

Homework: 2024/9/11

1.

1. Assume $Y_1 = X$, $Y_2 = X^2$, $X \sim N(0,1)$

$$\begin{aligned} \text{Cov}(Y_1, Y_2) &= E[Y_1 Y_2] - E[Y_1] E[Y_2] \\ &= E[X^3] - E[X] E[X^2] = 0 \end{aligned}$$

$\text{Cov}(Y_1, Y_2) = 0$ means Σ is a diagonal matrix, but Y_1 and Y_2 are not independent

Assume $Y_1 = X + Z$, $Y_2 = X - Z$, $X, Z \sim N(0,1)$

$$\text{Cov}(Y_1, Y_2) = E[X^2 - Z^2] = 0$$

\therefore In this case, Σ is a diagonal matrix, but Y_1 and Y_2 are also not independent

2.

2. The distribution of Y can be written as

$$g(y, \theta) = \frac{1}{(2\pi)^{m/2} \det(\Sigma)^{1/2}} \exp\left(-\frac{1}{2}(y-\mu)' \Sigma^{-1} (y-\mu)\right)$$

and the distribution of Y_i respectively is

$$f(y_i) = \frac{1}{(2\pi)^{1/2} \sigma} \exp\left(-\frac{1}{2}\left(\frac{y_i - \mu}{\sigma}\right)^2\right)$$

$\therefore g(y, \theta) = \prod_{i=1}^m f(y_i)$, which means Y_i, Y_j are independent for all $i \neq j$, i.e. Y is a vector of independent variables, $\theta \in \mathbb{D}$.

3.

3. Since $Y'Y = \sum_{i=1}^m Y_i^2$, and $Y_i^2 \sim \sigma^2 \chi^2$

$$\therefore E\left[\sum_{i=1}^m Y_i^2\right] = m\sigma^2$$

$$\text{For } (Y'Y)^2 = \left(\sum_{i=1}^m Y_i^2\right)^2 = \sum_{i=1}^m Y_i^4 + 2 \sum_{i=1}^m \sum_{j=1}^m Y_i^2 Y_j^2$$

$$E[Y_i^4] = 3\sigma^4, \quad E[2 \sum_{i=1}^m \sum_{j=1}^m Y_i^2 Y_j^2] = 2m\sigma^4$$

$$E[2 \sum_{i=1}^m \sum_{j=1}^m Y_i^2 Y_j^2] = 2 \cdot \binom{m}{2} \cdot E[Y_i^2 Y_j^2], \text{ and } Y_i, Y_j \text{ are independent}$$

$$= m(m-1) \cdot E[Y_i^2] E[Y_j^2]$$

$$= m(m-1) \sigma^4$$

$$\therefore E[(Y'Y)^2] = m^2 \sigma^4 + 2m\sigma^4$$

4.

Apply the law of total probability, we can count each probability that satisfies the condition:

```
[lin1214@archlinux hw2]$ Rscript homework2.r
[1] "alpha_1 = 0.909"
[1] "alpha_2 = 0.091"
[1] "beta_11 = 0.056"
[1] "beta_12 = 0.036"
[1] "beta_21 = 0.478"
[1] "beta_22 = 0.431"
[1] "gamma_1 = 0.135"
[1] "gamma_2 = 0.865"
```

5.

5.

$\beta = \arg \min_{\beta} (E[(m(X) - X'\beta)^2])$. to find β , we can calculate $\frac{d}{d\beta}(E[(m(X) - X'\beta)^2]) = 0$

$$\therefore \frac{d}{d\beta} (E[m(X)^2 - 2m(X)X'\beta + (X'\beta)^2]) = 2E[XX'\beta] - 2E[m(X)X'] = 0$$

Apply the Law of iterated expectations, we have $E[m(X)X'] = E[E[Y|X]X'] = E[XY]$

$$\therefore \beta = E[XX']^{-1} E[XY] \quad QED.$$

6.

6.

The linear projection coefficient β can be written as

$$\beta = \frac{E[XY]}{E[X^2]}, \quad X \sim t(3), \quad Y = \frac{1}{1+X^2}$$

\therefore To have β be calculated properly, we must compute the second momentum of X , which

$$\text{can be written as } E[X^2] = \frac{\nu}{\nu-2}, \quad X \sim t(\nu)$$

\rightarrow the degree-of-freedom must be set to $\nu > 2$, in this case, ν is set to 3.

We can then calculate beta by constructing the linear model with X and Y :

```
33 ✓ # Q6-2 =====
34 x <- rt(1e5, df = 3)
35 y <- 1 / (1 + x^4)
36
37 # (b)
38 model <- lm(y ~ x)
39 beta <- model$coefficients[2]
40 paste("beta =", beta)
```

7. Source Code

[Source Code](#)