The Inverse of a Function (TB P532)

The functions *f* and *g* are said to be *inverses* of each other. More generally, we have the following definition.

DEFINITION Inverse Functions

A function g is the inverse of the function f if

$$f[g(x)] = x \text{ for every } x \text{ in the domain of } g$$

and

$$g[f(x)] = x \text{ for every } x \text{ in the domain of } f$$

Equivalently, g is the inverse of f if the following condition is satisfied:

$$y = f(x) \qquad \text{if and only if} \qquad x = g(y)$$

for every x in the domain of f and for every y in its range.

The inverse of f is normally denoted by f^{-1}.

$$\boxed{\text{Do not confuse } f^{-1}(x) \text{ with } [f(x)]^{-1} = \frac{1}{f(x)}}$$

Example 1 [TB P534]

Show that the function $f(x) = x^{1/3}$ and $g(x) = x^3$ are inverses of each other.

$$f \circ g(x) = f(g(x)) = f(x^3) = (x^3)^{1/3} = x.$$

$$g \circ f(x) = g(f(x)) = g(x^{1/3}) = (x^{1/3})^3 = x$$

Therefore f and g are inverses of each other.

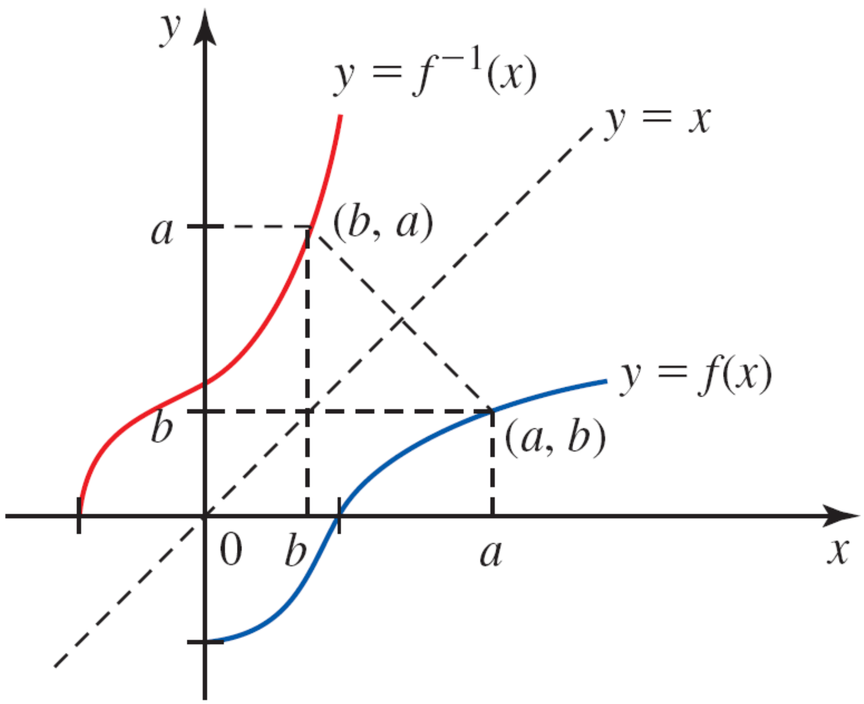
The Graphs of Inverse Function

Suppose that (a, b) is any point on the graph of a function f.

Then $b = f(a)$, and we have

$$f^{-1}(b) = f^{-1}[f(a)] = a$$

This shows that (b, a) is on the graph of f^{-1}.



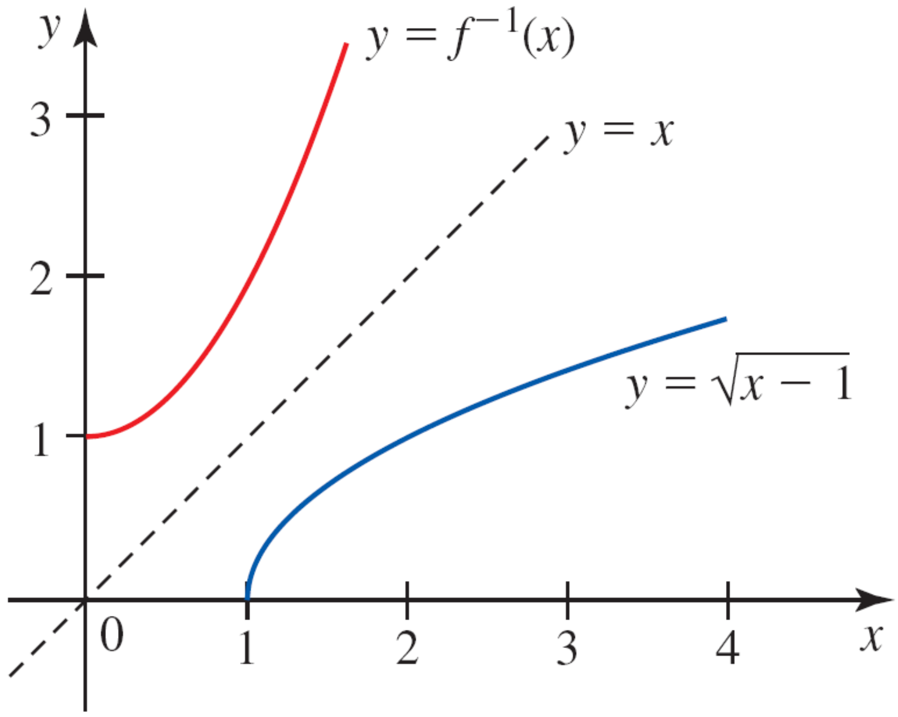
The Graphs of Inverse Functions

The graph of f^{-1} is the reflection of the graph of f with respect to the line $y = x$ and vice versa.

Example 2 [TB P534]

Sketch the graph of $f(x) = \sqrt{x - 1}$. Then reflect the graph of f with respect to the line $y = x$ to obtain the graph of f^{-1}.

Solution:



The graph of f^{-1} is obtained by reflecting the graph of f with respect to the line $y = x$.

Which Functions Have Inverses

Does every function have an inverse ?

Consider for exmaple, the function f defined by $y = x^2$ with

D_f =(-∞,∞) and R_f =[0,∞).

From the graph of f, each $y > 0$ is associated with exactly two values of x in D_f.
Any horizontal line $y = c$, where $c > 0$, cuts the graph of f more than once.

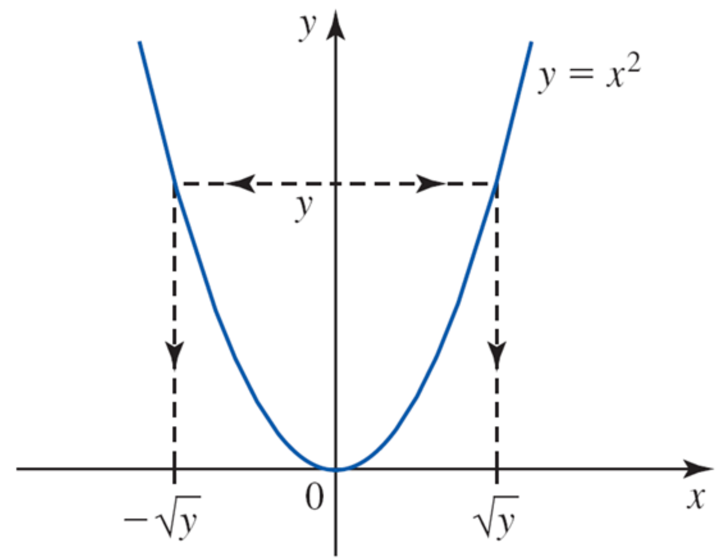


Figure 6

Each value of *y* is associated with two values of *x*.

This is called <u>many-to-one</u>.

This implies that <u>*f* does not have an inverse</u>, since the uniqueness requirement of a function cannot be satisfied in this case.

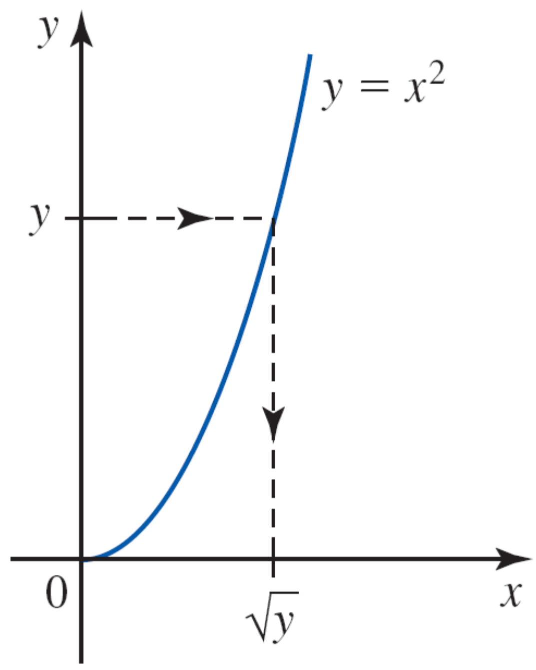
Which Functions Have Inverses

Next, consider the function g defined by the same rule as that of f, namely, $y = x^2$, but with domain restricted to $[0, \infty)$.

From the graph of g (Figure 7), each value of y in $R_g = [0, \infty)$ is the image of exactly *one* number $x = \sqrt{y}$ in D_g .

Any horizontal line $y = c$, cuts the graph of g at most once.

This is called <u>one-to-one</u>.



Which Functions Have Inverses

DEFINITION **One-to-One Function**

A function f with domain D is **one-to-one** if no two numbers in D have the same image; that is,

$$f(x_1) \neq f(x_2) \qquad \text{whenever} \qquad x_1 \neq x_2$$

Geometrically, a function is one-to-one if every horizontal line intersects its graph at no more than one point. This is called the *horizontal line test.*

We have the following important theorem concerning the existence of an inverse function.

THEOREM 1 **The Existence of an Inverse Function**

A function has an inverse if and only if it is one-to-one.

Example 3 [TB P535]

Determine whether the function has an inverse.

a. $f(x) = x^{1/3}$ **b.** $f(x) = x^3 - 3x + 1$

Solution:

a. Refer to Figure 3.

Using the horizontal line test,

we see that f is one-to-one

on $(-\infty, \infty)$.

Therefore, f has an inverse on

$(-\infty, \infty)$.

$$y = x^{\frac{1}{3}}$$
$$x^{\frac{1}{3}} = y$$
$$x = y^3$$
$$f^{-1}(x) = x^3$$

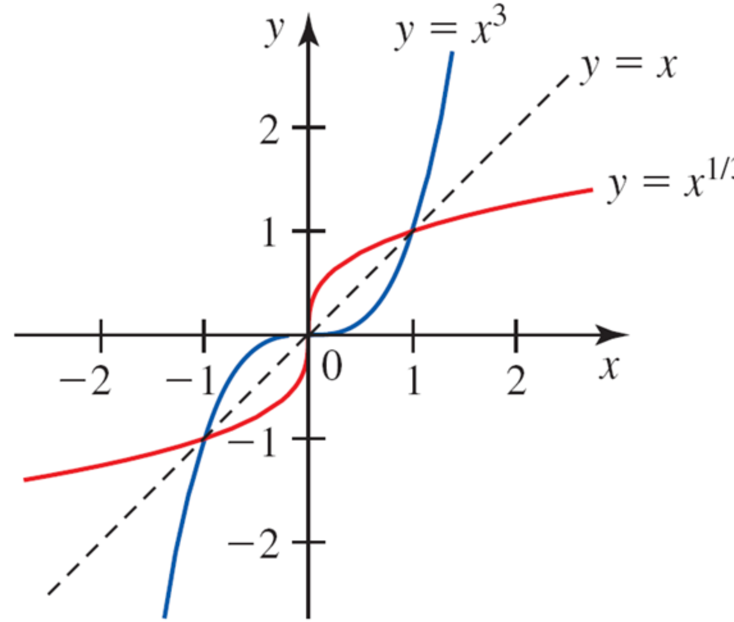


Figure 3

The functions $y = x^{1/3}$ and $y = x^3$ are inverses of each other.

Example 3 [TB P535]

Alternative method of proving one-to-one when no graph is given

Start wtih $f(x_1) = f(x_2)$, then use algebra to get $x_1 = x_2$.

Then f is one-to-one.

$$f(x_1) = f(x_2)$$

$$x_1^{\frac{1}{3}} = x_2^{\frac{1}{3}}$$

$$\left(x_1^{\frac{1}{3}}\right)^3 = \left(x_2^{\frac{1}{3}}\right)^3$$

$$x_1 = x_2$$

$\therefore f$ is one-to-one and has an inverse

Example 3 [TB P535]

b. The graph of f is shown in Figure 8.

Observe that the horizontal line $y = 1$ intersects the graph of f at three points, so f does not pass the horizontal line test.

Therefore, f is not one-to-one and does not have an inverse.

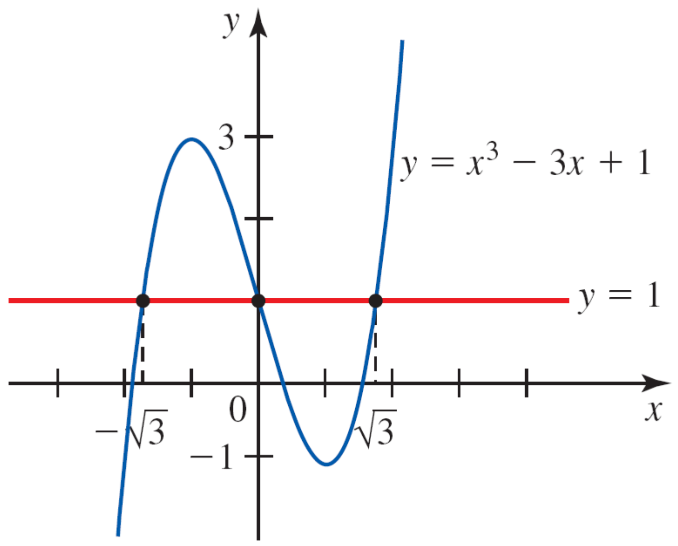


Figure 8

f is not one-to-one because it fails the horizontal line test.

Example 3 [TB P535]

Alternative method of proving one-to-one when no graph is given

Start wtih $f(x_1) = f(x_2)$,

$$f(x_1) = f(x_2)$$
$$x_1^{\,3} - 3x_1 + 1 = x_2^{\,3} - 3x_2 + 1$$
$$x_1^{\,3} - 3x_1 = x_2^{\,3} - 3x_2$$
$$x_1^{\,3} - x_2^{\,3} = 3\left(x_1 - x_2\right)$$
$$\left(x_1 - x_2\right)\left(x_1^{\,2} + x_1x_2 + x_2^{\,2}\right) = 3\left(x_1 - x_2\right)$$
$$\left(x_1^{\,2} + x_1x_2 + x_2^{\,2}\right) = 3$$

Choose $x_1 = 0$,

$$\left(0^2 + 0x_2 + x_2^{\,2}\right) = 3$$
$$x_2^{\,2} = 3$$
$$x_2 = \pm\sqrt{3}$$

$x_1 \neq x_2, \therefore f$ is not one-to-one.

Finding the Inverse of a Function

Let's summarize the steps for finding the inverse of a function, assuming that it exists.

Guidelines for Finding the Inverse of a Function

1. Write $y = f(x)$.
2. Solve this equation for x in terms of y (if possible).
3. Interchange x and y to obtain $y = f^{-1}(x)$.

Example 4 [TB P536]

Find the inverse of the function defined by $f(x) = \dfrac{1}{\sqrt{2x-3}}$.

Solution:

The graph of *f* shown in Figure 9 shows that *f* is one-to-one and so f^{-1} exists.

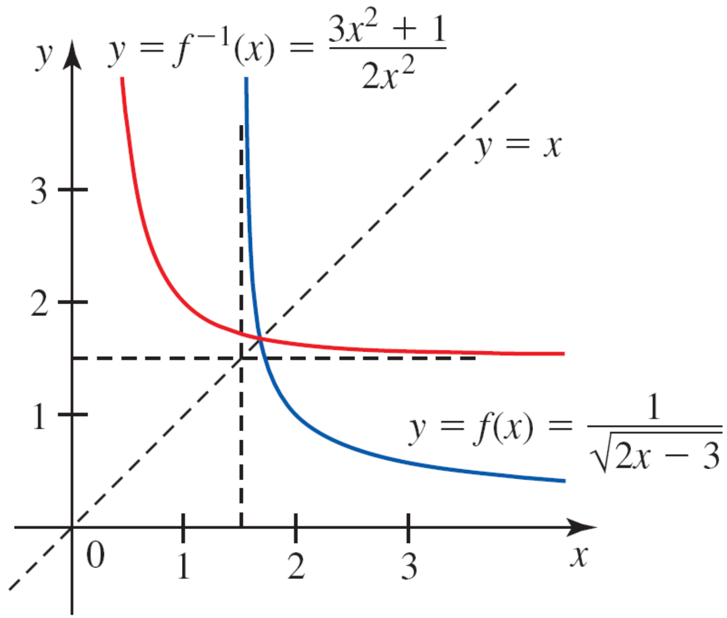


Figure 9

The graphs of *f* and f^{-1}. Notice that they are reflections of each other with respect to the line $y = x$.

Example 4 [TB P536]

Alternatively

$$f(x_1) = f(x_2)$$

$$\frac{1}{\sqrt{2x_1 - 3}} = \frac{1}{\sqrt{2x_2 - 3}}$$

$$\sqrt{2x_2 - 3} = \sqrt{2x_1 - 3}$$

$$x_2 = x_1$$

$\Rightarrow$ function f is 1-1

Example 4 [TB P536]

To find the rule for this inverse, write

$$y = \frac{1}{\sqrt{2x - 3}}$$

and then solve the equation for x:

$$y^2 = \frac{1}{2x - 3}$$

Square both sides.

$$2x - 3 = \frac{1}{y^2}$$

Take reciprocals.

$$2x = \frac{1}{y^2} + 3$$

$$= \frac{3y^2 + 1}{y^2}$$

Example 4 [TB P536]

Finally, interchanging *x* and *y*, we obtain

$$y = \frac{3x^2 + 1}{2x^2}$$

giving the rule for f^{-1} as

$$f^{-1}(x) = \frac{3x^2 + 1}{2x^2}$$

Inequalities and Absolute Value (TB PA1)

- interpret and write inequalities using appropriate notation and symbols
- list the properties of inequalities
- use properties of inequalities to solve linear inequalities, combined inequalities and inequalities involving absolute value

Inequalities

Properties of Inequalities

If $a, \; b$ and c are any real numbers, then

Property 1 If $a < b$ and $b < c$, then $a < c$

Property 2 If $a < b$, then $a + c < b + c$

Property 3 If $a < b$ and $c > 0$, then $ac < bc$

Property 4 If $a < b$ and $c < 0$, then $ac > bc$

Example 1 [TB PA3]

Find the set of real numbers that satisfy $-1 \le 2x - 5 < 7$.

$$-1 \le 2x - 5 < 7.$$

Add 5

$$4 \le 2x < 12$$

Divide by 2

$$2 \le x < 6$$

The solution is the set of all values of x in $[2, 6)$.

(Don't leave answer in inequalities if question asked for 'set'.)

Example 2 [TB PA3]

Solve the inequality $x^2 + 2x - 8 < 0$.

$$x^2 + 2x - 8 < 0$$

$$(x+4)(x-2) < 0$$

$\Rightarrow$ either $(x+4) > 0$ and $(x-2) < 0$

or $(x+4) < 0$ and $(x-2) > 0$

case 1

$(x+4) > 0$ and $(x-2) < 0$

$x > -4$ and $x < 2$

$\Rightarrow \quad -4 < x < 2$

case 2

$(x+4) < 0$ and $(x-2) > 0$

$x < -4$ and $x > 2$

$\Rightarrow$ no solution

$\therefore \quad -4 < x < 2$

$\Rightarrow x \in (-4, 2)$

Example 3 [TB PA4]

Solve the inequality $\dfrac{x+1}{x-1} \geq 0.$

$\Rightarrow$ either $(x+1) \geq 0$ and $(x-1) > 0$

or $(x+1) \leq 0$ and $(x-1) < 0$

case 1

$(x+1) \geq 0$ and $(x-1) > 0$

$x \geq -1$ and $x > 1$

$\Rightarrow x > 1$

case 2

$(x+1) \leq 0$ and $(x-1) < 0$

$x \leq -1$ and $x < 1$

$\Rightarrow x \leq -1$

$\therefore \quad x \leq -1, x > 1$

$\Rightarrow x \in (-\infty, -1] \cup (1, \infty)$

Extra example

Solve the inequality $\dfrac{1+x}{1-x} \leq 1.$

Bring over to LHS so as to make RHS zero

$$\frac{1+x}{1-x} - 1 \leq 0$$

$$\frac{1+x}{1-x} - \frac{1-x}{1-x} \leq 0$$

$$\frac{2x}{1-x} \leq 0$$

$\Rightarrow$ either $2x \leq 0$ and $(1-x) > 0$

or $2x \geq 0$ and $(1-x) < 0$

Extra example

Solve the inequality $\dfrac{1+x}{1-x} \le 1.$

case 1

$$2x \le 0 \text{ and } (1-x) > 0$$

$$x \le 0 \text{ and } x < 1$$

$$\Rightarrow \; x \le 0$$

case 2

$$2x \ge 0 \text{ and } (1-x) < 0$$

$$x \ge 0 \text{ and } x > 1$$

$$\Rightarrow \; x > 1$$

$$\therefore \quad x \le 0, x > 1$$

$$\Rightarrow x \in (-\infty, 0] \cup (1, \infty)$$

Absolute Value (TB PA4)

The absolute value of a number a is denoted by $|a|$ and is defined by

$$|a| = \begin{cases} a & \text{if } a \ge 0 \\ -a & \text{if } a < 0 \end{cases}$$

Absolute Value Properties

If a and b are any real numbers, then

Property 5 $\quad |-a| = |a|$

Property 6 $\quad |ab| = |a||b|$

Property 7 $\quad \left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}$

Property 8 $\quad |a+b| \le |a| + |b|$

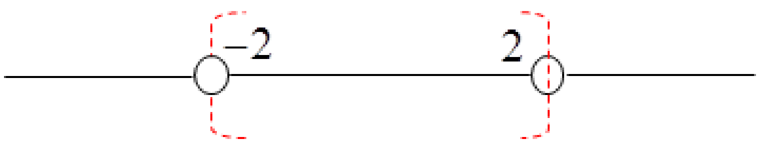
Inequalities Involve Absolute Value

Inequality involving modulus

The modulus of a number is the magnitude of the number regardless of its sign. Vertical lines enclosing the number denote a modulus.

For examples, $|5| = 5$ and $|-5| = 5$

The inequality $|t| < 2$ means that all numbers numerically (can be positive or negative) are less than 2, that is, any numbers between -2 and 2.

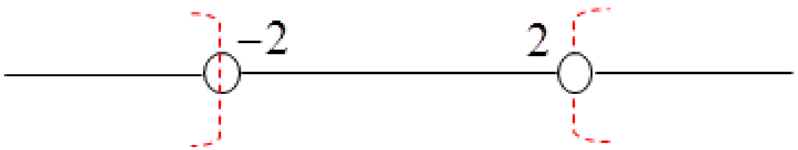


Thus $|t| < 2$ means $-2 < t < 2$

In general $|x| < a \Rightarrow -a < x < a$

Inequalities Involve Absolute Value

Similarly, the inequality $|t| > 2$ means that all numbers numerically (can be positive or negative) are greater than 2, that is, any numbers less than -2 or more than 2.



$$\boxed{\text{In general } |x| > a \Rightarrow x < -a \text{ or } x > a}$$

Example 4 [TB PA5]

Evaluate $a.\ |\pi - 5| + 3 \quad b.\ |\sqrt{3} - 2| + |2 - \sqrt{3}|.$

$a.\ |\pi - 5| + 3 = -(\pi - 5) + 3 = -\pi + 8$

$b.\ |\sqrt{3} - 2| + |2 - \sqrt{3}| = -(\sqrt{3} - 2) + 2 - \sqrt{3} = 4 - 2\sqrt{3}$

Example 6 [TB PA5]

Solve the inequalities $|2x-3| \leq 1$.

$$|2x-3| \leq 1$$

$$-1 \leq 2x-3 \leq 1$$

$$2 \leq 2x \leq 4$$

$$1 \leq x \leq 2$$

Example 7 [TB PA5]

Solve $|2x+3| \geq 5$

$2x+3 \geq 5$ or $2x+3 \leq -5$

$x \geq 1$ or $x \leq -4$

so the solution is $\{x \mid x \leq -4 \text{ or } x \geq 1\} = (-\infty, -4] \cup [1, \infty)$

Example 8 [TB PA6]

If $|x-2|<0.1$ and $|y-3|<0.2$, find an upper bound for $|x+y-5|$.

$$|x+y-5| = |(x-2)+(y-3)|$$

$$\begin{aligned} |(x-2)+(y-3)| &\le |x-2|+|y-3| \qquad \text{Property 8} \qquad |a+b| \le |a|+|b| \\ &< 0.1+0.2 \\ &< 0.3 \end{aligned}$$

$$\therefore \quad |x+y-5|<0.3$$