned}$$

We obtain that

$$v_1 = \begin{bmatrix} 1 \\ \frac{\sqrt{6}}{2}j \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ -\frac{\sqrt{6}}{2}j \end{bmatrix} \tag{1}$$

Let two complex numbers w_1 and w_2 such that $x = w_1 v_1 + w_2 v_2$. It is equivalent to

$$x = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

By solving the foregoing matrix equation, we obtain $w_1 = 1/2 - \sqrt{6}/3j$, $w_2 = 1/2 + \sqrt{6}/3j$. As a result, x can be expressed by a linear combination of eigenvectors of A.