1.

(1) let $C_{m\times n} = A + B$

$$C^T = (C_{ij}^T)_{n\times m} = (C_{ji})_{n\times m}$$

$$= (a_{ji} + b_{ji})_{n\times m}$$

$$= (a_{ij}^T + b_{ij}^T)_{n\times m} = (a_{ij}^T)_{n\times m} + (b_{ij}^T)_{n\times m}$$

$$= A^T + B^T$$ #

(2) Show $(AB)^T = B^T A^T$, A, B should be square matrix with the same dimension

let $C_{m\times m} = A_{m\times m} B_{m\times m}$

$$C^T = (C_{ij}^T)_{m\times m} = (C_{ji})_{m\times n}$$

$$C_{ji} = \sum_{k=1}^{m} a_{jk} b_{ki} = \sum_{k=1}^{m} (a_{kj}^T)(b_{ik}^T)$$

$$= \sum_{k=1}^{m} (b_{ik}^T)(a_{kj}^T)$$

$= (i\text{th row in } B^T)(j\text{th column in } A^T)$

Hence, $C^T = B^T A^T$

(3) 1° For $n=2$, from (2), $(AB)^T = B^T A^T$ ... (a)

2° If $n=k$ is true, that is

$$(A_1 A_2 \cdots A_k)^T = A_k^T A_{k-1}^T \cdots A_1^T$$

Then for $n = k+1$, let $B = A_1 A_2 \cdots A_k$

$$(A_1 A_2 \cdots A_k A_{k+1})^T = (B A_{k+1})^T = A_{k+1}^T B^T$$

Since $B^T = (A_1 A_2 \cdots A_k)^T = A_k^T A_{k-1}^T \cdots A_1^T$

$\Rightarrow A_{k+1}^T B^T = A_{k+1}^T A_k^T \cdots A_1^T$

$\Rightarrow (A_1 A_2 \cdots A_k A_{k+1})^T = A_{k+1}^T A_k^T \cdots A_1^T$, for all $k > 2$ ...(b)

From (a), (b), by deduction,

$\Rightarrow (A_1 A_2 \cdots A_n)^T = A_n^T \cdots A_2^T A_1^T$ #

2. A and B are symmetric $\Rightarrow A^T = A,\ B^T = B$ ... ①

Let $C = AB$.

Want to show that if C is symmetric

That is to show if $C = C^T$,

$\Rightarrow$ Show that $C^T = (AB)^T = B^T A^T \stackrel{?}{=} AB = C$ ... ②

From ①, ②, It is very clear that if $BA = AB$

then $C = AB$ is symmetric.

Here is an example:

Let $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \\ 3 & 7 \end{bmatrix}$, A, B symmetric

$C = AB = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 3 & 7 \end{bmatrix} = \begin{bmatrix} 4 & 10 \\ -2 & -4 \end{bmatrix}$ not symmetric

$BA = \begin{bmatrix} 1 & 3 \\ 3 & 7 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 10 & -4 \end{bmatrix} \neq AB$

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3.

let $C = A + jB$, A, B real

C is Hermitian $\Rightarrow$ $C^* = C$

$C^* = (A + jB)^* = A^* - jB^*$

$= A^T - jB^T$ (A, B real)

$C^* = C$

$A^T - jB^T = A + jB$

$\Rightarrow A^T = A, \ B^T = -B$

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4.

let $A_{n \times n} = \begin{bmatrix} A_1 & B \\ 0 & A_2 \end{bmatrix}$, where A_1 is with size of $p \times p$, whereas A_2 $q \times q$

$p + q = n$

By definition, $\det(A) = \sum_{(j_1 \cdots j_n)} S(j_1, \cdots j_n) a_{1j_1} \cdots a_{(p+1)j_{p+1}} \cdots a_{nj_n}$

If at least one of $a_{(p+1)j_{p+1}}, \cdots a_{nj_n}$ is not chosen from A_2, then $S(j_1, \cdots j_n) a_{1j_1} \cdots a_{(p+1)j_{p+1}} \cdots a_{nj_n}$ will be zero.

That is $j_{p+1}, j_{p+2} \cdots j_n$ are not chosen from $\{p+1, p+2, \cdots n\}$

then $S(j_1, \cdots j_n) a_{1j_1} \cdots a_{nj_n} = 0$

$$\Rightarrow \det A = \sum_{\substack{(j_1 \cdots j_p) \in \{1, \cdots p\} \\ (j_{p+1} \cdots j_n) \in \{p+1, \cdots n\}}} S(j_1, \cdots j_n)\, a_{1j_1} a_{2j_2} \cdots a_{nj_n}$$

$$= \sum_{\substack{(j_1 \cdots j_p) \in \{1, \cdots p\} \\ (j_{p+1} \cdots j_n) \in \{p+1, \cdots n\}}} S(j_1, \cdots j_p)\, a_{1j_1} \cdots a_{pj_p}\, S(j_{p+1}, \cdots j_n)\, a_{p+1\,j_{p+1}} \cdots a_{nj_n}$$

$$= \left(\sum_{(j_1 \cdots j_p)} S(j_1, \cdots j_p)\, a_{1j_1} \cdots a_{pj_p} \right) \left(\sum_{\substack{(j_{p+1}, \cdots j_n) \\ \in \{p+1 \cdots n\}}} S(j_{p+1}, \cdots j_n)\, a_{p+1\,j_{p+1}} \cdots a_{nj_n} \right)$$

$$= (\det A_1) \cdot (\det A_2)$$

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