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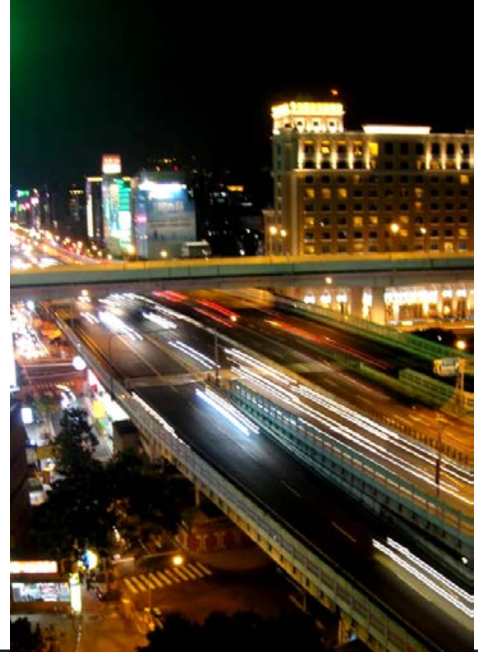

# 電路學 Circuit Theory

## Lecture 3

# Responses of RL, RC, and RLC Circuits


Week 9, Fall 2019

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# Contents

## Lecture 3:
## Responses of RL, RC, and RLC Circuits



# Contents

## 3.1 Capacitors and Inductors

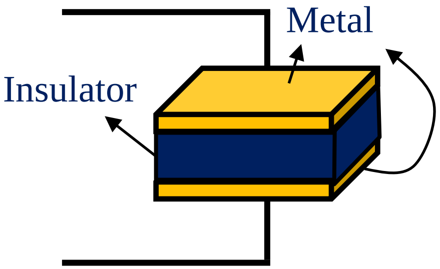
# Two New Passive Components: C & L

## Resistors:

- They dissipate energy

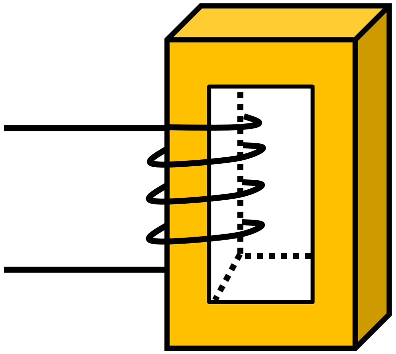
## Capacitors:

- They store energy
- The energy is stored at electric field
- Whenever electrical conductors are separated by a insulator, the capacitance occurs

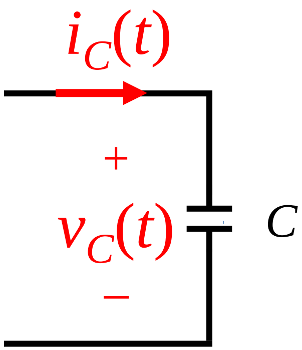
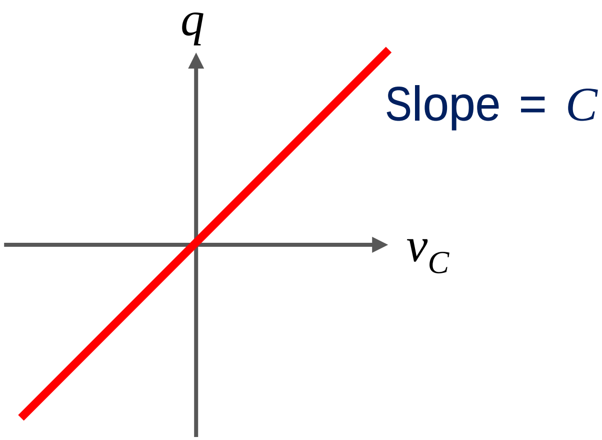



## Inductors

- They store energy
- The energy is stored at magnetic field
- Inductance results from a conductor linking a magnetic field



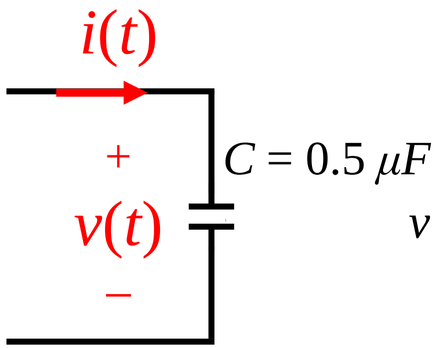
# Component Model of a Capacitor

| Symbol | Model |
| --- | --- |
|   |   |

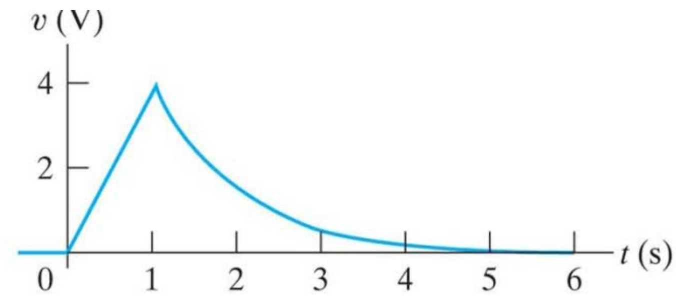
- $q = C \times v_C$;  where  $q$: charge, or "electric flux", in coulombs
  $C$: capacitance in F (Farad)
- On the other hand, $q(t) = \int_{-\infty}^{t} i_C(\tau)\,d\tau$

1. If $v_C$ is given, then $i_C(t) = \dfrac{dq}{dt} = C\dfrac{dv_C(t)}{dt}$
2. If $i_C$ is given, then $v_C(t) = \dfrac{q}{C} = \dfrac{1}{C}\int_{-\infty}^{t} i_C(\tau)\,d\tau = v_C(t_0) + \dfrac{1}{C}\int_{t_0}^{t} i_C(\tau)\,d\tau$

# EX 3.1 Circuit Variables of a Capacitor (1/5)




$$v(t)=\begin{cases}0 & t\le 0\text{ s}\\ 4t\text{ V} & 0<t\le 1\text{ s}\\ 4e^{-(t-1)}\text{ V} & t>1\text{ s}\end{cases}$$




1. Find the expressions for the current, power, and energy on the capacitor
2. Determine the interval of time when energy is being stored and delivered in the capacitor, respectively

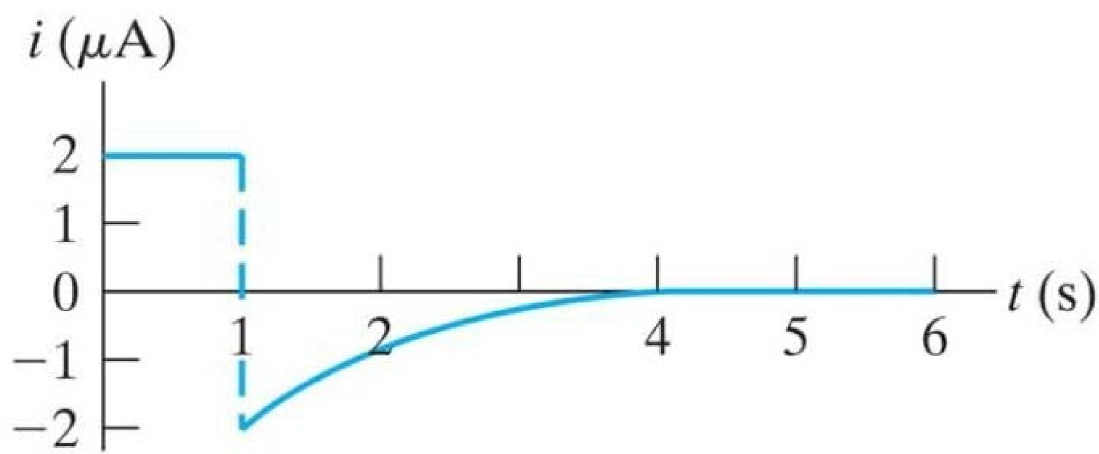


J. Nilsson and S. Riedel, *Electric Circuits*, 9th Edition, Pearson, NJ, 2011

# EX 3.1 Circuit Variables of a Capacitor (2/5)

- The current run through the capacitor: $i(t) = C\dfrac{dv(t)}{dt}$

$$\Rightarrow\; i(t) = \begin{cases} & t \le 0\ \text{s} \\ & 0 < t \le 1\ \text{s} \\ & t > 1\ \text{s} \end{cases}$$

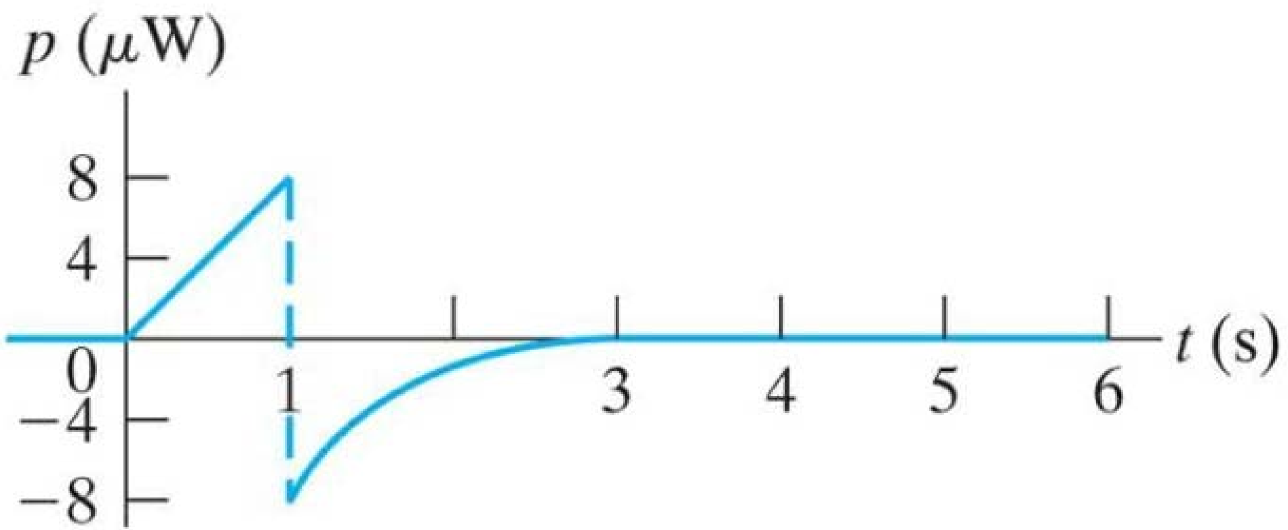



J. Nilsson and S. Riedel, *Electric Circuits*, 9th Edition, Pearson, NJ, 2011

# EX 3.1 Circuit Variables of a Capacitor (3/5)

- The power of the capacitor: $p(t)=v(t)i(t)$

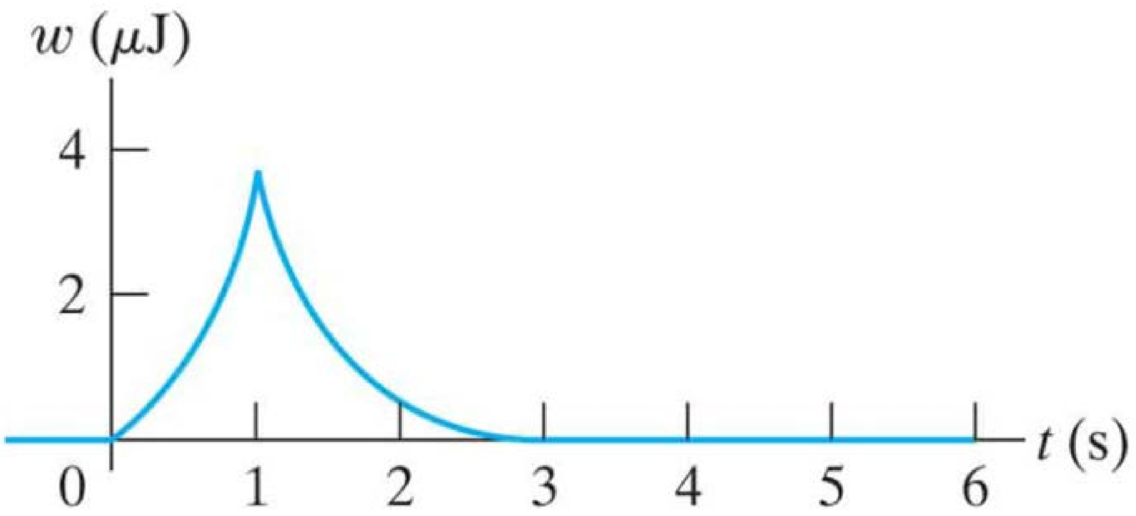
$$\Rightarrow p(t)=\begin{cases} & t\le 0\text{ s} \\ & 0<t\le 1\text{ s} \\ & t>1\text{ s} \end{cases}$$



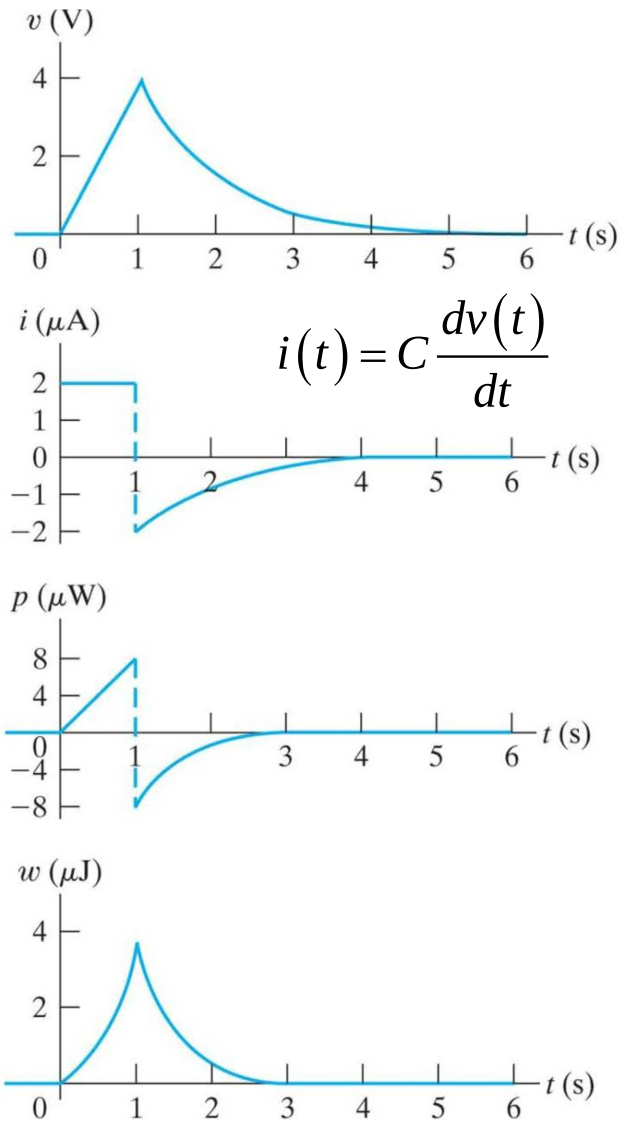
# EX 3.1 Circuit Variables of a Capacitor (4/5)

- The energy of the capacitor: $W(t)=\int_{-\infty}^{t} p(\tau)d\tau=\int_{-\infty}^{t} Cv(\tau)\frac{dv(\tau)}{d\tau}d\tau=\frac{1}{2}Cv^2(t)$

➡ $$W(t)=\begin{cases} & t\le 0\text{ s} \\ & 0<t\le 1\text{ s} \\ & t>1\text{ s} \end{cases}$$



# Circuit Variables of a Capacitor (5/5)




$$i(t) = C\frac{dv(t)}{dt}$$

- Summary:
  - Voltage on a capacitor must be continuous; it cannot change abruptly across the terminals of the capacitor
  - If the voltage across the terminals is constant, $i_c = 0$ (equivalent to open circuit)
- When does the capacitor store energy?
  - Storing energy: $w(t)$ increases
  - This is when the power is positive

$$\int_0^1 p(t)\,dt = \int_0^1 (8t)\,dt = 4\ uJ$$

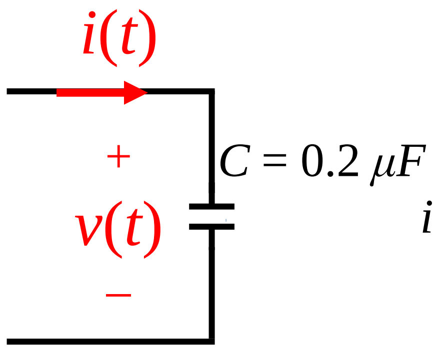
- When does the capacitor dissipate energy?
  - Dissipating energy: $w(t)$ decreases
  - This is when the power is negative

$$\int_1^\infty p(t)\,dt = \int_1^\infty \left(-8e^{-2(t-1)}\right)dt = -4\ uJ$$

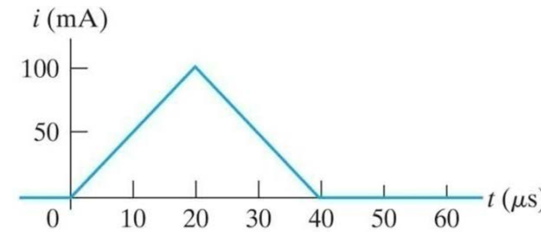


J. Nilsson and S. Riedel, *Electric Circuits*, 9th Edition, Pearson, NJ, 2011

# Circuit Variables of a Capacitor (1/4)




$$i(t)=\begin{cases}0 & t\le 0\ \mu s\\ 5000t\ \text{A} & 0<t\le 20\ \mu s\\ 0.2-5000t\ \text{A} & 20<t\le 40\ \mu s\\ 0 & t>40\ \mu s\end{cases}$$

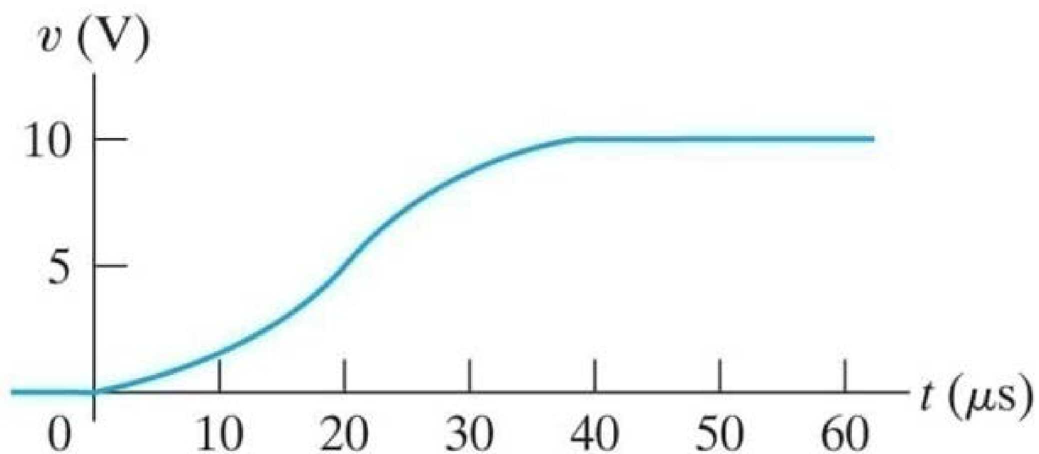



1. Let $v(0) = 0$. Find the expressions for the voltage, power, and energy on the capacitor

# EX 3.2 Circuit Variables of a Capacitor (2/4)

- The voltage dropped across the capacitor: $v(t)=v(t_0)+\frac{1}{C}\int_{t_0}^{t} i(\tau)\,d\tau$

$$\Rightarrow \quad v(t)=\begin{cases} 0 & t \le 0\ \mu s \\ & 0 < t \le 20\ \mu s \\ & 20 < t \le 40\ \mu s \\ & t > 40\ \mu s \end{cases}$$

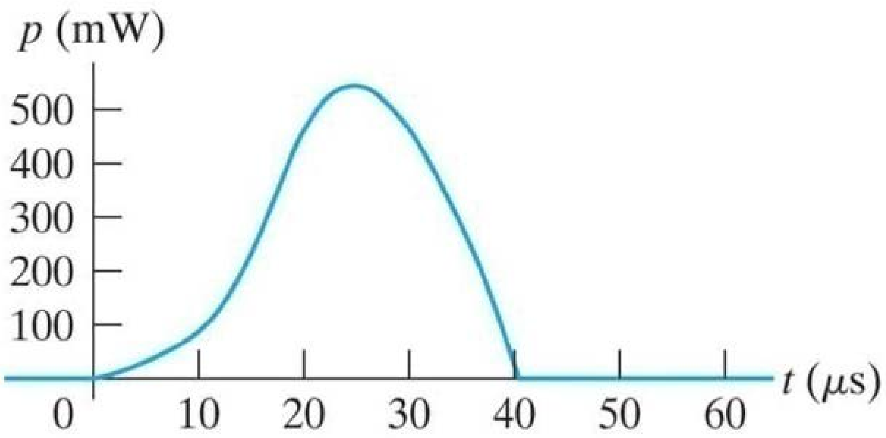



J. Nilsson and S. Riedel, *Electric Circuits*, 9th Edition, Pearson, NJ, 2011

# EX 3.2 Circuit Variables of a Capacitor (3/4)

- The power in the capacitor: $p(t) = v(t)i(t)$

$$p(t) = \begin{cases} 0 & t \le 0\ \mu s \\ \left(12.5\times10^{9}t^{2}\right)\times\left(5000t\right) = 62.5\times10^{12}t^{3}\ W & 0 < t \le 20\ \mu s \\ \left(10^{6}t - 12.5\times10^{9}t^{2} - 10\right)\times\left(0.2 - 5000t\right) = 62.5\times10^{12}t^{3} - 7.5\times10^{9}t^{2} + 2.5\times10^{5}t - 2\ W & 20 < t \le 40\ \mu s \\ 0 & t > 40\ \mu s \end{cases}$$

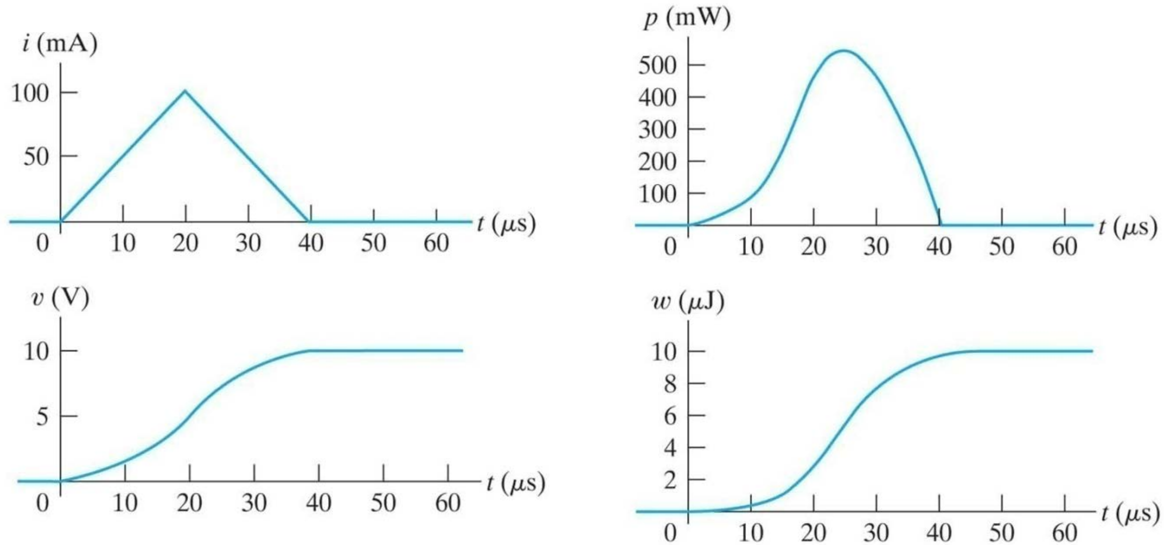



- The power > 0 all the time
- So the capacitor stores energy continuously

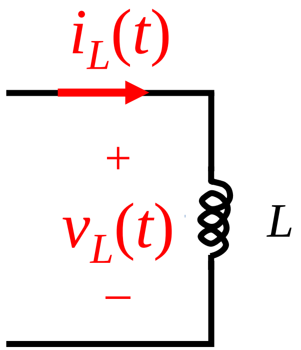
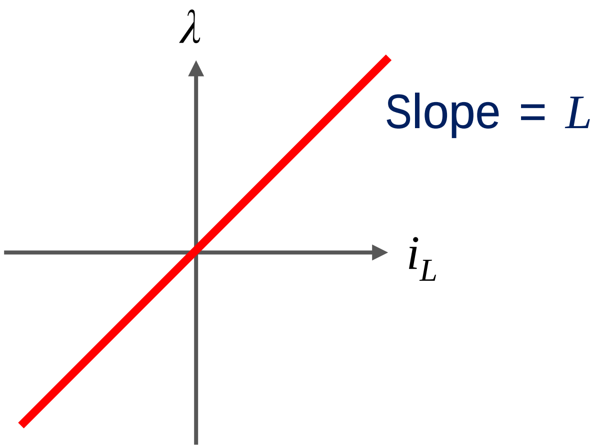
# EX 3.2 Circuit Variables of a Capacitor (4/4)

- The energy in the capacitor: $W(t) = \frac{1}{2}Cv^2(t)$

$$W(t) = \begin{cases} 0 & t \le 0\ \mu s \\ \frac{1}{2}\left(0.2\times10^{-6}\right)\times\left(12.5\times10^{9}t^{2}\right)^{2} = 15.625\times10^{12}t^{4}\ J & 0 < t \le 20\ \mu s \\ \frac{1}{2}\left(0.2\times10^{-6}\right)\times\left(10^{6}t - 12.5\times10^{9}t^{2} - 10\right)^{2} = 15.625\times10^{12}t^{4} - 2.5\times10^{9}t^{3} + 0.125\times10^{6}t^{2} - 2t + 10^{-5}\ J & 20 < t \le 40\ \mu s \\ \frac{1}{2}\left(0.2\times10^{-6}\right)\times(10)^{2} = 10\ \mu J & t > 40\ \mu s \end{cases}$$



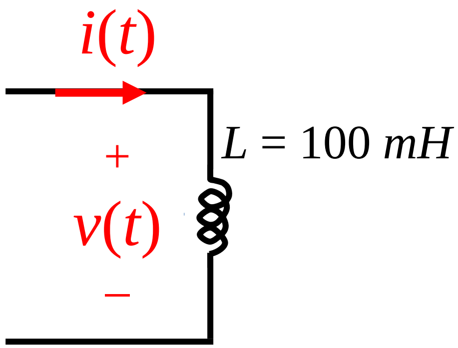
# Component Model of an Inductor

| Symbol | Model |
|---|---|
|   |   |

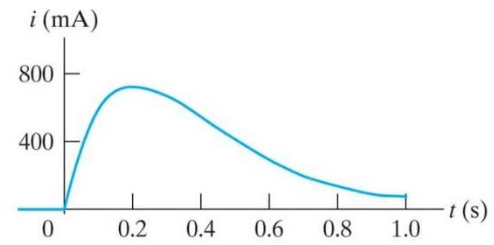
- $\lambda = L \times i_L$;  where  $\lambda$: "magnetic flux" in weber
  $L$: inductance in H (Henry)
- On the other hand, $\lambda(t) = \int_{-\infty}^{t} v_L(\tau)\,d\tau$

1. If $i_L$ is given, then $v_L(t) = \dfrac{d\lambda}{dt} = L\dfrac{di_L(t)}{dt}$
2. If $v_L$ is given, then $i_L(t) = \dfrac{\lambda}{L} = \dfrac{1}{L}\int_{-\infty}^{t} v_L(\tau)\,d\tau = i_L(t_0) + \dfrac{1}{L}\int_{t_0}^{t} v_L(\tau)\,d\tau$

# Circuit Variables of an Inductor (1/5)




$$i(t)=\begin{cases}0 & t\le 0\text{ s}\\ 10te^{-5t}\text{ A} & t>0\text{ s}\end{cases}$$

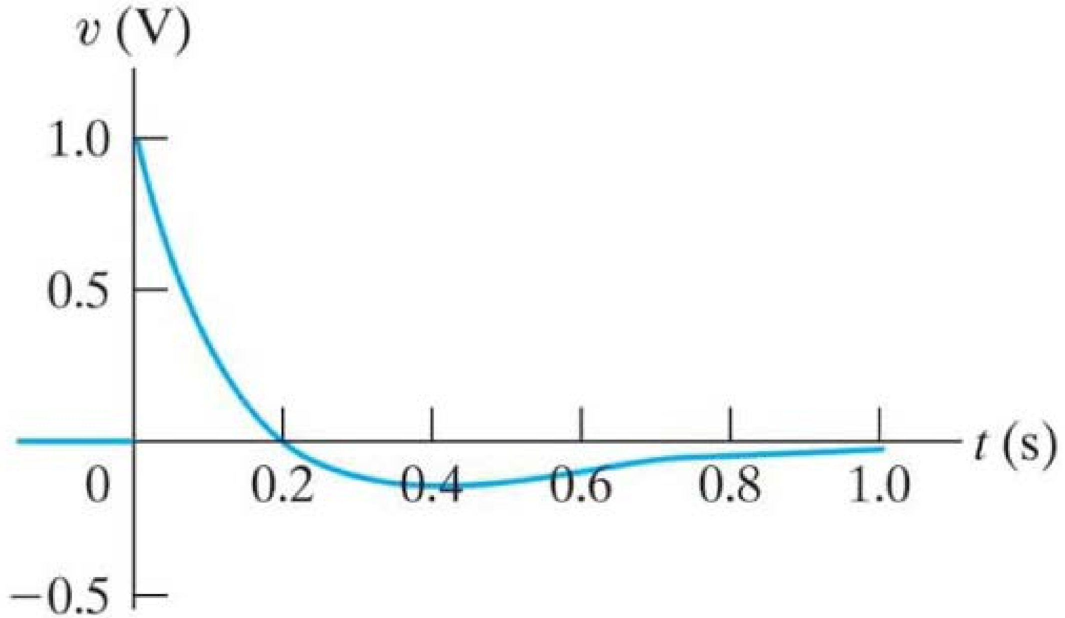



1. Find the expressions for the voltage, power, and energy on the inductor
2. Determine the interval of time when energy is being stored and delivered in the inductor, respectively

# Circuit Variables of an Inductor (2/5)

- The voltage dropped on the inductor: $v(t) = L\dfrac{di(t)}{dt}$

➡ $v(t) = \begin{cases} & t \le 0\ \text{s} \\ & t > 0\ \text{s} \end{cases}$

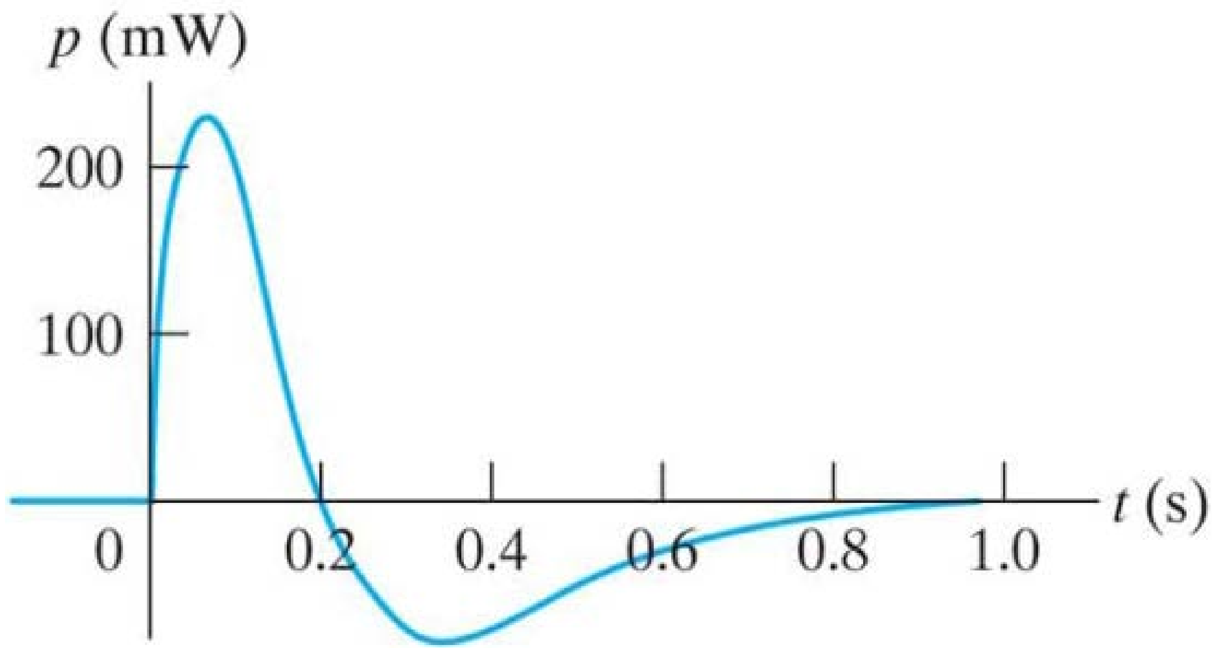


EX 3.3

# Circuit Variables of an Inductor (3/5)

- The power in the inductor: $p(t) = v(t)i(t)$

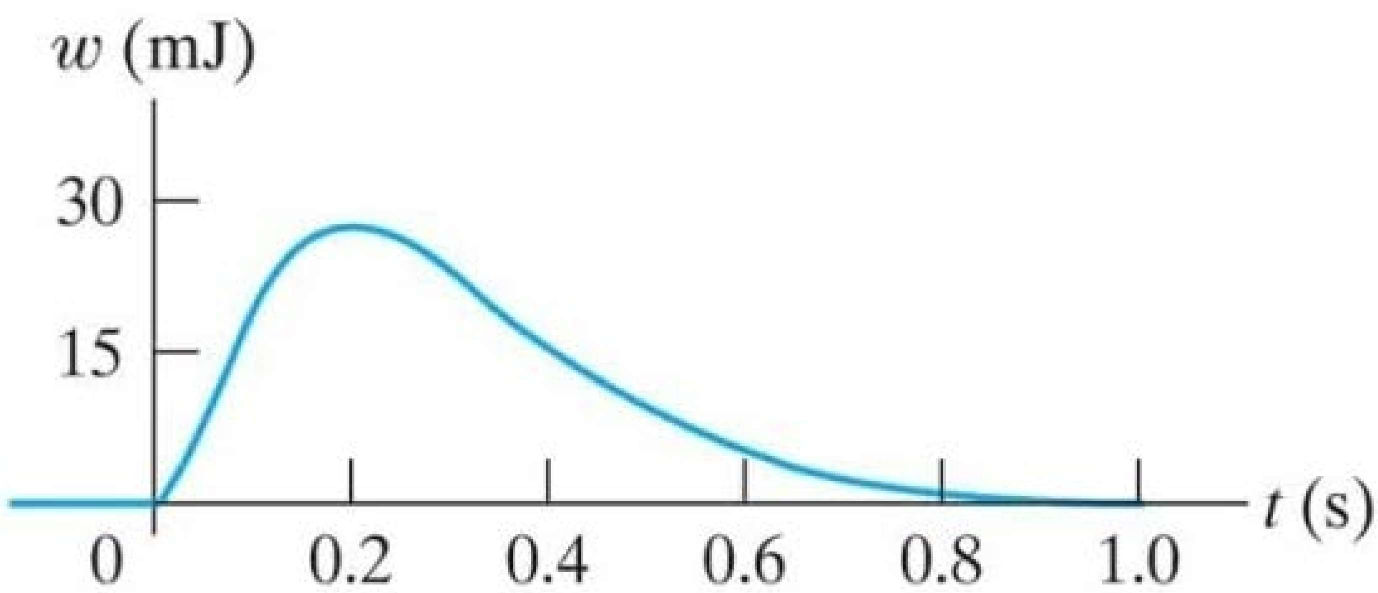
➡ $p(t) = \begin{cases} & t \le 0 \text{ s} \\ & t > 0 \text{ s} \end{cases}$



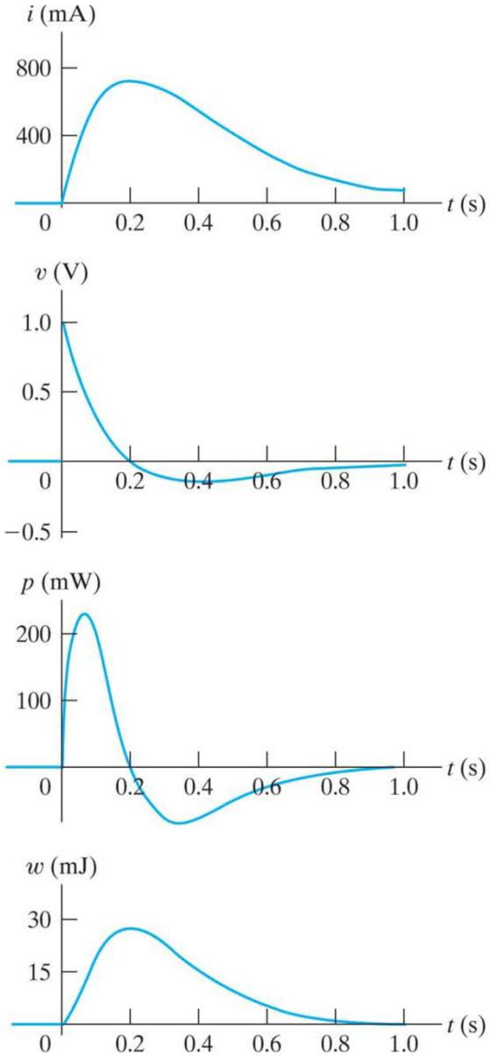
# EX 3.3 Circuit Variables of an Inductor (4/5)

- The energy in the inductor: $W(t)=\int_{-\infty}^{t} p(\tau)d\tau=\int_{-\infty}^{t} Li(\tau)\frac{di(\tau)}{d\tau}d\tau=\frac{1}{2}Li^2(t)$

➡ $W(t)=\begin{cases} & t\le 0\text{ s} \\ & t>0\text{ s} \end{cases}$



# EX 3.3 Circuit Variables of an Inductor (5/5)



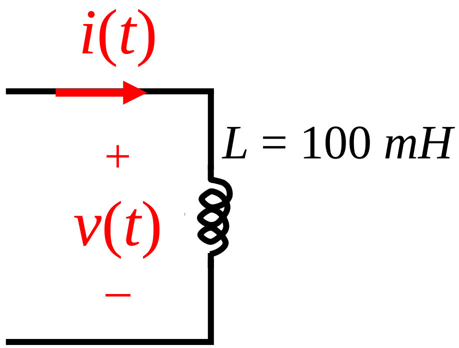

- Summary:
  - Current through an inductor must be continuous; it cannot change abruptly in an inductor
  - If the current run through an inductor is constant, $v_L = 0$ (equivalent to short circuit)
- When does the inductor store energy?
  - Storing energy: $w(t)$ increases
  - This is when the power is positive

$$\int_0^{0.2} p(t)\,dt = 27.07\ mJ$$

- When does the inductor dissipate energy?
  - Dissipating energy: $w(t)$ decreases
  - This is when the power is negative

$$\int_{0.2}^{\infty} p(t)\,dt = -27.07\ mJ$$

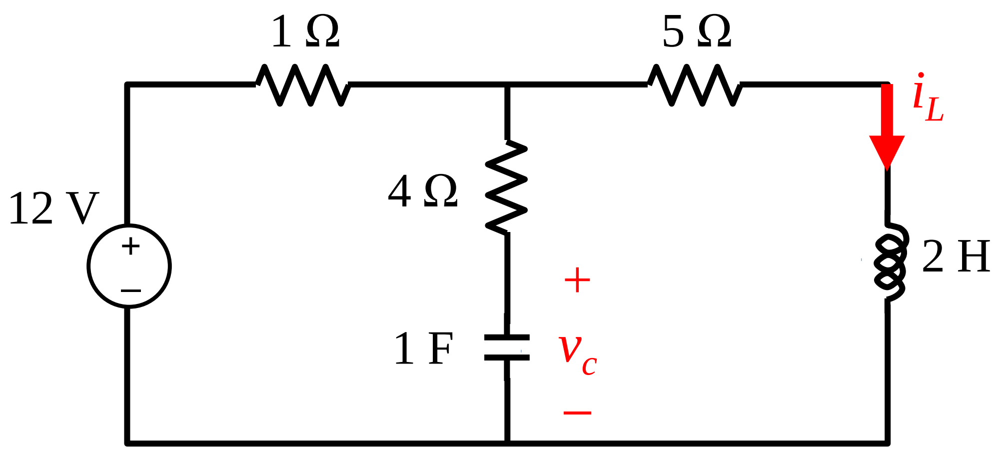
# EX 3.4 Circuit Variables of an Inductor




$$v(t) = 20te^{-10t}\ \text{V} \qquad t > 0\ \text{s}$$

1. Let $i(0) = 0$. Find the expressions for the current through the inductor

# DC Condition




- Under DC condition, find:
  1. the voltage across the capacitor
  2. the current through the inductor
  3. the energy in the capacitor
  4. the energy in the inductor

# Remarks

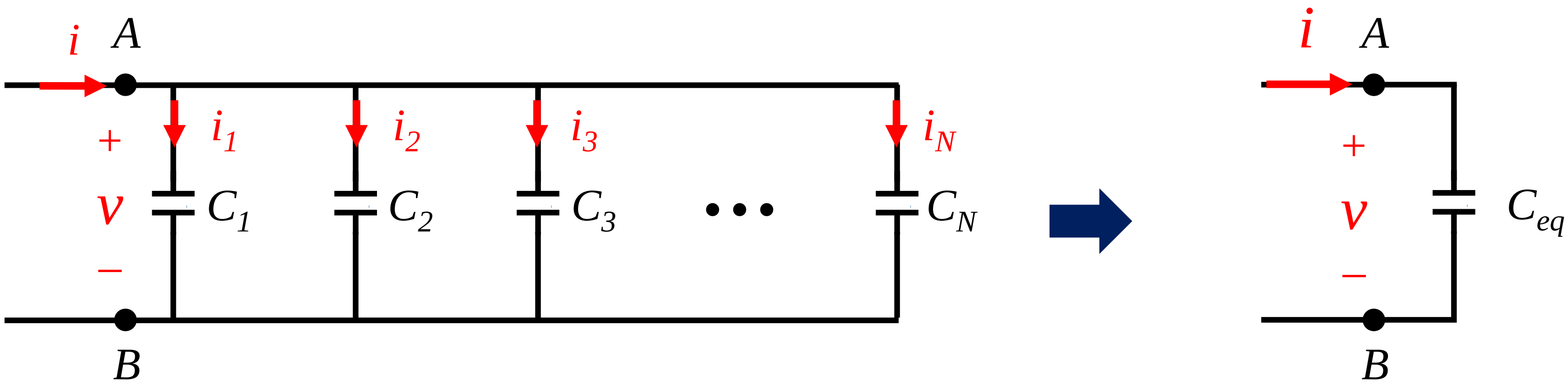
- C and L are capable of storing energy, so they can be used for generating a large amount of voltage or current for a short period of time
- They can also be used as temporary voltage or current sources
- The frequency sensitive property of L and C makes them useful for frequency discrimination
  - Low pass filters
  - High pass filters
  - Band pass filters



## Contents

# 3.2 Combinations of C and L

# *N* Capacitors in Parallel




Left: $i = i_1 + i_2 + i_3 + \ldots + i_N$

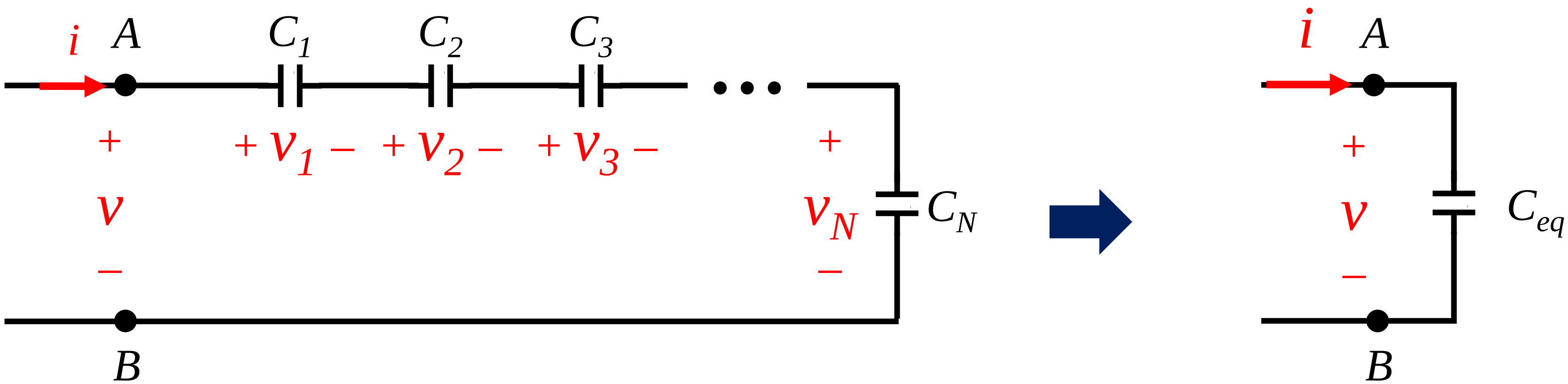
$$= C_1 \frac{dv}{dt} + C_2 \frac{dv}{dt} + C_3 \frac{dv}{dt} + \ldots + C_N \frac{dv}{dt}$$

$$= \left(C_1 + C_2 + C_3 + \ldots + C_N\right) \frac{dv}{dt}$$

Right: $i = C_{eq} \dfrac{dv}{dt}$

$$\boxed{C_{eq} = C_1 + C_2 + C_3 + \ldots + C_N}$$

# $N$ Capacitors in Series




Left: $v = v_1 + v_2 + v_3 + \ldots + v_N$

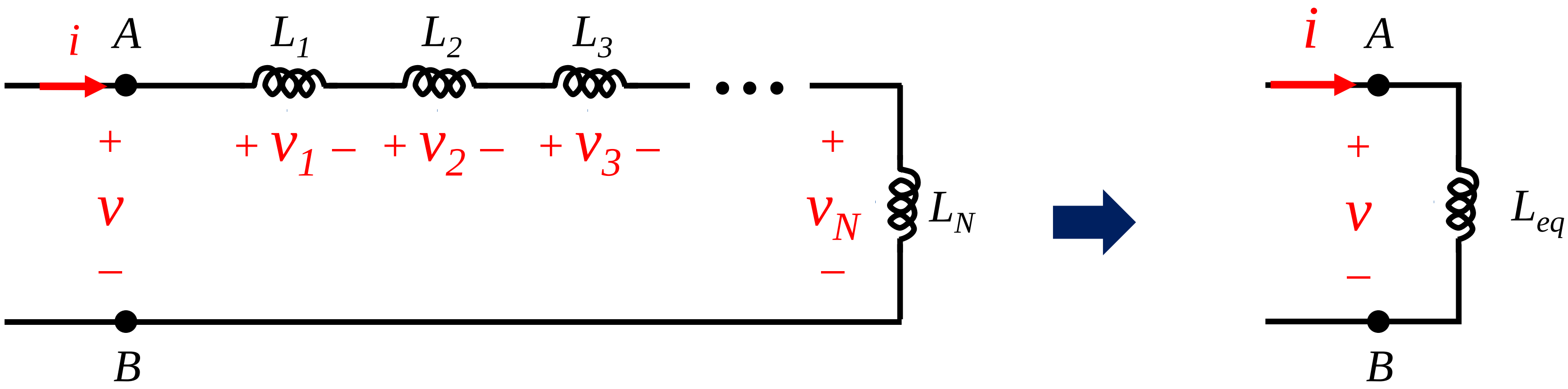
$$= \left[ v_1(t_0) + \frac{1}{C_1}\int_{t_0}^{t} i(\tau)\,d\tau \right] + \left[ v_2(t_0) + \frac{1}{C_2}\int_{t_0}^{t} i(\tau)\,d\tau \right] + \ldots + \left[ v_N(t_0) + \frac{1}{C_N}\int_{t_0}^{t} i(\tau)\,d\tau \right]$$

$$= \left( v_1(t_0) + v_2(t_0) + \ldots + v_N(t_0) \right) + \left( \frac{1}{C_1} + \frac{1}{C_2} + \ldots + \frac{1}{C_N} \right) \int_{t_0}^{t} i(\tau)\,d\tau$$

Right: $v = v(t_0) + \dfrac{1}{C_{eq}}\displaystyle\int_{t_0}^{t} i(\tau)\,d\tau$

$$\boxed{\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \ldots + \frac{1}{C_N}}$$

# *N* Inductors in Series




Left: $v = v_1 + v_2 + v_3 + \ldots + v_N$

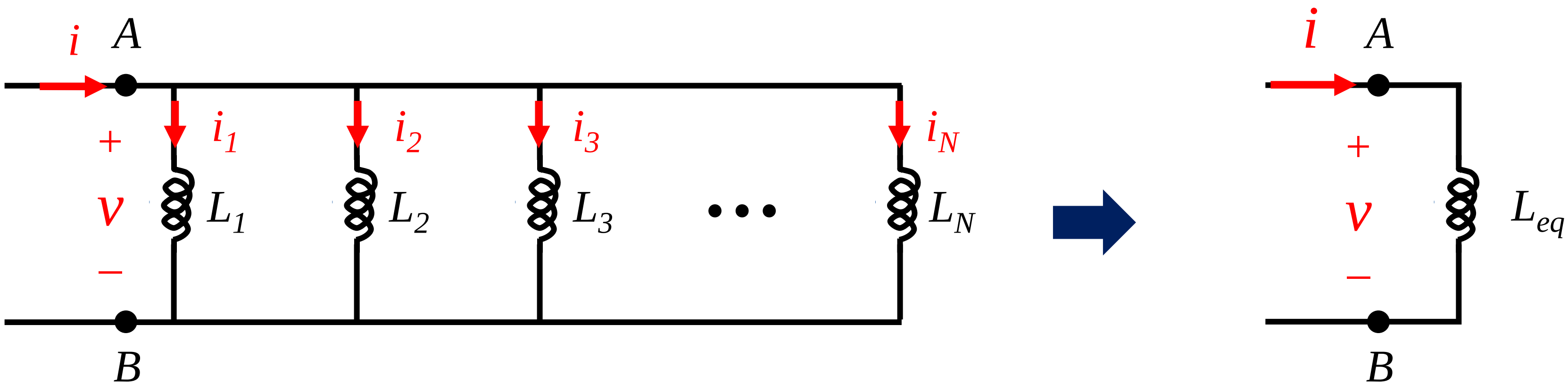
$$= L_1 \frac{di}{dt} + L_2 \frac{di}{dt} + L_3 \frac{di}{dt} + \ldots + L_N \frac{di}{dt}$$

$$= \left( L_1 + L_2 + L_3 \ldots + L_N \right) \frac{di}{dt}$$

Right: $v = L_{eq} \dfrac{di}{dt}$

$$\boxed{L_{eq} = L_1 + L_2 + L_3 + \ldots + L_N}$$

# *N* Capacitors in Parallel




Left: $i = i_1 + i_2 + i_3 + \ldots + i_N$

$$= \left[ i_1(t_0) + \frac{1}{L_1}\int_{t_0}^{t} v(\tau)\,d\tau \right] + \left[ i_2(t_0) + \frac{1}{L_2}\int_{t_0}^{t} v(\tau)\,d\tau \right] + \ldots + \left[ i_N(t_0) + \frac{1}{L_N}\int_{t_0}^{t} v(\tau)\,d\tau \right]$$

$$= \left( i_1(t_0) + i_2(t_0) + \ldots + i_N(t_0) \right) + \left( \frac{1}{L_1} + \frac{1}{L_2} + \ldots + \frac{1}{L_N} \right) \int_{t_0}^{t} v(\tau)\,d\tau$$

Right: $i = i(t_0) + \frac{1}{L_{eq}}\int_{t_0}^{t} v(\tau)\,d\tau$

$$\boxed{\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \ldots + \frac{1}{L_N}}$$

# Summary

| | Resistor | Capacitor | Inductor |
|---|---|---|---|
| Give $i$, find $v$ | $v = iR$ | $v(t) = v(t_0) + \frac{1}{C}\int_{t_0}^{t} i(\tau)\,d\tau$ | $v = L\frac{di}{dt}$ |
| Give $v$, find $i$ | $i = \frac{v}{R}$ | $i = C\frac{dv}{dt}$ | $i(t) = i(t_0) + \frac{1}{L}\int_{t_0}^{t} v(\tau)\,d\tau$ |
| Power or energy | $P = i^2 R = \frac{v^2}{R}$ | $W = \frac{1}{2}Cv^2$ | $W = \frac{1}{2}Li^2$ |
| Series connection | $R_{eq} = R_1 + R_2$ | $C_{eq} = \frac{C_1 C_2}{C_1 + C_2}$ | $L_{eq} = L_1 + L_2$ |
| Parallel connection | $R_{eq} = \frac{R_1 R_2}{R_1 + R_2}$ | $C_{eq} = C_1 + C_2$ | $L_{eq} = \frac{L_1 L_2}{L_1 + L_2}$ |
| DC case | **The same** | **Open circuit** | **Short circuit** |



Contents

# 3.3 Natural Responses of First-Order Circuits

# From Resistive Circuits to RC and RL Circuits

**Resistive circuits**

- Algebraic equations
- Solution techniques:
  Nodal analysis
  Mesh analysis

**RC and RL circuits**

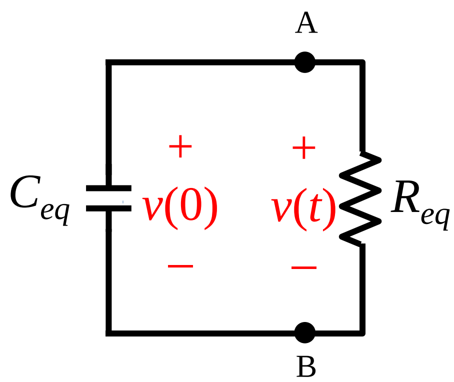
- Differential equations
- Solution techniques:
  Nodal analysis
  Mesh analysis

- The solution of RC and RL circuits:
  **Natural response + Forced response**
- Or in the form of
  **Transient response + Steady response**
- Or in the mathematical terms of
  **Homogeneous solution + Particular solution**

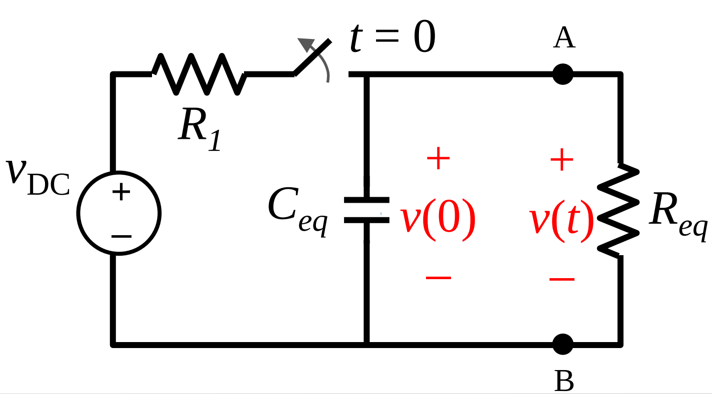
When input sources are set as DC inputs, the “forced response” is called a “step response”

# Natural Responses of First-Order Circuits

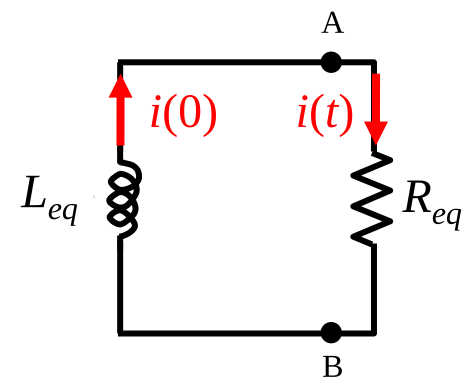
## First-order RC circuit



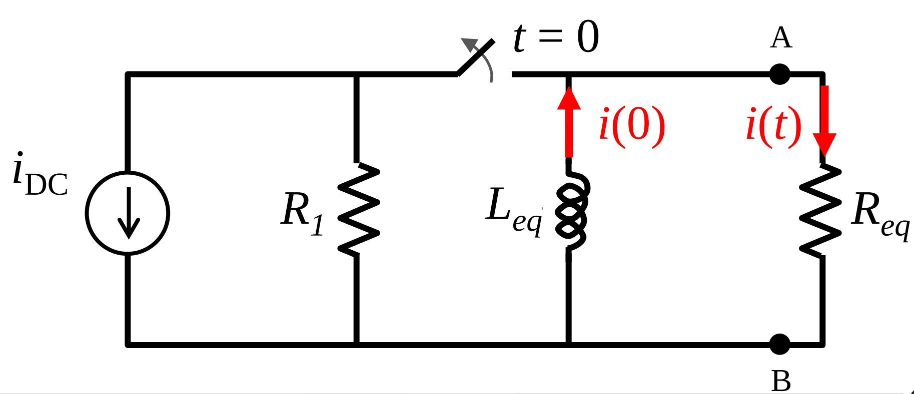

- Objective: Find $v(t)$ from a given $v(0)$
- How to create $v(0)$?



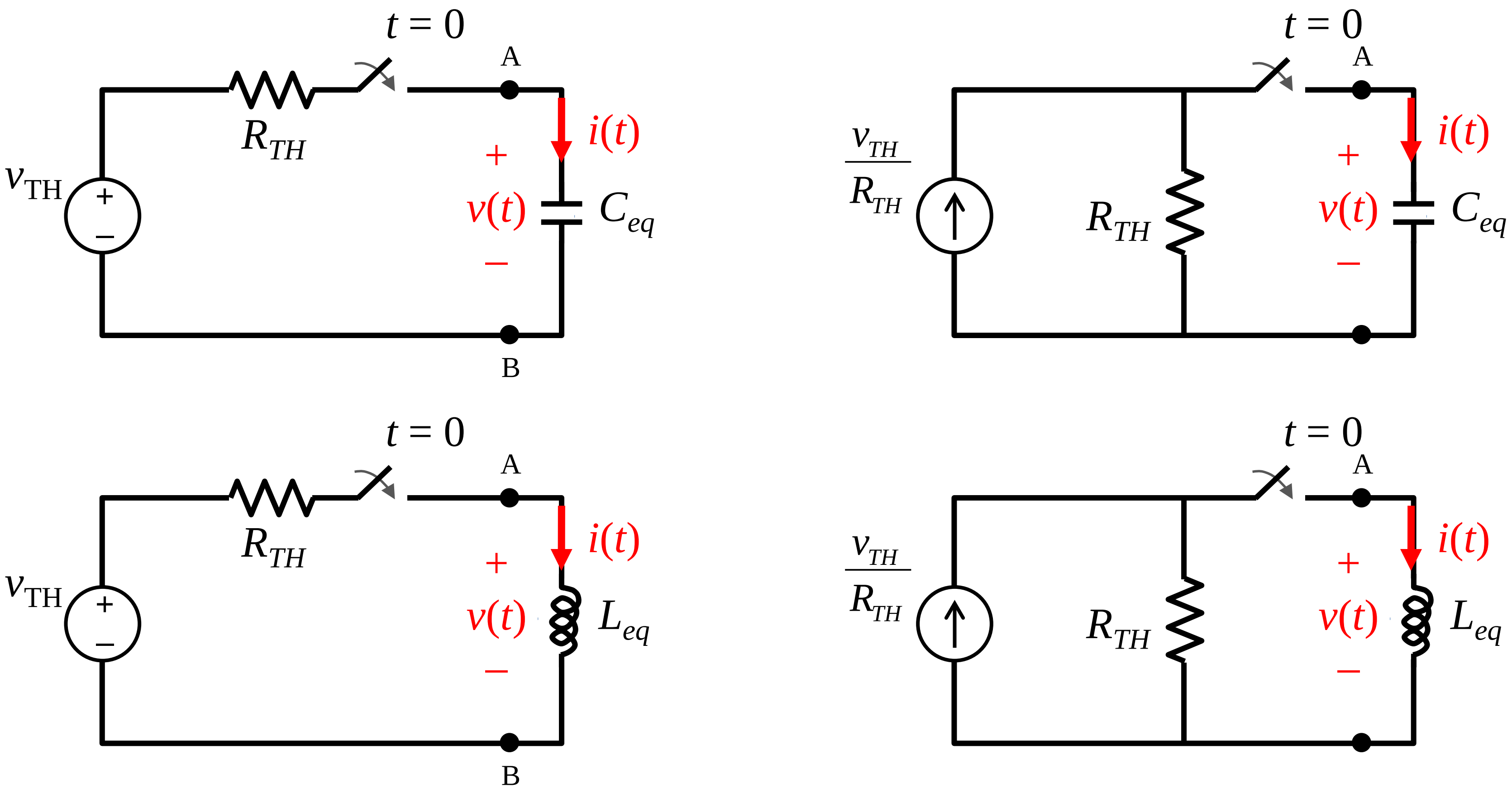

## First-order RL circuit




- Objective: Find $i(t)$ from a given $i(0)$
- How to create $i(0)$?

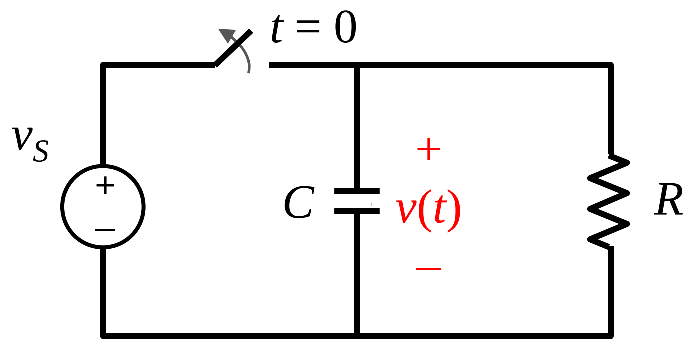


# Step Responses of First-Order Circuits



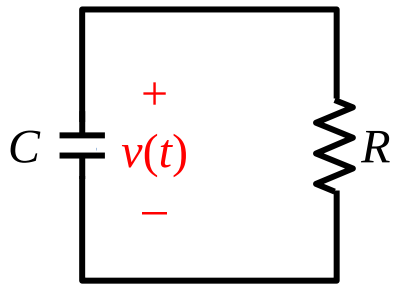

- Objective: Find $v(t)$ and $i(t)$ after $t = 0$
- Note that the voltage source and current source provide DC inputs

# EX 3.6 An RC Circuit




1. Find $v(t)$ for $t \geq 0^+$

# Time Constant in RC Circuits




$$v(t) = v_S e^{-\frac{t}{RC}}$$

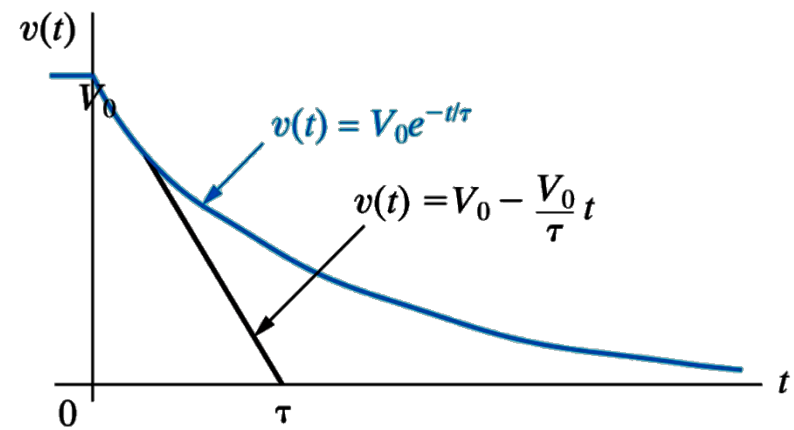
- After finishing the computation, we find an interesting parameter

$$\tau = RC \longrightarrow v(t) = v_S e^{-\frac{t}{\tau}}$$

① $t = \tau$: $v(t) = 0.368\ v_S$ (37 %)

② $t = 3\tau$: $v(t) = 0.050\ v_S$ (5 %)

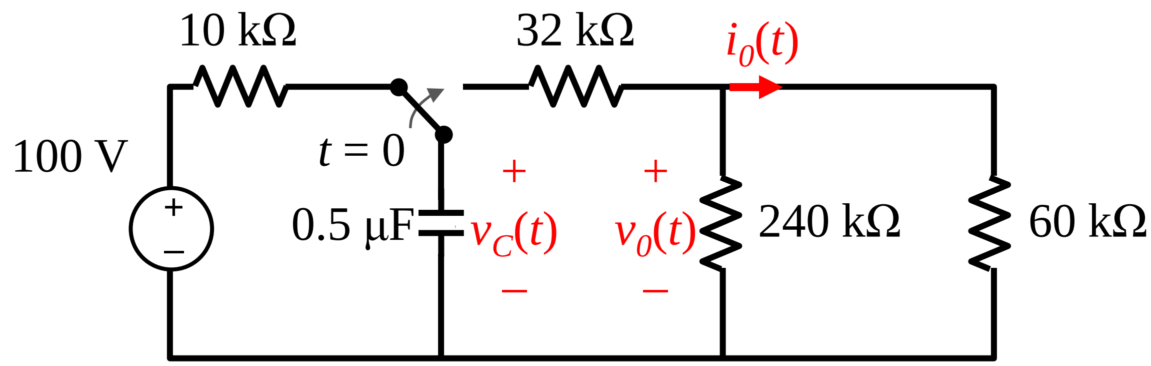
③ $t = 5\tau$: $v(t) = 0.007\ v_S \approx 0$ (1 %)




- When $t \geq 5\tau$, we call it a "steady state"
- The capacitor is fully discharged after $5\tau$

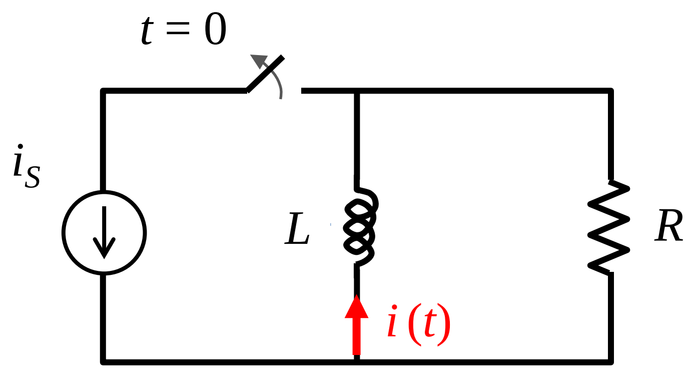
EX 3.7

# Example: Another RC Circuit



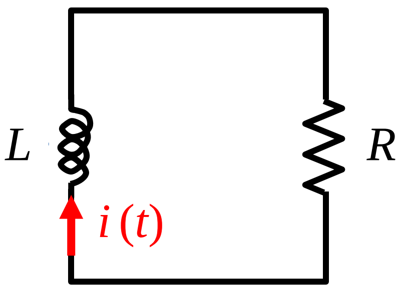

1. Find $v_C(t)$, $v_0(t)$, and $i_0(t)$ for $t \geq 0^+$
2. Find the total energy dissipated in the 60-kΩ resistor

# EX 3.8 An RL Circuit




1. Find $i(t)$ for $t \geq 0^+$

# Time Constant in RL Circuits




$$i(t) = i_S e^{-\frac{R}{L}t}$$

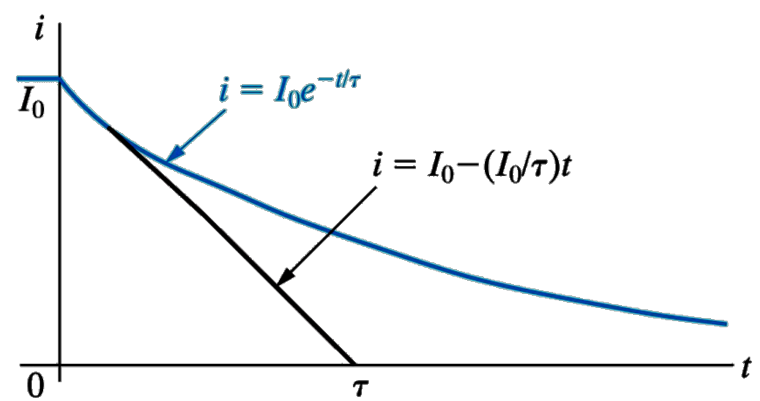
- In the above procedure, we find an interesting parameter again

$$\tau = \frac{R}{L} \longrightarrow i(t) = i_S e^{-\frac{t}{\tau}}$$

① $t = \tau$: $i(t) = 0.368\ i_S$ (37 %)

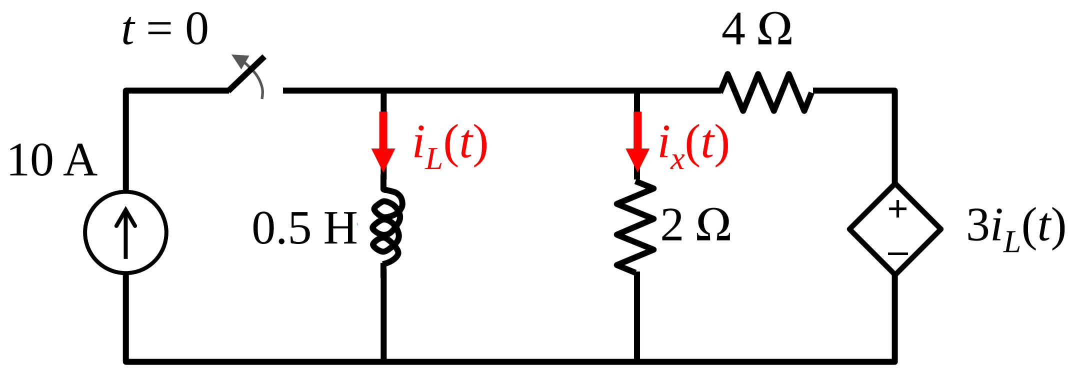
② $t = 3\tau$: $i(t) = 0.050\ i_S$ (5 %)

③ $t = 5\tau$: $i(t) = 0.007\ i_S \approx 0$ (1 %)



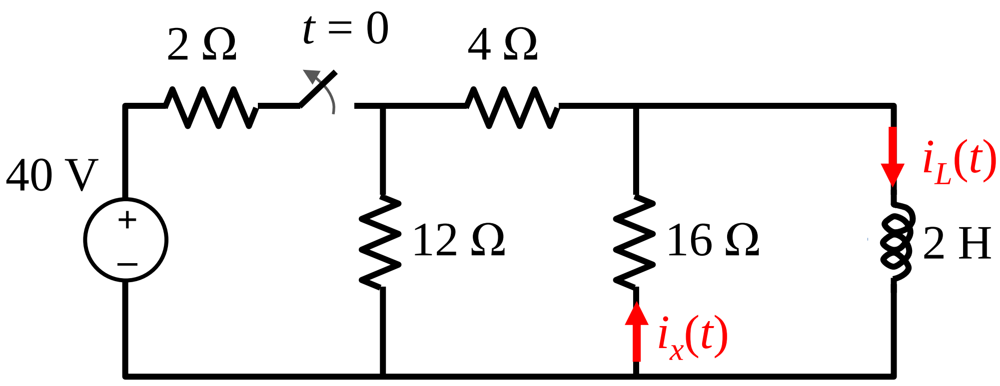

- When $t \geq 5\tau$, we call it a "steady state"
- The inductor is fully discharged after $5\tau$

# Example: Another RL Circuit




1. Find $i_L(t)$ and $i_x(t)$ for $t \geq 0^+$

# EX 3.10 Another Example of RL Circuits




- The switch has been turned on for a long time
- At $t = 0$, it's turned off

1. Find $i_x(t)$ for $t \geq 0^+$
2. Find $i_L(t)$ for $t \geq 0^+$



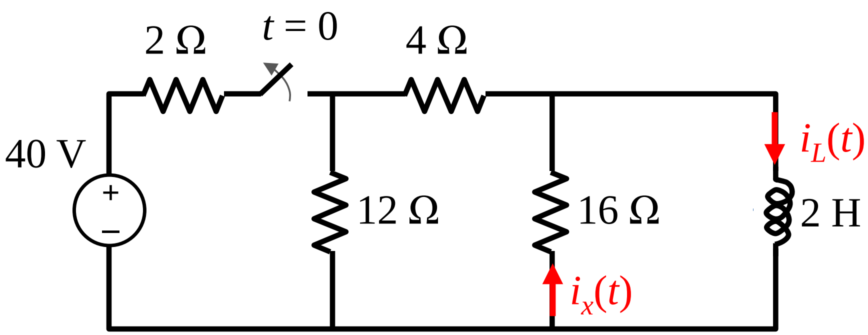
## Contents

# 3.4 Step Responses of First-Order Circuits

# Characteristics of Step Response

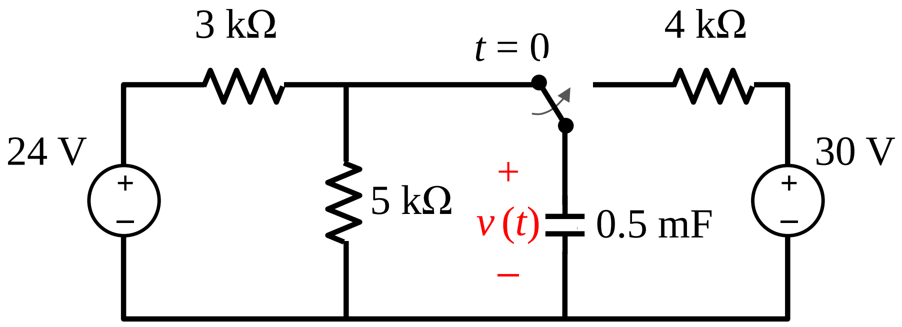
## Natural response:

- When the state of the switch is changed, the new state has no other sources

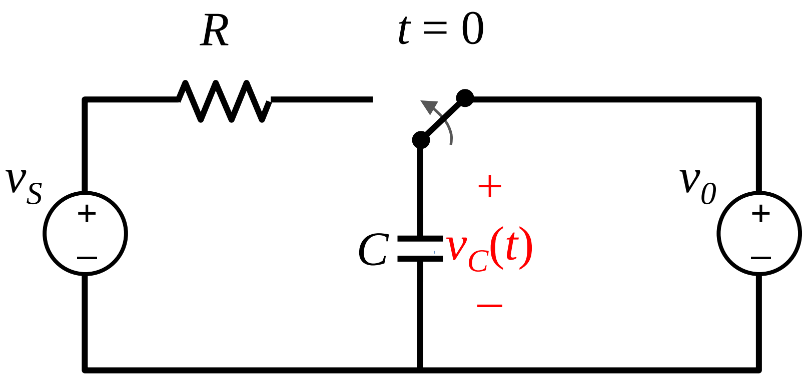



## Step response:

- When the state of the switch is changed, the new state has new sources
- The new sources cause new responses—**forced responses**
- If the new sources are DC inputs, then the force response is called the "**step response**"



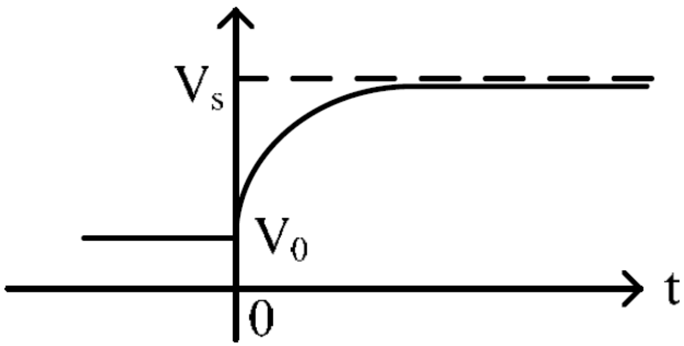
# EX 3.11 Basic Case of Step Response (1/2)

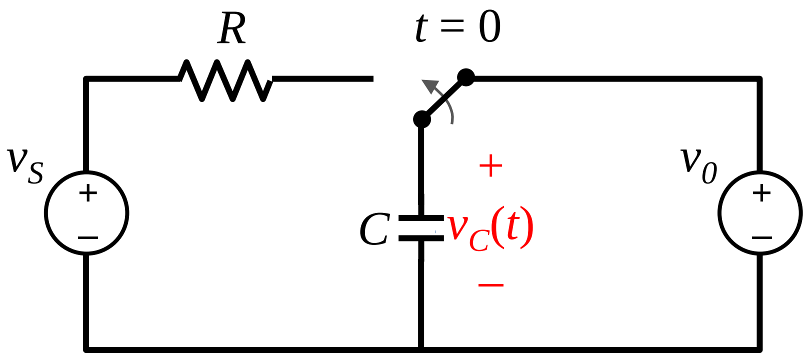



1. Find $v_C(t)$ for $t \geq 0^+$

# Basic Case of Step Response (2/2)

The complete solution: $v_C(t) = (V_0 - V_S)e^{-\frac{t}{RC}} + V_S$



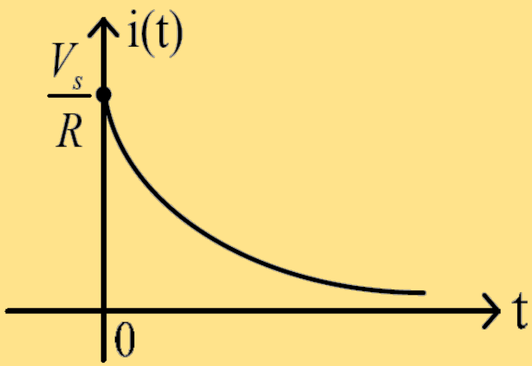




How about the current over $C$?

$$i_C(t) = C\frac{dv_C}{dt} = C\left[\frac{-1}{RC}(v_0 - v_S)e^{-\frac{t}{RC}}\right] = \frac{v_S - v_0}{R}e^{-\frac{t}{RC}}$$

If $v_0 = 0$, for $t = 0^+$, $i(0^+) = v_S/R$

$C$ is initially short circuited!



# Particular Solutions (1/2)

$$a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \cdots + a_1 y'(x) + a_0 y = g(x)$$

解的形式為$y_h + y_p$：

**$g(x)$ 長什麼樣子，particular solution 就是什麼樣子**

| | |
|---|---|
| **1.** 1 (any constant) | $A$ |
| **2.** $5x + 7$ | $Ax + B$ |
| **3.** $3x^2 - 2$ | $Ax^2 + Bx + C$ |
| **4.** $x^3 - x + 1$ | $Ax^3 + Bx^2 + Cx + E$ |
| **5.** $\sin 4x$ | $A \cos 4x + B \sin 4x$ |
| **6.** $\cos 4x$ | $A \cos 4x + B \sin 4x$ |
| **7.** $e^{5x}$ | $Ae^{5x}$ |
| **8.** $(9x - 2)e^{5x}$ | $(Ax + B)e^{5x}$ |
| **9.** $x^2e^{5x}$ | $(Ax^2 + Bx + C)e^{5x}$ |
| **10.** $e^{3x} \sin 4x$ | $Ae^{3x} \cos 4x + Be^{3x} \sin 4x$ |
| **11.** $5x^2 \sin 4x$ | $(Ax^2 + Bx + C) \cos 4x + (Ex^2 + Fx + G) \sin 4x$ |
| **12.** $xe^{3x} \cos 4x$ | $(Ax + B)e^{3x} \cos 4x + (Cx + E)e^{3x} \sin 4x$ |

# Particular Solutions (2/2)

## Example

$$y'' - 2y' - 3y = \underline{4x - 5} + \underline{6xe^{2x}}$$

Step 1: Find the solution of

$$y'' - 2y' - 3y = 0.$$

$$y_c = c_1 e^{3x} + c_2 e^{-x}$$

Step 2: Particular solution

$$y'' - 2y' - 3y = 4x - 5 \qquad\qquad y'' - 2y' - 3y = 6xe^{2x}$$

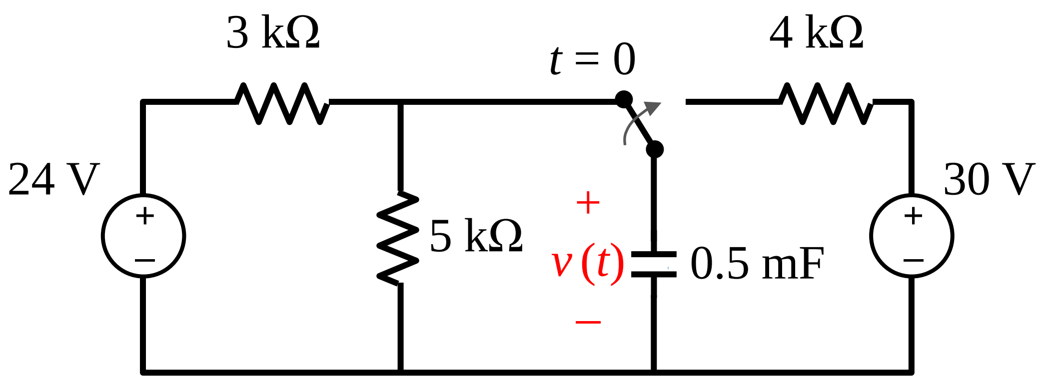
guess

guess

$$\boxed{y_{p_1} = Ax + B} \qquad\qquad \boxed{y_{p_2} = Cxe^{2x} + Ee^{2x}}$$

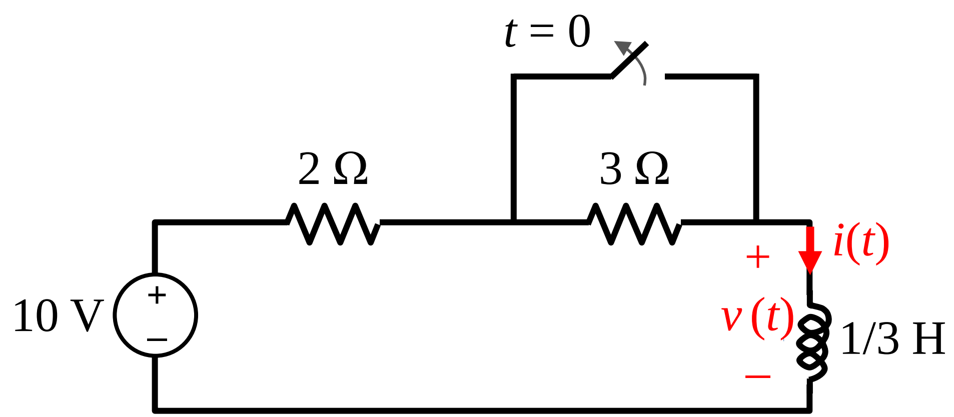
$$\boxed{y_{p_1} = -\frac{4}{3}x + \frac{23}{9}} \qquad\qquad \boxed{y_{p_2} = -(2x + \frac{4}{3})e^{2x}}$$

# An RC Circuit with Step Responses



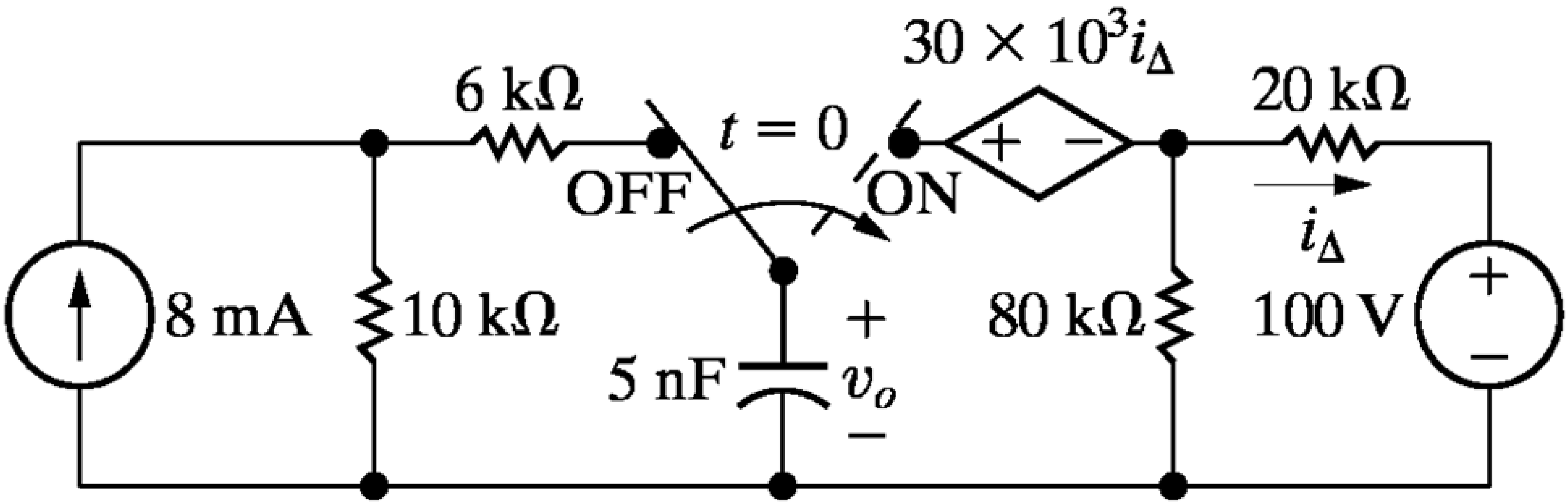

- $t \leq 0^-$ : the circuit is under steady state
  1. Find $v(t)$ for $t \geq 0^+$

# EX 3.13 An RL Circuit with Step Responses



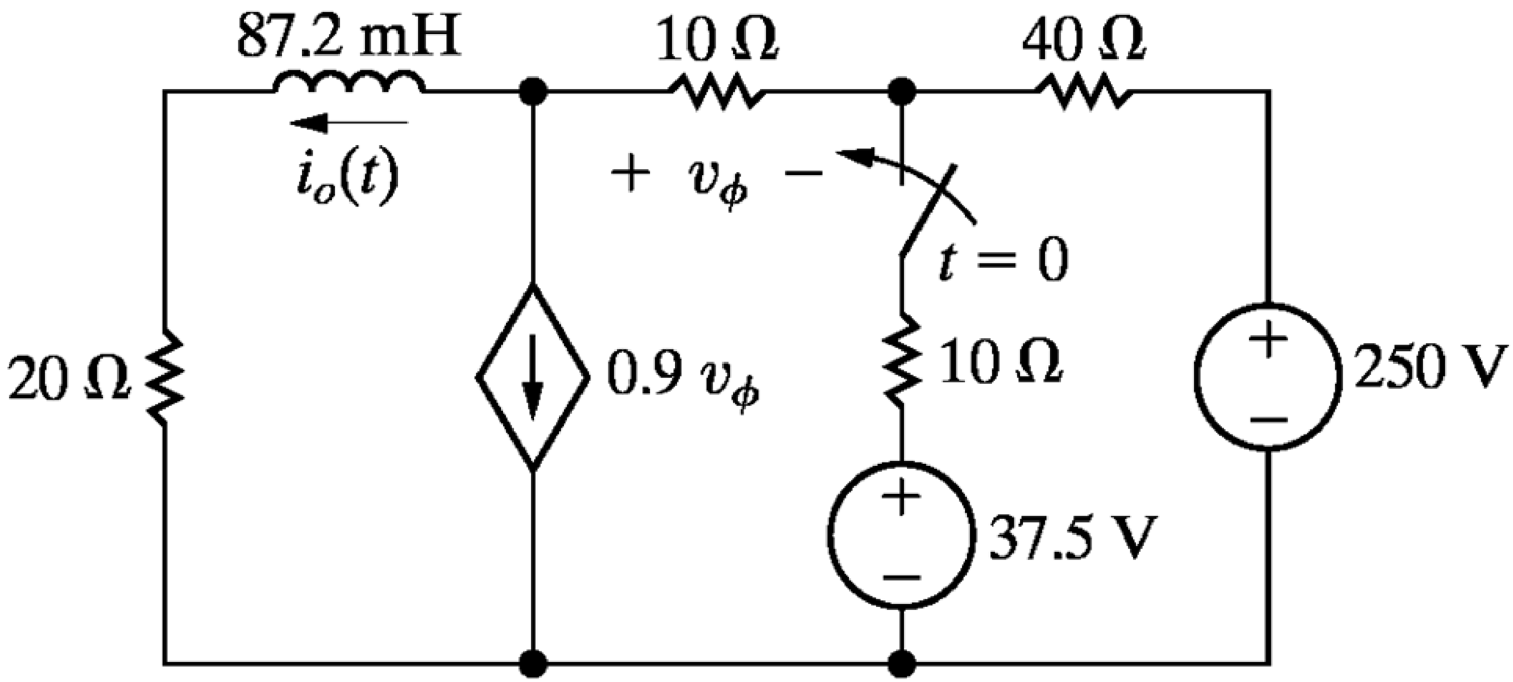

- $t \leq 0^-$ : the circuit is under steady state
  1. Find $v(t)$ for $t \geq 0^+$
  2. Find $i(t)$ for $t \geq 0^+$

# EX 3.14 First-Order RC Circuit



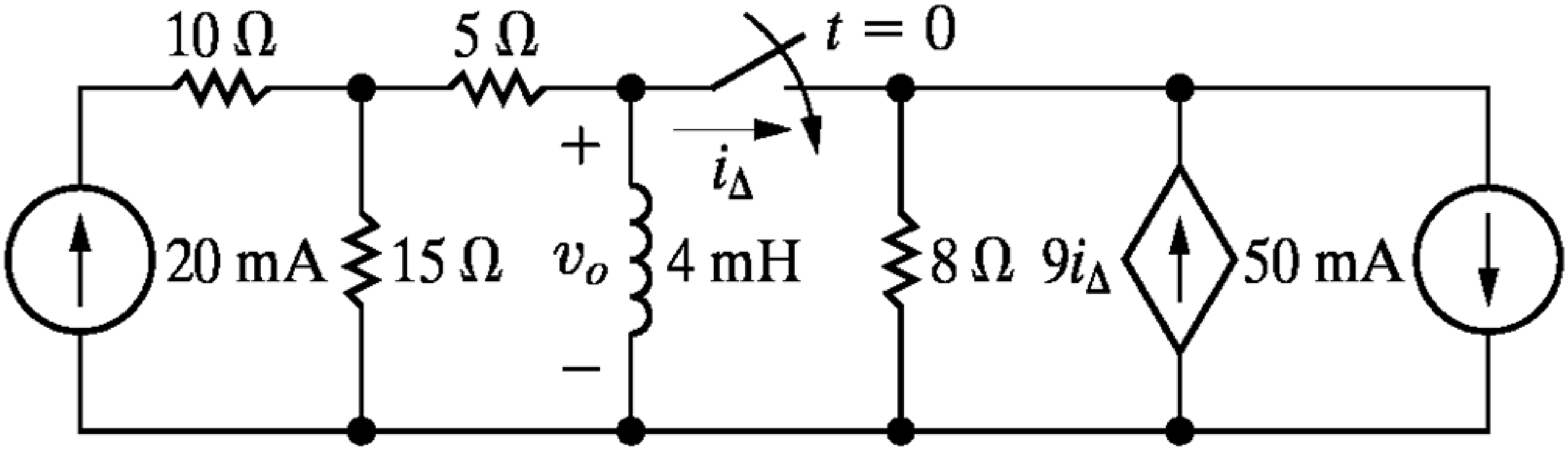

- The switch in the circuit has been in the OFF position for a long time
  1. Find $v_o(t)$ for $t \geq 0^+$
  2. Find $i_\Delta(t)$ for $t \geq 0^+$

## EX 3.15 First-Order RL Circuit




- The switch in the circuit has been open a long time before closing at $t = 0$
  1. Find $i_o(t)$ for $t \geq 0^+$

# First-Order RL Circuit




- The switch in the circuit has been open a long time before closing at $t = 0$
  1. Find $v_o(t)$ for $t \geq 0^+$



J. Nilsson and S. Riedel, *Electric Circuits*, 9th Edition, Pearson, NJ, 2011



# Contents

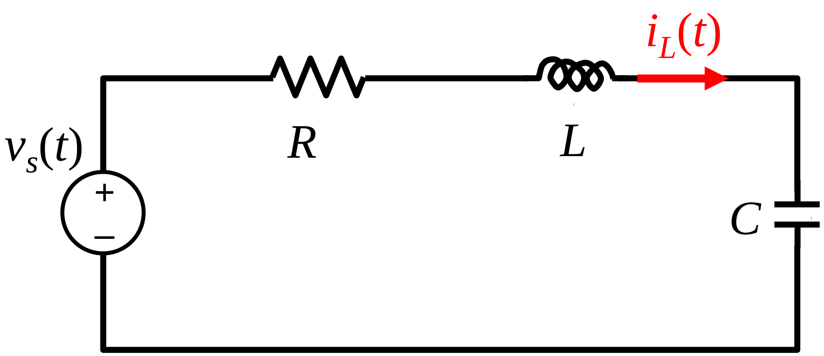
## 3.5 Linear Second-Order Circuits

# Examples of Linear 2nd-Order Circuits (1/2)

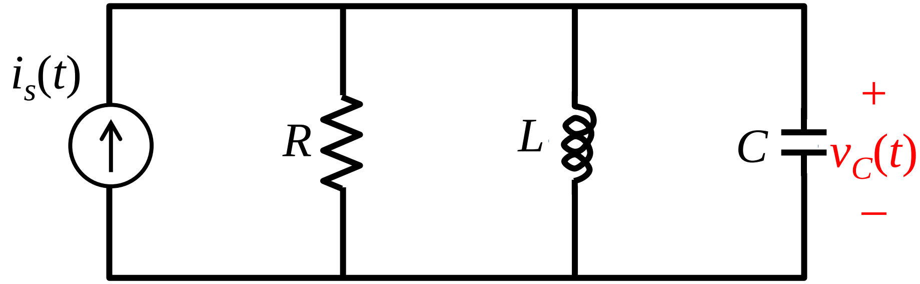
- One energy storage element
- $1^{st}$-order differential equations
- **<u>Need one initial condition to get the unique solution</u>**

- Two energy storage elements
- $2^{nd}$-order differential equations
- **<u>Need two initial conditions to get the unique solution</u>**

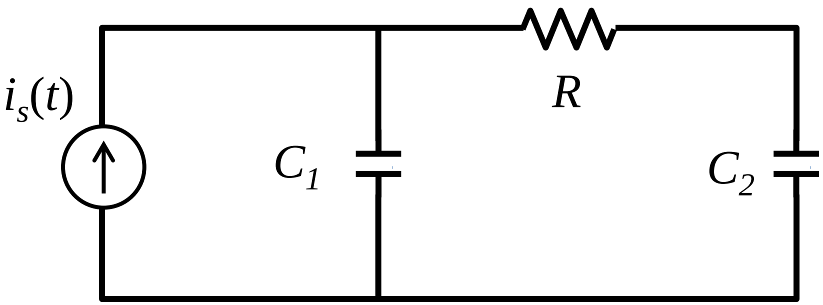
## Series RLC Circuit

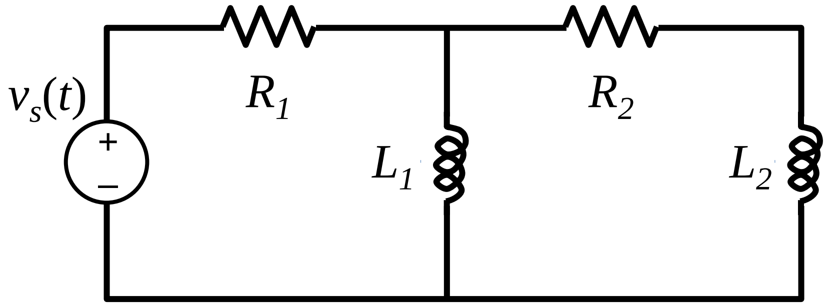



## Parallel RLC Circuit



# Examples of Linear 2nd-Order Circuits (2/2)

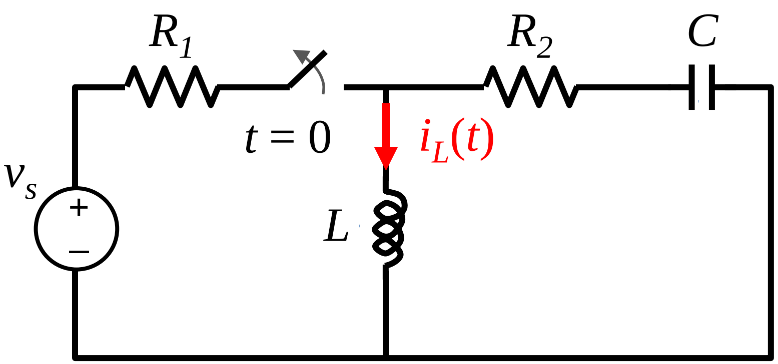


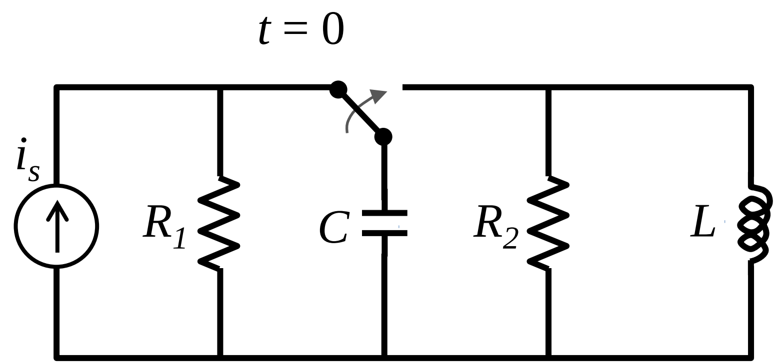

- We can't combine $C_1$ and $C_2$ ($L_1$ and $L_2$) together because there is a resistor between them

# Examples of Natural Response

## Series RLC Circuit



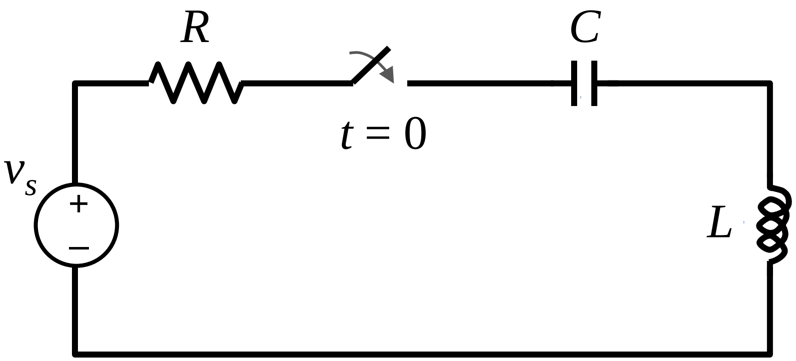

## Parallel RLC Circuit



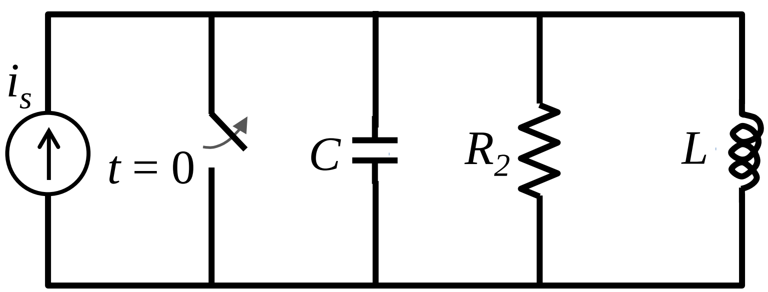

- When the switch is turned off, the new circuit has no external sources

# Examples of Step Response

Series RLC Circuit

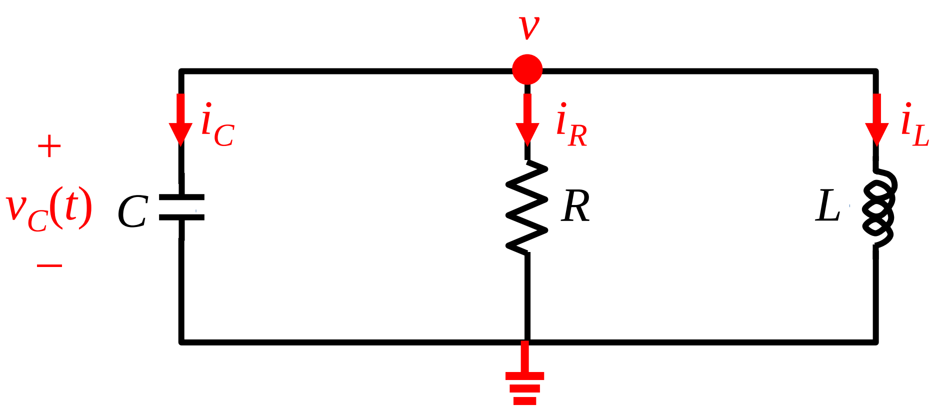



Parallel RLC Circuit




- When the switch is turned off, the new circuit has external sources

# How to Solve 2nd-Order RLC Circuits? (1/4)




Objective: find $v_C(t)$

1. Select the nodal analysis or mesh analysis and write down the equation

$$\left(C\frac{dv(t)}{dt}\right)+\left(\frac{v(t)}{R}\right)+\left(\frac{1}{L}\int_{0^+}^{t} v\,d\tau+i\left(0^+\right)\right)=0$$

2. Differentiate the equation as many times as required to get the standard form of a 2nd-order differential equation

$$C\frac{d^2v(t)}{dt^2}+\frac{1}{R}\frac{dv(t)}{dt}+\frac{1}{L}v(t)=0$$

# How to Solve 2nd-Order RLC Circuits? (2/4)

3. Solve the differential equation
   ① Homogeneous solutions $v_h(t)$
   ② Particular solution $v_p(t)$ (if the RLC circuit has external sources)

$$C\frac{d^2v(t)}{dt^2}+\frac{1}{R}\frac{dv(t)}{dt}+\frac{1}{L}v(t)=0$$

Considering that the RLC circuit has no external sources (as this example):

① Homogeneous solutions $x_h(t)$:

Suppose that the solutions have the form of $e^{mt}$

$$Cm^2e^{mt}+\frac{m}{R}e^{mt}+\frac{1}{L}e^{mt}=0 \rightarrow \left(Cm^2+\frac{m}{R}+\frac{1}{L}\right)e^{mt}=0$$

# How to Solve 2nd-Order RLC Circuits? (3/4)

Solving $Cm^2 + \frac{m}{R} + \frac{1}{L} = 0$:

$$m_1 = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$m_2 = -\frac{1}{2RC} - \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

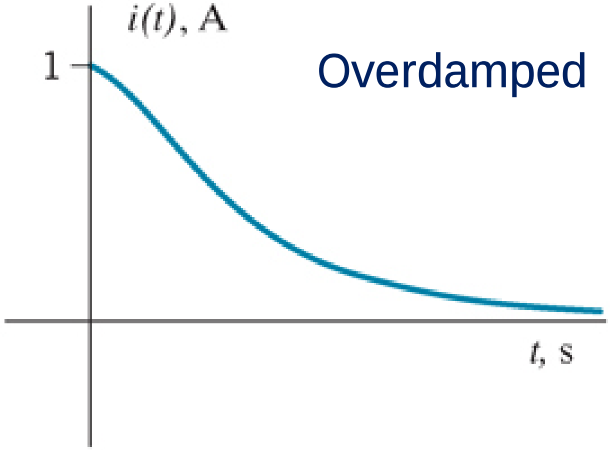
**First difference between 1st-order and 2nd-order circuits**

$\sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}} > 0,\ < 0,$ or $= 0$ leads to three different situations

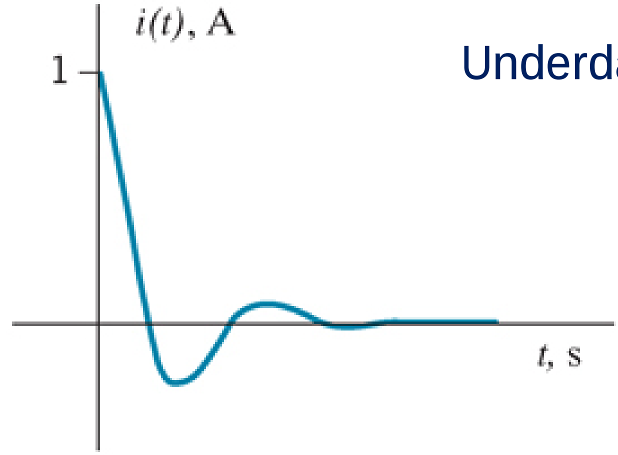
- Three new terminologies: overdamped, underdamped, & critically damped

# Overdamped, Underdamped, & Critically damped

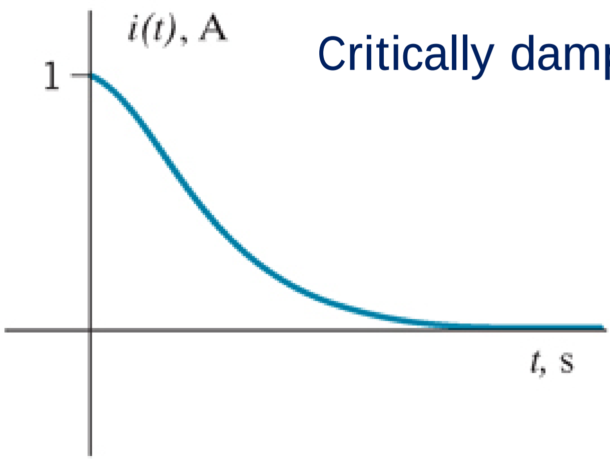
Overdamped



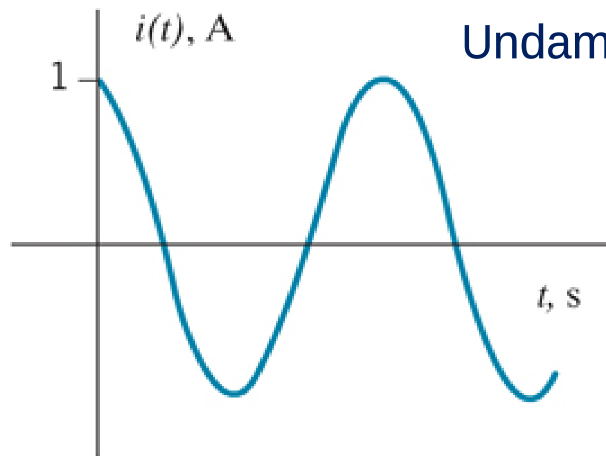

Underdamped




Critically damped




Undamped




R. C. Dorf and J. A. Svoboda, *Introduction to Electric Circuits*, 8th Edition, Wiley, NJ, 2011

# How to Solve 2nd-Order RLC Circuits? (4/4)

4. Express the final solution $v(t) = v_h(t) + v_p(t)$

(Supposing $\sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}} > 0$ )

$$\Rightarrow v_C(t) = c_1 e^{m_1 t} + c_2 e^{m_2 t}$$

**Second difference between 1st-order and 2nd-order circuits**

- In order to get the unique solution, we need to solve $c_1$ and $c_2$
- Two initial conditions (I. C.) are required (Lectures 3-2 and 3-3 only need one I. C.)
- What I. C. do we need?

$$v_C(0^+), \quad \frac{dv_C(0^+)}{dt}$$



# Contents

## 3.6 Responses of Second-Order Circuits

# Solution Procedure

All the problems in these two lectures can be casted into:

Step 1: ■Draw the circuit under $t \le 0^-$
■Find $i_L(0^-)$ on the inductor and $v_C(0^-)$ on the capacitor

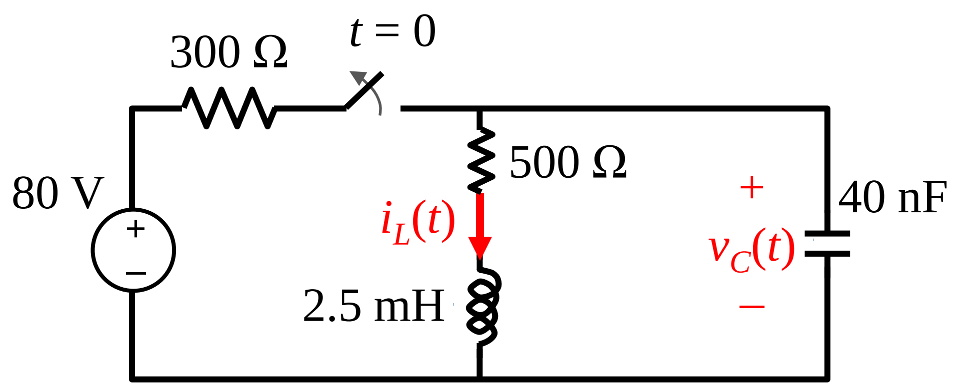
Step 2: ■Draw the circuit for $t \ge 0^+$
■Formulate the problem by nodal analysis or mesh analysis
■Differentiate the equation as many times as required to get the standard form of a 2nd order D. E.

$$a\frac{d^2x(t)}{dt^2}+b\frac{dx(t)}{dt}+x(t)=y(t)$$

Step 3: ■Solve the D. E. to get $x(t) = x_h(t) + x_p(t)$
■Find the initial conditions $x(0^+)$ and $dx(0^+)/dt$ and then get the unique solution

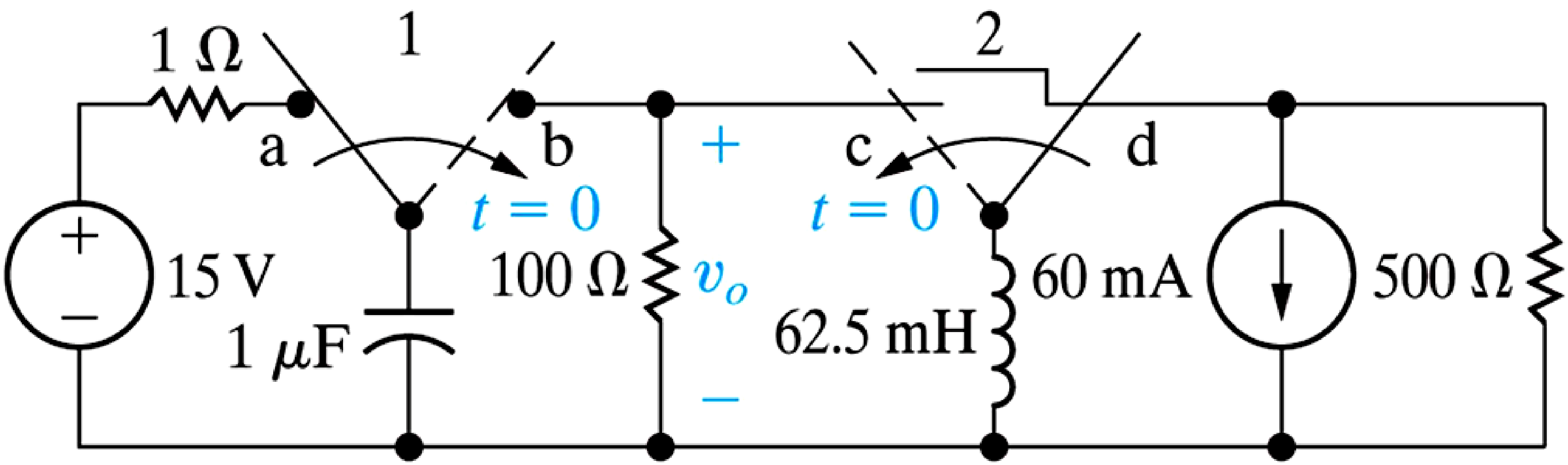
EX 3.17

# Natural Response of Series RLC Circuits




- $t \leq 0^-$: the circuit is under steady state
  1. Find $i_L(t)$ for $t \geq 0^+$

# Natural Response of Parallel RLC Circuits



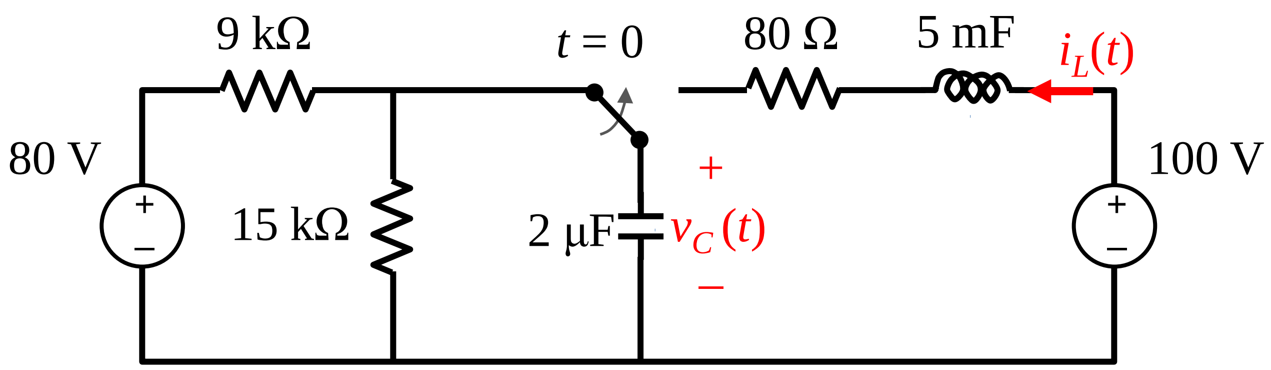

- $t \leq 0^-$: the circuit is under steady state

1. Find $v_o(t)$ for $t \geq 0^+$



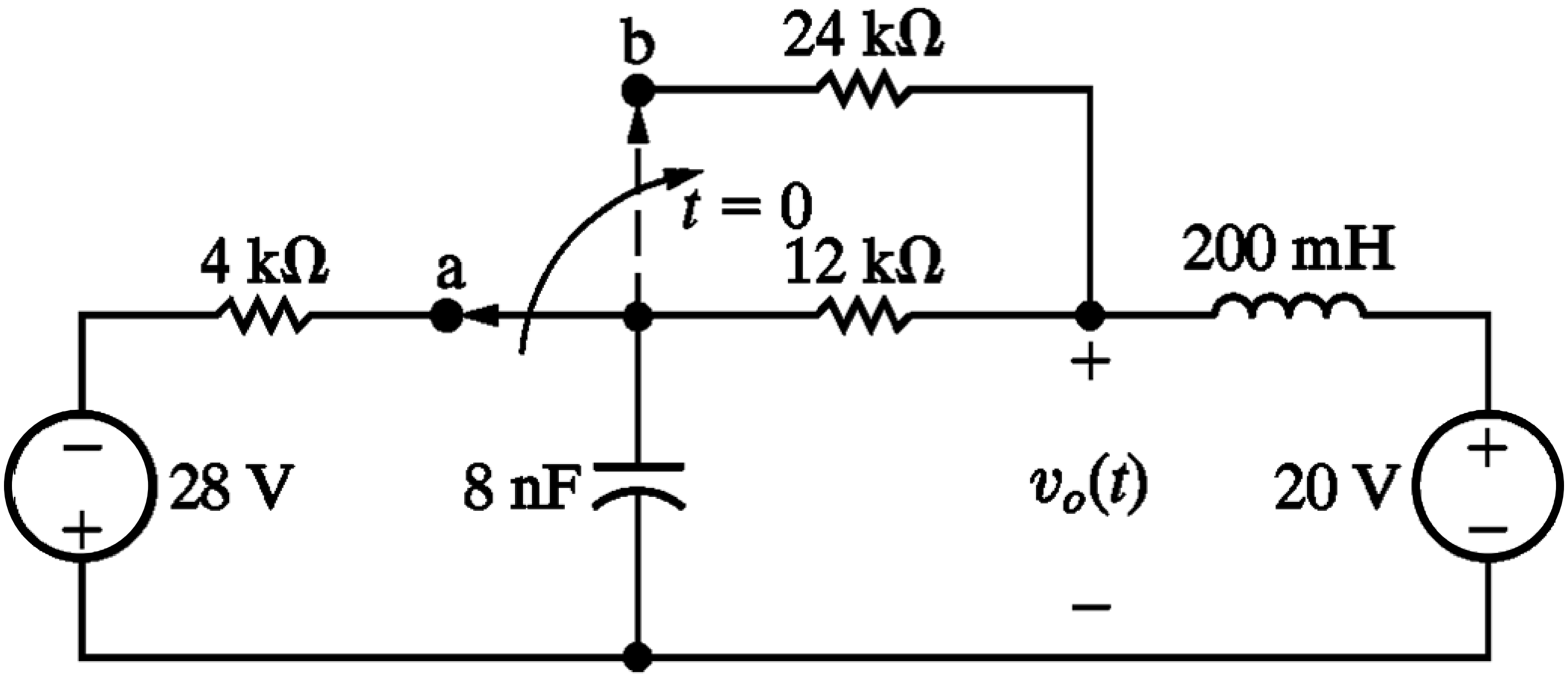
J. Nilsson and S. Riedel, *Electric Circuits*, 9th Edition, Pearson, NJ, 2011

# EX 3.19 Step Response of Series RLC Circuits



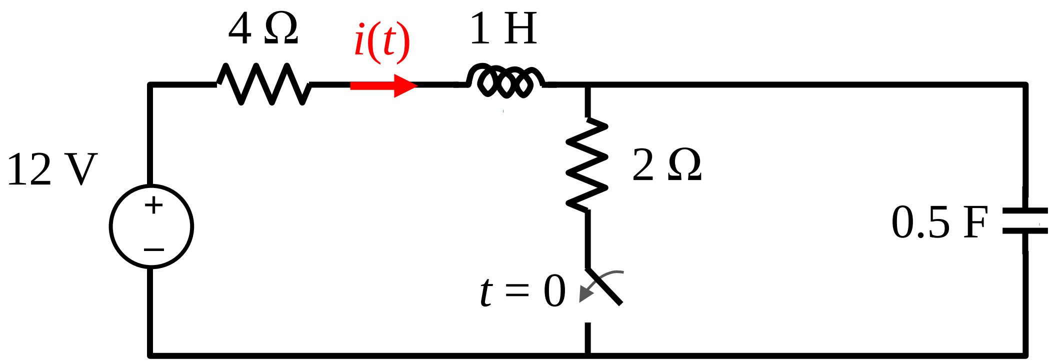

- $t \le 0^-$: the circuit is under steady state
  1. Find $i_L(0^+)$ for $t \ge 0^+$
  2. Find $di_L(0^+)/dt$ for $t \ge 0^+$
  3. Find $i_L(t)$ for $t \ge 0^+$

# EX 3.20 Second-Order RLC Responses




- The switch in the circuit has been in position *a* for a long time
  1. Find $i_L(t)$ for $t \geq 0^+$

# EX 3.21 A More Complex Example




- $t \le 0^-$: the circuit is under steady state
  1. Find $i(0^+)$ for $t \ge 0^+$
  2. Find $di(0^+)/dt$ for $t \ge 0^+$
  3. Find $i(t)$ for $t \ge 0^+$