① Given a singular Matrix:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Characteristic Polynomial:

$$(\lambda-1)^2 - 1 = 0$$
$$\lambda^2 - 2\lambda = 0$$
$$\lambda(\lambda-2) = 0$$

Two distinct eigenvalues: $\lambda = 0$, $\lambda = 2$

When $\lambda=0$,

$$\begin{bmatrix} \lambda-1 & -1 \\ -1 & \lambda-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-x_1 - x_2 = 0$$
$$-x_1 - x_2 = 0$$

Hence, the eigenvector corresponding to the eigenvalue, $\lambda = 0$, is $\begin{bmatrix} -x_2 \\ x_2 \end{bmatrix}$

Its basis for eigenspace is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

When $\lambda=2$,

$$\begin{bmatrix} \lambda-1 & -1 \\ -1 & \lambda-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 - x_2 = 0$$
$$-x_1 + x_2 = 0$$

Hence, the eigenvector corresponding to the eigenvalue, $\lambda = 2$, is $\begin{bmatrix} x_2 \\ x_2 \end{bmatrix}$

Its basis for eigenspace is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Clearly, the singular matrix, A, has two independent eigenvectors.

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$$\det(\lambda I - A) = \det\left[(\lambda I - A)^T\right] = \det\left[(\lambda I)^T - A^T\right]$$
$$= \det(\lambda I - A^T)$$

The characteristic polynomial corresponding to the matrix A is the same as the characteristic polynomial corresponding to the transpose of A. Hense, A and A^T have the same eigenvalues.

3.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

$$BA = \begin{bmatrix} a_{11}b_{11} + a_{21}b_{12} & a_{12}b_{11} + a_{22}b_{12} \\ a_{11}b_{21} + a_{21}b_{22} & a_{12}b_{21} + a_{22}b_{22} \end{bmatrix}$$

① $$\det(\lambda I - AB) = \det\begin{bmatrix} \lambda - (a_{11}b_{11} + a_{12}b_{21}) & -(a_{11}b_{12} + a_{12}b_{22}) \\ -(a_{21}b_{11} + a_{22}b_{21}) & \lambda - (a_{21}b_{12} + a_{22}b_{22}) \end{bmatrix}$$

$$= \left[\lambda - (a_{11}b_{11} + a_{12}b_{21})\right]\left[\lambda - (a_{21}b_{12} + a_{22}b_{22})\right] - (a_{11}b_{12} + a_{12}b_{22})(a_{21}b_{11} + a_{22}b_{21})$$

$$= \lambda^2 - (a_{11}b_{11} + a_{12}b_{21} + a_{21}b_{12} + a_{22}b_{22})\lambda + (a_{11}b_{11} + a_{12}b_{21})\cdot(a_{21}b_{12} + a_{22}b_{22})$$
$$- (a_{11}b_{12} + a_{12}b_{22})(a_{21}b_{11} + a_{22}b_{21})$$

$$= \lambda^2 - (a_{11}b_{11} + a_{12}b_{21} + a_{21}b_{12} + a_{22}b_{22})\lambda + a_{12}a_{21}(b_{12}b_{21} - b_{11}b_{22})$$
$$+ a_{11}a_{22}(b_{11}b_{22} - b_{12}b_{21})$$

(2) $\det(\lambda I - BA) = \det \begin{bmatrix} \lambda - (a_{11}b_{11} + a_{21}b_{12}) & -(a_{12}b_{11} + a_{22}b_{12}) \\ -(a_{11}b_{21} + a_{21}b_{22}) & \lambda - (a_{12}b_{21} + a_{22}b_{22}) \end{bmatrix}$

$= [\lambda - (a_{11}b_{11} + a_{21}b_{12})][\lambda - (a_{12}b_{21} + a_{22}b_{22})] - (a_{12}b_{11} + a_{22}b_{12}) \cdot (a_{11}b_{21} + a_{21}b_{22})$

$= \lambda^2 - (a_{11}b_{11} + a_{21}b_{12} + a_{12}b_{21} + a_{22}b_{22})\lambda + (a_{11}b_{11} + a_{21}b_{12})(a_{12}b_{21} + a_{22}b_{22})$
$- (a_{12}b_{11} + a_{22}b_{12}) \cdot (a_{11}b_{21} + a_{21}b_{22})$

$= \lambda^2 - (a_{11}b_{11} + a_{12}b_{21} + a_{21}b_{12} + a_{22}b_{22})\lambda + a_{12}a_{21}(b_{12}b_{21} - b_{11}b_{22})$
$+ a_{11}a_{22}(b_{11}b_{22} - b_{12}b_{21})$

Both $\det(\lambda I - AB)$ and $\det(\lambda I - BA)$ give the same characteristic polynomial. Hence, AB & BA have the same characteristic equation.

4.

(1) let $B = \begin{bmatrix} 3/2 & 1/2 \\ 1/2 & 3/2 \end{bmatrix}$

$$\det(\lambda I - B) = 0 \Rightarrow (\lambda - \tfrac{3}{2})^2 - \tfrac{1}{4} = 0$$

$$(\lambda - 2)(\lambda - 1) = 0, \quad \lambda = 2, 1$$

$\lambda = 2$, corresponding eigenvector $P_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = 1$, corresponding eigenvector $P_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$P = [P_1 \ P_2] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad P^{-1} = -\frac{1}{2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$P^{-1}BP = -\frac{1}{2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3/2 & 1/2 \\ 1/2 & 3/2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

(2) let $D = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$,

$$\det(\lambda I - D) = 0 \Rightarrow (\lambda - 1)(\lambda - 2) = 0$$

$$\lambda = 1, 2.$$

$\lambda = 2$, corresponding eigenvector $P_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = 1$, corresponding eigenvector $P_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

$$P = [P_1 \ P_2] = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$$

$$P^{-1}DP = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

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