1.

First show that Rayleigh Principle.

H is Hermitian. $\min_{\|u\|=1} \langle Hu, u\rangle = \lambda_n$

where λ_n is the least eigenvalue of H

H is Hermitian, The eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_n$ of H are real and we can find the corresponding orthonormal eigenvectors $u^1, u^2, \cdots u^n$.

$$\|u^i\| = \sqrt{\langle u^i, u^i\rangle} = 1$$

$$\langle u^i, u^j\rangle = 0 \quad \text{for } i \neq j$$

$$\langle Hu^i, u^i\rangle = \lambda_i \quad , \quad \langle Hu^i, u^j\rangle = 0. \text{ for } i \neq j$$

For every unit vector u can be written as the linear combination of $u^1, u^2, \cdots u^n$

$$u = c_1 u^1 + c_2 u^2 + \cdots + c_n u^n$$

where $|c_1|^2 + |c_2|^2 + \cdots + |c_n|^2 = 1$

$$\langle Hu, u\rangle = \langle H(c_1 u^1 + c_2 u^2 + \cdots + c_n u^n), u\rangle$$

$$= \sum_i |c_i|^2 \lambda_i \|u^i\|^2 \geq \sum_i |c_i|^2 \lambda_n = \lambda_n$$

(λ_n is the least eigenvalue)

Thus, $\min_{\|u\|=1} \langle Hu, u\rangle = \lambda_n \quad \cdots \quad ①$

equality iff $u = u^n$.

Show that $\lambda_n = \min\limits_{X \neq 0} \dfrac{\langle HX, X \rangle}{\langle X, X \rangle}$

$X \neq 0$, let $u = \dfrac{X}{\|X\|}$ be a unit vector

From ①, $\min\limits_{\|u\|=1} \langle Hu, u \rangle = \lambda_n$

$$\min_{X \neq 0} \left\langle H \frac{X}{\|X\|}, \frac{X}{\|X\|} \right\rangle = \lambda_n$$

$$\Rightarrow \min_{X \neq 0} \frac{\langle HX, X \rangle}{\langle X, X \rangle} = \lambda_n$$

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2. $H = \begin{bmatrix} 1 & 2 & 3 \\ 2 & \mu & 4 \\ 3 & 4 & 5 \end{bmatrix}$, H is a Hermitian matrix

If H is positive definite

$$\Rightarrow \begin{cases} \det\left(\begin{bmatrix} 1 & 2 \\ 2 & \mu \end{bmatrix}\right) > 0 & -① \\ \det\left(\begin{bmatrix} 1 & 2 & 3 \\ 2 & \mu & 4 \\ 3 & 4 & 5 \end{bmatrix}\right) > 0 & -② \end{cases}$$

From ①, $\mu - 4 > 0$, $\mu > 4$ $-$ ③

From ②, $5\mu + 24 + 24 - 9\mu - 16 - 20 > 0$

$4\mu < 12$, $\mu < 3$ $-$ ④

From ③, ④ $\mu > 4$ and $\mu < 3$ Contradiction!

There is no solution of μ.

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3. $|x|_\infty = \max_k |x_k|$, $|x|_1 = \sum_k |x_k|$

(1) Show that $|x|_\infty$ is a norm.

Check ~~three~~ properties.

① $x = 0$, $|x|_\infty = \max_k |x_k| = 0$

$x \neq 0$, $|x|_\infty = \max_k |x_k| > 0$

② $\alpha \neq 0$, $|\alpha x|_\infty = \max_k |\alpha x_k| = |\alpha| \cdot \max_k |x_k| = |\alpha| \cdot |x|_\infty$

③ $|x+y|_\infty = \max_k |x_k + y_k| \leq \max_k |x_k| + |y_k| = |x|_\infty + |y|_\infty$

Hence, $|x|_\infty$ is a norm

(2) Show that $|x|_1$ is a norm

Check three properties.

① $x \neq 0$, $|x|_1 = \sum_k |x_k| > 0$

$x = 0$, $|x|_1 = \sum_k |x_k| = 0$

② $\alpha \neq 0$, $|\alpha x|_1 = \sum_k |\alpha x_k| = |\alpha| \sum_k |x_k| = |\alpha| \cdot |x|_1$

③ $|x+y|_1 = \sum_k |x_k + y_k| \leq \sum_k |x_k| + |y_k| = |x|_1 + |y|_1$

Hence, $|x|_1$ is a norm.

(3) Find the matrix norm for $|x|_1$

$$|x|_1 = \sum_k |x_k| ,$$

$$\|A\|_1 = \max_{x \neq 0} \frac{|Ax|_1}{|x|_1} , \quad \text{where } A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$|Ax|_1 = \sum_k | a_{k1} x_1 + a_{k2} x_2 + \cdots + a_{kn} x_n |$$

$$\leq \sum_k (|a_{k1} x_1| + |a_{k2} x_2| + \cdots + |a_{kn} x_n|)$$

$$= \sum_k |a_{k1}||x_1| + |a_{k2}||x_2| + \cdots + |a_{kn}||x_n|$$

$$= S_1 |x_1| + S_2 |x_2| + \cdots + S_n |x_n|$$

where $S_j = \sum_{i=1}^{n} |a_{ij}|$

$$\frac{|Ax|_1}{|x|_1} = \frac{S_1|x_1| + S_2|x_2| + \cdots + S_n|x_n|}{|x_1| + |x_2| + \cdots + |x_n|} \leq \frac{S_M (|x_1| + |x_2| + \cdots + |x_n|)}{|x_1| + |x_2| + \cdots + |x_n|}$$

$$S_M = \max \{ S_1, \cdots S_n \}$$

$$\|A\|_1 = \max_{x \neq 0} \frac{|Ax|_1}{|x|_1} = \max_j \sum_{i=1}^{n} |a_{ij}|$$

(4) Find the matrix norm for $|x|_\infty$

$$|x|_\infty = \max_k |x_k|$$

$$\|A\|_\infty = \max_{x \neq 0} \frac{|Ax|_\infty}{|x|_\infty}, \quad \text{where } A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$|Ax|_\infty = \max_k |a_{k1} x_1 + a_{k2} x_2 + \cdots + a_{kn} x_n|$$

$$\leq \max_k \left(|a_{k1}||x_1| + |a_{k2}||x_2| + \cdots + |a_{kn}||x_n| \right)$$

$$\leq \max_k \left(|a_{k1}| + |a_{k2}| + \cdots + |a_{kn}| \right) \cdot S_M$$

where $S_M = \max_k |x_k| = |x|_\infty$

$$\frac{|Ax|_\infty}{|x|_\infty} \leq \max_k \left(|a_{k1}| + |a_{k2}| + \cdots + |a_{kn}| \right) = \max_i \sum_{j=1}^{n} |a_{ij}|$$

$$\|A\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}|$$

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