113-1 ENGINEERING MATHEMATICS HW2

Part I: : Use reduction of order to find a second solution $y_2(x)$.

1. $y'' + 9y = 0, \; y_1 = \sin 3x$

Define $y = u(x)\sin 3x$ so

$$y' = 3u\cos 3x + u'\sin 3x, \quad y'' = u''\sin 3x + 6u'\cos 3x - 9u\sin 3x,$$

and

$$y'' + 9y = (\sin 3x)u'' + 6(\cos 3x)u' = 0 \quad \text{or} \quad u'' + 6(\cot 3x)u' = 0.$$

If $w = u'$ we obtain the linear first-order equation $w' + 6(\cot 3x)w = 0$, which has the integrating factor $e^{6\int \cot 3x\,dx} = \sin^2 3x$. Now

$$\frac{d}{dx}[(\sin^2 3x)w] = 0 \quad \text{gives} \quad (\sin^2 3x)w = c.$$

Therefore $w = u' = c\csc^2 3x$ and $u = c_1 \cot 3x$. A second solution is $y_2 = \cot 3x \sin 3x = \cos 3x$.

2. $y'' - 25y = 0, \; y_1 = e^{5x}$

Define $y = u(x)e^{5x}$ so

$$y' = 5e^{5x}u + e^{5x}u', \quad y'' = e^{5x}u'' + 10e^{5x}u' + 25e^{5x}u$$

and

$$y'' - 25y = e^{5x}(u'' + 10u') = 0 \quad \text{or} \quad u'' + 10u' = 0.$$

If $w = u'$ we obtain the linear first-order equation $w' + 10w = 0$, which has the integrating factor $e^{10\int dx} = e^{10x}$. Now

$$\frac{d}{dx}[e^{10x}w] = 0 \quad \text{gives} \quad e^{10x}w = c.$$

Therefore $w = u' = ce^{-10x}$ and $u = c_1 e^{-10x}$. A second solution is $y_2 = e^{-10x}e^{5x} = e^{-5x}$.

3. $x^2y'' - 3xy' + 4y = 0, \ y_1 = x^2$

Sol :

$$y'' - \frac{3}{x}y' + \frac{4}{x^2} = 0$$

$$y_2(x) = x^2 \int \frac{e^{3\int \frac{1}{x}dx}}{x^4} dx = x^2 \int \frac{1}{x} dx = x^2 \ln x$$

Part II: Solve the given differential equation by undetermined coefficients .

4. $\frac{1}{4}y'' + y' + y = x^2 - 2x$

Sol:

From $\frac{1}{4}m^2 + m + 1 = 0$ we find $m_1 = m_2 = -2$. Then $y_c = c_1e^{-2x} + c_2xe^{-2x}$ and we assume $y_p = Ax^2 + Bx + C$. Substituting into the differential equation we obtain $A = 1$, $2A + B = -2$, and $\frac{1}{2}A + B + C = 0$. Then $A = 1$, $B = -4$, $C = \frac{7}{2}$, $y_p = x^2 - 4x + \frac{7}{2}$, and

$$y = c_1e^{-2x} + c_2xe^{-2x} + x^2 - 4x + \frac{7}{2}.$$

5. $y'' + 2y' + y = \sin x + 3\cos 2x$

Sol :

From $m^2 + 2m + 1 = 0$ we find $m_1 = m_2 = -1$. Then $y_c = c_1e^{-x} + c_2xe^{-x}$ and we assume $y_p = A\cos x + B\sin x + C\cos 2x + D\sin 2x$. Substituting into the differential equation we obtain $2B = 0$, $-2A = 1$, $-3C + 4D = 3$, and $-4C - 3D = 0$. Then $A = -\frac{1}{2}$, $B = 0$, $C = -\frac{9}{25}$, $D = \frac{12}{25}$, $y_p = -\frac{1}{2}\cos x - \frac{9}{25}\cos 2x + \frac{12}{25}\sin 2x$, and

$$y = c_1e^{-x} + c_2xe^{-x} - \frac{1}{2}\cos x - \frac{9}{25}\cos 2x + \frac{12}{25}\sin 2x.$$

6. $y'' - 2y' + 5y = e^x \cos 2x$

Sol:

From $m^2 - 2m + 5 = 0$ we find $m_1 = 1 + 2i$ and $m_2 = 1 - 2i$. Then $y_c = e^x(c_1 \cos 2x + c_2 \sin 2x)$ and we assume $y_p = Axe^x \cos 2x + Bxe^x \sin 2x$. Substituting into the differential equation we obtain $4B = 1$ and $-4A = 0$. Then $A = 0$, $B = \frac{1}{4}$, $y_p = \frac{1}{4} xe^x \sin 2x$, and

$$y = e^x(c_1 \cos 2x + c_2 \sin 2x) + \frac{1}{4} xe^x \sin 2x.$$

7. $\dfrac{d^3x}{dt^3} - \dfrac{d^2x}{dt^2} - 4x = e^{2t}$

Sol:

$m^3 - m^2 - 4 = 0 \Rightarrow m = 2, -\frac{1}{2} \pm \frac{\sqrt{7}}{2} i$

$\Rightarrow x_c = c_1 e^{2t} + e^{-\frac{t}{2}} \left[c_2 \cos\left(\frac{\sqrt{7}}{2} t\right) + c_3 \sin\left(\frac{\sqrt{7}}{2} t\right) \right]$

$x_p = Ate^{2t}$

$' \quad Ae^{2t} + 2Ate^{2t}$

$'' \quad 2Ae^{2t} + 2Ae^{2t} + 4Ate^{2t} = 4Ae^{2t} + 4Ate^{2t}$

$''' \quad 4Ae^{2t} + 4Ae^{2t} + 4Ae^{2t} + 8Ate^{2t} = 12Ae^{2t} + 8Ate^{2t}$

$\Rightarrow 12Ae^{2t} + 8Ate^{2t} - 4Ae^{2t} - 4Ate^{2t} - 4Ate^{2t} = e^{2t}$

$\Rightarrow 8Ae^{2t} = e^{2t}$

$\Rightarrow A = \frac{1}{8}$

$x = x_c + x_p = C_1 e^{2t} + e^{-\frac{t}{2}} \left[C_2 \cos\left(\frac{\sqrt{7}}{2} t\right) + C_3 \sin\left(\frac{\sqrt{7}}{2} t\right) \right] + \frac{1}{8} te^{2t}$

8. $3y'' + y = 0$

Sol:

From $3m^2 + 1 = 0$ we obtain $m_1 = i/\sqrt{3}$ and $m_2 = -i/\sqrt{3}$ so that

$$y = c_1 \cos(x/\sqrt{3}) + c_2 \sin(x/\sqrt{3}).$$

Part III: Solve the given initial-value problem.

9. $y'' + 4y' + 4y = (3 + x)e^{-2x}$, y(0)=2, y'(0)=5

Sol:

We have $y_c = c_1 e^{-2x} + c_2 x e^{-2x}$ and we assume $y_p = (Ax^3 + Bx^2)e^{-2x}$. Substituting into the differential equation we find $A = \frac{1}{6}$ and $B = \frac{3}{2}$. Thus $y = c_1 e^{-2x} + c_2 x e^{-2x} + \left(\frac{1}{6}x^3 + \frac{3}{2}x^2\right) e^{-2x}$. From the initial conditions we obtain $c_1 = 2$ and $c_2 = 9$, so

$$y = 2e^{-2x} + 9xe^{-2x} + \left(\frac{1}{6}x^3 + \frac{3}{2}x^2\right) e^{-2x}.$$

Part IV: Solve the given boundary-value problem

10. $y'' + y = x^2 + 1,\ y(0) = 5, y(1) = 0$

Sol:

We have $y_c = c_1 \cos x + c_2 \sin x$ and we assume $y_p = Ax^2 + Bx + C$. Substituting into the differential equation we find $A = 1$, $B = 0$, and $C = -1$. Thus $y = c_1 \cos x + c_2 \sin x + x^2 - 1$. From $y(0) = 5$ and $y(1) = 0$ we obtain

$$\begin{aligned} c_1 - 1 &= 5 \\ (\cos 1)c_1 + (\sin 1)c_2 &= 0. \end{aligned}$$

Solving this system we find $c_1 = 6$ and $c_2 = -6 \cot 1$. The solution of the boundary-value problem is

$$y = 6 \cos x - 6(\cot 1) \sin x + x^2 - 1.$$