# Ellipses

An ellipse is made up of points whose total distance from two fixed points (foci) are the same.

Suppose the foci are at ( $c$, $0$ ) and ( $-c$, $0$ ) as shown in Figure 15.

For a random point $P(x, y)$ on the ellipse,

it can be shown that :

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

where the total distance ( $d_1 + d_2$ ) from the foci is $2a$ and $b^2 = a^2 - c^2$ .

Observe that $b < a$.

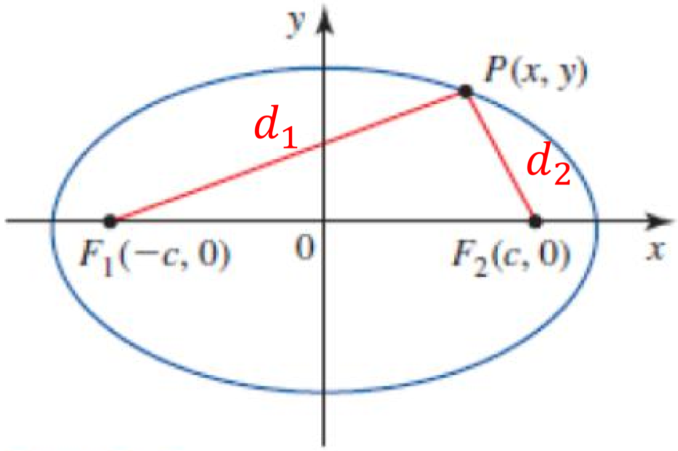



**FIGURE 15**
The ellipse with foci $F_1(-c, 0)$ and $F_2(c, 0)$

# Ellipses

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

When $x=0, \frac{0^2}{a^2}+\frac{y^2}{b^2}=1$

$$y=\pm b$$

$\therefore (0,b)$ and $(0,-b)$ lie on the ellipse.

When $y=0, \frac{x^2}{a^2}+\frac{0^2}{b^2}=1$

$$x=\pm a$$

$\therefore (a,0)$ and $(-a,0)$ lie on the ellipse.

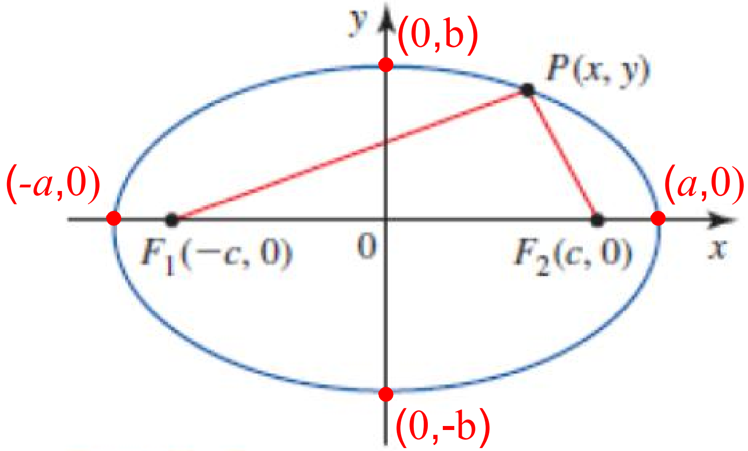



**FIGURE 15**
The ellipse with foci $F_1(-c, 0)$ and $F_2(c, 0)$

# Ellipses

- The vertices are the two points where the line joining the foci intersects the ellipse.
- The line segment joining the vertices is called the **major axis**, and its midpoint is called the **center** of the ellipse.
- The line segment passing through the center of the ellipse and perpendicular to the major axis is called the **minor axis** of the ellipse.

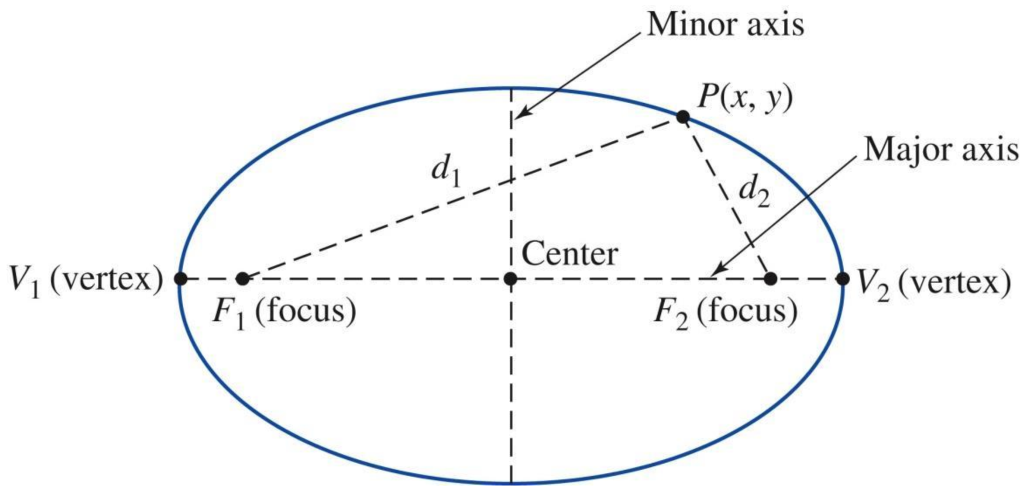



**Figure 13**

# We have the following standard equation of an ellipse.

**Standard Equation of an Ellipse**

An equation of the ellipse with foci $(\pm c, 0)$ and vertices $(\pm a, 0)$ is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad a \geq b > 0$$

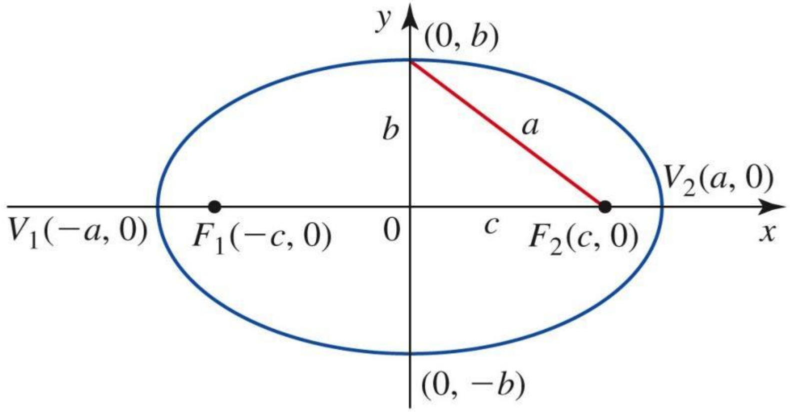
If # under $x^2$ > # under $y^2$
$\Rightarrow$ major axis is along $x$-axis

and an equation of the ellipse with foci $(0, \pm c)$ and vertices $(0, \pm a)$ is

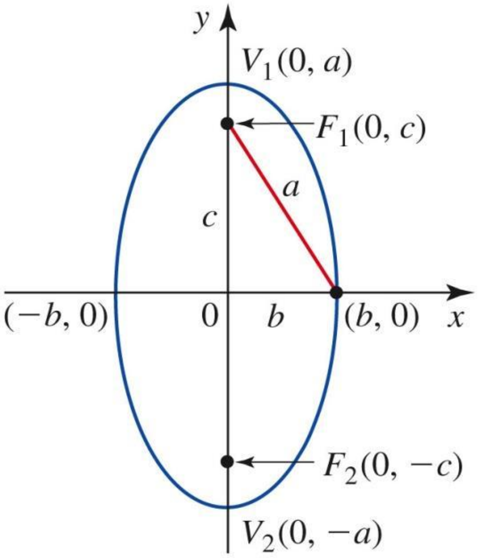
$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \qquad a \geq b > 0$$

If # under $y^2$ > # under $x^2$
$\Rightarrow$ major axis is along $y$-axis

where $c^2 = a^2 - b^2$. (See Figure 16.)




(**a**) The major axis is along the $x$-axis.




(**b**) The major axis is along the $y$-axis.

# Example 4 [TB P835]

Sketch the ellipse $\dfrac{x^2}{16}+\dfrac{y^2}{9}=1$. What are the foci and vertices?

Solution:

$\#$ under $x^2 > \#$ under $y^2$

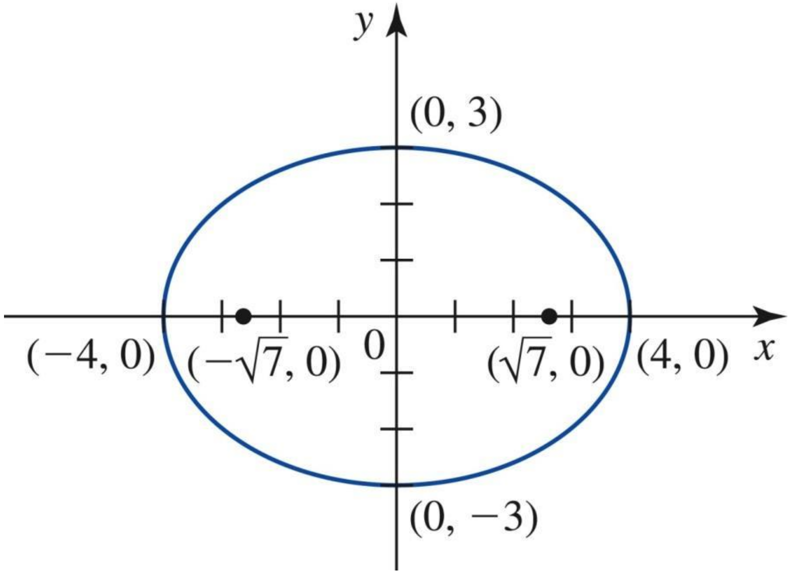
$\Rightarrow$ major axis is along $x$-axis

Given $a^2=16$ and $b^2=9$,

$$c^2 = a^2 - b^2$$
$$= 16 - 9$$
$$c = \pm\sqrt{7}$$

Foci are $(\pm c, 0) = (\pm\sqrt{7}, 0)$

Vertcies are $(\pm a, 0) = (\pm 4, 0)$



# Example 5 [TB P835]

Find an equation of the ellipse with foci $(0, \pm 2)$ and vertices $(0, \pm 4)$.

Solution:

Major axis is $y$-axis

$c = 2$ and $a = 4$, so

$$\begin{aligned} b^2 &= a^2 - c^2 \\ &= 16 - 4 \\ &= 12 \end{aligned}$$

Therefore the standard form of the equation for the ellipse is $\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1$

$$\frac{x^2}{12} + \frac{y^2}{16} = 1$$

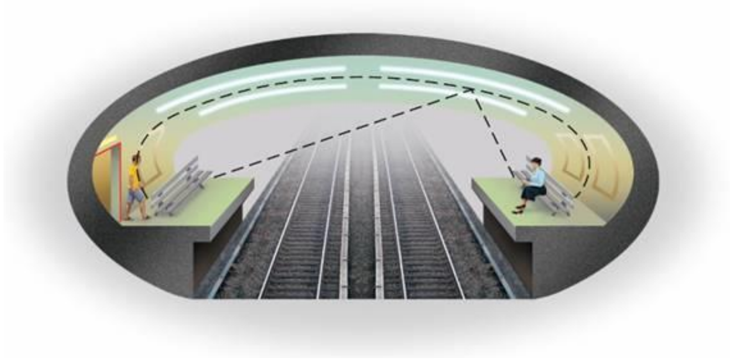
$$4x^2 + 3y^2 = 48$$

# Reflective Property of the Ellipse

The reflective property of the ellipse is used to design *whispering galleries*—rooms with elliptical-shaped ceilings, in which a person standing at one focus can hear the whisper of another person standing at the other focus.

A whispering gallery can be found in the rotunda of the Capitol Building in Washington, D.C.

Also, Paris subway tunnels are almost elliptical, and because of the reflective property of the ellipse, whispering on one platform can be heard on the other. (See Figure 19.)



A cross section of a Paris subway tunnel is almost elliptical.

**Figure 19**

# Hyperbolas

The definition of a hyperbola is similar to that of an ellipse. The *sum* of the distances between the foci and a point on an ellipse is fixed, whereas the *difference* of these distances is fixed for a hyperbola.

Suppose the foci are at ( $c$, $0$ ) and ( -$c$, $0$ ) as shown.

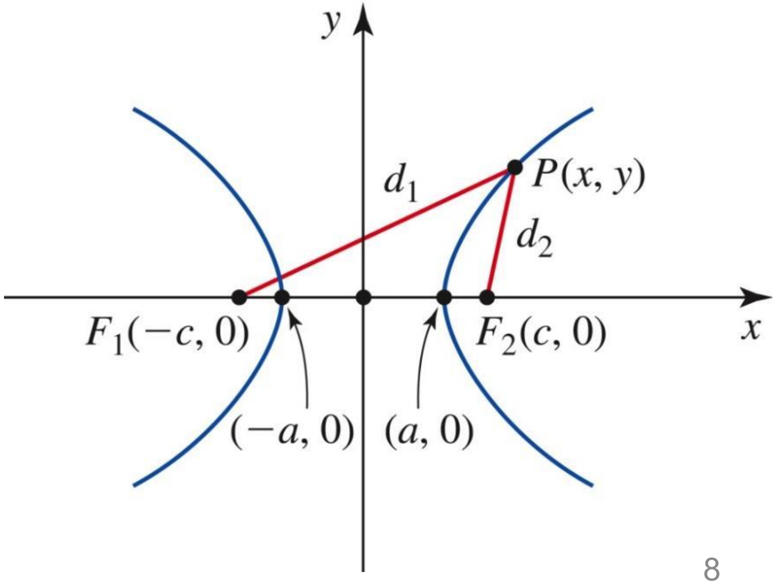
For a random point $P$( $x$, $y$ ) on the hyperbola,

it can be shown that

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

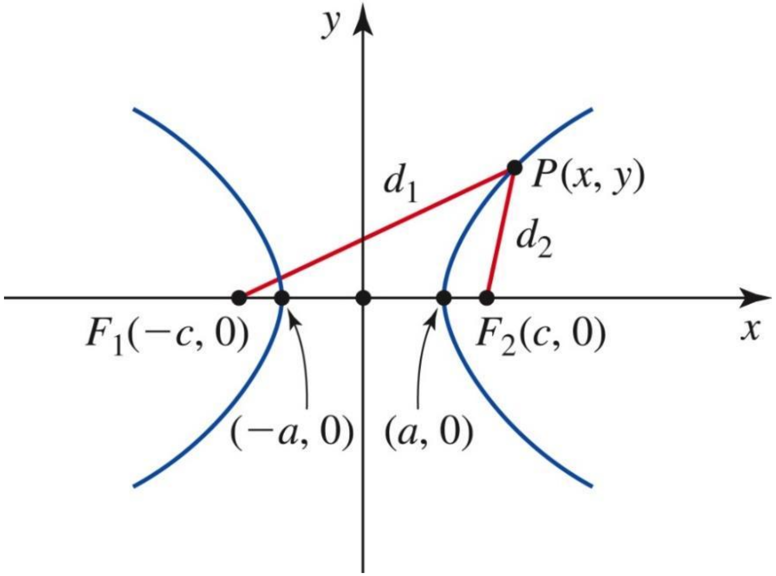
where $d_1 - d_2 = 2a$ and $b^2 = c^2 - a^2$.

$$\left[\text{or } c^2 = a^2 + b^2.\right]$$



# Hyperbolas

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$




When $x=0, \frac{0^2}{a^2}-\frac{y^2}{b^2}=1$

$$y^2=-b^2<0$$

$\therefore$ no points on $y$-axis.

When $y=0, \frac{x^2}{a^2}-\frac{0^2}{b^2}=1$

$$x=\pm a$$

$\therefore (a,0)$ and $(-a,0)$ lie on the hyperbola.

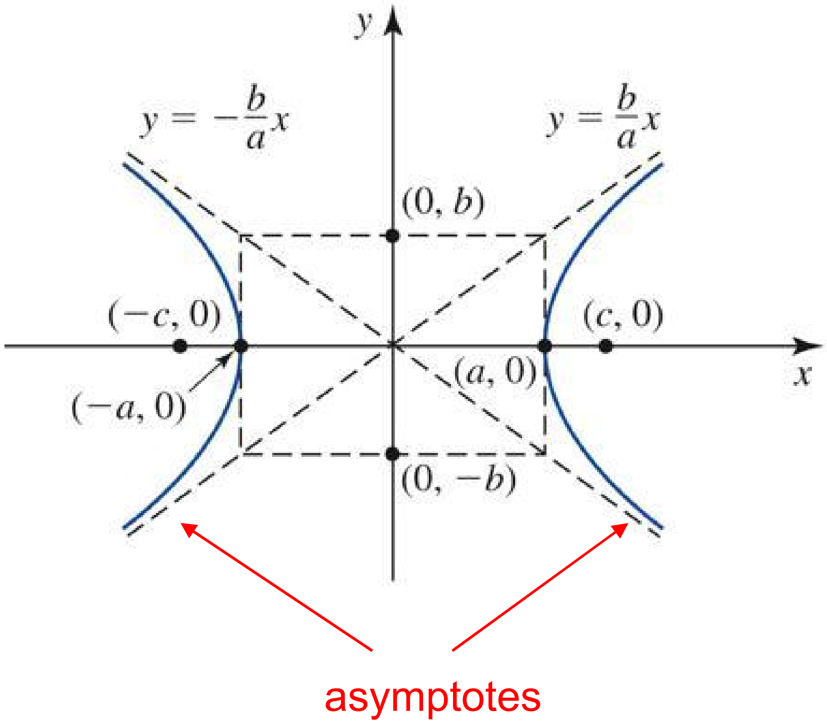
# Hyperbolas

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \Rightarrow \frac{x^2}{a^2} = \frac{y^2}{b^2} + 1$$

When $x$ is very large in magnitude,
$y$ is also very large in magnitude.

$$\frac{x^2}{a^2} \approx \frac{y^2}{b^2} \quad \Rightarrow \frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$$

$$y \approx \pm\frac{b}{a}x$$




$\therefore$ points on the hyperbola with large magnitude of $x$ and $y$ are near the lines $y = \pm\frac{b}{a}x$.

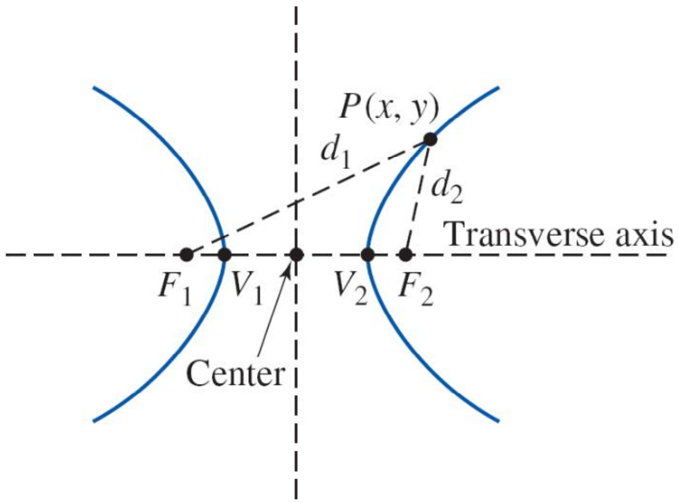
These lines are called asymptotes.

# Hyperbolas

The line passing through the foci intersects the hyperbola at two points, $V_1$ and $V_2$, called the **vertices** of the hyperbola.

The line segment joining the vertices is called the **transverse axis** of the hyperbola.

The midpoint of the transverse axis is called the **center** of the hyperbola.

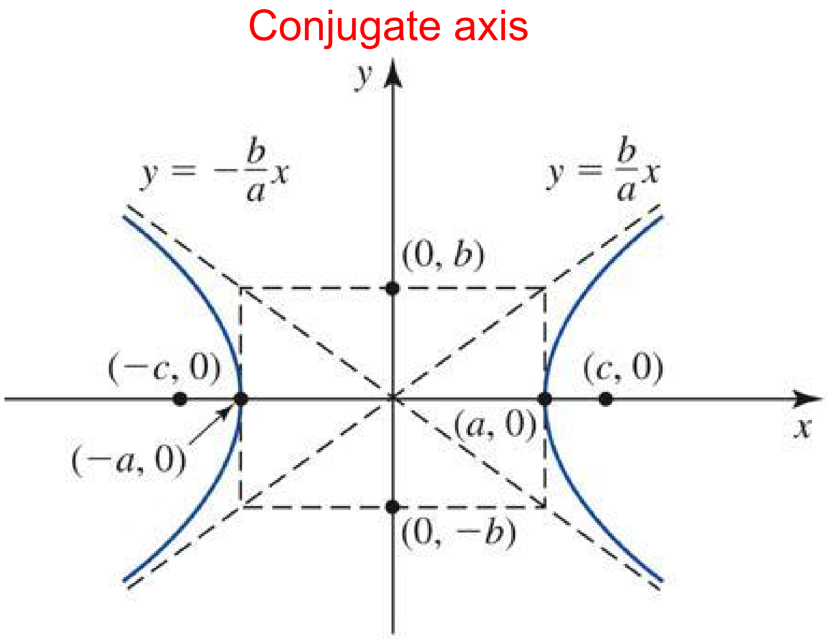



A hyperbola with foci $F_1$ and $F_2$. A point $P(x, y)$ is on the hyperbola if and only if $|d_1 - d_2|$ is a constant.

**Figure 20**

# Hyperbolas

The line segment passing through the center and perpendicular to the transverse axis, joining ( $0$ , $b$ ) and ( $0$, -$b$ ) is the **conjugate axis**.

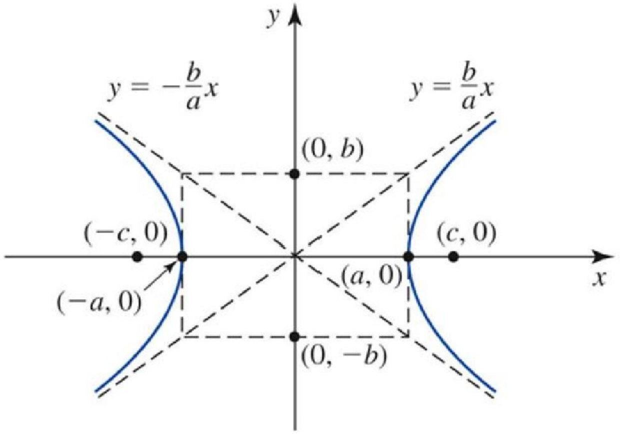


# Hyperbolas

Standard equation of a hyperbola with foci ( $\pm$ c, 0 ) and vertices ( $\pm$ a, 0 ) is $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1,$ where $b^2=c^2-a^2.$

Asymptotes are $y=\pm\dfrac{b}{a}x.$

RHS =1, start with $x^2 \Rightarrow$ cuts $x$-axis.



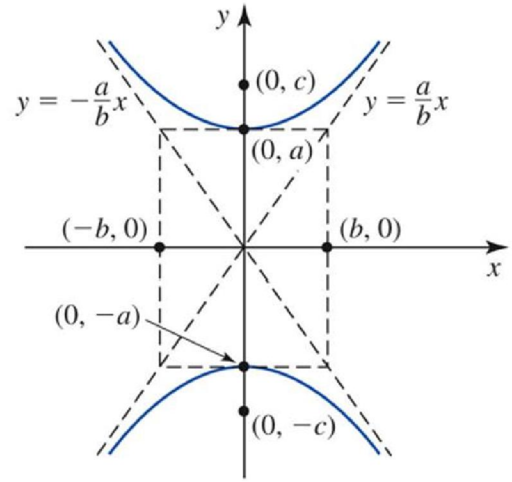

(a) $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ (The transverse axis is along the $x$-axis.)

# Hyperbolas

Standard equation of a hyperbola with foci ( 0, ± c ) and vertices ( 0, ± a ) is $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$, where $b^2 = c^2 - a^2$.

Asymptotes are $y = \pm \frac{a}{b} x$.

RHS =1, start with $y^2 \Rightarrow$ cuts $y$-axis.




(**b**) $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ (The transverse axis is along the $y$-axis.)

# Hyperbolas

Steps to sketch hyperbola :

1. Get equations of asymptotes by setting RHS '$1$' to '$0$', then rearrange.
2. Draw dotted box ($2a$ by $2b$).
   I. Join opposite corners of box for asymptotes.
   II. Vertices at middle of left/right or up/down sides of the box.
3. Complete hyperbola by drawing from near asymptotes to vertices $4$ times.
4. Label vertices, foci, asymptotes and equation of hyperbola.

# Example 7 [TB P838]

A hyperbola has vertices $(0, \pm 3)$ and passes through the point $(2, 5)$. Find an equation of the hyperbola. What are its foci and asymptotes?

Solution:

The vertices lie on the y-axis, so the standard equation of the hyperbola has the form

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

Vertices at $(0, \pm a)$, so $a = 3$, $\dfrac{y^2}{3^2} - \dfrac{x^2}{b^2} = 1$

$$\frac{y^2}{9} - \frac{x^2}{b^2} = 1$$

Substitute $x = 2,\ y = 5,\ \dfrac{5^2}{9} - \dfrac{2^2}{b^2} = 1 \to b = \dfrac{3}{2}$

[As (2, 5) is on the hyperbola]

# Example 7 [TB P838]

Substitute $b = \dfrac{3}{2}$ into eqn, $\dfrac{y^2}{9} - \dfrac{x^2}{\left(\dfrac{3}{2}\right)^2} = 1 \rightarrow y^2 - 4x^2 = 9$

$$c^2 = a^2 + b^2 = 9 + \frac{9}{4} = \frac{45}{4}$$

$$c = \pm\frac{3\sqrt{5}}{2}$$

Foci $(0, \pm c) = \left(0, \pm\dfrac{3\sqrt{5}}{2}\right)$

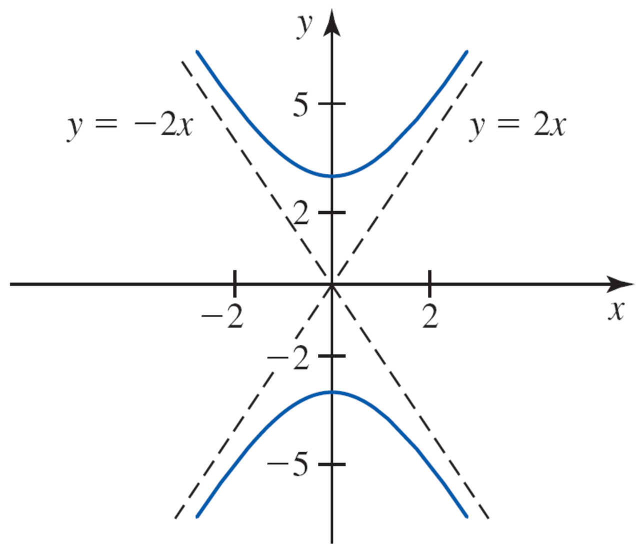
Asymptotes $\dfrac{y^2}{9} - \dfrac{x^2}{9/4} = \cancel{1}\ 0$

$$y^2 = 4x^2$$

$$y = \pm 2x$$

# Example 7 [TB P838]

The graph of the hyperbola is shown in Figure 24.




The graph of the hyperbola $y^2 - 4x^2 = 9$

**Figure 24**

# Shifted Conics

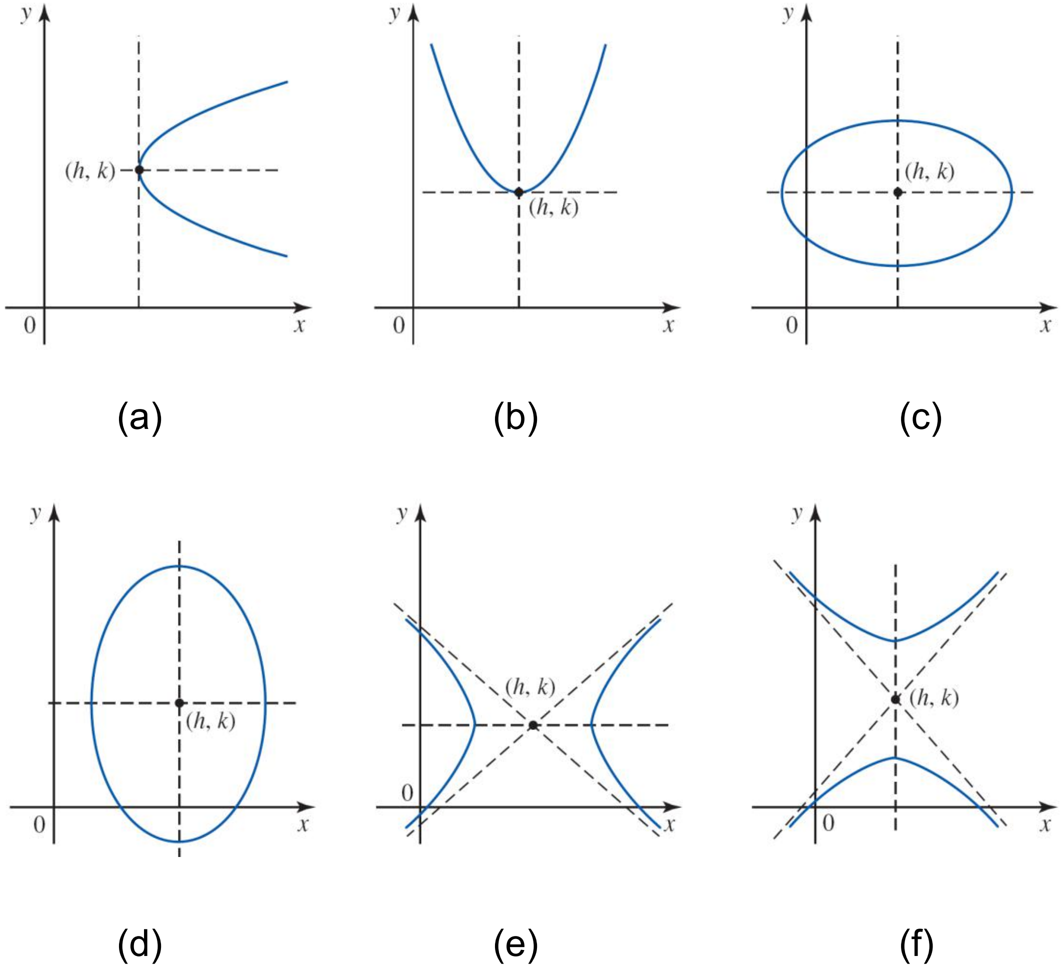



**Figure 26**

Shifted conics with centers at $(h, k)$

# Shifted Conics

| Conic | Orientation of axis | Equation of conic |
|---|---|---|
| Parabola | Axis horizontal | $(y-k)^2 = 4p(x-h)$ |
| Parabola | Axis vertical | $(x-h)^2 = 4p(y-k)$ |
| Ellipse | Major axis horizontal | $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$ |
| Ellipse | Major axis vertical | $\dfrac{(x-h)^2}{b^2} + \dfrac{(y-k)^2}{a^2} = 1$ |
| Hyperbola | Transverse axis horizontal | $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$ |
| Hyperbola | Transverse axis vertical | $\dfrac{(y-k)^2}{a^2} - \dfrac{(x-h)^2}{b^2} = 1$ |

$x \to x - h$

$y \to y - k$

# Example 9 [TB P840]

Find the standard equation of the ellipse with foci at $(1, 2)$ and $(5, 2)$ and major axis of length $6$. Sketch the ellipse.

Solution:

Since the foci $(1, 2)$ and $(5, 2)$ have the same $y$-coordinate, we see that they lie along the line $y = 2$ parallel to the $x$-axis.

The midpoint of the line segment joining $(1, 2)$ to $(5, 2)$ is $(3, 2)$, and this is the center of the ellipse.

From this we can see that the distance from the center of the ellipse to each of the foci is $2$, so $c = 2$.

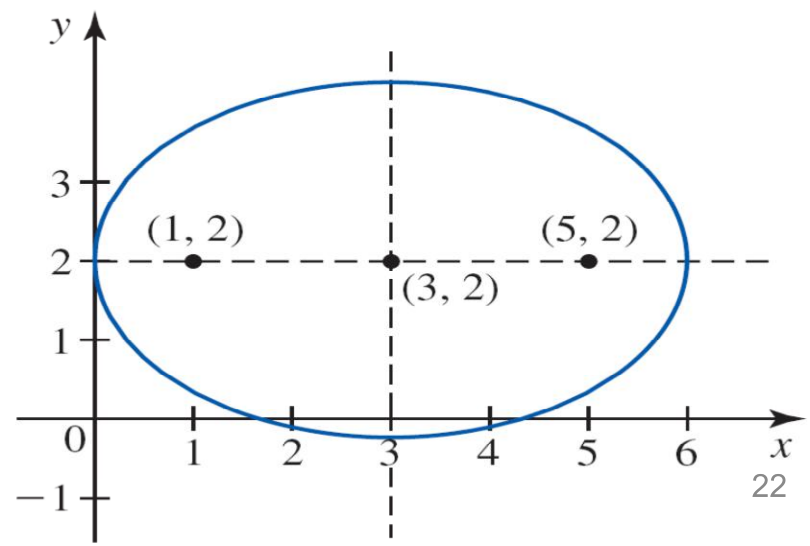
# Example 9 [TB P840]

Next, since the major axis of the ellipse is known to have length 6, we have $2a = 6$, or $a = 3$.

Finally, from the relation $c^2 = a^2 - b^2$, we obtain $4 = 9 - b^2$, or $b^2 = 5$.

Therefore, using Equation (9) from Table 1 with $h = 3$, $k = 2$, $a = 3$, and $b = \sqrt{5}$, we obtain the equation:

$$\frac{(x-3)^2}{9} + \frac{(y-2)^2}{5} = 1$$



# Shifted Conics

If you expand and simplify each equation in Table 1, you will see that these equations have the general form

$$Ax^2 + By^2 + Dx + Ey + F = 0$$

where the coefficients are real numbers.

Conversely, given such an equation, we can obtain an equivalent equation in the form listed in Table 1 by using the technique of completing the square.

# Example 10 [TB P841]

Find the standard equation of the hyperbola

$$3x^2 - 4y^2 + 6x + 16y - 25 = 0$$

Find its centre, foci, vertices and asymptotes and sketch its graph.

$$3(x^2 + 2x) - 4(y^2 - 4y) = 25$$

$$3\left[x^2 + 2x + (1)^2 - (1)^2\right] - 4\left[y^2 - 4y + (-2)^2 - (-2)^2\right] = 25$$

$$3(x+1)^2 - 3 - 4(y-2)^2 + 16 = 25$$

$$3(x+1)^2 - 4(y-2)^2 = 12$$

$$\frac{(x+1)^2}{4} - \frac{(y-2)^2}{3} = 1$$

$$\therefore a^2 = 4, \quad b^2 = 3$$

$$\Rightarrow c^2 = a^2 + b^2 = 7$$

# Example 10 [TB P841]

center = $(-1,\ 2\ )$

transverse axis is paralle to $x$-axis

vertices = $(\ a+h,\ 0+k\ )\ =\ (\pm 2-1,\ 2\ )$

foci = $(\ c+h,\ 0+k\ )\ =\ (\pm\sqrt{7}-1,\ 2\ )$

asymptotes: $y\ = 2\ \pm\dfrac{\sqrt{3}}{2}(x+1)$