CO240 - Models of Computation — Tutorial Sheets

Tutorial 1 - Expressions

1. d Consider the **big-step** operational semantics for the language *SimpleExp* given in the lectures. Find a number n such that

$$(4 + 1) + (2 + 2) \Downarrow n$$

Give the full derivation tree.

$$\text{(B-ADD)}\ \dfrac{\text{(B-ADD)}\ \dfrac{\text{(B-NUM)}\ \dfrac{}{4 \Downarrow 4} \qquad \text{(B-NUM)}\ \dfrac{}{1 \Downarrow 1}}{(4 + 1) \Downarrow 5} \qquad \text{(B-ADD)}\ \dfrac{\text{(B-NUM)}\ \dfrac{}{2 \Downarrow 2} \qquad \text{(B-NUM)}\ \dfrac{}{2 \Downarrow 2}}{(2 + 2) \Downarrow 2}}{(4 + 1) + (2 + 2) \Downarrow 9}$$

2. The big-step operation semantics for *SimpleExp* was only given for addition. Extend it to include *multiplication*. Give a proof that $((3 + 2) \times (1 + 4)) \Downarrow 25$

To do this, we need to add an additional rule as follows;

$$\text{(B-MUL)}\ \dfrac{E_1 \Downarrow n_1 \qquad E_2 \Downarrow n_2}{E_1 \times E_2 \Downarrow n_3}\ n_3 = n_1 \times n_2$$

Hence we can do the following;

$$\text{(B-MUL)}\ \dfrac{\text{(B-ADD)}\ \dfrac{\text{(B-NUM)}\ \dfrac{}{3 \Downarrow 3} \qquad \text{(B-NUM)}\ \dfrac{}{2 \Downarrow 2}}{(3 + 2) \Downarrow 5} \qquad \text{(B-ADD)}\ \dfrac{\text{(B-NUM)}\ \dfrac{}{1 \Downarrow 1} \qquad \text{(B-NUM)}\ \dfrac{}{4 \Downarrow 4}}{(1 + 4) \Downarrow 5}}{((3 + 2) \times (1 + 4)) \Downarrow 25}$$

3. Extend the **big-step** semantics further to include *subtraction*. Remember that the numbers in the syntax of the language are $0, 1, 2, \dots$ (no negative numbers).

How is an expression such as $(3 - 7)$ handled in your semantics? Have you made any arbitrary decisions about this? If so, what other options were available?

Note that this question has multiple valid options; we can either introduce a `NaN` concept, representing an "invalid" operation, which has to be propagated in all rules, or we could have it be some value. The latter can lead to ambiguity, because if we had $(3 - 7) \Downarrow 0$, and also $(4 - 7) \Downarrow 0$, we may unexpected results.

4. Recall the **small-step** operational semantics of *SimpleExp*.

   (a) Give the full derivation of the first step of evaluation of $((1 + 2) + (4 + 3))$ - give the derivation tree of the step (for some expression E);

$$((1 + 2) + (4 + 3)) \to E$$

   For the first step, we have the following;

$$\text{(S-LEFT)}\ \dfrac{\text{(S-ADD)}\ \dfrac{}{(1 + 2) \to 3}}{((1 + 2) + (4 + 3)) \to (3 + (4 + 3))}$$

   (b) Write down all the steps of evaluation needed to reduce the above expression to 10. Give the full derivation for each of these steps.

   Note that the **evaluation path** is;

$$((1 + 2) + (4 + 3)) \to (3 + (4 + 3)) \to (3 + 7) \to 10$$

The derivation tree for each step is as follows;

$$\text{(S-RIGHT)}\ \dfrac{\text{(S-ADD)}\ \dfrac{}{(4+3) \to 7}}{(3+(4+3)) \to (3+7)}$$

Followed by;

$$\text{(S-ADD)}\ \dfrac{}{(3+7) \to 10}$$

5. Here is the abstract syntax for a simple language *Bool* of boolean expressions:

$$B \in \mathit{Bool} ::= \texttt{true} \mid \texttt{false} \mid B\&B \mid \neg B \mid \texttt{if}\ B\ \texttt{then}\ B\ \texttt{else}\ B$$

Intuitively, every expression evaluates to either `true` or `false`.

(a) Give a **small-step** operational semantics for *Bool*.

$$\frac{B_1 \to B_1'}{B_1\&B_2 \to B_1'\&B_2}$$

$$\frac{B_2 \to B_2'}{\texttt{true}\&B_2 \to \texttt{true}\&B_2'}$$

$$\frac{B_2 \to B_2'}{\texttt{false}\&B_2 \to \texttt{false}\&B_2'}$$

$$\frac{}{\texttt{true}\&\texttt{true} \to \texttt{true}}$$

$$\frac{}{\texttt{true}\&\texttt{false} \to \texttt{false}}$$

$$\frac{}{\texttt{false}\&\texttt{true} \to \texttt{false}}$$

$$\frac{}{\texttt{false}\&\texttt{false} \to \texttt{false}}$$

$$\frac{B \to B'}{\neg B \to \neg B'}$$

$$\frac{}{\neg\texttt{true} \to \texttt{false}}$$

$$\frac{}{\neg\texttt{false} \to \texttt{true}}$$

$$\frac{B_1 \to B_1'}{\texttt{if}\ B_1\ \texttt{then}\ B_2\ \texttt{else}\ B_3}$$

$$\frac{}{\texttt{if true then}\ B_2\ \texttt{else}\ B_3 \to B_2}$$

$$\frac{}{\texttt{if false then}\ B_2\ \texttt{else}\ B_3 \to B_3}$$

Note that these are all evaluated right-to-left.

(b) Write down all the steps of evaluation needed to reduce the following expression to a result:

$$\begin{aligned}
&\neg(\texttt{if (false\&true) then (if true then (false\&true) else false) else }\neg\texttt{true})\\
\to\ &\neg(\texttt{if false then (if true then (false\&true) else false) else }\neg\texttt{true})\\
\to\ &\neg(\neg\texttt{true})\\
\to\ &\neg\texttt{false}\\
\to\ &\texttt{true}
\end{aligned}$$

6. The syntax of *SimpleExp* is extended with a new operator ?, as follows;

$$E \in \mathit{SimpleExp} ::= \ldots \mid (E?E)$$

This operator allows the implementation to choose to give the result of E_1, or E_2, when given $E_1?E_2$.

(a) Extend the **big-step** operational semantics with rules for ? that capture this meaning.

$$\text{(B-CHOICE-1)}\ \frac{E_1 \Downarrow n_1}{E_1?E_2 \Downarrow n_1} \qquad \text{(B-CHOICE-2)}\ \frac{E_2 \Downarrow n_2}{E_1?E_2 \Downarrow n_2}$$

(b) For what values of n does $(0?1) + (2?3) \Downarrow n$?

$$\text{(B-ADD)}\ \dfrac{\text{(B-CHOICE-1)}\ \dfrac{\text{(B-NUM)}\ \dfrac{}{0 \Downarrow 0}}{(0?1) \Downarrow 0} \quad \text{(B-CHOICE-1)}\ \dfrac{\text{(B-NUM)}\ \dfrac{}{2 \Downarrow 2}}{(2?3) \Downarrow 2}}{(0?1) + (2?3) \Downarrow 2}$$

$$\text{(B-ADD)}\ \dfrac{\text{(B-CHOICE-1)}\ \dfrac{\text{(B-NUM)}\ \dfrac{}{0 \Downarrow 0}}{(0?1) \Downarrow 0} \quad \text{(B-CHOICE-2)}\ \dfrac{\text{(B-NUM)}\ \dfrac{}{3 \Downarrow 3}}{(2?3) \Downarrow 3}}{(0?1) + (2?3) \Downarrow 3}$$

$$\text{(B-ADD)}\ \dfrac{\text{(B-CHOICE-2)}\ \dfrac{\text{(B-NUM)}\ \dfrac{}{1 \Downarrow 1}}{(0?1) \Downarrow 1} \quad \text{(B-CHOICE-1)}\ \dfrac{\text{(B-NUM)}\ \dfrac{}{2 \Downarrow 2}}{(2?3) \Downarrow 2}}{(0?1) + (2?3) \Downarrow 3}$$

$$\text{(B-ADD)}\ \dfrac{\text{(B-CHOICE-2)}\ \dfrac{\text{(B-NUM)}\ \dfrac{}{1 \Downarrow 1}}{(0?1) \Downarrow 1} \quad \text{(B-CHOICE-2)}\ \dfrac{\text{(B-NUM)}\ \dfrac{}{3 \Downarrow 3}}{(2?3) \Downarrow 3}}{(0?1) + (2?3) \Downarrow 4}$$

(c) Is the semantics deterministic? Is it total?

It is not deterministic as we have $0?1 \Downarrow 0$, as well as $0?1 \Downarrow 1$ - but $0 \neq 1$. It is total as it is applies to every expression (for something to be total, we need some number n for every expression E such that $E \Downarrow n$).

7. (a) Extend the **small-step** semantics for *SimpleExp* to handle the ? operator by adding appropriate derivation rules for $\rightarrow$.

$$\text{(S-CHOICE-1)}\ \frac{}{E_1?E_2 \rightarrow E_1} \qquad \text{(S-CHOICE-2)}\ \frac{}{E_1?E_2 \rightarrow E_2}$$

(b) Give all possible derivations of the first step of evaluation of $(0?1) + (2?3)$.

$$\text{(S-LEFT)}\ \dfrac{\text{(S-CHOICE-1)}\ \dfrac{}{0?1 \rightarrow 0}}{(0?1) + (2?3) \rightarrow 0 + (2?3)} \qquad \text{(S-LEFT)}\ \dfrac{\text{(S-CHOICE-2)}\ \dfrac{}{0?1 \rightarrow 1}}{(0?1) + (2?3) \rightarrow 1 + (2?3)}$$

(c) Give all of the possible evaluation paths for $(0?1) + (2?3)$.

$$\begin{aligned}
(0?1) + (2?3) &\rightarrow 0 + (2?3) \rightarrow 0 + 2 \rightarrow 2 \\
(0?1) + (2?3) &\rightarrow 0 + (2?3) \rightarrow 0 + 3 \rightarrow 3 \\
(0?1) + (2?3) &\rightarrow 1 + (2?3) \rightarrow 1 + 2 \rightarrow 3 \\
(0?1) + (2?3) &\rightarrow 1 + (2?3) \rightarrow 1 + 3 \rightarrow 4
\end{aligned}$$

(d) Is the semantics confluent?

We've shown $(0?1) + (2?3) \rightarrow^* 2$ and also $(0?1) + (2?3) \rightarrow^* 3$. Therefore, for the semantics to be confluent, there must be some E' such that $2 \rightarrow^* E'$ and $3 \rightarrow^* E'$ - however, since they are both in normal forms, they can only evaluate to themselves. $2 \neq 3$, hence it is not confluent.

(e) Is the semantics normalising?

Yes, there are no infinite sequences of expressions, hence any evaluation path will eventually reach a normal form.

8. Suppose that instead of the *SimpleExp* small-step rule (S-RIGHT), we had the following;

$$\text{(S-RIGHT}')\ \frac{E_2 \rightarrow E_2'}{(E_1 + E_2) \rightarrow (E_1 + E_2')}$$

(a) Given an evaluation path using the S-RIGHT rule, is it also an evaluation path using the S-RIGHT′ rule?

Yes, as the original rule constrained E_1 to be in a normal form, but the new rule doesn't. This means that the new rule covers all the cases of the original rule.

(b) Find an expression that has an evaluation path using the S-RIGHT′ rule that it did not have with the S-RIGHT rule.

$$(0+1)+(2+3) \to (0+1)+5 \to 1+5 \to 6$$

(c) Is $\to$ deterministic?

No, starting with $(0+1)+(2+3)$, we can go to either $1+(2+3)$ S-LEFT, or $(0+1)+5$ with S-RIGHT′ - however the two expressions are not equal.

(d) Is $\to$ confluent?

Yes, the rule allows for different evaluation order, but doesn't change the result of the evaluation.