ECE-GY 6303, Probability & Stochastic Processes

Solution to Homework # 9

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Problem 1

Which among the following represent auto-correlation function of a stochastic process?

a.) $\max(t_i, t_j)$;

b.) $t_i^2\, t_j^2$;

c.) $t_i + t_j$;

d.) $1/(t_i + t_j)$.

Solution:

a.) It is not an auto-correlation function. To see this, consider the 2-by-2 matrix R. Let $t_1 \le t_2$.

$$R = \begin{bmatrix} t_1 & t_2 \\ t_2 & t_2 \end{bmatrix}.$$

Then, $|R| = t_1 t_2 - t_2^2 \le 0$.

b.) It is an auto-correlation function as

$$R = \begin{bmatrix} t_1^2 \\ t_2^2 \\ \vdots \\ t_n^2 \end{bmatrix} \begin{bmatrix} t_1^2 & t_2^2 & \cdots & t_n^2 \end{bmatrix}$$

is a non-negative matrix.

c.) It is not an auto-correlation function. For example, let

$$R = \begin{bmatrix} 2t_1 & t_1 + t_2 \\ t_2 + t_1 & 2t_2 \end{bmatrix} .$$

and $|R| = 4t_1 t_2 - (t_1 + t_2)^2 = -(t_1 - t_2)^2 \le 0$.

d.) See Pillai "Cauchy Matrix and One of its Applications": https://www.youtube.com/watch?v=RPqKoxjhG

Problem 2

Given

$$R_1 = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \quad \text{and} \quad R_2 = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix},$$

define

$$R = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix},$$

where $c_{ij} = a_{ij}b_{ij}$. Show that if R_1 and R_2 are positive definite, then R is also positive definite.

Solution:

By eigen-decomposition,

$$R_1 = \sum_{i=1}^{n} \lambda_i u_i u_i^* = \sum_{i=1}^{n} A_i > 0, \quad R_2 = \sum_{i=1}^{n} \mu_i v_i v_i^* = \sum_{i=1}^{n} B_i > 0.$$

Then,

$$R = R_1 \circ R_2 = \left(\sum_{i=1}^{n} A_i\right) \circ \left(\sum_{i=1}^{n} B_i\right) = \sum_{i,j=1}^{n} \lambda_i \mu_j (u_i u_i^*) \circ (v_j v_j^*).$$

Note that

$$(u_i u_i^*) \circ (v_j v_j^*) = (u_i v_j) \circ (u_i v_j)^* = z_{ij} z_{ij}^* \geq 0.$$

Thus, $R \geq 0$.

Problem 3

a.) $X(t)$ is a W.S.S. process. Define

$$Y(t) = X(t) + aX(t-T) + bX(t+T).$$

Is $Y(t)$ W.S.S.?

b.) $X(t)$ is a zero mean Gaussian process with auto-correlation function $R_{XX}(t_1 - t_2)$. Let

$$Y(t) \ = \ X^2(t) + X(t-T).$$

Find $R_{YY}(t_1, t_2)$. Is $Y(t)$ stationary in any sense? Is $Y(t)$ Gaussian?

Solution:

a.)

$$Y(t) = X(t) + aX(t-T) + bX(t+T) := h(t) \star X(t),$$

where

$$h(t) = \delta(t) + a\delta(t-T) + b\delta(t+T).$$

Hence, the system is LTI and $Y(t)$ is W.S.S.

b.) As $X(t)$ is zero-mean Gaussian process, we have that

$$E[X_1X_2X_3] = 0,$$

and

$$E[X_1X_2X_3X_4] = E[X_1X_2]E[X_3X_4] + E[X_1X_3]E[X_2X_4] + E[X_2X_3]E[X_1X_4].$$

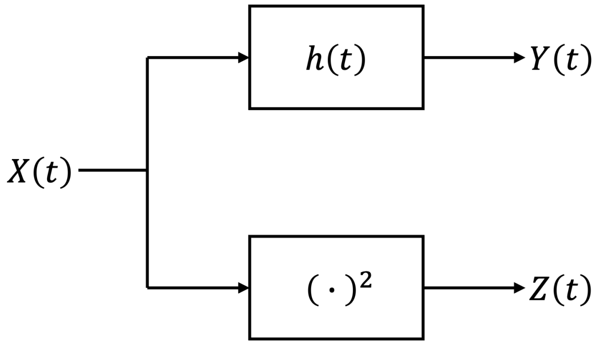
Using above two formulas to $R_{YY}(t_1, t_2)$ in $R_{YY}(t_1, t_2)$, we get

$$\begin{aligned}
R_{YY}(t_1, t_2) &= E\left[\left(X^2(t_1) + X(t_1 - T)\right)\left(X^2(t_2) + X(t_2 - T)\right)\right] \\
&= E[X^2(t_1)X^2(t_2)] + E[X(t_1 - T)X^2(t_2)] + E[X^2(t_1)X(t_2 - T)] + E[X(t_1 - T)X(t_2 - T)] \\
&= E[X^2(t_1)]E[X^2(t_2)] + 2E[X(t_1)X(t_2)]E[X(t_1)X(t_2)] + E[X(t_1 - T)X^2(t_2)] \\
&\qquad + E[X^2(t_1)X(t_2 - T)] + E[X^2(t_1)X(t_2 - T)] + E[X(t_1 - T)X(t_2 - T)] \\
&= (R_{XX}(0))^2 + 2(R(\tau))^2 + R(\tau).
\end{aligned}$$

with $\tau = t_2 - t_1$. Since $X(t)$ is a zero-mean Gaussian process and $X^2(t)$ is the output of a memoryless system, $X^2(t)$ also is S.S.S. Since $X(t-T)$ is a shift in time, it is also S.S.S. Since $Y(t) = X^2(t) + X(t-T)$ is generated from the same stationary Gaussian process $X(t)$, the sum is also a S.S.S. process. However $Y(t)$ is not Gaussian since $X^2(t)$ is not Gaussian.

Problem 4

$X(t)$ is a zero mean stationary Gaussian process with auto-correlation function $R_{XX}(\tau)$.



a.) Find $R_{YY}(t_1, t_2)$ and $R_{ZZ}(t_1, t_2)$.

b.) Is $Y(t)$ or $Z(t)$ stationary in any sense?

Solution:

a.)

$$Y(t) = h(t) \star X(t) = \int X(t-\alpha)h(\alpha)\,d\alpha.$$

Hence,

$$\begin{aligned}
R_{YY}(t_1, t_2) = E[Y(t_1)Y^*(t_2)] &= E\left[\int X(t_1-\alpha)h(\alpha)\,d\alpha \cdot \int X^*(t_2-\beta)h^*(\beta)\,d\beta\right] \\
&= \int\int E\left[X(t_1-\alpha)X^*(t_2-\beta)\right] h(\alpha)\,d\alpha\, h^*(\beta)\,d\beta \\
&= \int\int R_{XX}(t_1-t_2-\alpha+\beta)h(\alpha)\,d\alpha\, h^*(\beta)\,d\beta \\
&= \int R_{XX}(\tau+\beta) \star h(\tau+\beta) \cdot h^*(\beta)\,d\beta \\
&= R_{XX}(\tau) \star h(\tau) \star h^*(-\tau).
\end{aligned}$$

Here, $\tau = t_1 - t_2$.

$$\begin{aligned}
R_{ZZ}(t_1, t_2) = E[Z(t_1)Z^*(t_2)] &= E[X^2(t_1)(X(t_2))^2] \\
&= 2\left(E[X(t_1)X(t_2)]\right)^2 + E[X^2(t_1)]E[(X(t_2))^2] \\
&= 2R_{XX}^2(\tau) + R_{XX}^2(0).
\end{aligned}$$

b.) $Y(t)$ and $Z(t)$ are both W.S.S.

Problem 5

Suppose X_n conditional on X_{n-1} is Poisson distributed with parameter λX_{n-1}. Let

$$\mu_n = E[X_n], \quad \sigma_n^2 = \mathrm{Var}(X_n).$$

Find μ_n and σ_n^2 in terms of λ, given $\mu_1 = 1$ and $\sigma_1^2 = 1$.

Solution:

$$\mu_n = E[X_n] = E[E[X_n|X_{n-1}]] = E[\lambda X_{n-1}] = \lambda \mu_{n-1} = \lambda^2 \mu_{n-2} = \cdots = \lambda^{n-1}, \text{ since} \mu_1 = 1.$$

Also, from previous homework, we know that

$$\mathrm{Var}(X) = E[\mathrm{Var}(X|Y)] + \mathrm{Var}(E[X|Y]).$$

$$\begin{aligned}
\sigma_n^2 &= \mathrm{Var}(X_n) = E[\mathrm{Var}(X_n|X_{n-1})] + \mathrm{Var}(E[X_n|X_{n-1}]) \\
&= E[\lambda X_{n-1}] + \mathrm{Var}(\lambda X_{n-1}) \\
&= \lambda \mu_{n-1} + \lambda^2 \sigma_{n-1}^2 \\
&= \lambda \cdot \lambda^{n-2} + \lambda^2(\lambda^{n-2} + \lambda^2 \sigma_{n-2}^2) \\
&= \lambda^{n-1} + \lambda^n + \lambda^4 \sigma_{n-2}^2 \\
&= \lambda^{n-1} + \lambda^n + \lambda^4(\lambda^{n-3} + \lambda^2 \sigma_{n-3}^2) \\
&= \lambda^{n-1} + \lambda^n + \lambda^{n+1} + \lambda^6 \sigma_{n-3}^2 \\
&= \cdots = \lambda^{n-1} + \lambda^n + \lambda^{n+1} + \cdots + \lambda^{2(n-1)} \sigma_1^2 \\
&= \lambda^{n-1} + \lambda^n + \lambda^{n+1} + \cdots + \lambda^{2(n-1)} \\
&= \lambda^{n-1}(1 + \lambda + \lambda^2 + \cdots + \lambda^{n-1}) \\
&= \lambda^{n-1} \frac{\lambda^n - 1}{\lambda - 1}.
\end{aligned}$$

We have made use the fact that $\mu_1 = \sigma_1^2 = 1$.

Problem 6

X(t) is a zero mean Gaussian process with auto-correlation function $R_{XX}(t_i, t_j) = \min(t_i, t_j)$. Define

$$Y(t) = e^{j(X(t)+\theta)},$$

where $\theta \sim U(-\pi, \pi)$ and independent of $X(t)$.

a.) Find the mean and auto-correlation function of $Y(t)$.

b.) Is $Y(t)$ W.S.S.?

Solution:

a.)

$$E[Y(t)] = E[e^{jX(t)}]E[e^{j\theta}] = E[e^{jX(t)}]E[\cos\theta + j\sin\theta] = 0,$$
$$R_{YY}(t_1, t_2) = E[Y(t_1)Y^*(t_2)] = E\left[e^{j(X(t_1)+\theta)}e^{-j(X(t_2)+\theta)}\right] = E\left[e^{j(X(t_1)-X(t_2))}\right].$$

Since $X(t)$ is a Gaussian process, $Z = X(t_1) - X(t_2)$ is a Gaussian random variable with zero mean and variance

$$\sigma_Z^2 = E[Z^2] = E[X^2(t_1)] + E[X^2(t_2)] - 2E[X(t_1)X(t_2)] = t_1 + t_2 - 2\min(t_1, t_2).$$

Therefore,

$$R_{YY}(t_1, t_2) = e^{-\frac{1}{2}(t_1+t_2-2\min(t_1,t_2))}.$$

b.) When $t_1 \le t_2$,

$$R_{YY}(t_1, t_2) = e^{-\frac{1}{2}(t_1+t_2-2t_1)} = e^{-\frac{1}{2}(t_2-t_1)}.$$

Similarly, when $t_1 > t_2$,

$$R_{YY}(t_1, t_2) = e^{-\frac{1}{2}(t_1-t_2)}.$$

So $Y(t)$ is W.S.S.