HW1

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Part 1 : State the order, degree, linear or non-linear

1) $t^5 y^{(4)} - t^3 y'' + 6y = 0$

$\Rightarrow$ Linear 4th order, 1st degree O.D.E. #

2) $\frac{d^2y}{dx^2} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$

$\Rightarrow$ 有理化 : $\left(\frac{d^2y}{dx^2}\right)^2 = 1+\left(\frac{dy}{dx}\right)^2$

$\therefore$ Non-linear 2nd order, 2nd degree O.D.E. #

3) $(1-e^y)\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = \cos 2x$

$\Rightarrow \frac{d^2y}{dx^2} - e^y \cdot \frac{d^2y}{dx^2} - 2\frac{dy}{dx} = \cos 2x$

$\therefore$ Non-linear 2nd order, 1st degree O.D.E. #

Part 2 : Determine if it is guaranteed to have a unique solution

4) $y' = e^{xy^2}$, $y(0) = 1$

$\Rightarrow f(x,y) = e^{xy^2}$ —— ①

$\Rightarrow \frac{\partial f(x,y)}{\partial y} = 2xy e^{xy^2}$ —— ②

$\Rightarrow$ when $x=0$, $y=1$

↳ ① exists

↳ ② exists

$\therefore$ It has an unique solution (guaranteed)

5) $y' = \sqrt{y}$, $y(0) = 0$

$\Rightarrow f(x,y) = y^{\frac{1}{2}}$ —— ①

$\Rightarrow \frac{\partial f(x,y)}{\partial y} = \frac{1}{2}y^{-\frac{1}{2}}$

$= \frac{1}{2\sqrt{y}}$ —— ②

$\Rightarrow$ when $x=0$, $y=0$

↳ ① exists ↳ ② does not exist $\because$ 分母不為 0

$\therefore$ Not sure that whether it has an unique solution

Part 3 : Solve the given differential equation.

6) $\underbrace{\left(2y^2-\frac{3}{x}\right)}_{M}dx+\underbrace{\left(2xy+\frac{4}{x}\right)}_{N}dy=0$

$\Rightarrow \frac{\partial M}{\partial y}=4y \quad , \quad \frac{\partial N}{\partial x}=2y-\frac{4}{x^2}$

$\because \frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x} \quad \therefore$ 非正合

$\Rightarrow \frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}=2y-\frac{4}{x^2}-4y=-2y-\frac{4}{x^2}$

$\Rightarrow$ 猜 $I(x)$

$\hookrightarrow \dfrac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{-N}dx=\dfrac{-2y-\frac{4}{x^2}}{-2xy-\frac{4}{x}}dx$

$=\frac{1}{x}dx$

$\because \frac{dx}{x}=\frac{dI}{I}$

$I=e^{\int\frac{1}{x}dx}=e^{\ln x}=x$

$\therefore (2xy^2-3)dx+(2x^2y+4)dy=0$

$\Rightarrow M=\frac{\partial u}{\partial x}$ $\qquad \Rightarrow N=\frac{\partial u}{\partial y}$

$\hookrightarrow u=\int(2xy^2-3)dx$ $\qquad \hookrightarrow u=\int(2x^2y+4)dy$

$=x^2y^2-3x+f(y)$ —① $\qquad =x^2y^2+4y+g(x)$ —②

$\hookrightarrow$ By comparing ① and ②,

$\Rightarrow f(y)=4y$

$\Rightarrow g(x)=-3x$

$\therefore u(x,y)=x^2y^2-3x+4y=C$ #

7) $(x-y^3+y^2\sin x)dx=(3xy^2+2y\cos x)dy$

$\Rightarrow \underbrace{(x-y^3+y^2\sin x)}_{M}dx\underbrace{-(3xy^2+2y\cos x)}_{N}dy=0$

$\Rightarrow \frac{\partial M}{\partial y}=-3y^2+2y\sin x \quad , \quad \frac{\partial N}{\partial x}=-3y^2+2y\sin x$

$\because \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x} \quad \therefore$ 正合

$\Rightarrow M=\frac{\partial u}{\partial x}$ $\qquad \Rightarrow N=\frac{\partial u}{\partial y}$

$u=\int(x-y^3+y^2\sin x)dx$ $\qquad u=-\int(3xy^2+2y\cos x)dy$

$\therefore u=\frac{1}{2}x^2-xy^3-y^2\cos x+f(y)$ $\qquad \therefore u=-xy^3-y^2\cos x+g(x)$

$$U=\begin{cases}\frac{1}{2}x^2-xy^3-y^2\cos x+f(y) & \text{—①}\\ -xy^3-y^2\cos x+g(x) & \text{—②}\end{cases}$$

$\hookrightarrow$ By comparing ① and ②,

$\Rightarrow f(y)=0$

$\Rightarrow g(x)=\frac{1}{2}x^2$

$\therefore u(x,y)=\frac{1}{2}x^2-xy^3-y^2\cos x=C$ #

8) $(1+\ln x+\frac{y}{x})dx=(1-\ln x)dy$

$\Rightarrow \underbrace{(1+\ln x+\frac{y}{x})}_{M}dx\underbrace{-(1-\ln x)}_{N}dy=0$

$\Rightarrow \frac{\partial M}{\partial y}=\frac{1}{x} \quad , \quad \frac{\partial N}{\partial x}=\frac{1}{x}$

$\because \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x} \quad \therefore$ 正合

$\Rightarrow M=\frac{\partial u}{\partial x}$ $\qquad$ * $\int\ln x\,dx=x\ln x-x+C$

$\hookrightarrow u=\int(1+\ln x+\frac{y}{x})dx$

$=x\ln x+y\ln x+f(y)$ —①

$\Rightarrow N=\frac{\partial u}{\partial y}$

$\hookrightarrow u=-\int(1-\ln x)dy$

$=-y+y\ln x+g(x)$ —②

$\hookrightarrow$ By Comparing ① and ②,

$\Rightarrow f(y)=-y$

$\Rightarrow g(x)=x\ln x$

$\therefore u(x,y)=x\ln x+y\ln x-y=C$ #

9) $\underbrace{6xy}_{M}dx+\underbrace{(4y+9x^2)}_{N}dy=0$

$\Rightarrow \frac{\partial M}{\partial y}=6x \quad , \quad \frac{\partial N}{\partial x}=18x$

$\because \frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x} \quad \therefore$ 非正合

$\Rightarrow \frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}=18x-6x=12x$

$\Rightarrow$ 猜 $I(y)$

$\hookrightarrow \dfrac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M}dy=\frac{12x}{6xy}dy=\frac{2}{y}dy$

$\therefore \frac{2dy}{y}=\frac{dI}{I}$

$\Rightarrow I=e^{2\int\frac{1}{y}dy}=e^{2\ln y}=y^2$

$\therefore 6xy^3dx+(4y^3+9x^2y^2)dy=0$

$\Rightarrow M=\frac{\partial u}{\partial x}$ $\qquad \Rightarrow N=\frac{\partial u}{\partial y}$

$u=\int 6xy^3dx$ $\qquad u=\int(4y^3+9x^2y^2)dy$

$=3x^2y^3+f(y)$ —① $\qquad =y^4+3x^2y^3+g(x)$ —②

$\hookrightarrow$ By Comparing ① and ②,

$\Rightarrow f(y)=y^4$

$\Rightarrow g(x)=0$

$\therefore u(x,y)=y^4+3x^2y^3=C$ #

Part 4: Solve the given initial-value problem

10) $\underbrace{(e^x + y)}_{M}\,dx + \underbrace{(2 + x + ye^y)}_{N}\,dy = 0$, $y(0) = 1$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{-e^x - y}{2 + x + ye^y}$

$\Rightarrow f(x,y) = \dfrac{-e^x - y}{2 + x + ye^y}$ —— ①

$\Rightarrow \dfrac{\partial f(x,y)}{\partial y} = \dfrac{-(2+x+ye^y) + (e^x + y)(ye^y + e^y)}{(2 + x + ye^y)^2}$ —— ②

$\Rightarrow$ when $x = 0$, $y = 1$

↳ ① exists

↳ ② exists

$\therefore$ It has an unique solution (guaranteed)

2) $\dfrac{\partial M}{\partial y} = 1$, $\dfrac{\partial N}{\partial x} = 1$

$\because \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ $\therefore$ 正合

$\Rightarrow M = \dfrac{\partial u}{\partial x}$

$u = \int (e^x + y)\,dx$

$= e^x + xy + f(y)$ — ①

$\Rightarrow N = \dfrac{\partial u}{\partial y}$

$u = \int (2 + x + ye^y)\,dy$

$= 2y + xy + ye^y - e^y + g(x)$ — ②

y	$(+)$	e^y
1	$(-)$	e^y
0		e^y

↳ By Comparing ① and ②,

$\Rightarrow f(y) = 2y + ye^y - e^y$

$g(x) = e^x$

$\therefore u(x,y) = e^x + xy + 2y + ye^y - e^y = C$

↳ when $x = 0$, $y = 1$

$\Rightarrow u(0,1) = 1 + 0 + 2 + e - e$

$= 3$

$\therefore u(x,y) = e^x + xy + 2y + ye^y - e^y = 3$ #