EL6303 Probability and Stochastic Processes

Midterm Exam, Spring 2018

1. a. Suppose X is a Geometric random variable with parameter p. Show that $P(X > m + n \mid X > m)$ is *not* a function of m.

   b. Suppose X and Y are zero mean jointly normal random variables with equal variances and correlation coefficient $\rho \neq 0$. Is there a value for the coefficient a for which the random variables $aX + Y$ and $X - Y$ are independent? Express your answer in terms of ρ. [10]

2. a. Given $X \sim U[0,1]$ and

$$Y = \frac{X}{1-2X},$$

   Find the probability density function (pdf) of Y and plot it. [15]

   b. The probability density function of a random variable X is given by

$$f_X(x) = \begin{cases} \frac{3x^2}{2\pi^2}, & -\pi \leq x \leq \pi, \\ 0, & \text{otherwise.} \end{cases}$$

   Evaluate the conditional probability $P(X < \frac{\pi}{2} | X \geq 0)$. [15]

3. a. Given

$$f_{X,Y}(x,y) = \begin{cases} x+y, & 0 \leq x \leq 1, 0 \leq y \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

   Define $Z = X + Y$. Find the probability density function of Z. [15]

   b. Given the joint density function

$$f_{X,Y}(x,y) = \begin{cases} xye^{-(x+y)}, & x, y \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

   and

$$Z = \max(X, Y),$$

   Determine the probability density function of Z. [20]

4. X and Y are random variables with joint probability density function

$$f_{X,Y}(x,y) = \begin{cases} 2xye^{-(x+y)}, & 0 < y < x < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

   Define

$$Z = X + Y, \qquad W = \frac{X}{Y}.$$

   a. Find $f_{Z,W}(z,w)$, the joint probability density function of Z and W.

   b. Are Z and W independent random variables?

   c. Are Z and W uncorrelated random variables? [25]

Some useful probability density functions

1. Bernoulli

$$P(X = 1) = p; \quad P(X = 0) = q; \quad p + q = 1. \tag{1}$$

2. Binomial:

$$P(X = k) = \binom{n}{k} p^k q^{n-k}; \quad k = 0, 1, 2, \ldots n, \quad p + q = 1 \tag{2}$$

3. Geometric:

$$P(X = k) = p\ q^{k-1}; \quad k = 1, 2, \ldots \infty, \quad p + q = 1 \tag{3}$$

4. Negative Binomial:

$$P(X = n) = \binom{n-1}{k-1} p^k q^{n-k}; \quad n = k, k+1, \ldots, \infty \tag{4}$$

5. Poisson:

$$P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}; \quad k = 0, 1, 2, \ldots, \infty \tag{5}$$

6. Exponential:

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

7. Gaussian:

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\Big(- \frac{(x - \mu)^2}{2\sigma^2} \Big); \quad -\infty < x < \infty. \tag{7}$$