ECE-GY 6303, Probability & Stochastic Processes

Solution to Homework # 10

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Problem 1

Given system $H(\omega)$ with input $X(t)$ and output $Y(t)$, show that

a.) if $X(t)$ is W.S.S. and $R_{XX}(\tau) = e^{j\alpha\tau}$, then

$$R_{YX}(\tau) = e^{j\alpha\tau} H(\alpha), \quad \text{and} \quad R_{YY}(\tau) = e^{j\alpha\tau} |H(\alpha)|^2.$$

b.) if $R_{XX}(t_1, t_2) = e^{j(\alpha t_1 - \beta t_2)}$, then

$$R_{YX}(t_1, t_2) = e^{j(\alpha t_1 - \beta t_2)} H(\alpha), \quad \text{and} \quad R_{YY}(t_1, t_2) = e^{j(\alpha t_1 - \beta t_2)} H(\alpha) H^*(\beta).$$

Solution:

a.) Note that

$$Y(t) = \int_{-\infty}^{\infty} X(t-\tau) h(\tau) d\tau.$$

Then,

$$\begin{aligned}
R_{YX}(\tau) &= E[Y(t)X^*(t+\tau)] = E\left[\int_{-\infty}^{\infty} X(t-\beta)X^*(t+\tau)h(\beta)d\beta\right] \\
&= \int_{-\infty}^{\infty} R_{XX}(\tau+\beta)h(\beta)d\beta \\
&= \int_{-\infty}^{\infty} e^{j\alpha(\tau+\beta)}h(\beta)d\beta = e^{j\alpha\tau}\int_{-\infty}^{\infty} e^{j\alpha\beta}h(\beta)d\beta \\
&= e^{j\alpha\tau}H(\alpha),
\end{aligned}$$

and

$$\begin{aligned}
R_{YY}(\tau) &= E[Y(t)Y^*(t+\tau)] \\
&= E\left[\int_{-\infty}^{\infty} X(t-\beta)h(\beta)d\beta \int_{-\infty}^{\infty} X^*(t+\tau-\gamma)h^*(\gamma)d\gamma\right] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(\tau-\gamma+\beta)h(\beta)h^*(\gamma)d\beta d\gamma \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{j\alpha(\tau-\gamma+\beta)}h(\beta)h^*(\gamma)d\beta d\gamma \\
&= e^{j\alpha\tau}\int_{-\infty}^{\infty} e^{-j\alpha\gamma}\int_{-\infty}^{\infty} e^{j\alpha\beta}h(\beta)d\beta h^*(\gamma)d\gamma \\
&= e^{j\alpha\tau}|H(\alpha)|^2.
\end{aligned}$$

b.)

$$\begin{aligned}
R_{YX}(t_1,t_2) &= E[Y(t_1)X^*(t_2)] = E\left[\int_{-\infty}^{\infty} X(t_1-\gamma)X^*(t_2)h(\gamma)d\gamma\right] \\
&= \int_{-\infty}^{\infty} R_{XX}(t_1-\gamma,t_2)h(\gamma)d\gamma \\
&= \int_{-\infty}^{\infty} e^{j(\alpha(t_1-\gamma)-\beta t_2)}h(\gamma)d\gamma \\
&= e^{j(\alpha t_1-\beta t_2)}H(\alpha).
\end{aligned}$$

and

$$\begin{aligned}
R_{YY}(t_1,t_2) &= E[Y(t_1)Y^*(t_2+\tau)] \\
&= E\left[\int_{-\infty}^{\infty} X(t_1-\eta)h(\eta)d\eta \int_{-\infty}^{\infty} X^*(t_2+\tau-\gamma)h^*(\gamma)d\gamma\right] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(t_1-\eta,t_2+\tau-\gamma)h(\eta)h^*(\gamma)d\eta d\gamma \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{j(\alpha(t_1-\eta)-\beta(t_2+\tau-\gamma))}h(\eta)h^*(\gamma)d\eta d\gamma \\
&= e^{j(\alpha t_1-\beta t_2)}H(\alpha)H^*(\beta).
\end{aligned}$$

Problem 2

Show that

a.) if $R_{XX}[m_1, m_2] = q[m_1]\delta[m_1 - m_2]$ and $S = \sum_{n=0}^{N} a_n X[n]$, then $E[S^2] = \sum_{n=0}^{N} a_n^2 q[n]$.

b.) if $R_{XX}(t_1, t_2) = q(t_1)\delta(t_1 - t_2)$ and $S = \int_0^T a(t)X(t)dt$, then $E[S^2] = \int_0^T a^2(t)q(t)dt$.

Solution:

a.)

$$
\begin{aligned}
E[S^2] = E\left[\left(\sum_{n=0}^{N} a_n X[n]\right)^2\right] &= E\left[\sum_{n=0}^{N} a_n^2 X^2[n] + 2\sum_{m<n} a_m X[m] a_n X[n]\right] \\
&= \sum_{n=0}^{N} a_n^2 R_{XX}[n,n] + 2\sum_{m<n} a_m a_n R[m,n] \\
&= \sum_{n=0}^{N} a_n^2 q[n].
\end{aligned}
$$

b.)

$$
\begin{aligned}
E[S^2] = E\left[\left(\int_0^T a(t)X(t)dt\right)^2\right] &= E\left[\left(\int_0^T a(t_1)X(t_1)dt_1\right)\left(\int_0^T a(t_2)X(t_2)dt_2\right)\right] \\
&= E\left[\int_0^T \int_0^T a(t_1)a(t_2)X(t_1)X(t_2)dt_1 dt_2\right] \\
&= \int_0^T \int_0^T a(t_1)a(t_2)R_{XX}(t_1,t_2)dt_1 dt_2 \\
&= \int_0^T \int_0^T a(t_1)a(t_2)q(t_1)\delta(t_1 - t_2)dt_1 dt_2 \\
&= \int_0^T a^2(t_1)q(t_1)dt_1.
\end{aligned}
$$

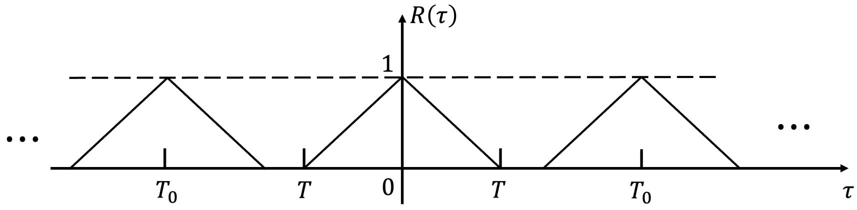
Problem 3

Find the power spectrum of the following autocorrelation functions

a.) $e^{-\alpha|\tau|}\cos(\omega_0\tau)$;

b.) $e^{-\alpha\tau^2}\cos(\omega_0\tau)$;

c.) $R(\tau)$ is a periodic function with period $T_0 > 2T$ as shown below



Solution: Denote Fourier transform by $\mathcal{F}[\,\cdot\,]$.

a.)

$$\begin{aligned} S(\omega) &= \mathcal{F}[e^{-\alpha|\tau|}\cos(\omega_0\tau)] = \frac{1}{2}\mathcal{F}[e^{-\alpha|\tau|}]_{\omega\to\omega-\omega_0} + \frac{1}{2}\mathcal{F}[e^{-\alpha|\tau|}]\Big|_{\omega\to\omega+\omega_0} \\ &= \frac{\alpha}{\alpha^2+\omega^2}\Big|_{\omega\to\omega-\omega_0} + \frac{\alpha}{\alpha^2+\omega^2}\Big|_{\omega\to\omega+\omega_0} \\ &= \frac{\alpha}{\alpha^2+(\omega-\omega_0)^2} + \frac{\alpha}{\alpha^2+(\omega+\omega_0)^2} \end{aligned}.$$

b.)

$$\begin{aligned} S(\omega) &= \mathcal{F}[e^{-\alpha\tau^2}\cos(\omega_0\tau)] = \frac{1}{2}\mathcal{F}[e^{-\alpha\tau^2}]_{\omega\to\omega-\omega_0} + \frac{1}{2}\mathcal{F}[e^{-\alpha\tau^2}]\Big|_{\omega\to\omega+\omega_0} \\ &= \sqrt{\frac{\pi}{\alpha}}e^{-\frac{\omega^2}{4\alpha}}\Big|_{\omega\to\omega-\omega_0} + \sqrt{\frac{\pi}{\alpha}}e^{-\frac{\omega^2}{4\alpha}}\Big|_{\omega\to\omega+\omega_0} \\ &= \sqrt{\frac{\pi}{\alpha}}e^{-\frac{(\omega-\omega_0)^2}{4\alpha}} + \sqrt{\frac{\pi}{\alpha}}e^{-\frac{(\omega+\omega_0)^2}{4\alpha}}. \end{aligned}$$

c.) As $R(\tau)$ is periodic with period T_0,

$$R(\tau) = \sum_{n=-\infty}^{\infty} a_n e^{-\gamma n\omega_0}, \quad \text{with} \quad \omega_0 = \frac{2\pi}{T_0}.$$

and hence,

$$S(\omega) = \sum_{n=-\infty}^{\infty} a_n \delta(\omega + n\omega_0),$$

where the n-th coefficient of Fourier series is

$$a_n = \frac{2}{T_0}\int_{-T}^{T} R(\tau)\cos(n\omega_0\tau)d\tau = \frac{4}{T_0}\int_0^T R(\tau)\cos(n\omega_0\tau)d\tau, \quad \omega_0 = \frac{2\pi}{T_0}.$$

When $n = 0$,

$$a_0 = \frac{4}{T_0}\cdot\frac{T}{2} = \frac{2T}{T_0};$$

When $n \geq 1$,

$$a_n = \frac{4}{T_0}\int_0^T \left(-\frac{\tau}{T}+1\right)\cos(n\omega_0\tau)d\tau = -\frac{4}{TT_0}\int_0^T \tau\cos(n\omega_0\tau)d\tau + \frac{4}{T_0}\sin\left(\frac{2n\pi T}{T_0}\right),$$

where

$$\begin{aligned}\int_0^T \tau\cos(n\omega_0\tau)d\tau &= \frac{1}{n\omega_0}\tau\sin(n\omega_0\tau)\Big|_0^T - \frac{1}{n\omega_0}\int_0^T \sin(n\omega_0\tau)d\tau \\ &= \frac{T\sin(n\omega_0 T)}{n\omega_0} - \frac{1}{n\omega_0}(\cos(n\omega_0 T)-1).\end{aligned}$$

Problem 4

Let $X(t)$ be W.S.S. with auto-correlation function $R_{XX}(\tau)$ and power spectrum $S_{XX}(\omega)$. Let $Y(t) = X(t+a) - X(t-a)$.

a.) Find the auto-correlation function of $Y(t)$.

b.) Find the power spectral density of $Y(t)$.

Solution:

a.)

$$\begin{aligned} R_{YY}(t_1, t_2) &= E\left[(X(t_1+a) - X(t_1-a))(X^*(t_2+a) - X^*(t_2-a))\right] \\ &= 2R_{XX}(\tau) - R_{XX}(\tau+2a) - R_{XX}(\tau-2a) \\ &= R_{YY}(\tau), \end{aligned}$$

where $\tau = t_1 - t_2$.

b.)

$$S_{YY}(\omega) = 2S_{XX}(\tau) - e^{j\omega 2a} S_{XX}(\tau) - e^{-j\omega 2a} S_{XX}(\tau) = 2S_{XX}(\tau)(1 - \cos 2a\omega).$$

Problem 5

Let

$$Y(t) = e^{j(\pi X(t)+\theta)},$$

where $X(t)$ is a Poisson process with parameter λt. Here, $\theta \sim U(0, 2\pi)$ and independent of $X(t)$.

a.) Is $Y(t)$ a W.S.S.?

b.) If so, find its power spectral density.

Solution:

a.)

$$E[Y(t)] = E\left[e^{j(\pi X(t)+\theta)}\right] = 0,$$
$$R_{YY}(t_1, t_2) = E\left[e^{j(\pi X(t_1)+\theta)}e^{-j(\pi X(t_2)+\theta)}\right] = E\left[e^{j\pi(X(t_1)-X(t_2))}\right].$$

Note that $X(t_2) - X(t_1)$ is a Poisson random variable as follows:

$$Z = X(t_2) - X(t_1) = n(t_1, t_2) \sim P(\lambda(t_2 - t_1)) = P(\lambda|t_2 - t_1|), \text{ with } \tau = t_2 - t_1 \text{ and } t_2 > t_1,$$

The characteristic function of $Z \sim P(\lambda|\tau|)$ is

$$\Phi_Z(\omega) = E[e^{j\omega Z}] = e^{\lambda|\tau|(e^{j\omega}-1)}.$$

Thus,

$$R_{YY}(t_1, t_2) = E\left[e^{-j\pi Z}\right] = \Phi_Z(-\pi) = e^{\lambda|\tau|(e^{j\omega}-1)}\Big|_{\omega=-\pi} = e^{-2\lambda|\tau|},$$

and $Y(t)$ is W.S.S.

b.)

$$S_{YY}(\omega) = \int_0^\infty e^{-2\lambda\tau}e^{-j\omega\tau}\,d\tau = \frac{4\lambda}{\omega + 4\lambda^2}.$$

Problem 6

$X(t)$ is a wide sense stationary zero mean Gaussian process with auto-correlation function $R_{XX}(\tau)$ and power spectrum $S_{XX}(\omega)$. Consider the process

$$Z(t) = X(t)\cos(\omega_0 t + \theta) + X^2(t-T)\sin(\omega_0 t + \theta)$$

and $\theta \sim U(0, 2\pi)$ independent of $X(t)$. Find the power spectrum of $Z(t)$ and express it in terms of $S_{XX}(\omega)$ and ω_0.

Solution:

The auto-correlation function of $Z(t)$ is given by

$$\begin{aligned}
R_{ZZ}(t_1, t_2) &= E[Z(t_1)Z^*(t_2)] \\
&= E\left[\left(X(t)\cos(\omega_0 t + \theta) + X^2(t-T)\sin(\omega_0 t + \theta)\right)\left(X(t)\cos(\omega_0 t + \theta) + X^2(t-T)\sin(\omega_0 t + \theta)\right)\right] \\
&= E[X(t_1)X(t_2)\cos(\omega_0 t_1 + \theta)\cos(\omega_0 t_2 + \theta) + X(t_1)X^2(t_2)\cos(\omega_0 t_1 + \theta)\sin(\omega_0 t_2 + \theta) \\
&\qquad + X^2(t_1)X(t_2)\sin(\omega_0 t_1 + \theta)\cos(\omega_0 t_2 + \theta) + X^2(t_1)X^2(t_2)\sin(\omega_0 t_1 + \theta)\sin(\omega_0 t_2 + \theta)] \\
&= R_{XX}(t_1, t_2)\cos(\omega_0 t_1 + \theta)\cos(\omega_0 t_2 + \theta) + E[X(t_1)X^2(t_2)]\cos(\omega_0 t_1 + \theta)\sin(\omega_0 t_2 + \theta) \\
&\quad + E[X^2(t_1)X(t_2)]\sin(\omega_0 t_1 + \theta)\cos(\omega_0 t_2 + \theta) + E[X^2(t_1)X^2(t_2)]\sin(\omega_0 t_1 + \theta)\sin(\omega_0 t_2 + \theta)].
\end{aligned}$$

Here, $E[X(t_1)X^2(t_2)] = E[X^2(t_1)X(t_2)] = 0$ as they are Gaussian odd moments. From joint moments of Gaussian random variables.

$$E[X^2(t_1)X^2(t_2)] = E[X^2(t_1)X^2(t_2)] + 2(E[X(t_1)X(t_2)])^2 = R_{XX}(t_1, t_1)R_{XX}(t_2, t_2) + 2R^2_{XX}(t_1, t_2)$$

Hence,

$$R_{ZZ}(\tau) = \frac{1}{2}\cos\omega_0\tau\,\left(R_{XX}(\tau) + (R_{XX}(0))^2 + 2(R_{XX}(\tau))^2\right).$$

Denote Fourier transform by $\mathcal{F}[\,\cdot\,]$.

$$\begin{aligned}
S_{ZZ}(\omega) &= \frac{1}{2}(R_{XX}(0))^2\mathcal{F}[\cos\omega_0\tau] + \frac{1}{2}\mathcal{F}[\cos\omega_0\tau R_{XX}(\tau)] + \mathcal{F}[\cos\omega_0\tau(R_{XX}(\tau))^2] \\
&= \frac{1}{2}(R_{XX}(0))^2\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] + \frac{1}{4}(S_{XX}(\omega - \omega_0) + S_{XX}(\omega + \omega_0)) \\
&\qquad + \frac{1}{2}\mathcal{F}[(R_{XX}(\tau))^2]\Big|_{\omega\to\omega+\omega_0} + \frac{1}{2}\mathcal{F}[(R_{XX}(\tau))^2]\Big|_{\omega\to\omega-\omega_0} \\
&= \frac{1}{2}(R_{XX}(0))^2\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] + \frac{1}{4}(S_{XX}(\omega - \omega_0) + S_{XX}(\omega + \omega_0)) \\
&\qquad + \frac{1}{4\pi}[S_{XX}(\omega) \star S_{XX}(\omega)]\Big|_{\omega\to\omega+\omega_0} + \frac{1}{4\pi}[S_{XX}(\omega) \star S_{XX}(\omega)]\Big|_{\omega\to\omega-\omega_0}.
\end{aligned}$$