

# ECE-GY 6303, PROBABILITY & STOCHASTIC PROCESSES

## Solution to Homework # 6

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### Problem 1

Let

$$f_{XY}(x, y) = \begin{cases} 2e^{-(x+y)} & 0 < x < y < \infty \\ 0 & \text{otherwise} \end{cases}$$

Define

$$Z = X + Y \quad W = Y/X$$

a.) Find  $f_{ZW}(z, w)$ .

b.) Are  $Z$  and  $W$  independent random variables? Prove your answer.

**Solution:**

a.)

$$\begin{cases} z = x + y \\ w = y/x \end{cases} \Rightarrow \begin{cases} y_1 = wz/(w+1) \\ x_1 = z/(w+1) \end{cases}$$

Then,

$$\left| \begin{array}{cc} \frac{\partial x_1}{\partial z} & \frac{\partial x_1}{\partial w} \\ \frac{\partial y_1}{\partial z} & \frac{\partial y_1}{\partial w} \end{array} \right| = \left| \begin{array}{cc} \frac{1}{w+1} & -\frac{z}{(w+1)^2} \\ \frac{w}{w+1} & \frac{z}{(w+1)^2} \end{array} \right| = \frac{z}{(w+1)^2}$$

$$f_{ZW}(z, w) = \frac{z}{(w+1)^2} f_{XY}(x_1, y_1) = \frac{2e^{-z}z}{(w+1)^2}, \quad (z, w) \in \mathcal{A}.$$

The region  $\mathcal{A}$  is described as follows:

$$\begin{aligned} x < y < \infty &\Rightarrow \frac{z}{w+1} < \frac{wz}{w+1} < \infty \Rightarrow 1 < w < \infty, \\ 0 < x < y &\Rightarrow 0 < \frac{z}{w+1} < \frac{wz}{w+1} \Rightarrow 0 < z < wz \Rightarrow 0 < z < \infty. \end{aligned}$$

b.) Marginal distributions:

$$\begin{aligned} f_Z(z) &= \int_1^\infty f_{ZW}(z, w) dw = ze^{-z}, \quad z \geq 0, \\ f_W(w) &= \int_0^\infty f_{ZW}(z, w) dz = \frac{2}{(w+1)^2}, \quad w \geq 1. \end{aligned}$$

They are independent.

## Problem 2

Given the joint density function

$$f_{XY}(x, y) = \begin{cases} 2e^{-(2x-y)} & 0 < y < x < \infty, \\ 0 & \text{otherwise,} \end{cases}$$

and the two functions

$$Z = 2X - Y, \quad W = Y/X.$$

a.) Find  $f_{ZW}(z, w)$ .

b.) Are  $Z$  and  $W$  independent random variables? Prove your answer.

**Solution:**

a.)

$$\begin{cases} 2x - y = z \\ y/x = w \end{cases} \Rightarrow \begin{cases} y_1 = zw/(2-w) \\ x_1 = z/(2-w) \end{cases}$$

Then,

$$\left| \begin{array}{cc} \frac{\partial x_1}{\partial z} & \frac{\partial x_1}{\partial w} \\ \frac{\partial y_1}{\partial z} & \frac{\partial y_1}{\partial w} \end{array} \right| = \left| \begin{array}{cc} \frac{1}{2-w} & \frac{z}{(2-w)^2} \\ \frac{w}{2-w} & \frac{2z}{(2-w)^2} \end{array} \right| = \frac{z}{(2-w)^2},$$

and

$$f_{ZW}(z, w) = \frac{z}{(2-w)^2} f_{XY}(x_1, y_1) = \frac{2e^{-z}z}{(2-w)^2}, \quad (z, w) \in \mathcal{A}.$$

The region  $\mathcal{A}$  is described as follows:

$$\begin{aligned} x < y < \infty &\Rightarrow z > 0, \\ 0 < x < y &\Rightarrow 0 < \frac{wz}{2-w} < \frac{z}{2-w} \Rightarrow 0 < w < 1. \end{aligned}$$

b.) Marginal distributions:

$$\begin{aligned} f_Z(z) &= \int_0^1 \frac{2e^{-z}z}{(2-w)^2} dw = ze^{-z}, \quad z \geq 0, \\ f_W(w) &= \int_0^\infty \frac{2e^{-z}z}{(2-w)^2} dz = \frac{2}{(2-w)^2}, \quad 0 < w < 1. \end{aligned}$$

They are independent.

### Problem 3

The joint p.d.f of  $X$  and  $Y$  is given by

$$f_{XY}(x, y) = \begin{cases} 2xye^{-(x+y)} & 0 < y < x < \infty \\ 0 & \text{otherwise} \end{cases}$$

Define

$$Z = X + Y \quad W = X/Y$$

a.) Find  $f_{ZW}(z, w)$ .

b.) Are  $Z$  and  $W$  independent random variables?

c.) Are  $Z$  and  $W$  uncorrelated random variables?

**Solution:**

a.)

$$\begin{cases} x + y = z \\ x/y = w \end{cases} \Rightarrow \begin{cases} y_1 = z/(w+1) \\ x_1 = wz/(w+1) \end{cases}$$

Then,

$$\left| \begin{array}{cc} \frac{\partial x_1}{\partial z} & \frac{\partial x_1}{\partial w} \\ \frac{\partial y_1}{\partial z} & \frac{\partial y_1}{\partial w} \end{array} \right| = \left| \begin{array}{cc} \frac{w}{w+1} & \frac{z}{(w+1)^2} \\ \frac{1}{w+1} & -\frac{z}{(w+1)^2} \end{array} \right| = -\frac{z}{(w+1)^2},$$

and

$$f_{ZW}(z, w) = \frac{z}{(w+1)^2} f_{XY}(x_1, y_1) = \frac{z}{(w+1)^2} \frac{2wz^2}{(w+1)^2} e^{-z} = \frac{2wz^3 e^{-z}}{(w+1)^4}, \quad (z, w) \in \mathcal{A}.$$

The region  $\mathcal{A}$  is described as follows:

$$\begin{aligned} x < y < \infty &\Rightarrow \frac{z}{w+1} < \frac{wz}{w+1} < \infty \Rightarrow 1 < w < \infty, \\ 0 < x < y &\Rightarrow 0 < \frac{z}{w+1} < \frac{wz}{w+1} \Rightarrow 0 < z < wz \Rightarrow 0 < z < \infty. \end{aligned}$$

b.) Marginal distributions:

$$\begin{aligned} f_Z(z) &= \int_1^\infty f_{ZW}(z, w) dw = \frac{1}{6} z^3 e^{-z}, \quad z \geq 0, \\ f_W(w) &= \int_0^\infty f_{ZW}(z, w) dz = \frac{12w}{(w+1)^4}, \quad w \geq 1. \end{aligned}$$

They are independent.

## Problem 4

The joint p.d.f of  $X$  and  $Y$  is given by

$$f_{XY}(x, y) = \begin{cases} \frac{3}{4}(x+y)^2 & 0 < x < 1, -1 < y < 1, \\ 0 & \text{otherwise} \end{cases}$$

Define

$$Z = X + Y \quad W = X - Y$$

- Find  $f_{ZW}(z, w)$ ,  $f_Z(z)$  and  $f_W(w)$ .
- Are  $Z$  and  $W$  independent random variables?
- Are  $Z$  and  $W$  uncorrelated random variables?
- Are  $Z$  and  $W$  orthogonal random variables ( $E[ZW] = 0$ )? Prove your answers.

**Solution:**

a.)

$$\begin{cases} x + y = z \\ x - y = w \end{cases} \Rightarrow \begin{cases} y_1 = (z - w)/2 \\ x_1 = (z + w)/2 \end{cases}$$

Then,

$$f_{ZW}(z, w) = f_{XY}(x_1, y_1) \cdot \left| \frac{\partial x_1}{\partial z} \frac{\partial y_1}{\partial w} \right| = \frac{3}{4} z^3 \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = \frac{3}{8} z^2, \quad (z, w) \in \mathcal{A}.$$

The region  $\mathcal{A}$  is described as follows:

$$\mathcal{A} = \{(z, w) : 0 < z + w < 2, -2 < z - w < 2\}.$$

b.) Marginal distributions:

$$f_Z(z) = \begin{cases} \frac{3}{4} z^2(z+1) & -1 < z < 0, \\ \frac{3}{4} z^2 & 0 \leq z < 1, \\ \frac{3}{4} z^2(z-2) & 1 \leq z < 2, \end{cases} \quad f_W(w) = \begin{cases} \frac{w^3+3w^2+6w+4}{4} & -1 < w < 0, \\ \frac{3w^2-6w+4}{4} & 0 \leq w < 1, \\ \frac{1}{4}(2-w)^3 & 1 \leq w < 2. \end{cases}$$

They are not independent.

c.)

$$E[ZW] = \int_{-1}^0 \int_{-w}^{w+2} \frac{3}{8} z^3 w dz dw + \int_0^1 \int_{-w}^{2-w} \frac{3}{8} z^3 w dz dw + \int_1^2 \int_{w-2}^{2-w} \frac{3}{8} z^3 w dz dw$$

Upon solving we get  $E[ZW] = 0$ .

$$E[Z] = \int_{-1}^0 \frac{3}{4} z^3(z+1) dz + \int_0^1 \frac{3}{4} z^3 dz + \int_1^2 \frac{3}{4} z^3(z-2) dz = \frac{9}{8}.$$

Similarly,

$$E[W] = \frac{1}{8}$$

Since as  $E[ZW] - E[Z]E[W] \neq 0$ , they are correlated.

d.) Yes.

## Problem 5

$X, Y$  are independent, identical geometric random variables with common parameter  $p$ , i.e., with  $q = 1 - p$ ,

$$P(X = k) = P(Y = k) = pq^k, \quad k = 0, 1, 2, \dots$$

a.)  $Z = X + Y$ ,  $W = \min\{X, Y\}$ , find  $f_{ZW}(z, w)$ ,  $f_Z(z)$  and  $f_W(w)$ .

b.)  $Z = \min\{X, Y\}$ ,  $W = X - Y$ , find  $f_{ZW}(z, w)$ ,  $f_Z(z)$  and  $f_W(w)$ .

**Solution:**

a.) Look at 16:30 on  $Z = X + Y$ ,  $W = \max(X, Y)$ :

<https://youtu.be/TU3Y9bagw9w>

b.)  $Z = \min(X, Y)$ ,  $W = X - Y$ :

<https://youtu.be/V1EyqL1cqTE>

## Problem 6

$X$  and  $Y$  are independent Geometric random variables with common parameter  $p$ , i.e.,  $P(X = k) = P(Y = k) = pq^k$  with  $q = 1 - p$ . Define

$$Z = X + Y, \quad W = |X - Y|.$$

Find

- a.)  $P(Z = m, W = k)$
- b.)  $P(Z = m)$
- c.)  $P(W = k)$

**Solution:**

- a.) When  $X \geq Y$ ,

$$\begin{cases} X + Y = m, \\ X - Y = k, \end{cases} \Rightarrow \begin{cases} X = (m + k)/2, \\ Y = (m - k)/2. \end{cases}$$

$$\begin{aligned} X \geq Y &\Rightarrow (m + k)/2 \geq (m - k)/2 \Rightarrow k \geq 0, \\ X \geq 0 &\Rightarrow (m + k)/2 \geq 0 \Rightarrow m \geq -k, \\ Y \geq 0 &\Rightarrow (m - k)/2 \geq 0 \Rightarrow m \geq k. \end{aligned}$$

When  $X < Y$ ,

$$\begin{cases} X + Y = m, \\ Y - X = k, \end{cases} \Rightarrow \begin{cases} X = (m - k)/2, \\ Y = (m + k)/2. \end{cases}$$

$$\begin{aligned} X < Y &\Rightarrow (m - k)/2 < (m + k)/2 \Rightarrow k > 0, \\ X \geq 0 &\Rightarrow (m - k)/2 \geq 0 \Rightarrow m \geq k, \\ Y \geq 0 &\Rightarrow (m + k)/2 \geq 0 \Rightarrow m \geq -k. \end{aligned}$$

$$\begin{aligned} P(Z = m, W = k) &= P(Z = m, W = k, X \geq Y) + P(Z = m, W = k, X < Y) \\ &= P(X + Y = m, X - Y = k, X \geq Y) + P(X + Y = m, Y - X = k, X < Y) \\ &= \begin{cases} P(X = \frac{m+k}{2})P(Y = \frac{m-k}{2}) & k = 0, m = 0, 2, 4, \dots \\ P(X = \frac{m+k}{2})P(Y = \frac{m-k}{2}) + P(X = \frac{m-k}{2})P(Y = \frac{m+k}{2}) & k = 0, 1, 2, \dots, m \geq k \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} p^2 q^m & k = 0, m = 0, 2, 4, \dots, \\ 2p^2 q^m & k = 0, 1, 2, \dots, m = k + 2, k + 4, \dots, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

- b.)

$$P(Z = m) = \sum_k P(Z = m, W = k) = \begin{cases} p^2 & m = 0 \\ p^2 q^m + \sum_{k=0,2,4,\dots}^m 2p^2 q^m & m = 2, 4, 6, \dots \\ \sum_{k=1,3,5,\dots}^m 2p^2 q^m & m = 1, 3, 5, \dots \end{cases}$$

- c.)

$$P(W = k) = \sum_m P(Z = m, W = k) = \begin{cases} p^2 & k = 0 \\ \sum_{m=k,k+2,k+4,\dots}^{\infty} 2p^2 q^m & k > 0 \end{cases}$$