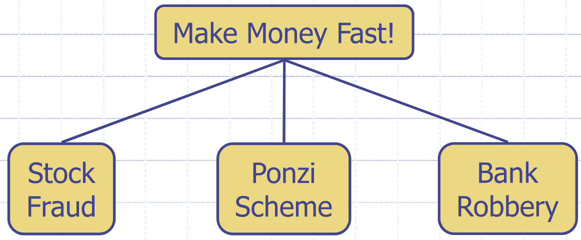
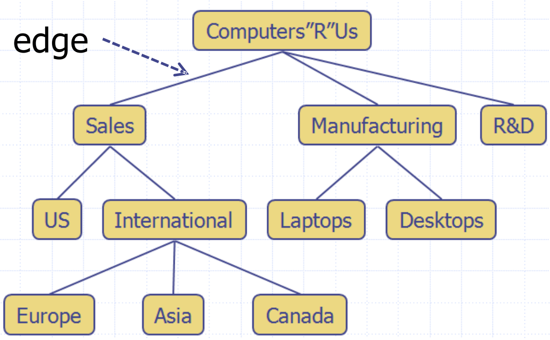
Trees

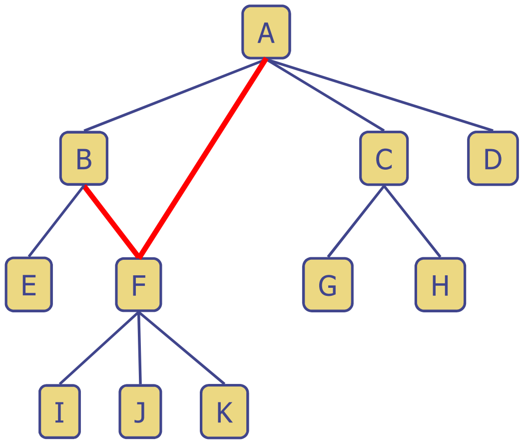


What is a Tree

- A tree is an abstract model of a hierarchical structure
- A tree T is a collection of nodes with nonlinear structure, called a ***parent-child*** relation
  - If nonempty, it has a special node, called the ***root*** of T, that has no parent
  - Each node v of T, except root, has a unique ***parent*** node w; every node with parent w is a ***child*** of w
- A unique ***path*** exists from the root to every other node.
- Applications:
  - Organization charts
  - File systems

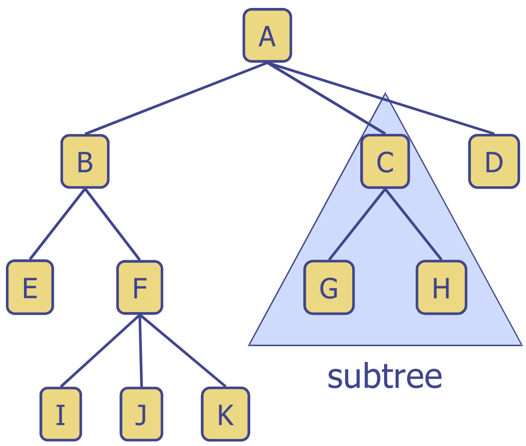


Not a Tree



Tree Terminology

- **Root**: node without parent (A)
- **Internal node**: node with at least one child (A, B, C, F)
- **External node** (a.k.a. **leaf**): node without children (E, I, J, K, G, H, D)
- **Ancestors** of a node: parent, grandparent, grand-grandparent, etc.
- **Descendants** of a node: child, grandchild, grand-grandchild, etc.
- **Subtree**: tree consisting of a node and its descendants
- **Ordered tree**: linear ordering defined for children of each node



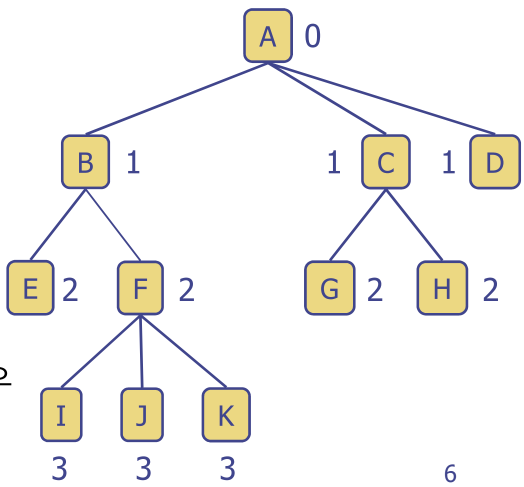
Tree ADT

- We use positions to abstract nodes (same as node in tree)
- Generic methods:
  - integer size()
  - boolean isEmpty()
  - Iterator iterator()
  - Iterable positions()
- Accessor methods:
  - position root()
  - position parent(p)
  - Iterable children(p)
- Query methods:
  - boolean isInternal(p)
  - boolean isExternal(p)
  - boolean isRoot(p)
- Update method:
  - element replace (p, o)
- Additional update methods may be defined by data structures implementing the Tree ADT

Depth of a Node

- The depth of a node is the number of its ancestors, excluding itself
  - depth(A) = 0, depth(B) = 1, depth(J) = 3

Algorithm *depth*(T, *v*)
    if *T.isRoot(v)*
        return *0*
    else
        return *1+depth(T, T.parent(v))*

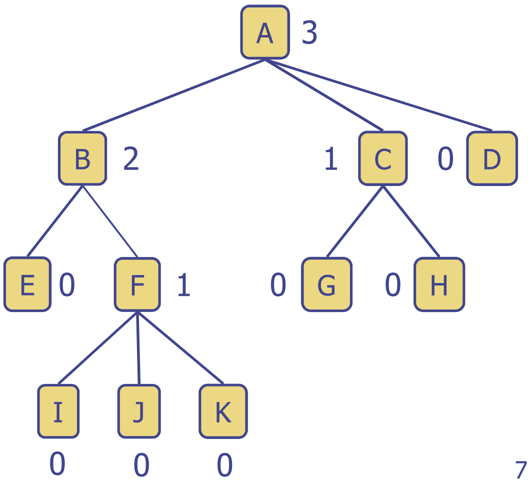


트리가 n 개의 노드로 구성되어 질 때,
worst case는 모든 노드의 child 가 하나일 경우 O(n)이다.

Height of a Node

- The height(v) in a tree T is
  - If v is an external node, then height(v) = 0
  - Otherwise, height(v) = 1 + max. height of its children

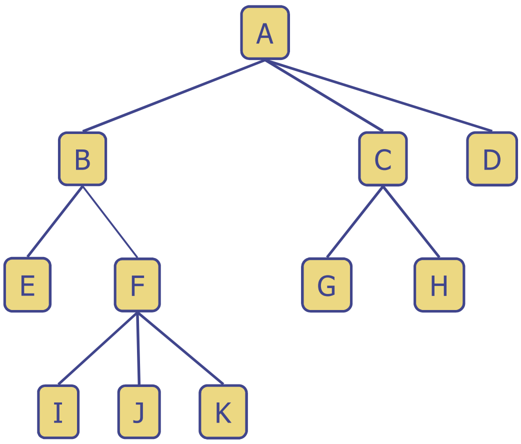
```
Algorithm height(T, v)
    if T.isExternal(v)
      return 0
    else
      h ← 0
      for each child w of v in T do
        h ← max(h, height(T, w))
      return 1+h
```



Height of a Tree

- The height of a tree T is the ***height of the root***
- The height of a tree T is equal to the ***maximum depth of a external node*** of T

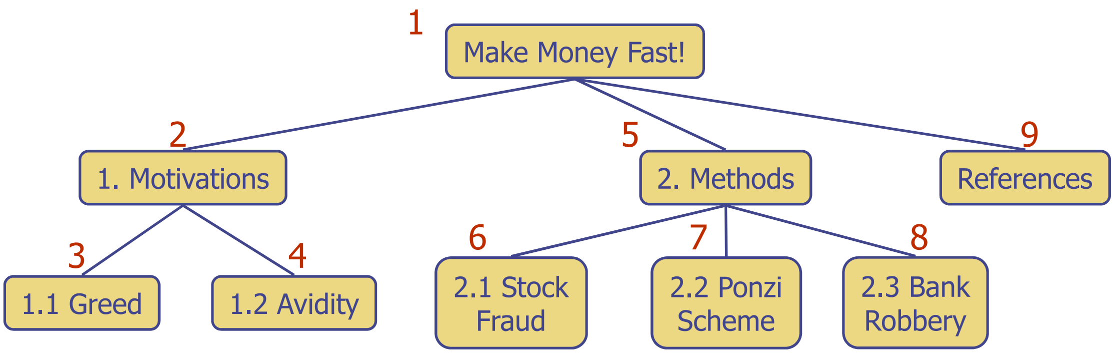
```
Algorithm height(T)
    h ← 0
    for each node v in T do
       if T.isExternal(v) then
         h ← max(h, depth(T, v))
      return h
```



Preorder Traversal

- A traversal visits the nodes of a tree in a systematic manner
- In a preorder traversal, a node is visited before its descendants
- Application: print a structured document

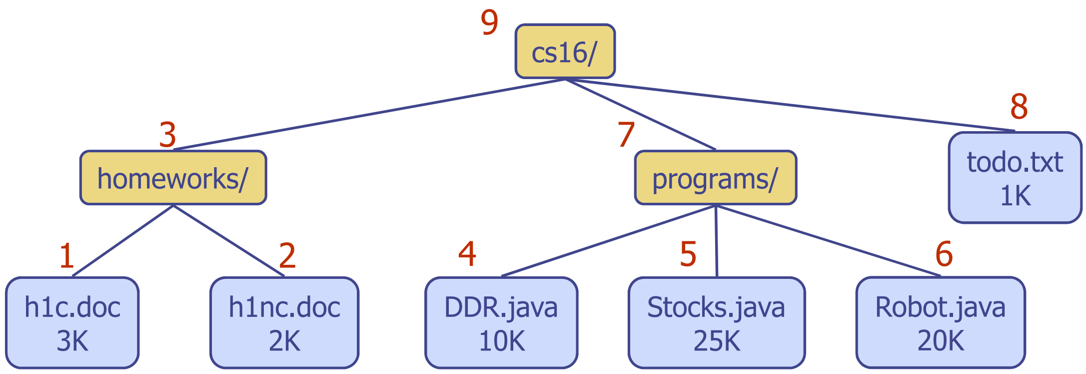
Algorithm *preOrder(v)*
    visit(v)
    for each child ***w*** of ***v***
        preorder (*w*)



Postorder Traversal

- In a postorder traversal, a node is visited after its descendants
- Application: compute space used by files in a directory and its subdirectories

Algorithm *postOrder(v)*
    for each child ***w*** of ***v***
        postOrder (*w*)
    visit(v)



Breadth-First (or Level-order) Traversal

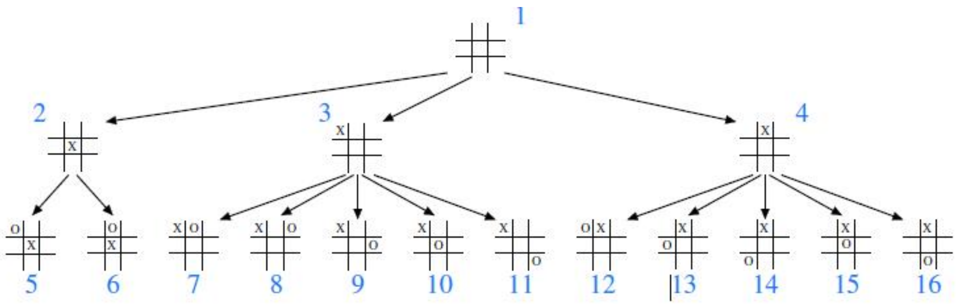


Figure 8.15: Partial game tree for Tic-Tac-Toe when ignoring symmetries; annotations denote the order in which positions are visited in a breadth-first tree traversal.

Algorithm breadthfirst():
  Initialize queue Q to contain root()
  while Q not empty **do**
    $p = Q$.dequeue()   { p is the oldest entry in the queue }
    perform the "visit" action for position p
    for each child c in children(p) **do**
      Q.enqueue(c)   { add p's children to the end of the queue for later visits }

Code Fragment 8.14: Algorithm for performing a breadth-first traversal of a tree.

Table of Contents?

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Paper          Paper               Electronics R'Us
Title            Title               1 R&D
Abstract         Abstract            2 Sales
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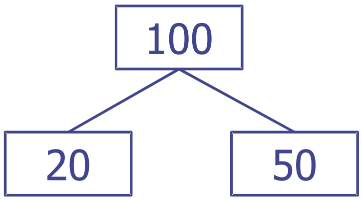
§1.1               §1.1                2.2 International
§1.2               §1.2                  2.2.1 Canada
§2               §2                      2.2.2 S. America
§2.1               §2.1
...                ...
(a)            (b)
```

```
/** Prints preorder representation of subtree of T rooted at p having depth d. */
public static <E> void printPreorderIndent(Tree<E> T, Position<E> p, int d) {
  System.out.println(spaces(2*d) + p.getElement());      // indent based on d
  for (Position<E> c : T.children(p))
    printPreorderIndent(T, c, d+1);                      // child depth is d+1
}
```

Computing Disk Space?

```java
/** Returns total disk space for subtree of T rooted at p. */
public static int diskSpace(Tree<Integer> T, Position<Integer> p) {
  int subtotal = p.getElement();        // we assume element represents space usage
  for (Position<Integer> c : T.children(p))
    subtotal += diskSpace(T, c);
  return subtotal;
}
```

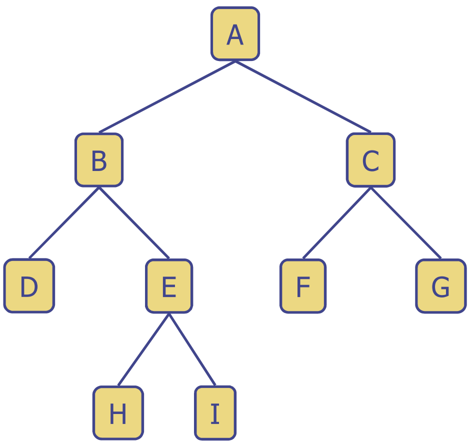
Code Fragment 8.25: Recursive computation of disk space for a tree. We assume that each tree element reports the local space used at that position.



Binary Trees

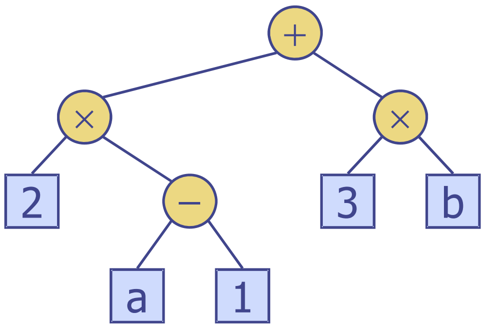
- A binary tree is an ***ordered tree*** with the following properties:
  - Each internal node has at most two children (exactly two for proper binary trees)
  - The children of a node are an ***ordered pair***
- We call the children of an internal node left child and right child
- Alternative recursive definition: a binary tree is either
  - a tree consisting of a single node, or
  - a tree whose root has an ordered pair of children, each of which is a binary tree

- Applications:
  - arithmetic expressions
  - decision processes
  - searching



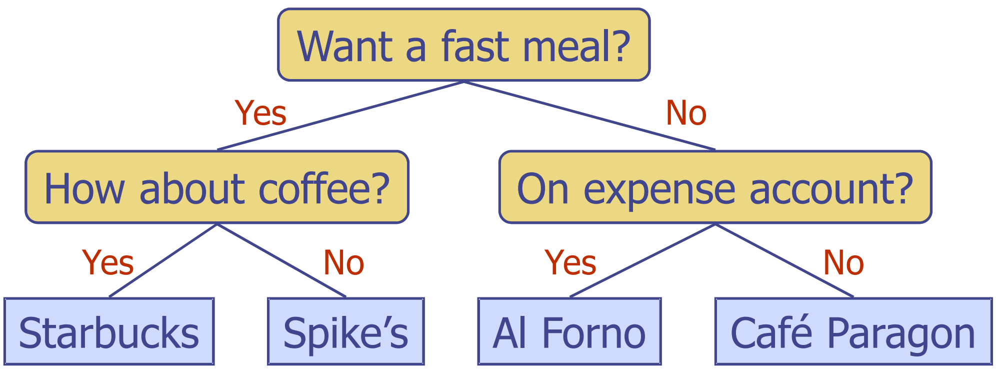
Arithmetic Expression Tree

- Binary tree associated with an arithmetic expression
  - internal nodes: operators
  - external nodes: operands
- Example: arithmetic expression tree for the expression $(2 \times (a - 1) + (3 \times b))$

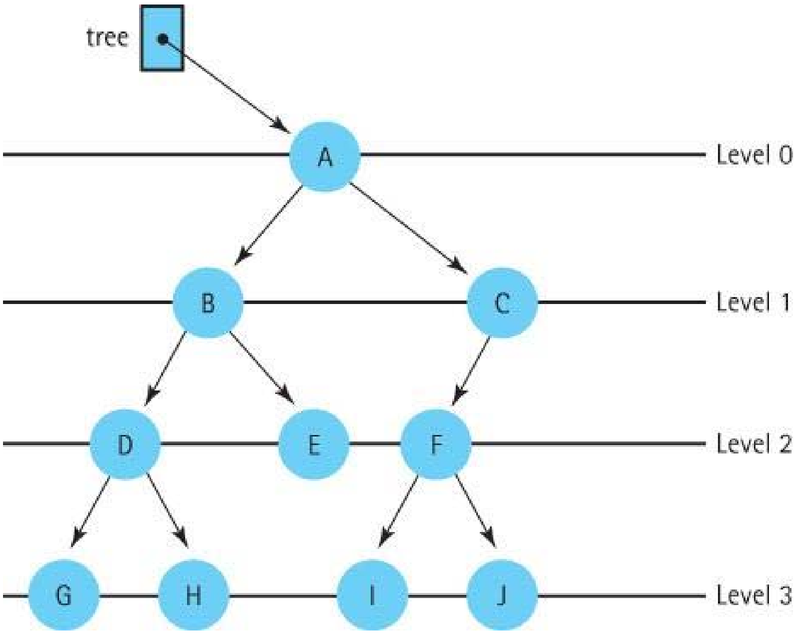


Decision Tree

- Binary tree associated with a decision process
  - internal nodes: questions with yes/no answer
  - external nodes: decisions
- Example: dining decision

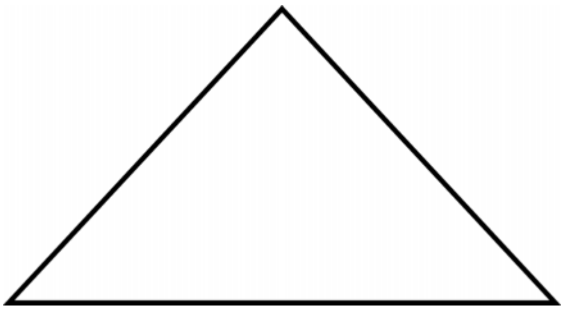


A Binary Tree and Levels



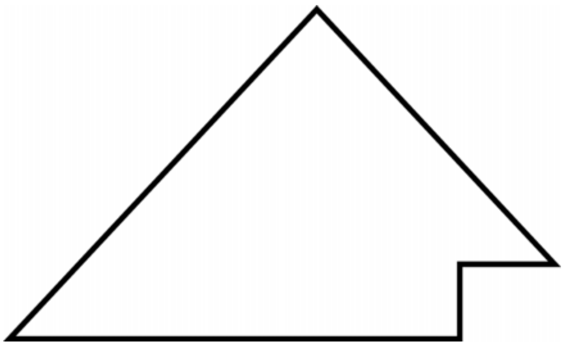
Full Binary Tree

- **Full Binary Tree:** A binary tree in which all of the leaves are on the same level and every nonleaf node has two children

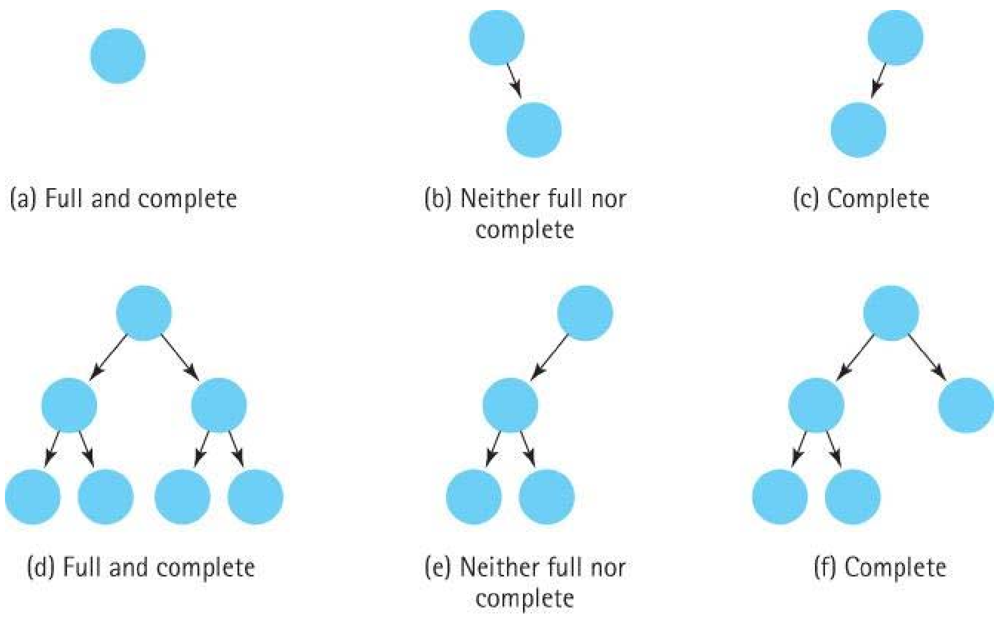


Complete Binary Tree

- **Complete Binary Tree:** A binary tree that is either full or full through the next-to-last level, with the leaves on the last level as far to the left as possible

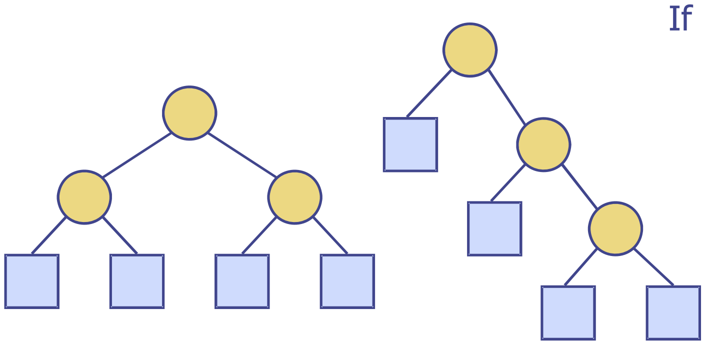


Examples of Different Types of Binary Trees



Properties of Binary Trees

- Notation
  - n number of nodes
  - n_e number of external nodes
  - n_i number of internal nodes
  - h height



- Properties:
  - $h+1 \le n \le 2^{h+1} - 1$
  - $1 \le n_e \le 2^h$
  - $h \le n_i \le 2^h - 1$
  - $\log_2 (n+1) - 1 \le h \le (n-1)$

If proper trees:

- $2h+1 \le n \le 2^{h+1} - 1$
- $h+1 \le n_e \le 2^h$
- $h \le n_i \le 2^h - 1$
- $\log_2 (n+1) - 1 \le h \le (n-1)/2$
- $n_e = n_i + 1$

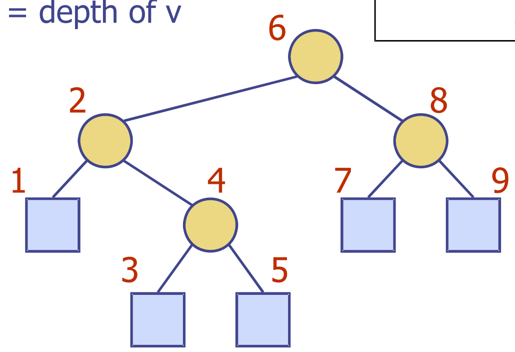
BinaryTree ADT

- The BinaryTree ADT extends the Tree ADT, i.e., it inherits all the methods of the Tree ADT
- Additional methods:
  - position left(p)
  - position right(p)
  - boolean hasLeft(p)
  - boolean hasRight(p)
- Update methods may be defined by data structures implementing the BinaryTree ADT

Inorder Traversal

- In an inorder traversal a node is visited after its left subtree and before its right subtree
- Application: draw a binary tree
  - x(v) = inorder rank of v
  - y(v) = depth of v

Algorithm *inOrder*(*v*)
    if *hasLeft* (*v*)
        inOrder (*left* (*v*))
    visit(*v*)
    if *hasRight* (*v*)
        inOrder (*right* (*v*))



Tree Drawing?

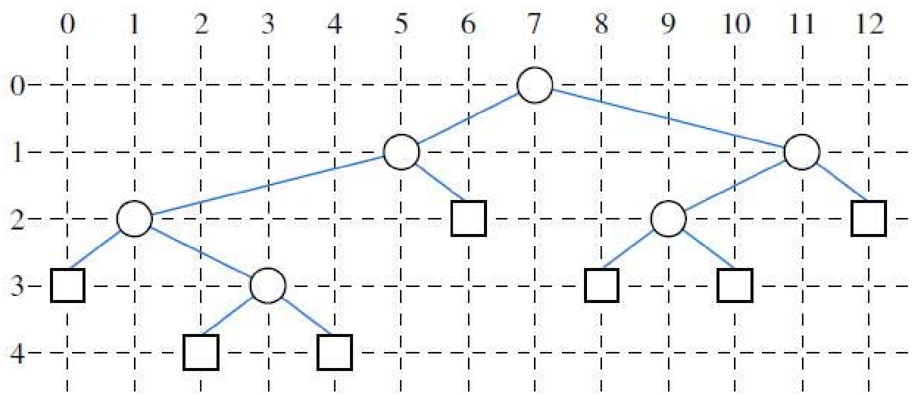
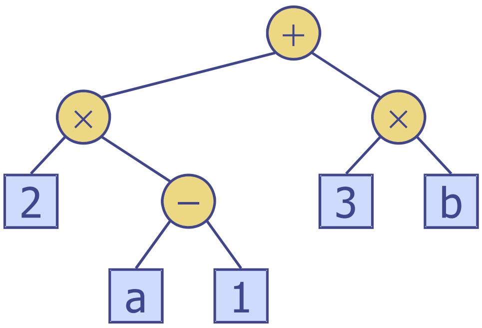


Figure 8.19: An inorder drawing of a binary tree.

Print Arithmetic Expressions

- Specialization of an inorder traversal
  - print operand or operator when visiting node
  - print "(" before traversing left subtree
  - print ")" after traversing right subtree



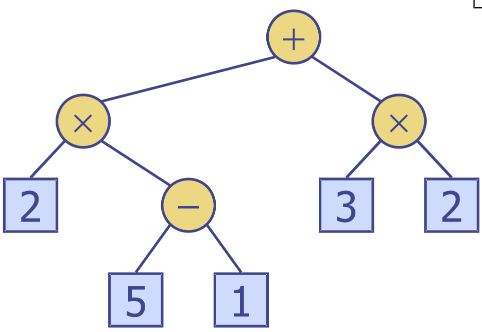
Algorithm *printExpression*(*v*)
    if *hasLeft* (*v*)
        print("(")
        inOrder (*left*(*v*))
    print(*v.element* ())
    if *hasRight* (*v*)
        inOrder (*right*(*v*))
        print (")")

$$((2 \times (a - 1)) + (3 \times b))$$

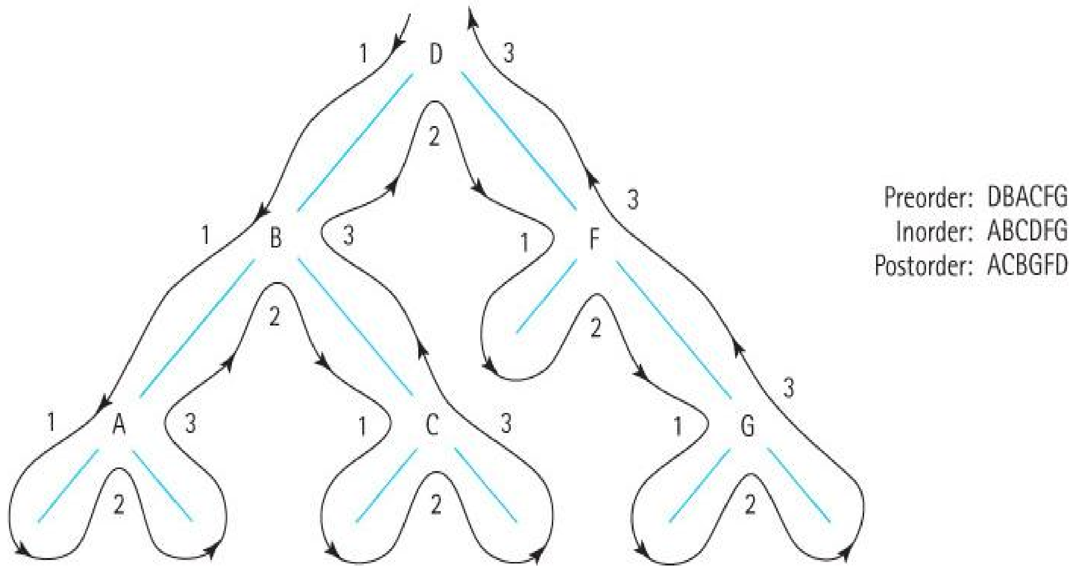
Evaluate Arithmetic Expressions

- Specialization of a postorder traversal
  - recursive method returning the value of a subtree
  - when visiting an internal node, combine the values of the subtrees

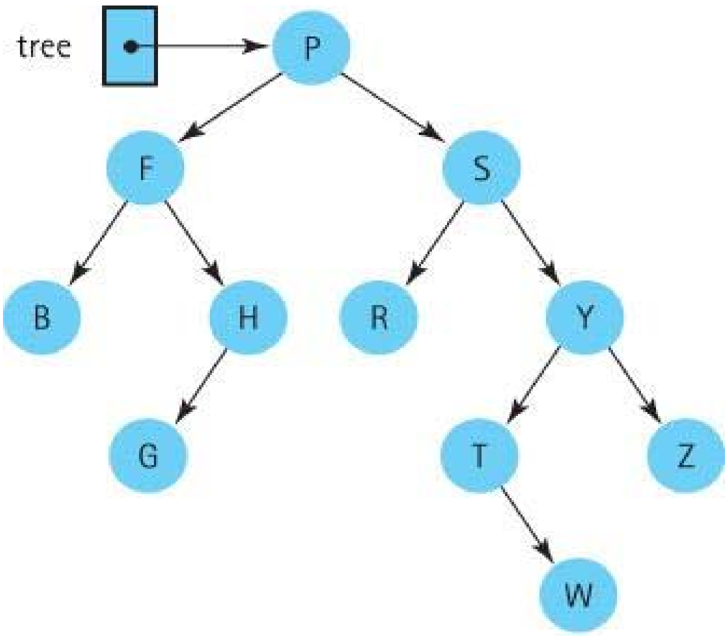
Algorithm *evalExpr*(*v*)
    if *isExternal* (*v*)
        return *v.element* ()
    else
        x ← *evalExpr*(*leftChild* (*v*))
        y ← *evalExpr*(*rightChild* (*v*))
        ◊ ← operator stored at *v*
        return *x* ◊ *y*



Euler Tour Traversal



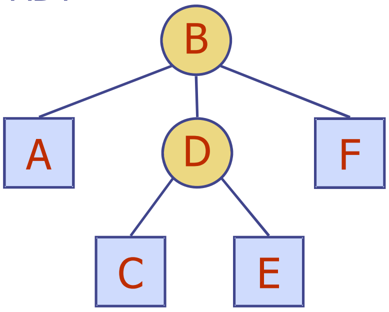
Three Binary Tree Traversals

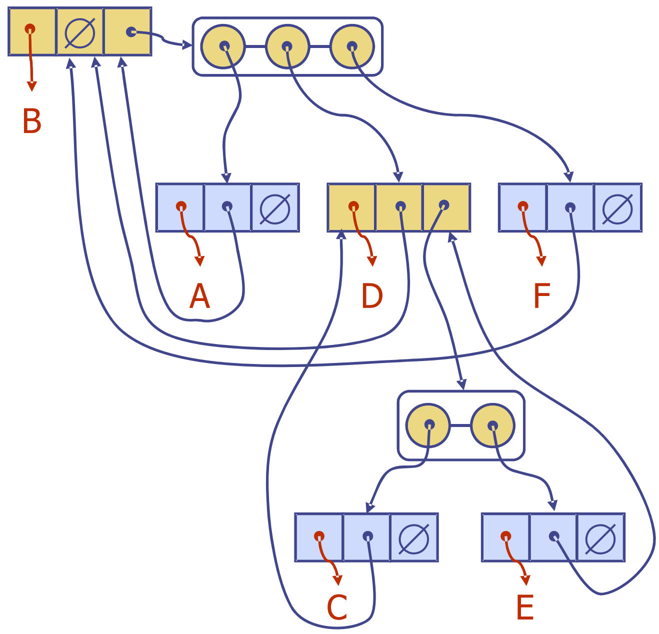


Inorder: B F G H P R S T W Y Z
Preorder: P F B H G S R Y T W Z
Postorder: B G H F R W T Z Y S P

Linked Structure for Trees

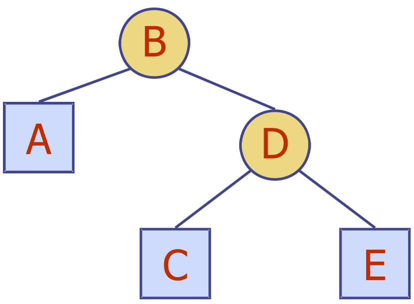
- A node is represented by an object storing
  - Element
  - Parent node
  - Sequence of children nodes
- Node objects implement the Position ADT

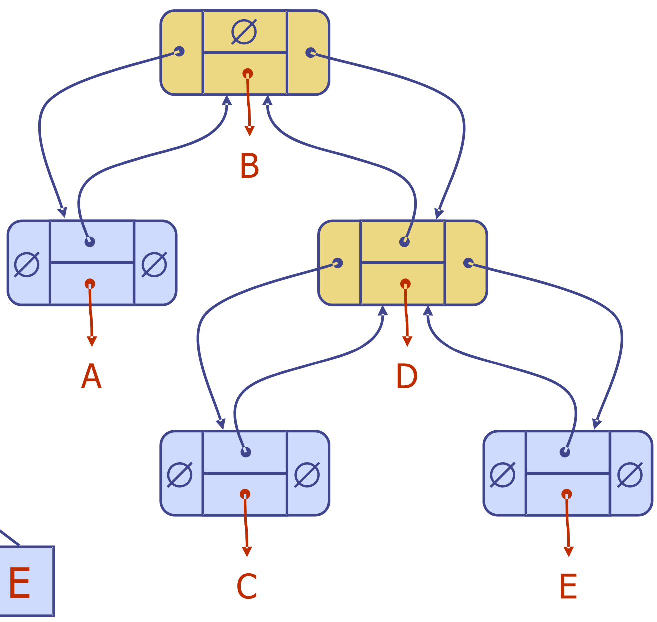




Linked Structure for Binary Trees

- A node is represented by an object storing
  - Element
  - Parent node
  - Left child node
  - Right child node
- Node objects implement the Position ADT

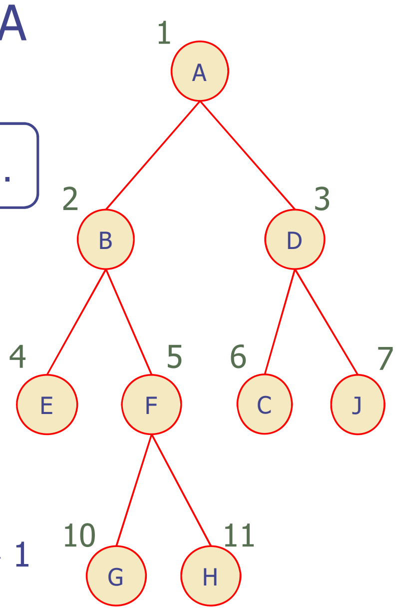




Array-Based Representation of Binary Trees

- Nodes are stored in an array A

	A	B	D	…	G	H	…
0	1	2	3		10	11	



- Node v is stored at A[rank(v)]
  - rank(root) = 1
  - if node is the left child of parent(node), rank(node) = 2 $\cdot$ rank(parent(node))
  - if node is the right child of parent(node), rank(node) = 2$\cdot$ rank(parent(node)) + 1