CO343 - Operations Research (60016)

Lecture 1

Operations research is the science of taking decisions, it's a branch of applied mathematics where we attempt to model problems where need to make a decision. The decisions aren't arbitrary, and we want to attempt to score each decision based on some metric (such as time, cost, etc.), to find the optimal solution.

The course focuses on formulating a mathematical model to represent the problem, and then developing a computer-based procedure for deriving solutions to the problem from the model. Assume our goal was the following;

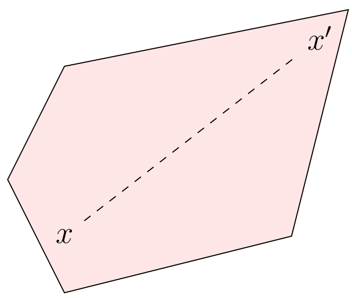
$$\min_{\boldsymbol{x}} z = f(\boldsymbol{x}) \qquad \text{subject to } \boldsymbol{x} \in \mathcal{X}$$

- **decision variables** $\boldsymbol{x} \in \mathbb{R}^n$
- **objective function** $f : \mathbb{R}^n \to \mathbb{R}$
- **feasible set** (set of admissible decisions) $\mathcal{X} \subseteq \mathbb{R}^n$
- **optimal solution** (any vector that minimises f) $\boldsymbol{x}^*$
- **optimal value** $z^* = f(\boldsymbol{x}^*)$

Linear Programming

A linear program optimises a **linear objective function**, where a feasible set is described by linear equality / inequality constraints. Compared to non-linear problems, where a **local** maximum may vary (and therefore be sub-optimal) depending on the starting search position, this isn't a concern for linear problems.

We can say the polygon representing a two dimensional feasible set is convex if the points on the line joining two points in the feasible set are also in the polygon. If this region is convex and linear, it can be proven that a local optimum is also a global optimum. For example, take x and x';



Linear Programming Example

A manufacturer produces A (acid) and C (caustic) and wants to decide a production plan. The ingredients for A and C are X (a sulphate) and Y (sodium).

- each ton of A requires 2 tons of X and 1 ton of Y
- each ton of C requires 1 ton of X and 3 tons of Y
- supply of X is limited to 11 tons per week
- supply of Y is limited to 18 tons per week
- A sells for £1000 per ton

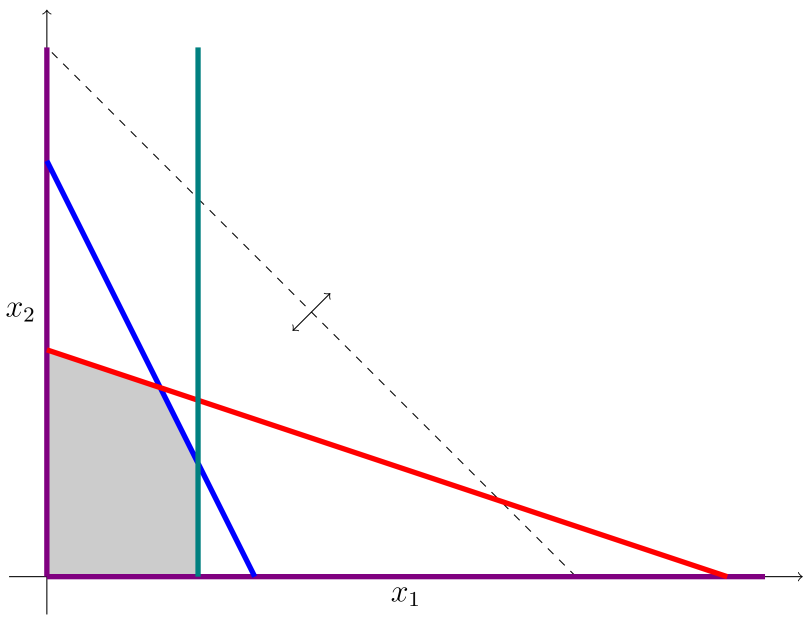
- C sells for £1000 per ton
- a maximum of 4 tons of A can be sold per week

Our goal is to maximise weekly value of sales of A and C. To determine how much A and C to produce, we need to formulate a **mathematical programming model**;

- **decision variables**
  - weekly production of A (tons) x_1
  - weekly production of B (tons) x_2
- **objective function** (weekly profit in £1000s) $z = f(x_1, x_2)$
- **feasible set** $\boldsymbol{x} = (x_1, x_2) \in \mathcal{X}$

A **production plan** is represetnable as $\boldsymbol{x} = (x_1, x_2)$. The objective function can be written as $z = x_1 + x_2$. Another constraint is that $x_1 \geq 0$ and $x_2 \geq 0$; we cannot produce a negative amount of a product. x_1 tons of A and x_2 tons of C requires $2x_1 + x_2$ tons of X, and we know that is limited to 11 tons per week; therefore we have the constraint $2x_1 + x_2 \leq 11$. Similarly, we also have the limitation of $x_1 + 3x_2 \leq 18$, because of the limitations of Y. Finally, we have another restriction that we cannot sell more than 4 tons of A, therefore $x_1 \leq 4$.

To get the overall feasible set, we intersect the feasible set of all the constraints to get the following;



Each of the following vertices is the intersection of constraints, which can be obtained by solving the linear equation of each line;

$$O = (0, 0)$$
$$P = (0, 6)$$
$$Q = (3, 5)$$
$$R = (4, 3)$$
$$S = (4, 0)$$

By moving the objective function (the dashed line), in the direction of the arrows, we can see that the z value increases further away from the origin, and therefore the graphical result that results in the highest value is Q. Typically the optimal solution lies on a vertex, however in some cases, there can be multiple solutions (an edge when the objective function is parallel to the constraint, or all the points in the feasible set in the case of a constant objective function).

The simplest algorithm is to enumerate all the vertices (intersections) of the feasible set, however this can have exponential complexity in the worst case and the number of vertices grow quite quickly in higher dimensions. The **Simplex Algorithm** finds an optimal vertex, often inspecting a **small subset** of the total.

We can vary this example, for example if we wanted to minimise $z = 3x_1 - x_2$ over the feasible set, we can examine the objective function at each of the vertices;

$O = (0,0)$	$P = (0,6)$	$Q = (3,5)$	$R = (4,3)$	$S = (4,0)$
0	-6	4	9	12

This therefore gives us $P = (x_1, x_2) = (0, 6)$ as the optimal.

On the other hand, if we were to maximise $z = 2x_1 + x_2$, any point on the line segment QR would be optimal; this tells us that points other than the vertices can be optimal, but there is at least one optimal vertex.

Additionally, if we were to set a production goal of 7 tons of A, we'd have an empty feasible set, since $x_1 \geq 7$ would cause an empty set with $x_1 \leq 4$. In this case, the LP is **infeasible**. Similarly, if the constraints on X and Y were removed, the objective function could grow to $+\infty$, hence the LP is **unbounded**.