① Given:

$$\begin{bmatrix} \dot{X}_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 6(t+1)^{-2} & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} e^{-t} \\ \sqrt{t} \end{bmatrix}, \quad X(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Fundamental Solutions to Time-varying system:

$$\mathbb{X}_1(t) = \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix}, \quad \mathbb{X}_1(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\mathbb{X}_2(t) = \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix}, \quad \mathbb{X}_2(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Case 1:

$$\dot{X}_1 = X_2$$

$$\ddot{X}_1 = \dot{X}_2 = 6(t+1)^{-2} X_1$$

Initial guess for $X_1 = (t+1)^3$ and $(t+1)^{-2}$

$$X_1 = C_1(t+1)^3 + C_2(t+1)^{-2}$$

$$X_1(0) = 1 = C_1 + C_2$$

$$X_2(0) = \dot{X}_1(0) = 0 = \left[3C_1(t+1)^2 - 2C_2(t+1)^{-3}\right]\Big|_{t=0} = 3C_1 - 2C_2$$

$$\left.\begin{aligned} C_1 + C_2 &= 1 \\ 3C_1 - 2C_2 &= 0 \end{aligned}\right\} \Rightarrow C_1 = \frac{2}{5}, \; C_2 = \frac{3}{5} \Rightarrow X_1(t) = \frac{2}{5}(t+1)^3 + \frac{3}{5}(t+1)^{-2}$$

Case 2:

$$X_1 = C_1(t+1)^3 + C_2(t+1)^{-2}$$

$$X_1(0) = 0 = C_1 + C_2$$

$$X_2(0) = \dot{X}_1(0) = 1 = \left[3C_1(t+1)^2 - 2C_2(t+1)^{-3}\right]\Big|_{t=0} = 3C_1 - 2C_2$$

$$\left.\begin{aligned} C_1 + C_2 &= 0 \\ 3C_1 - 2C_2 &= 1 \end{aligned}\right\} \Rightarrow C_1 = \frac{1}{5}, \; C_2 = \frac{-1}{5} \Rightarrow X_1(t) = \frac{1}{5}(t+1)^3 - \frac{1}{5}(t+1)^{-2}$$

Therefore,

$$\mathbf{X}_1(t) = \begin{bmatrix} \frac{2}{5}(t+1)^3 + \frac{3}{5}(t+1)^{-2} \\ \frac{6}{5}(t+1)^2 - \frac{6}{5}(t+1)^{-3} \end{bmatrix}$$

$$\mathbf{X}_2(t) = \begin{bmatrix} \frac{1}{5}(t+1)^3 - \frac{1}{5}(t+1)^{-2} \\ \frac{3}{5}(t+1)^2 + \frac{2}{5}(t+1)^{-3} \end{bmatrix}$$

$$x_h(t) = \mathbf{X}(t)\, x(0)$$

$$= \begin{bmatrix} \mathbf{X}_1(t) & \mathbf{X}_2(t) \end{bmatrix} x(0)$$

$$= \begin{bmatrix} \frac{2}{5}(t+1)^3 + \frac{3}{5}(t+1)^{-2} & \frac{1}{5}(t+1)^3 - \frac{1}{5}(t+1)^{-2} \\ \frac{6}{5}(t+1)^2 - \frac{6}{5}(t+1)^{-3} & \frac{3}{5}(t+1)^2 + \frac{2}{5}(t+1)^{-3} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$x_h(t) = \begin{bmatrix} \frac{4}{5}(t+1)^3 + \frac{1}{5}(t+1)^{-2} \\ \frac{12}{5}(t+1)^2 - \frac{2}{5}(t+1)^{-3} \end{bmatrix}$$ (solution to homogeneous part)

$$x(t) = \mathbf{X}(t)\, x(0) + \mathbf{X}(t) \int_0^t \mathbf{X}^{-1}(s)\, f(s)\, ds$$

$$= x_h(t) + \begin{bmatrix} \frac{2}{5}(t+1)^3 + \frac{3}{5}(t+1)^{-2} & \frac{1}{5}(t+1)^3 - \frac{1}{5}(t+1)^{-2} \\ \frac{6}{5}(t+1)^2 - \frac{6}{5}(t+1)^{-3} & \frac{3}{5}(t+1)^2 + \frac{2}{5}(t+1)^{-3} \end{bmatrix} \int_0^t \begin{bmatrix} \frac{3}{5}(s+1)^2 + \frac{2}{5}(s+1)^{-3} & \frac{-1}{5}(s+1)^3 + \frac{1}{5}(s+1)^{-2} \\ \frac{-6}{5}(s+1)^2 + \frac{6}{5}(s+1)^{-3} & \frac{2}{5}(s+1)^3 + \frac{3}{5}(s+1)^{-2} \end{bmatrix}$$

$$\begin{bmatrix} e^{-s} \\ \sqrt{s} \end{bmatrix} ds$$

$$= x_h(t) + \mathbf{X}(t) \int_0^t \begin{bmatrix} e^{-s}\left(\frac{3}{5}(s+1)^2 + \frac{2}{5}(s+1)^{-3}\right) + \sqrt{s}\left(\frac{-1}{5}(s+1)^3 + \frac{1}{5}(s+1)^{-2}\right) \\ e^{-s}\left(\frac{-6}{5}(s+1)^2 + \frac{6}{5}(s+1)^{-3}\right) + \sqrt{s}\left(\frac{2}{5}(s+1)^3 + \frac{3}{5}(s+1)^{-2}\right) \end{bmatrix} ds$$

2. $$\frac{d\,y(t)}{dt} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix} y(t), \qquad y(0) = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$$

let $y(t) = [\,y_1(t) \;\; y_2(t) \;\; y_3(t) \;\; y_4(t) \;\; y_5(t)\,]^T$

Hence,
$$\begin{bmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \\ \dot{y}_3(t) \\ \dot{y}_4(t) \\ \dot{y}_5(t) \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ y_4(t) \\ y_5(t) \end{bmatrix}$$

The differential equation decoupled into two sets of different equations with smaller size.

$$\begin{cases} \dot{y}_1(t) = 2\,y_1(t) \\ \dot{y}_2(t) = y_1(t) + 2\,y_2(t) \end{cases} \quad \cdots \text{(A)}$$

$$\begin{cases} \dot{y}_3(t) = 3\,y_3(t) \\ \dot{y}_4(t) = y_3(t) + 3\,y_4(t) \\ \dot{y}_5(t) = y_4(t) + 3\,y_5(t) \end{cases} \quad \cdots \text{(B)}$$

with $y(0) = [\,y_1(0) \;\; y_2(0) \;\; y_3(0) \;\; y_4(0) \;\; y_5(0)\,]^T = [\,1 \;\; 2 \;\; 3 \;\; 4 \;\; 5\,]^T$

From set (A), $\begin{cases} y_1(t) = e^{2t} \\ y_2(t) = (2+t)\,e^{2t} \end{cases}$

From set (B),

$$\begin{cases} y_3(t) = 3e^{3t} \\ y_4(t) = (4+3t)e^{3t} \\ y_5(t) = \left(5 + 4t + \frac{3}{2}t^2\right)e^{3t} \end{cases}$$

$$\Rightarrow \begin{cases} y_1(t) = e^{2t} \\ y_2(t) = (2+t)e^{2t} \\ y_3(t) = 3e^{3t} \\ y_4(t) = (4+3t)e^{3t} \\ y_5(t) = \left(5+4t+\frac{3}{2}t^2\right)e^{3t} \end{cases}$$

#