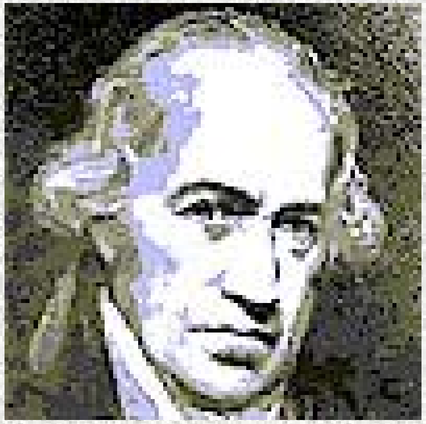
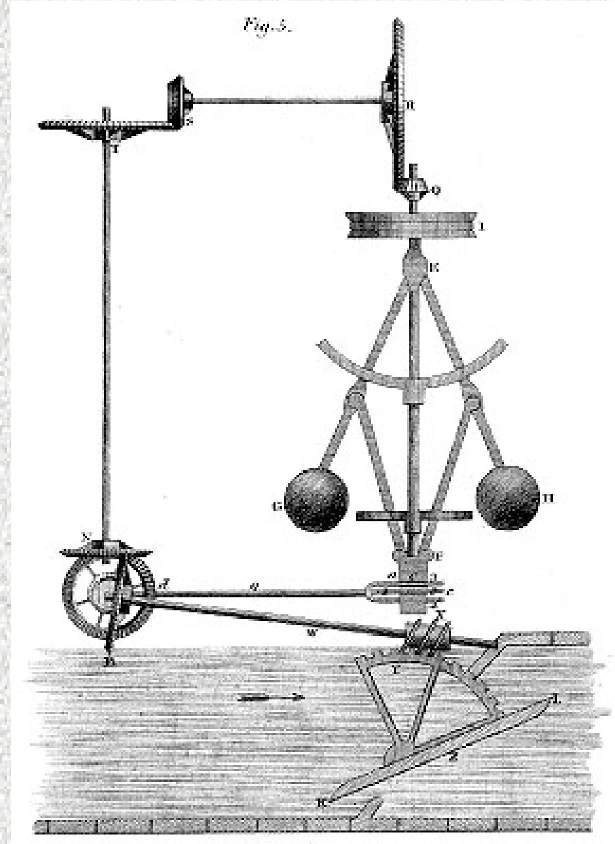
Lecture XI
Stability of Linear Systems

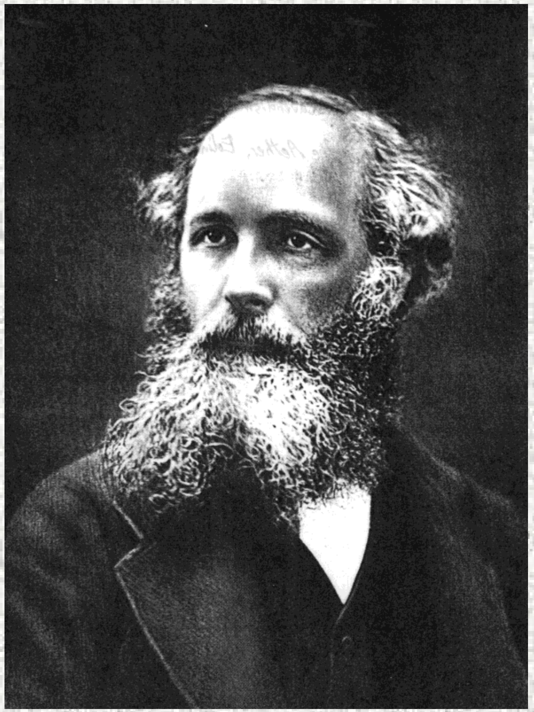
- Linearization
- Definition of stability
- Necessary and sufficient conditions for stability

Classical Example in Stability





J. Watt (1736-1819)



J. C. Maxwell (1868)
"**On Governors**"

Mathematical Modeling

Finite-dimensional differential equations:

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n$$

Equilibrium x_e if it satisfies $f(x_e) = 0$.

Without loss of generality, assume $x_e = 0$.

If not, consider $y = x - x_e$. Then,

$$\dot{y} = f\left(y + x_e\right)$$

has an equilibrium at the origin, i.e. $y_e = 0$.

Linearization

From nonlinear to linear systems:

$$\dot{x}_l = A x_l, \quad x_l \in \mathbb{R}^n$$

where

$$A = \left. \frac{\partial f(x)}{\partial x} \right|_{x=0} \doteq \frac{\partial f}{\partial x}(0) \in \mathbb{R}^{n \times n}$$

Often, the eq. is called "first-order approximation", or *linearization*, of the original nonlinear equation around the equilibrium point $x_e = 0$.

Comment

The linearized model only represents a good (local!) approximation of the nonlinear model near the equilibrium of interest:

$$\ddot{\theta} = -k_1 \sin\theta - k_2 \dot{\theta} \quad \text{(Rotational Pendulum)}$$

Equilibria:

$$\begin{pmatrix} \dot{\theta} \\ \theta \end{pmatrix} = \begin{pmatrix} 0 \\ 2m\pi \end{pmatrix}, \quad \begin{pmatrix} \dot{\theta} \\ \theta \end{pmatrix} = \begin{pmatrix} 0 \\ (2m+1)\pi \end{pmatrix}$$

where $m \in \mathbb{Z}$.

Comment (cont'd)

Around the first set of equilibria, the (local) linearized model of

$$\ddot{\theta} = -k_1 \sin\theta - k_2\dot{\theta}$$

becomes:

$$(\text{S1}) \quad \ddot{\theta} = -k_1\theta - k_2\dot{\theta}$$

However, around the second set of equilibria, the linearized model is totally different:

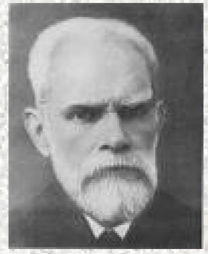
$$(\text{S2}) \quad \ddot{\theta} = +k_1\theta - k_2\dot{\theta}$$

Why Linearization Useful?

(Poincare-Lyapunov Theorem)

If the linearized system is stable, then the original nonlinear system is also stable.

Stability (Lyapunov, 1892)



We are only interested in "asymptotic stability".

Roughly speaking, we want to study the following two properties:

- continuity of the solution $x(t)$ w.r.t. $x(0)$:

  $|x(0)| < \delta \quad \Rightarrow \quad |x(t)| < \varepsilon < \infty, \forall t.$

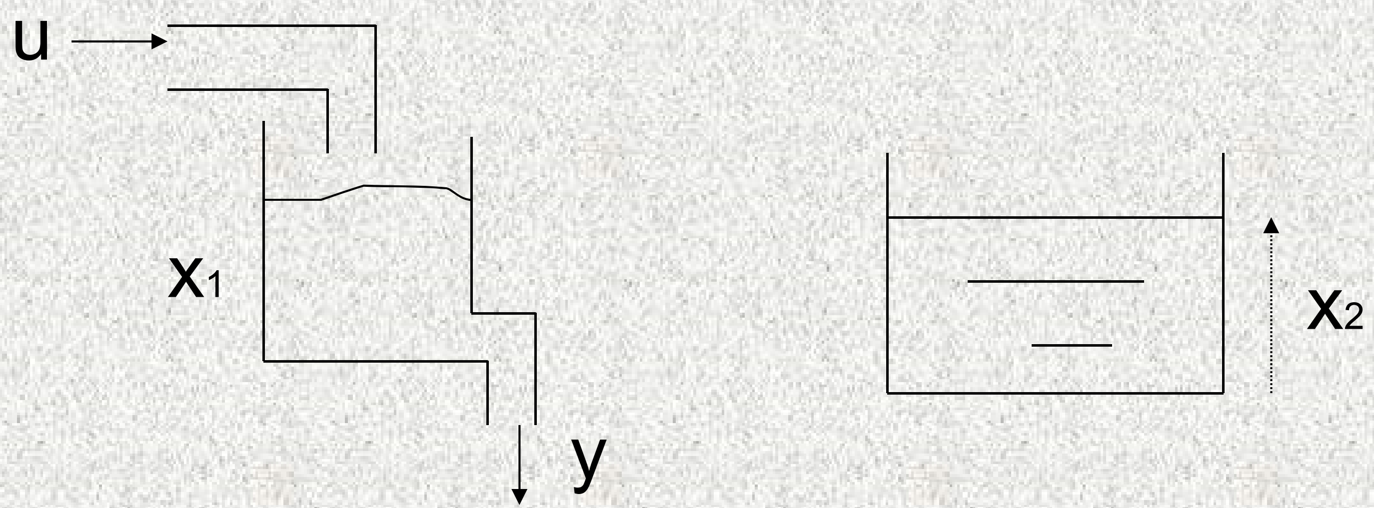
- attractiveness: $x(t) \to 0$ as $t \to \infty$.

Must be stable (1173 – now) !

Example: neutral vs. asymptotic stability

$$\dot{x} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$



Lyapunov's First Theorem (1892)

- If the linearized model $\dot{x}_l = Ax_l$ is (asymptotically) stable, then the original nonlinear system $\dot{x} = f(x)$ is also (asymptotically) stable at $x_e = 0$.

- If $\dot{x}_l = Ax_l$ is unstable (i.e. not stable), then $\dot{x} = f(x)$ is also unstable.

Remark: No conclusion can be drawn for "marginal stability".

Simple Examples

(1) $\dot{x} = -x + 2x^2 \doteq f(x)$

$$\dot{x}_l = -x_l \doteq Ax_l$$

are both (asymptotically) stable at the origin.

(2) $\dot{x} = \sin x \doteq f(x)$

$$\dot{x}_l = x_l \doteq Ax_l$$

are both unstable at the origin.

Comment

Neutral stability (or, marginal stability) of a linearized model does not imply neutral stability of its original nonlinear system.

Example: Both the nonlinear systems

$$\dot{x} = x^3 \text{ (unstable) and } \dot{x} = -x^3 \text{ (stable)}$$

share the same neutrally, but not asymptotically, stable linear model

$$\dot{x} = 0$$

A Necessary and Sufficient Condition for Stability

Consider the linear time-invariant system

$$\dot{x} = Ax, \quad x(0) = x_o \in \mathbb{R}^n.$$

It is (asymptotically) stable <u>if and only if</u> A is Hurwitz, i.e. all its eigenvalues have negative real part.

Proof : Using the Jordan canonical form.

Remarks on Jordan form

$$(1)\quad e^{At} = P \times blockdiag\left(e^{J_i t}\right) \times P^{-1}$$

$$(2)\ e^{J_i t} = \begin{pmatrix} 1 & t & \frac{t^2}{2!} & & \cdots & \frac{t^{n_i-1}}{(n_i-1)!} \\ & \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & t \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} e^{\lambda_i t}$$

Lyapunov Matrix Equation

If A is a Hurwitz matrix, then the solution to

$$A^T P + PA = -I$$

is symmetric and positive definite. Indeed,

$$P = \int_0^\infty e^{A^T t} e^{At} dt.$$

Sketch of the Proof

- The solution of $\dot{X} = A^T X + XA,\ X(0) = C$

  is: $X(t) = e^{A^T t} C e^{At}$.

- Integrating both sides from 0 to ∞ leads to:

$$-C = A^T \left(\int_0^\infty X(s)\, ds \right) + \left(\int_0^\infty X(s)\, ds \right) A$$

$$\textit{Thus},\ \text{when } C = I,\ P = \int_0^\infty X(s)\, ds := \int_0^\infty e^{A^T t} e^{At} dt.$$

Another Proof of Stability

Now, let's prove the stability of

$$\dot{x} = Ax, \quad x(0) = x_o$$

where A is Hurwitz.

Consider the function $V(x) = x^T P x$.

Differentiating $V(x(t))$ with respect to time yields

$$\dot{V} = x^T(t)\left(A^T P + PA\right)x(t) = -x^T(t)x(t)$$

$$\leq -x^T(t)Px(t)/\lambda_{\max}(P) \doteq -\mu V$$

Another Proof of Stability (cont'd)

From the fact

$$\dot{V} \leq -\mu V, \quad \mu \doteq 1/\lambda_{\max}(P) > 0,$$

we have

$$V(t) \leq e^{-\mu t} V(0)$$

So, $V(t) = x^T(t) P x(t),$ *and* thus $x(t),$

converge to 0 at an exponential rate.

$V = x^T P x$ is often called a Lyapunov function.

Test for stability of *A*

Let P be determined by the matrix equation

$$A^T P + PA = -I.$$

Then, A is a stable matrix (Hurwitz) iff P is positive definite.

Pictorially,

Stability of linear systems

$$\dot{x} = Ax$$

known

Stability of matrix A
Or A is Hurwitz

$\xrightarrow{?}$ $\xleftarrow{?}$

$P = P^T > 0$, solution to
Lyapunov matrix eq.

Proof

$\Rightarrow$: $P = \int_0^\infty e^{A^T t} e^{At} dt$ is a positive definite matrix.

$\Leftarrow$: Assume P is positive definite. Let $x(t)$ be solution to $\dot{x} = Ax,\ x(0) = x_o$.

Then, direct computation gives

$$\frac{d}{dt}\left(x^T(t)Px(t)\right) = -x^T(t)x(t).$$

Integrating both sides from 0 to t_1 implies:

Proof (cont'd)

Integrating both sides from 0 to t_1 implies

$$\int_0^{t_1} \left\| x(t) \right\|^2 dt = x^T(0)Px(0) - x^T(t_1)Px(t_1)$$

$$\leq x^T(0)Px(0)$$ because P positive definite

$$\Rightarrow \int_0^{\infty} \left\| x(t) \right\|^2 dt < \infty.$$

So, for any $x(0)$, $x(t) \to 0$,

leading to stability of A, as wished.

Extension:

Discrete-Time Equations & Systems

- Solutions of an inhomogeneous linear equation

  $$x(k+1) = A(k)x(k) + f(k),$$

  with given $x(0) = x_o \in \mathbb{R}^n$.

- Stability of linear difference equations

  $$x(k+1) = Ax(k), \quad x(k) \in \mathbb{R}^n$$

Solutions of Discrete-Time Equations

Solutions of an inhomogeneous linear equation

$$x(k+1) = A(k)x(k) + f(k)$$

with given $x(k_0) = x_o \in \mathbb{R}^n$.

Clearly, it holds

$$
\begin{aligned}
&x(k)\\
&= A(k-1)x(k-1) + f(k-1)\\
&= A(k-1)A(k-2)x(k-2) + A(k-1)f(k-2) + f(k-1)\\
&\ \vdots\\
&= \underbrace{A(k-1)A(k-2)\cdots A(j)}_{\Phi(k,j)}x(j)\\
&+\sum_{l=j}^{k-1}\underbrace{A(k-1)A(k-2)\cdots A(l+1)}_{\Phi(k,l+1)}f(l), \quad \Phi(k,k) \triangleq I
\end{aligned}
$$

So, the general solution with $x(k_0) = x_o$ is:

$$x(k) = \Phi\left(k, k_0\right) x_o + \sum_{l=k_0}^{k-1} \Phi\left(k, l+1\right) f(l)$$

where $\Phi\left(k, k_0\right)$ is called "transition matrix".

Comment

Unlike the continuous-time case, the discrete-time transition matrix

$$\Phi(k,j)=\begin{cases} A(k-1)\,A(k-2)\cdots A(j), & \forall k\geq j+1 \\ I, & k=j \end{cases}$$

may *not* be invertible! Here is such a simple example:

$$x(k+1)=\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} x(k)$$

Stability of Discrete-Time Linear Systems

A linear system taking the discrete-time form

$$x(k+1) = Ax(k)$$

is (asymptotically) stable if and only if all eigenvalues of A have magnitude strictly less than unity.

Example from Economics

Notations:

$$y(k) \; = \; \text{national income in year } k;$$

$$c(k) \; = \; \text{consumer expenditure;}$$

$$i(k) \; = \; \text{private investment;}$$

$$g(k) \; = \; \text{government expenditure.}$$

Example from Economics

A simplified classical model in economics:

$$
\begin{aligned}
y(k) &= c(k) + i(k) + g(k), \\
c(k+1) &= \alpha y(k), \quad 0 < \alpha < 1, \\
i(k+1) &= \beta\left[c(k+1) - c(k)\right], \quad \beta > 0.
\end{aligned}
$$

Example from Economics

$$x(k+1)=\underbrace{\begin{pmatrix}\alpha & \alpha\\ \beta(\alpha-1) & \beta\alpha\end{pmatrix}}_{A}x(k) + \underbrace{\begin{bmatrix}\alpha\\ \beta\alpha\end{bmatrix}}_{B}g(k),$$

$$y(k)=\underbrace{\begin{bmatrix}1 & 1\end{bmatrix}}_{C}x(k) + g(k)$$

where

$$x(k)=\begin{bmatrix}x_1(k)\\ x_2(k)\end{bmatrix}\triangleq\begin{bmatrix}c(k)\\ i(k)\end{bmatrix},\ g(k)=\text{input},\ y(k)=\text{output}.$$

Exercise

- Compute the transition matrix of the economic model.
- Study the stability of the economic model.

Sylvester Equation

A generalization of the Lyapunov matrix equation.

Given a triplet of matrices $A \in \mathbb{R}^{p\times p},\ B \in \mathbb{R}^{q\times q},\ C \in \mathbb{R}^{p\times q}$.

When will the following Sylvester equation have a (unique)

solution $X \in \mathbb{R}^{p\times q}$?

Sylvester equation: $AX - XB = C$

A Necessary and Sufficient Condition

Let $\alpha_1, \ldots, \alpha_p$ and $\beta_1, \ldots, \beta_q$ denote the (possibly repeated) eigenvalues of A and $B,$ resp. and define the linear mapping:

$$T : X \in \mathbb{R}^{p\times q} \to AX - XB \in \mathbb{R}^{p\times q}$$

Then, the following implication holds:

$$N_T = \left\{ O_{p\times q} \right\} \Leftrightarrow \alpha_i \neq \beta_j$$

for any pair of $i = 1, \ldots, p \ and \ j = 1, \ldots, q.$

Proof of the Necessity "=>"

Let $Au_i = \alpha_i u_i$ and $B^T v_j = \beta_j v_j$ for some pair of nonzero vectors u_i, v_j. Also let $X = u_i v_j^T \in \mathbb{R}^{p\times q}$. Then, the formula

$$TX = Au_i v_j^T - u_i v_j^T B = \left(\alpha_i - \beta_j\right) u_i v_j^T$$

implies that $\alpha_i - \beta_j \neq 0$ is necessary for

$$N_T \overset{\text{def}}{=} \left\{X : TX = O_{p\times q}\right\} = \{O_{p\times q}\}.$$

Proof of the Sufficiency “<=”

Assume now that $\alpha_i \neq \beta_j$ for any pair (i, j). We want to show

$N_T = \{O_{p\times q}\}$. Apply the Jordan decompositions:

$A = UJU^{-1}$, $B = V\tilde{J}V^{-1}$. Then,

$$AX - XB = O \Leftrightarrow UJU^{-1}X - XV\tilde{J}V^{-1} = O$$

$$\Leftrightarrow J\left(U^{-1}XV\right) - \left(U^{-1}XV\right)\tilde{J} = O$$

The proof is completed by letting $Y = U^{-1}XV$, and

writing $J, \tilde{J}$ using their Jordan blocks $J_1, \ldots, J_k$ and $\tilde{J}_1, \ldots, \tilde{J}_l$.

Proof of the Sufficiency “<=” (Cont’d)

$$JY - Y\tilde{J} = O \Leftrightarrow J_i Y_{ij} - Y_{ij}\tilde{J}_j = O, \text{ for } i = 1,\dots,k,\ j = 1,\dots,l.$$

Rewrite J_i and $\tilde{J}_j$ as $J_i = \alpha_i I_{p_i} + N, \tilde{J}_j = \beta_j I_{q_j} + \tilde{N}$. Then,

$$J_i Y_{ij} - Y_{ij}\tilde{J}_j = Y_{ij}\left(\alpha_i I_{q_j} - \tilde{J}_j\right) + NY_{ij}.$$

When $\alpha_i \neq \beta_j$, the matrix $\alpha_i I_{q_j} - \tilde{J}_j = \left(\alpha_i - \beta_j\right) I_{q_j} - \tilde{N}$

is invertible. So, $Y_{ij} = NY_{ij}M$, with $M = -\left(\alpha_i I_{q_j} - \tilde{J}_j\right)^{-1}$.

Iteratively, $Y_{ij} = N^k Y_{ij} M^k$, for $k = 2,3,\dots$

Proof of the Sufficiency “<=” (Cont’d)

Iteratively, $Y_{ij} = N^k Y_{ij} M^k$, for $k = 1, 2, 3, \ldots$

For large enough k, $N^k = O$. Then, $Y_{ij} = O$.

Therefore, $X = UYV^{-1} = O$.

i.e., $N_T = \left\{ O_{p \times q} \right\}$.

Unique Solution to the Sylvester Equation

Let $\alpha_1, \ldots, \alpha_p$ and $\beta_1, \ldots, \beta_q$ denote the (possibly repeated) eigenvalues of A and B, resp.

Then, the Sylvester equation

$$AX - XB = C$$

has a unique solution $X \in \mathbb{R}^{p \times q}$ if and only if $\alpha_i \neq \beta_j$ for any pair of $i = 1, \ldots, p$ *and* $j = 1, \ldots, q.$

Outline of the Proof

- The uniqueness follows from the prior result.
- The existence follows from the

"***Principle of Conservation of Dimension***":

For any linear mapping $T : U \to V$ between two vector spaces.

$$\dim N_T + \dim R_T = \dim U.$$

Exercise

Are the following systems asymptotically stable at the origin?

(1) $$\begin{cases} \dot{x}_1 = -x_1 + x_1^2 x_2 \\ \dot{x}_2 = -x_2 + x_1 \end{cases}$$

(2) $$\begin{cases} \dot{x}_1 = -x_1 - x_2 \\ \dot{x}_2 = x_1 - x_2^3 \end{cases}$$

Homework 10

Consider $\dot{x} = Ax + g(x),\ x(0) = x_o \in \mathbb{R}^n,$ where

- A is a stable matrix.
- $\|g(x)\| / \|x\| \to 0,$ as $\|x\| \to 0.$
- $\|x_o\|$ is sufficiently small.

Can you try to prove that the solution $x(t)$ of the nonlinear equation converges to 0, as $t \to \infty$?