HW2

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Part 1: Use reduction of order to find a second solution

$\Rightarrow y'' + p(x)y' + q(x)y = 0$

1) $y'' + \underbrace{9}_{q} y = 0$, $y_1 = \sin 3x$

$\Rightarrow y_2 = y_1 \int \frac{e^{-\int p\,dx}}{y_1^2} dx$

$\Rightarrow \because p = 0$
$\therefore -\int p\,dx = 0$

$\Rightarrow y_2 = \sin 3x \int (\sin 3x)^{-2} dx$
$\therefore y_2 = \sin 3x [-\frac{1}{3} \cot 3x]$
$\therefore y_2 = -\frac{1}{3} \cos 3x$ #

$\because (\sin 3x)^{-2} = \operatorname{cosec}^2 3x$

2) $y'' \underbrace{-25}_{q} y = 0$, $y_1 = e^{5x}$

$\Rightarrow \because p = 0$
$\therefore -\int p\,dx = 0$

$\Rightarrow y_2 = e^{5x} \int e^{-10x} dx$
$\therefore y_2 = e^{5x} \cdot e^{-10x} \cdot -\frac{1}{10}$
$\therefore y_2 = -\frac{1}{10} e^{-5x}$ #

3) $x^2 y'' - 3xy' + 4y = 0$, $y_1 = x^2$

$\Rightarrow y'' \underbrace{-\frac{3}{x}}_{p} y' + \underbrace{\frac{4}{x^2}}_{q} y = 0$

$\Rightarrow p = -3x^{-1}$
$\therefore -\int p\,dx = \int 3x^{-1} dx$
$= 3\ln|x|$

$\Rightarrow y_2 = x^2 \int \frac{e^{3\ln|x|}}{x^4} dx$
$y_2 = x^2 \int \frac{x^3}{x^4} dx$
$y_2 = x^2 \int \frac{1}{x} dx$
$\therefore y_2 = x^2 \cdot \ln|x|$ #

Part 2: Solve the given differential equation by undetermined coefficients

4) $\frac{1}{4} y'' + y' + y = x^2 - 2x$

$y_h \Rightarrow \frac{1}{4} y_h'' + y_h' + y_h = 0$
$\therefore \frac{1}{4}\lambda^2 + \lambda + 1 = 0$
$\Rightarrow \lambda^2 + 4\lambda + 4 = 0$
$\Rightarrow (\lambda + 2)^2 = 0$
$\therefore \lambda = -2$ (Two same real roots)
$\therefore y_h = C_1 e^{-2x} + C_2 x e^{-2x}$

$y_p \Rightarrow$ 猜 $y_p = k_0 x^2 + k_1 x + k_2$
$\Rightarrow y_p' = 2k_0 x + k_1$
$\Rightarrow y_p'' = 2k_0$

代入原 O.D.E.
$\Rightarrow \frac{1}{4}(2)k_0 + (2k_0)x + k_1 + (k_0)x^2 + (k_1)x + k_2 = x^2 - 2x$
$\Rightarrow x^2(k_0) + x(2k_0 + k_1) + (\frac{1}{2}k_0 + k_1 + k_2) = x^2 - 2x$
$\therefore k_0 = 1$
$\therefore k_1 = -2 - 2 = -4$
$\therefore k_2 = -\frac{1}{2} - (-4) = \frac{7}{2}$
$\therefore y_p = x^2 - 4x + \frac{7}{2}$
$\Rightarrow y = y_h + y_p$
$= C_1 e^{-2x} + C_2 x e^{-2x} + x^2 - 4x + \frac{7}{2}$ #

5) $y'' + 2y' + y = \sin x + 3\cos 2x$

$y_h \Rightarrow y_h'' + 2y_h' + y_h = 0$
$\therefore \lambda^2 + 2\lambda + 1 = 0$
$(\lambda + 1)^2 = 0$
$\therefore \lambda = -1$ (Two same real roots)

$\therefore y_h = C_1 e^{-x} + C_2 x e^{-x}$

$y_p \Rightarrow$ 猜 $y_p = k_0 \cos x + k_1 \sin x + k_2 \cos 2x + k_3 \sin 2x$
$y_p' = -k_0 \sin x + k_1 \cos x - 2k_2 \sin 2x + 2k_3 \cos 2x$
$y_p'' = -k_0 \cos x - k_1 \sin x - 4k_2 \cos 2x - 4k_3 \sin 2x$

代入原 O.D.E.
$\Rightarrow \cos x(-k_0 + 2k_1 + k_0) + \sin x(k_1 - 2k_0 - k_1) + \cos 2x(k_2 + 4k_3 - 4k_2)$
$+ \sin 2x(k_3 - 4k_2 - 4k_3) = \sin x + 3\cos 2x$
$\therefore k_0 = -\frac{1}{2}$
$k_1 = 0$

$\begin{cases} -3k_2 + 4k_3 = 3 & — ① \\ -4k_2 - 3k_3 = 0 & — ② \end{cases}$

①×3 + ②×4 :
$\Rightarrow -9k_2 - 16k_2 = 9$
$\therefore k_2 = -\frac{9}{25}$
$\therefore k_3 = (\frac{9}{25})(\frac{4}{3}) = \frac{12}{25}$
$\therefore y_p = -\frac{1}{2}\cos x - \frac{9}{25}\cos 2x + \frac{12}{25}\sin 2x$
$\Rightarrow y = y_h + y_p$
$= C_1 e^{-x} + C_2 x e^{-x} - \frac{1}{2}\cos x - \frac{9}{25}\cos 2x + \frac{12}{25}\sin 2x$ #

6) $y'' - 2y' + 5y = e^x \cos 2x$

$y_h \Rightarrow y_h'' - 2y_h' + 5y_h = 0$
$\Rightarrow \lambda^2 - 2\lambda + 5 = 0$
$\Rightarrow \lambda = \frac{-(-2) \pm \sqrt{4 - 20}}{2}$
$= 1 \pm 2i$ (Two different imaginary roots)

$\therefore y_h = e^x (C_1 \cos 2x + C_2 \sin 2x)$

$y_p \Rightarrow$ 猜 $y_p = xe^x (A\cos 2x + B\sin 2x)$
$\Rightarrow y_p' = xe^x(-2A\sin 2x + 2B\cos 2x) + (A\cos 2x + B\sin 2x)(xe^x + e^x)$
$= e^x(A\cos 2x + B\sin 2x) + xe^x(-2A\sin 2x + B\sin 2x + A\cos 2x + 2B\cos 2x)$
$\Rightarrow y_p'' = e^x(-2A\sin 2x + 2B\cos 2x + A\cos 2x + B\sin 2x)$
$+ xe^x(-4A\cos 2x + 2B\cos 2x - 2A\sin 2x - 4B\sin 2x)$
$+ (-2A\sin 2x + B\sin 2x + A\cos 2x + 2B\cos 2x)(xe^x + e^x)$
$= e^x(-4A\sin 2x + 2B\sin 2x + 2A\cos 2x + 4B\cos 2x)$
$+ xe^x(-4A\sin 2x - 3B\sin 2x - 3A\cos 2x + 4B\cos 2x)$

$\Rightarrow$ 代入原本 O.D.E.
$\Rightarrow$ By comparing coeff.
$\Rightarrow e^x \cos 2x = e^x(-2A\cos 2x - 2B\sin 2x - 4A\sin 2x + 2B\sin 2x + 2A\cos 2x + 4B\cos 2x)$
$= e^x[\cos 2x(4B) + \sin 2x(-4A)]$
$\therefore A = 0$, $B = \frac{1}{4}$
$\therefore y_p = xe^x(\frac{1}{4}\sin 2x)$
$\Rightarrow y = y_h + y_p$
$= e^x(C_1\cos 2x + C_2\sin 2x) + \frac{1}{4}xe^x(\sin 2x)$ #

7) $\frac{d^3x}{dt^3} - \frac{d^2x}{dt^2} - 4x = e^{2t}$

$\Rightarrow x''' - x'' - 4x = e^{2t}$
$x_h \Rightarrow x_h''' - x_h'' - 4x_h = 0$
$\Rightarrow \lambda^3 - \lambda^2 - 4 = 0$
$(\lambda - 2)(\lambda^2 + \lambda + 2) = 0$
$\therefore \lambda = 2$ or $\lambda = \frac{-1 \pm \sqrt{1-8}}{2}$
$= -\frac{1}{2} \pm \frac{\sqrt{7}}{2} i$
$\therefore x_h = C_1 e^{2t} + e^{-\frac{1}{2}t}(C_2 \cos \frac{\sqrt{7}}{2} t + C_3 \sin \frac{\sqrt{7}}{2} t)$

$x_p \Rightarrow$ 猜 $x_p = Ate^{2t}$
$\Rightarrow x_p' = 2Ate^{2t} + Ae^{2t}$
$\Rightarrow x_p'' = 4Ate^{2t} + 2Ae^{2t} + 2Ae^{2t} = 4Ate^{2t} + 4Ae^{2t}$
$\Rightarrow x_p''' = 8Ate^{2t} + 4Ae^{2t} + 8Ae^{2t} = 8Ate^{2t} + 12Ae^{2t}$
$\Rightarrow$ 代入原本 O.D.E.
$\Rightarrow e^{2t}(12A - 4A) + te^{2t}(8A - 4A - 4A) = e^{2t}$
$\therefore A = \frac{1}{8}$
$\therefore x_p = \frac{1}{8}te^{2t}$
$\Rightarrow x = x_h + x_p$
$= C_1 e^{2t} + e^{-\frac{1}{2}t}(C_2 \cos \frac{\sqrt{7}}{2} t + C_3 \sin \frac{\sqrt{7}}{2} t) + \frac{1}{8}te^{2t}$ #

8) $3y'' + y = 0$

$y_h \Rightarrow 3y_h'' + y_h = 0$
$\Rightarrow 3\lambda^2 + 1 = 0$
$\therefore \lambda = \pm \frac{\sqrt{3}}{3} i$
$\therefore y_h = C_1 \cos \frac{\sqrt{3}}{3} x + C_2 \sin \frac{\sqrt{3}}{3} x$

$\because r(x) = 0$
$\therefore y = y_h$
$= C_1 \cos \frac{\sqrt{3}}{3} x + C_2 \sin \frac{\sqrt{3}}{3} x$ #

Part 3: Solve the given initial-value problem.

9) $y'' + 4y' + 4y = (3 + x)e^{-2x}$, $y(0) = 2$, $y'(0) = 5$

$y_h \Rightarrow y_h'' + 4y_h' + 4y_h = 0$
$\Rightarrow \lambda^2 + 4\lambda + 4 = 0$
$(\lambda + 2)^2 = 0$
$\therefore \lambda = -2$ (Two same real roots)
$\therefore y_h = C_1 e^{-2x} + C_2 x e^{-2x}$

$y_p \Rightarrow$ 猜 $y_p = x^2(Ax + B)e^{-2x} = (Ax^3 + Bx^2)e^{-2x}$
$\Rightarrow y_p' = -2(Ax^3 + Bx^2)e^{-2x} + (3Ax^2 + 2Bx)e^{-2x} = e^{-2x}[x^3(-2A) + x^2(-2B + 3A) + x(2B)]$
$\Rightarrow y_p'' = 4(Ax^3 + Bx^2)e^{-2x} + (-2)(3Ax^2 + 2Bx)e^{-2x} - 2(3Ax^2 + 2Bx)e^{-2x} + (6Ax + 2B)e^{-2x}$
$= e^{-2x}[x^3(4A) + x^2(4B - 6A - 6A) + x(-4B - 4B + 6A) + 2B]$
$= e^{-2x}[x^3(4A) + x^2(4B - 12A) + x(-8B + 6A) + 2B]$
$\Rightarrow$ 代入原 O.D.E.
$\Rightarrow$ By comparing coeff. in e^{-2x}
↳ For x^3 part, $\Rightarrow 4A + 4(-2A) + 4(A) = 0 \Rightarrow 0 = 0$
↳ For x^2 part, $\Rightarrow 4B - 12A + 4(-2B + 3A) + 4(B) = 0 \Rightarrow 0 = 0$
↳ For x part, $\Rightarrow -8B + 6A + 4(2B) = 1 \Rightarrow 6A = 1 \therefore A = \frac{1}{6}$
↳ For const. part, $\Rightarrow 2B = 3 \therefore B = \frac{3}{2}$
$\therefore y_p = \frac{1}{6}x^3 e^{-2x} + \frac{3}{2}x^2 e^{-2x}$
$\Rightarrow y = y_h + y_p$
$\therefore y = C_1 e^{-2x} + C_2 x e^{-2x} + \frac{1}{6}x^3 e^{-2x} + \frac{3}{2}x^2 e^{-2x}$

$\Rightarrow$ As $y(0) = 2$, $y'(0) = 5$
$\Rightarrow y = C_1 e^{-2x} + C_2 x e^{-2x} + \frac{1}{6}x^3 e^{-2x} + \frac{3}{2}x^2 e^{-2x}$
↳ when $x = 0$,
$\Rightarrow y = C_1 = 2$
$\Rightarrow y' = -4e^{-2x} - 2C_2 x e^{-2x} + C_2 e^{-2x} + \cdots$
↳ when $x = 0$,
$\Rightarrow y' = -4 - 0 + C_2 + 0 = 5 \Rightarrow \therefore C_2 = 9$

$\therefore y = 2e^{-2x} + 9xe^{-2x} + \frac{1}{6}x^3 e^{-2x} + \frac{3}{2}x^2 e^{-2x}$ #

Part 4: Solve the given boundary-value problem

10) $y'' + y = x^2 + 1$, $y(0) = 5$, $y(1) = 0$

$y_h \Rightarrow y_h'' + y_h = 0$
$\Rightarrow \lambda^2 + 1 = 0$
$\therefore \lambda = \pm i$
$\therefore y_h = C_1 \cos x + C_2 \sin x$

$y_p \Rightarrow$ 猜 $y_p = Ax^2 + Bx + C$
$\Rightarrow y_p' = 2Ax + B$
$\Rightarrow y_p'' = 2A$

$\Rightarrow$ 代入原本 O.D.E.
$\Rightarrow$ By comparing coeff.
↳ For x^2 : $\Rightarrow \therefore A = 1$
↳ For x : $\Rightarrow \therefore B = 0$
↳ For const. : $\Rightarrow 2A + C = 1$
$\therefore C = -1$

$\therefore y_p = x^2 - 1$

$\Rightarrow y = y_h + y_p$
$= C_1 \cos x + C_2 \sin x + x^2 - 1$

$\Rightarrow$ As $y(0) = 5$, $y(1) = 0$
$\Rightarrow$ when $x = 0$,
↳ $y = C_1 - 1 = 5$
$\therefore C_1 = 6$

$\Rightarrow$ when $x = 1$,
↳ $y = 6\cos(1) + C_2 \sin(1) = 0$
$\therefore C_2 = \frac{-6\cos(1)}{\sin(1)}$
$= -6\cot(1)$

$\therefore y = 6\cos x - 6\cot(1) \cdot \sin x + x^2 - 1$ #