Conic Sections

Conic Sections, Plane Curves, and Polar Coordinates

- describe a parabola, a hyperbola, and an ellipse geometrically and algebraically
- represent conic sections in standard, parametric and graphical forms
- write the equation of a conic given the graph
- write the equation and draw graph of a conic from given information
- describe and apply properties of conic sections in the physical world

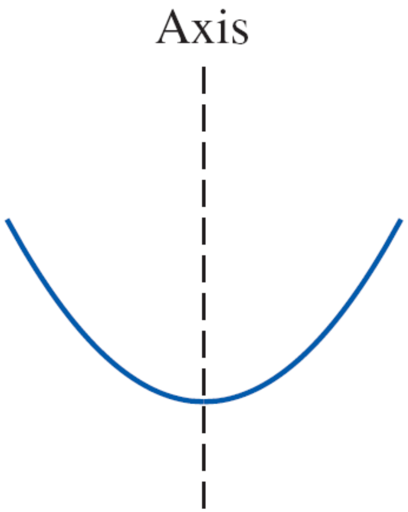
Conic Sections

Figure 1 shows the reflector of a radio telescope. The shape of the surface of the reflector is obtained by revolving a plane curve called a *parabola* about its axis of symmetry. (See Figure 2a.)

Photodisc/Getty Images

The reflector of a radio telescope

Figure 1



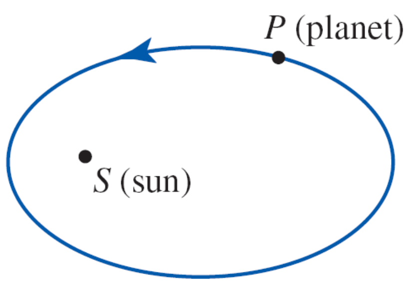
The cross section of a radio telescope is part of a parabola.

Figure 2(a)

Conic Sections

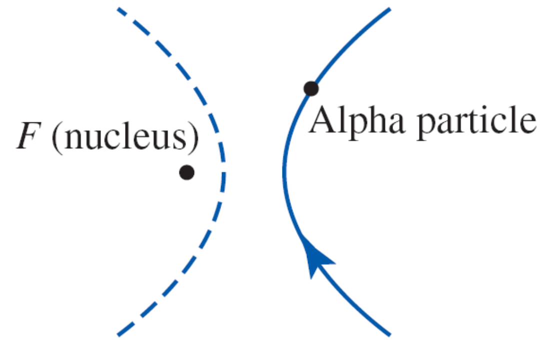
Figure 2b depicts the orbit of a planet P around the sun, S. This curve is called an *ellipse*.

Figure 2c depicts the trajectory of an incoming alpha particle heading toward and then repulsed by a massive atomic nucleus located at the point F. The trajectory is one of two branches of a *hyperbola*.



The orbit of a planet around the sun is an ellipse.

Figure 2(b)



The trajectory of an alpha particle in a Rutherford scattering is part of a branch of a hyperbola.

Figure 2(c)

Conic Sections

These curves—parabolas, ellipses, and hyperbolas—are called *conic sections* or, more simply, *conics* because they result from the intersection of a plane and a double-napped cone, as shown in Figure 3.

Parabola
(a)

Ellipse
(b)

Hyperbola
(c)

Figure 3
The conic sections

Parabola

A parabola is made up of points whose distances from a fixed point (focus) and a fixed line (directrix) are the same. Suppose the vertex, focus and directrix are as shown in Figure 5.

To find an equation of a parabola,

Focus $F = (0, p)$

Directrix : $y = -p$.

For a random point $P(x, y)$ on the parabola,

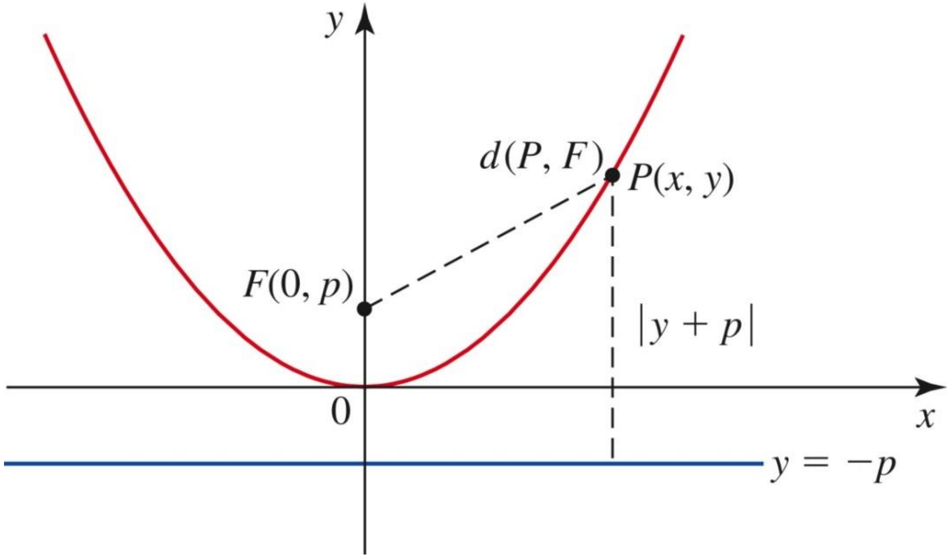


Figure 5

$$\sqrt{(x-0)^2 + (y-p)^2} = y + p$$

$$x^2 + (y-p)^2 = (y+p)^2$$

$$x^2 + y^2 - 2py + p^2 = y^2 + 2py + p^2$$

$$x^2 = 4py$$

Parabola

Standard Equation of a Parabola

An equation of the parabola with focus $(0, p)$ and directrix $y = -p$ is

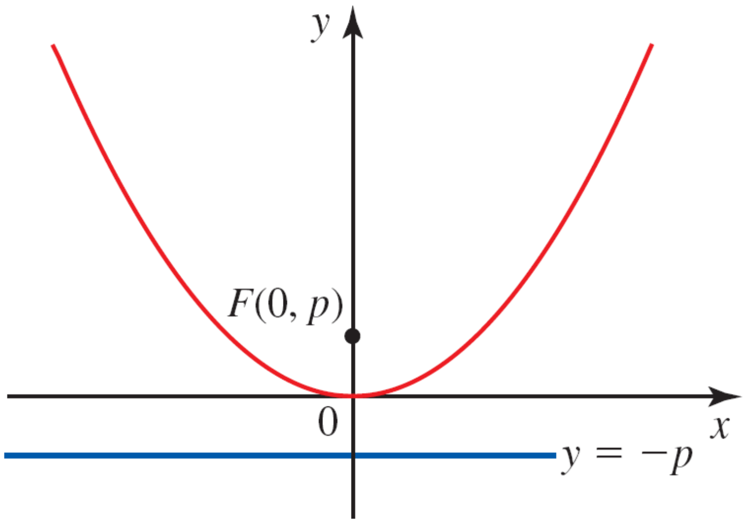
$$x^2 = 4py \tag{1}$$

Parabola

- By definition the point halfway between the focus and directrix lies on the parabola. This point V is called the **vertex** of the parabola.

- The line passing through the focus and perpendicular to the directrix is called the **axis** of the parabola.

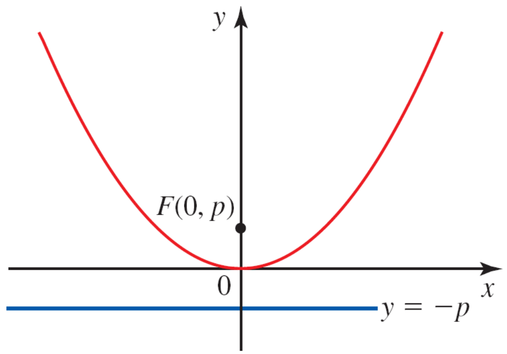
- Observe that the parabola is **symmetric** with respect to its axis.



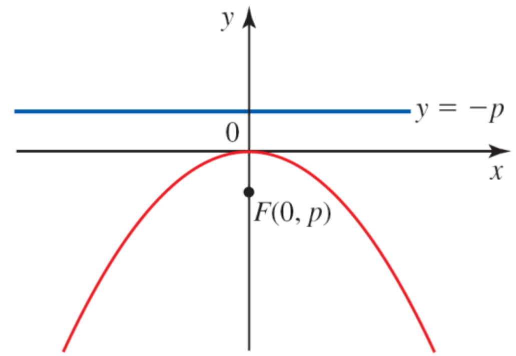
Parabola

- Observe that the parabola opens upward if $p > 0$ and opens downward if $p < 0$.

$$\boxed{x^2 = 4py}$$



$p > 0$

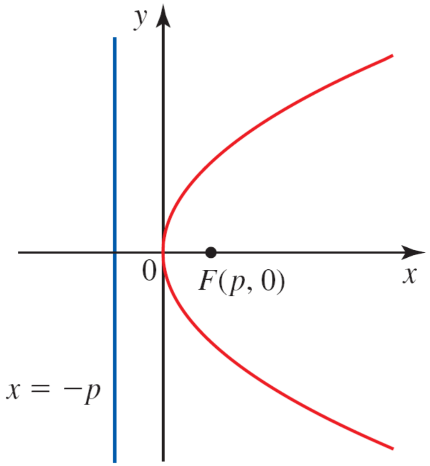


$p < 0$

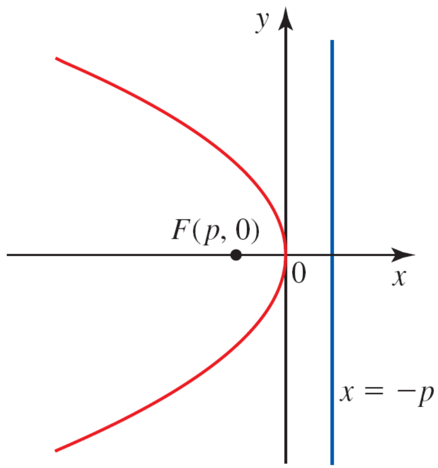
- The axis of the parabola coincides with the y-axis, since $x^2 = 4py$ remains unchanged if we replace x by $-x$.

Parabola

$$\boxed{y^2 = 4px}$$



$p > 0$



$p < 0$

- The parabola opens to the right if $p > 0$ and opens to the left if $p < 0$. In both cases the axis of the parabola coincides with the x-axis.

Example 1 [TB P830]

Find the focus and directrix of the parabola $y^2+6x=0$, and make a sketch of the parabola.

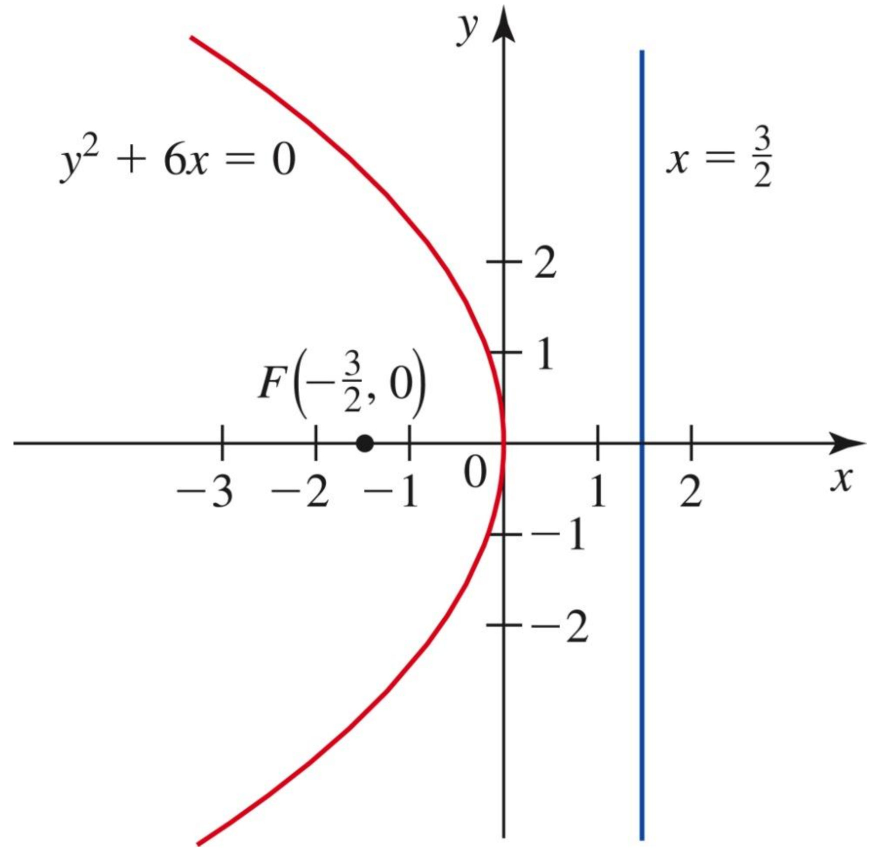
Solution:

$$y^2 = -6x \Rightarrow 4p = -6, \quad p = -\frac{3}{2}$$

axis of sym = x-axis

$$\Rightarrow \text{Vertex } (0,0), \text{Focus } (-\frac{3}{2},0),$$

and directrix, $x = \dfrac{3}{2}$



Example 2 [TB P830]

Find an equation of the parabola that has its vertex at the origin with axis of symmetry lying on the y-axis, and passes through the point $P(3, -4)$. What are the focus and directrix of the parabola?

Solution:

An equation of the parabola has the form $x^2 = 4py$.

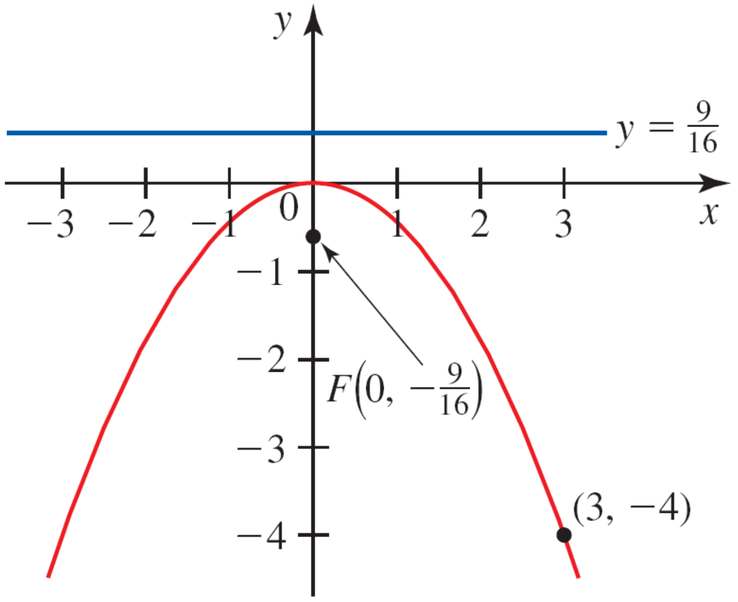
Since it passes thru $(3, -4)$,

$$3^2 = 4p(-4) \Rightarrow p = -\frac{9}{16}$$

$$x^2 = -\frac{9}{4}y$$

Focus: $F(0, p) = (0, -\frac{9}{16})$

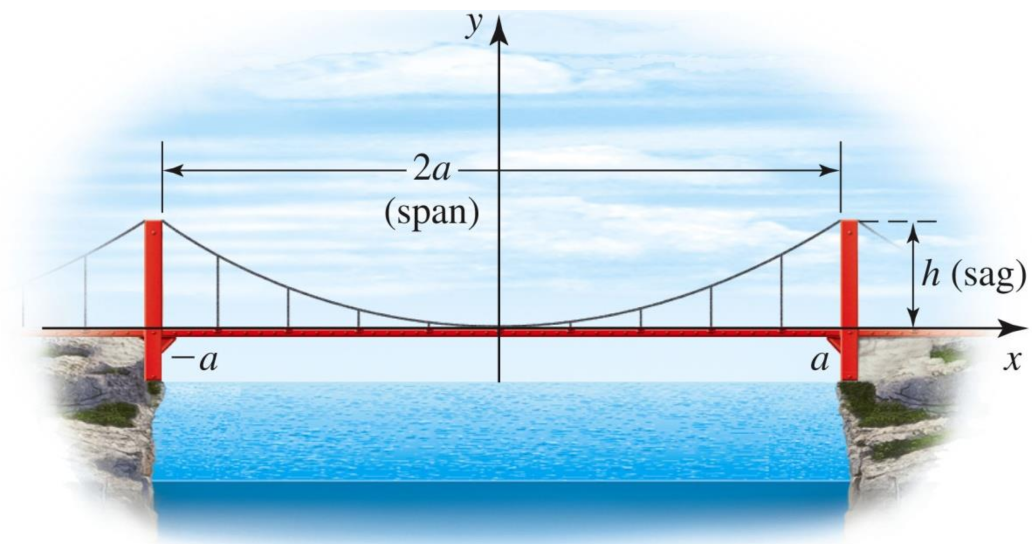
Directrix : $y = -p$, i.e. $y = \frac{9}{16}$



Example 3 [TB P831]

A bridge suspended by a flexible cable. If we assume the weight of the cable is neglibible in comparison to the weight of the bridge, then it can be shown that the shape of the cable is described by the equation $y = \frac{Wx^2}{2H}$ where W is the weight of the bridge in pounds per foot and H is the tension. Suppose that the span of the cable is $2a$ ft and the sag is h ft.

a) Find the equation of the cable in terms of a and h.



(a) Given $y = \dfrac{Wx^2}{2H}$ and (a, h) lies on the parabola,

$$h = \frac{Wa^2}{2H} \Rightarrow \frac{W}{2H} = \frac{h}{a^2}$$

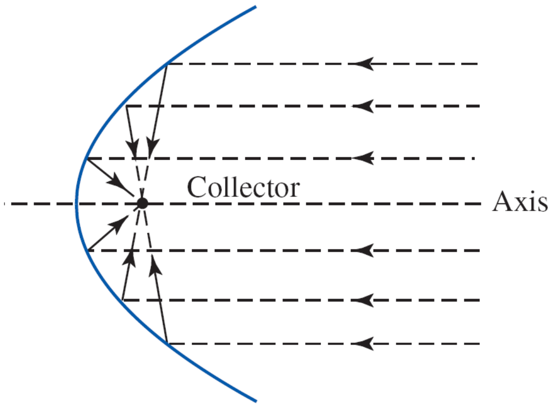
$$\therefore y = \frac{Wx^2}{2H} = \frac{W}{2H}x^2 = \frac{h}{a^2}x^2 = \frac{hx^2}{a^2}$$

Reflective Property of the Parabola

As was mentioned earlier, the reflector of a radio telescope has a shape that is obtained by revolving a parabola about its axis.

Figure on the right shows a cross section of such a reflector.

A radio wave coming in from a great distance may be assumed to be parallel to the axis of the parabola.



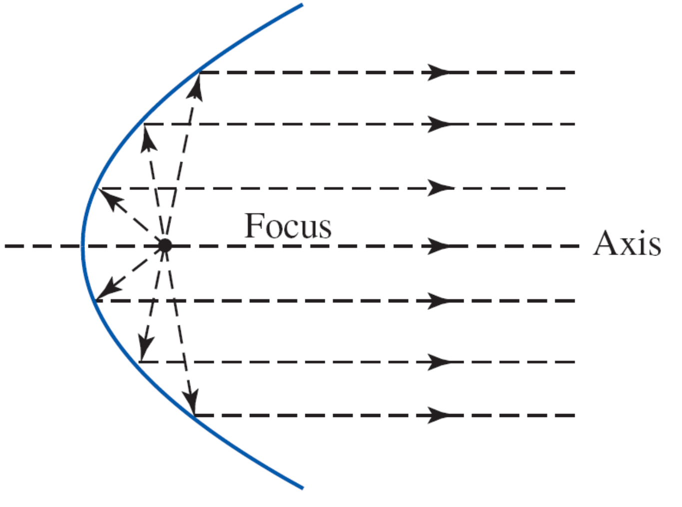
A cross section of a radio telescope

This wave will strike the surface of the reflector and be reflected toward the focus F, where a collector is located. (The angle of incidence is equal to the angle of reflection.)

Reflective Property of the Parabola

The reflective property of the parabola is also used in the design of headlights of automobiles.

Here, a light bulb is placed at the focus of the parabola. A ray of light emanating from the light bulb will strike the surface of the reflector and be reflected outward along a direction parallel to the axis of the parabola.



A cross section of a headlight