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Section 3.2

2) $\Rightarrow$ $\boxed{\text{Criteria for Subspaces}}$
- Must contain zero vector
- Must be closed under addition
- Must be closed under scalar multiplication

a) $S = \{ (x_1, x_2, x_3)^T \mid x_1 + x_3 = 1 \}$

$\Rightarrow$ Since the zero vector $(0,0,0)^T$ is not in the set because. $x_1 + x_3 \neq 1$

as when $x_1, x_3 = 0$.

$\therefore$ The set S does not form subspace of $\mathbb{R}^3$ because it does not contain

zero vector #

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b) $S = \{ (x_1, x_2, x_3)^T \mid x_1 = x_2 = x_3 \}$

$\Rightarrow$ $\mathbf{z} = (0,0,0)^T \in S$, the zero vector is in the set

$\Rightarrow$ i) If $x = (a,a,a)^T \in S$, then $\alpha x = (\alpha a, \alpha a, \alpha a)^T \in S$

ii) If $x = (a,a,a)^T \in S$ and $y = (b,b,b)^T \in S$,

then $x + y = (a+b, a+b, a+b)^T \in S$

$\therefore$ The set S is a subspace of $\mathbb{R}^3$ #

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c) $S = \{ (x_1, x_2, x_3)^T \mid x_3 = x_1 + x_2 \}$

$\Rightarrow$ $\mathbf{z} = (0,0,0)^T \in S$ as $0 = 0 + 0$, $\therefore S$ contains zero vector

$\Rightarrow$ i) If $x = (a, b, a+b)^T \in S$, then $\alpha x = (\alpha a, \alpha b, \alpha(a+b))^T \in S$

$\Rightarrow \alpha a + \alpha b = \alpha(a+b)$

ii) If $x = (a, b, a+b)^T \in S$ and $y = (c, d, c+d)^T \in S$,

then $x + y = (a+c, b+d, a+b+c+d)^T \in S$

as $a+c+b+d = a+b+c+d$

$\therefore$ The set S is a subspace of $\mathbb{R}^3$ #

d) $S = \{(x_1, x_2, x_3)^T \mid x_3 = x_1 \text{ or } x_3 = x_2\}$

$\Rightarrow \mathbf{0} = (0,0,0)^T \in S$, as $0 = 0$, $0 = 0$

$\Rightarrow$ i) If $\boxed{x = (a,b,a)^T \in S}$, then $\alpha x = (\alpha a, \alpha b, \alpha a)^T \in S$, if $\boxed{x = (a,b,b)^T \in S}$ $\Rightarrow$ then $\alpha x = (\alpha a, \alpha b, \alpha b) \in S$

ii) If $\boxed{x = (a,b,a)^T \in S}$ and $\boxed{y = (c,d,c)^T \in S}$,

then $x + y = (a+c, b+d, a+c)^T$ satisfies $x_3 = x_1$

$\Rightarrow$ However if $\boxed{y = (c,d,d)^T}$ that satisfies $x_3 = x_2$

then $x + y = (a+c, b+d, a+d)^T \notin S$, as $x_3 \neq x_1$, $x_3 \neq x_2$

$\therefore S$ is not a subspace of $\mathbb{R}^3$ #

13) $x_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $x_2 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$, $x = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$, $y = \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix}$

a) $\Rightarrow$ let $x = \alpha_1 x_1 + \alpha_2 x_2$

$$\begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$$

$$\Rightarrow \begin{cases} -1 = \alpha_1 + \alpha_2 & -① \\ 2 = 2\alpha_1 + 3\alpha_2 & -② \\ -1 = -\alpha_1 - 2\alpha_2 & -③ \end{cases}$$

$\Rightarrow$ As ① = ③ : $\alpha_1 + \alpha_2 = -\alpha_1 - 2\alpha_2$

$2\alpha_1 = -3\alpha_2$ — ④

Subs ④ into ②,

$\Rightarrow 2 = -3\alpha_2 + 3\alpha_2 = 0$

$\Rightarrow$ As $2 \neq 0$ $\therefore x \notin \text{Span}(x_1, x_2)$ #

b) $\Rightarrow$ let $y = \alpha_1 x_1 + \alpha_2 x_2$

$$\begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$$

$$\Rightarrow \begin{cases} -1 = \alpha_1 + \alpha_2 & — ① \\ 2 = 2\alpha_1 + 3\alpha_2 & — ② \\ -3 = -\alpha_1 - 2\alpha_2 & — ③ \end{cases}$$

$\Rightarrow$ ② − ① × 2 :

$\Rightarrow 4 = \alpha_2$ — ④

Subs ④ into ③

$\Rightarrow \alpha_1 = 3 - 8 = -5$

$\boxed{\therefore \alpha_1 = -5, \ \alpha_2 = 4}$

$\therefore y \in \text{Span}(x_1, x_2)$ #

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⇒ $\boxed{\text{Case 1 : } N(A) = \{0\}}$

⇒ If $N(A) = \{0\}$, so the null space of (A) contains only zero vector

⇒ ∴ Matrix A has full column rank (rank = 3)

∴ The columns of A are linearly independent

⇒ ∵ Matrix A is 4×3, ∴ 3 linearly independent columns

∴ $Ax = b$

⇒ Possible solutions : ① No solution, if b is not in the column space of A

② Exactly one solution, if b is in the column space of A

⇒ $\boxed{\text{Case 2 : } N(A) \neq \{0\}}$

⇒ It means the null space of A contains non-zero vectors, matrix A does not have full column rank (less than 3)

∴ The columns are not linearly independent

∴ $Ax = b$

⇒ Possible solutions : ① No solution, if b is not in the column space of A

② Infinitely many solutions, if b is in the column space of A, because there are free variables due to the non-zero null space

Section 3.3

7) $\Rightarrow$ Let $y_1 = ay_2 + by_3$

$\Rightarrow$ Given: $\begin{cases} y_1 = x_2 - x_1 \\ y_2 = x_3 - x_2 \\ y_3 = x_3 - x_1 \end{cases}$

$\Rightarrow x_2 - x_1 = a(x_3 - x_2) + b(x_3 - x_1)$

$\Rightarrow x_2 - x_1 = -bx_1 - ax_2 + (a+b)x_3$

$\Rightarrow -b = -1 \quad \therefore b = 1$

$-a = 1 \quad \therefore a = -1$

$\Rightarrow$ when $a = -1, b = 1$, $a+b = 0$ (which is satisfied)

$\therefore y_1$ can be expressed as a linear combination of y_2 and y_3.

$\therefore$ Therefore, y_1, y_2, y_3 are not linearly independent. #

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20)

$\Rightarrow$ Since $V_1, V_2, \ldots, V_n$ are linearly independent

$\Rightarrow C_1V_1 + C_2V_2 + \ldots + C_nV_n = 0$

$\Rightarrow$ such that $C_1 = C_2 = \ldots = C_n = 0$

$\Rightarrow$ Since $V_1, V_2, \ldots, V_n$ are linearly independent

$\therefore$ They form a basis for an n-dimensional subspace of V. In this subspace, any vector can be uniquely represented as a linear combination of $V_1, V_2, \ldots, V_n$.

$\Rightarrow$ For $V_2, V_3, \ldots, V_n$ to span V, they would have to form a basis of V, but, the number of these vectors is $n-1$.

$\Rightarrow$ Since it only has $n-1$ vectors $\therefore$

$\boxed{\therefore \text{They can at most span an } (n-1)\text{-dimensional subspace of } V}$

$\therefore$ It is impossible for $n-1$ vectors to span an n-dimensional space

$\therefore V_2, V_3, \ldots, V_n$ cannot span the vector space V #

Section 3.4

10) $\Rightarrow$ matrix $A = \begin{bmatrix} 1 & 2 & 1 & 2 & 0 \\ 0 & 2 & 2 & 0 & 4 \\ 3 & 1 & -2 & 6 & 5 \end{bmatrix} \xrightarrow{\times(-3)} \begin{bmatrix} 1 & 2 & 1 & 2 & 0 \\ 0 & 2 & 2 & 0 & 4 \\ 0 & -5 & -5 & 0 & 5 \end{bmatrix}$

$\xrightarrow{\times\frac{1}{2}} \begin{bmatrix} 1 & 2 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & -5 & -5 & 0 & 5 \end{bmatrix} \xrightarrow{\times 5} \begin{bmatrix} 1 & 2 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 15 \end{bmatrix} \xrightarrow{\times\frac{1}{15}} \begin{bmatrix} 1 & 2 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

$\Rightarrow$ The pivot colums are the first, second and fifth colums

$\therefore$ Vectors x_1, x_2, x_5 form a basis for R^3

$\therefore$ The basis for R^3 is : $\{x_1, x_2, x_5\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix} \right\}$

Section 3.5

5) $\Rightarrow$ $u_1 = (1,1,1)^T$, $u_2 = (1,2,2)^T$, $u_3 = (2,3,4)^T$

a) $\Rightarrow U = [u_1 \;\; u_2 \;\; u_3] = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix}$

$\Rightarrow U^{-1} = \dfrac{1}{\det(U)} \operatorname{adj}(U)$

$\Rightarrow \det(U) = \begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 2 & 4 \end{vmatrix} - \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix}$

$= 2 - 0 + (-1) = 1$

$\Rightarrow \operatorname{adj}(U) = \begin{bmatrix} \begin{vmatrix} 2 & 3 \\ 2 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} \\ -\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} \\ \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} \end{bmatrix}^T = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 2 & -1 \\ -1 & -1 & 1 \end{bmatrix}^T = \begin{bmatrix} 2 & 0 & -1 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$

$\boxed{\therefore U^{-1} = \begin{bmatrix} 2 & 0 & -1 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \quad \#}$

b) $\Rightarrow U^{-1} = \begin{bmatrix} 2 & 0 & -1 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$

i) $[3, 2, 5]^T$

$\Rightarrow 3\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} + 5\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \boxed{= \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}}$ #

ii) $[1, 1, 2]^T$

$\Rightarrow 1\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} + 2\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \boxed{= \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}}$ #

iii) $[2, 3, 2]^T$

$\Rightarrow 2\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} + 2\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \boxed{= \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}}$ #

Section 3.6

15) a) $U = \begin{bmatrix} 1 & 0 & 3 & 0 & -1 \\ 0 & 1 & 3 & 0 & -2 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow$ Solve $Ux = 0$

$\Rightarrow$ Let $x = [x_1\ x_2\ x_3\ x_4\ x_5]^T$

$\Rightarrow x_1, x_2, x_4$ are leading variables

$\Rightarrow x_3, x_5$ are free variables

$$\therefore \begin{cases} x_1 + 2x_3 - x_5 = 0 \\ x_2 + 3x_3 - 2x_5 = 0 \\ x_4 + 5x_5 = 0 \end{cases}$$

$\Rightarrow$ Let $x_3 = \alpha$, $x_5 = \beta$

$$\therefore \begin{aligned} x_1 &= -2\alpha + \beta \\ x_2 &= -3\alpha + 2\beta \\ x_3 &= \alpha \\ x_4 &= -5\beta \\ x_5 &= \beta \end{aligned}$$

$\therefore$ The general solution to $Ux = 0$ is

$$\Rightarrow x = \alpha \begin{bmatrix} -2 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 2 \\ 0 \\ -5 \\ 1 \end{bmatrix}$$

$\therefore$ A basis for $N(A)$ is $\left\{ \begin{bmatrix} -2 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ -5 \\ 1 \end{bmatrix} \right\}$ #

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b) i) $b = \begin{bmatrix} 0 \\ 5 \\ 3 \\ 4 \end{bmatrix}$, $x_0 = \begin{bmatrix} 3 \\ 2 \\ 0 \\ 2 \\ 0 \end{bmatrix}$

$\Rightarrow$ The general solution to $Ax = b$ is: $x = x_0 + x_n$

, where x_n is any vector in null space $N(A)$

$\Rightarrow$ From (a), we have $\Rightarrow x_n = \alpha \begin{bmatrix} -2 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 2 \\ 0 \\ -5 \\ 1 \end{bmatrix}$

$\therefore$ The general solution is: $x = \begin{bmatrix} 3 \\ 2 \\ 0 \\ 2 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} -2 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 2 \\ 0 \\ -5 \\ 1 \end{bmatrix}$

ii) $\Rightarrow U = \begin{bmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & 3 & 0 & -2 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow$ From U, we have

$\hookrightarrow a_3 = 2a_1 + 3a_2 = 2\begin{bmatrix} 2 \\ 1 \\ -3 \\ -2 \end{bmatrix} + 3\begin{bmatrix} -1 \\ 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \\ 3 \\ -1 \end{bmatrix}$

$\hookrightarrow a_5 = -a_1 - 2a_2 + 5a_4$

$$= -\begin{bmatrix} 2 \\ 1 \\ -3 \\ -2 \end{bmatrix} - 2\begin{bmatrix} -1 \\ 2 \\ 3 \\ 1 \end{bmatrix} + 5\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 5c_1 \\ 5c_2 - 5 \\ 5c_3 - 3 \\ 5c_4 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & 1 & c_1 & 5c_1 \\ 1 & 2 & 8 & c_2 & 5c_2 - 5 \\ -3 & 3 & 3 & c_3 & 5c_3 - 3 \\ -2 & 1 & -1 & c_4 & 5c_4 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 3 \\ 4 \end{bmatrix}$$

$\Rightarrow 6 - 2 + 0 + 2c_1 + 0 = 0$

$\boxed{\therefore c_1 = -2}$

$\Rightarrow -9 + 6 + 0 + 2c_3 + 0 = 3$

$\boxed{\therefore c_3 = 3}$

$\Rightarrow 3 + 4 + 0 + 2c_2 + 0 = 5$

$\boxed{\therefore c_2 = -1}$

$\Rightarrow -6 + 2 + 0 + 2c_4 + 0 = 4$

$\boxed{\therefore c_4 = 4}$

$\therefore \vec{a}_3 = \begin{bmatrix} 1 \\ 8 \\ 3 \\ -1 \end{bmatrix}, \vec{a}_4 \begin{bmatrix} -2 \\ -1 \\ 3 \\ 4 \end{bmatrix}, \vec{a}_5 = \begin{bmatrix} -10 \\ -10 \\ 12 \\ 20 \end{bmatrix}$

$$\boxed{\therefore A = \begin{bmatrix} 2 & -1 & 1 & -2 & -10 \\ 1 & 2 & 8 & -1 & -10 \\ -3 & 3 & 3 & 3 & 12 \\ -2 & 1 & -1 & 4 & 20 \end{bmatrix}} \quad \#$$

Matlab Exercise:

4)

a)

In general, if A is an m×n matrix with rank r, then r ≤ min(m,n). Why?

The rank of a matrix A is the maximum number of linearly independent rows or columns in A. It is also the dimension of the column space (or row space) of the matrix. For an m×n matrix A:

- The column space is a subspace of R^m
- The row space is a subspace of R^n

The dimension of the column space (or row space) (rank) cannot exceed the number of rows m or the number of columns n. Thus, r ≤ min(m,n).

If the entries of A are random numbers, we would expect that r = min(m,n). Why? Explain.

- For an m×n matrix with random entries, it is expected to have full rank, which is min(m,n), because random entries generally lead to a high probability of all rows and columns being linearly independent.

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>> HW4_2a
Matrix A1 is full rank
Matrix A2 is full rank
Matrix A3 is full rank
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b)

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>> HW4_2b
Matrix B1 is full rank
Matrix B2 is full rank
Matrix B3 is full rank
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c)

If x and y are nonzero vectors in R^m and R^n, respectively, then A = xy^T will be an m×n matrix with rank 1. Why? Explain.

- The matrix A=xy^T is an m×n matrix with rank 1 because it is constructed from the outer product of two nonzero vectors. This construction ensures that all rows are multiples of one another, and all columns are multiples of one another, leading to a column space and a row space of dimension 1, which defines the rank of the matrix as 1

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>> HW4_2c
Matrix A has rank 1
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d)

Given the dimensions and the properties of the matrices X and Y, we would expect the rank of A=XY^T to be 2, assuming that X and Y are full rank and their columns and rows are linearly independent. Hence, the rank of the resulting 8×6 matrix AA is expected to be 2.

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>> HW4_2d
Matrix A has rank 2
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e)

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>> HW4_2e
Matrix A has rank 3
Matrix B has rank 4
Matrix C has rank 5
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