CO142 - Discrete Structures

Prelude

The content discussed here is part of CO142 - Discrete Structures (Computing MEng); taught by Steffen van Bakel, in Imperial College London during the academic year 2018/19. The notes are written for my personal use, and have no guarantee of being correct (although I hope it is, for my own sake). This should be used in conjunction with the (extremely detailed) notes.

9th October 2018

Recommended Books

- K.H. Rosen. *Discrete Mathematics and its Applications*
- J.L. Gersting. *Mathematical Structures for Computer Science*
- J.K. Truss. *Discrete Mathematics for Computer Science*
- R. Johsonbaugh. *Discrete Mathematics*
- C. Schumacher. *Fundamental Notions of Abstract Mathematics*

However, these books don't cover the same content. Learn his notation.

Logical Formula, and Notation

This notation will be shared with **CO140**.

- $A \wedge B$ — A and B both hold
- $A \vee B$ — A or B holds (or both)
- $\neg A$ — A does not hold
- $A \Rightarrow B$ — if A holds, then so does B
- $A \Leftrightarrow B$ — A holds if and only if B holds
- $\forall x(A)$ — the predicate A holds for all x
- $\exists x(A)$ — the predicate A holds for some x
- $a \in A$ — the object a is in the set A (a is an element of A)
- $a \notin A$ — the object a is not in the set A
- $=_A$ — tests whether two elements of A are the same

Sets

Sets are like data types in Haskell: Haskell data type declaration;

- `data Bool = False | True`
- {`false, true`} — set of boolean values
- `[true, false, true, false]` — list of boolean values
- {`false, true`} = {`true, false`} — set equality (note that order doesn't matter)

A set is a collection of objects from a pool of objects. Each object is an *element*, or a *member* of the set. A set *contains* its elements. Sets can be defined in the following ways;

- $\{a_1, ..., a_2\}$ — as a collection of n distinct elements
- $\{x \in A \mid P(x)\}$ — for all the elements in A, where P holds
- $\{x \mid P(x)\}$ — for all elements, where P holds (dangerous - Russel's paradox)

Use of "triangleq"

The use of $\triangleq$ is for "is defined by". Hence the empty set, $\varnothing \triangleq \{\}$. The difference between $\triangleq$ and $=$, is that the former cannot be proven, it is fact, whereas the latter takes work to prove.

Russel's paradox

Not everything we write as $\{x \mid P(x)\}$ is automatically a set. Assume $R = \{X \mid X \notin X\}$ is a set, the set of all sets which don't contain themselves. As R is a set, then $R \in R$, or $R \notin R$ (law of excluded middle), and thus we can do a case by case analysis.

- Assume $R \in R$. By the definition of R, it then follows that $R \notin R$ (if $R \in R$, then it doesn't satisfy the definition of R) - which is a contradiction.
- Assume $R \notin R$. It then follows that $R \in R$, as it follows the definition of R, hence it is another contradiction.

As both assumptions lead to contradictions, it's possible to write sets which aren't defined. We should only select from a set that we know is defined; $\{x \in A \mid P(x)\}$ - where A is a well-defined set.

12th October 2018

Set Comparisons

We can define a set A, as being a subset of another set B if every element in A is an element in B. This can be formally written as; $A \subseteq B \triangleq \forall x \in A(x \in B)$. Note that we can also say $\forall x(x \in A \Rightarrow x \in B)$, and the two hold the same meaning. It's important to clarify in the latter that we're not specifying the domain of x, and we assume there is a universe of possible objects which forms a set. We're also able to define a strict subset such that $A \subset B \triangleq A \subseteq B \wedge A \neq B$.

We can say that any set is a trivial subset of itself, as we'd have $x \in A \Rightarrow x \in A$, which always evaluates to true, from propositional logic. Another trivial example is that $\varnothing$, the empty set, is a subset of every set. Using the second definition of subset, we can say that as $x \in \varnothing$ is false, by definition, and anything follows from falsity, whereas in the first definition we argue that all (0) elements of $\varnothing$ are in some other set.

We can also define set equality as $A = B \triangleq A \subseteq B \wedge B \subseteq A$. However, we can also consider the set composition notation for a set, such that $A = \{x \in C \mid P(x)\}$, and $B = \{x \in C \mid Q(x)\}$. If we're able to prove that $\forall x(P(x) \Leftrightarrow Q(x))$, it follows that $A = B$. This method can be quite powerful if we're familiar with logic, and equivalences. We can justify this by saying that $y \in A \Rightarrow P(y) \Rightarrow Q(y) \Rightarrow y \in B$, and also in the other direction; $y \in B \Rightarrow Q(y) \Rightarrow P(y) \Rightarrow y \in A$. This however requires both sets to be constructed on top of some known set C.

Set Composition

- $A \cup B \triangleq \{x \mid x \in A \vee x \in B\}$ — set union
- $A \cap B \triangleq \{x \in A \mid x \in B\}$ — set intersection
- $A \backslash B$ (or $A - B$) $\triangleq \{x \in A \mid x \notin B\}$ — set difference
- $A \triangle B \triangleq (A \backslash B) \cup (B \backslash A)$ — symmetric set difference
- $A \cap B = \varnothing$ — A, and B are disjoint sets

A Note on Proofs

Instead of writing out the formal definition, where we may lose the intuition, using a natural language (direct) proof is acceptable in this course.

Consider the following proof; $A \subseteq B$, and $B \subseteq C$, then show $A \subseteq C$. Here, we want to show that any element of A, is also an element of C. We can approach this intuitively by taking an arbitrary $a \in A$. By the the first assumption, we can say $a \in B$. Then, by the second assumption, $a \in C$. However, we've taken an arbitrary a, therefore this follows $\forall a \in A(a \in C)$, therefore $A \subseteq C$.

The crucial part of the aforementioned proof is the use of some **arbitrary** value. If we were to do a proof on the natural numbers, to show $\forall n \in \mathbb{N}[\text{even}(n)]$, and we proved even(2), it wouldn't prove it for all natural numbers.

We also want to aim for a direct proof, instead of a proof by contradiction, since we will often do the following; assume $\neg A$, then we somehow get A, which causes a contradiction ($\lightning$), and therefore A. However, we still did all the work to prove A.

Consider the proof to show that $C \cap D = D \cap C$. Let us first take some arbitrary $x \in (C \cap D)$. By definition of union, we know that $x \in C$, and $x \in D$. Therefore, it also fits the predicate for $(D \cap C)$. As such, $C \cap D \subseteq D \cap C$. To prove the other direction is trivial, and almost identical to this direction. Since we've proved both directions of $\subseteq$, we can conclude equality.

Prove that $A = (A \backslash B) \cup (A \cap B)$. I took the approach where we use predicate logic, since I assumed it would be much easier than proving both directions of $\subseteq$ (turns out that the proof is very similar as proving one direction, is proving the other). In order to keep my proof cleaner, let $a \triangleq x \in A$, $b \triangleq x \in B$, and the negations $\neg a \triangleq x \notin A$ (and similar for b). Let us now define $A = \{x \mid P(x)\}$, where $P(x) = a$, and $B = \{x \mid Q(x)\}$, where $Q(x) = (a \wedge \neg b) \vee (a \wedge b)$ - by definitions of set difference, union, and intersection. Since this proves equivalence between the two predicates, we can therefore prove that the sets are equal.

$$
\begin{aligned}
Q(x) &= (a \wedge \neg b) \vee (a \wedge b) \\
&= [(a \wedge \neg b) \vee a] \wedge [(a \wedge \neg b) \vee b] && (B \wedge C) \vee A \equiv (A \vee B) \wedge (A \vee C) \\
&= (a \vee a) \wedge (a \vee \neg b) \wedge (a \vee b) \vee (b \vee \neg b) && (B \wedge C) \vee A \equiv (A \vee B) \wedge (A \vee C) \text{ (twice)} \\
&= a \wedge (a \vee \neg b) \wedge (a \vee b) && A \vee A \equiv A,\ A \vee \neg A \equiv \top, \text{ and } A \wedge \top \equiv A \\
&= a && A \wedge (A \vee B) \equiv A \text{ (twice)} \\
&= P(x)
\end{aligned}
$$

16th October 2018

A Note on the Use of Venn Diagrams

While we can use a Venn diagram to aid in constructing a counter example, the diagram itself is not a counter example. We're also quite limited in the possible uses, as a diagram (in 2d) consisting of ≥ 4 sets doesn't represent all the possible combinations of sets.

Operator Properties

Similar to **CO140**, we have some properties which can be used on arbitrary sets. Note that these are not axioms, and therefore we are able to prove them.

- $A \cup A = A$ — idempotence
- $A \cap A = A$ — idempotence
- $A \cup B = B \cup A$ — commutativity
- $A \cap B = B \cap A$ — commutativity
- $A \triangle B = B \triangle A$ — commutativity
- $A \cup (B \cup C) = (A \cup B) \cup C$ — associativity
- $A \cap (B \cap C) = (A \cap B) \cap C$ — associativity

- $A \cup \varnothing = A$ — empty set
- $A \cap \varnothing = \varnothing$ — empty set
- $A \triangle A = \varnothing$ — empty set
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ — distributivity
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ — distributivity
- $A \cup (A \cap B) = A$ — absorption
- $A \cap (A \cup B) = A$ — absorption

Note that we are able to use the properties of logical connectives to aid us in our proofs, since those are fairly easy to prove with truth tables, as they have a finite number of configurations. For example, the proof of idempotence inherently uses the property $p \wedge p \equiv p$, and the same for $\vee$.

Cardinality

With some finite set A, we can say that the cardinality, $|A|$ is the number of distinct elements in A. Given two finite sets, we can then say that $|A \cup B| = |A| + |B| - |A \cap B|$. With the following set properties (and that for two disjoint finite sets, $|A \cup B| = |A| + |B|$), and knowing the RHSs are disjoint unions;

$$\begin{aligned}
A &= (A \backslash B) \cup (A \cap B) \\
B &= (B \backslash A) \cup (A \cap B) \\
A \cup B &= (A \backslash B) \cup (A \cap B) \cup (B \backslash A) \\
|A| &= |A \backslash B| + |A \cap B| \\
|B| &= |B \backslash A| + |A \cap B| \\
|A \cup B| &= |A \backslash B| + |A \cap B| + |B \backslash A| \\
&= |A| - |A \cap B| + |A \cap B| + |B| - |A \cap B| \\
&= |A| + |B| - |A \cap B|
\end{aligned}$$

19th October 2018

Powerset

Let us define the powerset of A, as $\wp A \triangleq \{x \mid x \subseteq A\}$. It's therefore important to note that $\wp\varnothing = \{\varnothing\}$, hence the powerset of the empty set has size 1. We can prove that $|\wp X| = 2^n$, for some set X, where $|X| = n$. This can be done (fairly) easily with mathematical induction, over natural numbers. Another approach it is to consider that each item in some arbitrary set, $A = \{a_1, a_2, ...a_n\}$, can either be in the powerset or not. Therefore, we can represent each subset of A as some n-bit binary number. Therefore, we can have a 2^n possible combinations, hence the size of $|\wp A| = 2^n$

Products

Let us define some **ordered** pair as $\langle a, b \rangle$, such that generally $\langle a, b \rangle \neq \langle b, a \rangle$.

Let there be some arbitrary sets A, and B. We can then define the Cartesian product as follows; $A \times B \triangleq \{\langle a, b \rangle \mid a \in A \wedge b \in B\}$. Since we'll often deal with binary relations, we use the shorthand $A^2 = A \times A$. We can define equality on ordered pairs as $\forall a, b, c, d[\langle a, b \rangle =_{A \times B} \langle c, d \rangle \triangleq a =_A c \wedge b =_B d]$. Note that in general, $\times$ is not a commutative operation.

Suppose that there are two finite sets $A = \{a_1, a_2, ..., a_n\}$, and $B = \{b_1, b_2, ..., b_m\}$, with sizes n, and m respectively - then it follows that $|A \times B| = |A| \cdot |B|$. We can justify this by constructing such a matrix R, of dimension $(A \times B)^{n,m}$ - thus having $n \cdot m$ elements;

$$R = \begin{matrix} \langle a_1, b_1 \rangle & \langle a_1, b_2 \rangle & \cdots & \langle a_1, b_m \rangle \\ \langle a_2, b_1 \rangle & \langle a_2, b_2 \rangle & \cdots & \langle a_2, b_m \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle a_n, b_1 \rangle & \langle a_1, b_2 \rangle & \cdots & \langle a_n, b_m \rangle \end{matrix}$$

We can also have an n-ary product, to construct an n-tuple $\langle a_1, a_2, ..., a_n \rangle$, when $n \geq 1$. Let there be some arbitrary sets, $A_1, A_2, ..., A_n$.

This is written as $A_1 \times ... \times A_n = \prod_{i=1}^{n} A_i$, and is defined as $\{\langle a_1, a_2, ..., a_n \rangle \mid \forall i \in [1, n][a_i \in A_i]\}$.

Partitions

Given some set S, we can define a **partition** of S to be a family of subsets $\{A_1, A_2, ..., A_n\}$ such that;

- none of them are empty (therefore $\forall i \in [1, n][A_i \neq \varnothing]$)
- the subsets cover S (therefore $S = \bigcup_{i=1}^{n} A_i$)
- they are pairwise disjoint (therefore $\forall i, j \in [1, n][i \neq j \Rightarrow A_i \cap A_j = \varnothing]$)

A partition of S is a set of non-empty subsets that are pairwise disjoint, and cover S.

Pigeonhole Principle

Given a set S of size n, partitioned into k sets such that $0 < k < n$, then at least one of the subsets must have at least 2 elements. We can prove this by contradiction (one of the few times we actually do this, in DS). Assume that there are k subsets, each of size 1 (therefore $\forall i \in [1, k][|A_k| = 1]$). By definition of a partition, we can form a cover of S, therefore (the last 2 steps are justified by the requirement of a partition being pairwise disjoint);

$n = |S| = |\bigcup_{i=1}^{k} A_i| = \sum_{i=1}^{k} |A_i| = \sum_{i=1}^{k} 1 = k$

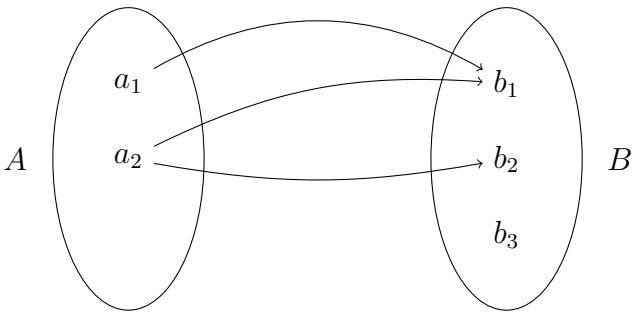
However, given the bounding condition $k < n$, there is no way that $k = n$, and the only assumption is that we made k sets of size 1.

Representing Relations

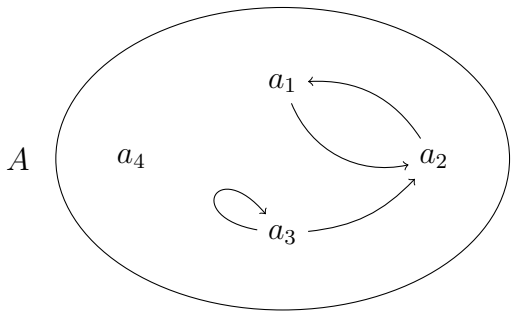
We define a relation between two sets A, and B (from A to B), as a subset of $A \times B$, such that $R \subseteq A \times B$. If we say that $R \subseteq A \times B$, it means that it has type $A \times B$. However, if $R \subseteq A^2$, it is a **binary** relation on A. Instead of writing $\langle a, b \rangle \in \mathbb{R}$, we will often shorten it to $a \; R \; b$.

A relation does not have to be meaningful; for a set of size $n = 2$, let it be $A = \{a, b\}$, it can have 16 (2^{n^2}) possible binary relations. For any set A, the possible binary relations can be generated by taking $\wp A^2$. A predicate over A is a 1-ary relation, which is just a subset of A. We also can say something along the lines of $\{\langle x, y, z \rangle \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$, as a ternary relation on the reals which covers the surface of a unit sphere at the origin.

Generally, writing out all pairs can become tedious, therefore there are numerous other ways of representing it. We can construct a diagram (a bipartite graph) for the following relation $A = \{a_1, a_1\}$, $B = \{b_1, b_2, b_3\}$, and $R = \{\langle a_1, b_1 \rangle, \langle a_2, b_1 \rangle, \langle a_2, b_2 \rangle\}$;



However, we might also want to represent a binary relation in a similar way, in which case we can draw a regular directed graph. Here we have $A = \{a_1, a_2, a_3, a_4\}$, and $R = \{\langle a_1, a_2\rangle, \langle a_2, a_1\rangle, \langle a_3, a_2\rangle, \langle a_3, a_3\rangle\}$;



It can also be represented as a matrix, such that we have

$$M_{i,j} = \begin{cases} \text{True} & \text{if } a_i \ R \ b_j \\ \text{False} & \text{otherwise} \end{cases}$$

Constructing Relations

Just like in sets, we can construct relations quite easily. Except, we now have a known set in they exist in (by the subset definition), hence (these examples use $R, S \subseteq A \times B$, and $T \subseteq B \times C$);

- $R \cup S \triangleq \{\langle a, b\rangle \in A \times B \mid \langle a, b\rangle \in A \vee \langle a, b\rangle \in B\}$ — relation union
- $R \cap S \triangleq \{\langle a, b\rangle \in A \times B \mid \langle a, b\rangle \in A \wedge \langle a, b\rangle \in B\}$ — relation intersection
- $\overline{R} \triangleq \{\langle a, b\rangle \in A \times B \mid \langle a, b\rangle \notin A\}$ — relation complement
- $R^{-1} \triangleq \{\langle b, a\rangle \in B \times A \mid a \ R \ b\}$ — inverse relation
- $\text{id}_A \triangleq \{\langle x, y\rangle \in A^2 \mid x =_A y\}$ — identity relation
- $R \circ T \triangleq \{\langle a, c\rangle \in A \times C \mid \exists b \in B[a \ R \ b \wedge b \ T \ c]\}$ — relation composition

  this is only defined when the types are matching

  we can define grandparentof $\triangleq$ parentof $\circ$ parentof

  therefore x gpo $y \triangleq \exists z(x \text{ po } z \wedge z \text{ po } y)$

23rd October 2018

To be honest, this lecture was basically just a tutorial. Some solutions are listed here;

Associativity of $\circ$

For arbitrary relations, $R \subseteq A \times B$, $S \subseteq B \times C$, and $T \subseteq C \times D$, show that $R \circ (S \circ T) = (R \circ S) \circ T$

Take some arbitrary $\langle x, y\rangle \in R \circ (S \circ T)$;

$$\begin{aligned} x \ R \circ (S \circ T) \ y &\triangleq \exists z[x \ R \ z \wedge z \ (S \circ T) \ y] \\ &\triangleq \exists z[x \ R \ z \wedge \exists w[z \ S \ w \wedge w \ T \ y]] \\ &\Leftrightarrow \exists w, z[x \ R \ z \wedge z \ S \ w \wedge w \ T \ y] \\ &\Leftrightarrow \exists w[x \ (R \circ S) \ w \wedge w \ T \ y] \\ &\triangleq x \ (R \circ S) \circ T \ y \end{aligned}$$

The key point to take from this proof is how we can use our knowledge of propositional logic, and apply it to sets. Since propositional logic is far easier to prove than an arbitrary set, we can reduce the work we do significantly.

Subsets of Inverse Relations

Given two binary relations $R, S \subseteq A^2$, prove that $R \subseteq S \Rightarrow R^{-1} \subseteq S^{-1}$

Take some arbitrary $\langle y, x\rangle \in R^{-1}$. In order to show the RHS, we want to show that this is also in S^{-1}. Let us also make the assumption (the LHS) that $R \subseteq S$, such that $k \in R \Rightarrow k \in S$, where k is any tuple. As we have some $\langle y, x\rangle \in R^{-1}$, it follows that there is a corresponding $\langle x, y\rangle \in R$. Because of our assumption, we can say that $\langle x, y\rangle \in S$, and therefore $\langle y, x\rangle \in S^{-1}$. Therefore, any arbitrary element of R^{-1} is also in S^{-1}, hence $R^{-1} \subseteq S^{-1}$ (given our assumption holds) - so $R \subseteq S \Rightarrow R^{-1} \subseteq S^{-1}$.

26th October 2018

The first part is just some stuff about how you should be doing proofs in natural language, as mathematics (and symbols) is just formalised human thinking. This then goes into (basically) natural deduction - so check **CO140** for techniques you can apply in proofs. Once again, we went through more questions in this lecture.

Relation Properties

Let there be $R \subseteq A^2$, such that R is a binary relation on A;

- R is reflexive $\triangleq \forall x \in A[\langle x, x\rangle \in R]$
  $\Leftrightarrow \mathrm{id}_A \subseteq R$
- R is symmetric $\triangleq \forall x, y \in A[\langle x, y\rangle \in A \Rightarrow \langle y, x\rangle \in R]$
  $\Leftrightarrow R = R^{-1}$
- R is transitive $\triangleq \forall x, z \in A[\exists y \in A[\langle x, y\rangle \in R \wedge \langle y, z\rangle \in R] \Rightarrow \langle x, z\rangle \in R]$
  $\Leftrightarrow R \circ R \subseteq R$
- R is an equivalence relation if it is reflexive, symmetric, and transitive

We consider something to be an equivalence if it has a weak equality, such that $a\ R\ b$ means that a is indistinguishable from b in some sense. We can write this as $a \sim_R b$.

30th October 2018

Equivalence Classes

Given $n \neq 0$, and $n \in \mathbb{N}$, the binary relation R_n on $\mathbb{Z}$ is defined by $a\ R_n\ b$ when n divides into $(b - a)$ is defined as; $R_n \triangleq \{\langle a, b\rangle \in \mathbb{Z}^2 \mid \exists q \in \mathbb{Z}[q \cdot n = (b - a)]\}$. This means that two numbers are in the same equivalence class given that they are an integer multiple of n apart. As such, they have the same result under modulo n.

Suppose we have some R, which is an equivalence relation on A. For any $a \in A$, we can define the equivalence class of a with respect to R as follows; $[a]_R \triangleq \{b \in A \mid a \sim_R b\}$. For brevity, we can omit the $_R$ when it's clear what equivalence relation we're referring to from the context. The set of equivalence classes is referred to as the **quotient set**; $\frac{A}{R}$; therefore with the example above, the set $\frac{\mathbb{Z}}{R_n}$ is the quotient set which represents integers which have modulo n.

Let us propose that the set of all equivalence classes, $\{[a] \mid a \in A\}$, forms a partition of A. This means that the equivalence classes aren't empty, they form a cover of A, and that they are pairwise disjoint.

We need to first show that no equivalence class is empty. First, let's take some arbitrary $x \in A$. By the reflexive nature of equivalences, we know that $x \sim_R x$, hence $x \in [x]$. As we took an arbitrary element of A, it's satisfied for all A, therefore none of the equivalence classes are empty.

Next we need to prove that it forms a cover of A, such that $A = \bigcup_{a \in A} [a]$; done by proving that $A \subseteq \bigcup_{a \in A} [a]$, and also $\bigcup_{a \in A} [a] \subseteq A$.

Doing the former, let us take some arbitrary $x \in A$. Now, it follows that it's in its own equivalence class $[x]$, under the same justification we gave for the first part of the proof ($x \sim_R x$ by reflexivity). Trivially, we can say that $[x] \subseteq \bigcup_{a \in A} [a]$. Hence $x \in \bigcup_{a \in A} [a]$, and as we took arbitrary x; $A \subseteq \bigcup_{a \in A} [a]$.

To prove the other direction, take some arbitrary equivalence class $[x] \in \bigcup_{a \in A} [a]$, and arbitrary $y \in [x]$. This then means we've taken arbitrary $y \in \bigcup_{a \in A} [a]$.

By our definition of an equivalence class, for $y \in [x]$, it must therefore mean $x \sim_R y$, and also that $y \in A$. Hence we get $\bigcup_{a \in A} [a] \subseteq A$. As we have both directions of $\subseteq$, we conclude the two sets are equal.

The last one can be done by proving two equivalence classes are equal, if they aren't pairwise disjoint. Suppose two arbitrary classes in the set of equivalence classes aren't pairwise disjoint, such that $[x] \cap [y] \neq \varnothing$. Therefore, this means that $w \in ([x] \cap [y])$, by definition of set union, we can then say that $w \in [x]$, and also $w \in [y]$. This then leads to $x \sim_R w$, and also $y \sim_R w$, by definition. However, by symmetry, we can rewrite the former as $w \sim_R x$. To establish equality, we need to show that they are subsets of each other (will only do one, since it's trivial to do the other way around). Take some arbitrary $v \in [x]$, then it follows that $x \sim_R v$. By transitivity, we can now say $w \sim_R v$, and therefore also $y \sim_R v$. It then follows that $v \in [y]$. As we took arbitrary $v \in [x]$, it follows that $[x] \subseteq [y]$. Hence the only way two items aren't disjoint in a set of equivalence classes, is when they are equal. Thus, the family of equivalence classes is pairwise disjoint, and is a partition of A.

Transitive Closure

Strange that I'm revising this **after** doing Warshall's algorithm in **CO150**. It's probably a better idea to learn the pre-requisites for a module, before doing it.

Suppose we have a binary relation R on A. We define the transitive closure of R^+, such that it's the smallest transitive relation that contains R. Defining R^k is required;

$$
\begin{aligned}
&R^1 \triangleq R \\
&R^2 \triangleq R \circ R \\
&R^3 \triangleq R \circ R \circ R = R \circ (R^2) \\
&\quad \cdots \\
&R^k \triangleq R \circ R \circ \cdots \circ R \quad (k \text{ times})
\end{aligned}
$$

We can then define $R^+ \triangleq \bigcup_{i \geq 1} R^i$, and thus we get $a \ R^+ \ b \Leftrightarrow \exists i \geq 1 [a \ R^i \ b]$.

Let R be some finite binary relation on A. If R is already transitive, then there is no more work we need to do. However, in the case that it's not transitive, it must mean that there is some $a, b, c \in A$ such that we have $a \ R \ b$, and also $b \ R \ c$, but not $a \ R \ c$. We then add the pair $\langle a, c \rangle$ to the relation. This is then repeated until we reach a point where the relation is transitive, and we are done.

Since every step was a requirement of transitivity, we have obtained the smallest possible relation containing R. In our proofs, we are allowed to create an infinite construction, but **not** an infinite step proof.

Consider some relation $<_1 \triangleq \{\langle m, n \rangle \in \mathbb{N}^2 \mid m = n + 1\}$ over $\mathbb{N}$. This is relation represents a difference of 1. Now, we can construct $<_2 = <_1 \circ <_1$ (fairly trivial to prove, since we know it covers the naturals, we can almost approach it inductively, however there is an easier solution where we use the properties of natural numbers (hint - if $n \in \mathbb{N}$, then $n + 1 \in \mathbb{N}$)). The transitive closure on $<_1$, $<_1^+$, is therefore $<$.

Note that I won't include most of the exercises, because it's surprisingly laborious to typeset them on LaTeX, especially when I'm sleep deprived. The ones included are ones that have specific techniques that we should remember for the exam.

2nd November 2018

Peano Arithmetic

Peano defined a set of **axioms** for the naturals, such that for the set $\mathbb{N}$, it must satisfy the properties;

1. $0 \in \mathbb{N}$
2. if $n \in \mathbb{N}$, then $\text{Succ}(n) \in \mathbb{N}$
3. for all $n \in \mathbb{N}$, $\text{Succ}(n) \neq 0$
4. for all $n, m \in \mathbb{N}$, if $\text{Succ}(n) = \text{Succ}(m)$, then $n = m$
5. suppose there is a set V, such that $0 \in V$, and for all $n \in \mathbb{N}$, if $n \in V$ then $\text{Succ}(n) \in V$, then $\mathbb{N} \subseteq V$

Note that the 5^{th} point is the principle of induction. Let us also define arithmetic as follows, on the successor function (will use $S(n)$ to represent $\text{Succ}(n)$, $A(m, n)$ to represent $\text{Add}(m, n)$, and $M(m, n)$ to represent $\text{Mult}(m, n)$;

- $A(0, n) = n$
- $A(S(m), 0) = S(A(m))$
- $M(0, n) = 0$
- $M(S(m), n) = A(M(m, n), n)$

Let us prove that $\forall n \in \mathbb{N}[A(n, S(0)) = S(n)]$. By the principle of induction, let us define some set $V \triangleq \{n \mid A(n, S(0)) = S(n)\}$. First, we have to show $0 \in V$, which means showing that $A(0, S(0)) = S(0)$. By the first case for addition, we can show the predicate holds trivially, hence $0 \in V$.

Now, let us make the assumption that some arbitrary $k \in \mathbb{N}$, $n \in V$, such that we are given $A(k, S(0)) = S(k)$. Our goal here is to prove that $S(k) \in V$, by showing $A(S(k), S(0)) = S(S(k))$. By the second case of addition, we can say that $A(S(k), S(0)) = S(A(k, S(0)))$. However, by our assumption, we can substitute the value for $S(k)$, therefore $S(S(k))$. Therefore, by our assumption, we can say $k \in V \Rightarrow S(k) \in V$. By the principle of induction, it follows that $\mathbb{N} \subseteq V$, hence $\forall n \in \mathbb{N}[A(n, S(0)) = S(n)]$.

While the idea of an infinite proof seems acceptable to us, intuitively, we cannot do an infinite proof. We are allowed to do infinite constructions, but not a proof with infinite steps, hence we must use induction.

Defining Natural Numbers with Sets

Assuming that our notion of sets is real, we can recursively define natural numbers as follows;

$$
\begin{aligned}
0 &\triangleq \varnothing \\
n &\triangleq n-1 \cup \{n-1\} \\
&= n-2 \cup \{n-2\} \cup \{n-1\} \\
&= \cdots \\
&= \{1, 2, 3, ..., n-2, n-1\} \\
0 &= \varnothing \\
1 &= \varnothing \cup \{\varnothing\} \\
&= \{\varnothing\} \\
2 &= \{\varnothing\} \cup \{\{\varnothing\}\} \\
&= \{\varnothing, \{\varnothing\}\}
\end{aligned}
$$

Defining Integers, and Rational Numbers

We can define the integers, $\mathbb{Z}$, as $\mathbb{Z} \triangleq \mathbb{N} \cup \{-n \mid n \in \mathbb{N}\}$. We can also define equality on the integers, $=_{\mathbb{Z}}$, as $=_{\mathbb{Z}} \triangleq \langle -0, 0\rangle \cup \{\langle n, m\rangle \mid n =_{\mathbb{N}} m\} \cup \{\langle -n, -m\rangle \mid n =_{\mathbb{N}} m\}$, which covers all the cases (note the specific inclusion of ± 0). By these definitions, we can see that $\mathbb{N} \subseteq \mathbb{Z}$.

With this, we're also able to define the rational numbers, $\mathbb{Q}$, as $\mathbb{Q} \triangleq \mathbb{Z} \times \mathbb{N}$, and we can define equality over the natural numbers, $=_{\mathbb{Q}}$, as $=_{\mathbb{Q}} \triangleq \{\langle\langle n_1, m_1\rangle, \langle n_2, m_2\rangle\rangle \in (\mathbb{Z} \times \mathbb{N})^2 \mid n_1 \cdot m_2 =_{\mathbb{Z}} n_2 \cdot m_1\}$. Note that the slides use $=_{\mathbb{N}}$, which is incorrect, as we have no guarantee that the product of a natural, and an integer is a natural, but we can guarantee it is an integer by the closure of multiplication on integers.

Defining $=_{\mathbb{R}}$ takes much more work. It's also established that every sequence of rational ($\mathbb{Q}$) numbers, e.g. $\{3, 3.1, 3.14, 3.141, 3.145, ...\}$ has an upper limit in $\mathbb{R}$.

6th November 2018

Functions

We define a function f, from A (the function domain to B (the function co-domain) as $f : A \to B$. **Every** element of A must map to a **unique** (exactly 1) element in B. We can formally write this as follows, where both conditions must hold;

1. $\forall a \in A \forall b_1, b_2 \in B[\langle a, b_1\rangle \in f \wedge \langle a, b_2\rangle \in f \Rightarrow b_1 = b_2]$
2. $\forall a \in A \exists b \in B[\langle a, b\rangle \in f]$

The set of all functions from A, to B is denoted as B^A, such that $B^A \subseteq \wp(A \times B)$. For brevity, we can use the following shorthands;

- $f : A \to B$ is short for $f \subseteq B^A$
- $\exists f : A \to B[\cdots]$ is short for $\exists f \in B^A[...]$
- $\forall f : A \to B[\cdots]$ is short for $\forall f \in B^A[...]$
- suppose that A is an n-ary product $A_1 \times ... \times A_n$, we can write $f(a_1, ..., a_n)$ instead of $f(\langle a_1, ..., a_n\rangle)$

We define equality on two functions $f, g : A \to B$, as $f =_{A \times B} g \triangleq \forall x \in A[f(x) =_B g(x)]$.

For any subset of A, $X \subseteq A$, the image of X under f is denoted $f[X] \triangleq \{f(x) \mid x \in X\}$. We can define the image set of A, as $f[A]$. For example, let there be sets $A = \{1, 2, 3\}$, $B = \{a, b, c\}$, and the function $f = \{\langle 1, a\rangle, \langle 2, b\rangle, \langle 3, a\rangle\}$. The image set of A, $f[A]$, is $\{a, b\}$, and the image set $f[\{1, 3\}] = \{a\}$.

These are some examples of functions;

- $f : \mathbb{N}^2 \to \mathbb{N}$ $\quad f(x, y) = x + y$
- $f : \mathbb{N} \to \mathbb{N}$ $\quad f(x) = x^2$
- $f : \mathbb{R} \to \mathbb{R}$ $\quad f(x) = x + 3$

However, the **relation** $R \triangleq \{\langle x, y\rangle \in \mathbb{R}^2 \mid x = y^2\}$ is not a function, as we can easily prove a one-to-many relationship, thus violating the first condition. For example; both $\langle 1, -1\rangle$, and $\langle 1, 1\rangle$ are in R, but clearly $-1 \neq_{\mathbb{R}} 1$. In order to verify a function is actually a function, we must prove that the LHS has 1 unique mapping.

Cardinality of Function Space

Let B^A represent the function space of all functions mapping from A to B, where both A, and B are finite sets, such that $|A| = m$, and $|B| = n$. For something to be a function, every item in A, must map to an item in B. For any item in A, there are n independent options to which it can map to. Hence it follows that there are n^m unique options; thus $|B^A| = n^m$.

Characteristic Function

Suppose we have a set A, and the characteristic function of $B \subseteq A$ is defined as $\chi_B : A \to \{0,1\}$, and for some n-ary relation R, such that $R \subseteq A_1 \times ... \times A_n$, we can define $\chi_R : A_1 \times ... \times A_n \to \{0,1\}$;

$$\chi_B(a) = \begin{cases} 1 & \text{if } a \in B \\ 0 & \text{if } a \in A \backslash B \end{cases}$$

$$\chi_R(a_1, ..., a_n) = \begin{cases} 1 & \text{if } \langle a_1, ..., a_n \rangle \in R \\ 0 & \text{if } \langle a_1, ..., a_n \rangle \notin R \end{cases}$$

Partial Functions

A partial function is a function **without** the second condition, therefore not all elements in A, must have a corresponding element in B. We denote an undefined value as $\perp$, therefore we can create a "function" from a partial function by saying $f : A \to (B \cup \{\perp\})$. If you are asked to give a function in an exam, **do not give a partial function**.

Properties of Functions

We will be working on some function $f : A \to B$.

- f is onto (surjective) when every element of B is in the image of A

  $\forall b \in B \exists a \in A[f(a) =_B b]$

  also $f[A] = B$ (?)
- f is one-to-one (injective) when every element of B has **at most one** $a \in A$ with $f(a) = b$

  $\forall a_1, a_2 \in A[f(a_1) =_B f(a_2) \Rightarrow a_1 =_A a_2]$
- f is bijective, if it is both surjective (onto), and injective (one-to-one)

The (Dual) Cantor-Bernstein Theorem states that if there exists $f : A \to B$, and $g : B \to A$, where they are both surjective, or both injective, then it follows that there is is a bijection $h : A \to B$. This theorem is extremely helpful when we want to prove that two infinite sets have the same cardinality.

These are some example functions, and their properties;

- $f : \mathbb{N}^2 \to \mathbb{N}$ $\qquad f(x, y) = x + y$

  we can prove it is surjective by taking some arbitrary $n \in \mathbb{N}$, and proving that $f(n, 0) = n + 0 = n$

  we can prove it is not injective by finding a counter example, such as $f(0,1) = 1 = f(1,0)$, but $\langle 0, 1 \rangle \neq \langle 1, 0 \rangle$
- $f : \mathbb{N} \to \mathbb{N}$ $\qquad f(x) = x^2$

  we can prove it is not surjective by finding a counter example; such as $f(x) = 3$, as there is no $x \in \mathbb{N}$ such that $x^2 = 3$
- $f : \mathbb{R} \to \mathbb{R}$ $\qquad f(x) = 4x + 3$

  this is bijective

Some Proof on Function Image?

No idea what this should actually be titled. But let there be a finite set A, a function $f : A \to B$, and $X \subseteq A$. We want to prove that $|f[X]| \leq |X|$.

By contradiction, let us assume that $|f[X]| > |X|$. Define a function p, such that $p : f[X] \to X$. Suppose we pick an arbitrary $b \in f[X]$, and some $a \in X$ (we know this exists by definition of image), such that we have $f(a) = b$, and define $p(b) = a$.

We take y, from $f[X]$, and place it into a pigeonhole labeled x, where $x \in X$, if $p(y) = x$. By the pigeonhole principle, there exists some $c \in X$, such that $d, d' \in f[X]$ (because the image set is larger). Hence, it follows that $p(d) = p(d') = c$. But, by definition of p, we have $f(c) = d$, and also $f(c) = d'$. Therefore, f isn't a function, hence we have a contradiction.

9th November 2018

An Improvement on Last Lecture's Proof

Using the same introduction as last lecture's proof, construct sets for all $x \in X$, $V_x \subseteq f[X]$. If for some $a \in X$, $f(a) = b$, then $b \in f[X]$ (by definition of the image), and $b \in V_a$ (by our definition). By making our assumption (which we want to be able to derive a contradiction from) $|f[X]| > |X|$. Therefore, $\exists c \in X[|V_c| > 1]$. So, we're able to say that there are distinct $d, d' \in V_c$. But by our definition of V_c, it follows that $f(c) = d$, and also $f(c) = d'$. This is not possible, as we defined f as a function, hence we have a contradiction. The main issue with last week's proof was assuming we had an inverse function, p.

Proposition on the Property of Functions

Given two **finite** sets (these do not always apply to infinite sets) A, and B, and a function $f : A \to B$, we can say that if...

- f is onto (surjective), then $|A| \geq |B|$

  note that if it is injective, then $f[A] = B$, therefore $|f[A]| = |B|$

  we can use what we just proved, that $|f[A]| \leq |A|$, so $|A| \geq |B|$
- f is one-to-one (injective), then $|A| \leq |B|$

  contraposition of the pigeonhole principle, or something
- f is bijective, then $|A| = |B|$

  this follows from the first two, trivially

Function Composition

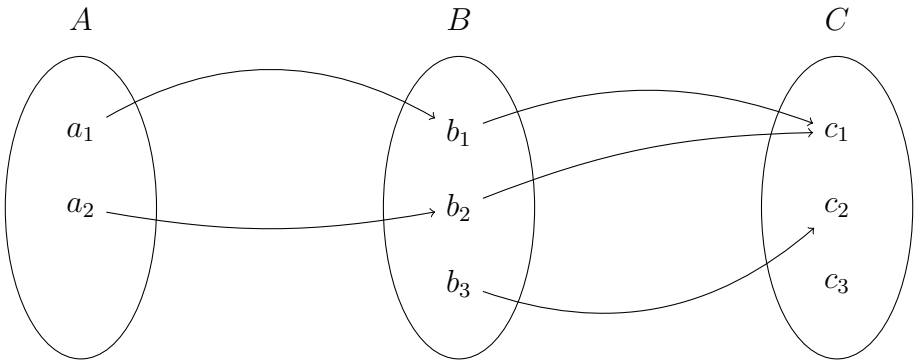
Suppose we have arbitrary sets, A, B, C, and let there be two functions $f : A \to B$, and $g : B \to C$. We can define the composition of f with g (meaning g applied to the image of f), as $\langle a, c\rangle \in (g \circ f) \triangleq \exists b \in B[\langle a, b\rangle \in f \wedge \langle b, c\rangle \in g]$ where $(g \circ f) : A \to C$. It's important to note that the order of the arguments is flipped, compared to relation composition. The crucial requirement is that the co-domain of f matches the domain of g.

We can prove that function composition is associative quite trivially. Recall that two functions i, j (let them both be functions from A to B) are equal if $\forall a \in A[i(a) =_B j(b)]$. We want to prove that $h \circ (g \circ f) = (h \circ g) \circ f$, and we are using the same definitions mentioned previously, plus a new set D, and $h : C \to D$. We first take an arbitrary $a \in A$;

$$\begin{aligned}(h \circ (g \circ f))(a) &= h((g \circ f)(a)) \\ &= h(g(f(a))) \\ &= (h \circ g)(f(a)) \\ &= ((h \circ g) \circ f))(a)\end{aligned}$$

As we have taken arbitrary $a \in A$, it follows that the two are equal.

When we're working with function compositions, especially when we're told to give a specific example, it's useful to draw out diagrams (wish I knew that during the Christmas exam). The example below has $f : A \to B$, such that $f = \{\langle a_1, b_1\rangle, \langle a_2, b_2\rangle\}$, and $g : B \to C$, such that $g = \{\langle b_1, c_1\rangle, \langle b_2, c_1\rangle, \langle b_3 c_2\rangle\}$. Thus the composed function $(g \circ f) : A \to B$, is $g \circ f = \{\langle a_1, c_1\rangle, \langle a_2, c_2\rangle\}$.



Proofs on Properties of Composed Functions

We want to show that if $g \circ f$ is an injective function, then it follows that f is an injective function.

Take some arbitrary $a_1, a_2 \in A$. Assume that $f(a_1) = f(a_2)$ (otherwise it doesn't help, since falsity implies anything). Now, by the definition of a function, we can say that $g(f(a_1)) = g(f(a_2))$, which means that $(g \circ f)(a_1) = (g \circ f)(a_2)$. However, we're assuming that $g \circ f$ is injective, hence it follows that $a_1 = a_2$. By assuming that $f(a_1) = f(a_2)$, we get $a_1 = a_2$, hence we have proven f is injective.

We want to show that if $g \circ f$ is a surjective function, then it follows that g is a surjective function.

Our goal is to show that $\forall c \in C \exists b \in B[g(b) = a]$. Knowing that $\forall c \in C \exists a \in A[(g \circ f)(a) = c]$, it implies that there exists some $b \in B$, such that $f(a) = b$ (by definition of function composition). Hence, g is surjective.

We want to show that if we have bijections $f : A \to B$, and $g : B \to C$, then $g \circ f$ is a bijection. It is sufficient to show that if f, g are both surjective, then so is $g \circ f$, and if f, g are both injective, then so is $g \circ f$.

The first part can be proven by assuming that they are both surjective. Therefore, it means that $\forall c \in C \exists b \in B[g(b) = c]$, and also $\forall b \in B \exists a \in A[f(a) = b]$. So, suppose that there is an arbitrary $c \in C$, therefore there exists some $b \in B$, such that $g(c) = b$. Now, with that b, we can say that there exists such an $a \in A$ where $f(a) = b$. Therefore, it follows that $\forall c \in C \exists a \in A[(g \circ f)(a) = c]$. Hence, the composed function is surjective.

The second part can be proven similarly, assuming that both functions are injective. Let us take some arbitrary $a_1, a_2 \in A$. We assume that $(g \circ f)(a_1) = (g \circ f)(a_2)$, hence $g(f(a_1)) = g(f(a_2))$. But we know g is injective, therefore $f(a_1) = f(a_2)$, but since f is also injective, it follows that $a_1 = a_2$. Hence, by assuming that $(g \circ f)(a_1) = (g \circ f)(a_2)$, we end up with $a_1 = a_2$, it follows that $g \circ f$ is injective. By proving it's also a surjection, it is therefore a bijective function.

Identity, and Inverse

Suppose we have a set A, the identity function on A, denoted $\mathrm{id}_A : A \to A$, is defined by $\forall a \in A[\mathrm{id}_A(a) = a]$. Let there be some arbitrary function $f : A \to B$, then the inverse of the function, $g : B \to A$ (normally written f^{-1}) has to fulfil the following criteria. $\forall a \in A[g(f(a)) = a] \wedge \forall b \in B[f(g(b)) = b]$. We can also write this, more succinctly, as $g \circ f = \mathrm{id}_A \wedge f \circ g = \mathrm{id}_B$.

Let there be a bijection $f : A \to B$, and its inverse $f^{-1} : B \to A$, defined as $f^{-1}(b) = a$, when $f(a) = b$. Take some arbitrary $b \in B$, we know that there must be a corresponding $a \in A$, since f is surjective. Therefore, we have shown that every item in the domain of f^{-1} maps to something (one of the criteria for a function). Knowing that f is also injective means that for arbitrary b, we have a single unique a that corresponds to it, therefore f^{-1} is a well defined function.

Suppose we have a function $f : A \to B$, with a well defined inverse g. We can prove that f is a bijection, and g is unique. We want to first prove that it is a surjection; take an arbitrary $b \in B$; now we know that $f(g(b)) = b$, by definition of inverse, hence it is onto. Now to prove that f is injective, take arbitrary $a_1, a_2 \in A$. Assume that $f(a_1) = f(a_2)$, then it follows that $g(f(a_1) = g(f(a_2))$. By

definition of inverse (with the identity), we can then say that $g(f(a_1)) = a_1 = a_2 = g(f(a_2))$, hence $a_1 = a_2$, thus f is injective, and since it is both injective, and surjective, it is a bijection.

Assume that there are two inverses of f; g, g'. Taking an arbitrary $b \in B$, we can say that $f(g(b)) = b$, and also $f(g'(b)) = b$, once again by using the identity definition. Hence, it follows that $g = g'$, as f is injective.

Cardinality of Sets

Let us define equivalence on sets (not equality), for **any** (they can be infinite) sets A, and B;
$A \approx B \triangleq \exists f : A \to B$ (where f is a bijection). Note that the previously mentioned (Dual) Cantor-Bernstein Theorem states that there is a bijection when there are two injective (or two surjective) functions $g : A \to B$, and $h : B \to A$.

We can also prove that it is an equivalence relation, such that it is reflexive, symmetric, and also transitive;

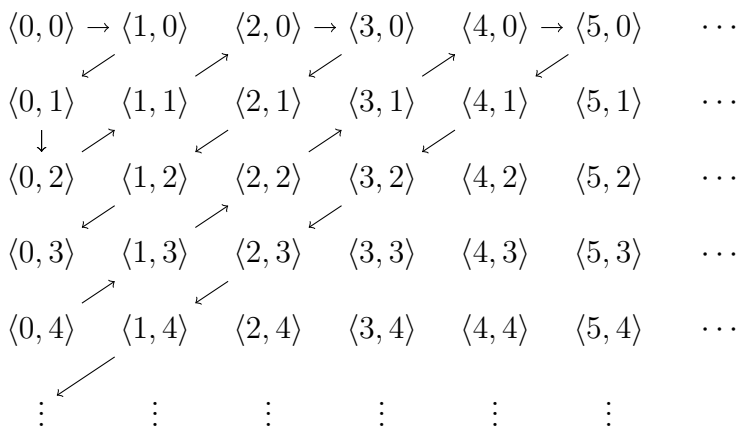
- $\approx$ is reflexive, hence $A \approx A$

  we have the identity relation $\mathrm{id}_A : A \to A$, which is a bijection
- $\approx$ is symmetric, hence if $A \approx B$, then $B \approx A$

  given that there is a bijection $f : A \to B$, there exists an inverse $f^{-1} : B \to A$, which is also a bijection (proven above)
- $\approx$ is transitive, hence if $A \approx B$, and $B \approx C$, then $A \approx C$

  given that there exists two bijections, $f : A \to B$, and $g : B \to C$, the composition $(g \circ f) : A \to C$ is also a bijection (proven above)

We can prove that $\mathbb{N} \approx \mathbb{N}^2$ by arranging the pairs in an infinite grid (remember that we're allowed infinite constructions, just not infinite step proofs). Each pair is visited once, and only once, therefore there exists a bijection.



Example of Bijection

At the start of the course, we considered the Cartesian product of sets, and considered whether it was associative. While they are not equal, we can build a bijection between them. For example, $f : (A \times B) \times C \to A \times (B \times C)$. In order to precisely define this, without using Haskell-style pattern matching, we can define $L(x, y) = x$, and $R(x, y) = y$. Therefore, to get $f(\langle a, b\rangle, c) = (a, \langle b, c\rangle)$, we define $f(x, y) = (L(x), \langle R(x), y\rangle)$, and similar for the inverse.

13th November 2018

Cardinality of Arbitrary Sets

We previously defined the cardinality of a finite set as the number of elements in it. Consider arbitrary finite sets A, and B, such that $|A| = |B| = n$. It is then trivial for us to construct a bijection $c_A : \{1, 2, ..., n\} \to A$, and similarly $c_B : \{1, 2, ..., n\} \to B$. Because the former is a bijection, we have a corresponding inverse function $c_A^{-1} : A \to \{1, 2, ..., n\}$. Now, we can therefore form a third bijection $(c_A^{-1} \circ c_B)A \to B$, hence $A \approx B$.

Cantor's Theorem

This proof is important in mathematics as it shows that there are different types of infinities. We want to prove that for any set A, $A \not\approx \wp A$.

To do this, we will do a proof by contradiction. Assume that there exists some bijection (therefore we also get the fact that it's surjective without doing any work) $f : A \to \wp A$. Define a set $B = \{x \in A \mid x \notin f(x)\}$. Note that this has a clear similarity to Russel's paradox. Now, it's clear that $B \subseteq A$, since we're constructing from a well defined set. Therefore, it follows that $B \in \wp A$, by the definition of $\wp$. However, from our assumption, we're given that it's a surjection, hence there exists some $b \in A$, such that $f(b) = B$.

We then get $b \in f(b)$ (same as $b \in B$), or $b \notin f(b)$. Consider the first case, when $b \in f(b) = B$, therefore, it means it satisfies the criteria that $b \notin f(b)$, thus $\lightning$. Consider the latter case, where $b \notin f(b)$, by the definition of B, $b \in B$, therefore $b \in f(b)$, hence $\lightning$. Because we get a contradiction on both cases, the assumption leads to falsity, hence f is not surjective, therefore it definitely isn't bijective. As there doesn't exist a bijection between the two sets, they do not have the same cardinality.

An important technique to pick out is that we don't care how the bijection is defined, just that it exists (for our purposes), and we can therefore use the properties for our proof.

Countable Sets

We can say some set A countable if it is finite, or $A \approx \mathbb{N}$. The elements of a countable set can be listed as a **finite, or infinite** sequence of distinct terms $A = \{a_1, a_2, a_3, ...\}$. For example, we can write the integers as $\mathbb{Z} = \{0, -1, 1, -2, 2, -3, 3, ...\}$, where the "counting bijection" $g : \mathbb{N} \to \mathbb{Z}$ is defined as follows;

$$g(x) = \begin{cases} \frac{x}{2} & x \text{ is even} \\ -\frac{x+1}{2} & x \text{ is odd} \end{cases}$$

As we're dealing with sets, there's no reason to construct a bijection that preserves an order. We can quite easily show that $\mathbb{Q}^+$ (the positive rationals) is countable, via the same technique we used for $\mathbb{N}^2$. By applying the method we used for showing countability of $\mathbb{Z}$, we can do the same for $\mathbb{Q}^+$ to $\mathbb{Q}$, and also union $\{0\}$, to make it complete.

In order to prove that the set $\{0, 1\} \times \mathbb{N}$ is countable, we can either construct a diagram similar to the one for $\mathbb{N}^2$, or we can define a bijection function $g : \mathbb{N} \to \{0, 1\} \times \mathbb{N}$, defined as follows;

$$g(x) = \begin{cases} \langle 0, \frac{x}{2} \rangle & x \text{ is even} \\ \langle 1, \frac{x+1}{2} \rangle & x \text{ is odd} \end{cases}$$

16th November 2018

Something on the Union of Countable Sets

Prove that if X, and Y are disjoint, countable sets, then the union $V = X \cup Y$ is also countable. This can be split into four cases, as follows;

1. Both X, and Y are finite sets. This is the simplest case, as we can say that $|X| = n$, and $|Y| = m$, where $n, m \in \mathbb{N}$, hence $|X \cup Y| = n + m \in \mathbb{N}$, due to the closure of addition on the natural numbers. We can take the sum, as proven at the start of the module.
2. X is a finite set, and Y is infinite. As Y is countable, we can say $Y \approx \mathbb{N}$, hence there exists some bijection $f : \mathbb{N} \to Y$. Since X is finite, we can say $|X| = n$, and also $X = \{x_1, x_2, ..., x_n\}$. Let us now construct another bijection, $h : \mathbb{N} \to X \cup Y$, defined as follows;

$$h(i) = \begin{cases} x_i & \text{if } i \leq |X| \\ f(i - |X|) & \text{otherwise} \end{cases}$$

   Since there exists such a bijection, it proves that the union is also countable.
3. X is an infinite set, and Y is finite. Same proof as above, but with the two sets swapped.
4. Both X, and Y, are countably infinite sets. This proof relies on the fact that $\{0, 1\} \times \mathbb{N}$ is countable. Let us make some bijection $k : \mathbb{N} \to \{0, 1\} \times \mathbb{N}$ defined as;

$$k(n) = \begin{cases} \langle 0, \frac{n}{2} \rangle & n \text{ is even} \\ \langle 1, \frac{n-1}{2} \rangle & n \text{ is odd} \end{cases}$$

   Since they are both countable, it follows that $X \approx \mathbb{N}$, and $Y \approx \mathbb{N}$, hence there exists a pair of bijections $f : \mathbb{N} \to X$, and $g : \mathbb{N} \to Y$. Let us also define an additional $h : \{0, 1\} \times \mathbb{N} \to X \cup Y$, defined as;

$$h(b, n) = \begin{cases} f(n) & b = 0 \\ g(n) & b = 1 \end{cases}$$

   Due to function composition, we could also create a bijection from $\mathbb{N} \to X \cup Y$, but proving that the Cartesian product is countable should be sufficient.

Diagonalisation

To clarify on the notation used; I am using v_i^j to denote the i^{th} element of V^j.

We want to prove that $\wp\mathbb{N}$ is uncountable, as $\mathbb{N} \not\approx \wp\mathbb{N}$. Any subset $V \subseteq \mathbb{N}$ can be represented as a list of 0s, and 1s. For example, let the set V^j be represented as $v_0^j, v_1^j, v_2^j, ...$, based on the characteristic function χ_{V^j}, such that $v_i^j = 1$ if $i \in V^j$, otherwise $v_i^j = 0$ if $i \notin V^j$ - it's important to note that $i \in \mathbb{N}$. Suppose there is a set of sets $V^0, V^1, V^2, ...$, let us define a new set W, where the i^{th} element, w_i, is defined as $w_i = 1 - v_i^i$. As we're still bounded to 0s, and 1s - it follows that $W \subseteq \mathbb{N}$. From the definition of W, we get that $\forall i \in \mathbb{N}[i \in W \Leftrightarrow i \notin V^i]$ therefore this new set is different from every other set V, in at least one item (hence $\forall i \in \mathbb{N}[W \neq V^i]$). The diagram below visualises this concept;

$$\begin{array}{lcccccc}
V^0 & = & \textcircled{v_0^0} & v_1^0 & v_2^0 & v_3^0 & v_4^0 & \cdots \\
V^1 & = & v_0^1 & \textcircled{v_1^1} & v_2^1 & v_3^1 & v_4^1 & \cdots \\
V^2 & = & v_0^2 & v_1^2 & \textcircled{v_2^2} & v_3^2 & v_4^2 & \cdots \\
V^3 & = & v_0^3 & v_1^3 & v_2^3 & \textcircled{v_3^3} & v_4^3 & \cdots \\
V^4 & = & v_0^4 & v_1^4 & v_2^4 & v_3^4 & \textcircled{v_4^4} & \cdots \\
 & & \vdots & \vdots & \vdots & \vdots & \vdots & \\
W & = & 1 - v_0^0 & 1 - v_1^1 & 1 - v_2^2 & 1 - v_3^3 & 1 - v_4^4 &
\end{array}$$

Proving the Rationals are Insignificant in the Reals

Here, we want to show that $\mathbb{Q}$ is insignificant (hence a zero-set) in $\mathbb{R}$. Since we've proven that the rationals are countable, we can write $\mathbb{Q} = \{q_0, q_1, q_2, ...\}$. We can construct intervals in $\mathbb{R}$, $V_\delta^i \subseteq \mathbb{R}$, as $V_\delta^i \triangleq (q_i - \delta \cdot 2^{-i}, q_i + \delta \cdot 2^{-i})$. Therefore, we can say that each interval therefore has a size of $\delta \cdot 2^{1-i}$, which can be written as $||V_\delta^i||$, for brevity.

We can also say that $V_\delta \triangleq \bigcup_{i=0}^{\infty} V_\delta^i$, hence $\mathbb{Q} \subseteq V_\delta$. As well as $0 < ||V_\delta|| \leq \sum_{i=0}^{\infty} ||V_\delta^i|| = 2\delta \sum_{i=0}^{\infty} 2^{-i} = 4\delta$.

We've therefore defined that the size of V_δ is bounded between 0, and 4δ. Since this is true for any δ, if we take $V = \lim_{\delta \to 0} V_\delta$, we approach 0, we show that V is negligible. Since $\mathbb{Q} \subseteq V$, $\mathbb{Q}$ is a null set in $\mathbb{R}$.

Orderings

Define R as a binary relation on A, we can then say the following properties;

- R is a pre-order — R is reflexive, and transitive
- R is anti-symmetric — $\forall a, b \in A[a \ R \ b \wedge b \ R \ a \Rightarrow a =_A b]$
- R is a partial order (often written as $\leq$) — R is reflexive, transitive, and anti-symmetric
  $(A, \leq)$, or $\leq_A$ means a partial order $\leq$ on A
- R is irreflexive — $\forall a \in A[\neg(a \ R \ a)]$
- R is a strict partial order (often written as $<$) — R is irreflexive, and transitive
- R is a total order — a partial order that satisfies $\forall a, b \in A[a \ R \ b \vee b \ R \ a]$

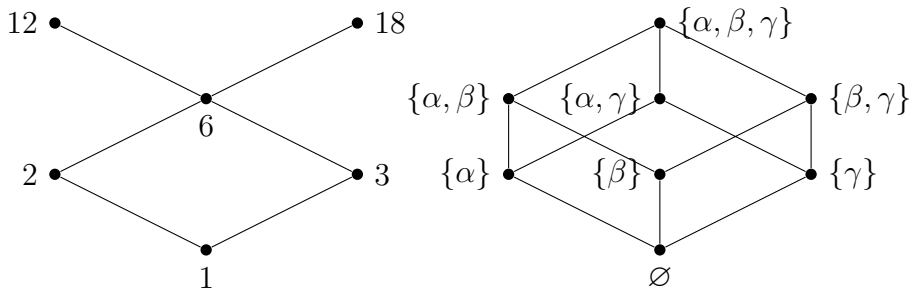
Given two partial orders $(A, \leq_A)$, and also $(B, \leq_B)$, the important orders on the product set $P = A \times B$ are;

- product order — $\langle a_1, b_1 \rangle \leq_P \langle a_2, b_2 \rangle \triangleq a_1 \leq_A a_2 \wedge b_1 \leq_B b_2$
- lexicographical order — $\langle a_1, b_1 \rangle \leq_P \langle a_2, b_2 \rangle \triangleq a_1 \leq_A a_2 \vee (a_1 =_A a_2 \wedge b_1 \leq_B b_2)$

If both $(A, \leq_A)$, and $(B, \leq_B)$ are total orders, the lexicographical order is also a total order, but in general the product order is partial.

Hasse Diagrams

By convention, Hasse diagrams only record the immediate predecessors (as the rest can be inferred). The direction of the lines are often omitted, and conventionally are read by looking up the page. For example, the Hasse diagrams for the binary relation 'divides' on the set $\{1, 2, 3, 6, 12, 18\}$, and the binary relation $\subseteq$ on $\wp\{\alpha, \beta, \gamma\}$ are as follows;



Properties of Partial Orders

Suppose there is a partial order $(A, \leq)$, and $a \in A$;

- a is minimal $\quad \forall b \in A[b \leq_A a \Rightarrow b =_A a]$
- a is maximal $\quad \forall b \in A[a \leq_A b \Rightarrow b =_A a]$
- a is least $\quad \forall b \in A[a \leq_A b]$
- a is greatest $\quad \forall b \in A[b \leq_A a]$

From this, we know that any least element is a minimal element, and any greatest element is also a maximal element. Note that the minimal element isn't necessarily unique; for example, given the set $W = \{V \subseteq \mathbb{N} \mid |V| \geq 1\}$, then every item which is just a set composed of a single natural number is minimal in $(W, \subseteq)$, and there exists no least element.

We can prove that a least element is unique. For example, let us take two 'least' elements, $a_1, a_2 \in A$. It then follows for all $b \in A$ that $a_1 \leq_A b$, and also $a_2 \leq_A b$. However, since we have this for all b, it follows that $a_1 \leq_A a_2$, and also $a_2 \leq_A a_1$, therefore $a_1 =_A a_2$. Note that in our first example, with the divisors, there isn't always a greatest element, but there are two maximal elements in 12, and 18.
By using logical equivalences, we can derive the condition for a to be minimal in a **strict partial order** $(A, <)$, note that equality is no longer considered, since we have irreflexivity);

$$\begin{aligned}
\forall b \in A[b \leq_A a \Rightarrow b =_A a] &\equiv \forall b \in A[\neg(b \leq_A a) \vee b =_A a] \\
&\equiv \forall b \in A[\neg(b \leq_A a \wedge b \neq_A a)] \\
&\equiv \neg\exists b \in A[b \leq_A a \wedge b \neq_A a] \\
&\equiv \neg\exists b \in A[b <_A a]
\end{aligned}$$

20th November 2018

Questions on the Divisor Order

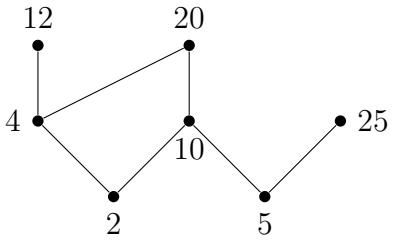
The following questions concern a divisor order $\leq_D$ defined on the positive natural numbers $(\mathbb{N}^+)$, defined such that $n \leq_D m \triangleq n$ divides m (or more formally as $\exists p \in \mathbb{Z}[m = p \cdot n]$).

The first question is to prove that this is a partial order, which means that it's transitive, anti-symmetric, and also reflexive. In order to prove that it's transitive, assume there are arbitrary $a, b, c \in \mathbb{N}^+$, such that $a \leq_D b$, and also $b \leq_D c$. The former implies that there is some integer, let it be p, such that $b = p \cdot a$, and the latter implies that there is some other integer q such that $c = q \cdot b$. By substitution, we can then say $c = q \cdot p \cdot a$. By closure of multiplication on the integers, we can say that $q \cdot p \in \mathbb{Z}$, therefore $a \leq_D c$, hence it is transitive (as we took arbitrary values).

In order to prove that it's anti-symmetric, suppose we have arbitrary $n, m \in \mathbb{N}^+$, such that $n \leq_D m$, and also $m \leq_D n$. The former suggests that there's an integer p, where $m = p \cdot n$, and the latter suggests similar (let the integer be q); $n = q \cdot m$. Once again, by substitution, we can say $m = p \cdot q \cdot m$. As we know $m \neq 0$ by the positive restriction, we can divide through, hence we have $p \cdot q = 1$. As they are both integers, we can conclude that $p = q = 1$, therefore $m = n$. Hence $\leq_D$ is also anti-symmetric.

Finally, we want to prove that it's reflexive. Take an arbitrary $n \in \mathbb{N}^+$. Trivially, we can say $n = 1 \cdot n$, and as $1 \in \mathbb{Z}$, we have $n \leq_D n$, hence $\leq_D$ is also reflexive. However, as we've proven the other properties, it follows that it is a partial order.

The next part requires us to draw the Hasse diagram of $(S, \leq_D)$, where $S = \{2, 4, 5, 10, 12, 20, 25\}$. The minimal elements here are $\{2, 5\}$, and the maximal elements are $\{12, 20, 25\}$.



The above is not a total order, as we don't have any relations (including implied ones by transitivity) between 20, and 25, nor 5, and 12, and so on. We can find a total order $\leq_T$, on S, by simply taking $\leq_{\mathbb{N}^+}$, which is the standard less-than-or-equal-to on the positive naturals.

Well-founded Partial Orders

A well-founded partial order is defined as a partial order that has no infinite **decreasing** chain of elements. For every infinite sequence, written as $a_1, a_2, a_3, ...$, such that $a_1 \geq a_2 \geq ...$, there exists some point $m \in \mathbb{N}$, such that $a_n = a_m$ for any $n \geq m$.

If we have two well-founded partial orders $(A, \leq_A)$, and $(B, \leq_B)$, then the lexicographical order of $L = A \times B$ is also well-founded. By contradiction, let us consider a chain $\langle a_1, b_1\rangle \geq_L \langle a_2, b_2\rangle \geq_L \langle a_3, b_3\rangle \geq_L ....$ Then, by lexicographical ordering, it follows that $a_1 \geq_A a_2 \geq_A a_3 \geq_A ...$, until some point, where it's just repeating a_m (since we're told it's well-founded). After this point, since $a_n = a_m$ for all $n \geq m$, we start checking b values. Now, we have $b_m \geq_B b_{m+1} \geq_B b_{m+2} \geq_B b_{m+3} \geq_B ...$,to another limit, let it be r, where it continues to repeat values. As we now have repeating $\langle a_r, b_r\rangle$, it follows that the lexicographical ordering must also be well-founded.

Ackermann Function

We can prove that the Ackermann function (will be referred to as A for brevity) terminates. This relies on the previous proof that the lexicographical ordering of the product set is also well-founded if the 'factors' are. Note that the Ackermann function is defined as $A : \mathbb{N}^{\rightarrow}\mathbb{N}$, and concerns the following cases;

$$\begin{aligned} A(0, y) &= y + 1 \\ A(x + 1, 0) &= A(x, 1) \\ A(x + 1, y + 1) &= A(x, A(x + 1, y)) \end{aligned}$$

We define the lexicographical ordering of natural number pairs as $\langle x, y\rangle < \langle x', y'\rangle \triangleq x < x' \vee (x = x' \wedge y < y')$. Applying this to the inputs of the functions, we obtain the following results;

$$\begin{aligned} \langle x + 1, 0\rangle &> \langle x, 1\rangle \\ \langle x + 1, y + 1\rangle &> \langle x, A(x + 1, y)\rangle \\ \langle x + 1, y + 1\rangle &> \langle x + 1, y\rangle \end{aligned}$$

Therefore, the function must terminate.

Induction

Refer to **CO141 - Reasoning About Programs** for a detailed introduction to induction, including mathematical, structural, as well as strong induction.