~~EL5~~25: Applied Matrix Theory

HW #5

① Given:

$$X^1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \quad X^2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \quad X^3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

We have to use Gram-Schmidt process **to find a set of mutually orthonormal vectors** u^1, u^2, u^3

By definition,

$$y^i := x^i + \sum_{k=1}^{i-1} a_{(i-1)k} X^k$$

So, three mutually orthogonal vectors:

$$y^1 = X^1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

$$y^2 = X^2 + a_{11} X^1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} + a_{11} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} a_{11} \\ 1 \\ 0 \\ -1-a_{11} \end{bmatrix}$$

$$y^3 = X^3 + a_{21} X^1 + a_{22} X^2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} + a_{21} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + a_{22} \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} a_{21} \\ a_{22} \\ 1 \\ -1-a_{21}-a_{22} \end{bmatrix}$$

With the scalars $a_{(i-1)k}$ chosen, it achieves the mutual orthogonality condition:

$$< y^i, y^j > = 0 \quad \forall i \neq j$$

Therefore, we know:

$$<y^1, y^2> = <y^2, y^1> = 0$$

$$<y^1, y^3> = <y^3, y^1> = 0$$

$$<y^2, y^3> = <y^3, y^2> = 0$$

$\langle y^1, y^2 \rangle = 0$

$$\begin{bmatrix} 1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} a_{11} \\ 1 \\ 0 \\ -1-a_{11} \end{bmatrix} = 0$$

$$a_{11} - (-1 - a_{11}) = 0$$

$$2a_{11} = -1$$

$$a_{11} = \frac{-1}{2} \Rightarrow y^2 = \begin{bmatrix} -1/2 \\ 1 \\ 0 \\ -1/2 \end{bmatrix}$$

$\langle y^1, y^3 \rangle = 0$

$$\begin{bmatrix} 1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} a_{21} \\ a_{22} \\ 1 \\ -1-a_{21}-a_{22} \end{bmatrix} = 0$$

$$a_{21} - (-1 - a_{21} - a_{22}) = 0$$

$$2a_{21} + a_{22} = -1 \quad \text{—— (1)}$$

$\langle y^2, y^3 \rangle = 0$

$$\begin{bmatrix} -1/2 & 1 & 0 & -1/2 \end{bmatrix} \begin{bmatrix} a_{21} \\ a_{22} \\ 1 \\ -1-a_{21}-a_{22} \end{bmatrix} = 0$$

$$\frac{-1}{2}a_{21} + a_{22} - \frac{1}{2}(-1 - a_{21} - a_{22}) = 0$$

$$\frac{-1}{2}a_{21} + a_{22} + \frac{1}{2} + \frac{1}{2}a_{21} + \frac{a_{22}}{2} = 0$$

$$\frac{3}{2}a_{22} = \frac{-1}{2}$$

$$a_{22} = \frac{-1}{3}$$

Sub $a_{22} = \frac{-1}{3}$ **into** (1)

$$2a_{21} - \frac{1}{3} = -1$$

$$a_{21} = \frac{-1}{3} \Rightarrow y^3 = \begin{bmatrix} -1/3 \\ -1/3 \\ 1 \\ -1/3 \end{bmatrix}$$

Now we know

$$y^1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \quad y^2 = \begin{bmatrix} -1/2 \\ 1 \\ 0 \\ -1/2 \end{bmatrix}, \quad y^3 = \begin{bmatrix} -1/3 \\ -1/3 \\ 1 \\ -1/3 \end{bmatrix}$$

By definition,

Othonormal vectors $u^i \triangleq y^i / \|y^i\|$, $i = 1, 2, \ldots N$

Therefore,

$$u^1 = \frac{y^1}{\|y^1\|} = \begin{bmatrix} \sqrt{2}/2 \\ 0 \\ 0 \\ -\sqrt{2}/2 \end{bmatrix}, \text{ where } \|y^1\| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

$$u^2 = \frac{y^2}{\|y^2\|} = \begin{bmatrix} -\sqrt{6}/6 \\ \sqrt{6}/3 \\ 0 \\ -\sqrt{6}/6 \end{bmatrix}, \text{ where } \|y^2\| = \sqrt{\left(\frac{-1}{2}\right)^2 + 1^2 + \left(\frac{-1}{2}\right)^2} = \frac{\sqrt{6}}{2}$$

$$u^3 = \frac{y^3}{\|y^3\|} = \begin{bmatrix} -\sqrt{3}/6 \\ -\sqrt{3}/6 \\ \sqrt{3}/2 \\ -\sqrt{3}/6 \end{bmatrix}, \text{ where } \|y^3\| = \sqrt{\left(\frac{-1}{3}\right)^2 + \left(\frac{-1}{3}\right)^2 + 1^2 + \left(\frac{-1}{3}\right)^2} = \frac{2}{\sqrt{3}}$$