ECE-GY 6303, Probability & Stochastic Processes

Solution to Homework # 8

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Problem 1

a.) Find the correlation function of

$$X(t) = \sum_{k=1}^{n} a_k \cos(\omega_k t + \phi_k),$$

where a_k and ω_k are constants and ϕ_k are independent random variables that are uniformly distributed in $(0, 2\pi)$.

b.) Show that $R(t_1, t_2) = \min(t_1, t_2)$ is an auto-correlation function.

Solution:

a.)

$$\begin{aligned}
R(t_2, t_1) &= E[X(t_1)X^*(t_2)] \\
&= E\left[\left(\sum_{k=1}^{n} a_k \cos(\omega_k t_1 + \phi_k)\right)\left(\sum_{i=1}^{n} a_i^* \cos(\omega_i t_2 + \phi_i)\right)\right] \\
&= \sum_{k=1}^{n}\sum_{i=1}^{n} a_k a_i^* E\left[\cos(\omega_k t_1 + \phi_k)\cos(\omega_i t_2 + \phi_i)\right] \\
&= \frac{1}{2}\sum_{k=1}^{n}\sum_{i=1}^{n} a_k a_i^* E\left[\cos(\omega_k t_1 - \omega_i t_2 + \phi_k - \phi_i) + \cos(\omega_k t_1 + \phi_k + \omega_i t_2 + \phi_i)\right] \\
&= \frac{1}{2}\sum_{k=1}^{n} a_k^2 E[\cos(\omega_k(t_2 - t_1))],
\end{aligned}$$

as $E\left[\cos(\omega_k t_1 - \omega_i t_2 + \phi_k - \phi_i)\right] = E\left[\cos(\omega_k t_1 + \phi_k + \omega_i t_2 + \phi_i)\right] = 0$ for $i \neq k$.

b.) Let $t_0 = 0$. Define the matrix

$$T = \begin{bmatrix} t_1 & t_1 & \cdots & t_1 \\ t_1 & t_2 & \cdots & t_2 \\ \vdots & \vdots & \ddots & \vdots \\ t_1 & t_2 & \cdots & t_n \end{bmatrix} = \begin{bmatrix} t_1 & t_1 & \cdots & t_1 \\ t_1 & t_1 & \cdots & t_1 \\ \vdots & \vdots & \ddots & \vdots \\ t_1 & t_2 & \cdots & t_1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & t_2 - t_1 & \cdots & t_2 - t_1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & t_2 - t_1 & \cdots & t_n - t_1 \end{bmatrix}$$

$$= t_1 \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} + (t_2 - t_1) \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & \cdots & 1 \end{bmatrix} + \cdots + (t_n - t_{n-1}) \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$= \sum_{k=1}^{n} \mu_k a_k a_k^{\mathrm{T}},$$

where $\mu_k = t_k - t_{k-1}$. As $\det(T) > 0$, $R(t_1, t_2) = \min(t_1, t_2)$ is an auto-correlation function.

Problem 2

$X(t) = N(0,t)$ is a Poisson process with parameter λt, i.e., k arrivals in $(0,t)$ is governed by

$$P[X(t) = k] = e^{-\lambda t}\frac{(\lambda t)^k}{k!}.$$

a.) Show that "the duration of the first arrival" is an exponential random variable;

b.) Show that "the duration of the n-th arrival" is a Gamma distributed random variable. Find its parameters.

Solution:

a.) Let τ_1 denote the time to the first arrival. Then,

$$P[\tau_1 > t] = P[N(0,t) = 0] = e^{-\lambda t}.$$

Thus,

$$F_{\tau_1}(t) = 1 - P[\tau_1 > t] = 1 - e^{-\lambda t}, \quad f_{\tau_1}(t) = \frac{dF_{\tau_1}(t)}{dt} = \lambda e^{-\lambda t}, \quad t \geq 0.$$

So τ_1 is exponentially distributed.

b.) Similarly with *a.*), let τ_n denote the time to the n-th arrival.

$$P[\tau_n > t] = P[N(0,t) \leq n-1] = \sum_{k=0}^{n-1} P[X(t) = k].$$

Hence,

$$F_{\tau_n}(t) = 1 - P[\tau_n > t] = 1 - \sum_{k=0}^{n-1} e^{-\lambda t}\frac{(\lambda t)^k}{k!},$$

and

$$\begin{aligned} f_{\tau_n}(t) = \frac{dF_{\tau_n}(t)}{dt} &= \sum_{k=0}^{n-1}\left(-ke^{-\lambda t}\frac{\lambda^k t^{k-1}}{k!} +\right) + \lambda\sum_{k=0}^{n-1} e^{-\lambda t}\frac{(\lambda t)^k}{k!} \\ &= -\lambda\sum_{k=0}^{n-2} e^{-\lambda t}\frac{(\lambda t)^k}{k!} + \lambda\sum_{k=0}^{n-1} e^{-\lambda t}\frac{(\lambda t)^k}{k!} \\ &= \lambda^n e^{-\lambda t}\frac{t^{n-1}}{(n-1)!}. \end{aligned}$$

The parameters are $\alpha = n$ and $\beta = \lambda$.

Problem 3

Let

$$X(t) \; = \; e^{j(\omega_0 t + W(t))},$$

where $W(t)$ is a Wiener process and ω_0 is a constant. Calculate the mean and auocorrelation function of $X(t)$.

Solution:

$$\begin{aligned}
R_{XX}(t_1,t_2) &= E[X(t_1)X^*(t_2)] \\
&= E\left[e^{j(\omega_0 t_1 + W(t_1))} e^{-j(\omega_0 t_2 + W(t_2))}\right] \\
&= e^{j\omega_0(t_1-t_2)} E\left[e^{j(W(t_1)-W(t_2))}\right].
\end{aligned}$$

Let

$$z = W(t_1) - W(t_2) \sim \mathcal{N}(0, \sigma_Z^2), \quad \sigma_Z^2 = E[(W(t_1)-W(t_2))^2] = \begin{cases} t_2 - t_1 & t_2 \geq t_1, \\ t_1 - t_2 & \text{otherwise.} \end{cases}$$

Hence,

$$R_{XX}(t_1,t_2) = e^{j\omega_0(t_1-t_2)} e^{j\cdot j\frac{\sigma_Z^2}{2}} = e^{j\omega_0(t_1-t_2)} e^{-\frac{|t_2-t_1|}{2}}.$$

Problem 4

Show that

a.) all strict sense stationary processe are wide sense stationary;

b.) wide sense stationarity implies strict sense stationarity for Gaussian process.

Solution:

a.) $X(t)$ is S.S.S. means that with $t_1, t_2, ..., t_n$ and $X(t_i) = X_i$, $X(t_i + \Delta t) = \tilde{X}_i$,

$$f_X(x_1, x_2, ..., x_n; t_1, t_2, ..., t_n) = f_{\tilde{X}}(x_1, x_2, ..., x_n; t_1, t_2, ..., t_n), \quad \forall \, \Delta t.$$

Thus $f_X(x, t) = f_X(x)$, and

$$f_{X_1 X_2}(x_1, x_2; t_1, t_2) = f_{X_1 X_2}(x_1, x_2; \tau), \quad \tau = t_2 - t_1.$$

Hence,

$$E[X(t)] = \int x f_X(x) dx = \text{Constant},$$
$$R_{X_1 X_2}(t_1, t_2) = \int \int x_1 x_2 f_{X_1 X_2}(x_1, x_2; \tau) dx_1 dx_2 = \rho(\tau), \quad \tau = t_2 - t_1.$$

So $X(t)$ is also W.S.S.

b.) For a Gaussian process $X(t)$, its mean and auto-correlaition function define the process. Let $X_i = X(t_i)$, $i = 1, 2, ..., n$. The joint distribution is given by

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi)^{n/2} |\mathbf{R}|} e^{-\mathbf{x}^{\mathrm{T}} R^{-1} \mathbf{x}},$$

where

$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} R_{11} & R_{12} & \cdots & R_{1n} \\ R_{21} & R_{22} & \cdots & R_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ R_{n1} & R_{n2} & \cdots & R_{nn} \end{bmatrix}, \quad R_{ij} = R_{XX}(t_i, t_j).$$

If $X(t)$ is W.S.S., then $R_{XX}(t_i, t_j) = R_{XX}(t_j - t_i)$ and $E[X(t)] =$ constant. Hence, the Gaussian process $X(t)$ is also S.S.S.

Problem 5

Let $X(t)$ be a zero-mean real stochastic process with

$$E[(X(t)-X(s))^2] = |t-s|, \quad \text{and} \quad X(0)=0.$$

a.) Show that

$$R_{XX}(t,s) = E[X(t)X(s)] = \frac{1}{2}(|t|+|s|-|t-s|) = \min(t,s).$$

b.) Define

$$Z_n(t) = n\left[X\left(t+\frac{1}{n}\right) - X(t)\right], \quad n = 1,2,...$$

Given n, find the auto-correlation function of $Z_n(t)$ and plot it. Is $Z_n(t)$ W.S.S.?

c.) What happens to $Z_n(t)$ when $n \to \infty$? (Does it appear to have the characteristics of any known stochastic process?)

Solution:

a.)

$$\begin{aligned}
R_{XX}(t,s) &= E[X(t)X(s)] \\
&= \frac{1}{2}\left(E[(X(t))^2] + E[(X(s))^2] - E[(X(t)-X(s))^2]\right) \\
&= \frac{1}{2}\left(E[(X(t)-X(0))^2] + E[(X(s)-X(0))^2] - E[(X(t)-X(s))^2]\right) \\
&= \frac{1}{2}(|t|+|s|-|t-s|) \\
&= \begin{cases} t & s \geq t, \\ s & \text{otherwise} \end{cases} \\
&= \min(t,s).
\end{aligned}$$

b.)

$$\begin{aligned}
R_{ZZ}(t,s) &= E[Z(t)Z(s)] \\
&= n^2 E\left[\left(X\left(t+\frac{1}{n}\right) - X(t)\right)\left(X\left(s+\frac{1}{n}\right) - X(s)\right)\right] \\
&= n^2\left(R_{XX}\left(t+\frac{1}{n}, s+\frac{1}{n}\right) - R_{XX}\left(t+\frac{1}{n}, s\right) - R_{XX}\left(t, s+\frac{1}{n}\right) - R_{XX}(t,s)\right) \\
&= \frac{n^2}{2}(|t-s+1/n| + |t-s-1/n| - 2|t-s|) \\
&= R_{ZZ}(t-s).
\end{aligned}$$

Hence, $Z(t)$ is W.S.S.

c.) As $n \to \infty$,

$$R_{ZZ}(t-s) = \delta(t-s).$$

Thus,

$$\lim_{n\to\infty} Z_n(t) = W(t), \quad \text{White Noise.}$$