CO150 - Recurrence Relations Cribsheet

Prelude

The content discussed here is part of CO150 - Graphs and Algorithms (Computing MEng); taught by Iain Phillips, in Imperial College London during the academic year 2018/19. Raihaan wanted me to do these. Probably copied mostly from the notes.

We refer to functions $W : \mathbb{N} \to \mathbb{N}$, and $A : \mathbb{N} \to \mathbb{N}$, as complexity functions. These will normally be solved by repeated expansion.

Binary Search

$$W(1) = 1$$
$$W(n) = 1 + W(\lfloor \frac{n}{2} \rfloor)$$
$$= 1 + 1 + W(\lfloor \frac{n}{4} \rfloor)$$
$$\dots$$
$$= 1 + 1 + \dots + 1 + W(1)$$
$$= 1 + \lfloor \log_2(n) \rfloor$$

The number of 1s we get is determined by how many times we can divide n by 2. Allow us to bound n as $2^k \le n < 2^{k+1} \Leftrightarrow k \le \log_2(n) < k+1$, hence $k = \lfloor \log_2(n) \rfloor$, and since $W(1) = 1$, $W(n) = 1 + \lfloor \log_2(n) \rfloor$.

Strassen's Algorithm

$$A(0) = 1$$
$$A(k) = 7A(k-1) + 18(\frac{n}{2})^2$$
$$= 7(7A(k-2) + 18(\frac{n}{4})^2) + 18(\frac{n}{2})^2$$
$$= 7^k + 18\frac{n^2}{4} \sum_{i=0}^{k-1}(\frac{7}{4})^i$$
$$= 7^k + 18\frac{n^2}{4} \cdot \frac{(\frac{7}{4})^l - 1}{\frac{7}{4} - 1}$$
$$= 7^k + 6n^2((\frac{7}{4})^k - 1)$$
$$= 7^k + 6 \cdot 4^k((\frac{7}{4})^k - 1)$$
$$= (1 + 6)7^k - 6 \cdot 4^k$$
$$= 7 \cdot 7^k - 6 \cdot n^2$$
$$= 7n^{\log_2(n)} - 6 \cdot n^2$$

For this, we're assuming $n = 2^k$, so we can easily subdivide the matrix. If this isn't the case, we can easily pad the matrices with 0 rows, or columns. The standard result for the partial sum of a geometric series is applied here.

Merge Sort

$$W(1) = 0$$
$$W(n) = n - 1 + W(\lceil \frac{n}{2} \rceil) + W(\lfloor \frac{n}{2} \rfloor)$$
$$= n - 1 + 2W(\frac{n}{2})$$
$$= n - 1 + 2(\frac{n}{2} - 1) + 2^2 W(\frac{n}{2^2}))$$
$$= n + n - (1 + 2) + 2^2 W(\frac{n}{2^2}))$$
$$\cdots$$
$$= n + n + \cdots + n - (1 + 2 + 2^2 + \cdots + 2^{k-1}) + 2^k W(\frac{n}{2^k}))$$
$$= kn - (2^k - 1) + 0$$
$$= n\log_2(n) - (n - 1)$$
$$= n\log_2(n) - n + 1$$
$$= n\lceil \log_2(n) \rceil - 2^{\lceil \log_2(n) \rceil} + 1$$

Note that here we're assuming $n = 2^k$, as it makes it easier. The standard result for a partial sum of a geometric series is used in the penultimate lines, and we take the ceiling, in order to generalise it for all n.

Master Theorem

Not really a recurrence relation, but it fits here.

Given the general form of a divide, and conquer algorithm; $T(n) = aT(\frac{n}{b}) + f(n)$, and critical exponent $E = \log_b(a)$

- if $n^{E+\epsilon} = O(f(n))$ for some $\epsilon > 0$, then $T(n) = \Theta(f(n))$

    informally; if $O(n^E) < O(f(n)) \Rightarrow T(n) = \Theta(f(n))$
- if $f(n) = \Theta(n^E)$ then $T(n) = \Theta(f(n)\log(n))$

    informally; if $O(n^E) = O(f(n)) \Rightarrow T(n) = \Theta(f(n)\log(n))$
- if $f(n) = O(n^{E-\epsilon})$ for some $\epsilon > 0$ then $T(n) = \Theta(n^E)$

    informally; if $O(n^E) < O(f(n)) \Rightarrow T(n) = \Theta(n^E)$

Insertion Sort

Worst Case

$$W(1) = 0$$
$$W(n) = n - 1 + W(n - 1)$$
$$= \sum_{i=0}^{n-1} i$$
$$= \frac{n(n-1)}{2}$$

This is no better than the worst case for insertion sort. However, it's fairly rare for this to happen, so we consider the average case.

Average Case

$$A(0) = 0$$
$$A(1) = 0$$

$$A(n) = n - 1 + \frac{1}{n} \sum_{s=0}^{n-1} (A(s) + A(n - s - 1))$$

$$= n - 1 + \frac{2}{n} \sum_{i=2}^{n-1} A(i)$$

This can then be used to prove $A(n)$ is $\Theta(n\log(n))$, but I'm not going to do that, because it's tedious.

Word Split Problem

$$W_1(0) = 0$$

$$W_1(n) = n + \sum_{i=0}^{n-1} W_1(i)$$

$$= n + W_1(n - 1) - (n - 1) + W_1(n - 1)$$
$$= 1 + 2W_1(n - 1)$$
$$= 2^n - 1$$

Note that the second line of the recurrence relation is justified by observing how all the terms from 0 to $n - 2$ are already present in $W_1(n - 1)$. Not in the notes, just something I wanted to check.

Suppose $W_1(n - 1) = n - 1 + \sum_{i=1}^{n-2} W_1(i)$. By arithmetic, it follows that $\sum_{i=1}^{n-2} W_1(i) = W_1(n - 1) - n + 1$.

$$
\begin{aligned}
W_1(n) \quad &= n + \sum_{i=0}^{n-1} W_1(i) \\
&= n + \sum_{i=0}^{n-2} W_1(i) + W_1(n - 1) && \text{by def. of } \Sigma \\
&= n + W_1(n - 1) - n + 1 + W_1(n - 1) && \text{by substitution} \\
&= 1 + 2W_1(n - 1) && \text{by arithmetic}
\end{aligned}
$$