ECE-GY 5253 HW #10

Question 1.

Since $\boldsymbol{A}$ is stable, $\exists \boldsymbol{P}$ s.t.

$$\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} = -\boldsymbol{I}$$

let

$$\boldsymbol{V}(\boldsymbol{x}) = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}$$

then we have

$$\begin{aligned}
\dot{\boldsymbol{V}} &= 2\boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}\dot{\boldsymbol{x}} \\
&= 2\boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}(\boldsymbol{A}\boldsymbol{x} + g(\boldsymbol{x})) \\
&= \boldsymbol{x}^{\mathrm{T}}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A})\boldsymbol{x} + 2\boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}g(\boldsymbol{x}) \\
&= -\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x} + 2\boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}g(\boldsymbol{x}) \\
&\leq -||\boldsymbol{x}||^2 + 2||\boldsymbol{x}||\,||\boldsymbol{P}||\,||g(\boldsymbol{x})|| \\
&= -||\boldsymbol{x}||^2 + 2||\boldsymbol{x}||^2\,||\boldsymbol{P}||\,\frac{||g(\boldsymbol{x})||}{||\boldsymbol{x}||}
\end{aligned}$$

since $\boldsymbol{x_0}$ is sufficiently small (which means that $||\boldsymbol{x_0}|| \to 0$), and

$$\lim_{||\boldsymbol{x}|| \to 0} \frac{||g(\boldsymbol{x})||}{||\boldsymbol{x}||} = 0$$

we can say that around $\boldsymbol{x_0}$

$$\begin{aligned}
\dot{\boldsymbol{V}} &\leq -||\boldsymbol{x}||^2 + 2||\boldsymbol{x}||^2\,||\boldsymbol{P}||\,\frac{||g(\boldsymbol{x})||}{||\boldsymbol{x}||} \\
&\leq -||\boldsymbol{x}||^2(1 - 2||\boldsymbol{P}||\delta)
\end{aligned}$$

where $\delta > 0$ and $\delta \to 0$

therefore, for $||\boldsymbol{x}|| \leq \frac{1}{4\delta||\boldsymbol{P}||}$ we have

$$\begin{aligned}
\dot{\boldsymbol{V}} &\leq -\frac{1}{2}||\boldsymbol{x}||^2 \\
&\leq -\frac{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}}{2\lambda_{\max}(\boldsymbol{P})} \\
&= -\frac{\boldsymbol{V}}{2\lambda_{\max}(\boldsymbol{P})} \\
&\doteq -\mu\boldsymbol{V}
\end{aligned}$$

hence from the fact we can get

$$\boldsymbol{V}(t) \leq e^{-\mu t}\boldsymbol{V}(0)$$

so, $\boldsymbol{V}(t) = \boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{P}\boldsymbol{x}(t)$ and thus $\boldsymbol{x}(t)$ converges to 0 at an exponential rate, hence

$$\lim_{t\to+\infty}\boldsymbol{x}(t) = 0$$