Homework: 2024/09/18

1.

1.-(a)

Assume $k \le n$, and since $(X'X)^{-1}$ exists, X must be full rank

$\therefore \text{rank}(X) = k$

$$\text{trace}(P) = \text{trace}(X(X'X)^{-1}X')$$
$$= \text{trace}(X'X(X'X)^{-1})$$
$$= \text{trace}(I_k) = k$$

$$\text{trace}(I_n - P) = \text{trace}(I_n) - \text{trace}(P)$$
$$= n - k$$

1.-(b.)

Assume any vector v,

$$v'Pv = v'PPv = v'P'Pv = (Pv)'Pv = \|Pv\| \geq 0$$

$\therefore$ for any v, we have $v'Pv \geq 0 \rightarrow P$ is positive semi-definite, QED.

$$v'Mv = \|Mv\| \geq 0$$

$\therefore$ for any v, we have $v'Mv \geq 0 \rightarrow M$ is positive semi-definite, QED.

2.

2.

$$\hat{\sigma}_Y^2 = \frac{1}{n}\sum((Y_i - \mu_Y) - (\bar{Y}_i - \mu_Y))^2$$
$$= \frac{1}{n}\sum((Y_i - \mu_Y)^2 - 2(Y_i - \mu_Y)(\bar{Y}_i - \mu_Y) + (\bar{Y}_i - \mu_Y)^2)$$

$$E[\hat{\sigma}_Y^2] = E\left[\frac{1}{n}\sum(Y_i - \mu_Y)^2\right] - E\left[\frac{2}{n}\sum(Y_i - \mu_Y)(\bar{Y}_i - \mu_Y)\right] + E\left[\frac{1}{n}\sum(\bar{Y}_i - \mu_Y)^2\right]$$
$$= \sigma_Y^2 - E\left[\frac{2}{n}(\bar{Y}_i - \mu_Y)\cdot n\cdot(Y_i - \mu_Y)\right] + E\left[\frac{1}{n}\sum(\bar{Y}_i - \mu_Y)^2\right]$$
$$= \sigma_Y^2 - E[(\bar{Y}_i - \mu_Y)^2] = \sigma_Y^2 - \frac{1}{n^2}\cdot n\cdot\sigma_Y = \frac{n-1}{n}\sigma_Y^2$$

$\therefore$ bias $E[\hat{\sigma}_Y^2] - \sigma_Y^2 = -\frac{1}{n}\sigma_Y^2$ #

3.

caluculate beta using the formula $$(X'X)^{-1} (X'Y)$$

```r
beta_hat <- solve(t(x) %*% x) %*% t(x) %*% y
print("beta_hat =")
print(beta_hat)
```

```
[1] "beta_hat ="
             [,1]
ones    0.01079947
x_dfy   0.99791023
x_infl -0.55424292
x_svar -0.44950368
x_tms   0.23354924
x_tbl   0.14438939
x_dfr   0.06759062
```

calculate beta using the FWL theorem

```r
x1 <- cbind(ones, x_dfy, x_infl, x_svar)
x2 <- cbind(x_tms, x_tbl, x_dfr)

m1 <- diag(nrow(x1)) - x1 %*% solve(t(x1) %*% x1) %*% t(x1)
m2 <- diag(nrow(x2)) - x2 %*% solve(t(x2) %*% x2) %*% t(x2)

beta1 <- solve(t(m2 %*% x1) %*% m2 %*% x1) %*% (t(m2 %*% x1) %*% m2 %*% y)
beta2 <- solve(t(m1 %*% x2) %*% m1 %*% x2) %*% (t(m1 %*% x2) %*% m1 %*% y)

# Combine intercept and beta_hat_fwl
beta_hat <- matrix(c(beta1, beta2), ncol = 1)
rownames(beta_hat) <- c("ones", "x_dfy", "x_infl", "x_svar", "x_tms", "x_tbl", "x_dfr")

```

```
[1] "beta_hat (FWL) ="
             [,1]
ones    0.01079947
x_dfy   0.99791023
x_infl -0.55424292
x_svar -0.44950368
x_tms   0.23354924
x_tbl   0.14438939
x_dfr   0.06759062
```

4.

calculate each r-squared

```r
# List of x variables
x_vars <- c("ones", "x_dfy", "x_infl", "x_svar", "x_tms", "x_tbl", "x_dfr")
x <- c()
r_squared_results <- c()

# Iterate through x variables
for (var in x_vars) {
  if (var == "ones") {
    x <- cbind(x, ones)
  } else {
    x <- cbind(x, dt[[var]])
  }
  beta_hat <- calculate_beta_hat(y, x)
  y_hat <- x %*% beta_hat
  r_squared <- calculate_r_squared(y, y_hat)
  r_squared_results <- c(r_squared_results, r_squared)
}
```

result of r-squared

```
[1] "R-squared values:"
[1] 0.00000000 0.01537088 0.01545069 0.02152345 0.02250799 0.03008154 0.03059011
```

plot of r-squared