1° Find the eigenvalues and corresponding eigenvectors of H

$\det(\lambda I - H) = 0$ where $H = \begin{bmatrix} 0 & 2 & -1 \\ 2 & 5 & -6 \\ -1 & -6 & 8 \end{bmatrix}$

$$\det\left(\begin{bmatrix} \lambda & -2 & 1 \\ -2 & \lambda-5 & 6 \\ 1 & 6 & \lambda-8 \end{bmatrix}\right) = \lambda^2(\lambda-13) - (\lambda-13)$$

$$= (\lambda^2-1)(\lambda-13) = 0$$

$\lambda = \pm 1,\ 13.$

$\lambda = 1,$ $\begin{bmatrix} \lambda & -2 & 1 \\ -2 & \lambda-5 & 6 \\ 1 & 6 & \lambda-8 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \\ -2 & -4 & 6 \\ 1 & 6 & -7 \end{bmatrix}$

By observation, the corresponding vector is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

The normalized eigenvector $p_1 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$

$\lambda = -1,$ $\begin{bmatrix} \lambda & -2 & 1 \\ -2 & \lambda-5 & 6 \\ 1 & 6 & \lambda-8 \end{bmatrix} = \begin{bmatrix} -1 & -2 & 1 \\ -2 & -6 & 6 \\ 1 & 6 & -9 \end{bmatrix} \sim \begin{bmatrix} -1 & -2 & 1 \\ 0 & -2 & 4 \\ 0 & 4 & -8 \end{bmatrix}$

$\sim \begin{bmatrix} -1 & -2 & 1 \\ 0 & -2 & 4 \\ 0 & 0 & 0 \end{bmatrix},$

let $X_3 = 1$, Then $X_2 = 2$, $X_1 = -2X_2 + X_3 = -3$

The normalized eigenvector $p_2 = \begin{bmatrix} 1/\sqrt{14} \\ 2/\sqrt{14} \\ -3/\sqrt{14} \end{bmatrix}$

$\lambda = 13;\ \begin{bmatrix} \lambda & -2 & 1 \\ -2 & \lambda-5 & 6 \\ 1 & 6 & \lambda-8 \end{bmatrix} = \begin{bmatrix} 13 & -2 & 1 \\ -2 & 8 & 6 \\ 1 & 6 & 5 \end{bmatrix} \sim \begin{bmatrix} 0 & -80 & -64 \\ 0 & 20 & 16 \\ 1 & 6 & 5 \end{bmatrix}$

$\sim \begin{bmatrix} 0 & 0 & 0 \\ 0 & 20 & 16 \\ 1 & 6 & 5 \end{bmatrix}$, let $X_3 = 5$, Then $X_2 = -4$

$X_1 = -1$

The normalized eigenvector $P_3 = \begin{bmatrix} -1/\sqrt{42} \\ -4/\sqrt{42} \\ 5/\sqrt{42} \end{bmatrix}$

let $O = [P_1\ P_2\ P_3] = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{-3}{\sqrt{14}} & \frac{-1}{\sqrt{42}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{14}} & \frac{-4}{\sqrt{42}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{14}} & 5/\sqrt{42} \end{bmatrix}$

$O^{-1} = O^T = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{-3}{\sqrt{14}} & \frac{2}{\sqrt{14}} & \frac{1}{\sqrt{14}} \\ \frac{-1}{\sqrt{42}} & \frac{-4}{\sqrt{42}} & 5/\sqrt{42} \end{bmatrix}$

Then $O^T H O = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 13 \end{bmatrix}$

Check out :

$$\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{-3}{\sqrt{14}} & \frac{2}{\sqrt{14}} & \frac{1}{\sqrt{14}} \\ \frac{-1}{\sqrt{42}} & \frac{-4}{\sqrt{42}} & \frac{5}{\sqrt{42}} \end{bmatrix} \begin{bmatrix} 0 & 2 & -1 \\ 2 & 5 & -6 \\ -1 & -6 & 8 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{-3}{\sqrt{14}} & \frac{-1}{\sqrt{42}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{14}} & \frac{-4}{\sqrt{42}} \\ \frac{1}{\sqrt{3}} & \frac{-4}{\sqrt{14}} & \frac{5}{\sqrt{42}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{3}{\sqrt{14}} & \frac{-2}{\sqrt{14}} & \frac{-1}{\sqrt{14}} \\ \frac{-13}{\sqrt{42}} & \frac{-52}{\sqrt{42}} & \frac{65}{\sqrt{42}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{-3}{\sqrt{14}} & \frac{-1}{\sqrt{42}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{14}} & \frac{-4}{\sqrt{42}} \\ \frac{1}{\sqrt{3}} & \frac{-4}{\sqrt{14}} & \frac{5}{\sqrt{42}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 13 \end{bmatrix}$$

#

2.

H is Hermitian and positive definite.

There exists a unitary matrix such that

$$U^* H U = \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & 0 \\ 0 & & \ddots \\ & & & \lambda_n \end{bmatrix}, \qquad U^* U = I$$

where $\lambda_1, \lambda_2 \cdots \lambda_n$ are the eigenvalues of H

and $\lambda_1, \lambda_2 \cdots \lambda_n$ are all positive

let $\Lambda = \Lambda_1^2$, where $\Lambda_1 = \begin{bmatrix} \sqrt{\lambda_1} & & \\ & \sqrt{\lambda_2} & 0 \\ 0 & & \ddots \\ & & & \sqrt{\lambda_n} \end{bmatrix}$

$$U^* H U = \Lambda_1^2 = \Lambda_1 \Lambda_1^2 = \Lambda_1 I \Lambda_1 = \Lambda_1 U^* U \Lambda_1$$

$$H = U \Lambda_1 U^* U \Lambda_1 U^* = (U \Lambda_1 U^*)^2$$

let $P = U \Lambda_1 U^* = U_1^* \Lambda_1 U_1$, $\quad U_1^* U_1 = U U^* = I$

U_1 is a unitary matrix, and the diagonal elements of Λ_1 are all greater than zero (Λ_1 is a diagonal matrix)

$\Rightarrow$ P is a positive definite matrix

$H = P^2$

#