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Problem 1

a) i) If we can prove $\frac{1}{t_i+t_j}$ is an autocorrelation function, then by $\boxed{\text{Schur Product Theorem}}$, $\frac{1}{(t_i+t_j)^2}$ is also an autocorrelation function. The proof of $\frac{1}{t_i+t_j}$ is an autocorrelation function is on the next page.

ii) $\frac{t_i t_j}{t_i+t_j} = \frac{1}{\frac{1}{t_i}+\frac{1}{t_j}} = \int_0^\infty e^{-(\frac{1}{t_i}+\frac{1}{t_j})w} dw$. Let $Y = \begin{pmatrix} e^{-\frac{1}{t_1}w} \\ e^{-\frac{1}{t_2}w} \\ \vdots \\ e^{-\frac{1}{t_n}w} \end{pmatrix}$

Then $YY^T = \begin{pmatrix} e^{-(\frac{1}{t_1}+\frac{1}{t_1})w} & e^{-(\frac{1}{t_1}+\frac{1}{t_2})w} & \cdots \\ e^{-(\frac{1}{t_1}+\frac{1}{t_2})w} & \cdots & \vdots \\ \vdots & & \vdots \end{pmatrix}$

Thus $R = \int_0^\infty YY^T dw$. For a given $\underline{x} \neq 0$, $x^* R x = \int_0^\infty x^* YY^T x \, dw \geq 0$

So R is non-negative definite, $\frac{t_i t_j}{t_i+t_j}$ is an autocorrelation function.

iii) $R = \begin{pmatrix} t_1^2 & t_1^2 & \cdots & t_1^2 \\ t_1^2 & t_2^2 & \cdots & t_2^2 \\ \vdots & t_2^2 & t_3^2 \cdots & t_3^2 \\ t_1^2 & t_2^2 & t_3^2 & t_n^2 \end{pmatrix} = t_1^2 \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} (1,1,\cdots,1) + (t_2^2 - t_1^2) \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 1 \end{pmatrix} (0,1,\cdots,1)$

$+ (t_3 - t_2) \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 1 \end{pmatrix} (0,0,1,\cdots,1) + \cdots + (t_n^2 - t_{n-1}^2) \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} (0,0,\cdots,1)$

$= \sum \alpha_i \mu_i \mu_i^T, \quad \alpha_i \geq 0$

So $x^* R x = \sum \alpha_i x^* \mu_i \mu_i^T x \geq 0$. R is non-negative definite, $\min(t_i^2, t_j^2)$ is an autocorrelation function.

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b) ~~Let X be~~ $F_Y(y) = P(Y \le y) = 1 - P(\underline{Y > y})$

$= 1 - P(X(y) = 0)$

$P(X(y) = 0) = e^{-\lambda y}$

So $F_Y(y) = 1 - e^{-\lambda y}$, $f_Y(y) = \frac{dF_Y(y)}{dy} = y e^{-\lambda y}$

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a) i) $\frac{1}{t_i + t_j} = \int_0^\infty e^{-(t_i + t_j)w} dw$. let $Y = \begin{pmatrix} e^{-t_1 w} \\ e^{-t_2 w} \\ \vdots \\ e^{-t_n w} \end{pmatrix}$

Then $YY^T = \begin{pmatrix} & & \\ & e^{-(t_i + t_j)w} & \\ & & \end{pmatrix}$, $R = \int_0^\infty YY^T dw$.

So $x^* R x = \int_0^\infty x^* Y Y^T x \, dw \geq 0$, $\frac{1}{t_i + t_j}$ is an autocorrelation function.

By $\boxed{\text{Schur Product Theorem}}$, $\frac{1}{(t_i + t_j)^2}$ is an autocorrelation function.

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problem 2

2.a) $E[Y(t)] = E[e^{j(\omega_0 t + X(t) + \theta)}]$

$= e^{j\omega_0 t} \cdot E[e^{jX(t)}] \cdot E[e^{j\theta}]$

$= 0$

$R_{YY}(t_1, t_2) = E[e^{j(\omega_0 t_1 + X(t_1) + \theta)} \cdot e^{-j(\omega_0 t_2 + X(t_2) + \theta)}]$

$= e^{j\omega_0(t_1 - t_2)} E[e^{j\underbrace{(X(t_1) - X(t_2))}}]$

$= e^{j\omega_0(t_1-t_2)} \cdot e^{-\frac{1}{2}\cdot (t_1^2 + t_2^2 - 2t_1t_2)}$

$= e^{j\omega_0(t_1-t_2) - \frac{1}{2}(t_1-t_2)^2}$

b) $Y(t)$ is W.S.S.

c) $S_{YY}(\omega) = \int_{-\infty}^{\infty} e^{j\omega_0\tau - \frac{1}{2}\tau^2} \cdot e^{-j\omega\tau} d\tau$

$= \int_{-\infty}^{\infty} e^{-\frac{1}{2}\tau^2 + j(\omega_0 - \omega)\tau} d\tau = \int_{-\infty}^{\infty} e^{-\frac{1}{2}\tau^2} \cos(\omega_0\tau) e^{-j\omega\tau} d\tau$

$\cancel{\int_{-\infty}^{\infty} e^{-\frac{1}{2}\tau^2} \cos(\omega_0 - \omega)\tau \, d\tau}$

$\cancel{\frac{1}{2}\sqrt{2\pi}\{}$ $= \frac{1}{2}\sqrt{2\pi}\{e^{-(\omega-\omega_0)^2/2} + e^{-(\omega+\omega_0)^2/2}]$

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problem 3

3. a) $R_{xy} = E[s(s+n)] = E(s^2+sn) = \sigma_s^2$

$R_{yy} = E[(s+n)(s+n)] = E(s^2+n^2+2sn) = \sigma_s^2+\sigma_n^2$

$H = R_{xy}R_{yy}^{-1} = \frac{\sigma_s^2}{\sigma_s^2+\sigma_n^2}$

$\hat{s}_1 = H X_1 = \frac{\sigma_s^2}{\sigma_s^2+\sigma_n^2}(s+n)$

$\sigma_1^2 = R_{xx} - R_{xy}H^* = \sigma_s^2 - \sigma_s^2 \cdot \frac{\sigma_s^2}{\sigma_s^2+\sigma_n^2} = \frac{\sigma_s^2\sigma_n^2}{\sigma_s^2+\sigma_n^2}$

b) $R_{xy} = E\{s(s+n),\ s(s+n+w)\} = E[s^2+sn,\ s^2+sn+sw] = (\sigma_s^2,\ \sigma_s^2)$

$$R_{yy} = E\begin{pmatrix} (s+n)^2 & (s+n)(s+n+w) \\ (s+n)(s+n+w) & (s+n+w)^2 \end{pmatrix} = \begin{pmatrix} \sigma_s^2+\sigma_n^2 & \sigma_s^2+\sigma_n^2 \\ \sigma_s^2+\sigma_n^2 & \sigma_s^2+\sigma_n^2+\sigma_w^2 \end{pmatrix}$$

$$H = R_{xy}R_{yy}^{-1} = \sigma_s^2(1,1)\cdot\frac{1}{(\sigma_s^2+\sigma_n^2)(\sigma_s^2+\sigma_n^2+\sigma_w^2)-(\sigma_s^2+\sigma_n^2)^2}\begin{pmatrix} \sigma_s^2+\sigma_n^2+\sigma_w^2 & -\sigma_s^2-\sigma_n^2 \\ -\sigma_s^2-\sigma_n^2 & \sigma_s^2+\sigma_n^2 \end{pmatrix}$$

$$= \frac{\sigma_s^2}{(\sigma_s^2+\sigma_n^2)\sigma_w^2}\cdot(\sigma_w^2,\ 0)$$

$$\hat{s}_2 = H\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \frac{\sigma_s^2}{(\sigma_s^2+\sigma_n^2)\sigma_w^2}(\sigma_w^2, 0)\begin{pmatrix} s+n \\ s+n+w \end{pmatrix} = \frac{\sigma_s^2}{(\sigma_s^2+\sigma_n^2)}(s+n)$$

$$\sigma_2^2 = R_{xx} - R_{xy}H^* = \sigma_s^2 - \sigma_s^2(1,1)\cdot\frac{\sigma_s^2}{(\sigma_s^2+\sigma_n^2)\sigma_w^2}\cdot\begin{pmatrix} \sigma_w^2 \\ 0 \end{pmatrix} = \frac{\sigma_s^2\sigma_n^2}{\sigma_s^2+\sigma_n^2}$$

$\sigma_1^2 = \sigma_2^2$

~~c) They perform equally well since $\sigma_1^2 = \sigma_2^2$.~~

c) They perform equally well since $\sigma_1^2 = \sigma_2^2$

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4. a) $R_{YY}(t_1,t_2) = E\left\{\left[X(t_1+T)\cos(\omega_0 t_1+\Theta) + X^2(t_1+T)\sin(\omega_0 t_1+\Theta)\right]\cdot\left[X(t_2+T)\cos(\omega_0 t_2+\Theta) + X^2(t_2+T)\sin(\omega_0 t_2+\Theta)\right]\right\}$

$$= E\left[X(t_1+T)X(t_2+T)\right] E\left[\cos(\omega_0 t_1+\Theta)\cos(\omega_0 t_2+\Theta)\right]$$

$$+ E\left[X^2(t_1+T)X^2(t_2+T)\right] E\left[\sin(\omega_0 t_2+\Theta)\sin(\omega_0 t_1+\Theta)\right]$$

$$+ 0$$

$$+ 0$$

$$= \frac{1}{2}R_{XX}(\tau)\cos\omega_0(t_1-t_2) + \frac{1}{2}E\left[X^2(t_1+T)X^2(t_2+T)\right]\cos\omega_0(t_2-t_1)$$

We know $E\left\{X^2(t_1+T)X^2(t_2+T)\right\} = 2\sigma^4\rho^2 + \sigma^4$

$\sigma^2 = R_{XX}(0)$, $\rho = \dfrac{Cov(X(t_1),X(t_2))}{\sigma_{X(t_1)}\sigma_{X(t_2)}} = \dfrac{R_{XX}(\tau)}{R_{XX}(0)}$

So $E\left[X^2(t_1+T)X^2(t_2+T)\right] = 2\cdot\dfrac{R_{XX}^2(\tau)}{R_{XX}^2(0)}\cdot R_{XX}^2(0) + R_{XX}^2(0)$

$$= 2R_{XX}^2(\tau) + R_{XX}^2(0)$$

Therefore, $R_{YY} = \frac{1}{2}R_{XX}(\tau)\cos\omega_0\tau + \frac{1}{2}\left(2R_{XX}^2(\tau) + R_{XX}^2(0)\right)\cos\omega_0\tau$

$$= \frac{1}{2}\cos\omega_0\tau\left[R_{XX}(\tau) + 2R_{XX}^2(\tau) + R_{XX}^2(0)\right]$$

$S_{YY}(\omega) = \int_{-\infty}^{\infty}\frac{1}{2}\cos\omega_0\tau\left[R_{XX}(\tau) + 2R_{XX}^2(\tau) + R_{XX}^2(0)\right]e^{-j\omega\tau}\,d\tau$

$$= \frac{1}{2}R_{XX}^2(0)\int\cos\omega_0\tau\cdot e^{-j\omega\tau}\,d\tau + \frac{1}{2}\int\cos\omega_0\tau\, R_{XX}(\tau)\,e^{-j\omega\tau}\,d\tau$$

$$+ \int\cos\omega_0\tau\, R_{XX}^2(\tau)\,e^{-j\omega\tau}\,d\tau$$

$$= I_1 + I_2 + I_3$$

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Problem 4

$$I_1 = \frac{1}{2} R_{XX}(0) \int_{-\infty}^{\infty} \cos w_0\tau \, e^{-jw\tau} d\tau = \frac{1}{2} R_{XX}(0) \pi \left[\delta(w-w_0) + \delta(w+w_0) \right]$$

$$I_2 = \frac{1}{2} \int_{-\infty}^{\infty} \cos w_0\tau \, R_{XX}(\tau) e^{-jw\tau} d\tau$$

$$= \frac{1}{4} \int_{-\infty}^{\infty} R_{XX}(\tau) e^{jw_0\tau} e^{-jw\tau} d\tau + \frac{1}{4} \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-jw_0\tau} e^{-jw\tau} d\tau$$

$$= \frac{1}{4} \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-j(w-w_0)} d\tau + \frac{1}{4} \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-j(w+w_0)\tau} d\tau$$

$$= \frac{1}{4} S_{XX}(w-w_0) + \frac{1}{4} S_{XX}(w+w_0)$$

$$I_3 = \int_{-\infty}^{\infty} \cos w_0\tau \, R_{XX}^2(\tau) e^{-jw\tau} d\tau$$

$$= \int_{-\infty}^{\infty} \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(w) e^{jw\tau} dw \right]^2 \cos w_0\tau \, e^{-jw\tau} d\tau$$

Therefore,
$$S_{YY}(w) = \frac{1}{2} R_{XX}(0) \pi \left[\delta(w-w_0) + \delta(w+w_0) \right]$$

$$+ \frac{1}{4} S_{XX}(w-w_0) + \frac{1}{4} S_{XX}(w+w_0)$$

$$+ \int_{-\infty}^{\infty} \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(w) e^{jw\tau} dw \right]^2 \cos w_0\tau \, e^{-jw\tau} d\tau$$

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Problem 4

b) $R_{xx}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(w)\, e^{jw\tau}\, dw$

$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{jw\tau}\, dw$

$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos w\tau\, dw$

$= \frac{1}{2\pi\tau} \sin w\tau \Big|_{-\pi}^{\pi}$

$= \frac{1}{\pi\tau} \sin(\pi\tau)$

$= \operatorname{sinc}(\pi\tau)$

$f_{\underline{X}}(\underline{x}) = \frac{1}{(2\pi)\sqrt{|R|}} e^{-\frac{1}{2}\underline{x}^T R \underline{x}}$, $R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\underline{x}^T R \underline{x} = \cancel{\begin{pmatrix} X(t) \\ X(t+1) \end{pmatrix}} \begin{pmatrix} X(t) & X(t+1) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X(t) \\ X(t+1) \end{pmatrix} = X^2(t) + X^2(t+1)$

$f_{\underline{X}}(\underline{x}) = \frac{1}{2\pi} e^{-\frac{1}{2}(X^2(t) + X^2(t+1))}$

$\Pr(X(t) > 0, X(t+1) > 0) = \int_0^{\infty}\int_0^{\infty} \frac{1}{2\pi} e^{-\frac{1}{2}(X^2(t) + X^2(t+1))}\, dX(t)\, dX(t+1)$

$= \frac{1}{2} \cdot \frac{1}{2}$

$= \frac{1}{4}$