EL5253 HW8 Kunal Ninawe N12512995 kvn238

Monday, November 9, 2020 10:01 AM

1) From Rayleigh's theorem we get: Q1

1. If λ_n is the least eigenvalue of a Hermitian matrix H, show that $\lambda_n = \min_{x \neq 0} \frac{\langle Hx, x\rangle}{\langle x, x\rangle}$.

$$\underbrace{\lambda_*}_{\text{largest}} = \frac{\langle Hx, x\rangle}{\langle x, x\rangle}$$

largest.

H can be shown as a Diagonal matrix D:

$$U^* D^* U = H$$

$$\therefore \langle Hx, x\rangle = x^* Hx$$
$$= x^* U^* D U x$$
$$= (Ux)^* D U x$$

We know that $\because$ U is a unitary matrix,

$$((Ux)^* (Ux) = x^* x$$

Hence, $$\min_{x\neq 0} D(x) = \min \frac{\langle Dx, x\rangle}{\langle x, x\rangle}$$

$\because$ Diagonal matrices contains eigenvalues

$$\therefore \lambda_n = \min_{x\neq 0} \frac{\langle Dx, x\rangle}{\langle x, x\rangle} = \min_{x\neq 0} \frac{\langle Hx, x\rangle}{\langle x, x\rangle}$$

2) 2. Find all possible values of μ guaranteeing the positive-definiteness of

$$H = \begin{pmatrix} 1 & 2 & 3 \\ 2 & \mu & 4 \\ 3 & 4 & 5 \end{pmatrix}.$$

→ We know that a positive definite matrix is a symmetric matrix with all +ve eigenvalues (λ).

Given $H = \begin{bmatrix} 1 & 2 & 3 \\ 2 & \mu & 4 \\ 3 & 4 & 5 \end{bmatrix}$, we also know that eigenvalues can be computed

as: $|H - \lambda I| = \begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & \mu-\lambda & 4 \\ 3 & 4 & 5-\lambda \end{vmatrix} = 0$

$$= (1-\lambda)\left[(\mu-\lambda)(5-\lambda) - 16\right] - 2\left[10 - 2\lambda - 12\right]$$
$$+ 3\left[8 - 3\mu - 3\lambda\right]$$
$$= \lambda^3 - \lambda^2(6+\mu) + \lambda(6\mu - 2\mu) - (12 - 4\mu) = 0$$

We know that all roots of the equation would be +ve.

so : $(6+\mu) > 0$, $(6\mu - 2\mu) > 0$, $(12 - 4\mu) > 0$

$$\Rightarrow \boxed{\mu > -6, \quad \mu > 4, \quad \mu < 3}$$

Hence there are <u>no</u> such μ values for which H can be a positive definite matrix.

3) 3. Show that $|x| = \max_k |x_k|$, denoted as $|x|_\infty$,

and $|x| = \sum_k |x_k|$, denoted as $|x|_1$,

are both norms.

3)

3. Show that $|x| = \max_k |x_k|$, denoted as $|x|_\infty$,

and $|x| = \sum_k |x_k|$, denoted as $|x|_1$,

are both norms.

What are their associated matrix norms?

→ Given $|x|_\infty = \max_K |x_K|$

and $|x|_1 = \sum_K |x_K|$

To show that $|x|_\infty$ & $|x|_1$ are norms, we need to show that it satisfies <u>norm</u> properties.

Now, for $|x|_\infty = \max_K |x_K|$

① $|x|_\infty = 0$ iff $x = 0$, as $\max |x_K| = 0$

if $x_K = 0 \ \forall K = 0$

& $|x|_\infty \geq 0$ as $\max_K |x_K| > 0$ as absolute func. is always +ve.

② To show that $|\alpha x|_\infty = |\alpha| \cdot |x|_\infty$

Now $|\alpha \cdot x|_\infty = \max_K |\alpha x_K|$

$$= \max_K |\alpha| \cdot |x_K|$$

$$= |\alpha| \cdot \max_K \cdot |x_K|$$

$$= |\alpha| \cdot |x_K|_\infty \quad \#$$

$$= |\alpha| \cdot |x_k|_\infty \quad \#$$

③ To show that, $|x+y|_\infty \leq |x|_\infty + |y|_\infty$

$$|x+y|_\infty = \max_k |x_k + y_k|$$

$$\leq \max_k \left(|x_k| + |y_k|\right)$$

$$\leq \max_k |x_k| + \max_k |y_k|$$

$$\leq |x_k|_\infty + |y_k|_\infty$$

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$\boxed{\text{Hence } |x|_\infty \text{ is a norm}}$

→ Now for $|x|_1 = \sum_k |x_k|$

① $|x|_1 = 0$, iff $\sum_k |x_k| = 0$, when $|x_k| = 0, \forall k$

$|x|_1 \geq 0$ as $\sum_k |x_k| \geq 0$ as absolute func. is always tve.

② To show that $|\alpha x|_1 = |\alpha| \, |x|_1$

Now $|\alpha \cdot x|_1 = \sum_k |\alpha x_k|$

$$\text{Now } |\alpha \cdot x|_1 = \sum_k |\alpha x_k|$$

$$= \sum_k |\alpha| \cdot \sum_k |x_k|$$

$$= |\alpha| \cdot \sum_k |x_k|$$

$$= |\alpha| \cdot |x|_1$$

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(3) $|x+y|_1 = |x|_1 + |y|_1$

$$\text{Now } |x+y|_1 = \sum_k |x_k + y_k|$$

$$\leq \sum_k |x_k| + \sum_k |y_k|$$

$$\leq |x|_1 + |y|_1$$

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Hence, $|x|_1$ is also a norm.

Now, Associated matrix norm:

Let A be a matrix,

$$\|A\|_1 = \max_j \sum_{i=1}^{n} |a_{ij}|$$ elements of matrix A.

$$\|A\|_\infty = \max \sum^{n} |a_{ij}|$$

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