

Lecture III

Key issues:

- Eigenvalues, eigenvectors and the characteristic polynomial of a square matrix
- Similarity

Eigenvalue and Eigenvector

Definition:

Given a square $n \times n$ matrix A , the set of **eigenvalues** $\lambda \in \mathbb{C}$ are such that the **characteristic equation**

$$Ac = \lambda c, \text{ or equivalently } (\lambda I - A)c = 0$$

has a nonzero solution $c \neq 0$.

Such a vector $c \in \mathbb{C}^n$ is called an **eigenvector** associated with eigenvalue λ .

As a result, for any eigenvalue,

$$\det(\lambda I - A) = 0, \quad \forall \lambda \in \sigma(A).$$

How to compute an eigenvalue?

They are the roots of the characteristic polynomial of A :

The eigenvalues λ of a matrix $A \in \mathbb{R}^{n \times n}$ are the roots of its **characteristic polynomial**:

$$\begin{aligned}\rho(\lambda) &= \det(\lambda I - A) \\ &= \lambda^n + \alpha_1 \lambda^{n-1} + \cdots + \alpha_{n-1} \lambda + \alpha_n.\end{aligned}$$

So, in total there are n eigenvalues (possibly repeated and complex-valued).

Comment on Characteristic Polynomial

Indeed, $\det(\lambda I - A) = 0$ implies, by means of permutation:
$$\det(\lambda I - A) = \lambda^n + \alpha_1 \lambda^{n-1} + \cdots + \alpha_{n-1} \lambda + \alpha_n = 0.$$

Comment 1

For any given eigenvalue λ , the eigenvectors are **nonzero** solutions to the homogeneous equation:

$$(\lambda I - A)c = 0.$$

Clearly, any vector of the form $\mu \times c$ for a **nonzero scalar μ** is still an eigenvector.

Comment 2

For any given eigenvalue λ , the eigenvectors are **nonzero** solutions to the homogeneous equation:

$$(\lambda I - A)c = 0.$$

$$c \in \text{Null}(\lambda I - A), \quad c \neq 0.$$

So, the maximum number of **linearly independent eigenvectors** is equal to:

$$n - \text{rank}(\lambda I - A).$$

An Example

For the identity matrix $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, the eigenvalues are $\lambda_1 = \lambda_2 = 1$, for which two **linearly independent** eigenvectors are:

$$c^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad c^2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Other choices of independent eigenvectors are:

$$c^1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad c^2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

A Useful Observation

A square matrix A of dimension n is singular
if and only if it has a zero eigenvalue.

Proof

Necessity :

Assume that A is singular. Then, there is a nonzero solution $x \neq 0$ to the equation

$$(*) \quad Ax = 0, \text{ or equivalently, } Ax = 0 \bullet x.$$

So, $\lambda = 0$ is an eigenvalue.

Sufficiency :

If $\lambda = 0$ is an eigenvalue, then there is an eigenvector $x \neq 0$, which is a solution to (*).

Thus, A must be a singular matrix.

Exercise

Find the eigenvalues and the associated eigenvectors of the following matrix

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

More about Eigenvalues

- A real square matrix can have **complex** eigenvalues.

For example, the eigenvalues of

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

are $\lambda_1 = -j$, $\lambda_2 = j$.

More about Eigenvalues

- A real square matrix can have **multiple** eigenvalues.

For example, the eigenvalues of

$$A = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$$

are $\lambda_1 = \lambda_2 = 1$ for any $a \in \mathbb{R}$.

We say that $\lambda_1 = 1$ is an eigenvalue of multiplicity 2.

In this case, the characteristic polynomial is

$$\det(\lambda I - A) = (\lambda - 1)^2.$$

More about Eigenvalues

- More generally, the characteristic polynomial of a matrix $A \in \mathbb{R}^{n \times n}$ takes the form

$$\det(\lambda I - A) = (\lambda - \lambda_1)^{m_1} (\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_r)^{m_r}$$

where $\lambda_1, \dots, \lambda_r$ are different, with $\sum_i m_i = n$,

λ_i an eigenvalue of (algebraic) multiplicity m_i .

When $r = n$ and $m_1 = \cdots = m_n = 1$, the matrix A is said to have **distinct** eigenvalues $\{\lambda_i\}_{i=1}^n$.

More about Eigenvectors

- Back to the example of $A = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$. It has an eigenvalue $\lambda_1 = 1$ of (algebraic!) multiplicity 2, for any $a \in \mathbb{R}$.

Case 1: For $a = 0$, the associated eigenvectors are **distinct**:

$$c^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad c^2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

More about Eigenvectors

- Back to $A = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$. It has an eigenvalue $\lambda_1 = 1$ of (algebraic) multiplicity 2, for any $a \in \mathbb{R}$.

Case 2: For $a \neq 0$, there is **only one** distinct

eigenvector: $c^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ Its “**geometric**” multiplicity is 1

i.e., all other eigenvectors take the form $r \times c^1$, for some scalar $r \neq 0$.

Exercise

Do you know the eigenvalues of the following matrix

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} ?$$

Can you try to find the eigenvectors for each eigenvalue?

A General Result

Assume $A \in \mathbb{R}^{n \times n}$ has n **distinct** eigenvalues $\lambda_1, \dots, \lambda_n$.

Then,

(1) A must have n **linearly independent** eigenvectors

c^1, \dots, c^n .

(2) In addition, each eigenvector c^j associated with λ_j is **unique** apart from a nonzero scalar multiplier.

Proof of Statement 1

Let $c^1 \neq 0, \dots, c^n \neq 0$ be eigenvectors satisfying

$$Ac^i = \lambda_i c^i, \quad \text{for } i = 1, 2, \dots, n.$$

We prove the statement by contradiction.

Assume that $\{c^i\}_{i=1}^n$ are linearly dependent. Let $k \leq n$

be the **least** positive integer such that k of the c 's are dependent. Without loss of generality, assume that

$\{c^i\}_{i=1}^k$ are dependent, that is, $\exists \alpha_i$ **not all zero**,

$$\alpha_1 c^1 + \alpha_2 c^2 + \dots + \alpha_k c^k = 0.$$

Proof of Statement 1 (cont'd)

$\exists \alpha_i$ **not all zero**, $\alpha_1 c^1 + \alpha_2 c^2 + \cdots + \alpha_k c^k = 0$.

Thus, $k \geq 2$ and **all** $\alpha_i \neq 0$ (otherwise, contradiction with k being the least).

Now, multiply the eq. by $(A - \lambda_k I)$ leads to:

$$\alpha_1 (\lambda_1 - \lambda_k) c^1 + \cdots + \alpha_{k-1} (\lambda_{k-1} - \lambda_k) c^{k-1} = 0$$

which, in turn, implies that $\{c^i\}_{i=1}^{k-1}$ are dependent.

A contradiction.

Proof of Statement 2

We must show the "uniqueness" of c^i :

$$Ac = \lambda_i c, \quad c \neq 0 \Rightarrow c = \mu c^i, \text{ with } \mu \neq 0.$$

As it was proved in statement (1), $\{c^i\}_{i=1}^n$ are linearly independent and thus form a basis. So,

$$c = \beta_1 c^1 + \cdots + \beta_i c^i + \cdots + \beta_n c^n.$$

Multiplying the above eq. by $(A - \lambda_i I)$ gives:

$$0 = \beta_1 (\lambda_1 - \lambda_i) c^1 + \cdots + 0 + \cdots + \beta_n (\lambda_n - \lambda_i) c^n$$

Therefore: $\beta_k = 0, \forall k \neq i.$

In other words, $c = \beta_i c^i$, as wished.

Corollary

Under the above conditions, define matrix

$$P = \begin{bmatrix} c^1 & c^2 & \cdots & c^n \end{bmatrix}$$

which is nonsingular & implies

$$P^{-1}AP = \text{diag}(\lambda_i).$$

In this case, we say that A is **similar to** $\text{diag}(\lambda_i)$,
while P is a **similarity matrix**.

Denote $A \sim \text{diag}(\lambda_i)$. A is called "**diagonalizable**"

Remark 1

As we will see in Lecture IV, $\text{diag}(\lambda_i)$ is a canonical form for matrices, which have distinct eigenvalues, and for symmetric matrices, which may have repeated eigenvalues.

Remark 2

Any two similar matrices A and B must have the same eigenvalues.

Indeed, $A \sim B \Leftrightarrow \exists P$, such that $B = P^{-1}AP$.

It then follows that

$$\begin{aligned}\det(B - \lambda I) &= \det(P^{-1}AP - \lambda I) \\ &= \det P^{-1} (A - \lambda I) P = \det P^{-1} \det(A - \lambda I) \det P \\ &= \det(A - \lambda I),\end{aligned}$$

because $\det P^{-1} \det P = \det P^{-1}P = 1$.

Questions

Are you ready for some tricky questions?

Question 1

If the set of all eigenvalues of $A \in \mathbb{R}^{3 \times 3}$, or
the **spectrum** $\sigma(A) = \{1, 2, -3\}$,
what is $\sigma(A + A^3)$?

A General Result

For any polynomial $p(t) = \sum_{i=0}^k a_i t^i$, and any $n \times n$

matrix A , denote $p(A) = \sum_{i=0}^k a_i A^i$, with $A^0 \triangleq I$.

If (λ, \mathbf{x}) is a pair of eigenvalue and eigenvector of A , then $(p(\lambda), \mathbf{x})$ is a pair of eigenvalue and eigenvector of $p(A)$.

(Its proof is left as an exercise.)

“Matrix Polynomial”

Question 2

For any **idempotent** matrix A , that is, $A^2 = A$, what are the possible eigenvalues?

Idempotent Matrix: 2x2 case

If $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ is idempotent, then

$$\begin{cases} a = a^2 + bc, \\ b = ab + bd, \text{ implying } b = 0, \text{ or } d = 1 - a \\ c = ca + cd, \text{ implying } c = 0, \text{ or } d = 1 - a \\ d = bc + d^2. \end{cases}$$

Examples: Idempotent Matrix

$$(1) A = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \text{ is idempotent, if } a, d = 0, 1.$$

$$(2) A = \frac{1}{2} \begin{pmatrix} 1 - \cos \theta & \sin \theta \\ \sin \theta & 1 + \cos \theta \end{pmatrix}.$$

Answer

An idempotent matrix can only have 0 or 1 as its eigenvalues.

Question 3

A **nilpotent** matrix A is such that $A^q = 0$ for a positive integer q . Such a smallest q is called the **index of nilpotency**.

What are the eigenvalues of a nilpotent matrix A ?

Answer

All eigenvalues of a nilpotent matrix are 0.

Indeed, if $Ax = \lambda x$, $x \neq 0$, then,
using $A^q = 0$, we obtain $\lambda^q x = 0$
which, in turn, implies $\lambda = 0$.

Examples: Nilpotent Matrix

$$(1) M = \begin{pmatrix} 0 & * & \dots & * \\ \vdots & & \ddots & \vdots \\ 0 & & \cdot & * \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

$$(2) M = \begin{pmatrix} 5 & -3 & 2 \\ 15 & -9 & 6 \\ 10 & -6 & 4 \end{pmatrix}$$

Equivalence Relation: Nilpotent Matrix

The following statements are equivalent:

- (1) $M \in \mathbb{R}^{n \times n}$ is nilpotent.
- (2) The minimal polynomial of M is s^q , for some $q \leq n$.
- (3) The characteristic polynomial of M is s^n .
- (4) The only eigenvalue of M is 0.

Question 4

Compute the algebraic and geometric multiplicities of the eigenvalue $\lambda = 2$ for the matrix

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

Useful Identities about Matrix Eigenvalues

For any $n \times n$ matrix A , we have

$$\text{trace}(A) = \sum_{i=1}^n \lambda_i,$$

$$\det(A) = \prod_{i=1}^n \lambda_i.$$

**See p.42 of (Horn-Johnson, 1st ed., 1985),
or p. 50 of (Horn-Johnson, 2nd ed., 2013).**

Similarity

Two matrices A and B are said to be similar, if

$B = P^{-1}AP$, for some invertible matrix P .

Notation: $A \sim B$

The set of all similar matrices to a given square matrix A :

$$S = \{P^{-1}AP : P \text{ is invertible}\}$$

Note:

Similar matrices are just different basis representations of a single linear mapping.

Similarity: Physical meaning

Let $T : V \rightarrow V$ be a linear transformation, and

$B_1 = \{v_1, \dots, v_n\}$, $B_2 = \{w_1, \dots, w_n\}$ be two bases for V .

Denote

$$[x]_{B_1} = \text{col}(\alpha_1, \dots, \alpha_n), \text{ with } x = \alpha_1 v_1 + \dots + \alpha_n v_n.$$

Then, by linearity,

$$Tx = \alpha_1 Tv_1 + \dots + \alpha_n Tv_n$$

For any basis B_2 of V ,

$$[Tv_j]_{B_2} \triangleq \text{col}(t_{1j}, \dots, t_{nj}),$$

$$[Tx]_{B_2} = \sum_{j=1}^n \alpha_j [Tv_j]_{B_2} = (t_{ij})_{n \times n} \text{col}(\alpha_1, \dots, \alpha_n).$$

Similarity: Physical meaning

$$[Tx]_{B_2} = \sum_{j=1}^n \alpha_j [Tv_j]_{B_2} = \left(t_{ij}\right)_{n \times n} \text{col}(\alpha_1, \dots, \alpha_n).$$

It is important to note that the matrix $\left(t_{ij}\right)_{n \times n}$ depends on T and the choice of the bases B_1 and B_2 , but not x .

Define the B_1 - B_2 basis representation of T as:

$${}_{B_2} [T]_{B_1} \triangleq \left(t_{ij}\right)_{n \times n} = \left[[Tv_1]_{B_2}, \dots, [Tv_n]_{B_2} \right]$$

So, $[Tx]_{B_2} = {}_{B_2} [T]_{B_1} [x]_{B_1}, \forall x \in V$.

For the special case when $B_1 = B_2$,

${}_{B_1} [T]_{B_1}$ is called **the B_1 representation of T** .

Similarity: Identities

For the identity linear transformation $Ix = x$, $\forall x \in V$, it can be shown that

$${}_{B_2} [I]_{B_1 B_1} {}_{B_1} [I]_{B_2} = I_n, \quad {}_{B_1} [I]_{B_2 B_2} {}_{B_2} [I]_{B_1} = I_n,$$

and

$${}_{B_2} [T]_{B_2} = {}_{B_2} [I]_{B_1} {}_{B_1} [T]_{B_1} {}_{B_1} [I]_{B_2}.$$

In other words,

$$B = P^{-1}AP, \text{ where } P = {}_{B_1} [I]_{B_2}, \quad A = {}_{B_1} [T]_{B_1}, \quad B = {}_{B_2} [T]_{B_2}.$$

B_2 - B_1 change of basis matrix

For a proof, see (Horn & Johnson, 2nd ed, 2013, page 40).

Exercise

Find an invertible matrix P such that

$$P^{-1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix} P$$

is diagonal.

Homework #3

1. Give all the solutions of the system

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} x = \begin{pmatrix} 10 & 13 \\ 11 & 14 \\ 12 & 15 \end{pmatrix}.$$

2. Prove that the following eq. has no solution:

$$\begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix} x = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

Homework #3

3. Find a least-squares fit

$$b = x_0 + x_1 a^1 + x_2 a^2$$

for the data:

$$b = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad a^1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \quad a^2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Homework #3

4. Find independent eigenvectors for

$$A = \begin{pmatrix} 1 & 2 \\ -3 & 1 \end{pmatrix}.$$

Can you express $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ as a linear combination of these eigenvectors of A ?