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(1)

Problem 1.
Consider the initial-value problem

$$\dot{x}(t) = \begin{pmatrix} 3 & 2 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} x(t) \triangleq Ax(t), \text{ with } x(0) = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}.$$

1.1) Transform the matrix A into a Jordan form.
1.2) On the basis of the Jordan form, solve the initial-value problem for all $t \geq 0$.

→ (1.1) We can see the rank of the matrix A

$\text{rank}(A) = 3$.

(i) Computing Eigenvalues:

$$A - \lambda I = \begin{bmatrix} 3-\lambda & 2 & 1 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = -\lambda^3 + 9\lambda^2 - 27\lambda + 27$$

This is the characteric polynomial.

roots (eigenvalues) are:

$$\boxed{\lambda_1 = 3, \ \lambda_2 = 3}$$

(ii) Finding eigenvectors:

for $\lambda = 3$, $A - \lambda I = \begin{bmatrix} 3-\lambda & 2 & 1 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Reducing this to row echelon form:

$$(A - \lambda I) X = \begin{bmatrix} 0 & 1 & 1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

After solving we get eigenvectors as:

$$\boxed{v_1 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}$$

Jordan form is computed by:

$$\boxed{A = S \circ J \circ S^{-1}}$$

where $\underline{S}$ is the transformation matrix

& $\underline{J}$ is the Jordan form.

After solving, we get,

$$S = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1/2 \\ 2 & 0 & 0 \end{bmatrix}, \quad J = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}, \quad S^{-1} = \begin{bmatrix} 0 & 0 & 1/2 \\ 1 & 0 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

(1.2) To solve the initial value problem, we have to:

(i) Find general soln:

$$\boxed{x(t) = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t}} \rightarrow (1)$$

but $\because$ we are using the jordan form: $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

eigenvalues will be the same: $\lambda_1 = 3, \lambda_2 = 3$

eigenvectors after solving is:

$$v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Substituting these computed eigenvalues and eigenvectors in equ (1)

$$x(t) = c_1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} (e^{3t} + t)$$

but, we have to find the one that satisfies

when $t = 0$, $x(t) = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

when $t = 0$, $x(t) = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^0 + c_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^0$$

After solving

$$\boxed{\begin{aligned} c_2 &= 1 \\ c_1 &= 2 \end{aligned}}$$

substituting c_1 & c_2 in the general soln.

$$\boxed{X = 1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{3t} + 2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} (e^{3t} + t)}$$

soln. of the initial value problem.

(2)

Problem 2.

Suppose the n Gersgorin discs of an $n \times n$ matrix A are mutually disjoint. Show that every eigenvalue of A is real, if A is real.

→ We know that there would be atmost n eigenvalues

- since it is a $n \times n$ matrix
- and since each of the disk is disjoint,
- we can take $m=1$ in the Gerschgorin circle theorem

to see that each disk must contain exactly one of the eigenvalues.

- we also know that $\because$ B is real, then coef. of characteric polynomial of B are all real.

→ i.e.- either the eigenvalues are real or they occur in complex conjugate pairs.

Suppose we have a complex conjugate pair

$$\lambda_1 = a + bi \quad \& \quad \lambda_2 = a - bi$$

we can see that:

$$|a \pm bi - b_{ii}| = \sqrt{(a + b_{ii})^2 + b^2}$$

$$|a \pm bi - b_{ii}| = \sqrt{(a+b_{ii})^2 + b^2}$$

- from above, we can say that both of these would neccesarily lie in the same gerschgorin disc, we cannot have complex eigenvalues.

Conclusion: The eigenvalues must be real if A is real. as complex values must appear in conjugate pairs.

(3)

Problem 3.

Consider $A = \begin{pmatrix} 1 & 1 \\ -1.5 & 2 \end{pmatrix}$ and show that $\rho(A) < \min \|D^{-1}AD\|_\infty$ over all $D = \begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix}$, with $p_1 > 0$ and $p_2 > 0$. (Recall that $\|M\|_\infty$ stands for the *maximum row sum matrix norm* of $M \in \mathbb{R}^{n\times n}$, *i.e.* $\|M\|_\infty := \max\limits_{1\le i\le n} \sum\limits_{j=1}^{n} |a_{ij}|$.)

→ let λ be the eigenvalue of A.

we will s.t $\Rightarrow$ $Ax = \lambda x$.

then, λ is also the eigenvalue of $D^{-1}AD$

let $y = \left(\frac{x_1}{p_1}, \frac{x_2}{p_2} \right)^T$

then.

$$D^{-1}AD\, y = \begin{bmatrix} \frac{1}{p_1} & 0 \\ 0 & \frac{1}{p_2} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1.5 & 2 \end{bmatrix} \begin{bmatrix} p_1 & 0 \\ 0 & p_2 \end{bmatrix} \begin{bmatrix} \frac{x_1}{p_1} \\ \frac{x_2}{p_2} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \frac{1}{p_1} & 0 \\ 0 & \frac{1}{p_2} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1.5 & 2 \end{bmatrix} \begin{bmatrix} \frac{\cancel{p_1}\cdot x_1}{\cancel{p_1}} \\ \frac{\cancel{p_2}\cdot x_2}{\cancel{p_2}} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \frac{1}{p_1} & 0 \\ 0 & \frac{1}{p_2} \end{bmatrix} \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow \lambda \underbrace{\begin{bmatrix} \frac{x_1}{p_1} \\ \frac{x_2}{p_2} \end{bmatrix}}_{\text{we know this is } y.}$$

$$D^{-1} A D y = \lambda y.$$

Now, $\| \lambda \cdot y \|_\infty = \| D^{-1} A D y \|_\infty$

$$\| \lambda \| \cdot \| y \|_\infty = \| D^{-1} A D y \|_\infty$$

$\Rightarrow$ we know that $\|M\|_\infty$ is a matrix norm, and thus, it is homogeneus.

i.e. $\rightarrow$ $|\lambda| \cancel{\|y\|_\infty} \leq \|D^{-1} A D\|_\infty \cancel{\|y\|_\infty}$

$$|\lambda| \leq \|D^{-1} A D\|_\infty$$

$$\max |\lambda| \leq \|D^{-1} A D\|_\infty$$

$\Rightarrow$ we also know that $\max|\lambda|$ is $= \rho(A)$

where $\rho(A)$ is the spectral radius of A.

$$\Rightarrow \rho(A) \leq \|D^{-1}AD\|_\infty$$

and

$$\boxed{\rho(A) \leq \min \|D^{-1}AD\|_\infty}$$

which is what we wanted to prove.