Functions

1.1 Introduction

1.2 Composite functions and inverse functions

- identify functions by definition and graph and state their domain and range
- use of functional notation and evaluate functions
- compute the composition of two functions and its domain
- recognize functions given their composition
- discuss the condition for an inverse function to exist
- define the inverse of a given function and the use of the notation f^{-1}
- relate the domain, range and graph of a function and its inverse
- recognize whether two functions are inverses of each other
- find the inverse of a (domain-restricted) function

Definition of a Function (TB p17)

DEFINITION **Function**

A **function** f from a set A to a set B is a rule that assigns to each element x in A one and only one element y in B.

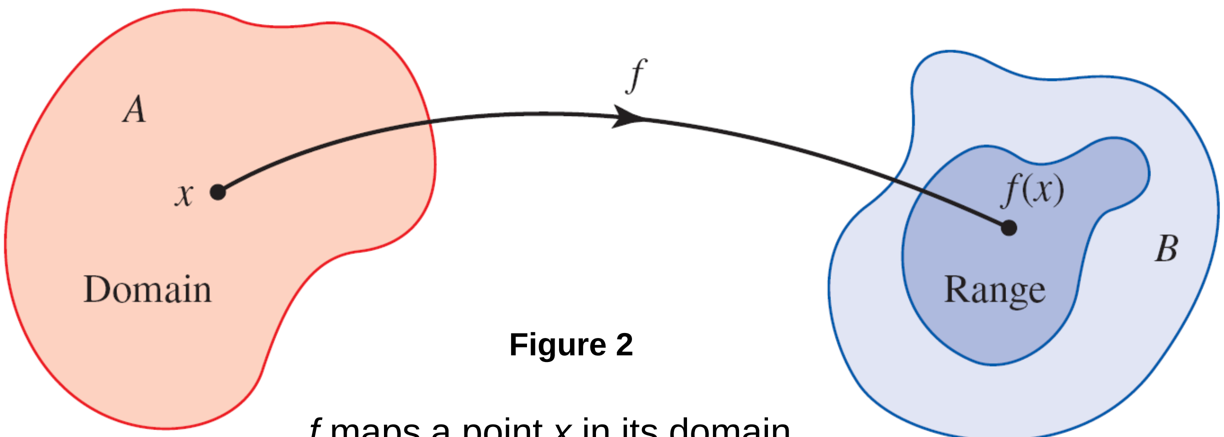


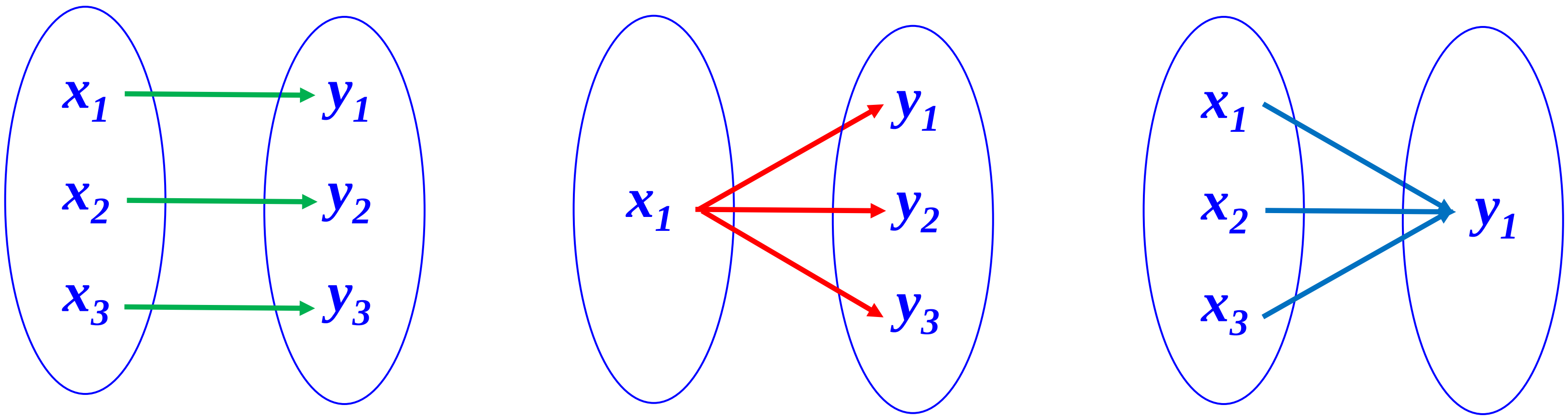
Figure 2

f maps a point x in its domain onto its image $f(x)$ in its range.

The set A is called the **domain** of the f, D_f. The element in B, called the **image** of x, is written $f(x)$ (and read "f of x"). The set of all values $y = f(x)$ as x varies over D_f is called the **range** of f, R_f.

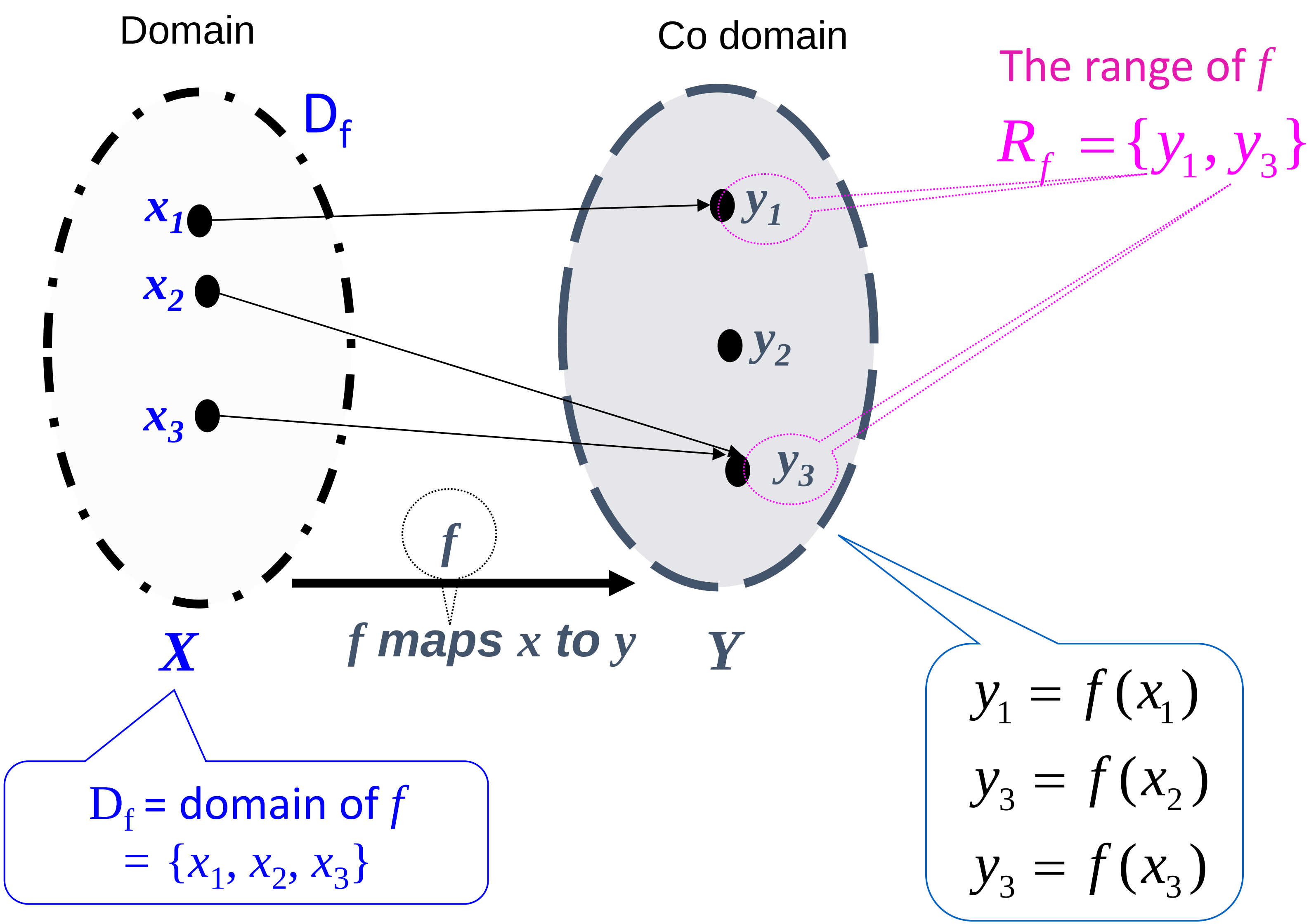
Definition of a Function (TB p17)

Function vs non-function



✓ **One-to-one** ✗ **One-to-many** ✓ **Many-to-one**

Definition of a Function (TB p17)



Describing functions (TB p17)

- Verbal

  Square root function

  $$f(x) = \sqrt{x}$$

- Table

TABLE 1 The function f giving the Manhattan hotel occupancy rate in year x

x (year)	$y = f(x)$ (percent)
0	81.1
1	83.7
2	74.5
3	75.0
4	75.9
5	83.2
6	84.9
7	85.1

Source: PricewaterhouseCoopers LLP.

- Graph

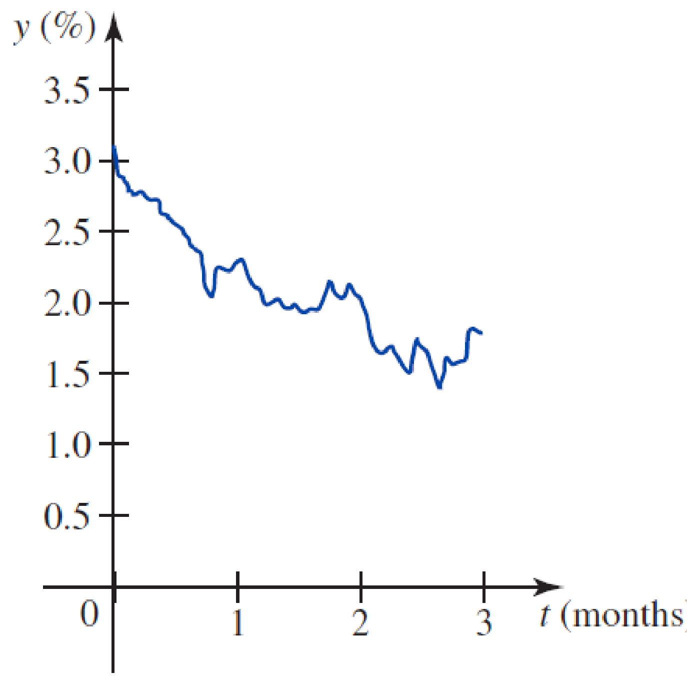


FIGURE 3
The function f gives the annual yield for two-year Treasury notes in the first three months of 2008.
Source: Financial Times.

Describing functions

- If a function f is described by an equation

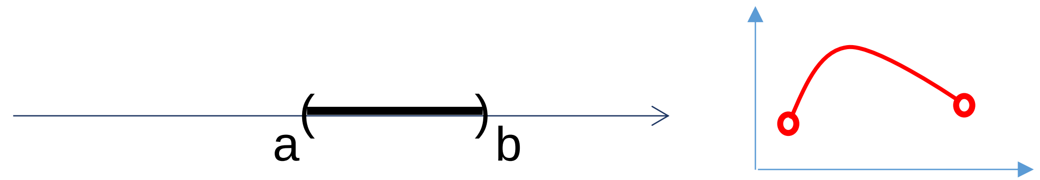
$$y = f(x),$$

  - x is the **independent variable**
  - y is the **dependent variable** because y depends upon the choice of x.

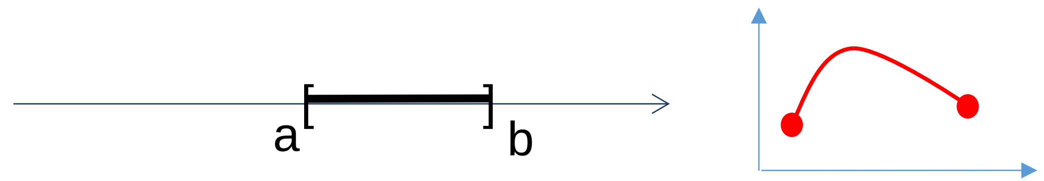
- x represents a number in D_f
- y is the unique number in R_f associated with x.

Some symbols to note

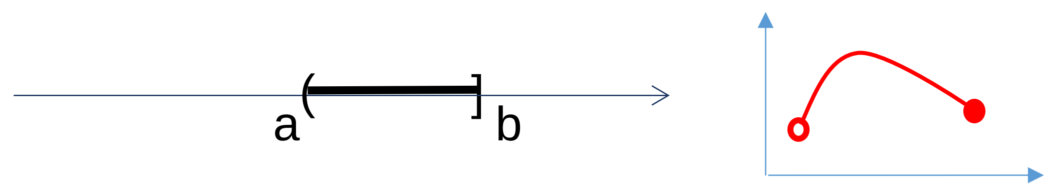
Open: (a, b)



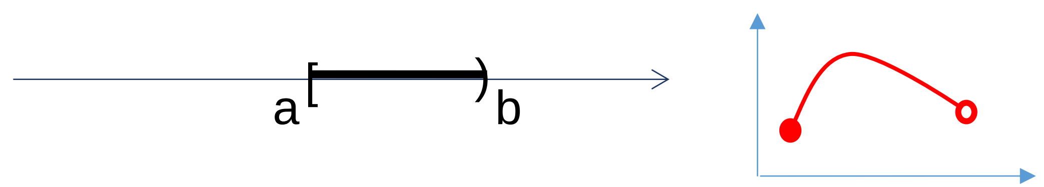
Closed: [a, b]



Half Open: (a, b]



Half Open: [a, b)



The difference is just whether the end-points are included

Example 1 [TB P18]

The function f defined by the function $y = \sqrt{x}$.

The domain of this function is the set of all values of x in the inteval $[0, \infty)$. The range of f consists of nonnegative numbers and is the set of all values in $[0, \infty)$.

$$f(x) = \sqrt{x}$$

$$Domain\ D_f = \{x : x \geq 0\} = [0, \infty)$$

$$Range\ R_f = \{x : x \geq 0\} = [0, \infty)$$

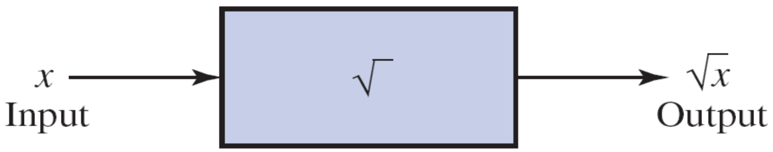


Figure 4a

The square root machine

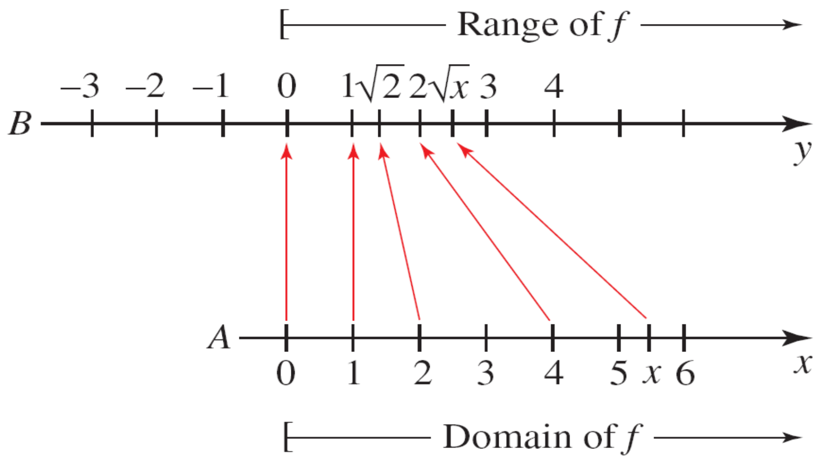


Figure 4b

The function *f* maps $\sqrt{x}$nto .

Example 2 [TB P19]

Let $f(x) = x^2 + 2x - 1$. Find

(a) f(-1)

(b) $f(\pi)$

(c) $f(t)$, where t is a real number

(d) $f(x+h)$, where h is a real number

(e) $f(2x)$

Solution

(a) $f(\text{-}1) = (-1)^2 + 2(-1) - 1 = -2$

(b) $f(\pi) = (\pi)^2 + 2(\pi) - 1 = \pi^2 + 2\pi - 1$

(c) $f(t) = t^2 + 2t - 1$

(d) $f(x+h) = (x+h)^2 + 2(x+h) - 1 = x^2 + 2xh + h^2 + 2x + 2h - 1$

(e) $f(2x) = (2x)^2 + 2(2x) - 1 = 4x^2 + 4x - 1$

Example 4 [TB P20]

Find the domain of the function:

(a) $f(x) = \dfrac{2x+1}{x^2 - x - 2}$ (b) $f(x) = \dfrac{x + \sqrt{x+1}}{2x-1}$

(a) Numerator has no constraint
Denominator cannot be 0

$$x^2 - x - 2 = 0$$
$$(x-2)(x+1) = 0$$
$$x = 2 \ or \ x = -1$$

Domain of $f(x)$ is of real numbers,
except -1 and 2

$D_f = (-\infty, -1) \cup (-1, 2) \cup (2, \infty)$

or $D_f = \{ x \mid x \in R, x \neq 2, x \neq -1 \}$

Example 4 [TB P20]

Find the domain of the function:

(a) $f(x) = \dfrac{2x+1}{x^2 - x - 2}$ (b) $f(x) = \dfrac{x + \sqrt{x+1}}{2x-1}$

(b) Expression under the square root must be ≥ 0.

$$x + 1 \geq 0$$
$$x \geq -1$$

Denominator cannot be 0

$$2x - 1 \neq 0$$
$$x \neq \frac{1}{2}$$

D_f = $[-1, \frac{1}{2}) \cup (\frac{1}{2}, \infty)$

or $D_f = \{ x \mid x \in R, x \geq -1, x \neq \frac{1}{2} \}$

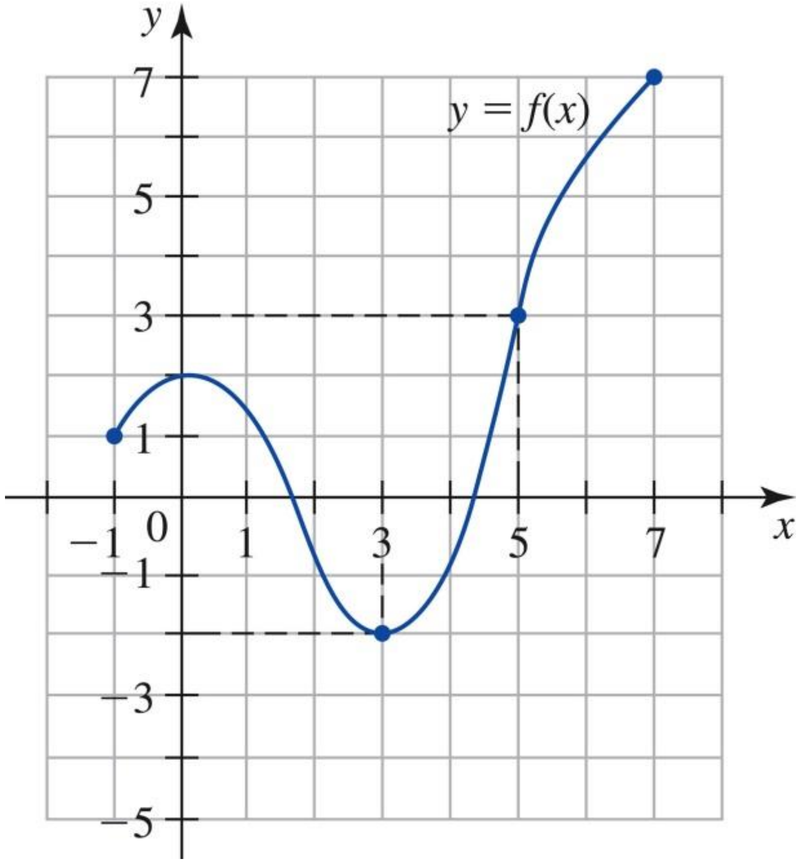
Example 5 [TB P20]

The graph of a function f is shown in Fig 7.

(a) What is $f(3)$, $f(5)$?

(b) What is the distance of the point $(3, f(3))$ from the x-axis?

(c) What is the domain and range of f ?



Solution

(a) $f(3) = -2$

$f(5) = 3$

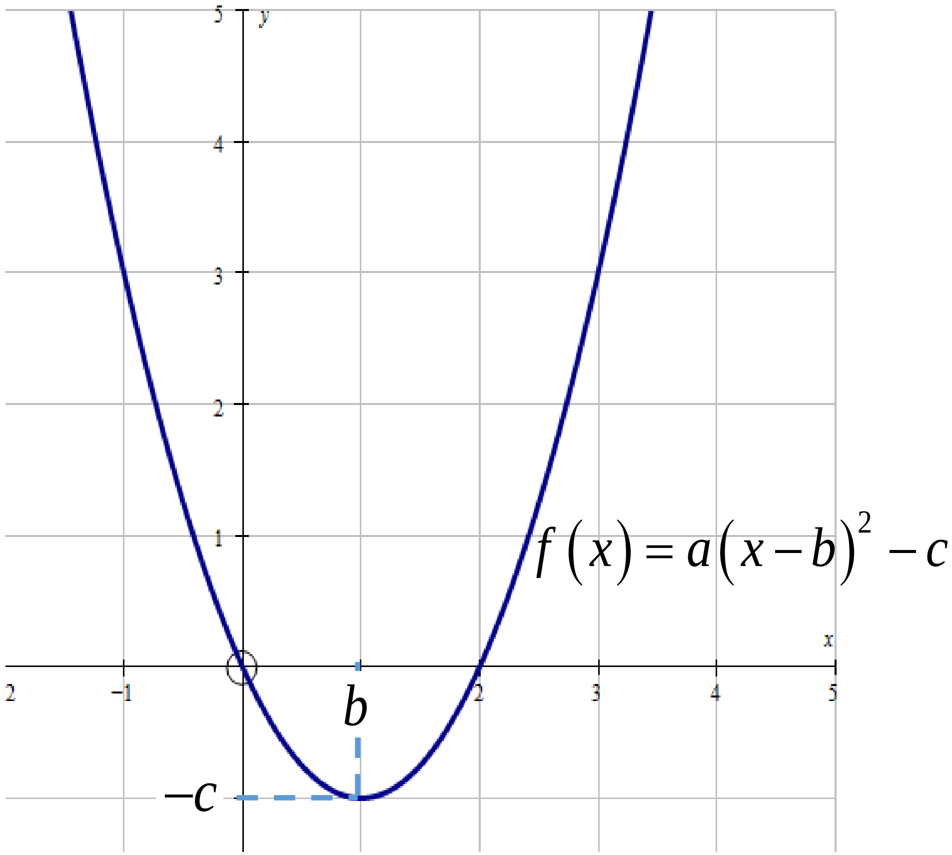
(b) Distance = 2

(c) Domain = $[-1, 7]$

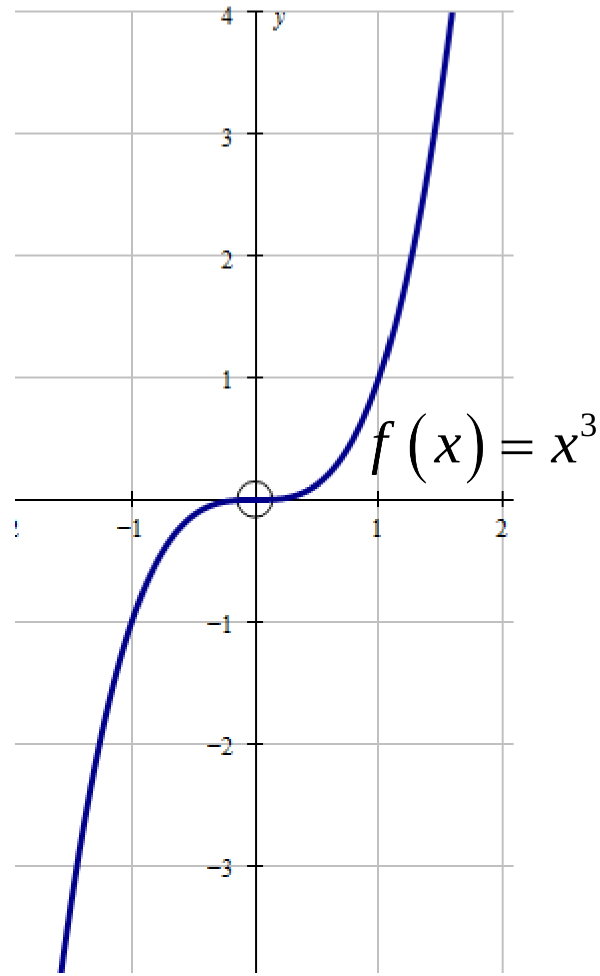
Range = $[-2, 7]$

Functions you may have seen

Graph of quadratic function

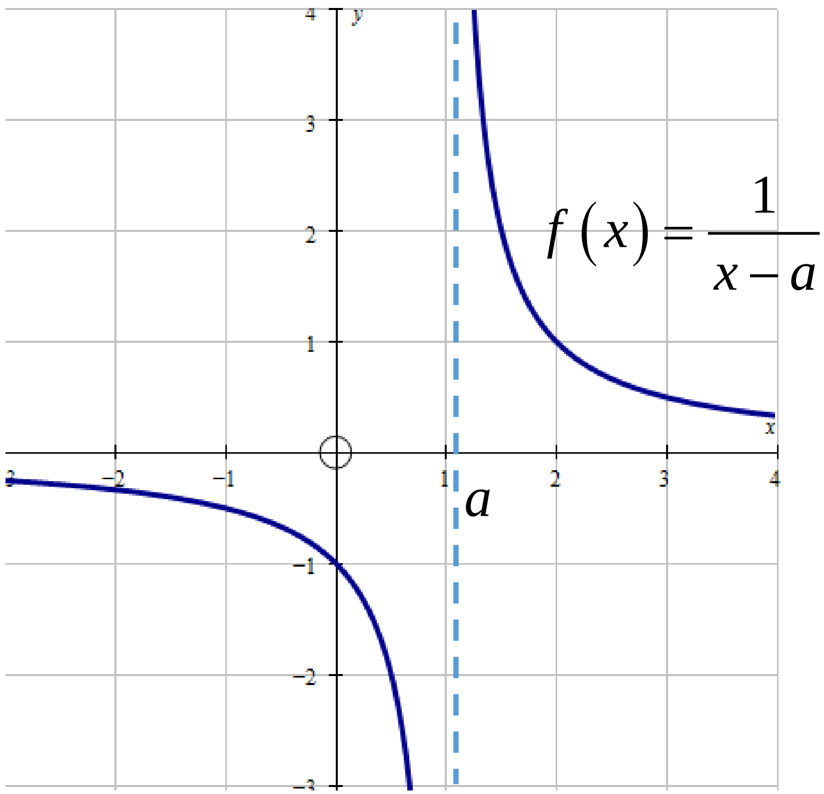


Graph of cubic function

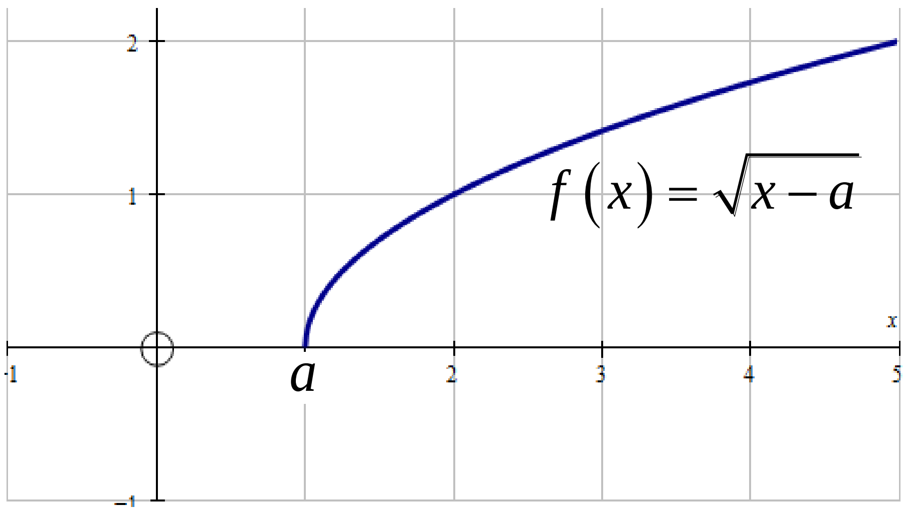


Functions you may have seen

Graph of reciprocal function



Graph of square root function



Transformation of Functions

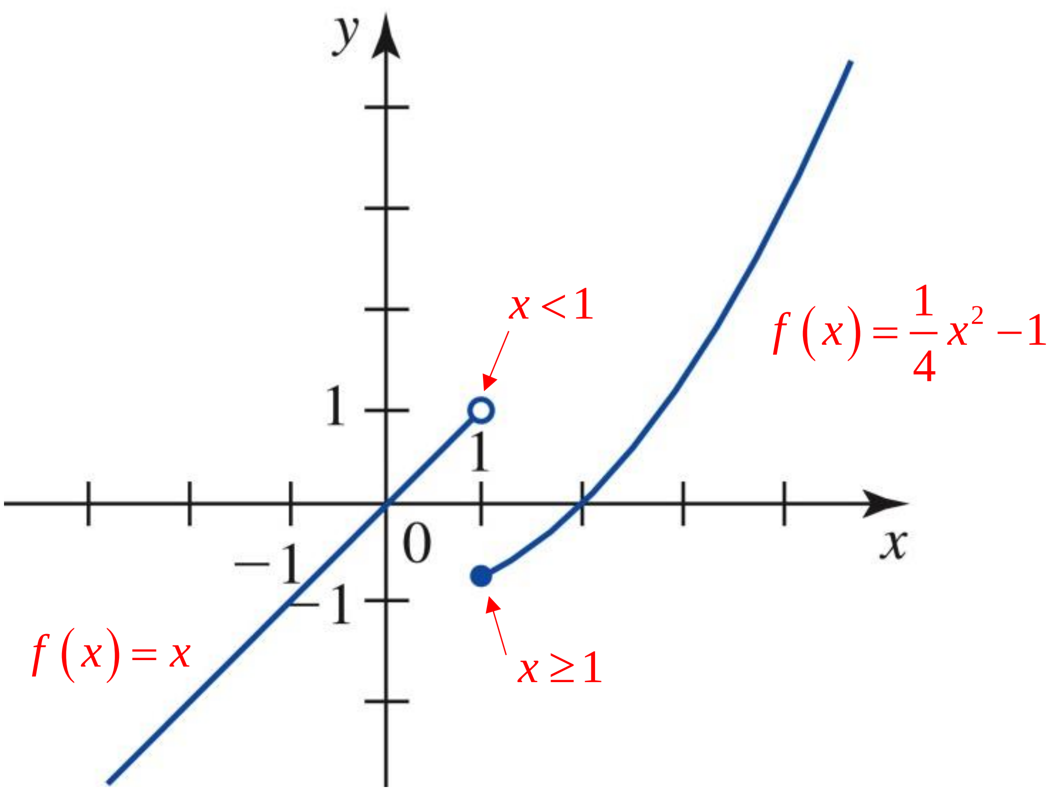
Starting with the graph of $y = f(x)$, the new graphs are obtained by these transformations.

$y = f(x+a)$	shift left by a units
$y = f(x-a)$	shift right by a units
$y = f(x)-b$	shift down by b units
$y = f(x)+b$	shift up by b units
$y = f(cx)$	compress c times horizontally
$y = c\,f(x)$	expand c times vertically
$y = f(-x)$	reflect about y-axis
$y = -f(x)$	reflect about x-axis

Example 8 [TB P22]

Sketch the graph of the function

$$f(x)=\begin{cases} x & \text{if } x<1 \\ \frac{1}{4}x^2-1 & \text{if } x\geq 1 \end{cases}$$



Piecewise Defined Function

Composition of Functions

DEFINITION **Composition of Two Functions**

Given two functions g and f, the composition of g and f, denoted by $g \circ f$, is the function defined by

$$(g \circ f)(x) = g(f(x)) \tag{2}$$

The domain of $g \circ f$ is the set of all x in the domain of f for which $f(x)$ is in the domain of g.

Figure 2 shows an interpretation of the composition $g \circ f$, in which the functions f and g are viewed as machines. **Notice that the output of f, $f(x)$, must lie in the domain of g for $f(x)$ to be an input for g.**

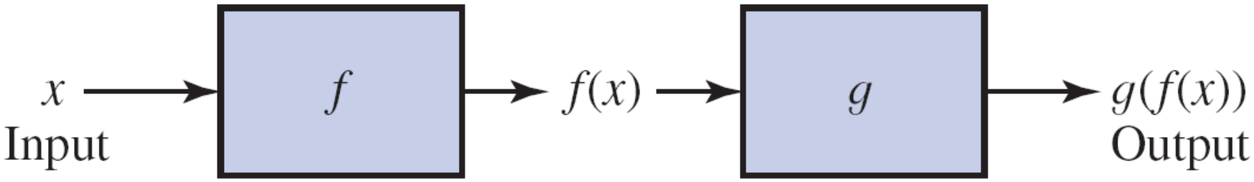


Figure 2 The output of f is the input for g (in this order)

Composition of Functions

Domain of $g \circ f$ is made of x in D_f and x with $f(x)$ in D_g.

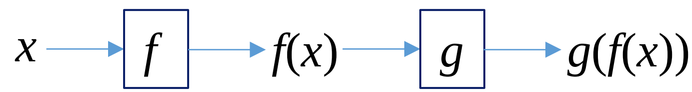
$$D_{g \circ f} = \left\{ x : x \in D_f \right\} \cap \left\{ x : f(x) \in D_g \right\}$$

Example 2 [TB P41]

Let f and g be functions defined by $f(x) = x + 1$ and $g(x) = \sqrt{x}$.

Find the functions $g \circ f$ and $f \circ g$. What is the domain of $g \circ f$?

$$(g \circ f)(x) = g\big(f(x)\big)$$
$$= g(x+1)$$
$$= \sqrt{x+1}$$



1. $x \in D_f = \mathbb{R}$

2. $f(x) \in D_g = [0, \infty)$

$$x + 1 \geq 0$$
$$x \geq -1 \Rightarrow [-1, \infty)$$

$$\therefore D_{g \circ f} = \mathbb{R} \cap [-1, \infty)$$
$$= [-1, \infty)$$

or $D_{g \circ f} = \{x \mid x \in \mathbb{R}, x \geq -1\}$

$$= \{x \mid -1 \leq x < \infty\}$$

Example 2 [TB P41]

Let f and g be functions defined by $f(x) = x + 1$ and $g(x) = \sqrt{x}$.

Find the functions $g \circ f$ and $f \circ g$. What is the domain of $g \circ f$?

$$
\begin{aligned}
(f \circ g)(x) &= f\left(g(x)\right) \\
&= f\left(\sqrt{x}\right) \\
&= \sqrt{x} + 1
\end{aligned}
$$

$$x \longrightarrow \boxed{g} \longrightarrow g(x) \longrightarrow \boxed{f} \longrightarrow f(g(x))$$

1. $x \in D_g = [0, \infty)$

2. $g(x) \in D_f = \mathbb{R}$

$$
\begin{aligned}
\therefore D_{f \circ g} &= [0, \infty) \cap \mathbb{R} \\
&= [0, \infty)
\end{aligned}
$$

or

$$
\begin{aligned}
D_{f \circ g} &= \{x \mid x \in \mathbb{R}, x \geq 0\} \\
&= \{x \mid 0 \leq x < \infty\}
\end{aligned}
$$

Example 3 [TB P42]

Let $f(x) = \sin x$ and $g(x) = 1 - 2x$. Find the functions $g \circ f$ and $f \circ g$. What are their domains?

Solution

$(g \circ f)(x) = g(f(x)) = 1 - 2f(x) = 1 - 2\sin x$

Range of f is [-1,1] and this interval lies in (-∞, ∞), the domain of g.

Therefore domain of $g \circ f$ is given by the domain of f, namely (-∞, ∞).

$$x \to \boxed{f} \to f(x) \to \boxed{g} \to g(f(x))$$

$$1.\ x \in D_f = \mathbb{R}$$

$$2.\ f(x) \in D_g = \mathbb{R}$$

$$\therefore D_{g \circ f} = \mathbb{R} \cap \mathbb{R}$$

$$= (-\infty, \infty)$$

Example 3 [TB P42]

Let $f(x) = \sin x$ and $g(x) = 1 - 2x$. Find the functions $g \circ f$ and $f \circ g$. What are their domains?

Solution

$(f \circ g)(x) = f(g(x)) = f(1-2x) = \sin(1-2x)$

Range of g is (-∞, ∞) and is also the domain of f.

Therefore domain of $f \circ g$ is given by the domain of g, namely (-∞, ∞).

$$x \longrightarrow \boxed{g} \longrightarrow g(x) \longrightarrow \boxed{f} \longrightarrow f(g(x))$$

$$1.\ x \in D_g = \mathbb{R}$$

$$2.\ g(x) \in D_f = \mathbb{R}$$

$$\therefore D_{f \circ g} = \mathbb{R} \cap \mathbb{R}$$

$$= (-\infty, \infty)$$

Example 4 [TB P42]

Find two functions f and g such that $F = g \circ f$ if $F(x) = (x+2)^4$.

Solution

The expression $(x+2)^4$ can be evaluated in 2 steps. First given any value x, add 2 to it. Second , raise the result to the 4th power.

$$f(x) = x + 2$$

$$g(x) = x^4$$

Alternatively

$$f(x) = (x+2)^2$$

$$g(x) = x^2$$

Composition of Functions

The composition of functions can be extended to include the composition of three or more functions.

For example, the composite function $h \circ g \circ f$ is found by applying f, g, and h in that order. Thus,

$$(h \circ g \circ f)(x) = h(g(f(x)))$$

Example 6 [TB P43]

Let $f(x) = x - \frac{\pi}{2}$, $g(x) = 1 + \cos^2 x$, and $h(x) = \sqrt{x}$. Find $h \circ g \circ f$.

Solution

$$\begin{aligned}(h \circ g \circ f)(x) = h(g(f(x))) &= h(g(x - \tfrac{\pi}{2})) \\ &= h(1 + \cos^2(x - \tfrac{\pi}{2})) \\ &= \sqrt{1 + \cos^2(x - \tfrac{\pi}{2})}\end{aligned}$$

Example 7 [TB P43]

Suppose $F(x) = \dfrac{1}{\sqrt{2x+3}+1}$. Find functions f, g and h such that $F = h \circ g \circ f$.

Solution

The role of F says that as a first step, we multiply x by 2 and add 3 to it.

$\Rightarrow f(x) = 2x + 3.$

Next we take square root of the results and add 1 to it $\Rightarrow g(x) = \sqrt{x} + 1.$

Finally, we take the reciprocal of the last result $\Rightarrow h(x) = \dfrac{1}{x}.$