CO142 - Discrete Structures

Prelude

The content discussed here is part of CO142 - Discrete Structures (Computing MEng); taught by Steffen van Bakel, in Imperial College London during the academic year 2018/19. The notes are written for my personal use, and have no guarantee of being correct (although I hope it is, for my own sake). This should be used in conjunction with the (extremely detailed) notes.

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Recommended Books

- K.H. Rosen. *Discrete Mathematics and its Applications*
- J.L. Gersting. *Mathematical Structures for Computer Science*
- J.K. Truss. *Discrete Mathematics for Computer Science*
- R. Johsonbaugh. *Discrete Mathematics*
- C. Schumacher. *Fundamental Notions of Abstract Mathematics*

However, these books don't cover the same content. Learn his notation.

Logical Formula, and Notation

This notation will be shared with **CO140**.

- $A \wedge B$ — A and B both hold
- $A \vee B$ — A or B holds (or both)
- $\neg A$ — A does not hold
- $A \Rightarrow B$ — if A holds, then so does B
- $A \Leftrightarrow B$ — A holds if and only if B holds
- $\forall x(A)$ — the predicate A holds for all x
- $\exists x(A)$ — the predicate A holds for some x
- $a \in A$ — the object a is in the set A (a is an element of
- A)
- $a \notin A$ — the object a is not in the set A
- $=_A$ — tests whether two elements of A are the same

Sets

Sets are like data types in Haskell: Haskell data type declaration;

- `data Bool = False | True`
- {`false, true`} — set of boolean values
- `[true, false, true, false]` — list of boolean values
- {`false, true`} `=` {`true, false`} — set equality (note that order doesn't matter)

A set is a collection of objects from a pool of objects. Each object is an *element*, or a *member* of the set. A set *contains* its elements. Sets can be defined in the following ways;

- $\{a_1, ..., a_2\}$ — as a collection of n distinct elements
- $\{x \in A \mid P(x)\}$ — for all the elements in A, where P holds
- $\{x \mid P(x)\}$ — for all elements, where P holds (dangerous - Russel's paradox)

Use of "triangleq"

The use of $\triangleq$ is for "is defined by". Hence the empty set, $\varnothing \triangleq \{\}$. The difference between $\triangleq$ and $=$, is that the former cannot be proven, it is fact, whereas the latter takes work to prove.

Russel's paradox

Not everything we write as $\{x \mid P(x)\}$ is automatically a set. Assume $R = \{X \mid X \notin X\}$ is a set, the set of all sets which don't contain themselves. As R is a set, then $R \in R$, or $R \notin R$ (law of excluded middle), and thus we can do a case by case analysis.

- Assume $R \in R$. By the definition of R, it then follows that $R \notin R$ (if $R \in R$, then it doesn't satisfy the definition of R) - which is a contradiction.
- Assume $R \notin R$. It then follows that $R \in R$, as it follows the definition of R, hence it is another contradiction.

As both assumptions lead to contradictions, it's possible to write sets which aren't defined. We should only select from a set that we know is defined; $\{x \in A \mid P(x)\}$ - where A is a well-defined set.

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Set Comparisons

We can define a set A, as being a subset of another set B if every element in A is an element in B. This can be formally written as; $A \subseteq B \triangleq \forall x \in A(x \in B)$. Note that we can also say $\forall x(x \in A \Rightarrow x \in B)$, and the two hold the same meaning. It's important to clarify in the latter that we're not the domain of x, an we assume there is a universe of possible objects which forms a set. We're also able to define a strict subset such that $A \subset B \triangleq A \subseteq B \wedge A \neq B$.

We can say that any set is a trivial subset of itself, as we'd have $x \in A \Rightarrow x \in A$, which always evaluates to true, from propositional logic. Another trivial example is that $\varnothing$, the empty set, is a subset of every set. Using the second definition of subset, we can say that as $x \in \varnothing$ is false, by definition, and anything follows from falsity, whereas in the first definition we argue that all (0) elements of $\varnothing$ are in some other set.

We can also define set equality as $A = B \triangleq A \subseteq B \wedge B \subseteq A$. However, we can also consider the set composition notation for a set, such that $A = \{x \mid P(x)\}$, and $B = \{x \mid Q(x)\}$. If we're able to prove that $\forall x(P(x) \Leftrightarrow Q(x))$, it follos that $A = B$. This method can be quite powerful if we're familiar with logic, and equivalences. We can justify this by saying that $y \in A \Rightarrow P(y) \Rightarrow Q(y) \Rightarrow y \in B$, and also in the other direcion; $y \in B \Rightarrow Q(y) \Rightarrow P(y) \Rightarrow y \in A$.

Set Composition

- $A \cup B \triangleq \{x \mid x \in A \vee x \in B\}$ set union
- $A \cap B \triangleq \{x \in A \mid x \in B\}$ set intersection
- $A \backslash B$ (or $A - B$) $\triangleq \{x \in A \mid x \notin B\}$ set difference
- $A \triangle B \triangleq (A \backslash B) \cup (B \backslash A)$ symmetric set difference))
- $A \cap B = \varnothing$ disjoint set

A Note on Proofs

Instead of writing out the formal definition, where we may lose the intuition, using a natural language (direct) proof is acceptable in this course.

Consider the following proof; $A \subseteq B$, and $B \subseteq C$, then show $A \subseteq C$. Here, we want to show that any element of A, is also an element of C. We can approach this intuitively by taking an arbitrary $a \in A$.

By the the first assumption, we can say $a \in B$. Then, by the second assumption, $a \in C$. However, we've taken an arbitrary a, therefore this follows $\forall a \in A(a \in C)$, therefore $A \subseteq C$.

The crucial part of the aforementioned proof is the use of some **arbitrary** value. If we were to do a proof on the natural numbers, to show $\forall n \in \mathbb{N}[\text{even}(n)]$, and we proved even(2), it wouldn't prove it for all natural numbers.

We also want to aim for a direct proof, instead of a proof by contradiction, since we will often do the following; assume $\neg A$, then we somehow get A, which causes a contradiction ($\lightning$), and therefore A. However, we still did all the work to prove A.

Consider the proof to show that $C \cap D = D \cap C$. Let us first take some arbitary $x \in (C \cap D)$. By definition of union, we know that $x \in C$, and $x \in D$. Therefore, it also fits the predicate for $(D \cap C)$. As such, $C \cap D \subseteq D \cap C$. To prove the other direction is trivial, and almost identical to this direction. Since we've proved both directions of $\subseteq$, we can conclude equality.