Geometric Assumption (Ricci Curvature) implies Topological Conclusion (Homology)

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Myers' Theorem

Suppose (M, g) is a complete Riemannian manifold.

$\mathrm{Ric} \geq \lambda g > 0$ $\longrightarrow$ $\pi_1(M) < \infty$ and M compact

The Splitting Theorem
(Cheeger-Gromoll)

If $\mathrm{Ric} \geq 0$ and M contains a line $\xrightarrow{\text{Cheeger-Gromoll Splitting Theorem}}$ $M^n = \mathbb{R} \times N$, $g = dr^2 + g_0$

M complete, noncompact, $\mathrm{Ric} \geq 0$ Implies Homology Result
(Shen-Sormani)

Let M^n be complete and noncompact.

$\mathrm{Ric} > 0$ $\longrightarrow$ $H_{n-1}(M, \mathbb{Z}) = 0$

$\mathrm{Ric} \geq 0$ $\longrightarrow$ $H_{n-1}(M, \mathbb{Z}) = 0$ or $\mathbb{Z}$

Bakry Émery Ricci Curvature

$$\mathrm{Ric}_\varphi^m = \mathrm{Ric} + \mathrm{Hess}\,\varphi - \frac{1}{m}(\nabla\varphi)^* \otimes (\nabla\varphi)^*,$$

where $\varphi : M^n \to \mathbb{R}$,
m is a constant.

$\mathrm{Ric}_\varphi^m \xrightarrow{\text{Plug in } \varphi\text{=constant}} \mathrm{Ric}$

Myers' Theorem for $\mathrm{Ric}_\varphi^\infty$
(Fernández-López and García-Río)

Let (M, g) be a complete Riemannian manifold.

$\mathrm{Ric}_\varphi^\infty \geq \lambda g > 0$ and $|\varphi| \leq A$ $\longrightarrow$ $\pi_1(M) < \infty$ and M compact

Splitting Theorem for $\mathrm{Ric}_\varphi^\infty$
(Fang-Li-Zhang)

If $\mathrm{Ric}_\varphi^\infty \geq 0$ and $\nabla\varphi \to 0$ at ∞ $\longrightarrow$ $M^n = \mathbb{R} \times N$, $g = dr^2 + g_0$

M complete, noncompact, $\mathrm{Ric}_\varphi^{-(n-1)} \geq 0$ Implies Homology Result
(L.)

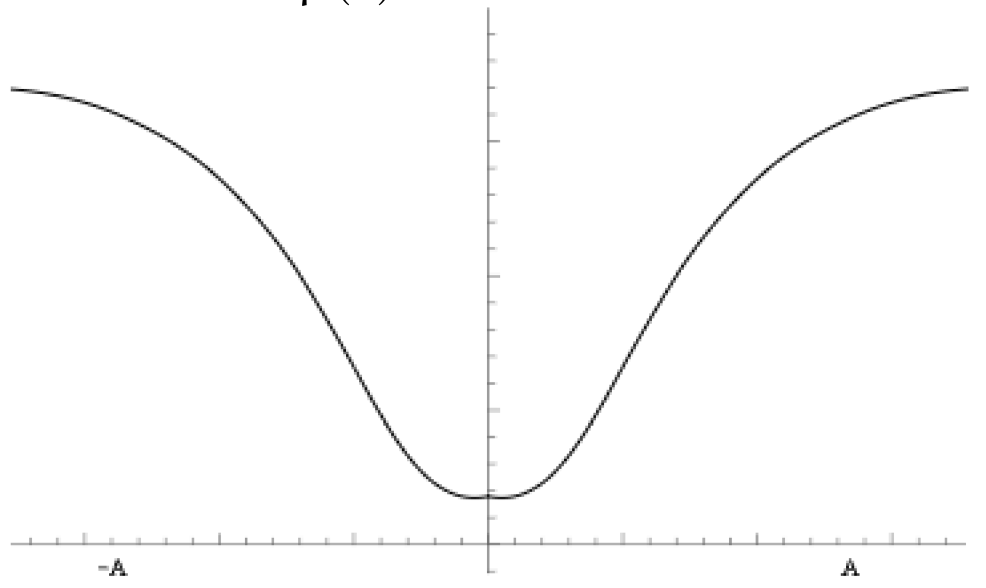
Let M^n be complete and noncompact.

$\mathrm{Ric}_\varphi^{1-n} > 0$, $\varphi \leq A$ $\longrightarrow$ $H_{n-1}(M, \mathbb{Z}) = 0$

$\mathrm{Ric}_\varphi^{1-n} \geq 0$, $|\varphi| \leq A$ $\longrightarrow$ $H_{n-1}(M, \mathbb{Z}) = 0$ or $\mathbb{Z}$

Example: $\nabla\varphi$ bounded, $\mathrm{Ric}_\varphi^\infty > 0$, Splitting Theorem doesn't hold, $H_{n-1}(M, \mathbb{Z}) = \mathbb{Z}$

$M = \mathbb{R} \times S^{n-1}$,
$g = dr^2 + \rho^2(r) g_N$.
Let V be a vector in the tangent space of S^{n-1}.
Let $\rho(r)$ look as follows:



We can choose $\varphi(r)$ such that the Splitting Theorem doesn't hold, $\nabla\varphi$ is bounded, $\mathrm{Ric}_\varphi^\infty > 0$.

Example: $\mathrm{Ric}_\varphi^m \geq 0$, φ unbounded, $H_{n-1}(M, \mathbf{Z}) \neq 0$

Consider $M = S^{n-1} \times \mathbb{R}$ with $m = 1 - n$.

Let $\varphi : \mathbb{R} \to \mathbb{R}$,
$r \to r^2$.

Then $\mathrm{Ric}_\varphi^m > 0$,
BUT $H_{n-1}(M, \mathbb{Z}) = \mathbb{Z}$.