HW3

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Part 1: Use Definition 4.1.1 to find $\mathcal{L}\{f(t)\}$

1) $f(t) = e^{-t}\sin t$

$\Rightarrow \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

$\Rightarrow \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}\cdot(e^{-t}\sin t)\,dt$

$= \int_0^\infty e^{-(s+1)t}\cdot \sin t\,dt$

↳ Let $u = \sin t$, $dv = e^{-(s+1)t}\,dt$

$\therefore du = \cos t\,dt$, $v = -\frac{1}{s+1}e^{-(s+1)t}$

$\int_0^\infty e^{-(s+1)t}\cdot\sin t\,dt = (\sin t)\left(-\frac{1}{s+1}e^{-(s+1)t}\right)\Big|_0^\infty + \int_0^\infty \frac{1}{s+1}e^{-(s+1)t}\cdot\cos t\,dt$

$= (0-0) + \int_0^\infty \frac{1}{s+1}e^{-(s+1)t}\cdot\cos t\,dt$, $\frac{1}{s+1} > 0$

$= \frac{1}{s+1}\int_0^\infty e^{-(s+1)t}\cdot\cos t\,dt$

↳ Let $u = \cos t$, $dv = e^{-(s+1)t}\,dt$

$du = -\sin t\,dt$, $v = -\frac{1}{s+1}e^{-(s+1)t}$

$= \frac{1}{s+1}\left[\cos t\cdot\left(-\frac{1}{s+1}e^{-(s+1)t}\right)\Big|_0^\infty - \int_0^\infty \frac{1}{s+1}\cdot e^{-(s+1)t}\cdot\sin t\,dt\right]$

$= \frac{1}{s+1}\left[\left(0+\frac{1}{s+1}\right) - \frac{1}{s+1}\int_0^\infty e^{-(s+1)t}\cdot\sin t\,dt\right]$

$\int_0^\infty e^{-(s+1)t}\cdot\sin t\,dt = \frac{1}{(s+1)^2} - \frac{1}{(s+1)^2}\int_0^\infty e^{-(s+1)t}\cdot\sin t\,dt$

$\therefore F(s) = \frac{1}{(s+1)^2} - \frac{1}{(s+1)^2}F(s)$

$F(s) + \frac{1}{(s+1)^2}F(s) = \frac{1}{(s+1)^2}$

$F(s)\left(1+\frac{1}{(s+1)^2}\right) = \frac{1}{(s+1)^2}$

$F(s) = \frac{1}{(s+1)^2}\times\frac{(s+1)^2}{(s+1)^2+1}$

$\therefore F(s) = \frac{1}{(s+1)^2+1}$ #

Part 2: Use Theorem 4.1.1 to find $\mathcal{L}\{f(t)\}$

2) $f(t) = e^t\sinh t$

$\Rightarrow \mathcal{L}\{\sinh kt\} = \frac{k}{s^2-k^2}$

$\Rightarrow \mathcal{L}\{e^t\cdot\sinh t\}$

↳ $\mathcal{L}\{\sinh t\} = \frac{1}{s^2-1}$

↳ $\mathcal{L}\{e^t\cdot\sinh t\} = \frac{1}{s^2-1}\Big|_{s\to s-1}$

$= \frac{1}{(s-1)^2-1}$ #

Part 3: Find $\mathcal{L}\{f(t)\}$ by first using a trigonometric identity

3) $f(t) = 2\sin^2\frac{t}{2}$

$\Rightarrow \cos 2t = 1-2\sin^2 t$

$\Rightarrow f(t) = 2\sin^2\frac{t}{2}$

$= 1-\cos t$

$\Rightarrow \mathcal{L}\{1-\cos t\} = \frac{1}{s} - \frac{s}{s^2+1}$ #

Part 4: Find the given inverse transform.

4) $\mathcal{L}^{-1}\left\{\frac{s+1}{s^2-4s}\right\}$

(Solution 1)

$\Rightarrow \frac{s+1}{s^2-4s} = \frac{s+1}{(s-2)^2-4}$

$= \frac{s-2}{(s-2)^2-4} + \frac{3}{2}\frac{2}{(s-2)^2-4}$

$\therefore \mathcal{L}^{-1}\left\{\frac{s+1}{s^2-4s}\right\} = \mathcal{L}^{-1}\left\{\frac{s-2}{(s-2)^2-4} + \frac{3}{2}\frac{2}{(s-2)^2-4}\right\}$

$\therefore f(t) = e^{2t}\cosh 2t + \frac{3}{2}e^{2t}\sinh 2t$ #

(Solution 2)

$\Rightarrow \frac{s+1}{s^2-4s} = \frac{s+1}{s(s-4)}$

$\Rightarrow \frac{s+1}{s(s-4)} = \frac{A}{s} + \frac{B}{s-4}$

$\therefore A = \frac{1}{-4} = -\frac{1}{4}$

$\therefore B = \frac{4+1}{4} = \frac{5}{4}$

$\therefore \frac{s+1}{s(s-4)} = -\frac{1}{4}\left(\frac{1}{s}\right) + \frac{5}{4}\left(\frac{1}{s-4}\right)$

$\therefore \mathcal{L}^{-1}\left\{\frac{s+1}{s^2-4s}\right\} = \mathcal{L}^{-1}\left\{-\frac{1}{4}\left(\frac{1}{s}\right) + \frac{5}{4}\left(\frac{1}{s-4}\right)\right\}$

$= -\frac{1}{4}H(t) + \frac{5}{4}e^{4t}$ #

5) $\mathcal{L}^{-1}\left\{\frac{6s+3}{s^4+5s^2+4}\right\}$

$\Rightarrow \frac{6s+3}{s^4+5s^2+4} = \frac{6s+3}{(s^2+4)(s^2+1)}$

$\Rightarrow$ Let $\frac{6s+3}{(s^2+4)(s^2+1)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+1}$

$6s+3 = (As+B)(s^2+1) + (Cs+D)(s^2+4)$

↳ By comparing coef.

s^3 : $A+C=0$ — ①

s^2 : $B+D=0$ — ②

s : $A+4C=6$ — ③

constant : $B+4D=3$ — ④

$\Rightarrow$ From ① : $A=-C$

$\therefore -C+4C=6$

$3C=6$ $\therefore C=2$ $\therefore A=-2$

$\Rightarrow$ From ② : $B=-D$

$\therefore -D+4D=3$

$\therefore D=1$ $\therefore B=-1$

$\therefore \frac{6s+3}{(s^2+4)(s^2+1)} = \frac{-2s-1}{s^2+4} + \frac{2s+1}{s^2+1}$

$\Rightarrow \frac{-2s-1}{s^2+4} = -2\left(\frac{s}{s^2+4}\right) - \frac{1}{2}\left(\frac{2}{s^2+4}\right)$ $\Rightarrow$ $\frac{2s+1}{s^2+1} = 2\left(\frac{s}{s^2+1}\right) + \frac{1}{s^2+1}$

$\therefore \mathcal{L}^{-1}\left\{\frac{6s+3}{s^4+5s^2+4}\right\} = \mathcal{L}^{-1}\left\{\frac{-2s-1}{s^2+4} + \frac{2s+1}{s^2+1}\right\}$

$\therefore f(t) = -2\cos 2t - \frac{1}{2}\sin 2t + 2\cos t + \sin t$ #

Part 5: Use the Laplace transform to solve the given initial-value problem

6) $y'' - 4y' + 4y = t^3e^{2t}$, $y(0)=0$, $y'(0)=0$

$\Rightarrow \mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0)$

$\Rightarrow \mathcal{L}\{y'\} = sY(s) - y(0)$

$\Rightarrow \mathcal{L}\{y\} = Y(s)$

$\Rightarrow \mathcal{L}\{t^3e^{2t}\} = \frac{3!}{s^4}\Big|_{s\to s-2}$

$= \frac{6}{(s-2)^4}$

$\Rightarrow \mathcal{L}\{y''-4y'+4y\} = \mathcal{L}\{t^3e^{2t}\}$

$[s^2Y(s) - sy(0) - y'(0)] - 4[sY(s) - y(0)] + 4Y(s) = \frac{6}{(s-2)^4}$

$s^2Y(s) - 4sY(s) + 4Y(s) = \frac{6}{(s-2)^4}$

$(s^2-4s+4)Y(s) = \frac{6}{(s-2)^4}$

$Y(s) = \frac{6}{(s-2)^4(s^2-4s+4)}$

$Y(s) = \frac{6}{(s-2)^6}$

$Y(s) = \frac{6}{5!}\left(\frac{5!}{(s-2)^6}\right)$

$\therefore y(t) = \frac{1}{20}e^{2t}t^5$ #

Part 6: Find either $F(s)$ or $f(t)$

7) $\mathcal{L}\{e^{3t}(9-4t+10\sin\frac{t}{2})\}$

$\Rightarrow \mathcal{L}\{9-4t+10\sin\frac{t}{2}\} = \frac{9}{s} - \frac{4}{s^2} + \frac{10\cdot\frac{1}{2}}{s^2+\frac{1}{4}}$

$= \frac{9}{s} - \frac{4}{s^2} + \frac{5}{s^2+\frac{1}{4}} = \frac{9}{s} - \frac{4}{s^2} + \frac{20}{4s^2+1}$

$\Rightarrow \mathcal{L}\{e^{3t}(9-4t+10\sin\frac{t}{2})\} = \frac{9}{s} - \frac{4}{s^2} + \frac{20}{4s^2+1}\Big|_{s\to s-3}$

$\therefore F(s) = \frac{9}{s-3} - \frac{4}{(s-3)^2} + \frac{20}{4(s-3)^2+1}$ #

8) $\mathcal{L}^{-1}\left\{\frac{(s+1)^2}{(s+2)^4}\right\}$

$\Rightarrow \frac{(s+1)^2}{(s+2)^4} = \frac{s^2+2s+1}{(s+2)^4}$

$= \frac{(s+2)^2-2(s+2)+1}{(s+2)^4}$

$= \frac{1}{(s+2)^2} - \frac{2}{(s+2)^3} + \frac{1}{(s+2)^4}$

$\therefore f(t) = e^{-2t}\left(t - t^2 + \frac{1}{6}t^3\right)$ #

Part 7: Use the Laplace transform to solve the given integral equation.

9) $f(t) = te^t + \int_0^t \tau f(t-\tau)\,d\tau$ (Find $F(s)$, then find $f(t)$)

$\Rightarrow$ Using convolution thm,

$\Rightarrow \int_0^t \tau f(t-\tau)\,d\tau = \tau\cdot f(t)$

$\therefore \mathcal{L}\{\tau\cdot f(t)\} = \frac{1}{s^2}F(s)$

$\therefore f(t) = te^t + \int_0^t \tau f(t-\tau)\,d\tau$

$F(s) = \frac{1}{(s-1)^2} + \frac{1}{s^2}F(s)$

$\left(1-\frac{1}{s^2}\right)F(s) = \frac{1}{(s-1)^2}$

$F(s) = \frac{1}{(s-1)^2}\times\frac{s^2}{s^2-1}$

$\therefore F(s) = \frac{s^2}{(s-1)^3(s+1)}$ #

$\Rightarrow F(s) = \frac{s^2}{(s-1)^3(s+1)}$

$\Rightarrow \frac{s^2}{(s-1)^3(s+1)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{(s-1)^3} + \frac{D}{s+1}$

$\Rightarrow D = \frac{(-1)^2}{(-2)^3} = -\frac{1}{8}$

$\Rightarrow C = F(s)(s-1)^3\Big|_{s=1} = \frac{1}{2}$

$\Rightarrow B = \frac{d}{ds}\left[F(s)(s-1)^3\right]\Big|_{s=1} = \frac{d}{ds}\left(\frac{s^2}{s+1}\right)\Big|_{s=1} = \frac{2s(s+1)-s^2}{(s+1)^2}\Big|_{s=1}$

$= \frac{4-1}{4} = \frac{3}{4}$

$\Rightarrow A = \frac{1}{2!}\cdot\frac{d^2}{ds^2}\left[F(s)(s-1)^3\right]\Big|_{s=1} = \frac{1}{2}\frac{d}{ds}\left[\frac{s^2+2s}{(s+1)^2}\right]\Big|_{s=1}$

$= \frac{1}{2}\cdot\frac{(2s+2)(s+1)^2-(s^2+2s)2(s+1)}{(s+1)^4}\Big|_{s=1}$

$= \frac{16-12}{2\cdot 16} = \frac{1}{8}$

$\Rightarrow \therefore \frac{s^2}{(s-1)^3(s+1)} = \frac{1}{8(s-1)} + \frac{3}{4(s-1)^2} + \frac{1}{2(s-1)^3} - \frac{1}{8(s+1)}$

$\therefore f(t) = \frac{1}{8}e^t + \frac{3}{4}te^t + \frac{1}{4}t^2e^t - \frac{1}{8}e^{-t}$ #

Part 8: Find the convolution $f*g$ of the given functions. After integrating, find the Laplace transform of $f*g$.

10) $f(t) = \cos 2t$, $g(t) = e^t$

$\Rightarrow f(t)*g(t) = \int_0^t f(\lambda)g(t-\lambda)\,d\lambda$

$\Rightarrow f(t)*g(t) = \int_0^t \cos 2\lambda\, e^{(t-\lambda)}\,d\lambda$

$= e^t\int_0^t \cos 2\lambda\cdot e^{-\lambda}\,d\lambda$

$= e^t\left[\frac{1}{5} + \frac{2}{5}e^{-t}\sin 2t - \frac{1}{5}e^{-t}\cos 2t\right)$

$= \frac{1}{5}e^t + \frac{2}{5}\sin 2t - \frac{1}{5}\cos 2t$ #

$\Rightarrow \mathcal{L}\{f(t)*g(t)\} = \frac{1}{5(s-1)} + \frac{4}{5(s^2+4)} - \frac{s}{5(s^2+4)}$

$= \frac{s}{(s-1)(s^2+4)}$ #

$\Rightarrow \int_0^t \cos 2\lambda\cdot e^{-\lambda}\,d\lambda = (\cos 2\lambda)(-e^{-\lambda})\Big|_0^t - 2\int_0^t e^{-\lambda}\cdot\sin 2\lambda\,d\lambda$

Let $u = \cos 2\lambda$, $dv = e^{-\lambda}\,d\lambda$

$du = -2\sin 2\lambda\,d\lambda$ $v = -e^{-\lambda}$

$\Rightarrow \int_0^t \cos 2\lambda\cdot e^{-\lambda}\,d\lambda = -e^{-t}\cos 2t + 1 - 2\int_0^t e^{-\lambda}\cdot\sin 2\lambda\,d\lambda$

Let $u = \sin 2\lambda$, $dv = e^{-\lambda}\,d\lambda$

$du = 2\cos 2\lambda\,d\lambda$ $v = -e^{-\lambda}$

$\Rightarrow \int_0^t \cos 2\lambda\cdot e^{-\lambda}\,d\lambda = -e^{-t}\cos 2t + 1 - 2\left[-\sin 2\lambda\cdot e^{-\lambda}\Big|_0^t + 2\int_0^t e^{-\lambda}\cos 2\lambda\,d\lambda\right]$

$\int_0^t \cos 2\lambda\cdot e^{-\lambda}\,d\lambda = -e^{-t}\cos 2t + 1 - 2\left[-\sin 2t\cdot e^{-t} + 2\int_0^t e^{-\lambda}\cos 2\lambda\,d\lambda\right]$

$\int_0^t \cos 2\lambda\cdot e^{-\lambda}\,d\lambda = -e^{-t}\cos 2t + 1 + 2\sin 2t\cdot e^{-t} - 4\int_0^t e^{-\lambda}\cos 2\lambda\,d\lambda$

$\therefore \int_0^t \cos 2\lambda\cdot e^{-\lambda}\,d\lambda = \frac{1}{5}\left(1 + 2\sin 2t\cdot e^{-t} - e^{-t}\cos 2t\right)$