

CO145 - Mathematical Methods

Prelude

The content discussed here is part of CO145 - Mathematical Methods (Computing MEng); taught by Michael Huth, and Mario Berta, in Imperial College London during the academic year 2018/19. The notes are written for my personal use, and have no guarantee of being correct (although I hope it is, for my own sake). This should be used in conjunction with the lecture notes. This module differs as there isn't as much new content, but it requires practice - as such, I will likely be including worked examples for my own benefit (which are probably incorrect).

Sequences

Formal Definition of a Limit

A sequence a_n , for $n \geq 1$, converges to some limit $l \in \mathbb{R}$ if, and only if, we can prove $\forall \epsilon > 0 [\exists N_\epsilon \in \mathbb{N} [\forall n > N_\epsilon [|a_n - l| < \epsilon]]]$.

To show convergence for the sequence $a_n = \frac{1}{n}$, we need to first make a guess for the limit - suppose $l = 0$. We can now attempt to find some N_ϵ . As $\frac{1}{n} - 0$ is positive for all $n \in \mathbb{N}$, we can drop the absolute, thus it's sufficient to find n such that $\frac{1}{n} < \epsilon$. Since both are positive (hence non-zero), we can take reciprocals on both sides, to get $n > \frac{1}{\epsilon}$. However, we are restricted by the fact that n must be an integer, hence it follows $N_\epsilon = \lceil \frac{1}{\epsilon} \rceil$. For any value of ϵ , we can get some N_ϵ with the function, thus it proves that a limit exists.

Common Converging Sequences

Note that for all of these, we are implicitly saying $\lim_{n \rightarrow \infty}$, and that $a_n \rightarrow 0$.

a_n	condition	N_ϵ
$\frac{1}{n^c}$	for some $c \in \mathbb{R}^+$	$\lceil \frac{1}{\epsilon^c} \rceil$
$\frac{1}{c^n}$	for some $c \in \mathbb{R}$, such that $ c > 1$	$\lceil \log_c(\frac{1}{\epsilon}) \rceil$
c^n	for some $c \in \mathbb{R}$, such that $ c < 1$	$\lceil \log_c(\epsilon) \rceil$
$\frac{1}{n!}$	$n > 1$	$\lceil e^{\frac{1}{\epsilon}} \rceil$
$\frac{1}{\ln(n)}$		

Combining Sequences

Suppose that $a_n \rightarrow a$, and $b_n \rightarrow b$, as $\lim_{n \rightarrow \infty}$:

- $\lim_{n \rightarrow \infty} \lambda a_n = \lambda a$ given $\lambda \in \mathbb{R}$
- $\lim_{n \rightarrow \infty} (a_n + b_n) = a + b$
- $\lim_{n \rightarrow \infty} (a_n b_n) = ab$
- $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{a}{b}$ given $b \neq 0$

For example, the sequence $a_n = \frac{4n^2+3n}{7n^2+3n-2}$, it's trivial to find the limit as $n \rightarrow \infty$ by inspection as $\frac{4}{7}$. However, if we divide every term by n^2 , we end up with $a_n = \frac{4+\frac{3}{n}}{7+\frac{3}{n}-\frac{2}{n^2}}$, which we can break into $a_n = \frac{b_n}{c_n}$, where $b_n = 4 + \frac{3}{n}$, and $c_n = 7 + \frac{3}{n} - \frac{2}{n^2}$. Using the rules from above, we can further break down the sequences (but I really cannot be bothered to do so), to a point where we get $a = \frac{4+0}{7+0-0} = \frac{4}{7}$.

Sandwich Theorem

In the sandwich theorem, where we want to prove that $\lim_{n \rightarrow \infty} a_n = l$, we need two sequences that form upper, and lower bounds for a_n , namely u_n , and l_n . If such sequences exist, and satisfy $\exists N \in \mathbb{N}[\forall n \geq N[l_n \leq a_n \leq u_n]]$, and both $\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} l_n = l$, then we get $\lim_{n \rightarrow \infty} a_n = l$.

For example, consider the sequence $a_n = \frac{\cos(n)}{n}$. We know that $-1 \leq \cos(n) \leq 1$, therefore $l_n = -\frac{1}{n} \leq a_n \leq \frac{1}{n} = u_n$. However, as both $u_n \rightarrow 0$, and $l_n \rightarrow 0$, when $n \rightarrow \infty$, it follows that $\lim_{n \rightarrow \infty} a_n = 0$.

The sandwich theorem can be proven by finding $N_{\epsilon l}$, and $N_{\epsilon u}$ for l_n , and u_n respectively. As they both converge to the same limit, we can justify that for some $N_\epsilon = \max(N_{\epsilon l}, N_{\epsilon u})$, any $n > N_\epsilon$, we have $|l_n - l| < \epsilon$, and $|u_n - l| < \epsilon$. By removing the modulus signs, we get $-\epsilon < l_n - l < \epsilon$, and $-\epsilon < u_n - l < \epsilon$. By rearranging this, and knowing $l_n < u_n$ by definition, we have $l - \epsilon < l_n < u_n < l + \epsilon$. In order to apply the sandwich theorem, we have to assume $l_n \leq a_n \leq u_n$, hence it follows $l - \epsilon < l_n \leq a_n \leq u_n < l + \epsilon$. This can then be arranged to get $-\epsilon < a_n - l < \epsilon$, thus for all $n > N_\epsilon$, $|a_n - l| < \epsilon$.

Ratio Tests

The follow tests allow us to check if a sequence a_n converges to 0, or diverges. If we want to verify a sequence $\lim_{n \rightarrow \infty} b_n = l$, where $l \neq 0$, we can reformulate it as $a_n = b_n - l$, and prove it for a_n .

If $|\frac{a_{n+1}}{a_n}| \leq c < 1$, for $c \in \mathbb{R}$, where n is sufficiently large, then $\lim_{n \rightarrow \infty} a_n = 0$. Conversely, if $|\frac{a_{n+1}}{a_n}| \geq c > 1$, then a_n diverges.

Suppose $a_n = 2^{-n}$, in order to show it converges, we want to find some c , such that $|\frac{2^{-(n+1)}}{2^{-n}}| \leq c < 1$. By arithmetic, we can say $|\frac{2^{-(n+1)}}{2^{-n}}| = \frac{1}{2} \leq c < 1$. As there exists real $c = \frac{1}{2} < 1$, we can conclude that a_n converges to 0.

However, we cannot prove that $a_n = \frac{1}{n!}$ converges, despite it being a fairly straightforward proof with just the standard ration tests. For example, we would end up with $|\frac{a_{n+1}}{a_n}| = \frac{1}{n+1}$. While this is fairly conclusive that it's less than 1, we don't get a constant c which sits between it, and 1.

Instead of analysing the behaviour of consecutive terms, we take the limit as $r = \lim_{n \rightarrow \infty} |\frac{a_{n+1}}{a_n}|$. If $r < 1$, it converges to 0, but if $r > 1$, it then diverges. Applying this to the previous factorial example, we can clearly see that $\lim_{n \rightarrow \infty} \frac{1}{n+1} = 0$, therefore a_n converges to 0.

Manipulating Absolute Values

A useful decomposition with absolute limits is $|x| < a \Leftrightarrow -a < x < a$. Further useful properties are;

- $|xy| = |x| \cdot |y|$
- $|\frac{x}{y}| = \frac{|x|}{|y|}$
- $|x + y| \leq |x| + |y|$ triangle inequality
- $|x - y| \geq ||x| - |y||$

Properties of Real Numbers

Suppose we have some set S of real numbers, then the following properties hold;

- u is an upper bound on S if $\forall s \in S[u \geq s]$
- l is a lower bound on S if $\forall s \in S[l \leq s]$
- S is bounded above if it has an upper bound, bounded below if it has a lower bound, and bounded if it has both
- a set may have no bounds, or many, therefore we say $\sup(S)$ is the least upper bound of S , and $\inf(S)$ to denote the greatest upper bound of S

- if a set has no upper bound, then $\sup(S) = \infty$, and similarly if there is no lower bound, then $\inf(S) = -\infty$

The fundamental axiom of analysis states that if an increasing sequence of reals is bounded above, then it must converge.

Series

A series is a sum of a sequence - we consider an infinite series as a summation in the form $S = \sum_{n=1}^{\infty} a_n$.

We can also take a partial sum, where we sum from 1 to n instead of ∞ as $S_n = \sum_{i=1}^n a_i$.

Since we're dealing with finite numbers at this point, we can consider S_n as a sequence - thus whether a series converges, or diverges, depends on whether the sequence of partial sums converge, or diverge. If we know that $\forall i \geq 1 [a_i > 0]$, then the partial sum is an increasing sequence where $S_1 < S_2 < S_3 < \dots$, if this is bounded above, then it must converge (from the axiom of analysis).

For a series to have a chance of converging, we require $\lim_{n \rightarrow \infty} a_n = 0$. However, just proving that is insufficient for concluding a series converges. It's important to note that we don't really care about the first values of a series, only how it behaves on large numbers, as it tends to infinity.

If S converges, or diverges, then $\sum_{n=N}^{\infty} a_n$ also converges, or diverges, as it's equivalent to $S - S_{N-1}$.

Geometric Series

To determine whether a limit G exists for the geometric series, we need to determine that the partial sums converge. Hence we do the proof on the right hand side. It's important to note that the only term that changes based on n is x^{n+1} , therefore we can conclude that G must converge when $|x| < 1$ (from earlier results).

Let us represent the geometric series as $G = \sum_{n=1}^{\infty} x^n$. Should a limit G exist - see the left side;

$$\begin{aligned}
 G &= \sum_{n=1}^{\infty} x^n \\
 &= x + \sum_{n=2}^{\infty} x^n \\
 &= x + x \sum_{n=1}^{\infty} x^n \\
 &= x + xG \\
 &= \frac{x}{1-x}
 \end{aligned}
 \qquad
 \begin{aligned}
 G_n &= \sum_{i=1}^n x^i \\
 &= x + \sum_{i=2}^n x^i \\
 &= x + x \sum_{i=1}^{n-1} x^i \\
 &= x + x(G_n - x^n) \\
 &= \frac{x - x^{n+1}}{1-x}
 \end{aligned}$$

Harmonic Series

The harmonic series is often used in comparison tests to prove that another series diverges.

Let harmonic series be written as $S = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{> \frac{1}{4} + \frac{1}{4}} + \underbrace{\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)}_{> \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}} + \dots > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots$

Therefore, with partial sums, we get $S_n = \sum_{i=1}^{2^n} \frac{1}{i} > 1 + \frac{n}{2}$. As S_n clearly diverges, S must also diverge.

Inverse Squares

Suppose we have the sum of squares $S = \sum_{n=1}^{\infty} \frac{1}{n^2}$, and an auxiliary partial sum $T_n = \sum_{i=1}^n \frac{1}{i(i+1)}$.

To prove that S converges, we should consider $T_n = 1 - \frac{1}{n+1}$ (write out the first few terms and a pattern should be spotted). Hence $\lim_{n \rightarrow \infty} T_n = 1$. Using this, we can argue how the partial sums of the series of inverse squares are bounded. It's trivial to say the following inequality; $\frac{1}{i(i+1)} < \frac{1}{i^2} < \frac{1}{i(i-1)}$.

If we sum this inequality from $i = 2$ to n , we can then justify the following - it's important to note that the sums on the left, and right, differ by only the first, and last term (therefore we can reduce it to a partial sum);

$$\begin{aligned} \frac{1}{2(3)} + \frac{1}{3(4)} + \frac{1}{4(5)} + \dots + \frac{1}{n(n+1)} &< \sum_{i=2}^n \frac{1}{i^2} < \frac{1}{1(2)} + \frac{1}{2(3)} + \frac{1}{3(4)} + \dots + \frac{1}{n(n-1)} \\ \Leftrightarrow T_n - \frac{1}{2} &< \sum_{i=2}^n \frac{1}{i^2} < T_{n-1} \\ \Leftrightarrow T_n + \frac{1}{2} &< \sum_{i=1}^n \frac{1}{i^2} < T_{n-1} + 1 \\ \Leftrightarrow \frac{3}{2} + \frac{1}{n+1} &< S_n < 2 + \frac{1}{n} \end{aligned}$$

Therefore, it follows that the partial sums are bounded by $\frac{3}{2}$, and 2. The existence of an upper bound suggests the sequence converges, therefore the series must also converge.

Important Series

These examples will be useful for the comparison tests covered later on. We can use these results to aid us in proving whether some other series converges, or diverges.

name	S	condition	diverges or converges
harmonic series	$\sum_{n=1}^{\infty} \frac{1}{n}$		diverges
harmonic primes	$\sum_{p:\text{prime}} \frac{1}{p}$		diverges
geometric series	$\sum_{n=1}^{\infty} x^n$	$ x \geq 1$	diverges
geometric series	$\sum_{n=1}^{\infty} x^n$	$ x < 1$	converges
inverse squares series	$\sum_{n=1}^{\infty} \frac{1}{n^2}$		converges
???	$\sum_{n=1}^{\infty} \frac{1}{n^c}$	$c > 1$	converges

Convergence Tests

For brevity, suppose we have the following;

- $S = \sum_{i=1}^{\infty} a_i$ a series we want to reason about, and $a_i \geq 0$
- $\sum_{i=1}^{\infty} c_i$ a series we have already established converges to c
- $\sum_{i=1}^{\infty} d_i$ a series we have already established diverges

In the **comparison test**, we have some $\lambda > 0$, and a $N \in \mathbb{N}$. If $\forall i > N [a_i \leq \lambda c_i]$, then S converges. On the other hand, if $\forall i > N [a_i \geq \lambda d_i]$, then S diverges.

To avoid having to find such a λ , we can use the **limit comparison test**, in which we can say if $\lim_{i \rightarrow \infty} \frac{a_i}{c_i}$ exists, then S converges, but if $\lim_{i \rightarrow \infty} \frac{d_i}{a_i}$ exists, then S diverges.

Another useful test for finding convergence, without needing a pre-established series, is **D'Alembert's ratio test**. Once again, we consider this from some point $N \in \mathbb{N}$, where the cases are;

1. $\forall i \geq N[\frac{a_{i+1}}{a_i} \geq 1]$, then S diverges
2. there exists a k such that $\forall i \geq N[\frac{a_{i+1}}{a_i} \leq k < 1]$, then S converges

This is similar to the comparison test for sequences, and once again the requirement of finding some k between the ratio, and 1 can be avoided by taking limits. This leads to **D'Alembert's limit ratio test**, in which we have the following cases;

1. $\lim_{i \rightarrow \infty} \frac{a_{i+1}}{a_i} > 1$, then S diverges
2. $\lim_{i \rightarrow \infty} \frac{a_{i+1}}{a_i} = 1$, then it is inconclusive
3. $\lim_{i \rightarrow \infty} \frac{a_{i+1}}{a_i} < 1$, then S converges

The second case, in which it is inconclusive, can be illustrated by setting $a_n = 1$, which obviously causes S to diverge, or by setting $a_n = \frac{1}{n^2}$ which causes S to converge. However, both lead to a limit test of 1.

The **integral test** for a sequence, where $a_n = f(n)$ is a decreasing function relies on the following idea;

$$\sum_{n=1}^{\infty} a_{n+1} < \int_1^{\infty} f(x) dx < \sum_{n=1}^{\infty} a_n$$

In practice, if the integral were to diverge once evaluated, the series would also diverge by the right hand inequality, but if it were to converge, then the series would converge with the left hand inequality.

Absolute Convergence

Consider a series with negative a_n values, and another series $S' = \sum_{n=1}^{\infty} |a_n|$. If S' converges, so does S .

However, the same isn't true the other way around, for example, when we have $a_n = (-1)^{n-1} \frac{1}{n}$, S converges to $\ln(2)$, however, doing this with the absolute series, S' , it would diverge, as if it were the normal harmonic series. In order to test for **absolute** convergence - we have the following cases;

1. if $\lim_{n \rightarrow \infty} |\frac{a_{n+1}}{a_n}| > 1$ then $\sum_{n=1}^{\infty} a_n$ diverges
2. if $\lim_{n \rightarrow \infty} |\frac{a_{n+1}}{a_n}| = 1$ then $\sum_{n=1}^{\infty} a_n$ can converge, or diverge
3. if $\lim_{n \rightarrow \infty} |\frac{a_{n+1}}{a_n}| < 1$ then $\sum_{n=1}^{\infty} a_n$ converges absolutely, and therefore also converges

Power Series, and Radius of Convergence

We can represent a function as a power series; $f(x) = \sum_{n=0}^{\infty} a_n x^n$.

Consider the case where $f(x) = \frac{x}{1-x}$, which has the expansion $\sum_{n=1}^{\infty} x^n$, which only converges for $|x| < 1$.

Take the power series $S = \sum_{n=1}^{\infty} n^2 x^n$, and apply the **D'Alembert ratio test**, we'd get the following;

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)^2 x^{n+1}}{n^2 x^n} \right| \\ &= \lim_{n \rightarrow \infty} \left| x \left(1 + \frac{1}{n} \right)^2 \right| \\ &= |x| \end{aligned}$$

As we want to establish convergence, we set the result $|x| < 1$, therefore it follows that the ratio of convergence for this particular power series is 1.

Power Series

As previously mentioned, a function can be represented in a series expansion on x ; $f(x) = \sum_{n=0}^{\infty} a_n x^n$.

Maclaurin Series

Let some function $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$, then we can easily work out a_0 , by setting $x = 0$, such that $f(0) = a_0$. By differentiating the function, we get $f'(x) = a_1 + 2a_2x + 3a_3x^2 + \dots$, therefore it follows that $f'(0) = a_1$. By differentiating again, we get $f''(x) = 2a_2 + 2 \cdot 3a_3x + \dots$, therefore it follows that $\frac{f''(0)}{2!} = a_2$. Hence, generally $a_n = \frac{f^{(n)}(0)}{n!}$, where $n \geq 0$.

A **Maclaurin series** is the representation of $f(x) = \sum_{i=0}^{\infty} a_i x^i$.

This function can then be differentiated n times to get $f^{(n)}(x) = \sum_{i=n}^{\infty} a_i (i-1)\dots(i-n+1)x^{i-n}$.

From this, we can see that $f^{(n)}(0) = n!a_n$, hence $f(x) = \sum_{n=0}^{\infty} f^{(n)}(0) \frac{x^n}{n!}$.

For example, if we take $f(x) = \ln(1+x)$, and differentiate repeatedly;

$$\begin{array}{ll} f(x) &= \ln(1+x) & f(0) &= 0 \\ f'(x) &= (1+x)^{-1} & f'(0) &= 1 \\ f''(x) &= (-1)(1+x)^{-2} & f''(0) &= -1 \\ f'''(x) &= (-2)(-1)(1+x)^{-3} & f'''(0) &= 2 \\ f^{(n)}(x) &= \frac{(-1)^{n-1}(n-1)!}{(1+x)^n} & f^{(n)}(0) &= (-1)^{n-1}(n-1)! \\ a_n &= \frac{(-1)^{n-1}}{n} \end{array}$$

Therefore, we can justify that $f(x) = \ln(1+x) = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$. By setting $x = -1$, we'd get $-\infty$, and the power series would become the harmonic series, which we know to diverge. However, by setting $x = 1$, we get $\ln(2)$, is the convergence of the alternating harmonic series (as previously mentioned, in the ratio test example).

Taylor Series

When we're interested in values around a certain point, we can use a more generalised version of a **Maclaurin series**. The general result, where we apply the series around a is;

$$f(x) = f(a) + \frac{f^{(1)}(a)}{1!}(x-a)^1 + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \dots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

For example, if we take $f(x) = \ln(x)$ around a , and differentiate repeatedly;

$$\begin{array}{ll} f(x) &= \ln(x) & f(2) &= \ln(2) \\ f'(x) &= \frac{1}{x} & f'(2) &= \frac{1}{2} \\ f''(x) &= (-1)\frac{1}{x^2} & f''(2) &= -\frac{1}{4} \\ f'''(x) &= (-2)(-1)\frac{1}{x^3} & f'''(2) &= \frac{1}{4} \\ f^{(n)}(x) &= \frac{(-1)^{n-1}(n-1)!}{x^n} & f^{(n)}(2) &= \frac{(-1)^{n-1}(n-1)!}{2^n} \\ a_n &= \frac{(-1)^{n-1}}{n2^n} \end{array}$$

For example, we can approximate $\ln(x)$ around 2 as; $\ln(x) = \ln(2) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n2^n}(x-2)^n$.

It's important to note the bounds for this, as we start from $n = 1$, and manually state the constant term, since that doesn't fall within the "pattern" we had. We use the same technique to calculate the radius of convergence, as follows;

$$\begin{aligned}
\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(x-2)^{n+1}}{\frac{(n+1)2^{n+1}}{(x-2)^n}} \right| \\
&= \lim_{n \rightarrow \infty} \frac{n}{n+1} \cdot \frac{|x-2|}{2} \\
&= \frac{|x-2|}{2}
\end{aligned}$$

And then, by setting $\frac{|x-2|}{2} < 1$, we have a radius of convergence of 2, around $x = 2$. Generally, the radius is limited by the closest singularity (where a point is not defined, or isn't well-behaved).

Taylor Series Error Term

In order to ensure termination, instead of summing to ∞ , we often truncate at some arbitrary k^{th} term.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n = \sum_{n=0}^k \frac{f^{(n)}(a)}{n!} (x-a)^n + \underbrace{\frac{f^{(k+1)}(c)}{(k+1)!} (x-a)^{k+1}}_{\text{Lagrange error term}}$$

The Lagrange error term replaces the tail of the infinite series, from, which would otherwise be a sum from $k+1$ to ∞ . The constant c lies between x , and a . By taking the bound $a < c < x$, or $x < c < a$, it can be used to generate the worst-case error.

The mean value theorem states that there is some point c between x , and a , such that $f'(c) = \frac{f(x)-f(a)}{x-a}$, therefore we can get $f(x) = f(a) + (x-a)f'(c)$.

Cauchy Error Term

Define the set of partial sums of the Taylor series as $F_k(t) = \sum_{n=0}^k \frac{f^{(n)}(t)}{n!} (x-t)^n$, where t is the offset.

It follows that $F_k(x) = f(x)$ for all k (justified as every term becomes 0, other than the first term, $n=0$, where it's $\frac{f(x)}{0!}$). Therefore the error term is derived as follows, where $R_k(x)$ is the remainder after k terms;

$$F_k(x) - F_k(a) = f(x) - \sum_{n=0}^k \frac{f^{(n)}(a)}{n!} (x-a)^n = R_k(x)$$

Power Series Solutions of ODE

When we have a differential equation in the form $\frac{dy}{dx} = ky$, for a constant k , we can consider the power series for y . By differentiating term by term, we can compare coefficients;

$$\frac{dy}{dx} \equiv \frac{d}{dx} \sum_{i=0}^{\infty} a_i x^i \equiv \sum_{i=1}^{\infty} a_i i x^{i-1} \equiv ky \equiv k \sum_{i=0}^{\infty} a_i x^i \equiv k \sum_{i=1}^{\infty} a_{i-1} x^{i-1} \equiv \sum_{i=1}^{\infty} k a_{i-1} x^{i-1}$$

By matching coefficients of x^{i-1} , where $i \geq 1$, we have $k a_{i-1} = a_i i$, therefore we end up with the relation $a_i = \frac{k}{i} a_{i-1}$. By "expanding" this, we end up with $a_i = \frac{k}{i} a_{i-1} = \frac{k}{i} \cdot \frac{k}{i-1} a_{i-2} = \dots = \frac{k^i}{i!} a_0$.

If we were to have some boundary condition, such as when $x=0$, $y=1$, we know that $y(0) = a_0$, hence $a_0 = 1$. Therefore, this can be solved to the following;

$$y = \sum_{i=0}^{\infty} \frac{(kx)^i}{i!} = e^{kx}$$

Linear Algebra

Linear Equation Systems

For brevity, instead of writing;

$$ax_1 + bx_2 + cx_3 = j \tag{1}$$

$$dx_1 + ex_2 + fx_3 = k \tag{2}$$

$$gx_1 + hx_2 + ix_3 = l \tag{3}$$

We will write $\mathbf{A}\vec{x} = \vec{y}$, where $\mathbf{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, and $\vec{y} = \begin{bmatrix} j \\ k \\ l \end{bmatrix}$

In general, for a real linear equation system, we have the following possible outcomes;

- no solution

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 2 & 0 & 3 \end{bmatrix}, \text{ and } \vec{y} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

By adding equations (1), and (2), we get $2x_1 + 3x_3 = 5$, which contradicts equation (3), hence it is **inconsistent**, and therefore has no solutions.

- unique solution

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 0 & 1 & 1 \end{bmatrix}, \text{ and } \vec{y} = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}$$

By doing equation (1) - equation (3), we get $x_1 = 1$. Adding equations (1), and (2), we end up with $2x_1 + 3x_3 = 5$, however, we know the value of x_1 , therefore it follows that $x_3 = 1$. Finally, by using this value in equation (3), we have $x_2 = 1$, therefore the only solution is;

$$\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- infinite solutions

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 2 & 0 & 3 \end{bmatrix}, \text{ and } \vec{y} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$$

Note that this is the same \mathbf{A} from the first example. However, this time adding equations (1), and (2), is consistent with equation (3). If we let $x_3 = \lambda$, such that x_3 is a free variable, with $\lambda \in \mathbb{R}$, we can get both x_1 , and x_2 in terms of λ . By doing equations (1) - (2), we get $2x_2 - x_3 = 1$, hence $x_2 = \frac{1}{2} + \frac{1}{2}\lambda$. By doing equations (1) + (2), we get $2x_1 + 3x_3 = 5$, which leads to $x_1 = \frac{5}{2} - \frac{3}{2}\lambda$. As such, we have an infinite set of solutions, given $\lambda \in \mathbb{R}$;

$$\vec{x} = \begin{bmatrix} \frac{5}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} -\frac{3}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

Groups

With a set G , and some operation $\otimes : G \mapsto G$, the criteria for (G, \otimes) being a **group** are (1 - 4), and (1 - 5) for an **Abelian group**;

1. **closure** $\forall x, y \in G[x \otimes y \in G]$
 2. **associativity** $\forall x, y, z \in G[(x \otimes y) \otimes z = x \otimes (y \otimes z)]$
 3. **neutral element** $\exists e \in G[\forall x \in G[x \otimes e = x \wedge e \otimes x = x]]$
 4. **inverse element** $\forall x \in G[\exists y \in G[x \otimes y = e \wedge y \otimes x = e]]$
- often written as $y = x^{-1}$, note that this does **not** mean $y = \frac{1}{x}$, defined with respect to \otimes
5. **commutative** $\forall x, y \in G[x \otimes y = y \otimes x]$

For example, we can say $(\mathbb{Z}, +)$ is a group, but $(\mathbb{N}, +)$ is not, since the latter doesn't have an inverse element; for example $-5 \notin \mathbb{N}$, so 5 doesn't have an inverse. While $(\mathbb{R} \setminus \{0\}, \cdot)$ is an Abelian group, (\mathbb{R}, \cdot) isn't a group as 0 doesn't have an inverse element ($\neg \exists x \in \mathbb{R}[0 \cdot x = 1]$).