1. *(15 points)*

a. Which among the following are autocorrelation functions?

i) $\frac{1}{(t_i+t_j)^2}$

ii) $\frac{t_i t_j}{t_i+t_j}$

iii) $\min(t_i^2, t_j^2)$

b. Let $X(t)$ represent a Poisson process with parameter λt. Let Y represent the waiting time to the first arrival. Find the probability density function of Y.

2. *(25 points)* Let $X(t)$ be a zero mean Gaussian process with auto-correlation function $R_{XX}(t_1, t_2) = t_1 t_2$. Define

$$Y(t) = e^{j(\omega_0 t + X(t) + \theta)},$$

where $\theta \sim U(0, 2\pi)$ is independent of $X(t)$.

a. Find the mean and autocorrelation function of $Y(t)$.

b. Is $Y(t)$ wide sense stationary (W.S.S.)?

c. If $Y(t)$ is wide sense stationary, find its power spectral density function $S_{YY}(\omega)$.

3. *(30 points)* Consider two sets of measurements to determine the unknown s:

$$X_1 = s + n, \quad X_2 = s + n + w,$$

where s, n, w are independent zero-mean random variables with variances

$$\text{Var}(s) = \sigma_s^2, \ \text{Var}(n) = \sigma_n^2, \text{Var}(w) = \sigma_w^2$$

a. Determine the best linear minimum mean square error (MMSE) for s based on X_1. Compute the minimum mean square error σ_1^2.

b. Determine the best linear mean square error (MMSE) estimator for s based on both X_1 and X_2. Compute the minium mean square error σ_2^2.

c. Between (a) and (b) above, which estimator is superior? Justify your answer by comparing σ_1^2 and σ_2^2.

4. *(30 points)*

a. Let $X(t)$ be a wide sense stationary (W.S.S) zero-mean Gaussian process with autocorrelation function $R_{XX}(\tau)$ and power spectrum $S_{XX}(\omega)$. Consider the process

$$Y(t) = X(t+T)\cos(\omega_0 t + \theta) + X^2(t-T)\sin(\omega_0 t + \theta),$$

and $\theta \sim U(0, 2\pi)$ is independent of X(t). Find the power spectrum of $Y(t)$ and express it in terms of $S_{XX}(\omega)$ and ω_0.

b. Let $X(t)$ be a zero mean stationary Gaussian process with power spectrum

$$S_{XX}(\omega) = \begin{cases} 1 & |\omega| < \pi, \\ 0 & \text{otherwise.} \end{cases}$$

Find the probability that both $X(t)$ and $X(t+1)$ are positive. That is, find $\Pr(X(t+1) > 0, X(t) > 0)$.

$$R(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) e^{j\omega\tau} \, d\omega \leftrightarrow S(\omega) = \int_{-\infty}^{\infty} R(\tau) e^{-j\omega\tau} \, d\tau$$

$$\delta(\tau) \leftrightarrow 1 \qquad\qquad 1 \leftrightarrow 2\pi\delta(\omega)$$

$$e^{j\beta\tau} \leftrightarrow 2\pi\delta(\omega - \beta) \qquad\qquad \cos\beta\tau \leftrightarrow \pi\delta(\omega - \beta) + \pi\delta(\omega + \beta)$$

$$e^{-\alpha|\tau|} \leftrightarrow \frac{2\alpha}{\alpha^2 + \omega^2} \qquad\qquad e^{-\alpha\tau^2} \leftrightarrow \sqrt{\frac{\pi}{\alpha}} e^{-\omega^2/4\alpha}$$

$$e^{-\alpha|\tau|} \cos\beta\tau \leftrightarrow \frac{\alpha}{\alpha^2 + (\omega - \beta)^2} + \frac{\alpha}{\alpha^2 + (\omega + \beta)^2}$$

$$2e^{-\alpha\tau^2} \cos\beta\tau \leftrightarrow \sqrt{\frac{\pi}{\alpha}} \left[e^{-(\omega-\beta)^2/4\alpha} + e^{-(\omega+\beta)^2/4\alpha} \right]$$

$$\begin{cases} 1 - \dfrac{|\tau|}{T} & |\tau| < T \\ 0 & |\tau| > T \end{cases} \leftrightarrow \frac{4\sin^2(\omega T/2)}{T\omega^2}$$

$$\frac{\sin\sigma\tau}{\pi\tau} \leftrightarrow \begin{cases} 1 & |\omega| < \sigma \\ 0 & |\omega| > \sigma \end{cases}$$