Homework: 2024/9/11

1.

1. Assume $Y_1 = X$, $Y_2 = X^2$, $X \sim N(0,1)$

$Cov(Y_1, Y_2) = E[Y_1 Y_2] - E[Y_1]E[Y_2]$

$= E[X^3] - E[X]E[X^2] = 0$

$Cov(Y_1, Y_2) = 0$ means Σ is a diagonal matrix, but Y_1 and Y_2 are not independent

Assume $Y_1 = X + Z$, $Y_2 = X - Z$, $X, Z \sim N(0,1)$

$Cov(Y_1, Y_2) = E[X^2 - Z^2] = 0$

∴ In this case, Σ is a diagonal matrix, but Y_1 and Y_2 are also not independent

2.

2. The distribution of Y can be written as

$$g(y, \theta) = \frac{1}{(2\pi)^{m/2} \det(\Sigma)^{1/2}} \exp\left(-\frac{1}{2}(y-\mu)' \Sigma^{-1} (y-\mu)\right)$$

and the distribution of Y_i respectively is

$$f(y_i) = \frac{1}{(2\pi)^{1/2} \sigma} \exp\left(-\frac{1}{2}\left(\frac{y_i - \mu}{\sigma}\right)^2\right)$$

∴ $g(y, \theta) = \prod_{i=1}^{m} f(y_i)$, which means Y_i, Y_j are independent for all $i \neq j$, i.e. Y is a vector of independent variables, QED.

3.

3. Since $Y'Y = \sum_{i=1}^{m} Y_i^2$, and $Y_i^2 \sim \sigma^2 \chi^2$

∴ $E[\sum_{i=1}^{m} Y_i^2] = m\sigma^2$

For $(Y'Y)^2 = (\sum_{i=1}^{m} Y_i^2)^2 = \sum_{i=1}^{m} Y_i^4 + 2\sum_{i=1}^{m}\sum_{j=1}^{i} Y_i^2 Y_j^2$

$E[Y_i^4] = 3\sigma^4$, $E[\sum Y_i^4] = 3m\sigma^4$

$E[2\sum\sum Y_i^2 Y_j^2] = 2 \cdot \binom{m}{2} \cdot E[Y_i^2 Y_j^2]$, and Y_i, Y_j are independent

$= m(m-1) \cdot E[Y_i^2]E[Y_j^2]$

$= m(m-1)\sigma^4$

∴ $E[(Y'Y)^2] = m^2\sigma^4 + 2m\sigma^4$

4.

Apply the law of total probability, we can count each probability that satisfies the condition:

```
[lin1214@archlinux hw2]$ Rscript homework2.r
[1] "alpha_1 = 0.091"
[1] "alpha_2 = 0.909"
[1] "beta_11 = 0.056"
[1] "beta_12 = 0.036"
[1] "beta_21 = 0.478"
[1] "beta_22 = 0.431"
[1] "gamma_1 = 0.135"
[1] "gamma_2 = 0.865"
[1] "beta = -0.00102822569589241"
```

5.

5.

$\beta = \underset{b}{\arg\min} \left(E[(m(X) - X'b)^2] \right)$. to find β, we can calculate $\frac{d}{db}\left(E[(m(X)-X'b)^2]\right) = 0$

$\therefore \frac{d}{db}\left(E[m(X)^2 - 2m(X)X'b + (X'b)^2] \right) = 2E[XX'b] - 2E[m(X)X'] = 0$

Apply the Law of iterated expectations, we have $E[m(X)X'] = E[E[Y|X]X'] = E[XY]$

$\therefore b = E[XX']^{-1} E[XY]$. QED.

6.

6.

The linear projection coefficient β can be written as

$$\beta = \frac{E[XY]}{E[X^2]}, \quad X \sim t(3), \quad Y = \frac{1}{1+X^4}$$

$\therefore$ To have β be caculated properly, we must compute the second momentum of X, which can be written as $E[X^2] = \frac{\nu}{\nu-2}$, $X \sim t(\nu)$

$\rightarrow$ the degree-of-freedoms must be set to $\nu > 2$, in this case, ν is set to 3 #

We can then calculate beta by constructing the linear model with X and Y:

```r
# Q6-2 =====================================================
x <- rt(1e5, df = 3)
y <- 1 / (1 + x^4)

# (b)
model <- lm(y ~ x)
beta <- model$coefficients[2]
paste("beta =", beta)
```

7. Source Code

Source Code