Homework 5 林業誠 E24105038 資訊114

Section 4.1

17 (a) $L(x) = (x_3, x_2, x_1)^T$

$\Rightarrow$ The kernel of L consists of all vectors x such that $L(x) = 0$

$\therefore L(x) = (x_3, x_2, x_1)^T = 0$

$\Rightarrow x_3 = 0,\ x_2 = 0,\ x_1 = 0$

$\boxed{\therefore \ker(L) = \{0\}}$ #

$\Rightarrow$ Since x_1, x_2, x_3 can be any real numbers

$\boxed{\therefore L(R^3) = R^3}$ #

b) $L(x) = (x_1, x_2, 0)^T$

$\Rightarrow$ The kernel of L consists of all vectors x such that $L(x) = 0$

$\therefore L(x) = (x_1, x_2, 0)^T = 0$

$\Rightarrow x_1 = 0,\ x_2 = 0$

$\therefore \ker(L) = \{(0, 0, x_3)^T : x_3 \in R\} \boxed{= \mathrm{Span}(e_3)}$ #

$\Rightarrow$ Since x_1, x_2 can be any real numbers

$\therefore L(R^3) \boxed{= \mathrm{Span}(e_1, e_2)}$ #

c) $L(x) = (x_1, x_1, x_1)^T$

$\Rightarrow$ L consists of all vectors x such that $L(x) = 0$

$\therefore L(x) = (x_1, x_1, x_1)^T = 0$

$\Rightarrow x_1 = 0$

$\therefore \ker(L) = \{(0, x_2, x_3)^T : x_2, x_3 \in R\} \boxed{= \mathrm{Span}(e_2, e_3)}$ #

$\Rightarrow$ Since x_1 can be any real number

$\therefore L(R^3) = \{(x, x, x)^T : x \in R\} \boxed{= \mathrm{Span}((1,1,1)^T)}$ #

Section 4.2

18) $u_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, u_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, u_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

$b_1 = (1,-1)^T, \quad b_2 = (2,-1)^T$

a) $L(x) = (x_3, x_1)^T$

$\Rightarrow L(u_1) = (-1, 1)^T$

$\Rightarrow L(u_2) = (1, 1)^T$

$\Rightarrow L(u_3) = (1, -1)^T$

$\Rightarrow L(u_1) = (-1, 1)^T$

$\hookrightarrow \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$\begin{cases} \alpha + 2\beta = -1 & - ① \\ -\alpha - \beta = 1 & - ② \end{cases}$

From ①: $\alpha = -1 - 2\beta$ — ③

Subs ③ into ②: $1 + 2\beta - \beta = 1$

$\boxed{\beta = 0}$

$\boxed{\therefore \alpha = -1}$

$\therefore$ The first column is $\boxed{\begin{bmatrix} -1 \\ 0 \end{bmatrix}}$ #

$\Rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$\begin{cases} \alpha + 2\beta = 1 & - ① \\ -\alpha - \beta = 1 & - ② \end{cases}$

$\Rightarrow$ From ①: $\alpha = 1 - 2\beta$ — ③

$\Rightarrow$ subs ③ into ②: $-1 + 2\beta - \beta = 1$

$\boxed{\therefore \beta = 2}$

$\boxed{\therefore \alpha = -3}$

$\therefore$ The second column is $\boxed{\begin{bmatrix} -3 \\ 2 \end{bmatrix}}$ #

$\Rightarrow \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$\boxed{\therefore \alpha = 1, \beta = 0}$

$\therefore$ Third column is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\boxed{\therefore \text{Matrix} = \begin{bmatrix} -1 & -3 & 1 \\ 0 & 2 & 0 \end{bmatrix}}$ #

b) $L(x) = (x_1 + x_2, x_1 - x_3)^T$

$\Rightarrow L(u_1) = (1+0, 1-(-1))^T = (1, 2)^T$

$\Rightarrow L(u_2) = (1+2, 1-1)^T = (3, 0)^T$

$\Rightarrow L(u_3) = (-1+1, -1-1)^T = (0, -2)^T$

$\Rightarrow L(u_1) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$\begin{cases} \alpha + 2\beta = 1 & - 1 \\ -\alpha - \beta = 2 & - 2 \end{cases}$ $\Rightarrow$ From 1 : $\alpha = 1 - 2\beta$ $-$ 3

$\Rightarrow$ Subs 3 into 2 :

$-1 + 2\beta - \beta = 2$

$\boxed{\therefore \beta = 3}$ $\boxed{\therefore \alpha = -5}$

(first column)

$\boxed{\therefore \text{Matrix} = \begin{bmatrix} -5 & -3 & 4 \\ 3 & 3 & -2 \end{bmatrix}}$ #

$\Rightarrow L(u_2) = \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$\begin{cases} \alpha + 2\beta = 3 & - 1 \\ -\alpha - \beta = 0 & - 2 \end{cases}$

From 1 : $\alpha = 3 - 2\beta$ $-$ 3

Subs 3 into 2 $\quad -3 + 2\beta - \beta = 0$

(second column) $\boxed{\therefore \beta = 3}$ $\boxed{\therefore \alpha = -3}$

$\Rightarrow L(u_3) = \begin{bmatrix} 0 \\ -2 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$\begin{cases} \alpha + 2\beta = 0 & - 1 \\ -\alpha - \beta = -2 & - 2 \end{cases}$

From 1 : $\alpha = -2\beta$ $-$ 3

Subs 3 into 2 : $2\beta - \beta = -2$

(third column) $\boxed{\therefore \beta = -2}$ $\boxed{\therefore \alpha = 4}$

c) $L(x) = (2x_2, -x_1)^T$

$\Rightarrow L(u_1) = (0, -1)^T$

$\Rightarrow L(u_2) = (4, -1)^T$

$\Rightarrow L(u_3) = (2, 1)^T$

$\Rightarrow L(u_1) = \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$\begin{cases} \alpha + 2\beta = 0 & -① \\ -\alpha - \beta = -1 & -② \end{cases}$

From ① : $\alpha = -2\beta$ $-③$

Subs ③ into ② : $2\beta - \beta = -1$

$\boxed{\therefore \beta = -1}$

$\boxed{\therefore \alpha = 2}$

$\Rightarrow L(u_2) = \begin{bmatrix} 4 \\ -1 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$\begin{cases} \alpha + 2\beta = 4 & -① \\ -\alpha - \beta = -1 & -② \end{cases}$

From ① : $\alpha = 4 - 2\beta$ $-③$

Subs ③ into ② : $-4 + 2\beta - \beta = -1$

$\boxed{\therefore \beta = 3}$

$\boxed{\therefore \alpha = -2}$

$\Rightarrow L(u_3) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$\begin{cases} \alpha + 2\beta = 2 & -① \\ -\alpha - \beta = 1 & -② \end{cases}$

From ① : $\alpha = 2 - 2\beta$ $-③$

Subs ③ into ② : $-2 + 2\beta - \beta = 1$

$\boxed{\therefore \beta = 3}$

$\boxed{\therefore \alpha = -4}$

$\boxed{\therefore \text{Matrix} = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 3 \end{bmatrix}}$ #

---

20) $\Rightarrow$ V and W are vector spaces with ordered bases E and F

$\Rightarrow L : V \to W$ is a linear transformation

$\Rightarrow$ A is the matrix representing L relative to E and F

a)

$\Rightarrow \boxed{v \in \ker(L) \Rightarrow [v]_E \in N(A)}$

↳ Suppose $v \in \ker(L)$. By definition, this means $L(v) = 0$

$\Rightarrow$ Since $L(v) = A[v]_E$

$\therefore L(v) = A[v]_E = 0$

$\therefore [v]_E \in N(A)$ because $A[v]_E = 0$

$\Rightarrow \boxed{[v]_E \in N(A) \Rightarrow v \in \ker(L)}$

↳ Suppose $[v]_E \in N(A)$, by definition $\therefore A[v]_E = 0$

$\Rightarrow$ Since $L(v) = A[v]_E$

$\therefore L(v) = 0$

$\therefore v \in \ker(L)$

$\boxed{\therefore v \in \ker(L) \text{ if and only if } [v]_E \in N(A)}$ (proved) #

b) $\boxed{w \in L(V) \Rightarrow [w]_F \text{ is in the column space of } A}$

$\Rightarrow$ Suppose $\boxed{w \in L(V)}$. By definition, this means there exists some $v \in V$ such that $L(v) = w$

$\Rightarrow$ Since $L(v) = A[v]_E$, $\therefore w = A[v]_E$

$\therefore \boxed{[w]_F = A[v]_E}$ $\Rightarrow$ This implies that $[w]_F$ is a linear combination of the columns of A, meaning it lies in the column space of A.

$\Rightarrow$ $\boxed{[w]_F \text{ is in the column space of } A \Rightarrow w \in L(V)}$

$\Rightarrow$ Suppose $[w]_F$ is in column space of A

$\Rightarrow$ By definition, this means there exists some $[v]_E$ such that

↳ $[w]_F = A[v]_E$

$\Rightarrow$ Since $w = A[v]_E$ and $L(v) = A[v]_E$

$\therefore w = L(v)$

$\boxed{\therefore w \in L(V)}$

$\boxed{\therefore w \in L(V) \text{ if and only if } [w]_F \text{ is in the column space of } A}$ (proved)

Section 4.3

4) $\Rightarrow L(x) = Ax$

$$\Rightarrow A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & -2 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$\Rightarrow V_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}, \quad V_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow V = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 3 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & 3 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 3 & -2 & -2 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 & -2 & 1 & 3 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -1 & -3 \\ 0 & 1 & 0 & -2 & 1 & 2 \\ 0 & 0 & 1 & -2 & 1 & 3 \end{array}\right]$$

$$\therefore V^{-1} = \begin{bmatrix} 3 & -1 & -3 \\ -2 & 1 & 2 \\ -2 & 1 & 3 \end{bmatrix}$$

$$\Rightarrow \because B = V^{-1} A V$$

$$= \begin{bmatrix} 3 & -1 & -3 \\ -2 & 1 & 2 \\ -2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ -1 & -2 & 1 \\ 1 & 1 & -1 \end{bmatrix} V$$

$$= \begin{bmatrix} 1 & -1 & 5 \\ -1 & 0 & -3 \\ 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 2 & 3 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\boxed{\therefore B = \begin{bmatrix} -1 & -8 & 6 \\ -1 & 3 & -4 \\ 2 & 7 & -4 \end{bmatrix}} \#$$

Matlab Exercies

1)

```
>> HW5_1
Matrix A:
     0     0     5     1
     1     0     4     2
     0     1     3     3
     0     0     2     4

Coordinate vector y with respect to F:
    -1
    -1
    -1
     4

Coordinate vector z of L(x) with respect to F:
    -1
     3
     8
    14

Coordinate vector of L(x) with respect to the standard basis:
    24
    25
    22
    14
```

- **The above output from matlab is for question a, b, c ,d respectively**