CO245 - Probability and Statistics

15th January 2020

Probability is a mathematical formalism used to describe and quantify uncertainty.

Sample Spaces and Events

- **sample space** S or Ω

  a set containing the possible outcomes of a random experiment

  for example; sample space of two coin tosses $S = \{(H,H),(H,T),(T,H),(T,T)\}$
- **event** E $(E \subseteq S)$

  any subset of the sample space (collection of some possible events)

  for example; event of the first coin being heads in two tosses $E = \{(H,H),(H,T)\}$

  the extremes are $\varnothing$ (the null event) which will never occur, or S (the universal event) which will always occur - there is only uncertainty when the events are strictly between the events, such that $\varnothing \subset E \subset S$
- **elementary event** singleton subset containing exactly one element from S

When performing a random experiment, the outcome will be a single element $s^* \in S$. Then an event $E \subseteq S$ has **occurred** iff $s^* \in E$. If it has not occurred, then $s^* \notin E \Leftrightarrow s^* \in \bar{E}$ (can be read as not E).

With a set of events $\{E_1, E_2, \dots\}$, we can have the following set operations;

- $\bigcup\limits_i E_i = \{s \in S \mid \exists i.\ [s \in E_i]\}$ will only occur if at least one of the events E_i occurs ("or")
- $\bigcap\limits_i E_i = \{s \in S \mid \forall i.\ [s \in E_i]\}$ will only occur if all of the events E_i occurs ("and")
- $\forall i, j.\ E_i \cap E_j = \varnothing$ $(i \neq j)$ if they are mutually exclusive (at most one can occur)

σ-algebra

In an uncountably infinite set, the event set you are assigning probabilities to cannot be every subset, as the probabilities cannot be made to sum to 1 under reasonable axioms.

We define the σ-algebra as the subset of events which we can assign probabilities to. We want to define a probability function P that corresponds to the subsets of S that we wish to **measure**. This set of subsets is referred to as $\mathfrak{S}$ (the event space), with the following three properties (corresponding to the axioms of probability);

- nonempty $S \in \mathfrak{S}$
- closed under complements $E \in \mathfrak{S} \Rightarrow \bar{E} \in \mathfrak{S}$
- closed under countable union (therefore any countable set is fine) $E_1, E_2, \dots \in \mathfrak{S} \Rightarrow \bigcup\limits_i E_i \in \mathfrak{S}$

A probability measure on the pair $(S, \mathfrak{S})$ is a mapping $P : \mathfrak{S} \to [0,1]$, satisfying the following three axioms;

- $\forall E \in \mathfrak{S}.\ [0 \leq P(E) \leq 1]$
- $P(S) = 1$
- countably additive, for **disjoint subsets** $E_1, E_2, \dots \in \mathfrak{S}$ $P\left(\bigcup\limits_i E_i\right) = \sum\limits_i P(E_i)$

From these, we can derive the following;

- $P(\bar{E}) = 1 - P(E)$

$$\underbrace{P(E) + P(E)}_{\text{disjoint}} = P(\underbrace{E \cup \bar{E}}_{E \cup \bar{E} = S}) = P(S) = 1$$

- $P(\varnothing) = 0$ special case of the above, when $E = S$
- for any events E and F $P(E \cup F) = P(E) + P(F) - P(E \cap F)$