CO150 - Recurrence Relations Cribsheet

Prelude

The content discussed here is part of CO150 - Graphs and Algorithms (Computing MEng); taught by Iain Phillips, in Imperial College London during the academic year 2018/19. Raihaan wanted me to do these. Probably copied mostly from the notes.

We refer to functions $W : \mathbb{N} \to \mathbb{N}$, and $A : \mathbb{N} \to \mathbb{N}$, as complexity functions. These will normally be solved by repeated expansion.

Binary Search

$$
\begin{aligned}
W(1) &= 1 \\
W(n) &= 1 + W(\lfloor \frac{n}{2} \rfloor) \\
&= 1 + 1 + W(\lfloor \frac{n}{4} \rfloor) \\
&\dots \\
&= 1 + 1 + \cdots + 1 + W(1) \\
&= 1 + \lfloor \log_2(n) \rfloor
\end{aligned}
$$

The number of 1s we get is determined by how many times we can divide n by 2. Allow us to bound n as $2^k \leq n < 2^{k+1} \Leftrightarrow k \leq \log_2(n) < k+1$, hence $k = \lfloor \log_2(n) \rfloor$, and since $W(1) = 1$, $W(n) = 1 + \lfloor \log_2(n) \rfloor$.

Strassen's Algorithm

$$
\begin{aligned}
A(0) &= 1 \\
A(k) &= 7A(k-1) + 18(\frac{n}{2})^2 \\
&= 7(7A(k-2) + 18(\frac{n}{4})^2) + 18(\frac{n}{2})^2 \\
&= 7^k + 18\frac{n^2}{4} \sum_{i=0}^{k-1} (\frac{7}{4})^i \\
&= 7^k + 18\frac{n^2}{4} \cdot \frac{(\frac{7}{4})^l - 1}{\frac{7}{4} - 1} \\
&= 7^k + 6n^2((\frac{7}{4})^k - 1) \\
&= 7^k + 6 \cdot 4^k((\frac{7}{4})^k - 1) \\
&= (1+6)7^k - 6 \cdot 4^k \\
&= 7 \cdot 7^k - 6 \cdot n^2 \\
&= 7n^{\log_2(n)} - 6 \cdot n^2
\end{aligned}
$$

For this, we're assuming $n = 2^k$, so we can easily subdivide the matrix. If this isn't the case, we can easily pad the matrices with 0 rows, or columns. The standard result for the partial sum of a geometric series is applied here.

Merge Sort

$$
\begin{aligned}
W(1) &= 0 \\
W(n) &= n - 1 + W(\lceil \frac{n}{2} \rceil) + W(\lfloor \frac{n}{2} \rfloor) \\
&= n - 1 + 2W(\frac{n}{2}) \\
&= n - 1 + 2(\frac{n}{2} - 1) + 2^2 W(\frac{n}{2^2})) \\
&= n + n - (1 + 2) + 2^2 W(\frac{n}{2^2})) \\
&\dots \\
&= n + n + \cdots + n - (1 + 2 + 2^2 + \cdots + 2^{k-1}) + 2^k W(\frac{n}{2^k})) \\
&= kn - (2^k - 1) + 0 \\
&= n\log_2(n) - (n - 1) \\
&= n\log_2(n) - n + 1 \\
&= n\lceil \log_2(n) \rceil - 2^{\lceil \log_2(n) \rceil} + 1
\end{aligned}
$$

Note that here we're assuming $n = 2^k$, as it makes it easier. The standard result for a partial sum of a geometric series is used in the penultimate lines, and we take the ceiling, in order to generalise it for all n.

Master Theorem

Not really a recurrence relation, but it fits here.

Given the general form of a divide, and conquer algorithm; $T(n) = aT(\frac{n}{b}) + f(n)$, and critical exponent $E = \log_b(a)$

- if $n^{E+\epsilon} = O(f(n))$ for some $\epsilon > 0$, then $T(n) = \Theta(f(n))$
  informally; if $O(n^E) < O(f(n)) \Rightarrow T(n) = \Theta(f(n))$
- if $f(n) = \Theta(n^E)$ then $T(n) = \Theta(f(n)\log(n))$
  informally; if $O(n^E) = O(f(n)) \Rightarrow T(n) = \Theta(f(n)\log(n))$
- if $f(n) = O(n^{E-\epsilon})$ for some $\epsilon > 0$ then $T(n) = \Theta(n^E)$
  informally; if $O(n^E) < O(f(n)) \Rightarrow T(n) = \Theta(n^E)$

Quicksort

Worst Case

$$
\begin{aligned}
W(1) &= 0 \\
W(n) &= n - 1 + W(n - 1) \\
&= \sum_{i=0}^{n-1} i \\
&= \frac{n(n-1)}{2}
\end{aligned}
$$

This is no better than the worst case for insertion sort. However, it's fairly rare for this to happen, so we consider the average case.

Average Case

$$
\begin{aligned}
A(0) &= 0 \\
A(1) &= 0 \\
A(n) &= n - 1 + \frac{1}{n}\sum_{s=0}^{n-1}(A(s) + A(n-s-1)) \\
&= n - 1 + \frac{2}{n}\sum_{i=2}^{n-1} A(i)
\end{aligned}
$$

This can then be used to prove $A(n)$ is $\Theta(n\log(n))$, but I'm not going to do that, because it's tedious.

Word Split Problem

$$
\begin{aligned}
W_1(0) &= 0 \\
W_1(n) &= n + \sum_{i=0}^{n-1} W_1(i) \\
&= n + W_1(n-1) - (n-1) + W_1(n-1) \\
&= 1 + 2W_1(n-1) \\
&= 2^n - 1
\end{aligned}
$$

Note that the second line of the recurrence relation is justified by observing how all the terms from 0 to $n-2$ are already present in $W_1(n-1)$. Not in the notes, just something I wanted to check.

Suppose $W_1(n-1) = n - 1 + \sum_{i=1}^{n-2} W_1(i)$. By arithmetic, it follows that $\sum_{i=1}^{n-2} W_1(i) = W_1(n-1) - n + 1$.

$$
\begin{aligned}
W_1(n) &= n + \sum_{i=0}^{n-1} W_1(i) && \\
&= n + \sum_{i=0}^{n-2} W_1(i) + W_1(n-1) && \text{by def. of } \Sigma \\
&= n + W_1(n-1) - n + 1 + W_1(n-1) && \text{by substitution} \\
&= 1 + 2W_1(n-1) && \text{by arithmetic}
\end{aligned}
$$