Midterm Solution

July 17, 2022

1 Problem 1

Let

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 2 & 3 \\ 2 & -5 & -6 \end{bmatrix} \text{ and } b = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix},$$

1. Find all the solutions of the linear equation $Ax = b$ with $x = [x_1, x_2, x_3]^T \in \mathbb{R}^3$.
2. Find the dimension of Null(A), i.e. $\{x \in \mathbb{R}^3 | Ax = 0\}$, and the rank of A^T.

2 Problem 2

Are the following statements true or false? If true, prove the statement. If false, give a counterexample.

1. A matrix $A \in \mathbb{R}^{n\times n}$ with n real orthonormal eigenvectors is symmetric.
2. Assume that $w \neq 0$ is an eigenvector for matrices $A, B \in \mathbb{R}^{n\times n}$, then $AB - BA$ is not invertible.
3. If the Jordan canonical form of A is J, then that of A^2 is J^2.

3 Problem 3

Given $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, $D \in \mathbb{R}^{m\times m}$, $a \in \mathbb{R}^n$, and $b \in \mathbb{R}^m$. Assume D and $(A - BD^{-1}C)$ are invertible. Let $X = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$.

1. Give the expression of the solution to $\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$. Also, give the expression of X^{-1}.
2. Prove that $\det(X) = \det(D)\det(A - BD^{-1}C)$.

Solution:

Problem 1

1. We first transform $[A, b]$ into a row-reduced echelon form. Here r_1, r_2 and r_3 refer to the first, second and third row of $[A, b]$, respectively. First multiplication of r_1 by $\frac{1}{2}$ gives

$$\begin{bmatrix} 1 & \frac{3}{2} & 2 & \frac{1}{2} \\ 4 & 2 & 3 & 3 \\ 2 & -5 & -6 & 3 \end{bmatrix},$$

and then we add $-4r_1$ to r_2, and add $-2r_1$ to r_3, and get

$$\begin{bmatrix} 1 & \frac{3}{2} & 2 & \frac{1}{2} \\ 0 & -4 & -5 & 1 \\ 0 & -8 & -10 & 2 \end{bmatrix}.$$

Then we multiply r_2 by $-\frac{1}{4}$ and add $8r_2$ to r_3 and get

$$\begin{bmatrix} 1 & \frac{3}{2} & 2 & \frac{1}{2} \\ 0 & 1 & \frac{5}{4} & -\frac{1}{4} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Finally we add $-\frac{3}{2}r_2$ to r_1 and get

$$\begin{bmatrix} 1 & 0 & \frac{1}{8} & \frac{7}{8} \\ 0 & 1 & \frac{5}{4} & -\frac{1}{4} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

To find the solution of $Ax = b$, from the echelon form,

$$\begin{cases} x_1 + \frac{1}{8}x_3 = \frac{7}{8} \\ x_2 + \frac{5}{4}x_3 = -\frac{1}{4} \end{cases}$$

gives a particular solution $x_p = [1, 1, -1]^T$. Then we need to find the solution of the homogeneous equation $Ax = 0$. From the echelon form,

$$\begin{cases} x_1 + \frac{1}{8}x_3 = 0 \\ x_2 + \frac{5}{4}x_3 = 0 \end{cases}$$

gives the basic solution $x_h = k[-1, -10, 8]^T$ with $k \in \mathbb{R}$. Therefore, all the solutions can be written as

$$x = x_p + x_h = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ -10 \\ 8 \end{bmatrix} k, \quad k \in \mathbb{R}.$$

2. Since the basic solution x_h spans a one dimensional subspace, dim Null$(A) = 1$. Since $\mathbb{R}^3 =$ Null$(A) \oplus$R(A^T) where R(A^T) is the image space of A^T, rank$(A^T) =$ dim R$(A^T) = 3-$dim Null$(A) = 2$.

Problem 2

1. True. Assuming matrix $A \in \mathbb{R}^{n\times n}$ has n real orthonormal eigenvectors $p_i \in \mathbb{R}^n$ and corresponding eigenvalue $\lambda_i \in \mathbb{R}$ for $i = 1, ..., n$, then if $P = [p_1, ..., p_n] \in \mathbb{R}^{n\times n}$ and $D = \text{diag}\{\lambda_1, \lambda_2, ..., \lambda_n\} \in \mathbb{R}^{n\times n}$ (D is a diagonal matrix and the diagnal elements are arranged in order), $A = PDP^T$, from which $A^T = PDP^T = A$ and A is symmetric.

2. True. As $w \neq 0$ is an eigenvector for both matrices $A, B \in \mathbb{R}^{n\times n}$, there exist $\lambda_1, \lambda_2 \in \mathbb{C}$, such that $Aw = \lambda_1 w$ and $Bw = \lambda_2 w$. Hence,

$$ABw = A(Bw) = A(\lambda_2 w) = \lambda_2 Aw = \lambda_1\lambda_2 w$$

$$BAw = B(Aw) = B(\lambda_1 w) = \lambda_1 Bw = \lambda_1\lambda_2 w$$

As a consequence, w is the eigenvector of AB and BA. As $(AB - BA)w = ABw - BAw = \lambda_1\lambda_2 w - \lambda_1\lambda_2 w = 0$, the nullspace of $(AB - BA)$ is not empty, and it cannot be invertible.

3. False. Assume $A = J = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, then $A^2 = J^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ which is similar to the Jordan canonical form J, as $A^2 = PJP^{-1}$ with $P = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$ and $P^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. Since $J \neq J^2$, the statement is false.

Problem 3

1. The linear equation can be rewritten as

$$\begin{aligned} Ax + By &= a \\ Cx + Dy &= b. \end{aligned}$$

As D is invertible,

$$y = D^{-1}(b - Cx).$$

Plugging this into the first equation, we get

$$Ax + BD^{-1}(b - Cx) = a.$$

This equation can be rewritten as

$$(A - BD^{-1}C)x = a - BD^{-1}b.$$

As $(A - BD^{-1}C)$ is invertible,

$$\begin{aligned} x &= (A - BD^{-1}C)^{-1}(a - BD^{-1}b) \\ y &= D^{-1}[b - C(A - BD^{-1}C)^{-1}(a - BD^{-1}b)]. \end{aligned}$$

That is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (A - BD^{-1}C)^{-1} & -(A - BD^{-1}C)^{-1}BD^{-1} \\ -D^{-1}C(A - BD^{-1}C)^{-1} & D^{-1} + D^{-1}C(A - BD^{-1}C)^{-1}BD^{-1} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$

As $\begin{bmatrix} x \\ y \end{bmatrix} = X^{-1} \begin{bmatrix} a \\ b \end{bmatrix}$, we have

$$X^{-1} = \begin{bmatrix} (A - BD^{-1}C)^{-1} & -(A - BD^{-1}C)^{-1}BD^{-1} \\ -D^{-1}C(A - BD^{-1}C)^{-1} & D^{-1} + D^{-1}C(A - BD^{-1}C)^{-1}BD^{-1} \end{bmatrix}$$

2. X can be factorized as

$$X = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & BD^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} I & 0 \\ D^{-1}C & I \end{bmatrix}$$

Then,

$$\begin{aligned} \det(X) &= \det\left(\begin{bmatrix} I & BD^{-1} \\ 0 & I \end{bmatrix} \right) \det\left(\begin{bmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{bmatrix} \right) \det\left(\begin{bmatrix} I & 0 \\ D^{-1}C & I \end{bmatrix} \right) \\ &= \det(A - BD^{-1}C)\det(D). \end{aligned}$$