CO233 - Computational Techniques

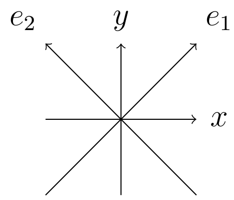
15th January 2020

Vector and Matrix Norms

An orthonormal basis of $\mathbb{R}^n$ are unit vectors that are pairwise mutually perpendicular; such that for $(e_1, \cdots, e_n)$;

- $e_i \cdot e_i = 1$
- $e_i \cdot e_j = 0$, if $i \neq j$

The standard canonical basis of $\mathbb{R}^3$ are the i, j, k vectors, and similar in $\mathbb{R}^2$. However, we can form another orthonormal basis of $\mathbb{R}^2$ by bisecting the angles as such;



If we take a vector $\boldsymbol{v} \in \mathbb{R}^n$, the Euclidean norm (or the ℓ_2-norm) is defined as such;

$$||\boldsymbol{v}||_2 = \sqrt{\sum_{i=1}^{n} v_i^2}$$

A norm, a mapping $|| \cdot || : \mathbb{R}^n \to \mathbb{R}^+$, must satisfy these 3 axioms;

(i) $||\boldsymbol{v}|| > 0$ given that $\boldsymbol{v} \neq \boldsymbol{0}$

(ii) $||\lambda \boldsymbol{v}|| = |\lambda| \; ||\boldsymbol{v}||$

(iii) $||\boldsymbol{v} + \boldsymbol{w}|| \leq ||\boldsymbol{v}|| + ||\boldsymbol{w}||$ (triangular inequality)

Some other (ℓ_p) norms are defined as follows;

$$\begin{aligned}
\ell_1\text{-norm } ||\boldsymbol{v}||_1 &= \sum_{i=1}^{n} |v_i| \\
\ell_\infty\text{-norm } ||\boldsymbol{v}||_\infty &= \max\{|v_i| : 1 \leq i \leq n\} \\
\ell_p\text{-norm } ||\boldsymbol{v}||_p &= (\sum_{i=1}^{n} |v_i|^p)^{\frac{1}{p}}
\end{aligned}$$