ECE-GY 6303, Probability & Stochastic Processes

Solution to Homework # 7

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Problem 1

The random variables X and Y are jointly distributed over the region $0 < x < y < 1$ as

$$f_{XY}(x,y) = \begin{cases} kx & 0 < x < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

for some k.

a.) Determine k.

b.) Find the variances of X and Y.

c.) What is the covariance between X and Y?

Solution:

a.)

$$1 = \int_0^1 \int_x^1 kxdydx = \frac{k}{6} \quad \Rightarrow \quad k = 6.$$

b.)

$$f_X(x) = \int_x^1 6xdy = 6(x - x^2), \quad f_Y(y) = \int_0^y 6xdx = 3y^2.$$

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2 = 6\int_0^1 x^3 - x^4 dx - \left(6\int_0^1 x^2 - x^3 dx\right)^2 = \frac{1}{20}.$$

$$\mathrm{Var}(Y) = E[Y^2] - (E[Y])^2 = 3\int_0^1 y^4 dy - \left(3\int_0^1 y^3 dy\right)^2 = \frac{3}{80}.$$

c.)

$$\mathrm{Cov}(X,Y) = E[XY] - E[X]E[Y] = 6\int_0^1 \int_x^1 x^2 ydydx - \int_0^1 6(x^2 - x^3)dx \cdot \int_0^1 3y^3 dy = \frac{1}{40}.$$

Problem 2

The random variables X and Y are jointly distributed over the region $0 < x < y < \infty$ as

$$f_{XY}(x,y) = \begin{cases} 2xye^{-(x+y)} & 0 < x < y < \infty \\ 0 & \text{otherwise.} \end{cases}$$

a.) Determine $E[X|Y]$ and $E[Y|X]$.

b.) Determine the correlation coefficient ρ bwteen X and Y.

Solution:

a.)

$$\begin{aligned} f_X(x) &= \int_x^\infty f_{XY}(x,y)dy = 2xe^{-x}\int_x^\infty ye^{-y}dy = 2xe^{-x}\left(-ye^{-y} - e^{-y}\right)\big|_x^\infty = 2x(1+x)e^{-2x},\ x \geq 0, \\ f_Y(y) &= \int_0^y f_{XY}(x,y)dx = 2ye^{-y}\int_0^y xe^{-x}dy = 2ye^{-y}(1 - e^{-y} - ye^{-y}),\ y \geq 0. \end{aligned}$$

Hence,

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{xe^{-x}}{1 - e^{-y} - ye^{-y}},\ y > x > 0, \\ f_{Y|X}(y|x) &= \frac{f_{XY}(x,y)}{f_X(x)} = \frac{ye^{x-y}}{x+1},\ y > x > 0, \\ E[X|Y=y] &= \int_0^y xf_{X|Y}(x|y)dx = \frac{1 - 2e^{-y} - 2ye^{-y} - y^2e^{-y}}{1 - e^{-y} - ye^{-y}}, \qquad q \\ E[Y|X=x] &= \int_x^\infty yf_{Y|X}(y|x)dy = \frac{2 + 2x + x^2}{1+x}. \end{aligned}$$

b.)

$$\begin{aligned} E[X] &= \int_0^\infty xf_X(x)dx = \int_0^\infty 2x^2(1+x)e^{-2x}dx = \frac{5}{4}, \\ E[X^2] &= \int_0^\infty x^2 f_X(x)dx = \int_0^\infty 2x^3(1+x)e^{-2x}dx = \frac{9}{4}, \\ \mathrm{Var}(X) &= E[X^2] - (E[X])^2 = \frac{11}{16}, \\ E[Y] &= \int_0^\infty yf_Y(y)dy = \int_0^\infty 2y^2e^{-y}(1 - e^{-y} - ye^{-y})dy = \frac{11}{4}, \\ E[Y^2] &= \frac{39}{4}, \\ \mathrm{Var}(Y) &= \frac{35}{16}, \\ E[XY] &= \int_0^\infty \int_y^\infty xyf_{XY}(x,y)dxdy = 4 \end{aligned}$$

Thus, $\mathrm{Cov}(X,Y) = 9/16$. Hence,

$$\rho_{XY} = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\sigma_X^2\sigma_Y^2}} = \frac{9}{\sqrt{385}}.$$

Problem 3

For any two random variables X and Y with $E[X^2], E[Y^2] < \infty$, show that

$$\mathrm{Var}(X) = \mathrm{Var}(E[X|Y]) + E[\mathrm{Var}(X|Y)].$$

Solution: Define

$$g(y) = E[X|y] = \int x f_{X|Y}(x|y)dx.$$

Then,

$$\mathrm{Var}(E[X|Y]) = E[(g(Y))^2] - (E[g(Y)])^2 = \int (g(y))^2 f_Y(y)dy - \left(\int g(y)f_Y(y)dy\right)^2,$$

and

$$\begin{aligned} E[\mathrm{Var}(X|Y)] &= E\left[E[X^2|Y] - (E[X|Y])^2\right] = E\left[E[X^2|Y] - (g(Y))^2\right] \\ &= E[E[X^2|Y]] - E[(g(Y))^2]. \end{aligned}$$

Note that by tower property,

$$E[g(Y)] = E[E[X|Y]] = E[X], \quad \text{and} \quad E[E[X^2|Y]] = E[X^2].$$

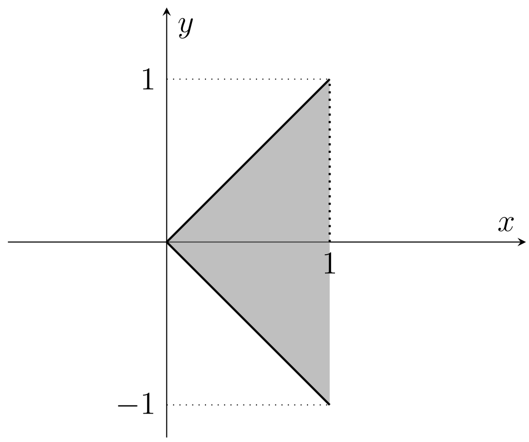
Thus,

$$\mathrm{Var}(E[X|Y]) + E[\mathrm{Var}(X|Y)] = E[X^2] - (E[X])^2 = \mathrm{Var}(X).$$

Problem 4

The random variables X and Y are jointly distribute

$$f_{XY}(x,y) = \begin{cases} \frac{3}{2}x & (x,y) \in \text{ shaded area}, \\ 0 & \text{otherwise}. \end{cases}$$



a.) Find $E[X|Y=y]$.

b.) Find the correlation coefficient ρ_{XY} between X and Y.

c.) Write MATLAB code to generate n-dimensional vectors i, $[x(i), y(i)]$, are distributed with the above distribution. (*Hint: Generate Y from $f_y(y)$, then generate X from $f_{X|Y}(x|y)$.*)

Solution:

a.)

$$\text{When } 0 \le y \le 1, \quad f_Y(y) = \int_y^1 \frac{3}{2}xdy = \frac{3}{4}(1-y^2),$$
$$\text{When } -1 \le y \le 0, \; f_Y(y) = \int_{-y}^1 \frac{3}{2}xdy = \frac{3}{4}(1-y^2).$$

Hence,

$$f_Y(y) = \frac{3}{4}(1-y^2), \quad -1 \le y \le 1, \quad f_{X|Y}(x|Y=y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{2x}{1-y^2}.$$

and

$$\text{When } 0 \le y \le 1, \quad E[X|Y=y] = \int_y^1 \frac{2x^2}{1-y^2}dx = \frac{2}{3} \cdot \frac{1+y+y^2}{1+y}.$$
$$\text{When } -1 \le y \le 0, \quad E[X|Y=y] = \int_{-y}^1 \frac{2x^2}{1-y^2}dx = \frac{2}{3} \cdot \frac{1-y+y^2}{1-y}.$$

b.)

$$f_X(x) = \int_{-x}^x \frac{3}{2}xdy = 3x^2, \quad E[X] = \int_0^1 3x^3dx = \frac{3}{4}, \quad E[Y] = \int_{-1}^1 \frac{3}{4}(y-y^3)dy = 0.$$
$$E[XY] = \int_0^1 \int_{-x}^x xy\frac{3}{2}xdydx = 0.$$

$$\mathrm{Cov}(X,Y) = E[XY] - E[X]E[Y] = 0 - \frac{3}{4} \cdot 0 = 0, \quad \rho_{XY} = \frac{\mathrm{Cov}(X,Y)}{\mathrm{Var}(X) \cdot \mathrm{Var}(Y)} = 0.$$

Problem 5

a.) Suppose X is a Geometric random variable with parameter p. Show that $P(X > m+n|X > m)$ is not a function of m.

b.) Suppose X and Y are zero mean jointly normal random variables with equal variances σ^2, and correlation coefficient $\rho \neq 0$.

i.) Is there a value for the coefficient a for which the random variables $aX + Y$ and $X - Y$ are independent?

ii.) Find the variance of $Z = \alpha X^2 + \beta Y^2$, where α and β are constants.

Solution:

a.)

$$P(X > m+n|X > m) = \frac{P(X > m+n, X > m)}{P(X > m)}$$

Since

$$(X > m+n) \bigcup (X > m) = (X > m+n),$$

$$P(X > m+n|X > m) = \frac{P(X > m+n)}{P(X > m)} = \frac{\sum_{k=m+n+1}^{\infty} pq^k}{\sum_{k=m+1}^{\infty} pq^k} = q^n.$$

b.) i.) Since $Z := aX + Y$ and $W = X - Y$ are linear combinations of X and Y, and X and Y are jointly Gaussian, Z and W are also jointly Gaussian.Hence, Z and W are only independent when $\rho_{ZW} = 0$. But,

$$\begin{aligned}
\rho_{ZW} &= \frac{E[ZW] - E[Z]E[W]}{\sigma_Z \sigma_W} = \frac{E[(aX+Y)(X-Y)]}{\sigma_Z \sigma_W} \\
&= \frac{a(E[X^2] - E[XY]) - (E[Y^2] - E[XY])}{\sigma_Z \sigma_W} \\
&= \frac{a(\sigma^2 - \rho\sigma^2) - (\sigma^2 - \rho\sigma^2)}{\sigma_Z \sigma_W}.
\end{aligned}$$

Hence, when $a = 1$, Z and W are independent.

ii.) Note that

$$E[Z] = (\alpha + \beta)\sigma^2.$$

$$\begin{aligned}
\text{Var}(Z) &= E[\alpha^2 X^4 + \beta^2 Y^4 + 2\alpha\beta X^2 Y^2] - (\alpha+\beta)^2\sigma^4 \\
&= 3(\alpha^2 + \beta^2)\sigma^4 + 2\alpha\beta E[X^2 Y^2] - (\alpha+\beta)^2\sigma^4 \\
&= (2\alpha^2 + 2\beta^2 - 2\alpha\beta)\sigma^4 + 2\alpha\beta(\sigma^4 + 2\rho^2\sigma^4) \\
&= 2(\alpha^2 + \beta^2 + 2\rho^2\alpha\beta)\sigma^4.
\end{aligned}$$