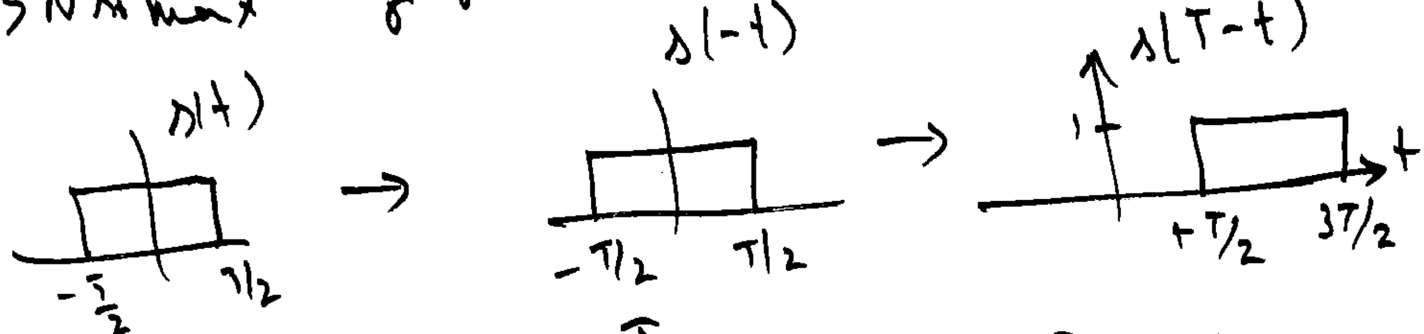
EL 6303
HW #11 Solutions

1) a) Best Filter $\equiv$ Matched Filter

$$SNR_{o,max} = \frac{1}{\sigma^2}\int_0^T (s(t))^2\, dt = \frac{1}{\sigma^2}\int_0^T (t/T)^2\, dt = \frac{T}{3\sigma^2}$$

b)



$$SNR_{o,max} = \frac{1}{\sigma^2}\int_0^T s^2(t)\, dt = \frac{T}{\sigma^2}$$

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2) $\hat{s} = H\underline{Y} = H\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$; $Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} s+n \\ sn \end{pmatrix}$

a) Best Estimator:

$$H = R_{sy} R_{yy}^{-1}$$

$$R_{yy} = E(YY^T) = \begin{pmatrix} E(s+n)^2 & E(s^2 n) \\ E(s^2 n) & E(s^2 n^2) \end{pmatrix}$$

$$= \begin{pmatrix} \sigma_s^2 + \sigma_n^2 & \mu\sigma_s^2 \\ \mu\sigma_s^2 & \sigma_n^2 \end{pmatrix}$$

2/

$$R_{YY}^{-1} = \frac{1}{(\sigma_s^2+\sigma_n^2)\sigma_n^2-\mu^2\sigma_n^4}\begin{pmatrix}\sigma_n^2 & -\mu\sigma_s^2\\ -\mu\sigma_s^2 & \sigma_s^2+\sigma_n^2\end{pmatrix}$$

$$R_{sY} = E[sY^T] = E(sY_1, sY_2)]$$
$$= [E(s(s+n)), E(s\cdot sn)] = [\sigma_s^2, \sigma_s^2\mu]$$

$$H = R_{sY}R_{YY}^{-1} = \frac{\sigma_s^2}{\Delta}[\sigma_n^2, -\mu\sigma_s^2]$$

where $\Delta = (\sigma_s^2+\sigma_n^2)\sigma_n^2 - \mu^2\sigma_s^4$

H represents the best linear estimator ~~for~~ s based on $Y = \begin{pmatrix}Y_1\\Y_2\end{pmatrix}$.

b) Minimum mean square error

$$\sigma_{min}^2 = R_{ss} - R_{sY}H^*$$
$$= \sigma_s^2 - \frac{\sigma_s^2}{\Delta}[\sigma_s^2\sigma_n^2 - \mu^2\sigma_s^4]$$
$$= \frac{\sigma_s^2}{\Delta}(\cancel{\sigma_s^2}\sigma_n^2 + \sigma_n^4 \cancel{-\mu^2\sigma_s^4} + \cancel{\mu^2\sigma_s^4} - \cancel{\sigma_s^2}\sigma_n^2] = (\sigma_s^2\sigma_n^2/\Delta).$$

Problem 3

Let $X_n = X(nT)$ represents a discrete-time W.S.S real stochastic process with auto-correlation function $r_k = E(X_{n+k}X_n)$. Suppose X_0 is known, and X_1 and X_2 are unknowns.

a. Determine the best estimator for the unknown X_1 in terms of X_0. Find the associated minimum mean square error σ_1^2.

b. Determine the best estimator for the unknown X_2 in terms of X_0. Find the associated minimum mean square error σ_2^2.

c. Show that $\sigma_1^2 > \sigma_2^2$ in some cases and $\sigma_1^2 < \sigma_2^2$ in other cases. Show an example where σ_2^2 is smaller than σ_1^2.

Solution:

a.

$$\hat{X}_1 = a_0 X_0$$
$$\epsilon = X_1 - \hat{X}_1$$
$$E(\epsilon X_0) = 0 \text{(Based on orthogonal principle:)}$$
$$E(X_1X_0) - a_0E(X_0X_0) = 0$$
$$a_0 = \frac{r_1}{r_0}$$
$$\therefore \text{ the best estimator for } X_1 \text{ is } \hat{X}_1 = \frac{r_1}{r_0}X_0$$
$$\sigma_1^2 = E(|\epsilon|^2) = E(X_1^2) - 2a_0E(X_1X_0) + a_0^2E(X_0^2) = r_0(\frac{r_1^2}{r_0^2} + 1) - 2\frac{r_1}{r_0} \cdot r_1 = r_0 - \frac{r_1^2}{r_0}$$

b.

$$
\begin{aligned}
&\hat{X}_2 = a_0X_0 + a_1X_1\\
&\epsilon = X_2 - \hat{X}_2\\
&\begin{cases} E(\epsilon X_0) = 0 \\ E(\epsilon X_1) = 0 \end{cases}\\
\Rightarrow &\begin{cases} r_2 - a_0r_0 - a_1r_1 = 0 \\ r_1 - a_0r_1 - a_1r_0 = 0 \end{cases}\\
\Rightarrow &\begin{cases} a_0 = \frac{r_1^2 - r_2r_0}{r_1^2 - r_0^2} \\ a_1 = \frac{r_1(r_2-r_0)}{r_1^2-r_0^2} \end{cases}
\end{aligned}
$$

$\therefore$the best estimator for X_2 is:

$$
\begin{aligned}
\hat{X}_2 &= \frac{r_1^2 - r_2r_0}{r_1^2 - r_0^2}X_0 + \frac{r_1(r_2 - r_0)}{r_1^2 - r_0^2}X_1\\
&= \frac{r_1^2 - r_2r_0}{r_1^2 - r_0^2}X_0 + \frac{r_1(r_2 - r_0)}{r_1^2 - r_0^2} \cdot \frac{r_1}{r_0}X_1\\
&= \frac{r_2}{r_0}X_0\\
\sigma_2^2 &= E(|\epsilon|^2)\\
&= E(\epsilon X_2) - a_0E(\epsilon X_0) - a_1E(\epsilon X_1)\\
&= E(\epsilon X_2)\\
&= E(X_2^2) - 2\frac{r_2}{r_0}E(X_2X_0) + \frac{r_2^2}{r_0^2}E(X_0^2)\\
&= r_0 - \frac{r_2^2}{r_0}
\end{aligned}
$$

c.

$$
\begin{aligned}
\sigma_1^2 - \sigma_2^2 &= \frac{r_2^2}{r_0} - \frac{r_1^2}{r_0}\\
&= \frac{1}{r_0}(r_2^2 - r_1^2)\\
&= \frac{1}{r_0}(r_2 + r_1)(r_2 - r_1)
\end{aligned}
$$

$\therefore$ when $r_2 > r_1$, σ_1^2 is larger and when $r_2 < r_1$, σ_2^2 is larger.

Problem 4

Different versions of noisy data vs. a different take on the noise. Which is better?

i. Consider different versions of data

$$
\begin{aligned}
Y_1 &= X + n\\
Y_2 &= X + w
\end{aligned}
$$

versus

ii. another look at the noise

$$Y_1 = X + n$$
$$Y_2 = n + w$$

Here X, n, w are zero mean uncorrelated random variables with variances σ_X^2, σ_n^2 and σ_w^2 respectively. X is the desired unknown.

a. Find the best estimator for X using Y_1 and Y_2 in the both cases (i) and (ii).

b. Find the mean square error in the both cases above. Which set of data is preferable for estimating X?

Solution:

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} X + n \\ X + w \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} X + \begin{pmatrix} n \\ w \end{pmatrix}$$

$$R_{XY} = E[X(X+n), X(n+w)] = \sigma_X^2[1, 1]$$

$$R_{YY} = \begin{bmatrix} \sigma_X^2 + \sigma_n^2 & \sigma_n^2 \\ \sigma_n^2 & \sigma_n^2 + \sigma_w^2 \end{bmatrix}$$

$$R_{YY}^{-1} = \frac{1}{(\sigma_X^2 + \sigma_n^2)\sigma_w^2 + \sigma_X^2\sigma_n^2} \begin{bmatrix} \sigma_X^2 + \sigma_w^2 & \sigma_X^2 \\ \sigma_X^2 & \sigma_X^2 + \sigma_n^2 \end{bmatrix}$$

$$H = R_{XY}R_{YY}^{-1} = \frac{\sigma_X^2}{(\sigma_w^2 + \sigma_n^2)\sigma_X^2 + \sigma_w^2\sigma_n^2} \begin{pmatrix} \sigma_w^2 \\ \sigma_n^2 \end{pmatrix}$$

$$\hat{X}_2 = HY = \frac{1}{(\sigma_n^2 + \sigma_w^2) + \frac{\sigma_n^2\sigma_w^2}{\sigma_X^2}}(\sigma_w^2 Y_1 + \sigma_n^2 Y_2)$$

$$\sigma_2^2 = R_{XX} - HR_{XY}$$

$$= \sigma_X^2 - \frac{\sigma_X^2(\sigma_w^2 + \sigma_n^2)\sigma_X^2}{\sigma_X^2(\sigma_n^2 + \sigma_w^2) + \sigma_n^2\sigma_w^2} = \frac{\sigma_X^2\sigma_w^2\sigma_n^2}{\sigma_X^2(\sigma_w^2 + \sigma_n^2) + \sigma_w^2\sigma_n^2}$$

$$= \frac{\sigma_X^2}{1 + \sigma_X^2(\frac{1}{\sigma_n^2} + \frac{1}{\sigma_w^2})} < \sigma_1^2 = \frac{\sigma_X^2}{1 + \frac{\sigma_X^2}{\sigma_n^2}}$$

$$Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} X + n \\ n + w \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} X + \begin{pmatrix} n \\ w + n \end{pmatrix}$$

$$R_{XY} = E[X(X+n), X(n+w)] = \sigma_X^2(1, 0)$$

$$R_{YY} = \begin{bmatrix} \sigma_X^2 + \sigma_n^2 & \sigma_n^2 \\ \sigma_n^2 & \sigma_n^2 + \sigma_w^2 \end{bmatrix}$$

$$R_{YY}^{-1} = \frac{1}{(\sigma_w^2 + \sigma_n^2)\sigma_X^2 + \sigma_n^2\sigma_w^2} \begin{bmatrix} \sigma_n^2 + \sigma_w^2 & \sigma_n^2 \\ \sigma_n^2 & \sigma_X^2 + \sigma_n^2 \end{bmatrix}$$

$$H = R_{XY}R_{YY}^{-1} = \frac{\sigma_X^2}{(\sigma_w^2 + \sigma_n^2)\sigma_X^2 + \sigma_w^2\sigma_n^2} \begin{pmatrix} \sigma_w^2 + \sigma_n^2 \\ -\sigma_n^2 \end{pmatrix}$$

$$\hat{X}_3 = HY = \frac{1}{1 + \frac{\sigma_n^2\sigma_w^2}{\sigma_X^2(\sigma_n^2 + \sigma_w^2)}}\left(Y_1 - \frac{\sigma_n^2}{\sigma_n^2 + \sigma_w^2} Y_2\right)$$

$$\begin{aligned} \sigma_3^2 &= R_{XX} - HR_{XY} \\ &= \sigma_X^2 - \frac{\sigma_X^2(\sigma_w^2 + \sigma_n^2)\sigma_X^2}{\sigma_X^2(\sigma_n^2 + \sigma_w^2) + \sigma_n^2\sigma_w^2} - \frac{\sigma_X^2\sigma_w^2\sigma_n^2}{\sigma_X^2(\sigma_w^2 + \sigma_n^2) + \sigma_w^2\sigma_n^2} \\ &= \frac{\sigma_X^2}{1 + \sigma_X^2\left(\frac{1}{\sigma_n^2} + \frac{1}{\sigma_w^2}\right)} = \sigma_2^2 < \sigma_1^2 \end{aligned}$$

Both estimators have identical performance

Problem 5

Let X, Y, Z be zero mean correlated random variables with common correlation coefficient equal to $-\frac{1}{2}$ and all of the variances are equal to 1.

a. Find the best linear estimate for Z in terms of X and Y.

b. Find the best linear estimate for X in terms of Y and Z.

c. What are the minimum mean square estimation errors in the above cases?

Solution:

a.

$\hat{Z} = aX + bY$

$\epsilon = Z - \hat{Z} = Z - aX - bY$

Othogonality principle says ϵ perpendicular to data

ϵ perpendicular to X and ϵ perpendicular to Y

ϵ perpendicular to X $\rightarrow E(\epsilon X) = 0$

$E(\epsilon X) = E\{(Z - aX - bY)X\} = 0$

$$aE(X^2) + bE(XY) = E(XZ) \tag{6}$$

Similarly ϵ perpendicular to Y $\rightarrow E(\epsilon Y) = 0$

$E(\epsilon Y) = E\{(Z - aX - bY)Y\} = 0$

$$aE(XY) + bE(Y^2) = E(ZY) \tag{7}$$

$E(X^2) = E(Y^2) = E(Z^2) = 1$

Also $\rho = \dfrac{E(XY)}{\sigma_X \sigma_Y} = -\dfrac{1}{2} \rightarrow \rho = -\dfrac{1}{2}$

$E(XY) = E(YZ) = E(ZX) = \rho = -\dfrac{1}{2}$

b.

Hence (6) and (7) becomes

$$a + \rho b = \rho \tag{8}$$

$$\rho a + b = \rho \tag{9}$$

Solving (8) and (9) we get

$b(1 - \rho^2) = \rho - \rho^2 = \rho(1 - \rho)$

$$b = \frac{\rho}{1+\rho} = a \tag{10}$$

$$\hat{Z} = \frac{\rho}{1+\rho}(X + Y) \tag{11}$$

$$\hat{X} = \frac{\rho}{1+\rho}(Y + Z) \tag{12}$$

(11) and (12) are the best linear estimators.

c.

$E(\epsilon^2) = E(\epsilon(Z - aX - bY)) = E(\epsilon Z)$
Since $E(\epsilon X) = E(\epsilon Y) = 0$
$E((Z - aX - bY)Z) = E(Z^2 - aXZ - bYZ)$
$= E(Z^2) - aE(XZ) - bE(YZ)$
$= 1 - (a+b)\rho = 1 - \dfrac{\rho^2}{1+\rho}$
$= \dfrac{1+\rho-\rho^2}{1+\rho} > 0$

Problem 6

Given the datasets

$$Y_1 = s + n$$

and

$$Y_2 = s + w,$$

where s, n, w are zero mean independent random variables with variances σ_x^2, σ_n^2 and σ_w^2.

a. Find the best estimator $\hat{s_1}$ for s using Y_1 and find the corresponding minimum mean sqaure error σ_1^2.

b. Find the best estimator $\hat{s_2}$ for s using Y_1 and Y_2, and find the corresponding minimum mean sqaure error σ_2^2. Which one is smaller, σ_1^2 or σ_2^2?

Solution:

a. Denote

$$Y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} s+n \\ s+w \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} s + \begin{bmatrix} n \\ w \end{bmatrix}. \quad (1)$$

We know $\mathbb{E}(w) = 0, \mathbb{E}(s) = 0, \mathbb{E}(n) = 0$, so

$$R_{sY_1} = \sigma_s^2,\ R_{Y_1Y_1} = \sigma_s^2 + \sigma_n^2,\ \Rightarrow H =; R_{sY_1}; R_{Y_1Y_1}^{-1} = \frac{1}{1+\frac{\sigma_n^2}{\sigma_s^2}}.$$

We know

$$s_1 = HY = \frac{1}{1+\frac{\sigma_n^2}{\sigma_s^2}} Y_1$$

and the minimum MSE

$$\sigma_1^2 = \frac{\sigma_s^2}{1+\frac{\sigma_s^2}{\sigma_n^2}} = \frac{\sigma_s^2\sigma_n^2}{\sigma_n^2+\sigma_s^2}$$

b. We know

$$
\begin{aligned}
R_{sY} &= [s(s+n), s(s+w)] = \sigma_s^2[1,1] \\
R_{YY} &= E(YY^*) \\
&= \begin{bmatrix} E((s+n)^2) & E((s+n)(s+w)) \\ E((s+w)(s+n)) & E((s+w)^2) \end{bmatrix} \\
&= \begin{bmatrix} \sigma_s^2+\sigma_n^2 & \sigma_s^2 \\ \sigma_s^2 & \sigma_s^2+\sigma_n^2\sigma_w^2 \end{bmatrix} \\
R_{YY}^{-1} &= \frac{1}{(\sigma_s^2+\sigma_n^2)(\sigma_s^2+\sigma_n^2\sigma_w^2)-\sigma_s^4} \begin{bmatrix} \sigma_s^2+\sigma_n^2 & \sigma_s^2 \\ \sigma_s^2 & \sigma_s^2+\sigma_w^2 \end{bmatrix} \\
H = R_{XY}R_{YY}^{-1} &= \frac{\sigma_s^2}{(\sigma_s^2+\sigma_n^2)(\sigma_s^2+\sigma_n^2\sigma_w^2)-\sigma_s^4} \begin{bmatrix} \sigma_w^2\sigma_n^2 & \sigma_n^2 \end{bmatrix} \\
\hat{s}_2 = HY &= \frac{\sigma_s^2}{(\sigma_n^2+\sigma_w^2)+\frac{\sigma_n^2\sigma_w^2}{\sigma_s^2}}[\sigma_w^2 Y_1 + \sigma_n^2 Y_2] \\
\sigma_2^2 &= \sigma_s^2 - HR_{sY} \\
&= \sigma_s^2 - \frac{\sigma_s^4(\sigma_w^2+\sigma_n^2)}{(\sigma_n^2+\sigma_w^2)+\frac{\sigma_n^2\sigma_w^2}{\sigma_s^2}} \\
&= \frac{\sigma_s^2\sigma_n^2\sigma_w^2}{\sigma_n^2\sigma_w^2+\sigma_s^2(\sigma_n^2+\sigma_w^2)} \\
&= \frac{\sigma_s^2}{1+\sigma_s^2\left(\frac{1}{\sigma_n^2}+\frac{1}{\sigma_w^2}\right)} < \frac{\sigma_s^2}{1+\frac{\sigma_s^2}{\sigma_n^2}} = \sigma_1^2
\end{aligned}
$$

Therefore σ_2^2 is smaller. So The second data estimator is better.