Lecture III

Key issues:

- Eigenvalues, eigenvectors and the characteristic polynomial of a square matrix
- Similarity

Eigenvalue and Eigenvector

Definition:

Given a square $n \times n$ matrix A, the set of eigenvalues $\lambda \in \mathbb{C}$ are such that the characteristic equation

$$Ac = \lambda c \text{ , or equivalently } (\lambda I - A)c = 0$$

has a nonzero solution $c \neq 0$.

Such a vector $c \in \mathbb{C}^n$ is called an eigenvector associated with eigenvalue λ.

As a result, for any eigenvalue,

$$\det(\lambda I - A) = 0, \ \forall \lambda \in \sigma(A).$$

How to compute an eigenvalue?

They are the roots of the characteristic polynomial of *A*:

The eigenvalues λ of a matrix $A \in \mathbb{R}^{n \times n}$ are the roots of its characteristic polynomial:

$$\rho(\lambda) = \det(\lambda I - A)$$
$$= \lambda^n + \alpha_1 \lambda^{n-1} + \cdots + \alpha_{n-1}\lambda + \alpha_n.$$

So, in total there are *n* eigenvalues (possibly repeated and/or complex-valued).

Comment on Characteristic Polynomial

Indeed, $\det(\lambda I - A) = 0$ implies, by means of permutation:

$$\det(\lambda I - A) = \lambda^n + \alpha_1 \lambda^{n-1} + \cdots + \alpha_{n-1}\lambda + \alpha_n = 0.$$

Comment 1

For any given eigenvalue λ, the eigenvectors are nonzero solutions to the homogeneous equation:

$$(\lambda I - A)c = 0.$$

Non-uniqueness of eigenvector:

Clearly, any vector of the form *μ x c for a nonzero scalar μ* is still an eigenvector.

Comment 2

For any given eigenvalue λ, the eigenvectors are nonzero solutions to the homogeneous equation:

(λI - *A*)*c*=0.

$$c \in Null(\lambda I - A),\ c \neq 0.$$

So, the maximum number of **linearly independent eigenvectors** is equal to:

n – rank(λI - *A*).

An Example

For the identity matrix $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, the eigenvalues are $\lambda_1 = \lambda_2 = 1$, for which two linearly independent eigenvectors are:

$$c^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad c^2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Other choices of independent eigenvectors are:

$$c^1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad c^2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

A Useful Observation

A square matrix A of dimension n is singular **if and only if** it has a zero eigenvalue.

Proof

Necessity :

Assume that A is singular. Then, there is

a nonzero solution $x \neq 0$ to the equation

(*) $\quad Ax = 0,$ or equivalently, $Ax = 0 \bullet x.$

So, $\lambda = 0$ is an eigenvalue.

Sufficiency :

If $\lambda = 0$ is an eigenvalue, then there is an eigenvector

$x \neq 0,$ which is a solution to (*).

Thus, A must be a singular matrix.

Exercise

Find the eigenvalues and the associated eigenvectors of the following matrix

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

More about Eigenvalues

- A real square matrix can have complex eigenvalues.

For example, the eigenvalues of

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

are $\lambda_1 = -j, \ \ \lambda_2 = j.$

More about Eigenvalues

- A real square matrix can have **multiple** eigenvalues.

For example, the eigenvalues of

$$A = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$$

are $\lambda_1 = \lambda_2 = 1$ for any $a \in \mathbb{R}$.

We say that $\lambda_1 = 1$ is an eigenvalue of multiplicity 2.

In this case, the characteristic polynomial is

$$\det(\lambda I - A) = (\lambda - 1)^2.$$

More about Eigenvalues

- More generally, the characteristic polynomial of a matrix $A \in \mathbb{R}^{n\times n}$ takes the form

$$\det(\lambda I - A) = (\lambda-\lambda_1)^{m_1}(\lambda-\lambda_2)^{m_2}\cdots(\lambda-\lambda_r)^{m_r}$$

where $\lambda_1, \ldots, \lambda_r$ are different, with $\sum_i m_i = n$,

λ_i an eigenvalue of (algebraic) multiplicity m_i.

When $r = n$ and $m_1 = \cdots = m_n = 1$, the matrix A is said to have distinct eigenvalues $\{\lambda_i\}_{i=1}^n$.

More about Eigenvectors

- Back to the example of $A = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$. It has an eigenvalue $\lambda_1 = 1$ of (algebraic!) multiplicity 2, for any $a \in \mathbb{R}$.

Case 1: For $a = 0$, the associated eigenvectors are distinct:

$$c^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad c^2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

More about Eigenvectors

- Back to $A = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$. It has an eigenvalue $\lambda_1 = 1$ of (algebraic) multiplicity 2, for any $a \in \mathbb{R}$.

Case 2: For $a \neq 0$, there is only one distinct

eigenvector: $c^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Its "geometric" multiplicity is 1

i.e., all other eigenvectors take the form $r \times c^1$, for some scalar $r \neq 0$.

Exercise

Do you know the eigenvalues of the following matrix

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}?$$

Can you try to find the eigenvectors for each eigenvalue?

A General Result

Assume $A \in \mathbb{R}^{n \times n}$ has n distinct eigenvalues $\lambda_1, \ldots, \lambda_n$.

Then,

(1) A must have n linearly independent eigenvectors

$c^1, \ldots, c^n$.

(2) In addition, each eigenvector c^j associated with λ_j is unique apart from a nonzero scalar multiplier.

Proof of Statement 1

Let $c^1 \neq 0, \; \ldots, \; c^n \neq 0$ be eigenvectors satisfying

$$Ac^i = \lambda_i c^i, \quad \text{for } i = 1, 2, \ldots, n.$$

We prove the statement by contradiction.

Assume that $\left\{c^i\right\}_{i=1}^n$ are linearly dependent. Let $k \leq n$ be the least positive integer such that k of the c's are dependent. Without loss of generality, assume that $\left\{c^i\right\}_{i=1}^k$ are dependent, that is, $\exists \alpha_i$ not all zero,

$$\alpha_1 c^1 + \alpha_2 c^2 + \cdots + \alpha_k c^k = 0.$$

Proof of Statement 1 (cont'd)

$\exists \alpha_i$ not all zero, $\alpha_1 c^1 + \alpha_2 c^2 + \cdots + \alpha_k c^k = 0.$

Thus, $k \geq 2$ and all $\alpha_i \neq 0$ (otherwise, contradiction with k being the least).

Now, multiply the eq. by $\left(A - \lambda_k I\right)$ leads to:

$$\alpha_1 \left(\lambda_1 - \lambda_k\right) c^1 + \cdots + \alpha_{k-1} \left(\lambda_{k-1} - \lambda_k\right) c^{k-1} = 0$$

which, in turn, implies that $\left\{c^i\right\}_{i=1}^{k-1}$ are dependent.

A contradiction.

Proof of Statement 2

We must show the "uniqueness" of c^i :

$$Ac = \lambda_i c, \quad c \neq 0 \;\Rightarrow\; c = \mu c^i, \text{ with } \mu \neq 0.$$

As it was proved in statement (1), $\left\{c^i\right\}_{i=1}^{n}$ are linearly independent and thus form a basis. So,

$$c = \beta_1 c^1 + \cdots + \beta_i c^i + \cdots + \beta_n c^n.$$

Multiplying the above eq. by $\left(A - \lambda_i I\right)$ gives:

$$0 = \beta_1\left(\lambda_1 - \lambda_i\right)c^1 + \cdots + 0 + \cdots + \beta_n\left(\lambda_n - \lambda_i\right)c^n$$

Therefore: $\beta_k = 0, \; \forall k \neq i.$

In other words, $c = \beta_i c^i$, as wished.

Corollary

Under the above conditions, define matrix

$$P = \begin{bmatrix} c^1 & c^2 & \cdots & c^n \end{bmatrix}$$

which is nonsingular & implies

$$P^{-1}AP = diag\left(\lambda_i\right).$$

In this case, we say that A is similar to $diag\left(\lambda_i\right)$,

while P is a similarity matrix.

Denote $A \sim diag\left(\lambda_i\right)$. A is called "diagonalizable"

Remark 1

As we will see in Lecture IV, $diag\left(\lambda_i\right)$ is a canonical form for matrices, which have distinct eigenvalues, and for symmetric matrices, which may have repeated eigenvalues.

Remark 2

Any two similar matrices A and B must have the same eigenvalues.

Indeed, $A \sim B \Leftrightarrow \exists P,$ such that $B = P^{-1}AP.$

It then follows that

$$\det\left(B - \lambda I\right) = \det\left(P^{-1}AP - \lambda I\right)$$

$$= \det P^{-1}\left(A - \lambda I\right)P = \det P^{-1} \det\left(A - \lambda I\right)\det P$$

$$= \det\left(A - \lambda I\right),$$

because $\det P^{-1} \det P = \det P^{-1}P = 1.$

Questions

Are you ready for some tricky questions?

Question 1

If the set of all eigenvalues of $A \in \mathbb{R}^{3\times 3}$, or the spectrum $\sigma(A) = \{1, 2, -3\}$, what is $\sigma\left(A + A^3\right)$?

A General Result

For any polynomial $p(t) = \sum_{i=0}^{k} a_i t^i$, and any $n \times n$ matrix A, denote $p(A) = \sum_{i=0}^{k} a_i A^i$, with $A^0 \triangleq I$.

If (λ, x) is a pair of eigenvalue and eigenvector of A, then $(p(\lambda), x)$ is a pair of eigenvalue and eigenvector of $p(A)$.

"**Matrix Polynomial**"

(Its proof is left as an exercise.)

Question 2

For any idempotent matrix A, that is, $A^2 = A$, what are the possible eigenvalues?

Idempotent Matrix: 2x2 case

If $A=\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ is idempotent, then

$$\begin{cases} a = a^2 + bc, \\ b = ab + bd, \text{ implying } b = 0, \text{ or } d = 1 - a \\ c = ca + cd, \text{ implying } c = 0, \text{ or } d = 1 - a \\ d = bc + d^2. \end{cases}$$

Examples: Idempotent Matrix

$$(1)\ A=\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \text{ is idempotent, if } a,d=0,1.$$

$$(2)\ A=\frac{1}{2}\begin{pmatrix} 1-\cos\theta & \sin\theta \\ \sin\theta & 1+\cos\theta \end{pmatrix}.$$

Answer

An idempotent matrix can only have 0 or 1 as its eigenvalues.

Question 3

A nilpotent matrix A is such that $A^q = 0$ for a positive integer q. Such a smallest q is called the index of nilpotency.

What are the eigenvalues of a nilpotent matrix A?

Answer

All eigenvalues of a nilpotent matrix are 0.

Indeed, if $Ax = \lambda x,\ x \neq 0,$ then,
using $A^q = 0,$ we obtain $\lambda^q x = 0$
which, in turn, implies $\lambda = 0.$

Examples: Nilpotent Matrix

$$(1)\ M = \begin{pmatrix} 0 & * & \cdots & * \\ \vdots & & \ddots\ \ * & \vdots \\ 0 & & & * \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$(2)\ M = \begin{pmatrix} 5 & -3 & 2 \\ 15 & -9 & 6 \\ 10 & -6 & 4 \end{pmatrix}$$

Equivalence Relation: Nilpotent Matrix

The following statements are equivalent:

(1) $M \in \mathbb{R}^{n\times n}$ is nilpotent.

(2) The minimal polynomial of M is s^q, for some $q \le n$.

(3) The characteristic polynomial of M is s^n.

(4) The only eigenvalue of M is 0.

Question 4

Compute the algebraic and geometric multiplicities of the eigenvalue $\lambda = 2$ for the matrix

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

Useful Identities about Matrix Eigenvalues

For any $n \times n$ matrix A, we have

$$trace(A) = \sum_{i=1}^{n} \lambda_i ,$$

$$\det(A) = \prod_{i=1}^{n} \lambda_i .$$

See p.42 of (Horn-Johnson, 1st ed., 1985), or p. 50 of (Horn-Johnson, 2nd ed., 2013).

Similarity

Two matrices *A* and *B* are said to be similar, if

$$B = P^{-1}AP, \text{ for some invertible matrix } P.$$

Notation: $A \sim B$

The set of all similar matrices to a given square matrix A:

$$S = \left\{ P^{-1}AP : P \text{ is invertible} \right\}$$

Note:

Similar matrices are just different basis representations of a single linear mapping.

Similarity: Physical meaning

Let $T: V \to V$ be a linear transformation, and

$B_1 = \{v_1, \ldots, v_n\}$, $B_2 = \{w_1, \ldots, w_n\}$ be two bases for V.

Denote

$[x]_{B_1} = col(\alpha_1, \ldots, \alpha_n)$, with $x = \alpha_1 v_1 + \ldots + \alpha_n v_n$.

$Then$, by linearity,

$$Tx = \alpha_1 T v_1 + \ldots + \alpha_n T v_n$$

For any basis B_2 of V,

$$[Tv_j]_{B_2} \triangleq col(t_{1j}, \ldots, t_{nj}),$$

$$[Tx]_{B_2} = \sum_{j=1}^{n} \alpha_j [Tv_j]_{B_2} = (t_{ij})_{nxn}\, col(\alpha_1, \ldots, \alpha_n).$$

Similarity: Physical meaning

$$\left[Tx\right]_{B_2} = \sum_{j=1}^{n} \alpha_j \left[Tv_j\right]_{B_2} = \left(t_{ij}\right)_{nxn} col\left(\alpha_1, \ldots, \alpha_n\right).$$

It is important to note that the matrix $\left(t_{ij}\right)_{nxn}$ depends on T and the choice of the bases B_1 and B_2, but not x.

Define the B_1-B_2 *basis representation of* T as:

$${}_{B_2}\left[T\right]_{B_1} \triangleq \left(t_{ij}\right)_{nxn} = \left[\left[Tv_1\right]_{B_2}, \ldots, \left[Tv_n\right]_{B_2}\right]$$

So, $\left[Tx\right]_{B_2} = {}_{B_2}\left[T\right]_{B_1}\left[x\right]_{B_1}, \forall x \in V.$

For the special case when $B_1 = B_2$,

${}_{B_1}\left[T\right]_{B_1}$ is called the B_1 representation of T.

Similarity: Identities

For the identity linear transformation $Ix = x, \ \forall x \in V$,

it can be shown that

$$_{B_2}[I]_{B_1\,B_1}[I]_{B_2} = I_n, \quad _{B_1}[I]_{B_2\,B_2}[I]_{B_1} = I_n,$$

and

$$_{B_2}[T]_{B_2} = {}_{B_2}[I]_{B_1}\ {}_{B_1}[T]_{B_1}\ {}_{B_1}[I]_{B_2}.$$

In other words,

$B = P^{-1}AP$, where $P = {}_{B_1}[I]_{B_2}$, $A = {}_{B_1}[T]_{B_1}$, $B = {}_{B_2}[T]_{B_2}$.

B_2-B_1 change of basis matrix

For a proof, see (Horn & Johnson, 2nd ed, 2013, page 40).

Exercise

Find an invertible matrix P such that

$$P^{-1}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}P$$

is diagonal.

Homework #3

1. Give all the solutions of the system

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} x = \begin{pmatrix} 10 & 13 \\ 11 & 14 \\ 12 & 15 \end{pmatrix}.$$

2. Prove that the following eq. has no solution:

$$\begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix} x = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

Homework #3

3. Find a least-squares fit

$$b = x_0 + x_1 a^1 + x_2 a^2$$

for the data:

$$b = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad a^1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \quad a^2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Homework #3

4. Find independent eigenvectors for

$$A = \begin{pmatrix} 1 & 2 \\ -3 & 1 \end{pmatrix}.$$

Can you express $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ as a linear combination

of these eigenvectors of A?