Homework 7 林業誠 E24105038 資訊114

7-12) Since X_1, X_2 are independent, the joint probability distribution is:

$$\Rightarrow f(x_1,x_2) = f(x_1)f(x_2) = (e^{-x_1})(e^{-x_2}) = e^{-(x_1+x_2)}, \quad x_1>0, x_2>0$$

$$\Rightarrow y_1 = x_1+x_2, \quad y_2 = \frac{x_1}{(x_1+x_2)}$$

$$\hookrightarrow x_1 = y_2(x_1+x_2), \quad \therefore \boxed{x_2 = y_1(1-y_2)}$$

$$\therefore \boxed{x_1 = y_1 y_2}, \quad \text{for } y_1>0,\ 0<y_2<1$$

$$\Rightarrow J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} y_2 & y_1 \\ 1-y_2 & -y_1 \end{vmatrix} = -y_1y_2 - y_1 + y_1y_2 \boxed{= -y_1}$$

$$\Rightarrow g(y_1,y_2) = f(\underbrace{y_1y_2}_{x_1}, \underbrace{y_1(1-y_2)}_{x_2})|J| = e^{-(y_1y_2+y_1-y_1y_2)}|-y_1|$$

$$\boxed{= y_1 e^{-y_1}, \text{ for } y_1>0,\ 0<y_2<1}$$

$$\Rightarrow g(y_1) = \int_0^1 y_1 e^{-y_1}\, dy_2 \boxed{= y_1 e^{-y_1}}, \quad y_1>0$$

$$\Rightarrow g(y_2) = \int_0^\infty y_1 e^{-y_1}\, dy_1 = \Gamma(2) = \boxed{1}, \quad 0<y_2<1$$

$$\boxed{\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx, \text{ for } \alpha>0}$$

$$\Rightarrow \boxed{g(y_1,y_2) = y_1 e^{-y_1} = g(y_1)g(y_2)} \ \#$$

$\hookrightarrow$ Since $g(y_1,y_2) = g(y_1)g(y_2)$, $\therefore$ random variables Y_1 and Y_2 are independent.

7-14) $\because y = x^2$

$\therefore x_1 = \sqrt{y},\ x_2 = -\sqrt{y}$

$$\Rightarrow f(x) = \begin{cases} \frac{1+x}{2}, & -1<x<1 \\ 0, & \text{elsewhere} \end{cases}$$

$$\Rightarrow J_1 = \frac{1}{2\sqrt{y}}, \quad J_2 = -\frac{1}{2\sqrt{y}}$$

$$\Rightarrow g(y) = f(\sqrt{y})|J_1| + f(-\sqrt{y})|J_2| = \frac{1+\sqrt{y}}{2}\left(\frac{1}{2\sqrt{y}}\right) + \frac{1-\sqrt{y}}{2}\left(\frac{1}{2\sqrt{y}}\right)$$

$$= \frac{2}{4\sqrt{y}} \quad \boxed{= \frac{1}{2\sqrt{y}},\ 0<y<1}$$

$$\boxed{\therefore g(y) = \begin{cases} \frac{1}{2\sqrt{y}}, & 0<y<1 \\ 0, & \text{elsewhere} \end{cases}} \ \#$$

7.18) $\Rightarrow$ moment-generating function of X is

$\hookrightarrow \boxed{M_X(t) = E(e^{tX})} = \sum_{x=1}^{\infty} e^{tx} f(x) = \sum_{x=1}^{\infty} e^{tx} pq^{x-1}$

$= \frac{p}{q} \sum_{x=1}^{\infty} e^{tx} q^x$

$= \frac{p}{q} \sum_{x=1}^{\infty} (e^t q)^x$ (geometric series)

$\hookrightarrow$ if $\sum_{x=0}^{\infty} (qe^t)^x \neq \infty$,

$\Rightarrow \therefore |qe^t| < 1$

$e^t < \frac{1}{q}$

$\boxed{\therefore t < \ln q^{-1} = -\ln q}$ $\boxed{\therefore t < -\ln q}$

$= \frac{p}{q}\left(\frac{qe^t}{1-qe^t}\right)$

$\boxed{\therefore M_X(t) = \frac{pe^t}{1-qe^t}, \ t < -\ln q, \quad \because 1-qe^t \neq 0 \ \therefore t \neq \ln\frac{1}{q}}$

$\Rightarrow \boxed{\mu = \left.\frac{dM_X(t)}{dt}\right|_{t=0}}$

$= \left.\frac{(1-qe^t)(pe^t) - (pe^t)(-qe^t)}{(1-qe^t)^2}\right|_{t=0} = \frac{(1-q)p + pq}{(1-q)^2} = \frac{p-pq+pq}{(1-q)^2} = \frac{p}{(1-q)^2}$

$\boxed{\because p = 1-q}$ $\boxed{\therefore \mu = \frac{1}{p}}$ #

$\Rightarrow \mu_2' = \left.\frac{d^2M_X(t)}{dt^2}\right|_{t=0} = \left.\frac{(1-qe^t)^2(pe^t) - (pe^t)(2)(1-qe^t)(-qe^t)}{(1-qe^t)^4}\right|_{t=0}$ ☆

$= \frac{(1-q)^2 p + 2pq(1-q)}{(1-q)^4}$

$= \frac{p - pq + 2pq}{(1-q)^3} = \frac{p+pq}{(1-q)^3} = \frac{p(1+q)}{p^3} \boxed{= \frac{1-q}{p^2}}$

$\therefore \sigma^2 = \mu_2' - \mu^2 = \frac{1-q}{p^2} - \frac{1}{p^2} \boxed{= \frac{q}{p^2}}$ #

7.22) $\Rightarrow M_X(t) = (1-2t)^{-\frac{v}{2}}$ (moment-generating function)

$\Rightarrow \mu = \left.\frac{dM_X(t)}{dt}\right|_{t=0} = \left.-\frac{v}{2}(1-2t)^{-\frac{v}{2}-1} \cdot (-2)\right|_{t=0}$

$= -\frac{v}{2} \cdot (-2) \boxed{= v}$ #

$\Rightarrow \mu_2' = \left.\frac{d^2M_X(t)}{dt^2}\right|_{t=0} = \left.\left(-\frac{v}{2}-1\right)(v)(1-2t)^{-\frac{v}{2}-2} \cdot (-2)\right|_{t=0}$

$= (v+2)(v) = v^2 + 2v$

$\Rightarrow \sigma^2 = \mu_2' - \mu^2 = v^2 + 2v - v^2 \boxed{= 2v}$ #

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1)

b)

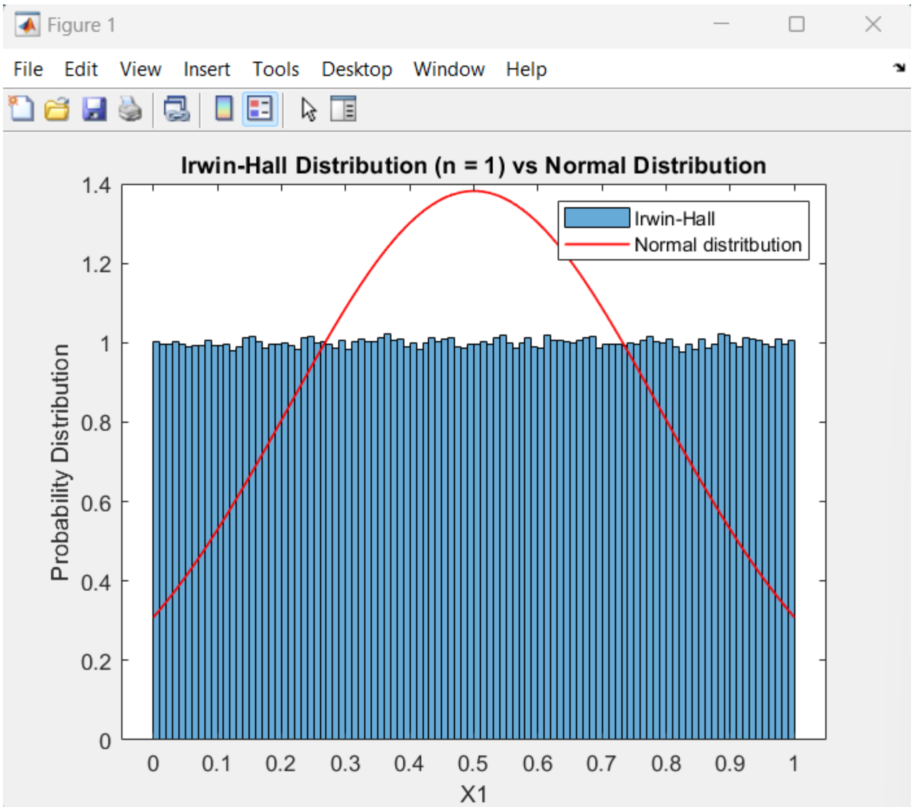

Figure 1
File Edit View Insert Tools Desktop Window Help
Irwin-Hall Distribution (n = 1) vs Normal Distribution
Irwin-Hall
Normal distritbution
Probability Distribution
X1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0.2
0.4
0.6
0.8
1.2
1.4

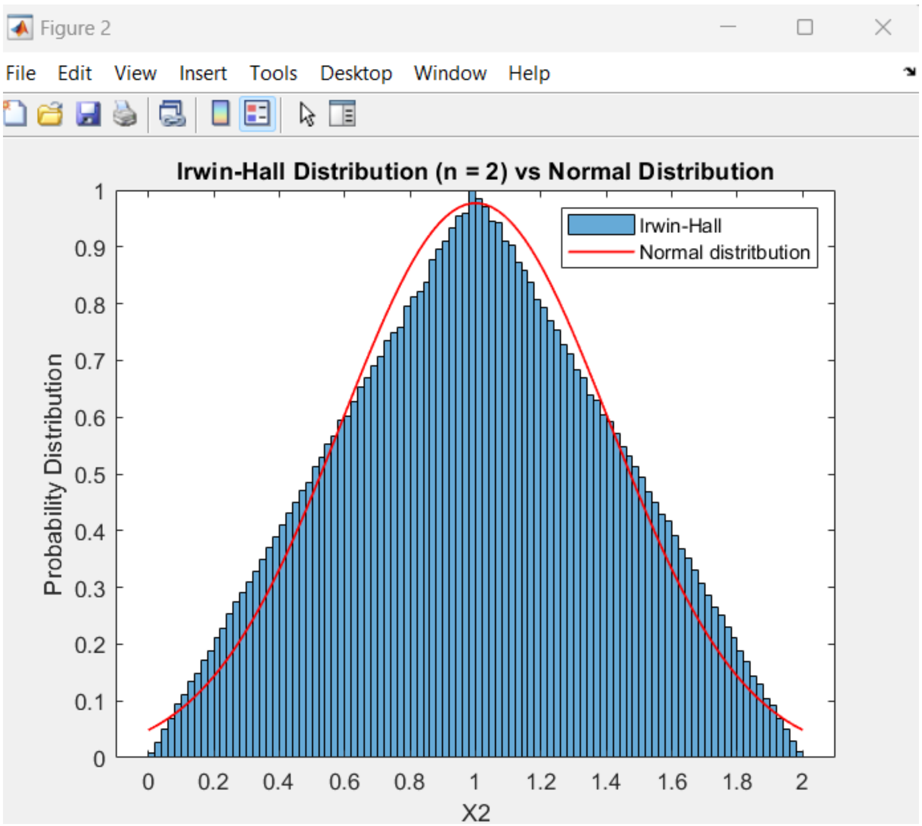

Figure 2
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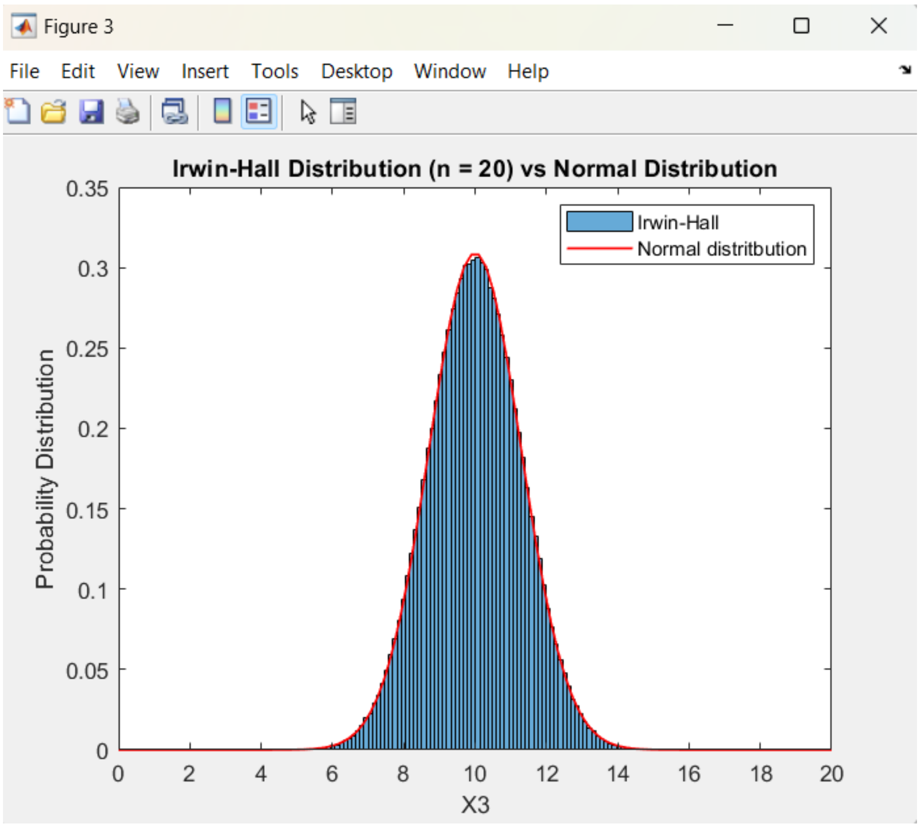
Irwin-Hall Distribution (n = 2) vs Normal Distribution
Irwin-Hall
Normal distritbution
Probability Distribution
X2
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9



We can observe that as n increases, the Irwin-Hall distribution becomes more approximate and less errors to a normal distribution. For n = 1, the Irwin-Hall distribution is a uniform distribution, and it deviates significantly from the normal distribution. However, as n increases to n = 2 and n = 20, the Irwin-Hall distribution becomes more bell-shaped and the approximation and the error rate to the normal distribution improves. The errors in approximating a normal distribution with the Irwin-Hall distribution decrease as n gets larger.