EL9343 Homework 1

Due: Sept. 14th 8:00 a.m.

1. Prove the following properties of asymptotic notation:

   (a) $n = \omega(\sqrt{n})$

$$\forall c > 0, \exists n_0 = c^2 + 1 > 0, \text{ such that } n > c\sqrt{n}, \forall n \geq n_0$$
$$\therefore n = \omega(\sqrt{n})$$

   (b) If $f(n) = \Omega(g(n))$, and $h(n) = \Theta(g(n))$, then $f(n) = \Omega(h(n))$

$$f(n) = \Omega(g(n)) \Leftrightarrow \exists c_1 > 0, n_1 > 0, \text{ such that } f(n) \geq c_1 g(n), \forall n \geq n_1$$
$$h(n) = \Theta(g(n)) \Leftrightarrow \exists c_2 > 0, n_2 > 0, \text{ such that } h(n) \leq c_2 g(n), \forall n \geq n_2$$
$$\therefore g(n) \geq \frac{1}{c_2} h(n), \forall n \geq n_2$$
$$\therefore \exists c_3 = \frac{c_1}{c_2} > 0, n_3 = \max(n_1, n_2) > 0, \text{ such that } \forall n \geq n_3, f(n) \geq c_1 g(n) \geq c_1 \frac{1}{c_2} h(n) = c_3 h(n)$$
$$\therefore f(n) = \Omega(h(n))$$

   (c) $f(n) = O(g(n))$ if and only if $g(n) = \Omega(f(n))$ (*Transpose Symmetry* property)

$$\text{Part1: } f(n) = O(g(n)) \implies g(n) = \Omega(f(n))$$
$$f(n) = O(g(n)) \Leftrightarrow \exists c_1 > 0, n_1 > 0, \text{ such that } f(n) \leq c_1 g(n), \forall n \geq n_1$$
$$\therefore c'_1 = \frac{1}{c_1}, \forall n \geq n_1, g(n) \geq c'_1 f(n) \Leftrightarrow g(n) = \Omega(f(n))$$
$$\text{Part2: } f(n) = O(g(n)) \impliedby g(n) = \Omega(f(n))$$
$$g(n) = \Omega(f(n)) \Leftrightarrow \exists c_2 > 0, n_2 > 0, \text{ such that } g(n) \geq c_2 f(n), \forall n \geq n_2$$
$$\therefore c'_2 = \frac{1}{c_2}, \forall n \geq n_2, f(n) \leq c'_2 g(n) \Leftrightarrow f(n) = O(g(n))$$

2. Indicate, for each pair of expressions (A, B) in the table below, whether A is O, o, Ω, ω, or Θ of B. Assume that $k \geq 1, \epsilon > 0$, and $c > 1$ are constants. Your answer should be in the form of the table with "yes" or "no" written in each box.

	A	B	O	o	Ω	ω	Θ
a	$\lg^k n$	n^ϵ	yes	yes	no	no	no
b	n^k	c^n	yes	yes	no	no	no
c	$\sqrt{n}$	$n^{\sin n}$	no	no	no	no	no
d	2^n	$2^{n/2}$	no	no	yes	yes	no
e	$n^{\lg c}$	$c^{\lg n}$	yes	no	yes	no	yes
f	$\lg(n!)$	$\lg(n^n)$	yes	no	yes	no	yes

3. You have 5 algorithms, A1 took $O(n)$ steps, A2 took $\Theta(n \log n)$ steps, and A3 took $\Omega(n^2)$ steps, A4 took $o(n^3)$ steps, A5 took $\omega(n^{3/2})$ steps. You had been given the exact running time of each algorithm, but unfortunately you lost the record. In your messy desk you found the following formulas:

   (a) $4(5^{3 \log_5 n}) + 12n + 9527$

   (b) $\sqrt[5]{3n!}$

   (c) $\frac{5^{\log_{16} n}}{6}^2 + 4n + 17$

   (d) $3n \log_3 n + (\log_2 n)^3$

   (e) $\log_4 \log_2 n + 61$

   (f) $2^{5 \log_4 n}$

   (g) $(\log_2 n)^2 + \log_3 \log_3 n$

   For each algorithm write down all the possible formulas that could be associated with it.

(a) $4(5^{3\log_5 n}) + 12n + 9527$

$$4(5^{3\log_5 n}) + 12n + 9527 = \Theta(5^{3\log_5 n} + n) = \Theta(n^3 + n) = \Theta(n^3)$$

(b) $\sqrt[5]{3n!}$

$$\text{By Stirling's approximation, } n! \approx \sqrt{2\pi n}(\frac{n}{e})^n$$

$$\sqrt[5]{3n!} \approx \sqrt[5]{3}(2\pi n)^{1/10}(\frac{n}{e})^{n/5} = \Omega(n^n)$$

(c) $\frac{5^{\log_{16} n^2}}{6} + 4n + 17$

$$\frac{5^{\log_{16} n^2}}{6} + 4n + 17 = \Theta(n^{\log_{16} 25} + n) = \Theta(n^{\log_{16} 25}) \approx \Theta(n^{1.20})$$

(d) $3n\log_3 n + (\log_2 n)^3$

$$3n\log_3 n + (\log_2 n)^3 = \Theta(n\log n)$$

(e) $\log_4 \log_2 n + 61$

$$\log_4 \log_2 n + 61 = \Theta(\log\log n)$$

(f) $2^{5\log_4 n}$

$$2^{5\log_4 n} = \Theta(n^{2.5})$$

(g) $(\log_2 n)^2 + \log_3 \log_3 n$

$$(\log_2 n)^2 + \log_3 \log_3 n = \Theta((\log_2 n)^2)$$

Thus,

A1: e, g

A2: d

A3: a, b, f

A4: c, d, e, f, g

A5: a, b, f

4. Show what is printed by the following algorithm when called with **MAXIMUM**$(A, 1, 5)$ where $A = [7, 3, 8, 5, 9]$? Where the function **PRINT** simple prints its arguments in some appropriate manner.

MAXIMUM(A, l, r)
if $(r - l == 0)$ **then**
  return $A[r]$
end if

$lmax =$ **MAXIMUM**$(A, l, \lfloor (l + r)/2 \rfloor)$
$rmax =$ **MAXIMUM**$(A, \lfloor (l + r)/2 \rfloor + 1, r)$
PRINT$(rmax, lmax)$
if $rmax < lmax$ **then**
  return $lmax$
else
  return $rmax$
end if

The printed result is,

$3, 7$

$8, 7$

$9, 5$

$9, 8$