1\. Solve the initial-value problem

$\dot{x}_1 = x_2 + e^{-t}, \qquad x_1(0) = 1$

$\dot{x}_2 = 6(t+1)^{-2} x_1 + \sqrt{t}, \quad x_2(0) = 2.$

Answer. The given func is $\dot{x}_1 = x_2 + e^{-t}$, so $\frac{dx_1}{dt} = x_2 + e^{-t}$

$x_1(0) = 1$

Solving the equations as:

$\frac{dx_1}{dt} = x_2 + e^{-t}$

$dx_1 = (x_2 + e^{-t})dt \quad x_1(t) = x_2 t - e^{t} + c \rightarrow ①$

Subsituting the initial condition,

$x_1(t) = x_2(t) - e^{t} + C$

$x_1(0) = x_2(0) - e^{0} + C \quad \Rightarrow C = 2$

Subsititute in equation ①

$x_1(t) = x_2(t) - e^{t} + 2$

next: Given : $\dot{x}_2 = 6(t+1)^{-2} x_1 + \sqrt{t}$

i.e: $\frac{dx_2}{dt} = 6(t+1)^{-2} x_1 + \sqrt{t}$

$x_2(0) = 2.$

Solving the equations as:

$\frac{dx_2}{dt} = 6(t+1)^{-2} x_1 + \sqrt{t}$

$dx_2 = (6(t+1)^{-1} x_1 + \sqrt{t})\, dt$

$x_2(t) = -6(t+1)^{-1} x_1 + \frac{2t\sqrt{t}}{3} + C \rightarrow ②$

Subsituting $x_2(0) = 2$ in ②

$x_2(0) = -6(1)^{-1} x_1 + 0 + C$

$2 = -6x_1 + C \quad \Rightarrow C = 2 + 6x_1$

Sustitute in equation ②

$x_2(t) = -6(t+1)^{-1} x_1 + \frac{2t\sqrt{t}}{3} + 2 + 6x_1$

$x_2(t) = [t(t+1)^{-1}]6x_1 + \frac{2t\sqrt{t}}{3} + 2$ » solution for 2nd equation

2. Solve the initial-value problem

$$\frac{dy(t)}{dt} = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 & 3 \end{pmatrix} y(t), \quad y(0) = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$$

Answer: $y' = Ay$

we know $|A - \lambda I| = 0$

$$\Rightarrow A - \lambda I = \begin{bmatrix} 2-\lambda & 0 & 0 & 0 & 0 \\ 1 & 2-\lambda & 0 & 0 & 0 \\ 0 & 0 & 3-\lambda & 0 & 0 \\ 0 & 0 & 1 & 3-\lambda & 0 \\ 0 & 0 & 0 & 1 & 3-\lambda \end{bmatrix} \quad |A - \lambda I| = (2-\lambda)^2 (3-\lambda)^3 = 0$$

i.e $\lambda = 2$ & $\lambda = 3$

$$\text{for } \lambda = 2 \Rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Geometric multiplicity $= 1$

$$\text{for } \lambda = 3 \Rightarrow A - \lambda I = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

similarly $G.M = 1$

$\therefore$ we have $H^{-1}AH = J$

$$H = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \quad J = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

General form $\Rightarrow$

$$y(t) = c_1 e^{2t}\begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix} + c_2 e^{2t}\left(t\begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix} + \begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}\right) + c_3 e^{3t}\begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix} +$$

$$c_4 e^{3t}\left[t\begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix} + \begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix}\right] + c_5 e^{3t}$$

If we have $y(t)$ when $t=0$ $\quad y(0) = \begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix}$

$$\begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix} = c_1 + c_2 + c_3 + c_4 + c_5$$

$$\begin{matrix}\downarrow & \downarrow & \downarrow & \downarrow & \downarrow\\ 1 & 2 & 3 & 4 & 5\end{matrix}$$