HW # 11

EL6303 Probability and Stochastic Processes

Final Exam, Spring 2018

1. Let $X(t)$ be a zero mean real stochastic process with $X(0) = 0$ and

$$E\Big[\,(X(t_1) - X(t_2))^2\,\Big] = |t_1 - t_2|$$

   a. Show that

$$R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)] = \frac{1}{2}\left(|t_1| + |t_2| - |t_1 - t_2|\right).$$

   b. Define $Z_n(t) = n\Big[X\left(t + \frac{1}{n}\right) - X(t)\Big], n = 1, 2, \ldots$. Keeping n fixed, find the autocorrelation function of $Z_n(t)$ and plot it. Is $Z_n(t)$ wide sense stationary (WSS)?

   c. What happens to $Z_n(t)$ as $n \to \infty$? (Does it appear to have the characteristics of white noise process? If so, why?) [25]

2. Consider two sets of measurements to determine the unknown X.

$$X_1 = s + n, \quad X_2 = as + n.$$

Here, s, n, and a are independent zero mean random variables with variances

$$\mathrm{Var}(s) = \sigma_s^2,\ \mathrm{Var}(n) = \sigma_n^2,\ \mathrm{Var}(a) = \sigma_a^2.$$

   a. Determine the best linear minimum mean square error (MMSE) estimator for s based on X_1. Compute the minimum mean square error σ_1^2.

   b. Determine the best linear minimum mean square error (MMSE) estimator for s based on both X_1 and X_2. Compute the minimum mean square error σ_2^2.

   c. Between (a) and (b) above, which estimator is superior? (Hint: Justify your answer by comparing σ_1^2 and σ_2^2.) [25]

3. $X(t)$ is a zero mean Gaussian process with autocorrelation function $R_{XX}(t_i, t_j) = \min(t_i, t_j)$. Define

$$Y(t) = e^{jX(t)}$$

   a. Find the mean and autocorrelation function of $Y(t)$.

   b. Is $Y(t)$ first or second order wide sense stationary (WSS)?

   c. If $Y(t)$ is second order stationary, find its power spectral density function $S_{YY}(\omega)$. [25]

4. $X(t)$ is a wide sense stationary (WSS) zero mean Gaussian process with autocorrelation function $R_{XX}(\tau)$ and power spectrum $S_{XX}(\omega)$. Consider the process

$$Z(t) = X(t)\cos(\omega_0 t + \theta) + X^2(t - T)\sin(\omega_0 t + \theta).$$

and $\theta \sim U(0, 2\pi)$ independent of $X(t)$.

   a. Find the power spectrum of $Z(t)$ and express it in terms of $S_{XX}(\omega)$ and ω_0.

   b. Find the power spectrum of the following autocorrelation functions

      (i) $e^{-\alpha|\tau|}\cos(\omega_0\tau)$,

      (ii) $e^{-\alpha\tau^2}\cos(\omega_0\tau)$.

[25]