

Lecture VII

- The Jordan Canonical Form
- Examples and Applications

Review of Canonical Forms

- If A is an $n \times n$ matrix with **distinct** eigenvalues, then there exists a nonsingular matrix P s.t.

$$P^{-1}AP = \text{diag}(\lambda_i).$$

- If A is **Hermitian** (possibly having repeated eigenvalues), \exists a unitary matrix U such that

$$U^*AU = \text{diag}(\lambda_i).$$

A Motivating Example

As stated previously, not every matrix can be transformed into a canonical diagonal form.

For example,

$$A = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \quad b \neq 0$$

cannot be transformed into a diagonal matrix.

Jordan Canonical Form

Theorem (Jordan):

Let A be an $n \times n$ matrix whose **different** eigenvalues are $\lambda_1, \dots, \lambda_s$ with multiplicities m_1, \dots, m_s :

$$\det(\lambda I - A) = \prod_{i=1}^s (\lambda - \lambda_i)^{m_i}$$

Then, A is transformable into a Jordan canonical form.

i.e., \exists **nonsingular** P such that

$$P^{-1}AP = \text{blockdiag}(\Lambda_i) \doteq J$$

Theorem (Jordan), **cont'd**:

$$P^{-1}AP = \text{blockdiag}(\Lambda_i) \doteq J$$

where

$$\Lambda_i = \begin{pmatrix} \lambda_i & 0 & \cdots & \cdots & 0 \\ 1 & \lambda_i & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & \lambda_i \\ 0 & 0 & \cdots & \cdots & 1 & \lambda_i \end{pmatrix}$$

Jordan
Block

Comments

- *In some texts, J^T is used as Jordan form.*
- Different Jordan blocks, say Λ_i, Λ_j may be associated with the same eigenvalues.
- \bar{s} = The total number of Jordan blocks: $s \leq \bar{s} \leq n$.

Illustration via 3x3 matrices

If a 3×3 matrix A has an eigenvalue λ_1 of multiplicity three, then it may be reduced into one of the following Jordan forms:

$$J_1 = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{pmatrix}, \quad J_2 = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 1 & \lambda_1 \end{pmatrix},$$

$$J_3 = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 1 & \lambda_1 & 0 \\ 0 & 1 & \lambda_1 \end{pmatrix}.$$

Remark 1

The distinct Jordan forms (J_i, J_k) , $i \neq k$, are **not** similar to each other.

Remark 2

When each Jordan block $\Lambda_i(\lambda_i)$ in the Jordan form J is one-dimensional (i.e. $n_i = 1$) and $s = n$, the Jordan matrix J becomes diagonal.

Application to Matrix Analysis of Differential Equations

Given a set of 1st-order differential equations

$$\dot{x}(t) = Ax(t), \quad x(0) \in \mathbb{R}^n,$$

applying the transformation $y = P^{-1}x$ yields:

$$\dot{y}(t) = \left(P^{-1}AP \right) y(t) := Jy(t).$$

$$\Leftrightarrow \dot{y}^i(t) = \Lambda_i y^i(t), \quad y^i \in \mathbb{R}^{m_i}, \quad y = \begin{bmatrix} y^1 \\ \vdots \\ y^{\bar{s}} \end{bmatrix}.$$

Comment

So, with the help of Jordan canonical form, solving differential equations can be reduced down to solving *lower-order (disjoint!)* differential equations.

(see a forthcoming lecture.)

Principal Vectors

In order to develop a constructive method for P resulting in Jordan form, let's introduce the notion of **principal vector**, or generalized eigenvector, which is a generalization of eigenvector.

Principal Vectors

A (possibly zero) vector p is a **principal vector of grade** $g \geq 0$ belonging to the eigenvalue λ_i if

$$(\lambda_i I - A)^g p = 0,$$

for which g is the smallest non-negative integer.

Examples

- The vector $p = 0$ is the principal vector of grade 0.
- The (nonzero) eigenvectors are the principal vectors of grade 1.

Motivating Question

In case of transformation to diagonal canonical form, i.e., $P^{-1}AP = \text{diag}(\lambda_i)$, the columns of P are linearly independent eigenvectors.

What about the matrix P in Jordan form?

How to construct P from principal vectors?

Linear Spaces

Define the linear space composed of all principal vectors of grade $\leq g$ belonging to λ_i :

$$P_g(\lambda_i) = \left\{ p \mid (\lambda_i I - A)^g p = 0 \right\}$$

i.e., the null space of $(\lambda_i I - A)^g$.

Clearly,

$$P_0(\lambda_i) \subset P_1(\lambda_i) \subset P_2(\lambda_i) \subset \dots$$

An Interesting Result

Let A be an $n \times n$ matrix with the distinct eigenvalues $\lambda_1, \dots, \lambda_s$, $1 \leq s \leq n$, with multiplicities m_1, \dots, m_s .

Then, **every** vector $x \in \mathbb{R}^n$ can be written as

$$x = p^1 + p^2 + \dots + p^s$$

where p^i is a uniquely defined principal vector associated with λ_i of grade $\leq m_i$.

Comment 1

A special, but interesting, case is when there are n linearly independent eigenvectors, say, c^1, \dots, c^n . In this case, $\exists \xi_i$ scalars s.t.

$$x = \xi_1 c^1 + \dots + \xi_n c^n := p^1 + \dots + p^n.$$

Comment 2

Its proof relies upon the well-known Cayley-Hamilton theorem; see any standard matrix or linear algebra textbook.

Example

Consider the matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ 7 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

- * Compute its eigenvalues and the associated eigenvectors.
- * Can each column be written as a linear combination of eigenvectors?
- * Show that each column can be written as a unique representation of principal vectors.

Answer

- $\lambda_1 = 1$ ($m_1 = 2$), $\lambda_2 = 2$ ($m_2 = 1$).
- *The* eigenvectors of λ_1 are of the form
 $\xi \times \text{col}(0, 1, 0)$, ξ any nonzero scalar.

The eigenvectors of λ_2 are of the form
 $\xi \times \text{col}(0, 0, 1)$, ξ any nonzero scalar.

- $P_2(\lambda_1) = \{p^1 \mid p^1 = \text{col}(\alpha, \beta, 0)\}$
 $P_1(\lambda_2) = \{p^2 \mid p^2 = \text{col}(0, 0, \gamma)\}.$

Cayley-Hamilton Theorem Revisited

For any $n \times n$ matrix A ,

$$\rho(A) = A^n + \alpha_1 A^{n-1} + \cdots + \alpha_{n-1} A + \alpha_n I = O$$

where $\rho(\lambda)$ is the characteristic polynomial of A , i.e.,

$$\rho(\lambda) = \det(\lambda I - A) = \lambda^n + \sum_{i=1}^n \alpha_i \lambda^{n-i}.$$

Example

Consider $A = \begin{pmatrix} -7 & -4 \\ 8 & 5 \end{pmatrix}$. Verify that

1) The characteristic polynomial $\rho_A(\lambda)$ is:

$$\rho_A(\lambda) = \lambda^2 + 2\lambda - 3.$$

$$2) \rho_A(A) = A^2 + 2A - 3I = 0 \triangleq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Another Proof

Define the $n \times n$ matrix of signed cofactors:

$$C(\lambda) = \text{cof}(\lambda I - A).$$

Then, using $M (\text{cof } M)^T = (\det M) I$,

$$(\lambda I - A) C^T(\lambda) = \rho(\lambda) I.$$

In addition,

$$C^T(\lambda) = \lambda^{n-1} C_0 + \cdots + \lambda C_{n-1} + C_n$$

for constant matrices C_i 's.

Proof (cont'd)

By identification of the coefficients of equal powers of λ gives

$$C_0 = I$$

$$C_1 - AC_0 = \alpha_1 I$$

$$\vdots$$

$$C_{n-1} - AC_{n-2} = \alpha_{n-1} I$$

$$-AC_{n-1} = \alpha_n I.$$

Multiplying the first eq. by A^n , the second by A^{n-1} , ...,
and then adding them up leads to: $O = \rho(A)I$.

Question

How to compute principal vectors for a given matrix?

A Motivating Example

Consider a 2×2 Jordan block $J = \begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix}$.

Denote $P = \begin{bmatrix} x^1 & x^2 \end{bmatrix}$ that transforms A into J .

Namely, $P^{-1}AP = J$. So, we have $AP = PJ$, or

$$A \begin{bmatrix} x^1 & x^2 \end{bmatrix} = \begin{bmatrix} x^1 & x^2 \end{bmatrix} \begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix}$$

$\Rightarrow Ax^2 = \lambda x^2$, so x^2 is an eigenvector;

$(A - \lambda I)x^1 = x^2$, so x^1 is a principal vector (of grade 2).

Comment

Usually, $\{x^1, x^2\}$ is called a **Jordan Basis** for this 2×2 matrix A . In other words, the JCF transformation matrix P is composed of a Jordan basis, or a set of **linearly independent** eigenvectors and principal vectors.

General Procedure

Step 1: Solve the characteristic equation

$$(A - \lambda I) z^1 = 0.$$

Step 2: For each independent z^1 , solve

$$(A - \lambda I) z^2 = z^1$$

where z^2 clearly solves $(A - \lambda I)^2 z^2 = 0$.

Collect only those z^2 which are **linearly independent** with the previously found eigenvectors z^1 .

General Procedure

Step 3: For each independent z^2 , solve

$$(A - \lambda I) z^3 = z^2$$

where z^3 clearly solves $(A - \lambda I)^3 z^3 = 0$.

Collect only those z^3 which are **linearly independent** with the previously found vectors z^1, z^2 .

Step 4: Continue in this way till the total number of independent eigenvectors and principal vectors equals to the (algebraic) multiplicity of λ .

General Procedure

Step 4 (cont'd): Denote

$$\begin{bmatrix} x^1, x^2, \dots, x^m \end{bmatrix} = \begin{bmatrix} z^m, z^{m-1}, \dots, z^1 \end{bmatrix}$$

and

$$P = \begin{bmatrix} x^1, x^2, \dots, x^m \end{bmatrix}.$$

Therefore,  eigenvector

$$P^{-1}AP = J \text{ (associated with eigenvalue } \lambda).$$

Comments

- Not any arbitrary choice of linearly independent principal vectors would lead to a correct transformation matrix P .

*For example, at **Step 2**, the linearly independent principal vectors z^2 are chosen according to*

$$(A - \lambda I) z^2 = z^1$$

but NOT :

$$(\lambda I - A) z^2 = z^1.$$

- See (the 1960 book of Gantmacher, Vol.1, Chap. VI, Section 8) for another general method of constructing a transformation matrix.

More on Jordan Basis

Without going into the full details in proving Jordan's Theorem, let's illustrate the concept of Jordan basis and its use in the canonical transformation.

Consider a principal vector v of grade $g = n = 4$. Define:

$$\left. \begin{aligned} x^1 &:= v \\ x^2 &:= (A - \lambda I) x^1 \\ x^3 &:= (A - \lambda I) x^2 \\ x^4 &:= (A - \lambda I) x^3 \end{aligned} \right\} \text{Jordan Basis}$$

Jordan Basis (cont'd)

Then, the 4×4 matrix A can be transformed into the Jordan canonical form:

$$J = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 1 & \lambda & 0 & 0 \\ 0 & 1 & \lambda & 0 \\ 0 & 0 & 1 & \lambda \end{pmatrix}$$

Principal vector
of grade 4

That is,

$$P^{-1}AP = J, \quad P = (x^1, x^2, x^3, x^4).$$

eigenvector

Comment

If we define $\tilde{P} = (x^4, x^3, x^2, x^1)$, then A is transformed into the Jordan canonical form J^T , i.e.:

$$\tilde{P}^{-1}A\tilde{P} = J^T = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}.$$

A More Complex Case

If $(A - \lambda I)$ has rank $n - 2$, i.e. its null space is of dimension 2, then \exists two linearly independent eigenvectors to $(A - \lambda I)q = 0$.

Thus, we need $n - 2$ linearly independent principal vectors. In this case, the Jordan basis takes the form $\{v^1, v^2, \dots, v^k\}$ and $\{u^1, u^2, \dots, u^l\}$, $k + l = n$.

So, A is transformed into the Jordan canonical form

$$P^{-1}AP = \text{diag} \{J_1, J_2\}.$$

Exercise 1

Find a transformation matrix P to bring the following matrix

$$M = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \quad b \neq 0$$

into the Jordan Canonical Form

$$J = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Exercise 2

Find a transformation matrix to bring the following matrix into a Jordan form:

$$A = \begin{pmatrix} 1 & -1 & 1 & -1 \\ -3 & 3 & -5 & 4 \\ 8 & -4 & 3 & -4 \\ 15 & -10 & 11 & -11 \end{pmatrix}$$

Solution:

$$P = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & -5 & 0 & -5 \\ 0 & 4 & 1 & 5 \\ -1 & 11 & 0 & 12 \end{pmatrix},$$

$$J = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

$\lambda I - A$ becomes (after elementary operations on rows and columns:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda + 1 & 0 \\ 0 & 0 & 0 & (\lambda + 1)^3 \end{pmatrix}.$$

Therefore, the matrix has two elementary divisors:

$$\lambda + 1 \text{ and } (\lambda + 1)^3,$$

which give two Jordan blocks, respectively:

$$J_1 = -1, \quad J_2 = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}.$$

See (the 1960 book of Gantmacher, Vol.1, pp.160-164) for the details.

Practicing Problems for Midterm

1. Compute the eigenvalues of the matrix

$$A = \begin{pmatrix} 7 & -2 \\ 4 & 1 \end{pmatrix}$$

and transform it to one of the canonical forms.

Practicing Problems for Midterm

2. Consider the block diagonal matrix

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \text{ with } A_i \in \mathbb{R}^{n_i \times n_i}, n_1 + n_2 = n.$$

Show that the eigenvalues of A are those of A_1 and A_2 .

Practicing Problems for Midterm

3. Assume A is a nonsingular matrix. If λ is an eigenvalue of A with eigenvector x , show that λ^{-1} is an eigenvalue of A^{-1} . In addition, give an eigenvector associated with λ^{-1} .

Practicing Problems for Midterm

4. Show that $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ cannot be transformed into a diagonal matrix under any similarity transformation.

Practicing Problems for Midterm

5. For any given 2×2 real orthogonal matrix U , one of the following must hold:

(i) $U = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$ for some θ ;

(ii) $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$ for some θ .

(Only for those who love math proof!)

Practicing Problems for Midterm

6. Show that J^T is similar to J . That is,

$$\begin{pmatrix} 0 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 0 \end{pmatrix} J^T \begin{pmatrix} 0 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 0 \end{pmatrix} = J.$$

Practicing Problems for Midterm

7. Assume that A , D are invertible matrices.

Show that

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}BD^{-1} \\ 0 & D^{-1} \end{pmatrix}.$$

Practicing Problems for Midterm

8. Assume that A , D are invertible matrices.

Show that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} + A^{-1}BECA^{-1} & -A^{-1}BE \\ -ECA^{-1} & E \end{pmatrix}$$

where E is the inverse of the **Schur complement**

of A : $E = (D - CA^{-1}B)^{-1}$.

Note: A Very Useful Identity.

Practicing Problems for Midterm

9. Reduce the following matrix into a canonical diagonal form:

$$A = \begin{pmatrix} M & 0_{2 \times 2} \\ 0_{2 \times 2} & M \end{pmatrix}$$

where

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Practicing Problems for Midterm

10. Reduce the following matrix into a Jordan canonical form:

$$A = \begin{pmatrix} 3 & 2 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Practicing Problems for Midterm

11. Rank Inequalities (See Horn-Johanson text, page 13)

- **Sylvester inequality**

$\forall A \in \mathbb{R}^{m \times k}, B \in \mathbb{R}^{k \times n}$, we have

$$(\text{rank}A + \text{rank}B) - k \leq \text{rank}AB \leq \min\{\text{rank}A, \text{rank}B\}.$$

- **Frobenius inequality**

$\forall A \in \mathbb{R}^{m \times k}, B \in \mathbb{R}^{k \times p}, C \in \mathbb{R}^{p \times n}$, we have

$$\text{rank}AB + \text{rank}BC \leq \text{rank}B + \text{rank}ABC$$

with equality iff there are matrices X and Y such that

$$B = BCX + YAB.$$

Homework #7

1. For the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

identify the spaces $P_g(\lambda)$ and the principal vectors of grade 2.

Homework #7

2. Express the following vectors as unique representations of principal vectors found in Problem 1:

$$x = \begin{bmatrix} \sqrt{2} \\ -9 \\ 84 \end{bmatrix}, \quad x = \begin{bmatrix} 0 \\ 9.3 \\ 0 \end{bmatrix}.$$

Homework #7

3. Can you transform the following matrix into a Jordan form:

$$A = \begin{pmatrix} \lambda & \lambda & \lambda \\ 0 & \lambda & \lambda \\ 0 & 0 & \lambda \end{pmatrix}, \quad \lambda \neq 0?$$