1.

$$A = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix},$$

Null space of $A \Rightarrow$ the solution set of $Ax=0$

$$\begin{cases} x_1 + 4x_2 + 7x_3 = 0 & - \text{①} \\ 2x_1 + 5x_2 + 8x_3 = 0 & - \text{②} \\ 3x_1 + 6x_2 + 9x_3 = 0 & - \text{③} \end{cases}$$

① $\times 3$ − ③ , $6x_2 + 12x_3 = 0$, $x_2 = -2x_3$ ... ④

Substitute ② with ④ , $2x_1 - 10x_3 + 8x_3 = 0$

$x_1 = x_3$ ... ⑤

From ④, ⑤ $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$, $t \in \mathbb{R}$

The dimension of the null space is 1.

The rank of A is 2.

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2.

(1) Show that $\det(AB) = (\det A)(\det B)$

let $C = AB$

$$C = [c_{ij}] = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & & & \\ c_{n1} & c_{n2} & \cdots & c_{n1} \end{bmatrix} = \begin{bmatrix} \sum_{k_1=1}^{n} a_{1k_1} b_{k_1 1} & \sum_{k_2=1}^{n} a_{1k_2} b_{k_2 2} & \cdots \sum_{k_n=1}^{n} a_{1k_n} b_{k_n n} \\ \sum_{k_1=1}^{n} a_{2k_1} b_{k_1 1} & \sum_{k_2=1}^{n} a_{2k_2} b_{k_2 2} & \sum_{k_n=1}^{n} a_{2k_n} b_{k_n n} \\ \vdots & \vdots & \\ \sum_{k_1=1}^{n} a_{nk_1} b_{k_1 1} & \sum_{k_2=1}^{n} a_{nk_2} b_{k_2 2} & \sum_{k_n=1}^{n} a_{nk_n} b_{k_n n} \end{bmatrix}$$

$$= \sum_{k_1, k_2 \cdots k_n = 1}^{n} \begin{bmatrix} a_{1k_1} b_{k_1 1} & a_{1k_2} b_{k_2 2} & \cdots & a_{1k_n} b_{k_n n} \\ a_{2k_1} b_{k_1 1} & a_{2k_2} b_{k_2 2} & & a_{2k_n} b_{k_n n} \\ \vdots & \vdots & & \vdots \\ a_{nk_1} b_{k_1 1} & a_{nk_2} b_{k_2 2} & & a_{nk_n} b_{k_n n} \end{bmatrix}$$

$$\det C = \sum_{k_1, k_2 \cdots k_n = 1}^{n} \det \begin{bmatrix} a_{1k_1} b_{k_1 1} & a_{1k_2} b_{k_2 2} & \cdots & a_{1k_n} b_{k_n n} \\ a_{2k_1} b_{k_2 1} & a_{2k_2} b_{k_2 2} & \cdots & a_{2k_n} b_{k_n n} \\ \vdots & \vdots & & \vdots \\ a_{nk_1} b_{k_1 1} & a_{nk_2} b_{k_2 2} & \cdots & a_{nk_n} b_{k_n n} \end{bmatrix}$$

$$= \sum_{k_1 \cdots k_n = 1}^{n} \det \begin{bmatrix} a_{1k_1} & a_{1k_2} & \cdots & a_{1k_n} \\ a_{2k_1} & a_{2k_2} & & a_{2k_n} \\ \vdots & \vdots & & \vdots \\ a_{nk_1} & a_{nk_2} & & a_{nk_n} \end{bmatrix} b_{k_1 1} b_{k_2 2} \cdots b_{k_n n}$$

$k_1, k_2 \cdots k_n$ are from 1 to n, if there are at least two that are equal, then $\det \begin{bmatrix} a_{1k_1} & a_{1k_2} & \cdots & a_{1k_n} \\ a_{2k_2} & a_{2k_2} & \cdots & a_{2k_n} \\ \vdots & & & \vdots \\ a_{nk_1} & a_{nk_2} & \cdots & a_{nk_n} \end{bmatrix} = 0$

$$\det \begin{bmatrix} a_{1k_1} & a_{1k_2} & \cdots & a_{1k_n} \\ a_{2k_2} & a_{2k_2} & \cdots & a_{2k_n} \\ \vdots & \vdots & & \vdots \\ a_{nk_n} & a_{nk_2} & \cdots & a_{nk_n} \end{bmatrix} = S(k_1, k_2 \cdots k_n) \det A$$

$$\Rightarrow \det C = \sum_{(k_1 k_2 \cdots k_n)} S(k_1, k_2 \cdots k_n) \det A \, b_{k_1 1} b_{k_2 2} \cdots b_{k_n n}$$

$$= (\det A) \left[\sum_{(k_1, \cdots k_n)} S(k_1, \cdots k_n) \, b_{k_1 1} b_{k_2 2} \cdots b_{k_n n} \right]$$

$$= (\det A)(\det B^T) = (\det A)(\det B)$$

(2) Following the same procedure, we can obtain

$$\det(BA) = (\det B)(\det A^T) = (\det B)(\det A)$$

$$= (\det A)(\det B)$$

(3) From (1), (2)

$$\Rightarrow \det(AB) = \det(BA) = (\det A)(\det B)$$

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Gored

3.

(1) $A=\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}, \quad B=\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}$

$$AB=\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}=\begin{bmatrix}3 & -1\\7 & -1\end{bmatrix}$$

$$BA=\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}=\begin{bmatrix}4 & 6\\-2 & -2\end{bmatrix}\neq AB$$

(2) $A=\begin{bmatrix}2 & 3\\5 & 7\end{bmatrix}, \quad B=\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$

$$AB=\begin{bmatrix}2 & 3\\5 & 7\end{bmatrix}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}=\begin{bmatrix}5 & 5\\12 & 12\end{bmatrix}$$

$$BA=\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}2 & 3\\5 & 7\end{bmatrix}=\begin{bmatrix}7 & 10\\7 & 10\end{bmatrix}\neq AB$$

(3) $A=\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix} \quad B=\begin{bmatrix}5 & 6\\7 & 8\end{bmatrix}$

$$AB=\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}5 & 6\\7 & 8\end{bmatrix}=\begin{bmatrix}19 & 22\\43 & 50\end{bmatrix}$$

$$BA=\begin{bmatrix}5 & 6\\7 & 8\end{bmatrix}\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}=\begin{bmatrix}23 & 34\\31 & 46\end{bmatrix}\neq AB$$

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4. $x_1 + 2x_2 + 3x_3 + 4x_4 = 0 \quad -①$

$2x_1 + 4x_2 + \lambda_1 x_3 + \lambda_2 x_4 = 0 \quad -②$

$① \times 2 - ②, \quad (6-\lambda_1)x_3 = -(8-\lambda_2)x_4$

1°, If $6-\lambda_1 = 8-\lambda_2 = 0$, That is $\lambda_1 = 6, \lambda_2 = 8$

$\Rightarrow x_1 + 2x_2 + 3x_3 + 4x_4 = 0$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2t_2-3t_3-4t_4 \\ t_2 \\ t_3 \\ t_4 \end{pmatrix} = t_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t_3 \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t_4 \begin{pmatrix} -4 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad t_2, t_3, t_4 \in \mathbb{R}$$

The dimension of the null space is 3.

2° If $\lambda_1 \neq 6, \lambda_2 = 8 \Rightarrow (6-\lambda_1)x_3 = 0, \quad x_3 = 0$

$\Rightarrow x_1 + 2x_2 + 4x_4 = 0$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2t_2-4t_4 \\ t_2 \\ 0 \\ t_4 \end{pmatrix} = t_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t_4 \begin{pmatrix} -4 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad t_2, t_4 \in \mathbb{R}$$

The dimension of the null space is 2.

3° If $\lambda_1 = 6, \lambda_2 \neq 8 \Rightarrow (8-\lambda_2)x_4 = 0 \quad x_4 = 0$

$\Rightarrow x_1 + 2x_2 + 3x_3 = 0$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2t_2-3t_3 \\ t_2 \\ t_3 \\ 0 \end{pmatrix} = t_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t_3 \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad t_2, t_3 \in \mathbb{R}$$

The dimension of the null space is 2.

4° If $\lambda_1 \neq 6$, $\lambda_2 \neq 8$ $\Rightarrow$ $X_4 = \dfrac{-(\lambda_1-6)}{(\lambda_2-8)} X_3$

$$\Rightarrow X_1 + 2X_2 + 3X_3 + 4\cdot\left(\frac{-(\lambda_1-6)}{\lambda_2-8}\right) X_3 = 0$$

$$X_1 + 2X_2 + \left(3 - \frac{4(\lambda_1-6)}{\lambda_2-8}\right) X_3 = 0$$

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix} = \begin{pmatrix} -2t_2 - \left(3 - \frac{4(\lambda_1-6)}{\lambda_2-8}\right) t_3 \\ t_2 \\ t_3 \\ -\frac{\lambda_1-6}{\lambda_2-8} t_3 \end{pmatrix}$$

$$= t_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t_3 \begin{pmatrix} \frac{4(\lambda_1-6)}{\lambda_2-8} - 3 \\ 0 \\ 1 \\ -\frac{\lambda_1-6}{\lambda_2-8} \end{pmatrix}, \quad t_2, t_3 \in \mathbb{R}$$

The dimension of the null space is 2 #