Tutorial 1 - Linear Maps and Norms

1. For $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ the transpose matrix $\boldsymbol{A}^\top \in \mathbb{R}^{n\times m}$ is defined by $(\boldsymbol{A}^\top)_{i,j} = \boldsymbol{A}_{j,i}$. Show that for $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ and $\boldsymbol{B} \in \mathbb{R}^{n\times p}$ we have $(\boldsymbol{AB})^\top = \boldsymbol{B}^\top\boldsymbol{A}^\top$.

   Recall $(\boldsymbol{AB})_{i,j} = \sum\limits_{k=1}^{n} \boldsymbol{A}_{i,k}\boldsymbol{B}_{k,j}$.

$$\begin{aligned}
(\boldsymbol{AB})_{i,j} &= \sum_{k=1}^{n} \boldsymbol{A}_{i,k}\boldsymbol{B}_{k,j} \\
((\boldsymbol{AB})^\top)_{i,j} &= (\boldsymbol{AB})_{j,i} \\
&= \sum_{k=1}^{n} \underbrace{\boldsymbol{A}_{j,k}}_{\in\mathbb{R}}\underbrace{\boldsymbol{B}_{k,i}}_{\in\mathbb{R}} \\
&= \sum_{k=1}^{n} \boldsymbol{B}_{k,i}\boldsymbol{A}_{j,k} \\
&= \sum_{k=1}^{n} (\boldsymbol{B}^\top)_{i,k}(\boldsymbol{A}^\top)_{k,j} \\
&= \boldsymbol{B}^\top\boldsymbol{A}^\top \qquad \blacksquare
\end{aligned}$$

2. An orthonormal set of vectors in a set of normalised vectors. (i.e. of Euclidean length 1) that are mutually orthogonal. Check that one of the two following pairs of vectors are orthogonal.

   (a) Dot product is 0, hence orthogonal.

$$\begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$$

   (b) Dot product is -4, hence not orthogonal.

$$\begin{bmatrix} 3 \\ 5 \\ 3 \\ -4 \end{bmatrix} \text{ and } \begin{bmatrix} 4 \\ -2 \\ 2 \\ 3 \end{bmatrix}$$

$$\begin{aligned}
\boldsymbol{v_1} &= \frac{1}{\sqrt{2^2+5^2+1^2}}\begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix} \\
&= \frac{1}{\sqrt{30}}\begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix} \\
\boldsymbol{v_2} &= \frac{1}{\sqrt{3^2+1^2+1^2}}\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} \\
&= \frac{1}{\sqrt{11}}\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} \\
\boldsymbol{v_3} &= \boldsymbol{v_1} \times \boldsymbol{v_2}
\end{aligned}$$

$$= \frac{1}{\sqrt{330}} \begin{bmatrix} 4 \\ -5 \\ 17 \end{bmatrix}$$

3. (a) For vectors $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^m$, we define $\text{proj}_{\boldsymbol{u}}(\boldsymbol{v}) = \frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\boldsymbol{u}\cdot\boldsymbol{u}}\boldsymbol{u}$ if $\boldsymbol{u} \neq \boldsymbol{0}$, and 0 otherwise. Explain geometrically, what $\text{proj}_{\boldsymbol{u}}(\boldsymbol{v})$ represents.

   (b) Now suppose we have any (not necessarily orthonormal) basis $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$ for $\mathbb{R}^3$, let

$$\boldsymbol{u_1} = \boldsymbol{v_1},\ \boldsymbol{u_2} = \boldsymbol{v_2} - \text{proj}_{\boldsymbol{u_1}}(\boldsymbol{v_2}),\ \boldsymbol{u_3} = \boldsymbol{v_3} - \text{proj}_{\boldsymbol{u_1}}(\boldsymbol{v_3}) - \text{proj}_{\boldsymbol{u_2}}(\boldsymbol{v_3}), \text{ and } \boldsymbol{w_i} = \frac{u_i}{\|\boldsymbol{u_i}\|}$$

   Check that $\{\boldsymbol{w_i} : i = 1, 2, 3\}$ is an orthonormal basis for $\mathbb{R}^3$.

4. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map, and let $\boldsymbol{e_1}, \boldsymbol{e_2}$ be a basis for $\mathbb{R}^2$, suppose;

$$f(\boldsymbol{e_1}) = \boldsymbol{5e_1 - 6e_2} \text{ and } f(\boldsymbol{e_2}) = \boldsymbol{e_2} + 3\boldsymbol{e_1}$$

   (a) Find the matrix $\boldsymbol{A}$ representing f with respect to the basis $\boldsymbol{e_1}, \boldsymbol{e_2}$.

$$\boldsymbol{A} = \begin{bmatrix} 5 & 3 \\ -6 & 1 \end{bmatrix}$$

   (b) If $\boldsymbol{v} \in \mathbb{R}^2$ is given by $\boldsymbol{v} = \boldsymbol{2e_1 - e_2}$. Find $f(\boldsymbol{v})$ and check that the matrix representing f correctly computes the coordinates of $f(\boldsymbol{v})$ with respect to the basis $\boldsymbol{e_1}, \boldsymbol{e_2}$.

$$\begin{aligned} f(\boldsymbol{v}) &= 2f(\boldsymbol{e_1}) - f(\boldsymbol{e_2}) \\ &= 2(5\boldsymbol{e_1} - 6\boldsymbol{e_2}) - (\boldsymbol{e_1} + 3\boldsymbol{e_1}) \\ &= 7\boldsymbol{e_1} - 13\boldsymbol{e_2} \end{aligned}$$

$$\begin{bmatrix} 5 & 3 \\ -6 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ -13 \end{bmatrix} \qquad \text{as expected}$$

   (c) Suppose now we have a new basis $\boldsymbol{d_1}, \boldsymbol{d_2}$ given by

$$\boldsymbol{d_1} = \boldsymbol{e_1} - \boldsymbol{e_2} \text{ and } \boldsymbol{d_2} = \boldsymbol{e_1} + \boldsymbol{e_2}$$

   Find the matrix representing f in the new basis $\boldsymbol{d_1}, \boldsymbol{d_2}$.

$$\begin{aligned} \boldsymbol{I}_{ED} &= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \\ \boldsymbol{I}_{DE} &= (\boldsymbol{I}_{DE})^{-1} \\ &= \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \\ f_{DD} &= \boldsymbol{I}_{DE} \underbrace{f_{EE}}_{\boldsymbol{A}} \boldsymbol{I}_{ED} \\ &= \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 5 & 3 \\ -6 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 9 & 13 \\ -5 & 3 \end{bmatrix} \end{aligned}$$