Spring 2022 Lecture 2

$$\log_b\left(a^{\log_b x}\right) \log_b\left(\left(x^{\log_b a}\right)\right) = \log_b a \log_b x$$

$$\log_b x \log_b a$$

$$\log_b x = \frac{\log_a x}{\log_a b}$$

$$\Leftrightarrow \quad a^{\log_a \cancel{b} \log_{\cancel{b}} x} = a^{\log_a x} = x^{\log_a a} = x$$

$$\left(a^{\log_a b}\right)^{\log_b x} = b^{\log_b x} = x$$

$$x(1 + x + x^2 + \cdots x^n) = Sx$$

$$\underline{x + x^2 + x^3 + \cdots x^n} + x^{n+1} = Sx$$

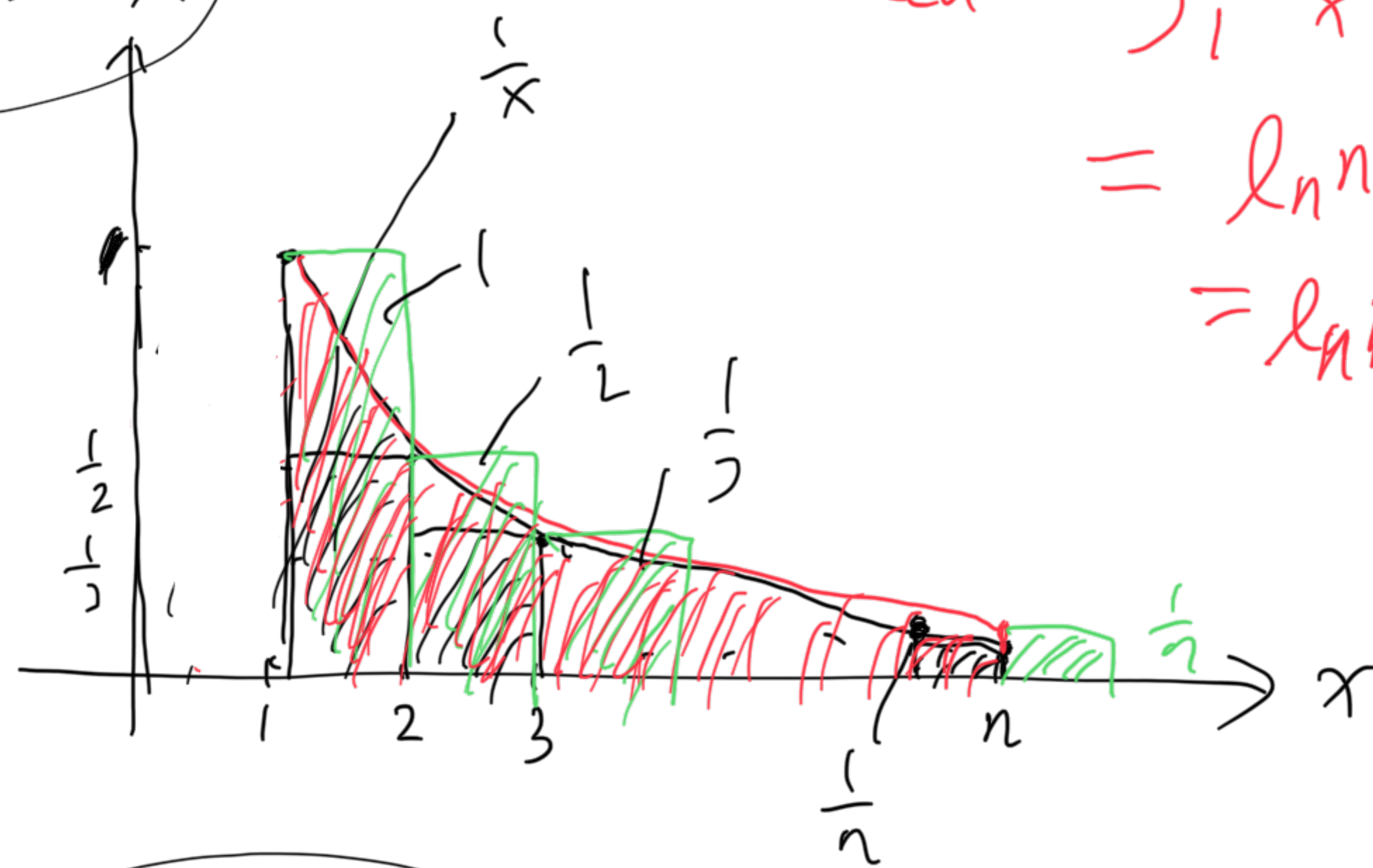
$$S - 1 + x^{n+1} = Sx$$

$$\Longrightarrow \quad S = \frac{x^{n+1} - 1}{x - 1}$$

$$1 + \frac{1}{2} + \cdots + \frac{1}{n} \approx \ln n$$

$f(x) = \frac{1}{x}$

$$\text{Red} = \int_1^n \frac{1}{x}\,dx = \ln n - \ln 1 = \ln n$$



$$1 + \frac{1}{2} + \cdots + \frac{1}{n} \le 1 + \int_1^n \frac{1}{x}\,dx = 1 + \ln n$$

$$\ge \int_1^n \frac{1}{x}\,dx = \ln n$$

$$f(1) > \int_1^2 f(x)\,dx > f(2)$$

$\sum_{k=1}^{n} f(k)$ $\qquad \int_1^n f(x)\,dx$

$$f(1) < \int_1^2 f(x)\,dx < f(2)$$

$$T(n) = 2T\left(\frac{n}{2}\right) + \Theta(n)$$

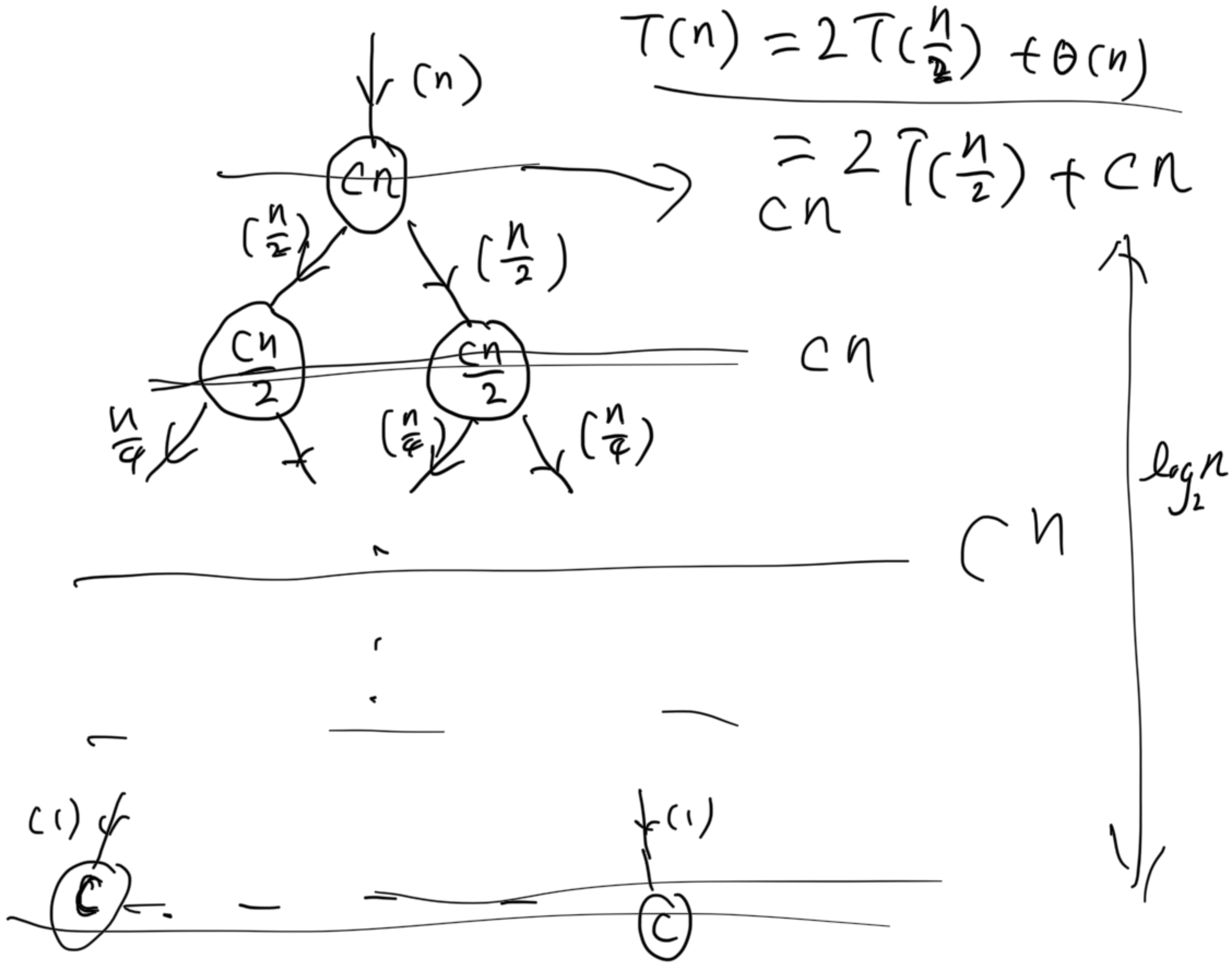
$$2\left(T\left(\frac{n}{2}\right) = 2T\left(\frac{n}{4}\right) + \Theta\left(\frac{n}{2}\right)\right)$$

$$4\left(T\left(\frac{n}{4}\right) = 2T\left(\frac{n}{8}\right) + \Theta\left(\frac{n}{4}\right)\right)$$

$$8\left(T\left(\frac{n}{8}\right) = 2T\left(\frac{n}{16}\right) + \Theta\left(\frac{n}{8}\right)\right)$$

$$T(n) = 2T\left(\frac{n}{2}\right) + \Theta(n)$$

$$= 2T\left(\frac{n}{2}\right) + cn$$



$$cn\log_2 n = \Theta(n\lg n)$$

$$T(n) = 2T\left(\frac{n}{2}\right) + \Theta(n)$$

$k = \frac{n}{2} < n$

Guess is $T(n) = \Theta(n\lg n)$

$$\Rightarrow c_2 n\lg n \le T(n) \le c_1 n\lg n$$

$S(n)$

$\forall n \geq n_0$

if $k < n$ $S(k)$ is true

$$T(\tfrac{n}{2})$$

$$T(n) \leq 2\left(c_1 \frac{n}{2} \log \frac{n}{2}\right) + cn \qquad \forall n$$

$$= c_1 n(\log n - \log 2) + cn$$

$$= c_1 n \log n + (c - c_1 \log 2)n$$

$$\leq c_1 n \log n$$

$$c - c_1 \log 2 \leq 0$$

$$\Rightarrow c_1 \geq \frac{c}{\log 2}$$

$$T(\tfrac{n}{2})$$

$$T(n) \geq 2\left(c_2 \frac{n}{2} \log\left(\frac{n}{2}\right)\right) + cn$$

$$= c_2 n \log n - c_2 n \log 2 + cn$$

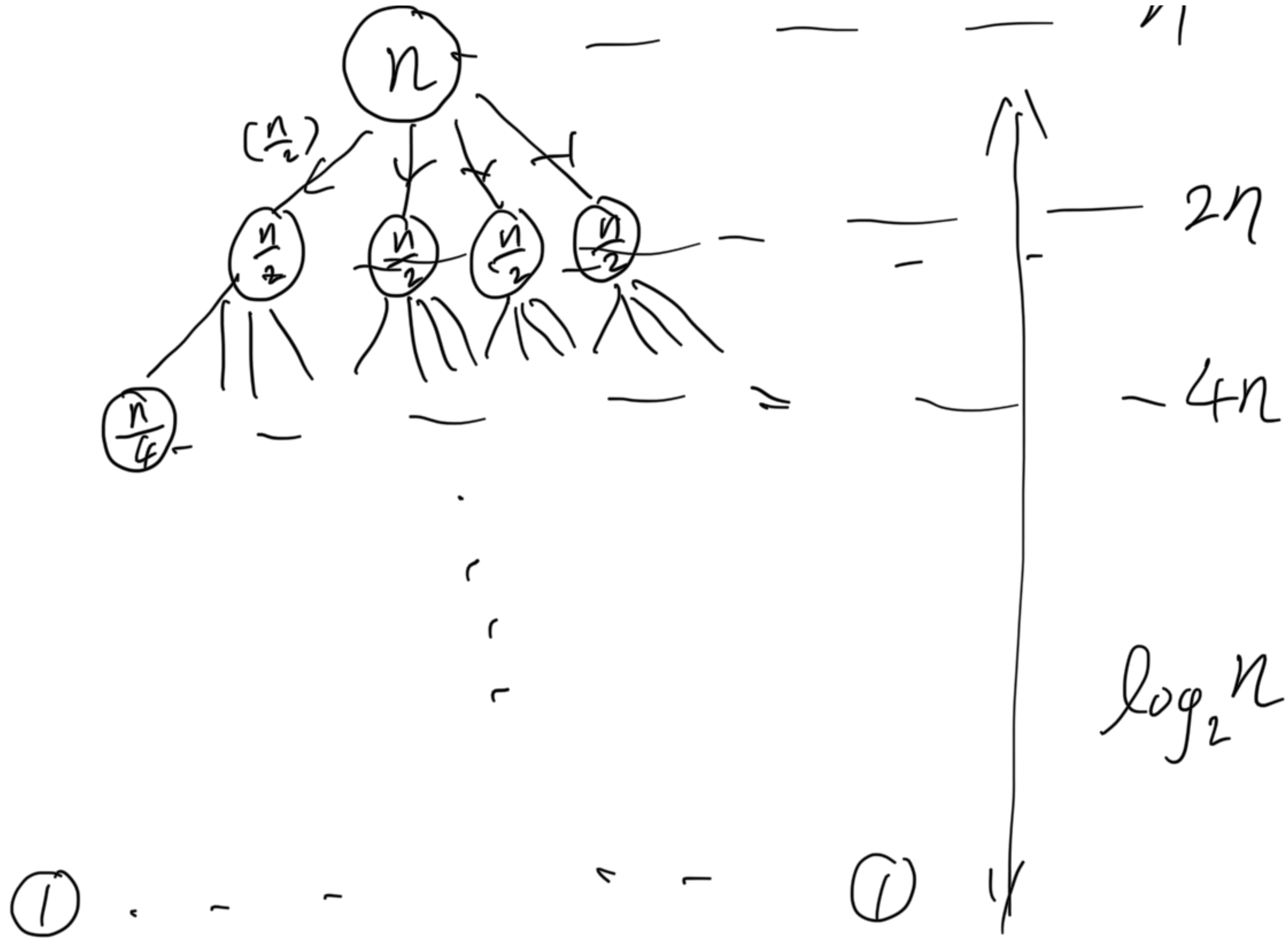
$$\geq c_2 n \log n$$

$$-c_2 \log 2 + c \geq 0$$

$$c_2 \leq \frac{c}{\log 2}$$

$$T(n) = 4T\left(\frac{n}{2}\right) + n \qquad T\left(\frac{n}{2}\right) = 4T\left(\frac{n}{4}\right) + \frac{n}{2}$$

$$f(n)$$



$$n + 2n + 4n + \cdots \quad 2^{\log_2 n} \times n \qquad 2^{\log_2 n + 1}$$

$$n\,(1+2+4 \cdots + 2^{\log_2 n}) = n \times \frac{2n-1}{2-1}$$

$$2^{\log_2 n} \to n$$

$$= n\,(2n-1)$$

$$= \Theta(n^2)$$

$$c_2 n^2 \le \boxed{T(n) \le c_1 n^2}, \quad \forall n \ge n_0$$

$$T(n) = 4T\left(\frac{n}{2}\right) + n \qquad \frac{n}{2} < n$$

$$T\left(\frac{n}{2}\right) \le c_1 \left(\frac{n}{2}\right)^2$$

$$\le 4 c_1 \left(\frac{n}{2}\right)^2 + n$$

$$= c_1 n^2 + n$$

$$\le c_1 n^2 \ ?$$

$$T(n) = \Theta(n^2) \Longleftrightarrow T(n) = \Theta(n^2 - n)$$

$$\downarrow$$

$$T(n) \le c_1 (n^2 - n)$$

$k < n$

$$T(n) = 4T\left(\frac{n}{2}\right) + n$$

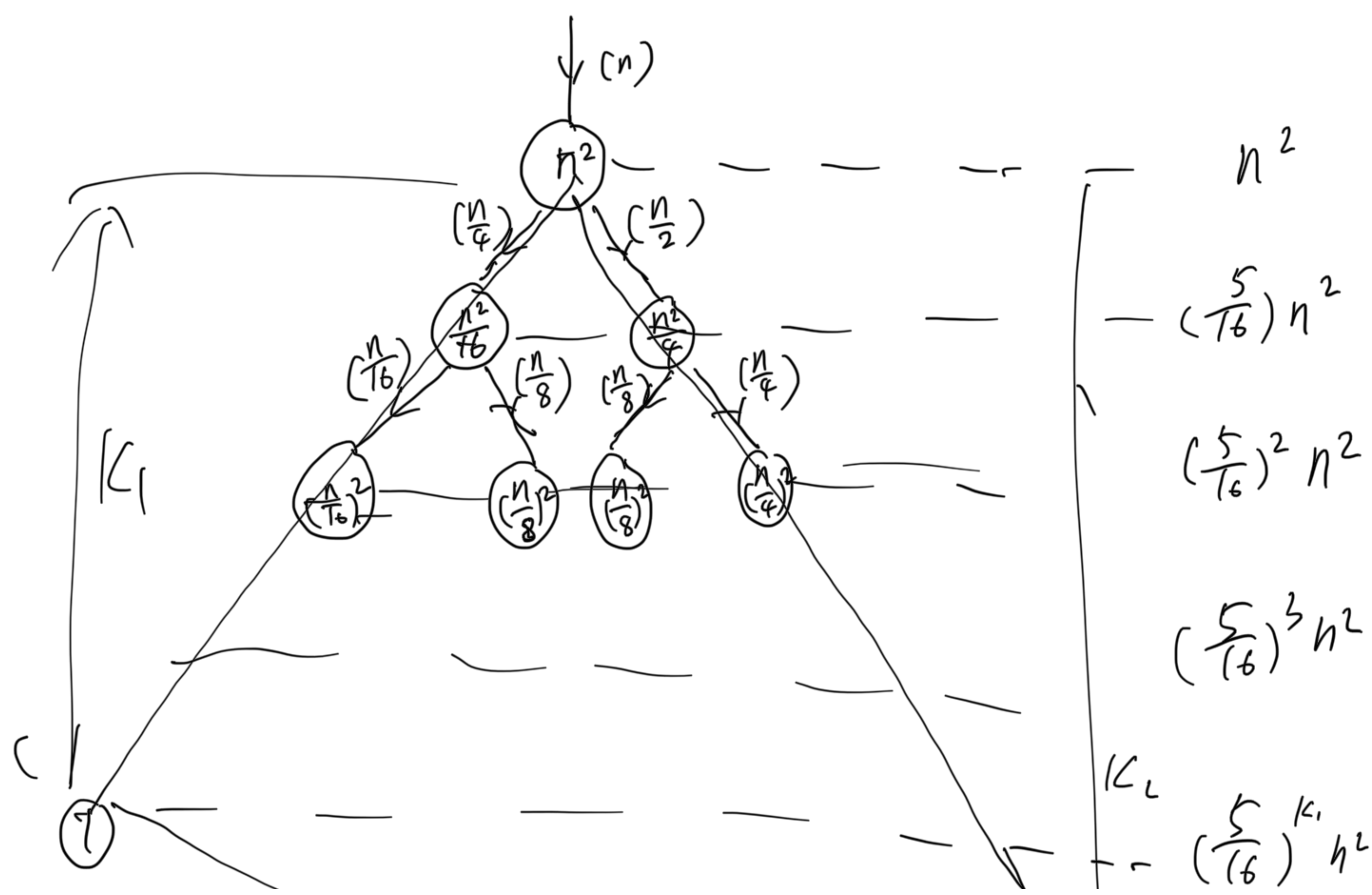
$$T\left(\frac{n}{2}\right) \le c_1 \left(\frac{n^2}{4} - \frac{n}{2}\right)$$

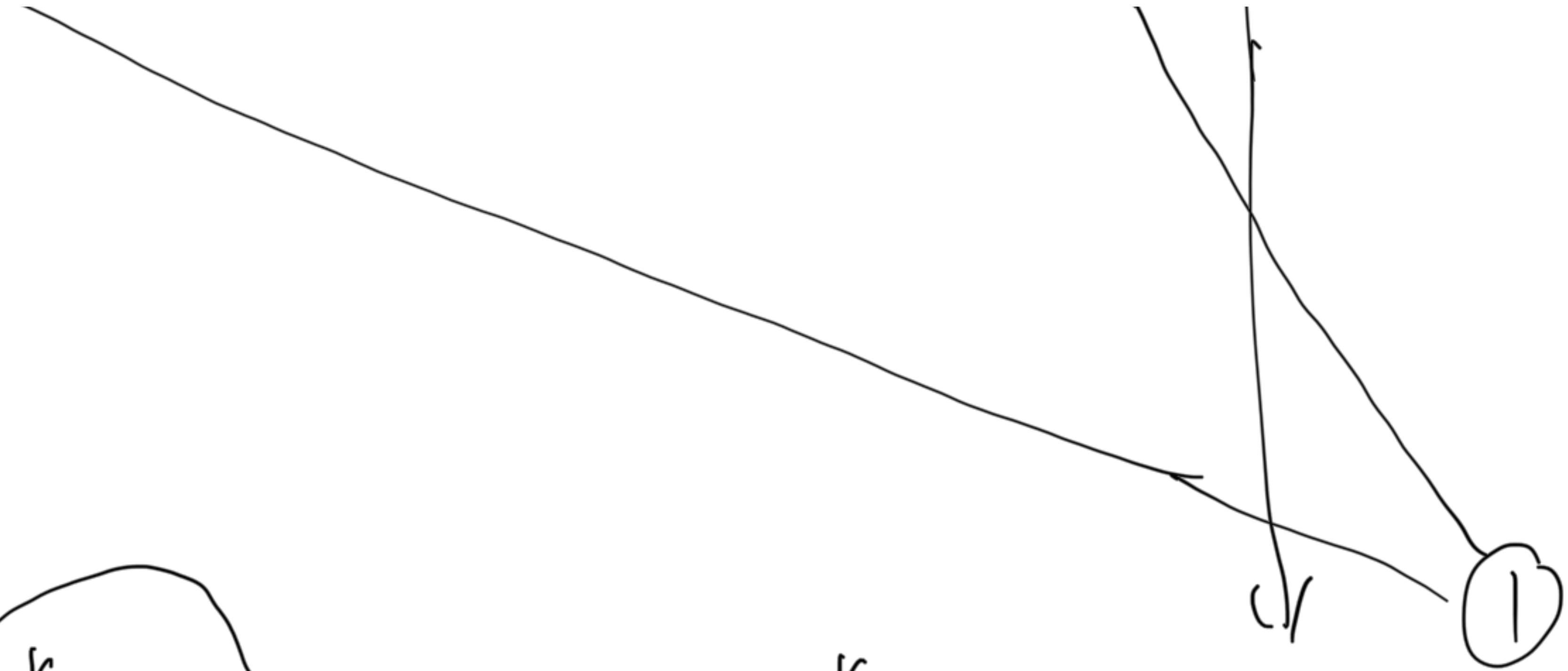
$$\le c_1 n^2 - 2c_1 n + n \le c_1 n^2$$

$$-2c_1 n + n \le 0$$

$$c_1 \ge \frac{1}{2} \quad \checkmark$$

$$T(n) = T\left(\frac{n}{4}\right) + T\left(\frac{n}{2}\right) + n^2$$





$$\sum_{k=1}^{k_1} \left(\frac{5}{16}\right)^k n^2 \le T(n) \le \sum_{k=1}^{k_2} \left(\frac{5}{16}\right)^k n^2$$

$k_1 = \log_4 n \qquad k_2 = \log_2 n$

$$1 \le \sum_{k=1}^{k_1} \left(\frac{5}{16}\right)^k \le \sum_{k=1}^{k_2} \left(\frac{5}{16}\right)^k \le \sum_{k=1}^{\infty} \left(\frac{5}{16}\right)^k$$

$$\frac{1}{1-\frac{5}{16}} = \frac{16}{11}$$

$$n^2 \le T(n) \le \frac{16}{11} n^2$$

$$T(n) = \Theta(n^2)$$

$$c_2 n^2 \le T(n) \le c_1 n^2$$

$$T(n) = T\left(\frac{n}{2}\right) + T\left(\frac{n}{4}\right) + n^2$$

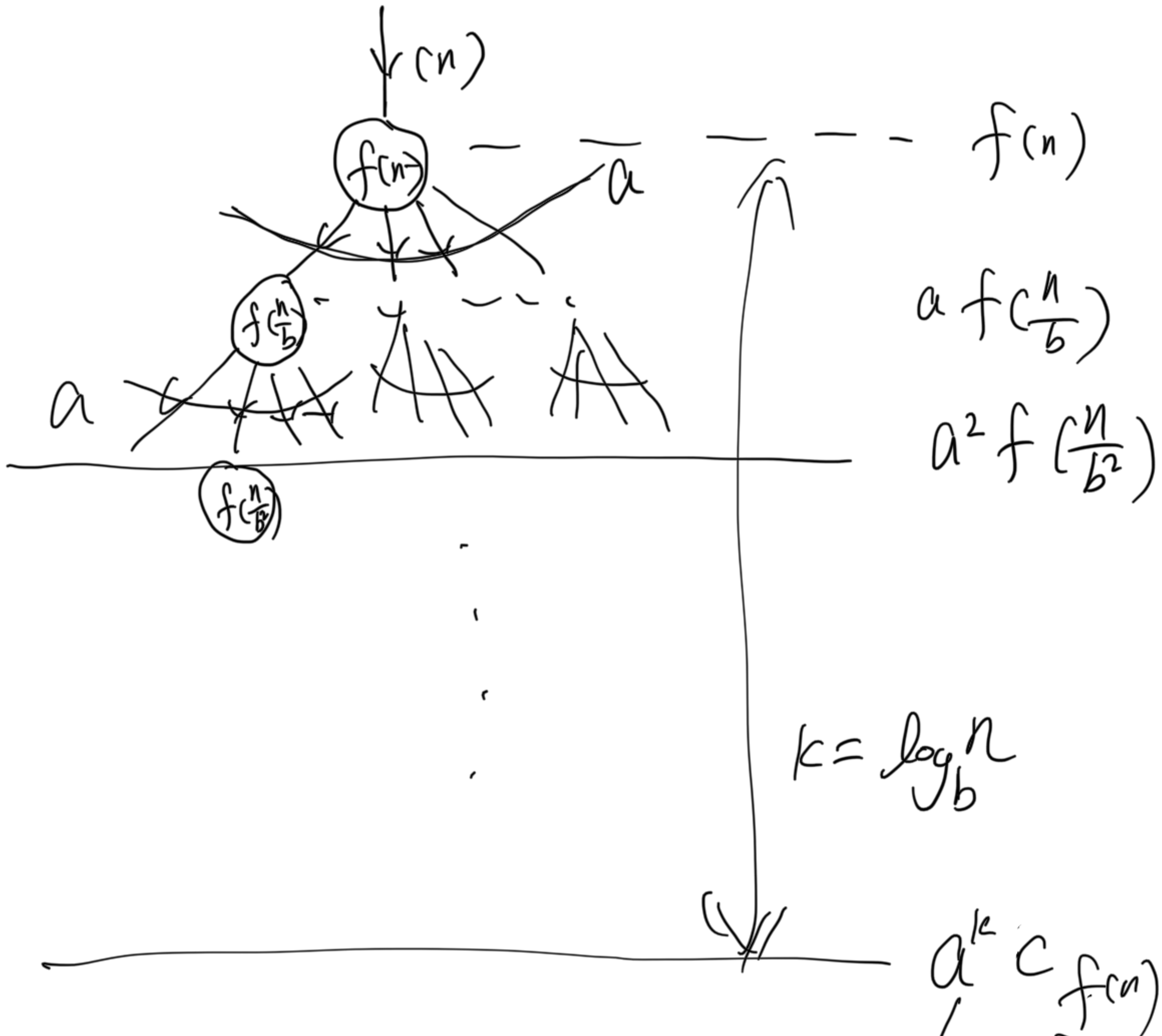
$$\le c_1 \left(\frac{n}{2}\right)^2 + c_1 \left(\frac{n}{4}\right)^2 + n^2$$

$$= \frac{c_1 5}{16} n^2 + n^2 \le c_1 n^2$$

$$1 \le c_1 \frac{11}{16}$$

$$\Rightarrow c_1 \ge \frac{16}{11}$$

$$T(n) = a\log\left(\frac{n}{b}\right) + f(n)$$



$$c\, a^{\log_b n} = c\, n^{\log_b a}$$