Homework 6 林業誠 B24105038 資訊114

Section 5.1

15) a) projection of a_2 onto a_1

$\Rightarrow \operatorname{proj}_{a_1} a_2 = \dfrac{a_1^T a_2}{\|a_1\|^2} a_1$

$\Rightarrow$ Vector $a_2 = \operatorname{proj}_{a_1} a_2 + a_2^\perp$

$\therefore a_2^\perp = a_2 - \operatorname{proj}_{a_1} a_2 = a_2 - \dfrac{a_1^T a_2}{\|a_1\|^2} a_1$

$\Rightarrow \|a_2^\perp\| = \sqrt{\|a_2\|^2 - \left(\dfrac{a_1^T a_2}{\|a_1\|}\right)^2}$

$\hookrightarrow h = \|a_2^\perp\|$

$\therefore h^2 = \|a_2\|^2 - \left(\dfrac{a_1^T a_2}{\|a_1\|}\right)^2$

$\boxed{\therefore h^2 \|a_1\|^2 = \|a_1\|^2 \|a_2\|^2 - (a_1^T a_2)^2}$ # (proved)

b) $\Rightarrow$ The area of parallelogram formed by vectors a_1 and a_2 can be found using cross product (in R^3) or the determinat (in R^2)

$\hookrightarrow$ For 2×2 matrix A with columns a_1 and a_2

$\Rightarrow A = [a_1 \quad a_2] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$\Rightarrow \boxed{|\det(A)| = a_{11}a_{22} - a_{12}a_{21}}$

$\Rightarrow$ Area $= |a_1 \cdot a_2^\perp| = \left| \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \right| = |a_{11}a_{22} - a_{12}a_{21}| = |\det(A)|$

$\Rightarrow \boxed{\therefore \text{Area of } P = |\det(A)|}$ # (proved)

Section 5.2

13) a)

$\Rightarrow$ if $x \in N(A^TA)$, it means $A^TAx=0$

$\Rightarrow A(A^TAx)=A(0)$

$(AA^T)Ax=0$

$\hookrightarrow$ let $y=Ax$, $\Rightarrow AA^Ty=0$, which implies $y \in N(A^T)$

$\boxed{\therefore Ax \in N(A^T)}$

$\hookrightarrow$ Since $y=Ax$ and Ax is the image of x under A

$\boxed{\therefore Ax \in R(A)}$

$\boxed{\therefore Ax \text{ is in both } R(A) \text{ and } N(A^T)}$ #

$\boxed{\therefore \text{ if } x \in N(A^TA)\text{, then } Ax \text{ is in both } R(A) \text{ and } N(A^T)}$ (proved)

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b) $\Rightarrow$ Suppose $x \in N(A)$ means $Ax=0$

$\Rightarrow Ax=0$

$A^TAx=A^T0$

$A^TAx=0$

$\therefore x \in N(A^TA)$

$\boxed{\therefore N(A) \subseteq N(A^TA)}$

$\Rightarrow$ Suppose $x \in N(A^TA)$ means $A^TAx=0$

$\Rightarrow A^TAx=0$

$x^TA^TAx=0$

$\because x^TA^T=(Ax)^T$

$\therefore (Ax)^T(Ax)=0$

$\therefore Ax=0$

$\therefore x \in N(A)$

$\boxed{\therefore N(A^TA) \subseteq N(A)}$

$\boxed{\therefore N(A)=N(A^TA)}$ #

(proved)

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c) $\Rightarrow$ The rank of a matrix is the dimension of its column space

$\Rightarrow$ $\because$ The column space of A^TA is the same as the column space of A, which implies $R(A^TA) = R(A)$

$\boxed{\therefore \text{rank}(A^TA) = \text{rank}(A)}$ # (proved)

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d) $\Rightarrow$ If A has linearly independent columns, A is of full column rank.

$\Rightarrow$ Let A has n columns, so A has n independent columns

$\therefore A^TA$ is an $n \times n$ matrix

$\Rightarrow$ Since the columns of A are linearly independent

$\therefore A^TA$ is invertible

$\therefore \det(A^TA) \neq 0$

$\boxed{\therefore A^TA \text{ is nonsingular}}$ #

$\boxed{\therefore \text{If } A \text{ has linearly independent columns, then } A^TA \text{ is nonsingular}}$ #

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Section 5.3

9) ⇒ Given $P = A(A^TA)^{-1}A^T$

a) ⇒ let $b \in R(A)$, so there exists some x such that $b = Ax$

⇒ $b = Ax$

$Pb = P(Ax)$

$Pb = A(A^TA)^{-1}A^T(Ax)$

$Pb = A(A^TA)^{-1}(A^TAx)$

↳ Since $(A^TA)^{-1}(A^TA) = I$

$\therefore Pb = Ax$

$\because Ax = b$

$\therefore Pb = Ax = b$

$\boxed{\therefore Pb = b}$ # (proved)

⇒ Explanation in terms of projections:

↳ The matrix P is a projection matrix that projects any vector b onto the column space of A. Since b is already in $R(A)$, projecting it onto $R(A)$ will leave it unchanged.

b) $\Rightarrow$ Let $b \in R(A)^{\perp}$, which means b is orthogonal to every vector in $R(A)$

$\Rightarrow$ Since $P = A(A^TA)^{-1}A^T$

$\Rightarrow P = A(A^TA)^{-1}A^T$

$Pb = A(A^TA)^{-1}A^Tb$

$\Rightarrow$ Since $b \in R(A)^{\perp}$, $\therefore A^Tb = 0$

$\therefore Pb = A(A^TA)^{-1} \cdot 0$

$\boxed{\therefore Pb = 0}$ # (proved)

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c)

(a) Any vector $b \in R(A)$ lies on the plane. The projection of b onto the plane $R(A)$ is just b itself. So, geometrically, projecting a vector already on the plane doesn't change its position, which the projection of b results in the same vector b.

(b) Any vector $b \in R(A)^{\perp}$ is orthogonal to the plane. The projection of b onto the plane $R(A)$ results in the zero vector because there is no component of b lying in the plane. Geometrically, projecting a vector orthogonal to the plane onto the plane collapses it to the origin (zero vector).

Section 5.4

16) $\Rightarrow x = (2,3,1)^T,\ y = (5,-6,2)^T$

$\Rightarrow \|x-y\|_1 = |2-5| + |3-(-6)| + |1-2|$

$= 3 + 9 + 1 \boxed{= 13}$ #

$\Rightarrow \|x-y\|_2 = (|2-5|^2 + |3-(-6)|^2 + |1-2|^2)^{\frac{1}{2}}$

$= \sqrt{9+81+1} \boxed{= \sqrt{91}}$ #

$\Rightarrow \|x-y\|_\infty = \max(|2-5|, |3-(-6)|, |1-2|) \boxed{= 9}$ #

$\boxed{\therefore \text{They are closest under infinity-norm, furthest apart under 1-norm}}$

30)

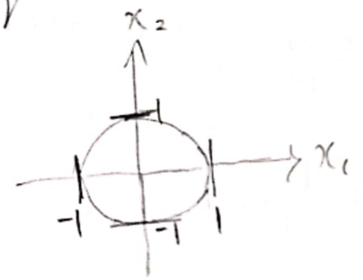
a) $\|x\|_2 = 1$

$\Rightarrow \|x\|_2 = (|x_1|^2 + |x_2|^2)^{\frac{1}{2}}$

$1^2 = x_1^2 + x_2^2$

$\therefore x_1^2 + x_2^2 = 1$

$\therefore$ The equation of a circle with radius $=1$ centered at origin

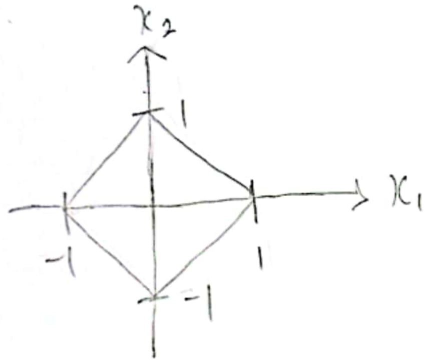


b) $\|x\|_1 = 1$

$\Rightarrow \|x\|_1 = |x_1| + |x_2|$

$\therefore |x_1| + |x_2| = 1$

$\Rightarrow$ This represents a diamond (rhombus) shape centered at the origin with vertices $(1,0)$, $(0,1)$, $(-1,0)$ and $(0,-1)$.

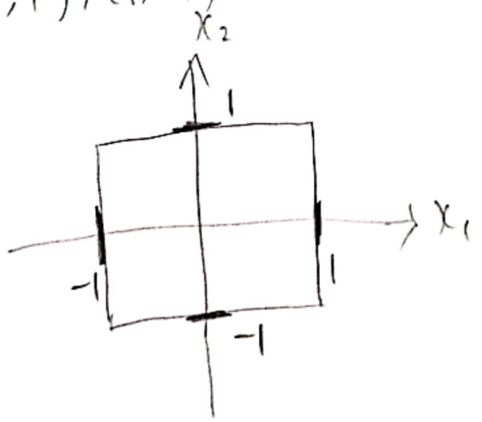


c) $\|x\|_\infty = 1$

$\|x\|_\infty = \max(|x_1|, |x_2|)$

$\therefore \max(|x_1|, |x_2|) = 1$

$\Rightarrow$ This represents a square centered at the origin with vertices at $(1,1)$, $(1,-1)$, $(-1,1)$ and $(-1,-1)$.



Matlab Exercise

(1):

Results:

1.2.2)

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lsqr converged at iteration 3 to a solution with relative residual 0.0016.
Affine Transformation Coefficients:
a11 = 1.0001, a12 = 0.0683, a13 = -27.6303
a21 = -0.0685, a22 = 0.9999, a23 = 18.7495
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1.2.3)

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Root Mean Square Error: 0.87 pixels
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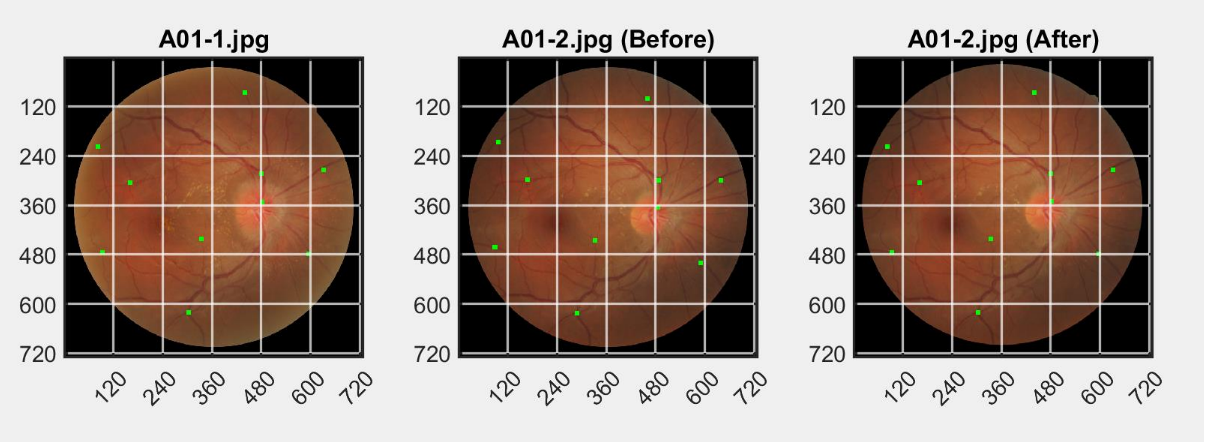
1.4.1)



Before Registration:

- The key points in A01-2.jpg are not aligned with the corresponding key points in A01-1.jpg.
- There is a visible discrepancy between the key points in the two images, as shown by the grid lines which are misaligned.

After Registration:

- The key points in the transformed A01-2.jpg are much closer to their corresponding key points in A01-1.jpg.
- The alignment of the key points shows significant improvement, with the green dots nearly overlapping the reference points.

Findings: The transformation has successfully aligned the key points from A01-2.jpg with those in A01-1.jpg. This visual alignment indicates that the affine transformation has effectively minimized the discrepancies between the key points.

1.4.2)

(2):

Accuracy of the Affine Transformation:

- To check how accurate the transformation is, we use a measurement called Root Mean Square Error (RMSE). This number tells us how close the points in the transformed image (A01-2.jpg) are to the points in the reference image (A01-1.jpg). A lower RMSE means the points are very close to where they should be, indicating that the transformation is accurate. If the RMSE is low, it means the transformation has done a good job of aligning the points correctly.

(3):

Effectiveness of the Image Registration Algorithm:

- The image registration algorithm for fundus images is effective because it aligns the key points in the images well. After using the algorithm, the green dots (key points) in the transformed image (A01-2.jpg) closely match the key points in the reference image (A01-1.jpg). This is confirmed by the low RMSE value, which indicates the points are very close to each other. This precise alignment is important for medical images to ensure accurate analysis and diagnosis.