

CO316 - Computer Vision

(60006)

Lecture 1 - Introduction

Computer vision tries to build a system that can understand the world in a similar way to a human. At a higher level, the pipeline for vision consists of sensing an image or video, processing it, and then understanding it. For a human, the sensor is the eyes, and the processor is done by the primary visual cortex. On the other hand, a sensor can be a camera, or some form of medical imaging device, and the processor is the computer itself (and more importantly, the algorithm).

A **classification** problem has the goal of determining the **label** of what is in the picture. Classification is considered to be successful if one of the labels the algorithm predicts matches the true label. On the other hand, object **detection** attempts to draw a bounding box around an object (where are objects in the picture). We can quantify the success of detection based on the following. Consider the following, where the region in **red** is drawn by a human, and the region in **blue** is predicted by the algorithm;



We consider the detection of the intersection over union (IoU) is above 0.5;

$$\text{IoU} = \frac{A \cap B}{A \cup B} > 0.5$$

Another more complex piece of information we can extract is to perform **image segmentation**, allowing us to draw contours for each object.

Applications

Computer vision is used in our lives daily;

- **face detection**

This can be noticed in most camera applications on modern smartphones, when a small box is drawn around faces. The algorithm first extracts **Haar** features from an image, and then determines (with these features) whether a region is a face or not.

One example of these features is checking the contrast between the eyes and nose (horizontally); as the eyes tend to be quite dark in comparison. Another contrast is checked, this time between your eyes, as the nose tends to be brighter.

- **automatic number plate recognition**

Automated barriers in parking lots can read number plates in order to calculate how long a car stays. Similarly, this can also be used to recognise building numbers, which is overlaid onto *Google Maps*, allowing for a large database of street numbers to be built in an automated fashion.

- **autonomous driving**

- **image style transfer**

Choi et al. StarGAN: Unified Generative Adversarial Networks for Multi-Domain Image-to-Image Translation - used for changing features on inputs. Related to face motion capture (see *Face2Face*). Also see *DeepFake*.

- **Kinect**

Works by taking a depth image, segmenting it into body parts, locating key points and building a skeleton.

- **design**

See *OpenAI's DALL-E*, combining NLP and computer vision by generating images based on the concepts of words in a sentence.

- **healthcare**

Medical image analysis can be used for disease diagnosis. For example, identifying breast cancer lesions from mammograms.

Lecture 2 - Image Formation

An image, in RGB format, can be represented as pixels, each being three numbers. A digital image is formed from a lighting source being reflected into an optics sensor (eyes, cameras, etc).

Light

A **point light source** originates from a single location in space, such as a small light bulb, or the sun. This can be described with three properties; location, intensity, and the spectrum.

On the other hand, an **area light source** is more complex. For example, this could be a ceiling light; a rectangle of point lights.

Reflectance

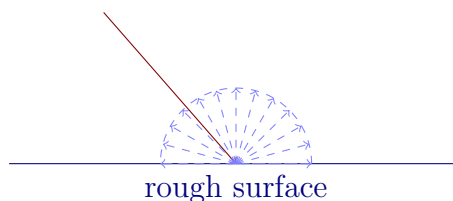
When light emitted from the source hits the surface of an object, it will be reflected. To describe this, we typically use the **bidirectional reflectance distribution function (BRDF)** to model this behaviour (where λ is the wavelength, L_r is the output power, and E_i is the input power);

$$f_r(\underbrace{\theta_i, \varphi_i}_{\text{incident}}, \underbrace{\theta_r, \varphi_r}_{\text{reflected}}, \lambda) = \frac{dL_r}{dE_i}$$

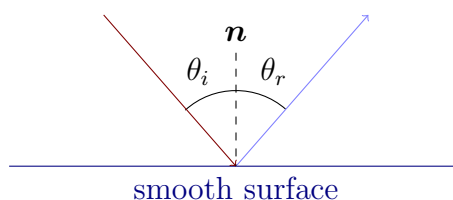
While this is a very general model, it is very complex.

As such, we can use **diffuse reflection**, where light is assumed to be scattered uniformly in all directions. This has a constant BRDF - this says that regardless of the incident or reflected directions, nor the wavelength, the power will be constant;

$$f_r(\theta_i, \varphi_i, \theta_r, \varphi_r, \lambda) = f_r(\lambda)$$



On the other hand, we can use **specular reflection** which performs reflections in a mirror-like fashion. The reflection and incident directions are symmetric with respect to the surface normal \mathbf{n} , such that $\theta_r = \theta_i$, with the same amount of power;



While these two are the **ideal** cases, the majority of cases, we see a combination of both of those, as well as **ambient** illumination. Ambient illumination accounts for general illumination which could be complicated to model. For example, these could be repeated reflections between walls (which would be very difficult to calculate), and we instead assume that there is some light that exists in the 3D space representing the room. Another example could be a distance source, such as the sky (which has atmosphere).

Combining these, we can use the **Phong** reflection model. This is an empirical model that describes how a surface reflects light as a combination of ambient, diffuse, and specular components.

‘Duality’ with Computer Graphics

Using the game engine to produce example images is useful, as we are able to directly obtain the labels of objects from the engine itself, as well as visual output. As such, we can use these images as training for a model, since we also have an associated label map. This synthetic data is complementary to time-consuming manual annotations.

Optics and Sensors

Both our eyes and cameras work in similar ways, with a lens governed by the thin lens equation, where f denotes the focal length of the lens, u denotes the distance from the subject to the lens, and v denotes the distance from the lens to the image;

$$\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$$

Our eyes work by light rays being focused by the cornea and lens onto the retina, where vision begins with two neural cells. The **cone** cells are responsible for colour vision, and function in bright light. On the other hand, the **rod** cells have little role in colour vision, but function in dim light.

Humans have three types of cone cells (**trichromatic vision**), which have different response curves. The short cone cells respond to short wavelength lights (**violet**, **blue**), whereas the medium cone cells respond to medium wavelength lights (**green**), and long cone cells respond to long wavelength lights (**red**). Occasionally, there may be two cone cells, or four, which are referred to as **dichromacy** or **tetrachromacy** respectively.

Note that colours are not objective physical properties of light or electromagnetic wave (which have a physical property of wavelength). Colour is a subjective feature, dependent on the visual perception of the observer. Since **rod** cells are more sensitive to light, they are the primary source of visual information at night.

On the other hand, camera sensors have two common types;

- **CCD** (charged-coupled device) often used in handheld cameras
- **CMOS** (complementary metal-oxide semiconductor) used by most smartphone cameras

These sensors convert incoming light into electron charges, which are then read. **Bayer** filter arrays are a way to arrange RGB filters on sensors, half of which are green, and the remaining two quarters are red and blue. This mimics the human eyes, which are most sensitive to green light;



CMOS works by having sensors underneath each of these filtered portions, which can report an electrical signal. However, note that only one colour is available at each pixel (therefore the rest must be interpolated from the neighbours, by using bilinear interpolation; which simply averages the 4 neighbours). For example, consider the following pixel (denoted as a white cross);



Note that the use of different filters, and this interpolation, can lead to slightly different colours between cameras.

Image Representation

The earliest colour space was described in 1931 by CIE, by performing a colour matching experiment. In this experiment, an observer attempts to match different levels of red, green, and blue lights to match a target light. This allows for colours to be represented in 3D space, as (X, Y, Z) , corresponding to the different levels. Colours can also be represented on a 2D plane, by normalising brightness;

$$\begin{aligned}
 x &= \frac{X}{X+Y+Z} \\
 y &= \frac{Y}{X+Y+Z} \\
 z &= \frac{Z}{X+Y+Z} \\
 &= 1 - x - y
 \end{aligned}$$

therefore redundant

Here X, Y, Z are primary colours (R, G, B), and x, y are chromacity / colour after removing brightness. This is much easier to draw. However, this colour space, also known as the **gamut** of human vision, was invented before computer screens.

The sRGB (standard RGB) space was created by *HP* and *Microsoft* in 1996 for use on monitors, printers, and the internet.

sRGB definition	x	y
red	0.64	0.33
green	0.30	0.60
blue	0.15	0.06

This is represented by a triangle (which is a subset) in the gamut of human vision. As this is a subset, it cannot produce all the colours visible by the human eye.

There are other colour spaces, such as HSV, CMYK, and so on. Note that CMYK is a **subtractive** colour model, starting from white, whereas RGB is an **additive** colour model, where we start from black. There can also be an alpha channel in RGB, which represents transparency. In a greyscale image, the three components are equal, hence only require one number.

Quantisation

Note that this is covered in lecture 3.

Quantisation maps a continuous signal to a discrete signal. The pictures from a camera are a continuous signal, but when it is stored on the camera, it is quantised to a discrete signal. Numerical errors can occur during this process, and the magnitude of the errors depends on the number of bits used (less error with more bits; 16 bits can store from 0 to 65535, compared to 8 bits storing from 0 to 255).

Physically, an analog-to-digital convert (ADC) is used to perform the conversion. The energy of photons are converted into voltage, amplified, and then converted.

Compression

In order to reduce the cost of storage to transmission, compression may be used. Lossy compression loses information after the compression (such as discrete cosine transform (DCT) in JPEG, often used for images or videos). However, lossless compression can also reduce the file size (less efficient compared to lossy), and is preferred for archival purposes or important imaging, where detail needs to be recovered.

Lecture 3 - Image Filtering I

Note that in this course, most of the examples will be done on greyscale images, but can be applied to the channels individually. Some examples of filters include;

- identity filter

Does nothing to the image.

- low-pass / smoothing (moving average, Gaussian)

Removes high-frequency signals, and keep low-frequency signals.

- high-pass / sharpening

Similar to the previous filter (keeps high-frequency signals, and removes low-frequency).

- denoise (median, non-local means, block-matching and 3D filtering)

Moving Average Filter

This is commonly used for 1D signal processing (time series), such as stocks, which can be quite noisy. To smooth out a noisy curve, it moves a window across the signal (and calculates the average value within the window) - the larger the window size, the smoother the result.

In a two dimensional case, we can use a **filter kernel** (for example, with a 3×3 kernel);

$$\frac{1}{9} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

At each pixel, we apply the kernel centred at the pixel, and take the average of the pixels around it (and itself) to create a new output image. When we blur with a larger kernel, such as 7×7 , we end up with a blurrier image. Note that the output image is smaller than the input image. We can pad the image with zeroes (anything outside of the picture is 0), or by mirroring the pixels (copying the boundary pixels).

Consider an image of size $N \times N$, and a kernel size of $K \times K$. At each pixel, we perform K^2 multiplications (by the kernel weights), and then $K^2 - 1$ summations. This has to be done for each pixel, hence N^2 times. Therefore this results in $N^2 K^2$ multiplications and $N^2(K^2 - 1)$ summations; giving a **complexity** of $O(N^2 K^2)$. However, we'd like to reduce this, if possible.

If a big filter can be separated as two filters (**separable filter**) applied consecutively, we can perform the first operation, and then the second. An average in a 2D window can be done as an average across rows (horizontal), and then an average across columns (vertical);

$$\begin{bmatrix} \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} * \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$$

Doing these two filters, we end up with an equivalent result to the original 2D filter. Note that $*$ is a convolution (see next lecture).

Consider the complexity of separable filtering. The image size remains as $N \times N$, however we have two kernels, of $1 \times K$ and $K \times 1$ respectively. At each pixel we do K multiplications followed by $K - 1$ summations. Again, this is done for N^2 pixels, and twice (once for each filter). Therefore, the total number of multiplications is $2N^2K$ multiplications and $2N^2(K - 1)$ summations. This is better, in contrast to the original complexity, as we have complexity of $O(N^2K)$ - which will make a difference for large K .

A moving average filter removes high frequency signals (noise or sharpness), which results in a smooth but blurry image.

Gaussian Filter

The kernel is a 2D Gaussian distribution;

$$h(i, j) = \frac{1}{2\pi\sigma^2} e^{-\frac{i^2+j^2}{2\sigma^2}}$$

Here we have $i, j = 0, 0$ as the centre of the kernel. While the support is infinite, small values outside the range $[-k\sigma, k\sigma]$ can be ignored (very small values, such as $k = 3$ or $k = 4$). Note that σ is a manually defined parameter. This is a separable filter, which is equivalent to two 1D Gaussian filters with the same σ , with one along the x -axis and the other along the y -axis;

$$h(i, j) = h_x(i) * h_y(j)$$

$$h_x(i) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{i^2}{2\sigma^2}}$$

High-pass Filter

One design is to do the following;

$$\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{identity}} + \underbrace{\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \end{bmatrix} \right)}_{\text{high-frequency}} = \begin{bmatrix} -\frac{1}{9} & -\frac{1}{9} & -\frac{1}{9} \\ -\frac{1}{9} & \frac{7}{9} & -\frac{1}{9} \\ -\frac{1}{9} & -\frac{1}{9} & -\frac{1}{9} \end{bmatrix}$$

We can add a high frequency signal to the identity, in order to enhance it.

Median Filter

This is a non-linear filter (not performing an average calculation by multiplication). This moves a sliding window, and replaces the centre pixel with the median value in the window - this is not a linear equation.

Lecture 4 - Image Filtering II

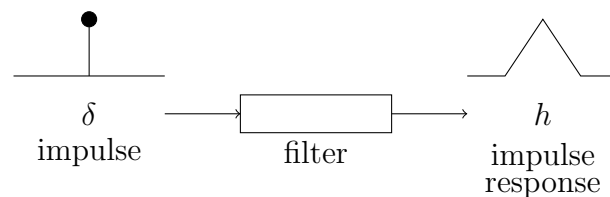
Mathematical Description

Consider a simple filter, of size 3, in 1D. With an input f , and an output g , it can be written as the weighted average in a window;

$$g[n] = \frac{1}{3}f[n-1] + \frac{1}{3}f[n] + \frac{1}{3}f[n+1]$$

In general, filtering takes in an input signal f , processes it and generates an output signal g . A filter is a device (or process) that removes unwanted components or features from a signal (keeps / enhances wanted filters).

In order to mathematically describe a filter, we need the concept of **impulse response**; the output of a filter when the input is an **impulse signal** (only have a signal at a single time point);



For a continuous signal, we treat an impulse as a Dirac delta function $\delta(x)$, whereas for a discrete signal, we treat an impulse as a Kronecker delta function $\delta[i]$;

$$\delta(x) = \begin{cases} \infty & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$
$$\int_{-\infty}^{\infty} \delta(x) dx = 1$$
$$\delta[i] = \begin{cases} 1 & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}$$

The impulse response h completely characterises a **linear time-invariant** filter. Note that we can consider a filter as time-invariant if, by shifting the input signal some number of time steps k , the output signal will remain the same (shape and values) but shifted by the **same** number of time steps. For example;

- $g[n] = 10 \cdot f[n]$ is time-invariant and amplifies the input by a constant
- $g[n] = n \cdot f[n]$ is **not** time-invariant since the amount it amplifies the input depends on the time step n

As long as we know h , and have an input x , we can calculate the output signal y . Since it uniquely describes a filter, we often denote a filter by its impulse response function h .

Additionally, a filter can be **linear**. If it is a linear system, when two input signals are combined linearly, their outputs will also be combined linearly. Let $f_1[n]$ lead to an output $g_1[n]$, and $f_2[n]$ to $g_2[n]$.

$$\text{output}(\alpha f_1[n] + \beta f_2[n]) = \alpha g_1[n] + \beta g_2[n]$$

Convolution

Most of the filters we've previously covered are linear time-invariant. Since h characterises how the system works (as it is linear time-invariant), it's possible for the output g to be described as the **convolution** between an input f and impulse response h ;

$$g[n] = f[n] * h[n]$$

We can describe an input signal $f[n]$ as the following, where each time step is a constant multiplied by a spike;

$$f[n] = f[0]\delta[n] + f[1]\delta[n-1] + f[2]\delta[n-2] + f[3]\delta[n-3] + \dots$$

However, since we know the output of $\delta[n]$ is $h[n]$, we can write the output as;

$$g[n] = f[0]h[n] + f[1]h[n-1] + f[2]h[n-2] + f[3]h[n-3] + \dots$$

The output mathematical operation is defined as a convolution, where a signal f and a filter with impulse response / convolution kernel h is defined as;

$$g[n] = f[n] * h[n] = \sum_{m=-\infty}^{\infty} f[m]h[n-m]$$

The continuous form (previously we have only considered the discrete form);

$$g(t) = f(t) * h(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau) d\tau$$

Focusing on the discrete case, we notice the following (when we replace m with $n-m$);

$$\sum_{m=-\infty}^{\infty} f[m]h[n-m] = \sum_{m=-\infty}^{\infty} f[n-m]h[m]$$

This shows that the convolution of f and h is equivalent to the convolution of h and f (commutativity);

$$f[n] * h[n] = h[n] * f[n]$$

By expanding the equations, we can also show that convolution satisfies associativity, such that;

$$f * (g * h) = (f * g) * h$$

We also have distributivity;

$$f * (g + h) = (f * g) + (f * h)$$

And also differentiation;

$$\frac{d}{dx}(f * g) = \frac{df}{dx} * g = f * \frac{dg}{dx}$$

With a more concrete example, we can visualise it as follows (this is the moving average of size 3);



In our case, we have the kernel $h[n]$, and the values;

$$\begin{aligned} h[-1] &= \frac{1}{3} \\ h[0] &= \frac{1}{3} \\ h[1] &= \frac{1}{3} \\ h[n] &= \left[\frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right] \end{aligned}$$

This can be expanded into the 2D case, which is used for image filtering;

$$g[m, n] = f[m, n] * h[m, n] = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} f[i, j] h[m - i, n - j]$$

This can also be written as the following, replacing $m - i, n - j$ by i, j ;

$$g[m, n] = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} f[m - i, n - j] h[i, j]$$

By using the property of associativity, if a big filter (call it f_b) can be written as the convolution of g and h (smaller filters), we can first convolve f with g , then with h - this is used for separable filtering;

$$f * f_b = f * (g * h) = (f * g) * h$$

For example (note that we have padded zeroes, but in code we do not need that);

$$\underbrace{\begin{bmatrix} \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \end{bmatrix}}_{f_b} = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 \end{bmatrix}}_g * \underbrace{\begin{bmatrix} 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & 0 \end{bmatrix}}_h$$

Lecture 5 - Edge Detection I

Importance of Edges

In computer vision an edge refers to lines where image brightness changes sharply with discontinuities. This may be due to different reasons such as discontinuities in colour, depth, surface normals, etc. Edges capture important properties of what we see in the world, and they are important features for image analysis (for example, we first capture edges, then facial features, and so on). *Hubel* and *Wiesel* performed vision experiments in 1959, finding that neuron cells in the primary visual cortex are orientation selective, responding strongly to lines or edges of a particular orientation. Humans can also read images even if reduced to simple line drawings. Edges are also heavily used in convolutional networks, with the first layer tending to learn edges, with the later layers learning increasingly complex patterns.

Detection

An image can be considered as a function of pixel positions (consider plotting the image on the $x - y$ axes, and having the z axis denote intensity). Mathematically, derivatives characterise function discontinuities, which can be used to help edge detection. For a continuous function, the derivative is;

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$$

On the other hand, for a discrete function (finite difference);

$$\underbrace{f'[x] = f[x + 1] - f[x]}_{\text{forward difference}} \quad \underbrace{f'[x] = f[x] - f[x - 1]}_{\text{backward difference}} \quad \underbrace{f'[x] = \frac{f[x + 1] - f[x - 1]}{2}}_{\text{central difference}}$$

Notice that these can also be performed with convolutions, using the following kernels;

$$\underbrace{h = [1, -1, 0]}_{\text{forward difference}} \quad \underbrace{h = [0, 1, -1]}_{\text{backward difference}} \quad \underbrace{h = [1, 0, -1]}_{\text{central difference}}$$

Prewitt and Sobel Filters

The Prewitt filters, along the x -axis (horizontal direction) and along the y -axis (vertical direction) are as follows;



Note that the Prewitt filter is a separable filter ([1] - moving average for smoothing);

$$\underbrace{\begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix}}_{\text{Prewitt filter}} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{[1]} * \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{finite difference}}$$

The Sobel filter is quite similar to the Prewitt filter;



Note that the Sobel filter is also a separable filter;

$$\underbrace{\begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ 1 & 0 & -1 \end{bmatrix}}_{\text{Sobel filter}} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{\text{smoothing}} * \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{finite difference}}$$

Note that the outputs of the filters are different, with the horizontal filter detecting changes in the horizontal direction; describing discontinuity along the x -axis (leading to vertical lines). These can be combined to describe two properties of edges; the magnitude and the orientation.

$$\begin{aligned} g_x &= f * h_x && \text{derivative along } x\text{-axis} \\ g_y &= f * h_y && \text{derivative along } y\text{-axis} \\ g &= \sqrt{g_x^2 + g_y^2} && \text{magnitude of the gradient} \\ \theta &= \arctan2(g_y, g_x) && \text{angle of the gradient} \end{aligned}$$

Derivative of Gaussian

Note that in both the filters above, there is some smoothing as derivatives are sensitive to noise (therefore the smoothing kernel helps to suppress noise). The Prewitt filters use the mean average kernel, whereas Sobel filters use the kernel $[1, 2, 1]$. Another option is to use the Gaussian kernel for smoothing, before calculating derivatives;

$$h[x] = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}$$

The derivative of Gaussian filter is an operation, performing Gaussian smoothing, before taking the derivative. However, recalling the rules for the differentiation of convolution, we have f (input signal) convolved with the derivative of the Gaussian kernel;

$$\begin{aligned}\frac{d}{dx}(f * h) &= f * \frac{dh}{dx} \\ &= f * \frac{-x}{\sqrt{2\pi}\sigma^3} e^{-\frac{x^2}{2\sigma^2}}\end{aligned}$$

In the 2D case, the Gaussian filter is as follows;

$$h[x, y] = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}$$

However, it is a separable filter (and therefore can be sped up) equivalent to the convolution of two 1D Gaussian filters. Notice there's a parameter σ in the derivative of Gaussian. With a small σ , there is more detail in the magnitude map; on the other hand, a large σ value suppresses noise and results in a smoother derivative. As such, different σ values help to find edges at different scales.

Lecture 6 - Edge Detection II

Canny Edge Detection

The results of the filters from the previous lectures (gradient magnitude map) have values ranging from black to white. However, we want either a 0 or 1 (black or white) - a **binary edge map**.

There should be good detection; a low probability of failing to mark real edge points and low probability of falsely marking non-edge points. In addition, there should be good localisation - the points marked as edges should be as close as possible to the centre of the edge. Finally, there should only be a single response to a single edge.

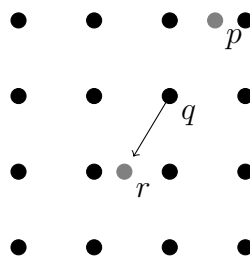
Canny edge detection is done in the following steps;

1. perform Gaussian filtering to suppress noise

The choice of σ depends on the type of edge we desire at the end. If we want large edges (such as buildings), σ should be set to a large value (such as 7). However, if we want to pay attention to fine features, σ can be set to a small value.

2. calculate gradient magnitudes and directions
3. apply non-maximum suppression (NMS) to get a single response for each edge

Non-maximum suppression aims to get a single response for each edge, by using the idea that the edge occurs where the gradient magnitude is maximum at the centre of the edge.



Compare the magnitude at q with the magnitudes of p and r , both along the gradient direction (in opposite directions). If q is the local maximum (larger than both p and r), we keep the magnitude at pixel q .

$$M(x, y) = \begin{cases} M(x, y) & \text{if local maximum} \\ 0 & \text{otherwise} \end{cases}$$

However, notice that there may be issues with p and r not being at exact pixel locations. For example, using pixel r (not at an integer position), we can perform interpolation. It's possible to use linear interpolation (weighted by distance), or a simpler algorithm such as nearest neighbour interpolation (better performance). Note that different interpolation algorithms will lead to different outcomes. Another approach is to round the gradient direction into 8 possible angles, in steps of 45° in the range $[0^\circ, 315^\circ]$, allowing us to check only along those directions to give exact pixels (intuitively quite similar to nearest neighbour).

After NMS, there are fewer points with bright values.

4. perform hysteresis thresholding to find potential edges

Many pixels that are local maxima may still have very low magnitudes; however we only want edges with high magnitudes. A simple thresholding would be to convert an intensity image to a binary image with a threshold t ;

$$\text{binary}(x, y) = \begin{cases} 1 & \text{if } I(x, y) \geq t \\ 0 & \text{otherwise} \end{cases}$$

On the other hand, hysteresis thresholding defines two thresholds t_{low} and t_{high} . If the magnitude is $\geq t_{\text{high}}$, it is accepted as an edge pixel, and if it is $< t_{\text{low}}$, it is rejected. However, if we have a value between the thresholds, we have a **weak edge** (which may or may not be an edge). If it is connected to existing edge pixels, it is accepted, whereas if it is not connected (adjacent to one strong edge) to an existing edge, it will be rejected.

The initial goals are satisfied as follows;

- **good detection**

False positives are reduced by using Gaussian smoothing to suppress noise. On the other hand, false negatives are reduced by using hysteresis thresholding to find weak edges.

- **good localisation**

NMS finds locations based on gradient magnitude and direction.

- **single response**

also done with NMS

Learning-based Edge Detection

Decades of effort have been made to improve detection accuracy. This includes using richer features such as colour and texture, enforcing smoother, as well as using machine learning (by learning mapping from an image to edge directly from data).

This machine learning algorithm assumes paired data (images x and manually defined edge maps y). The problem finds a model (with model parameters θ) that maps x to y , such that $y = f(x | \theta)$. This differs from Canny edge detector, as our example integrates from multiple scales (fine-scale edges) with coarse-scale edges to form a final output. On the other hand, Canny edge detector uses a single scale for edge detection, controlled by a single parameter σ .

An application for learning-based edge detection is to learn a mapping from a rough sketch to a simplified sketch. However, this isn't just an edge detection problem since it cannot be solved using NMS; when this is done by humans, a high-level understanding about the sketch is required.

Conclusion

Edge detection is a fundamental problem in image processing and computer vision; aiming to identify where discontinuities occur, or identifying points that are edges. Two solutions are proposed;

- if we know the explicit criteria, we can implement computational criteria

- otherwise, we can collect data pairs representing what we want to achieve and train a machine learning model

Edges provide important low-level features both human vision and computer vision for understanding images. These algorithms provide ideas for other detection algorithms.

We can also go in the other direction, taking edges to images. This aims to train a model that generates an image that looks close to real images, from edges. A model G (generator) learns the mapping from edge to image. For example, we take an edge image x to $G(x)$ (an image). A discriminator d then attempts to assess whether the image is real or fake (GAN).

Lecture 7 - Hough Transform

In the last lecture, we covered edge detection (1 for edge pixel, 0 for background). If we know these edges form some shape (such as a line), we want to obtain a parametric representation.

Line Parameterisation

A line can be represented by two parameters (e.g. m (slope) and b (y -intercept)), which is much more efficient than a lot of edge points;

- slope intercept form m (slope) and b (y -intercept)

$$y = mx + b$$

- double intercept form $(a, 0)$ and $(0, b)$ are on the line

$$\frac{x}{a} + \frac{y}{b} = 1$$

- normal form θ is angle and ρ is distance

$$x \cos(\theta) + y \sin(\theta) = \rho$$

Hough Transform

Hough transform transforms from image space (edge map) to parameter space (two parameters of a line). The output is a parametric model, from a n input of edge points. Each edge point ‘votes’ for possible models in the parameter space. One way is to fit a line model (m, b) to the edge points $(x_1, y_1), (x_2, y_2), \dots$;

$$\min_{m,b} \sum_i (y_i - \underbrace{(mx_i + b)}_{\hat{y}})^2$$

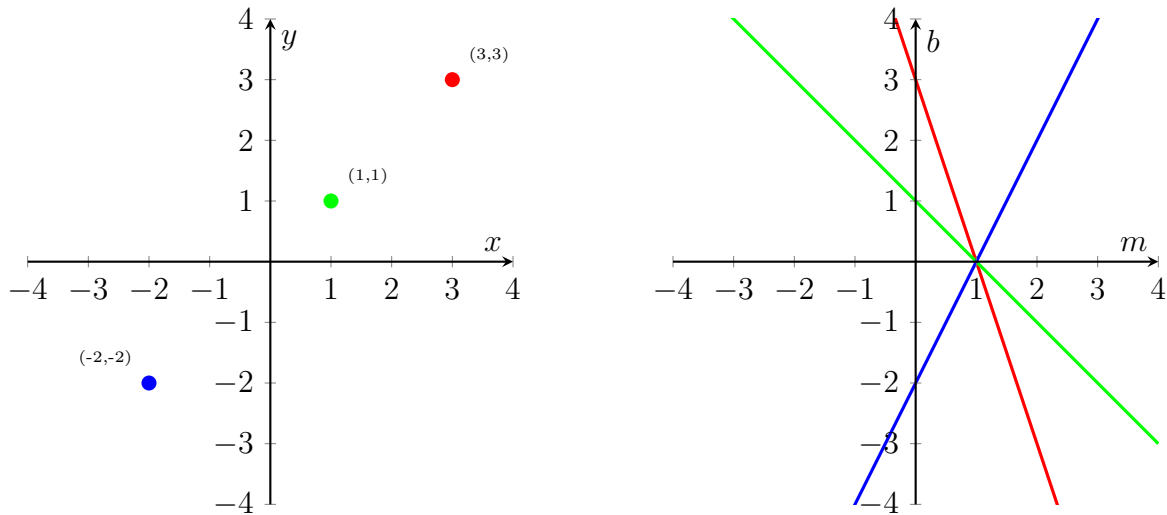
For the Hough transform we will use the slope intercept form.

$$y = mx + b \Leftrightarrow b = y - mx$$

Assume we have the edge points $(x_1, y_1), \dots$, each point will vote for a line model in the parameter space; for example, the first point votes for;

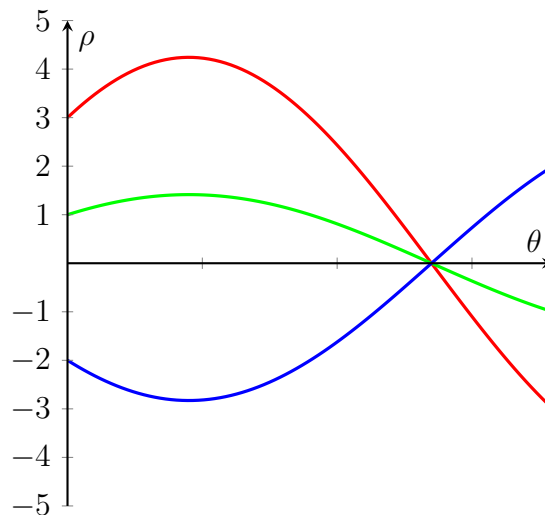
$$b = y_1 - mx_1$$

This can be seen graphically as the following (image space on the left, parameter space ($b = y - mx$) on the right);



The intersection will get 3 votes, the other points on the line get 1 vote each, and the empty spaces get 0 votes. This gives a result of $m = 1$ and $b = 0$, thus we get the line $y = x$. However, in practice this is divided into two dimensional bins, with each point incrementing the vote by 1 in one of the bins. One problem is that the parameter space is too large for m and b (both are infinite).

The solution to this is to use the normal form. While ρ can still be infinite (in theory, however we can actually limit this to the image size in practice), we have $\theta \in [0, \pi)$. The transform from the image space to the parameter space ($x \cos(\theta) + y \sin(\theta) = \rho$) will however look different;



This gives the resultant vote $\theta = \frac{3\pi}{4}$ and $\rho = 0$.

To perform this algorithm, we do the following;

1. initialise the bins $H(\rho, \theta)$ to zero in the parameter space
2. for each edge point (x, y) ;
 - a. for θ from 0 to π
 - i. calculate $\rho = x \cos \theta + y \sin \theta$
 - ii. accumulate $H(\rho, \theta) = H(\rho, \theta) + 1$
3. find (ρ, θ) where $H(\rho, \theta)$ is a local maximum and larger than a threshold

This is done in a similar way to the previous lecture. The local maximum allows multiple solutions to be detected, and the threshold reduces false positives.

4. the detected lines are given by $\rho = x \cos \theta + y \sin \theta$

In contrast to model fitting (minimisation), Hough transform can simultaneously detect multiple lines (as long as they are local maximums and above a threshold) whereas the former can only detect a single line. Hough transform is robust to noise for two reasons. First, the initial edge map is generated after image smoothing (already suppressing noise). Furthermore, the broken edge maps are still able to vote and contribute to line detection. Similarly, it is also robust to object occlusion (objects overlapping, such as a tree obstructing part of a face).

However, the computational complexity is high; we have to vote in a 2D or 3D parameter space for each edge point. We also need to set parameters carefully, such as parameters for the edge detector, the threshold for the accumulator, or the radius range (in the case of circles).

This can be generalised to other shapes (which can be analytically represented), such as ellipses or planes in a 3D space. We can also weigh the votes, by taking the gradient magnitude (such that stronger edge points are weighted higher).

In general, if it is a simple shape (such that there is no analytical equation), we can still vote as long as we have a model. For example, consider the case of pedestrian detection; we can vote for points above a patch where a foot is detected, and below where a head is detected.

Circles

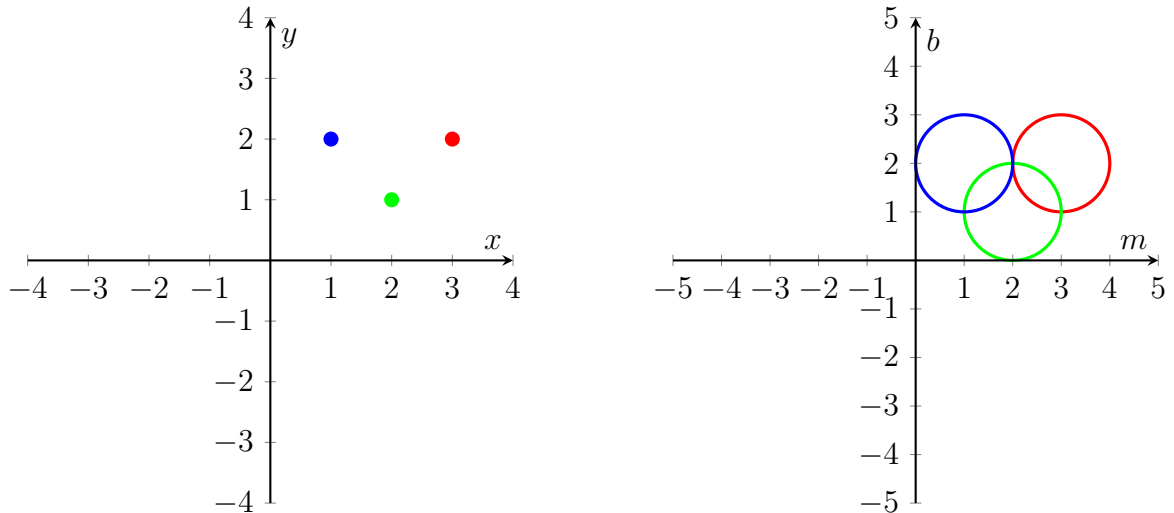
We can parameterise a circle as;

$$(x - a)^2 + (y - b)^2 = r^2$$

However, this is a very large parameter space (hence many bins). If we have some prior knowledge (such as the radius r), we can make the problem easier by reducing the search space - we can just vote for a, b (the centre of the circle). The votes are still circles in the parameter space $H(a, b)$ (assuming $r = 1$);

$$(a - x)^2 + (b - y)^2 = 1$$

This can be seen graphically as the following (image space on the left, parameter space $((a - x)^2 + (b - y)^2 = 1)$ on the right);



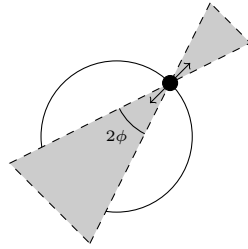
If we don't know the radius, we set a range $r \in [r_{\min}, r_{\max}]$, and then perform a similar algorithm to before.

Another representation of a circle can be done with trigonometric functions (parametric form);

$$x = a + r \cos \theta$$

$$y = b + r \sin \theta$$

However, if we know θ , we can vote along a direction. Since we know the direction of an edge point (obtained from edge detection), we can narrow it down (along the gradient or opposite the gradient). We assume an accuracy of $\pm\phi$, and vote within that area.



The algorithm can be done as follows;

1. initialise all bins $H(a, b, r)$ to zero
2. for each possible radius $r \in [r_{\min}, r_{\max}]$
 - a. for each edge point (x, y) ;
 - i. let θ be gradient direction / opposite gradient direction
 - ii. calculate $a = x - r \cos \theta$ and $b = y - r \sin \theta$
 - iii. accumulate $H(a, b, r) = H(a, b, r) + 1$
3. find (a, b, r) where $H(a, b, r)$ is a local maximum and larger than a threshold

Lecture 8 - Interest Point Detection I

An interest point is a point we are interested in, and are useful for subsequent processing and analysis (classification, matching, retrieval, etc). These are commonly corners or blobs, where local image structure is rich. These are also known as keypoints, landmarks, or low-level features.

Applications

We can define landmarks, on a face, which are most representative. Once we detect these landmarks, the locations can be used for purposes such as mood analysis, AR, etc.

Another application is **image matching**; detecting relationships between images (such as orientation) and finding spatial correspondence between two images. This can be done by pixels (by using all information), by edges (using some information), or by interest points (using only important information). Matching with pixels often works by optimising a similarity metric (based on all pixels), with a spatial transformation T ;

$$\max_T \text{Similarity}(I_A, I_B(T))$$

A more efficient approach is to find interest points and see how they are transformed between images (consider different perspectives of an art piece).

Harris Detector

Corners are intersection of edges (which are high magnitude of gradient). For corner detection, by just looking at the gradient magnitude of a single pixel, we can't tell if it is a corner or edge pixel. However, if we look at the small windows, we can tell a difference;

- flat change of intensity in **neither** direction
- edge change of intensity along just **one** direction
- corner change of intensity along **both** directions

We want to define a window W and a window function w (1 if it is in the window, 0 otherwise), where we have the **sum of squared difference**, **intensity in the shifted window** and the **original intensity**;

$$E(u, v) = \sum_{(x, y) \in W} w(x, y) [(I(x + u, y + v) - I(x, y))^2]$$

Note that the window function $w(x, y)$ is used as it isn't necessarily 1 inside, and 0 outside, we can use also use a Gaussian to put more focus on the middle of the window.

We can use the following Taylor expansion (note that I_x and I_y denote the image derivative along x and y respectively);

$$I(x + u, y + v) = I(x, y) + uI_x(x, y) + vI_y(x, y) + \dots$$

Using just the first order approximation;

$$\begin{aligned} E(u, v) &= \sum w(x, y) [I(x + u, y + v) - I(x, y)]^2 \\ &\approx \sum w(x, y) [\underbrace{u I_x(x, y)}_{I_x} + \underbrace{v I_y(x, y)}_{I_y}]^2 \\ &= \sum w(x, y) (u^2 I_x^2 + 2uv I_x I_y + v^2 I_y^2) \\ &= \sum w(x, y) \begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} I_x^2 I_x I_y & I_x I_y I_y^2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \quad \text{written as matrix formula} \\ &= \begin{bmatrix} u & v \end{bmatrix} \sum w(x, y) \begin{bmatrix} I_x^2 & I_x I_y \\ I_x I_y & I_y^2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \end{aligned}$$

Therefore, for a small shift (u, v) , we have the approximation (note that $E(u, v)$ will be large when image derivatives are large);

$$E(u, v) \approx \begin{bmatrix} u & v \end{bmatrix} \mathbf{M} \begin{bmatrix} u \\ v \end{bmatrix}$$

Where $\mathbf{M} \in \mathbb{R}^{2 \times 2}$ is computed from the window function and image derivatives (this matrix will also provide the direction of changes);

$$\mathbf{M} = \sum_{x, y} w(x, y) \begin{bmatrix} I_x^2 & I_x I_y \\ I_x I_y & I_y^2 \end{bmatrix}$$

We have the following simple cases (and examples) - note that these are **only** diagonals;

- **flat region** small values on diagonal

$$\mathbf{M} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

- **edge** diagonal, large on one of diagonal

$$\mathbf{M} = \begin{bmatrix} 10 & 0 \\ 0 & 0.1 \end{bmatrix}$$

In this case, we have a large change if shifting along u , but a small change if we shift along v , hence we have an edge.

- **corner** diagonal, large on both diagonal

$$\mathbf{M} = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$$

If we move along either direction, there will be a large change, hence it is a corner.

However, in the case that \mathbf{M} is more complex (such that it isn't a diagonal matrix), we can simplify it by performing eigen decomposition. Because \mathbf{M} is a real symmetric matrix, we can decompose it as;

$$\mathbf{M} = \mathbf{P}\mathbf{\Lambda}\mathbf{P}^\top$$

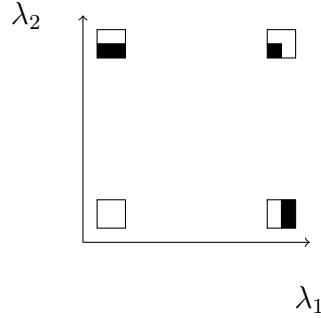
In the above, \mathbf{P} has eigenvectors (orthogonal to each other) as columns (and therefore \mathbf{P}^\top has eigenvectors as rows), and $\mathbf{\Lambda}$ is a diagonal matrix with eigenvalues;

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

Referring back to $E(u, v)$, we have the following;

$$E(u, v) \approx \begin{bmatrix} u & v \end{bmatrix} \mathbf{P}\mathbf{\Lambda}\mathbf{P}^\top \begin{bmatrix} u \\ v \end{bmatrix}$$

The same rules as above apply; however instead of moving along u , we instead move along the first eigenvector (and instead of moving along v , we move along the second eigenvector). The edges are no longer horizontal or vertical, they are in the directions of the eigenvectors. The eigenvalues can be interpreted as follows;



Here we have the following cases;

- $\lambda_1 \sim 0$ and $\lambda_2 \sim 0$ flat
- $\lambda_1 \gg \lambda_2$ vertical edge
- $\lambda_2 \gg \lambda_1$ horizontal edge
- λ_1 and λ_2 both large corner

However, we want to define a single number for ‘cornerness’ (high when both eigenvalues are large). This can be defined in multiple ways;

- *Harris and Stephens (1988)*

$$R = \lambda_1 \lambda_2 - k(\lambda_1 + \lambda_2)^2$$

Here k is a small number, such as 0.05. The small k ensures that the number does not become too small (when subtracted) when both eigenvalues are large.

- *Kanade and Tomasi (1994)*

$$R = \min(\lambda_1, \lambda_2)$$

- *Noble (1998)*

$$R = \frac{\lambda_1 \lambda_2}{\lambda_1 + \lambda_2 + \epsilon}$$

However; notice that we only want the values of $\lambda_1\lambda_2$ and $\lambda_1 + \lambda_2$, therefore we do not need to perform eigen decomposition. Using the following properties for matrix determinant and trace, we can obtain the following;

$$\begin{aligned}\det(\mathbf{M}) &= \det(\mathbf{P}\mathbf{\Lambda}\mathbf{P}^\top) \\ &= \det(\mathbf{\Lambda}) \\ &= \lambda_1\lambda_2 \\ \text{trace}(\mathbf{M}) &= \text{trace}(\mathbf{P}\mathbf{\Lambda}\mathbf{P}^\top) \\ &= \text{trace}(\mathbf{\Lambda}) \\ &= \lambda_1 + \lambda_2\end{aligned}$$

Therefore, we only need the determinant and trace for the original matrix to calculate R ;

$$R = \det(\mathbf{M}) - k(\text{trace}(\mathbf{M}))^2$$

This detector can also find strong responses at blobs and textures, as well as corners. The algorithm is as follows;

1. compute x and y derivatives of an image

G can be the Sobel filter

$$\begin{aligned}I_x &= G_x * I \\ I_y &= G_y * I\end{aligned}$$

2. compute the matrix \mathbf{M} at each pixel

$$\mathbf{M} = \sum_{x,y} w(x,y) \begin{bmatrix} I_x^2 & I_x I_y \\ I_x I_y & I_y^2 \end{bmatrix}$$

3. calculate the detector response

$$R = \lambda_1\lambda_2 - k(\lambda_1 + \lambda_2)^2$$

4. detect interest points which are local maxima (and R above a certain threshold)

Note that the Harris detector is **rotation-invariant**. We will still get the same change of intensities when shifting along a rotated direction - the eigenvalues of \mathbf{M} remain the same, but the eigenvectors will change. However, it is not invariant to scale. At a different zooming scale, an edge may be detected as a corner or vice versa (for example if the circle arc is large it will be detected as an edge, but if it is small, it will be detected as a corner).

Lecture 9 - Interest Point Detection II

Scale

The intuitive ideal for scaling is to check whether the Harris detector gives the highest response at a given scale. If the region looks most like a corner at scale σ , there should be a high response.

$$\mathbf{M} = \sum_{x,y} w(x,y) \begin{bmatrix} I_x^2(\sigma) & I_x(\sigma)I_y(\sigma) \\ I_x(\sigma)I_y(\sigma) & I_y^2(\sigma) \end{bmatrix}$$

The response is determined by the eigenvalues of \mathbf{M} , which are determined by the derivatives $I_x(\sigma)$ and $I_y(\sigma)$. These are inversely proportional to scale σ ; the larger the scale, the smaller the derivative magnitude.

Consider signals which only differ by scale s (and have same peak magnitude);

$$f(x) = g(sx)$$

However, when we look at the derivatives of the function, we have a different peak magnitude; since the derivative looks at how quickly the functions change. Since we have the relation above, we have;

$$f(x + \Delta x) - f(x) = g(sx + s\Delta x) - g(sx)$$

Looking at the Taylor expansion for the above, we have;

$$\begin{aligned} f(x + \Delta x) &= f(x) + \Delta x \cdot f'(x) + \dots \\ g(x + \Delta x) &= g(x) + \Delta x \cdot g'(x) + \dots \\ g(sx + s\Delta x) &= g(sx) + s\Delta x \cdot g'(sx) + \dots \end{aligned}$$

Substituting back into the first equation, it gives us the following

$$\begin{aligned} \Delta x \cdot f'(x) &= s\Delta x \cdot g'(sx) \\ f' &= sg'(sx) \end{aligned}$$

This gives us the following result;

$$\frac{df}{dx} = s \cdot \frac{dg}{dx} \Big|_{sx}$$

Therefore, we can multiply derivative by its scale s to allow for magnitude comparison across scales, with the scale adapted Harris detector;

$$\mathbf{M} = \sum_{x,y} w(x,y) \sigma^2 \begin{bmatrix} I_x^2(\sigma) & I_x(\sigma)I_y(\sigma) \\ I_x(\sigma)I_y(\sigma) & I_y^2(\sigma) \end{bmatrix}$$

We can apply the scale adapted detector at multiple scales - at each pixel we determine the scale with the largest response (at this scale, the region looks most like a corner). Something is detected as an interest point if it is a local maximum both along the scale dimension (most appropriate scale) and across space, therefore we can use (x, y, σ) for the interest point. This gives the following adapted algorithm;

1. for each scale σ
 - a. perform Gaussian smoothing with σ
 - b. calculate x and y derivatives of the smoothed image $I_x(\sigma)$ and $I_y(\sigma)$, respectively
 - c. at each pixel, compute the matrix \mathbf{M} (see above)
 - d. calculate the detector response $R = \lambda_1 \lambda_2 - k(\lambda_1 + \lambda_2)^2$
2. detect interest points which are local maxima across both scale and space, with R above a threshold

Laplacian of Gaussian (LoG)

This first performs Gaussian smoothing, followed by the Laplacian operator. The Laplacian is the sum of second derivatives, for a 2D image;

$$\Delta f = \nabla^2 = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

We have the following filter;

$$\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 1 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{\frac{\partial^2 f}{\partial x^2}} + \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{\frac{\partial^2 f}{\partial y^2}} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{\text{Laplacian filter}}$$

The second derivative is even more sensitive to noise than the first derivative.

The Laplacian of Gaussian filter is as follows (where h is the Gaussian kernel);

$$\Delta(f * h) = \frac{\partial^2(f * h)}{\partial x^2} + \frac{\partial^2(f * h)}{\partial y^2} = f * \left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} \right)$$

Since we know the formation of a 2D Gaussian, we can derive the following;

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = -\frac{1}{\pi\sigma^4} \left(1 - \frac{x^2 + y^2}{2\sigma^2} \right) e^{-\frac{x^2 + y^2}{2\sigma^2}}$$

LoG can also be a good interest point detector. We need to ensure it is comparable between scales (note that I_{xx} denotes the second derivative, we need to multiply by σ^2 rather than just σ ; without the multiplication we have the LoG response at scale σ);

$$\text{LoG}_{\text{norm}}(x, y, \sigma) = \sigma^2 (I_{xx}(x, y, \sigma) + I_{yy}(x, y, \sigma))$$

Difference of Gaussian (DoG)

This is defined as; the difference of Gaussians with different scales (note that I is the image, Lowe suggests $k = \sqrt{2}$);

$$\text{DoG}(x, y, \sigma) = I * G(k\sigma) - I * G(\sigma)$$

This approximates the normalised Laplacian of Gaussian, which provides convenience in calculating the response across different scales;

$$\text{DoG}(x, y, \sigma) \approx (k - 1)\sigma^2 \nabla^2 G(x, y, \sigma)$$

DoG filters are used in SIFT, which is a pipeline for detecting and describing interest points.

Interest Point Detectors

All of the interest point detectors are scale-invariant (other than the Harris detector, before normalisation);

detector	response
scale adapted Harris detector	$\lambda_1 \lambda_2 - k(\lambda_1 + \lambda_2)^2$
normalised Laplacian of Gaussian	$\sigma^2 (I_{xx}(x, y, \sigma) + I_{yy}(x, y, \sigma))$
difference of Gaussian	$I * G(k\sigma) - I * G(\sigma)$

All of these follow similar procedures, where we calculate the detector responses across values, providing a three-dimensional response map (x and y , as well as σ for scale). From here, we find local extrema across both space and scale.

Lecture 10 - Feature Description I

Pixel Intensity

A simple method for describing a feature is to use the intensity of a pixel. However, this is sensitive to absolute value (the intensity of the same point will change during different lighting conditions). This descriptor is not discriminative; many pixels can have the same intensity within an image. Furthermore, a single pixel cannot represent any local content (such as an edge, or an object).

Patch Intensities

This improves on a single pixel, as we can take some window representing the local pattern. If the images are of similar intensity range, and are roughly aligned, it can perform well. However, it will still be sensitive to absolute intensity values as before. Furthermore, it is not rotation-invariant; for example if we look at an image from a different rotation we can still tell what the image is.

Gradient Orientation

While gradient magnitude is sensitive to intensity changes, orientation isn't. However, this still isn't rotation-invariant.

Histogram

We can take the intensity histogram of a patch. This is done by binning the intensity values of a patch, and generating a histogram from these values. This has several benefits, where it is robust to rotation, as well as being robust to scaling (if the entire image is enlarged). However, it is still sensitive to intensity changes.

As such, we can consider combining the advantages of gradient orientation and the advantages of this histogram.

SIFT (Scale-invariant Feature Transform)

This is an algorithm for detecting and describing local features. It transforms an intensity image to a set of interest points, each of which is described by a feature vector.

1. detection of scale-space extrema

The first step is to search across scales and pixel locations, looking for interest points. This uses the difference of Gaussian filter, as previously encountered (in the SIFT paper, the different scales are multiplied by $\sqrt{2}$ continuously; $\sigma, \sqrt{2}\sigma, 2\sigma, 2\sqrt{2}\sigma, 4\sigma, \dots$);

$$\text{DoG}(x, y, \sigma) = I * G(k\sigma) - I * G(\sigma)$$

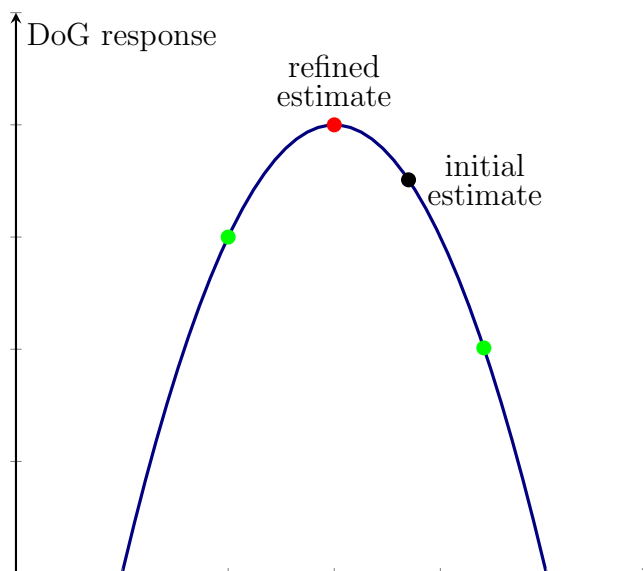
As before, a point is an interest point if it is a local extremum both along the scale dimension (most appropriate scale) and across space.

Note that extrema means both minima and maxima. If the input interest point is a very bright blob, DoG filter gives a negative response, and vice versa (a dark blob will lead to a positive response).

2. keypoint localisation

The initial version of SIFT simply locates the keypoints at the scale-space extrema (which we have from above). An improved version fits a model onto nearby data, improving accuracy for image matching.

A quadratic function can be [fitted](#) onto the DoG response of neighbouring pixels, and the location and scale of extremum for this (quadratic) function can be estimated.



Denote the DoG response as $D(x, y, \sigma)$ or $D(\mathbf{x})$, where $\mathbf{x} = (x, y, \sigma)^\top$ (containing both location and scale). By Taylor expansion we have (omitting higher order terms);

$$D(\mathbf{x} + \Delta\mathbf{x}) = D(\mathbf{x}) + \underbrace{\frac{\partial D}{\partial \mathbf{x}}^\top}_{(1)} \Delta\mathbf{x} + \frac{1}{2} \Delta\mathbf{x}^\top \underbrace{\frac{\partial^2 D}{\partial \mathbf{x}^2}}_{(2)} \Delta\mathbf{x}$$

(1) first derivatives can be estimated with finite difference

$$\frac{\partial D}{\partial \mathbf{x}} = \frac{D(x+1, y, \sigma) - D(x-1, y, \sigma)}{2}$$

(2) second derivatives / Hessian can also be estimated with finite difference

This gives us a quadratic function for $\Delta\mathbf{x}$, the shift to the refined estimate, from the initial estimate \mathbf{x} . Therefore, to find a refined extrema for the function;

$$\frac{\partial D(\mathbf{x} + \Delta\mathbf{x})}{\partial \Delta\mathbf{x}} = \frac{\partial D}{\partial \mathbf{x}} + \frac{\partial^2 D}{\partial \mathbf{x}^2} \Delta\mathbf{x} = 0 \Rightarrow \Delta\mathbf{x} = -\frac{\partial^2 D^{-1}}{\partial \mathbf{x}^2} \frac{\partial D}{\partial \mathbf{x}}$$

Shifting the initial estimate by $\Delta\mathbf{x}$ gives us the **refined estimate**.

Recall that \mathbf{x} is a 3D vector, and gives both location and scale. This means that we can get refined estimates for the location (sub-pixel accuracy), as well as a refined scale, which gives values between the scales of $\sigma, \sqrt{2}\sigma, 2\sigma, 2\sqrt{2}\sigma, 4\sigma, \dots$

3. orientation assignment

This step attempts to assign a consistent orientation to each keypoint, based on image properties. Feature descriptors can then be represented relative to this orientation, to achieve rotation-invariance. To do this, we need to know the orientation of the feature, and once we have this, we can perform sampling in a rotated coordinate system when we calculate features, giving us the same features.

We can calculate the gradient orientation for pixels in some neighbourhood and vote for the dominant orientation. We can create an orientation histogram with 36 bins (10° per bin), and each pixel in the neighbourhood votes for an orientation bin (weighted by the gradient magnitude) - the keypoint will be assigned an orientation, which would be the majority bin.

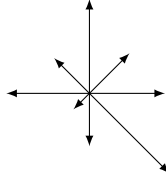
We now have the location (x, y) , scale σ , and dominant orientation θ . Using this, we can draw samples using a window size proportional to σ rotated by θ .

4. keypoint descriptor

Now that we have sample points, we need to describe the local image content. SIFT uses a histogram of gradient orientations.

A histogram of gradient orientations is made from the calculated gradient magnitudes and orientation for the sampling points. These orientations are calculated relative to the dominant orientation θ . In practice, subregions are used, with each subregion having 4×4 samples - each subregion will have an orientation histogram, which (when combined) describes what it looks like around a given keypoint.

For each subregion, we can construct an orientation histogram with 8 bins (each bin representing 45°). This can also be represented as 8 arrows (seen below), where the length of the arrow denotes the sum of the gradient magnitude along a given direction;



In Lowe's implementation, 16 subregions were used. This gives the descriptor a dimensionality of $128 = 16 \times 8$ (since each subregion has 8 bins); meaning each keypoint is described with a feature vector of 128 elements.

This descriptor is robust to rotation due to a consideration of a dominant orientation, similarly it is robust to scaling as we draw samples from a window proportional to scale. It is also robust to changes in illumination since we use gradient orientations for feature description.

Keypoint Matching

Keypoints between two images are matched by identifying the nearest neighbours (which is defined by the Euclidean distance of SIFT descriptors). Each keypoint in image A identifies its nearest neighbour in the database of keypoints for image B . However, we may not need to find the exact nearest neighbours for the sake of efficiency.

Suppose we find a keypoint (x, y) in A that corresponds to a keypoint (u, v) in B . We assume they are related with an affine transformation (**rotation, scaling, etc.** and **translation**);

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

However, with many pairs of corresponding keypoints, we can write the equation as;

$$\begin{array}{l} \text{keypoint 1} \\ \text{keypoint 2} \end{array} \begin{bmatrix} x_1 & y_1 & 0 & 0 & 1 & 0 \\ 0 & 0 & x_1 & y_1 & 0 & 1 \\ x_2 & y_2 & 0 & 0 & 1 & 0 \\ 0 & 0 & x_2 & y_2 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ t_x \\ t_y \end{bmatrix} = \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \vdots \end{bmatrix}$$

This can also be written as a linear system, where only \mathbf{m} is unknown;

$$\mathbf{A}\mathbf{m} = \mathbf{b}$$

However, we can obtain the least-square solution to the linear system with the **Moore-Penrose inverse**, minimising the squared difference $\|\mathbf{A}\mathbf{m} - \mathbf{b}\|^2$;

$$\mathbf{m} = (\mathbf{A}^\top \mathbf{A})^{-1} \mathbf{A}^\top \mathbf{b}$$

Once this is solved, we have the spatial transformation between the two images.

RANSAC

However, the squared difference can be sensitive to outliers (noise) - in our case, this would be points that are deemed to be corresponding but actually aren't. To ensure our solution is robust to the outliers we can use methods which consider outliers in the model fitting such as **RANSAC (Random Sample Consensus)**.

RANSAC aims to find the best fitting line to a given set of points. The algorithm to perform this is as follows (this is done repeatedly until we get a good fitting line);

1. randomly sample some points (the example shows 2 points in a 2D illustration)
2. fit a line along the sampled points
3. find the number of **inliers** within a threshold to the line
4. terminate if enough inliers have been found, or we have reached a certain number of iterations

Lecture 11 - Feature Description II

SIFT

As mentioned before, the SIFT descriptor at each interest point is formed by the concatenation of the gradient orientation histograms for 16 subregions (each having $128 = 16 \times 8$ values). Calculating magnitudes and orientations can be slow, especially if we want real-time performance (such as stitching on a phone camera, or for robotics).

One solution is to implement SIFT on FPGAs, which is faster than a CPU (once configured to perform this function). Another approach is to improve the algorithm, where we decompose the pipeline into several steps, where we evaluate how to improve each step. The pipeline for image matching is as follows;

1. feature detection (e.g. DoG)

There are a number of approaches for faster feature detection, including separable filtering, down-sampling the image, or cropping the Gaussian kernel.

2. feature description (e.g. SIFT)

Here we can consider whether we can evaluate the gradient orientation faster, or if we can use a different method of describing the local content.

3. interest point matching

Here we can approximate the nearest neighbour search, as well as use a lower-dimensional feature vector.

SURF (Speeded-Up Robust Features)

To accelerate computation, SURF only computes gradients along horizontal and vertical directions (rather than all of them) using **Haar wavelets**. Recall that the SIFT descriptor needs to calculate a histogram of gradient orientations, for each of the 16 subregions centred at an interest point.

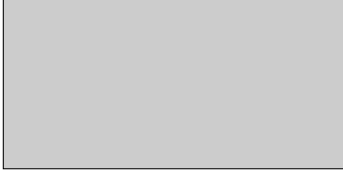
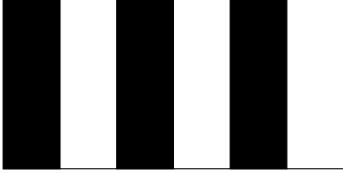

SURF still uses the same subregions and sample points, however we do not need to calculate gradients for arbitrary orientations. This applies two very simple filters (Haar wavelets, of size 2×2 or 4×4), d_y and d_x onto the sample points;

$$\begin{array}{cc} d_x & d_y \\ \begin{array}{|c|c|} \hline \text{black} & \text{white} \\ \hline -1 & +1 \\ \hline \end{array} & \begin{array}{|c|} \hline \text{black} \\ \hline -1 \\ \hline \text{white} \\ \hline +1 \\ \hline \end{array} \end{array}$$

Summing the pixel intensities with a weight of 1 or -1 is very fast, since it is either an addition or subtraction. For each subregion, we sum up the Haar wavelet responses over the sample points - the descriptor for a given subregion is defined by 4 elements (the sum of values and the sum of absolute values);

$$\left(\sum d_x, \sum d_y, \sum |d_x|, \sum |d_y| \right)$$

We can visualise the meanings of these descriptors as follows. If all four values are low, then the subregion is mostly homogeneous. If there are patterns similar to zebra stripes, a change from black to white, and then from white back to black would give a sum of 0 when we look at d_x (since they cancel out). However, if we look at $|d_x|$, the value doubles (since they will not cancel).

	homogeneous region	zebra pattern	gradually increasing intensities
			
$\sum d_x$	low	low	high
$\sum d_x $	low	high	high
$\sum d_y$	low	low	low
$\sum d_y $	low	low	low

The SURF descriptor still uses $16 = 4 \times 4$ subregions, each with 4 elements. This leads to a dimensionality of 64 for each interest point. Compared to SIFT, it is approximately 5 times faster due to the use of simple Haar wavelets.

If we know the ground truth (in this case, the transformation between image A and B), we can perform quantitative analysis on the performance of algorithms. The paper shows that SURF performs as well as SIFT in matching.

BRIEF (Binary Robust Independent Elementary Features)

In both SIFT and SURF, each dimension is a floating-point number (represented in 4 bytes), and our comparison of feature vectors are done with Euclidean distance. Ideally, we'd want to make feature description and matching even faster.

We can further shorten the descriptor by performing quantisation (converting a continuous number into a discrete number, hence 8 bits for the range $[0, 255]$), or even performing binarization (converting each into a binary number).

Haar wavelets in SURF compare a local region toe another, giving a difference (which is a float). In BRIEF, comparing points p, q will result in a binary value as an output (1 if q is brighter than p);

$$\tau(p, q) = \begin{cases} 1 & \text{if } I(p) < I(q) \\ 0 & \text{otherwise} \end{cases}$$

BRIEF randomly sample n_d pairs of points for binary tests; this random pattern is only determined once, and the same pattern will be applied to all interest points. If we set $n_d = 256$, we will perform 256 tests in total, each giving us a single bit (with a total of 32 bytes; SIFT uses 128 bytes, and SURF uses 64 bytes). The descriptor is then a n_d -dimensional bitstring.

Note that this is also fast to compute, as we can perform bit shifting (\ll n denotes a bit shift by n places). For example, if we performed 8 binary tests, we could do the following;

```

1 descriptor = ((I(p1) < I(q1)) << 7) + ((I(p2) < I(q2)) << 6)
2             + ((I(p3) < I(q3)) << 5) + ((I(p4) < I(q4)) << 4)
3             + ((I(p5) < I(q5)) << 3) + ((I(p6) < I(q6)) << 2)
4             + ((I(p7) < I(q7)) << 1) + ((I(p8) < I(q8)) << 0)
```

Not only does it use less memory, the computation only compares two numbers, without calculating gradient orientation like in SIFT, or the intensity difference like in SURF. We also avoid calculating the Euclidean distance, instead we use the Hamming distance. This allows for an efficient calculation by performing a bitwise XOR between two descriptors, and then taking a bit count - both of which can be done extremely quickly on modern CPUs in very few instructions. For example;

10001001 XOR 11000011 = 01001010, thus a Hamming distance of 3

However; BRIEF isn't rotation-invariant nor scale-invariant. It assumes that the images taken are from a camera where the movement **only involves translation**. By combining all of the above, BRIEF is approximately 40 times faster than SURF, hence being around 200 times faster than SIFT. Furthermore, if there is no rotation or scaling, BRIEF is comparable to SURF in terms of matching accuracy.

Image Similarity

Image similarity can be defined by the number of matched point. Intuitively, if two images are similar, there should be a higher number of interest points that can be matched together. See *VisualRank*.

HOG (Histograms of Oriented Gradient)

We can also extend the idea of feature descriptors (which describe local content centred at a point) to describe the feature of a large region, or even a whole image.

While HOG and SIFT both use gradient orientation histograms for feature descriptions, HOG differs in that it describes features for a large image region (rather than a point). HOG divides a large region into a (dense) grid of cells, describes each cell, and concatenates the local descriptions to form a global description.

In the example, we divide the image into equally spaced cells (each of $8 \times 8 \text{ pixels}$). 4 of these cells form a block; and we calculate the gradient orientation histogram for a block (thus forming a description of that block). Note that the description vector \mathbf{v} , which is the concatenation of the 4 histograms (from each cell), is normalised to form a locally normalised descriptor - note the inclusion of a small value ϵ to improve numerical stability. This is done to prevent issues between differing brightnesses;

$$\mathbf{v}_{\text{norm}} = \frac{\mathbf{v}}{\sqrt{\|\mathbf{v}\|_2^2 + \epsilon^2}}$$

The block is then shifted along horizontally by one cell (8 pixels in our case), note that there will be some overlap between the blocks. The HOG descriptor is then formed by concatenating all the normalised local descriptors for all blocks.

We can use HOG to perform feature extraction from an input image to a feature representation. Feature extraction transforms an input image into low-dimensional vectors for easier comparison or matching. A classifier (trained with a dataset consisting of images, and often labels) can be used to give an output label to the feature representation.

Lecture 12 - Image Classification I

This lecture starts with some motivation on image classification. The ImageNet project aims to collect 80,000 classes, with each class having between 500 to 1,000 images. The ILSVRC uses a subset of this, with only 1,000 (simpler) classes used; there are also no overlap between classes, since pairs of classes cannot be ancestors of each other. In ImageNet classes, all classes are organised as a tree. Below the root node, ImageNet has 12 subtrees (with each subtree expanding further into many classes);

- mammal
- bird
- fish
- reptile
- amphibian
- vehicle

- furniture
- musical instrument
- geological formation
- tool
- flower
- fruit

These classes are defined with concepts from our natural language defined by WordNet (a lexical database of English). WordNet groups nouns into sets of cognitive synonyms (synsets), each expressing a **distinct** concept - these synsets will form the leaves of the tree, with the concepts already being organised as a tree.

Classifier

This lecture is literally just going over stuff in **CO395**. For example, drawing a line as a classifier for data points in 2D space. When labels are not available for training data, we perform unsupervised learning; a common technique is to perform clustering for classification.

MNIST Example

The example we go through will use the handwritten digit recognition dataset from MNIST - there are 60,000 training samples and another 10,000 reserved for testing. Each sample is a 28×28 image, which is labelled with a class from 0 to 9.

However, we need to perform some pre-processing. We need to first find where the digit is in a large image, normalise the size of the digit to 28×28 , as well as normalising the location (place the mass centre in the image centre). Sometimes slant correction may be performed, shifting each row to force the principal axis to become vertical. MNIST provides both datasets.

There are two ways to extract features, either with hand-crafted techniques (such as HOG or pixel intensities), or learnt features such as CNNs. After which, we perform classification, with classifiers like KNN (once again, see **CO395**).

To define the neighbours, we need a distance metric between two data points \mathbf{x} and \mathbf{y} ;

- **Euclidean distance**

$$D(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2}$$

In our case, we have $784 = 28 \times 28$ dimensions. If each dimension has different scales, it can be better to normalise the feature vector (for example, weights and heights have different value ranges depending on the units). An example is to normalise each dimension to a Gaussian distribution $\mathcal{N}(0, 1)$, allowing for the dimensions to be treated fairly.

- **cosine distance**

$$D(\mathbf{x}, \mathbf{y}) = \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\| \|\mathbf{y}\|} = \frac{x_1 y_1 + \dots + x_n y_n}{\sqrt{\sum_{i=1}^n x_i^2} \sqrt{\sum_{i=1}^n y_i^2}}$$

- **generalised**

The Manhattan distance uses the ℓ_1 -norm;

$$D_1(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^n |x_i - y_i|$$

A more general form is the ℓ_p -norm (note that the Euclidean distance is the ℓ_2 -norm);

$$D_p(\mathbf{x}, \mathbf{y}) = \left(\sum_{i=1}^n |x_i - y_i|^p \right)^{\frac{1}{p}}$$

Using $K = 1$, it can already work quite well. However, we may still have some inaccuracies, where a 7 may be classified as a 3, a 7 being classified as 0, or an 8 being classified as a 5, and so on.

The lecture then goes over hyperparameter tuning for K , finding that $K = 3$ is ideal for the MNIST dataset, with a 2.4% error rate. KNN has no training step, is simple (but quite effective), and can also handle multi-class classification easily. However, it is expensive to use KNN in terms of both storage of training data and searching (for an extremely large set, the computational cost can become very expensive).

Suppose we have N training images, and M test images. While we have no training time, the computational cost to classify all M test images is $O(MN)$.

However, we are typically fine with slow training (since we can perform the training on high-powered hardware), but we want the test time to be fast (since it may be deployed on weaker hardware).

Features

For simple pre-processed images (like in our example), it might be fine to use the 28×28 pixel intensities as the feature vector. However, generally it may not be ideal to use pixel intensities, since it isn't invariant to scale nor rotation (and is generally a much higher dimension).

Lecture 13 - Image Classification II

Linear Classifier

A linear classifier, in 2D, has the form;

$$w_1x_1 + w_2x_2 + b = 0$$

The rule of the classifier assigns a class c on x based on the following;

$$c = \begin{cases} 1 & \text{if } w_1x_1 + w_2x_2 + b \geq 0 \\ -1 & \text{otherwise} \end{cases}$$

For a linear classifier, once we know \mathbf{w} and b , we can discard the training data (unlike in KNN). This is much faster in test time.

For a more general case, the linear model is as follows (where we have the **weights** / **normal**, **data** / **feature**, and **bias**);

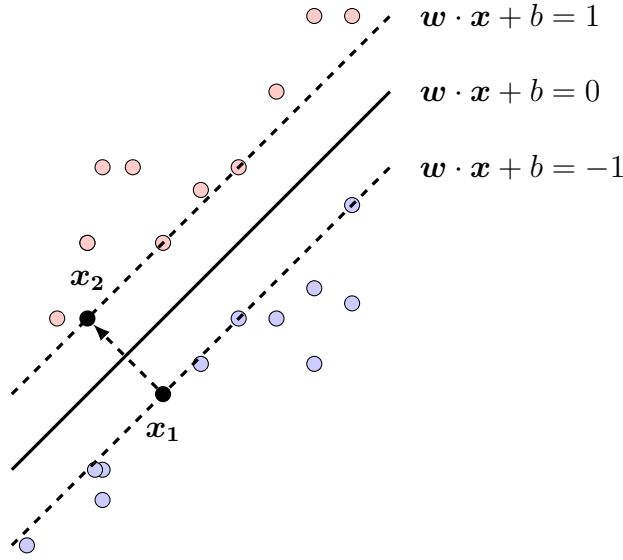
$$\mathbf{w} \cdot \mathbf{x} + b = 0$$

Similarly, we have the same rule to assign class c ;

$$c = \begin{cases} 1 & \mathbf{w} \cdot \mathbf{x} + b \geq 0 \\ -1 & \text{otherwise} \end{cases}$$

Linear SVM (Support Vector Machine)

To determine the maximum margin hyperplane, we only need to look at the inner most points, which are the **support vectors**. We would like the support vectors to fulfil the equations $\mathbf{w} \cdot \mathbf{x} + b = -1$ (blue points) and $= 1$ (red points), to allow other points to be easily classified. We want to maximise the margin between the two dashed lines.



We want the margin between the two following lines to be as large as possible, and we can also derive the distance as follows;

$$\begin{aligned}
 & \mathbf{w} \cdot \mathbf{x} + b = 1 \\
 & \mathbf{w} \cdot \mathbf{x} + b = -1 \\
 & \mathbf{w} \cdot \mathbf{x}_1 + b = -1 && \text{let this be a point on the } -1 \text{ plane} \\
 & \mathbf{w} \cdot (\mathbf{x}_1 + m\mathbf{n}) + b = 1 && \text{move the point along the normal by the size of the margin to } \mathbf{x}_2 \\
 & m\mathbf{w} \cdot \mathbf{n} = 2 && \text{from the previous two equations} \\
 & \mathbf{n} = \frac{\mathbf{w}}{\|\mathbf{w}\|}
 \end{aligned}$$

SVM then aims to solve optimise the following;

$$\max_{\mathbf{w}, b} \frac{2}{\|\mathbf{w}\|}$$

This is subject to the following constraints, for all $i = 1, \dots, N$ (note that y_i denotes the label of the point);

$$\begin{aligned}
 & \mathbf{w} \cdot \mathbf{x}_i + b \geq 1 && \text{if } y_i = 1 \\
 & \mathbf{w} \cdot \mathbf{x}_i + b \leq -1 && \text{if } y_i = -1
 \end{aligned}$$

However, we've assumed that all the points are linearly separable, which isn't always the case. In this case, we cannot solve the optimisation problem, since the constraints require the data to be linearly separable. This can be done with slack variables, $\xi_i \geq 0$ (for each data point). This formulates the optimisation problem as

$$\min_{\mathbf{w}, b} \|\mathbf{w}\|^2 + C \sum_{i=1}^N \xi_i$$

Subject to the following, for all data points ($\forall i$);

- $\xi_i \geq 0$
- $y_i(\mathbf{w} \cdot \mathbf{x}_i + b) \geq 1 - \xi_i$

Note that ξ_i will be 0 when the point is correctly classified or $0 < \xi_i \leq 1$ for points within the margin (but on the correct side of the hyperplane), otherwise it represents the distance from the margin. In the above, C is a regularisation term (a small C will allow constraints to easily be fulfilled, whereas a large C would make constraints difficult to ignore - typically leading to a tighter margin).

Optimisation

Note that we can formulate the slack variable as;

$$\xi_i = \underbrace{\max(0, 1 - y_i(\mathbf{w} \cdot \mathbf{x}_i + b))}_{\text{hinge loss}}$$

Therefore, we aim to optimise the following function;

$$\min_{\mathbf{w}, b} \|\mathbf{w}\|^2 + C \sum_{i=1}^N \max(0, 1 - y_i(\mathbf{w} \cdot \mathbf{x}_i + b))$$

However, note that both of these functions (a quadratic term and a hinge loss function) are **convex** functions (hence we know there is a minimum, and that a local minimum is also a global minimum). Note that the non-negative weighted sum of two convex functions is also a convex function. A convex function $f : X \rightarrow R$ satisfies the following;

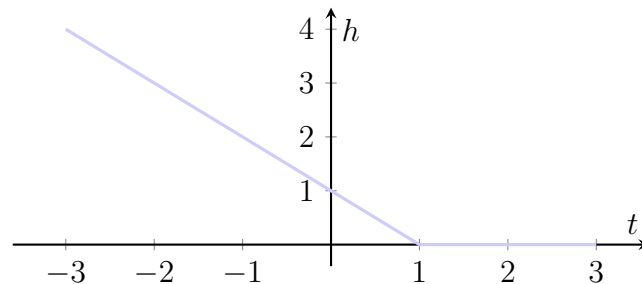
$$\forall x_1, x_2 \in R, \forall t \in [0, 1] [f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2)]$$

A line joining $(x_1, f(x_1))$ and $(x_2, f(x_2))$ will lie **above** the function curve. There are two approaches to optimise this;

- **gradient descent**

used in large scale-optimisation

Note that the hinge loss function looks like the following;



While it is not differentiable, we can use a more generalised concept for a gradient, the **subgradient**;

$$\nabla_{\mathbf{w}} h = \begin{cases} -y_i \mathbf{x}_i & \text{if } y_i(\mathbf{w} \cdot \mathbf{x}_i + b) < 1 \\ 0 & \text{otherwise} \end{cases}$$

This would be similar for $\nabla_b h$, however, we mainly focus on \mathbf{w} for our derivation. For each iteration, we move along the negative of the gradient by some step length η ;

$$\begin{aligned} \mathbf{w}^{(k+1)} &= \mathbf{w}^{(k)} - \eta \nabla_{\mathbf{w}} \left(\|\mathbf{w}\|^2 + C \sum_{i=1}^N \max(0, 1 - y_i(\mathbf{w} \cdot \mathbf{x}_i + b)) \right) \\ &= \mathbf{w}^{(k)} - \eta \left(2\mathbf{w}^{(k)} + C \sum_{i=1}^N \nabla_{\mathbf{w}} h \right) \end{aligned}$$

- **Lagrangian duality** (solves the dual problem)

often used for SVM libraries

This is beyond the scope of the course, but works by solving the **dual** problem (given the **primal** problem). See **CO343** for more details about duality.

In the dual problem, the result of many of the a_i are 0, except for the points located on the margin (the support vectors). The solution of the primal problem is then given by;

$$\mathbf{w} = \sum_{i=1}^N a_i y_i \mathbf{x}_i$$

b can also be estimated with the support vectors by substituting back into the equation.

Classification

At test time, classification can be done according to;

$$c = \begin{cases} 1 & \text{if } f(\mathbf{x}) = \mathbf{w} \cdot \mathbf{x} + b \geq 0 \\ -1 & \text{otherwise} \end{cases}$$

This works quite well if the data can be linearly separable. However, if the points are not linearly separable in the **original** feature space, it may be easier to transform the features. Consider a dataset which has two classes, the first being a circle of points centered around some point \mathbf{x}_c , and another class being a ring of points around the first class. Using the transformation $\mathbf{Phi}(\mathbf{x}) = (\mathbf{x} - \mathbf{x}_c)^2$, the first class would have a lower value (since it is closer to \mathbf{x}_c , which is then squared) after the transformation, whereas the second class would have a much higher value, thus separating the points.

Therefore, if we transform the feature vector using a function Φ , the classifier also must change;

$$f(\mathbf{x}) = \underbrace{\left(\sum_i a_i y_i \Phi(\mathbf{x}_i) \right)}_{\text{transformed } \mathbf{w}} \cdot \Phi \mathbf{x} + b$$

We can also define the kernel (different from image filtering) as;

$$\mathbf{k}(\mathbf{x}_i, \mathbf{x}_j) = \Phi(\mathbf{x}_i) \cdot \Phi(\mathbf{x}_j)$$

The classifier (which is now a non-linear function) only contains the kernel;

$$f(\mathbf{x}) = \sum_i a_i y_i \mathbf{k}(\mathbf{x}_i, \mathbf{x}) + b$$

The common kernels used for SVM are;

- **linear kernel**

$$\mathbf{k}(\mathbf{x}_i, \mathbf{x}_j) = \mathbf{x}_i \cdot \mathbf{x}_j$$

- **polynomial kernel**

$$\mathbf{k}(\mathbf{x}_i, \mathbf{x}_j) = (\mathbf{x}_i \cdot \mathbf{x}_j)^d \text{ or } (1 + \mathbf{x}_i \cdot \mathbf{x}_j)^d$$

- **Gaussian kernel / radial basis function (RBF) kernel**

$$\mathbf{k}(\mathbf{x}_i, \mathbf{x}_j) = e^{-\frac{\|\mathbf{x}_i - \mathbf{x}_j\|^2}{2\sigma^2}}$$

Note that we only have binary classification at the moment. For multi-class classification, we can take several approaches (considering the MNIST digit recognition example);

- **one versus rest**

In this, we train a classifier for each of the 10 digits, the first classifies between a digit that is 1 and not 1, the second being a classifier between 2 and not 2, and so on. The classifier which produces the highest response at test time determines the result;

$$c = \underset{k=1, \dots, K}{\operatorname{argmax}} f_k(\mathbf{x})$$

- **one versus one**

In contrast to before, we have a classifier for each pairing of digits - for example, we'd have a classifier for 1 versus 2, 1 versus 3, 2 versus 3, and so on. At test time, each classifier votes for the digit, and the majority vote wins. The number of classifiers we need for K classes is;

$$\frac{K(K-1)}{2}$$