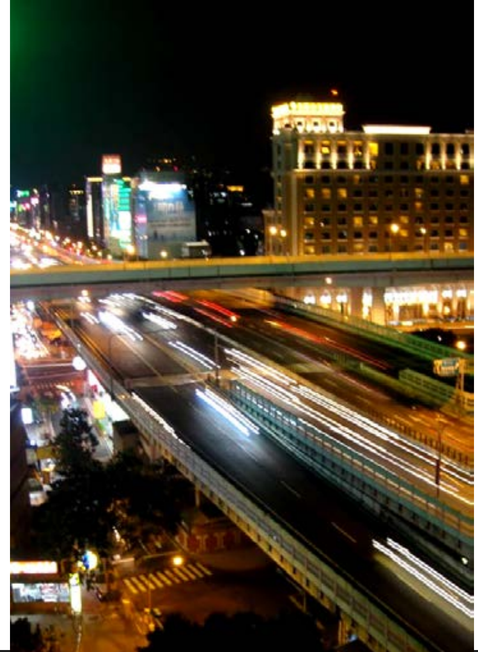
TAIPEI TECH

國立臺北科技大學



電路學 Circuit Theory

Lecture 2

Advanced Techniques for Circuit Analysis

Week 4, Fall 2019

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Lecture 2:
Advanced Techniques for Circuit Analysis

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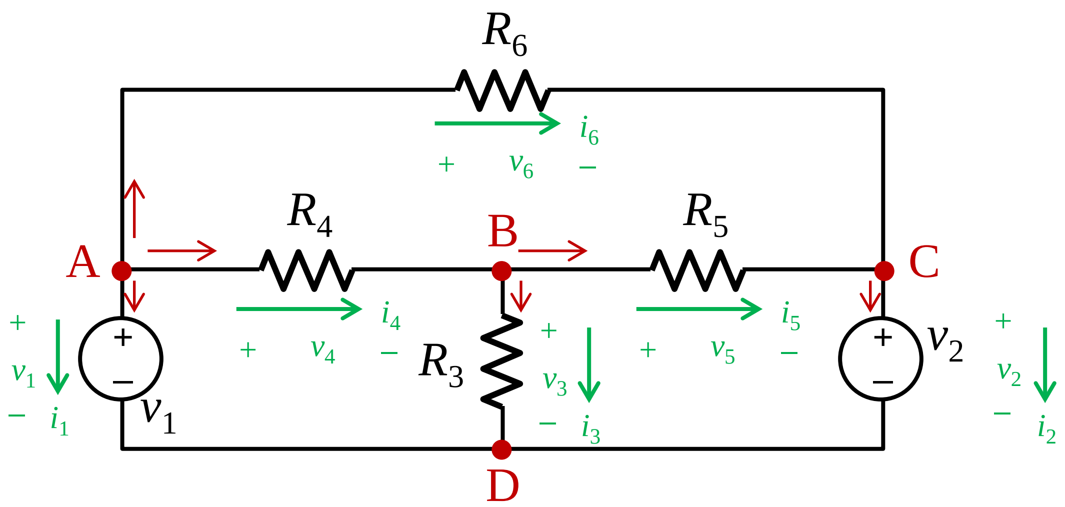
2.1 Nodal Analysis (Node-Voltage Method)

Too Many Variables in 2B Method! (1/2)



2B method: a direct algebraic approach

KCL:

- Node A: $i_1 + i_4 + i_6 = 0$
- Node B: $-i_4 + i_3 + i_5 = 0$
- Node C: $i_2 - i_5 - i_6 = 0$
- Node D: $-i_3 - i_2 - i_1 = 0$

3 KCL equations

KVL:

- Loop ABDA: $v_4 + v_3 - v_1 = 0$
- Loop BCDB: $v_5 + v_2 - v_3 = 0$
- Loop ACBA: $v_6 - v_5 - v_4 = 0$

3 KVL equations

Component models:

- v_1 = Given v_1
- v_2 = Given v_2
- $v_3 = R_3 i_3$
- $v_4 = R_4 i_4$
- $v_5 = R_5 i_5$
- $v_6 = R_6 i_6$

Too Many Variables in 2B Method! (2/2)

Objective of 2B method: find v_k and i_k, k = 1, 2, 3, ..., B

- The number of unknown: 2B
- The number of KCL equations: $N - 1$
- The number of KVL equations: $B - (N - 1)$
- The number of component model: B
- So there are 2B equations for solving 2B unknowns!

But it has serious problems:

- Too many variables
- Not efficient at all

➡ We need better methods to reduce the complexity!

Two IMPORTANT Methods in Midterm

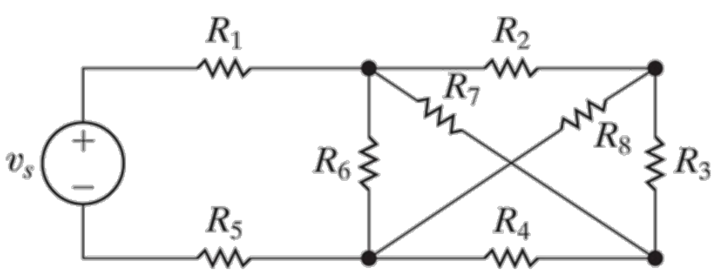
Nodal analysis

- Based on KCL
- Suitable for general cases

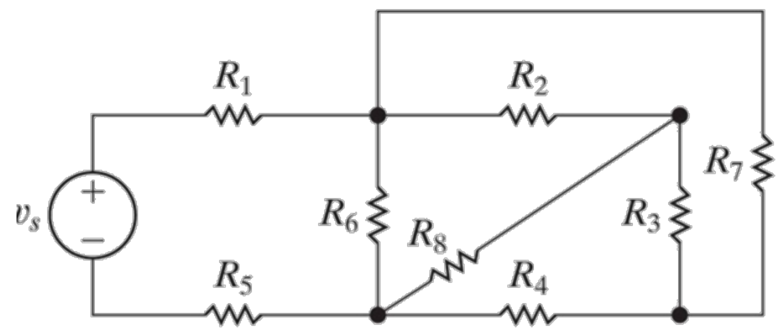
Mesh analysis

- Based on KVL
- Only suitable for planar circuits

Terminology: planar circuit

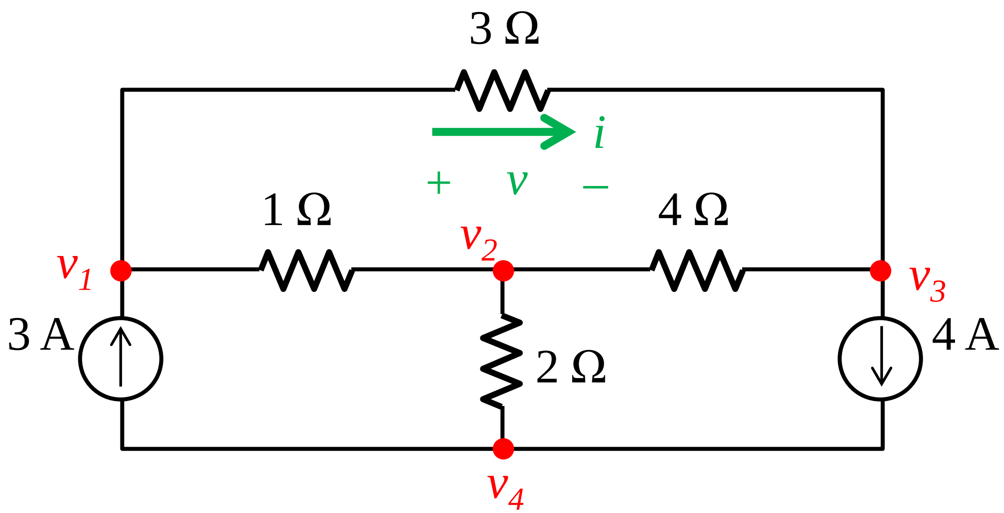


- Before putting it in order



- After rearrangement
- Definition: a circuit that can be drawn on a plane with no crossing branches

The Idea of Nodal Analysis



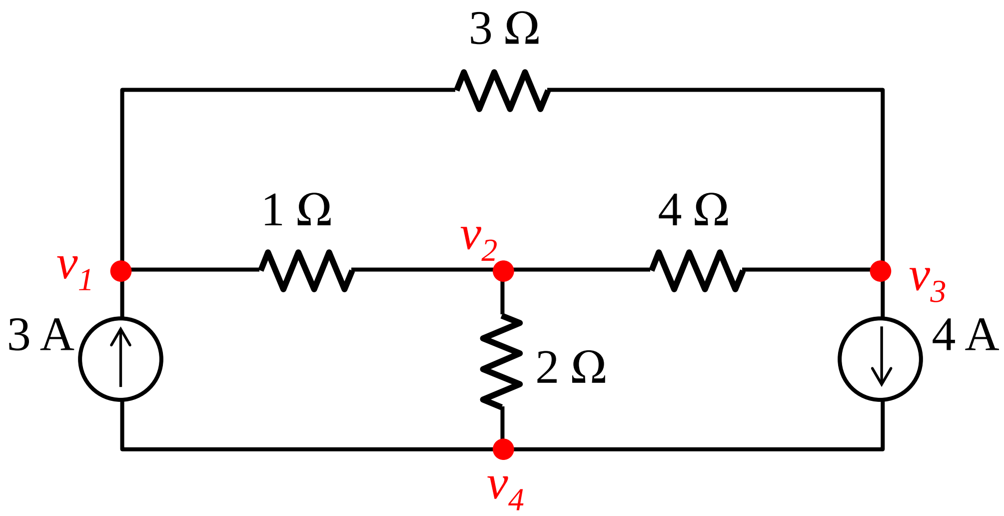
Previous method: use **branch** currents as the variable

- But the number of branch is much larger than the number of node

New method: use **node** voltages as the variable

- Each branch voltage can be obtained from nodal voltages
- The corresponding branch current can be calculated by its component model
- 12 branch variables → 3 node variables

Characteristics of Node-Voltage Method:



1. <u>**KVL is automatically satisfied in every loop (why?)**</u>

2. We only need to find the independent KCL equations
3. The component model is applied so that the branch current can be expressed

$$Gv = I$$

Five Cases in the Following Analysis

Only containing current sources

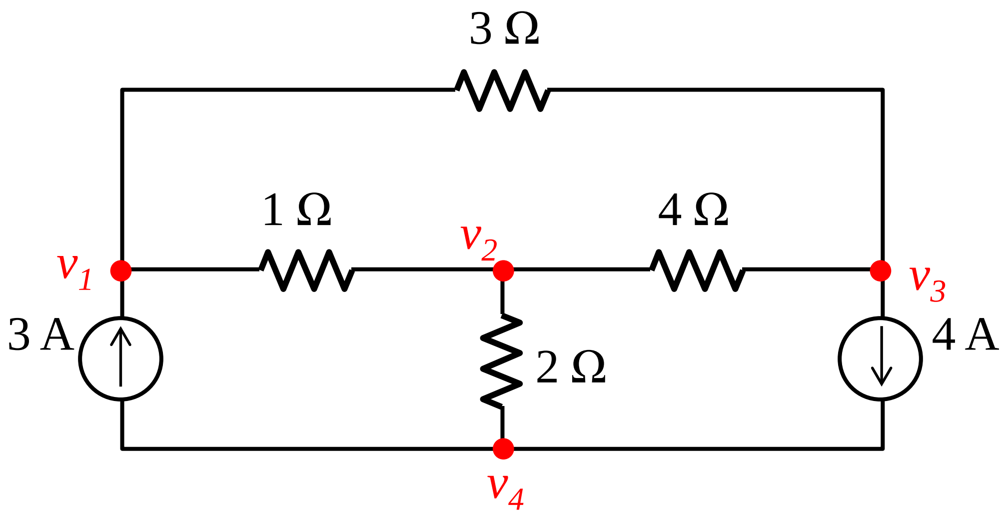
1. With independent current sources: basic case
2. With dependent current sources: expressing the controlling parameters in terms of nodal voltages

Containing voltage sources

3. Voltage sources connect to the reference node: the easiest case
4. Voltage sources do not contain the reference node: by supernode concept
5. With dependent voltage sources: expressing the controlling parameters in terms of the nodal voltages

Ex. 2.1 With Independent Current Source

CASE 1



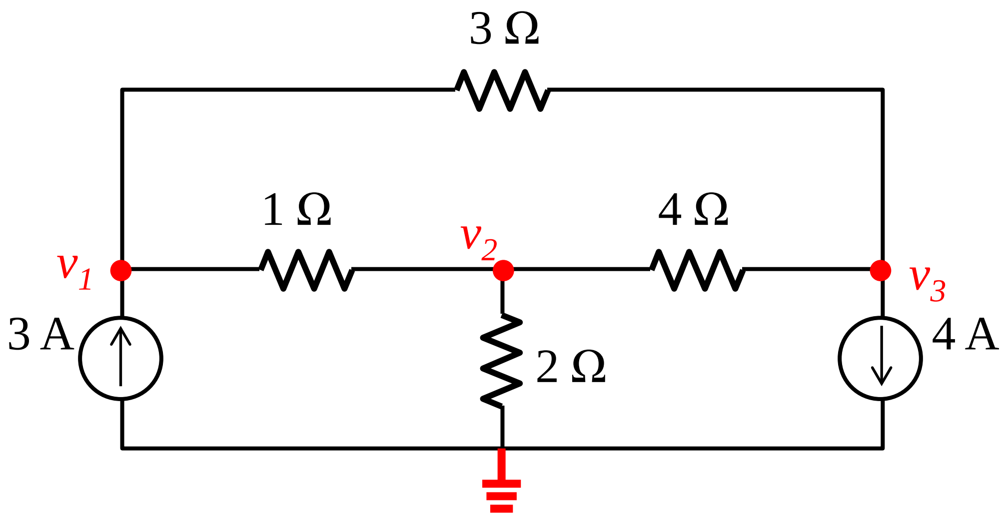
Step 1

- Find 4 nodes
- Select a node as reference node. Which one?

- Set v_1, v_2, and v_3 as unknowns

With Independent Current Source

CASE 1



Step 2

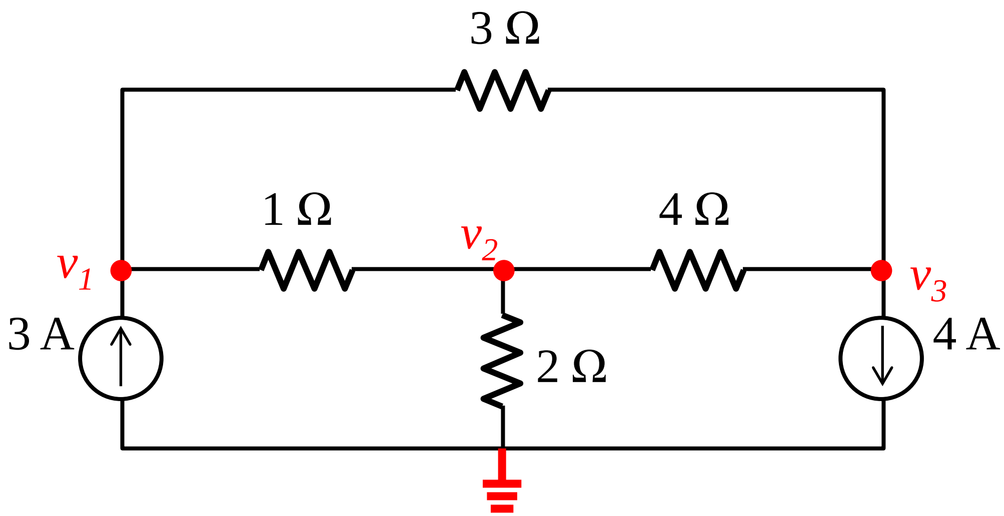
- Apply KCL to n – 1 nodes and use the component model (Ohm's law) to express the branch currents in terms of nodal voltages

$$\sum_{leaving} i_k = 0 \quad \Rightarrow$$

$\forall$ node

Ex. 2.1 With Independent Current Source

CASE 1



Step 3 (not necessary)

- Matrix expression

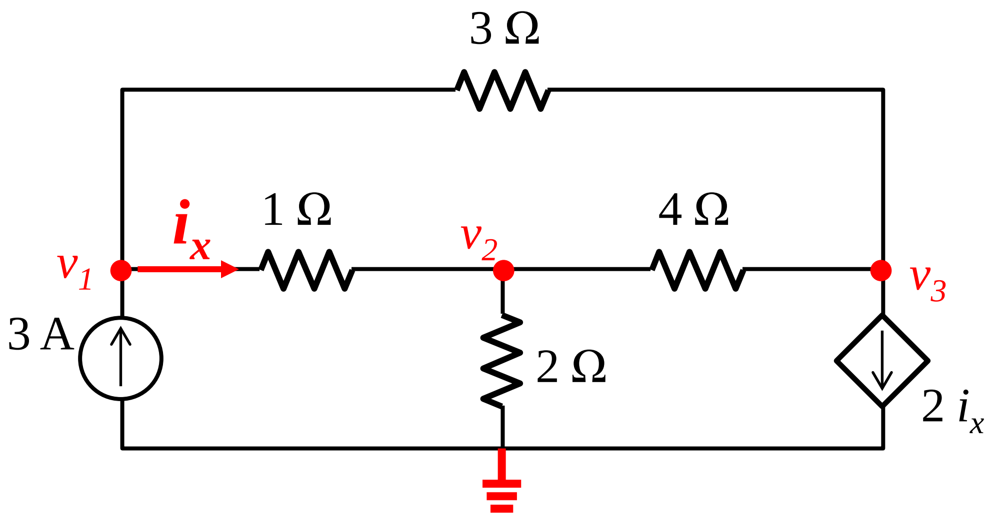
$$Gv = I$$

$$\begin{bmatrix} 1+\frac{1}{3} & -1 & -\frac{1}{3} \\ -1 & 1+\frac{1}{2}+\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{3} & -\frac{1}{4} & \frac{1}{3}+\frac{1}{4} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 3A \\ 0 \\ -4A \end{bmatrix}$$

G	Conductance matrix
G_{kk}	Sum of the conductances connected to node k
G_{kj}	Negative of the sum of conductances directly connecting nodes k and j ($k \neq j$)
v_k	Unknown; nodal voltage of the k^{th} node
I_k	Sum of independent current sources ***entering*** to node k

Ex. 2.2 With Dependent Current Source

CASE 2



Step 1

- Set v_1, v_2, and v_3 as unknowns; set a reference node
- How to express i_x?

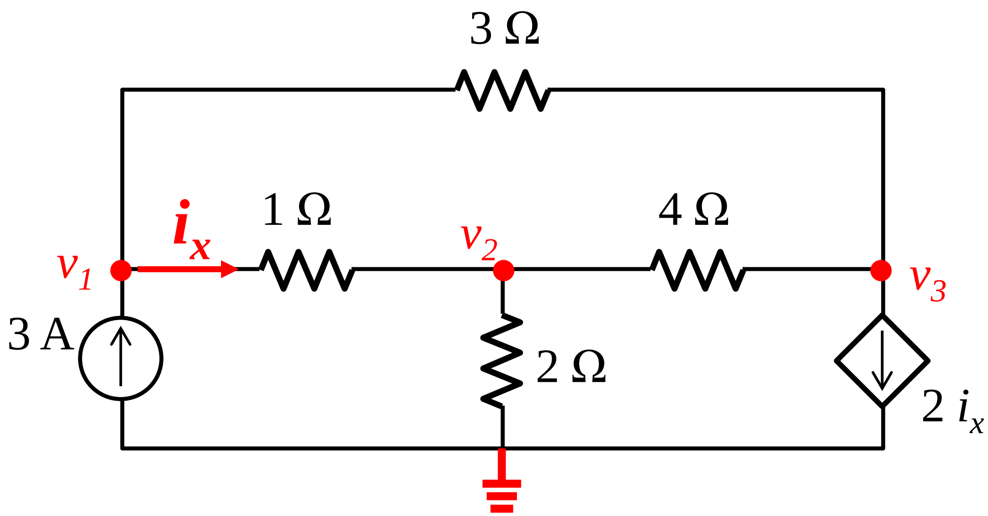
Watch out the current direction!

Step 2

- Apply KCL to n – 1 nodes

Ex. 2.2 With Dependent Current Source

CASE 2

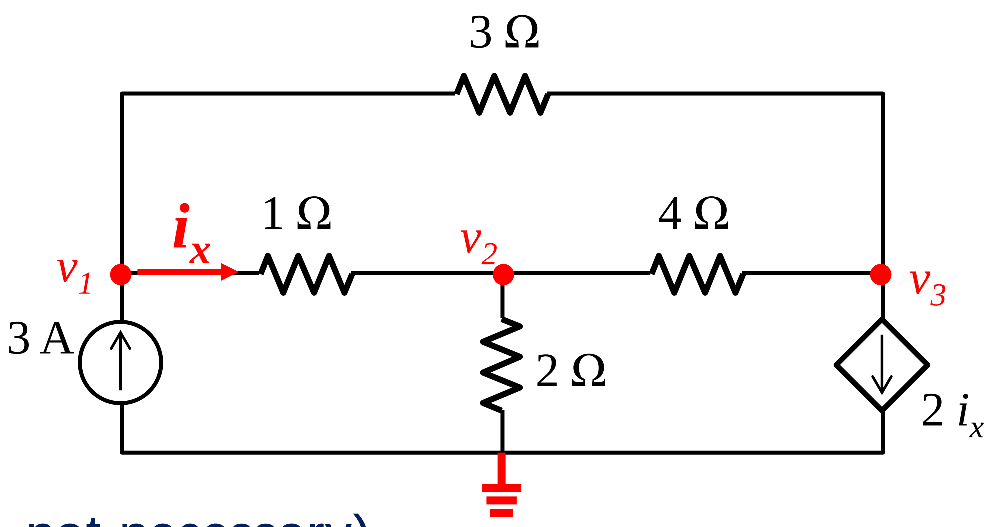


- Rearrange the equations and solve them!

Ex. 2.2

With Dependent Current Source

CASE 2



Step 3 (again, not necessary)

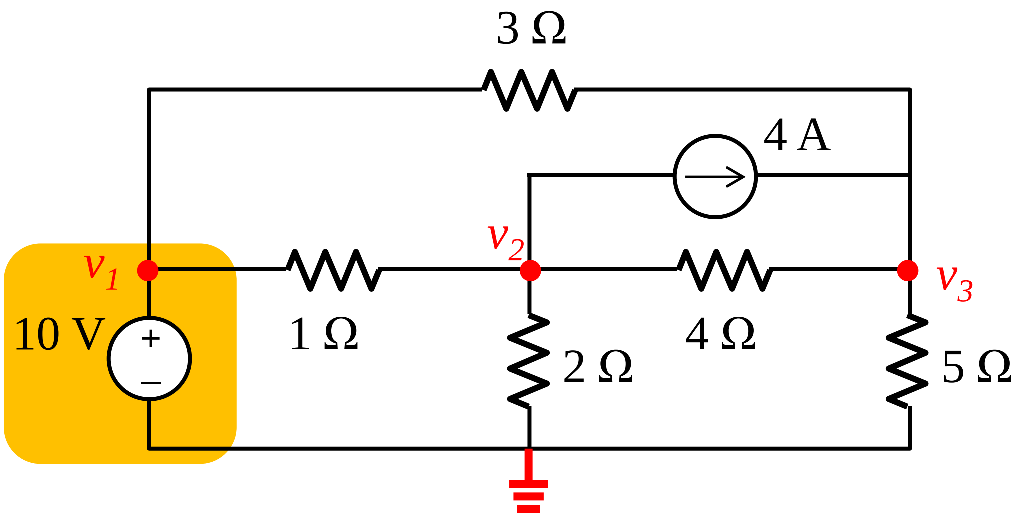
- Step 3: Matrix expression

$$\begin{bmatrix} 1+\frac{1}{3} & -1 & -\frac{1}{3} \\ -1 & 1+\frac{1}{2}+\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{3}+2 & -\frac{1}{4}-2 & \frac{1}{3}+\frac{1}{4} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 3A \\ 0 \\ 0 \end{bmatrix}$$

- No longer a symmetric matrix
- Off diagonal terms are not always negative

Ex. 2.3 With Voltage Source Connected to the Reference Node

CASE 3



Step 1

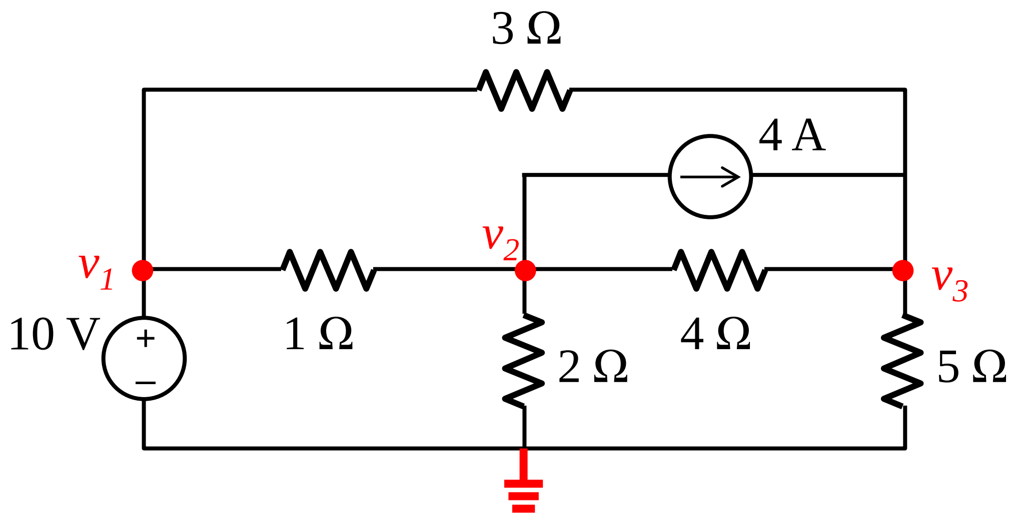
- Set v_1, v_2, and v_3 as unknowns; set a reference node
- How to express v_1?

Step 2

- Apply KCL to the 2nd and 3rd nodes

With Voltage Source Connected to the Reference Node

CASE 3



Step 3 (not necessary)

- Matrix expression

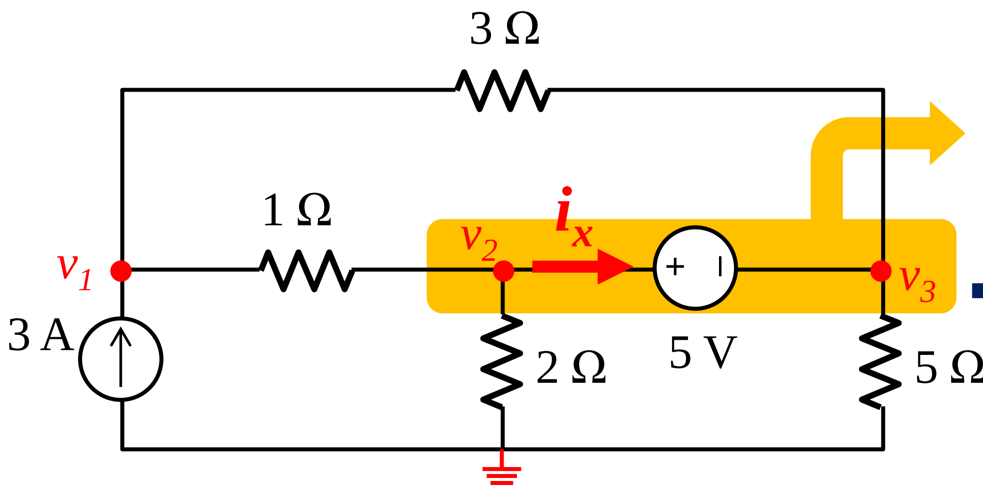
$$\begin{bmatrix} 1+\frac{1}{2}+\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{3}+\frac{1}{4}+\frac{1}{5} \end{bmatrix}\begin{bmatrix} v_2 \\ v_3 \end{bmatrix}=\begin{bmatrix} 10-4 \\ \frac{10}{3}+4 \end{bmatrix}$$

- Only two unknowns!

$$v_1 = 10\ V$$

With Voltage Source NOT Connected to the Reference Node

CASE 4



Component model of (+ −) :

v_s = 5 V

i_s = ?

➡ Define an extra unknown i_x

Step 1

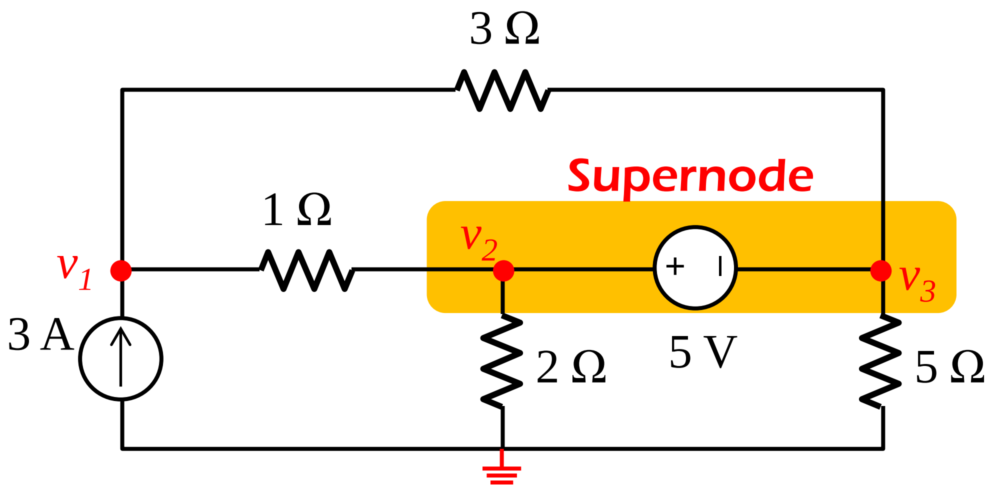
- Set v_1, v_2, and v_3 as unknowns; set a reference node
- What's the relation between v_2 and v_3?

Step 2

- Apply KCL to n – 1 nodes

With Voltage Source NOT Connected to the Reference Node

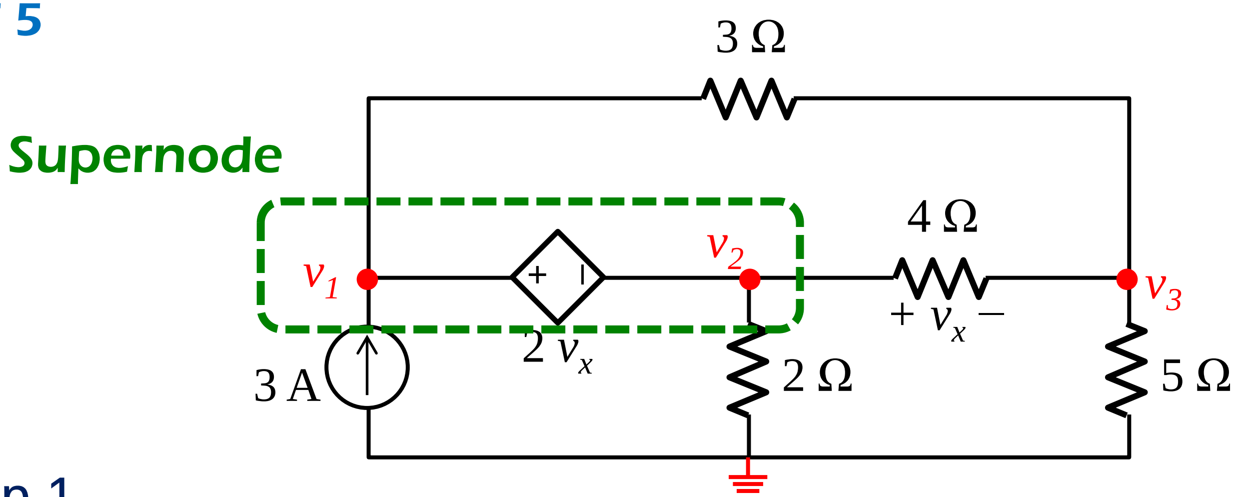
CASE 4



- And we already know that $v_2 - v_3 = 5$ V:

Ex. 2.5 With Dependent Voltage Source

CASE 5



Step 1

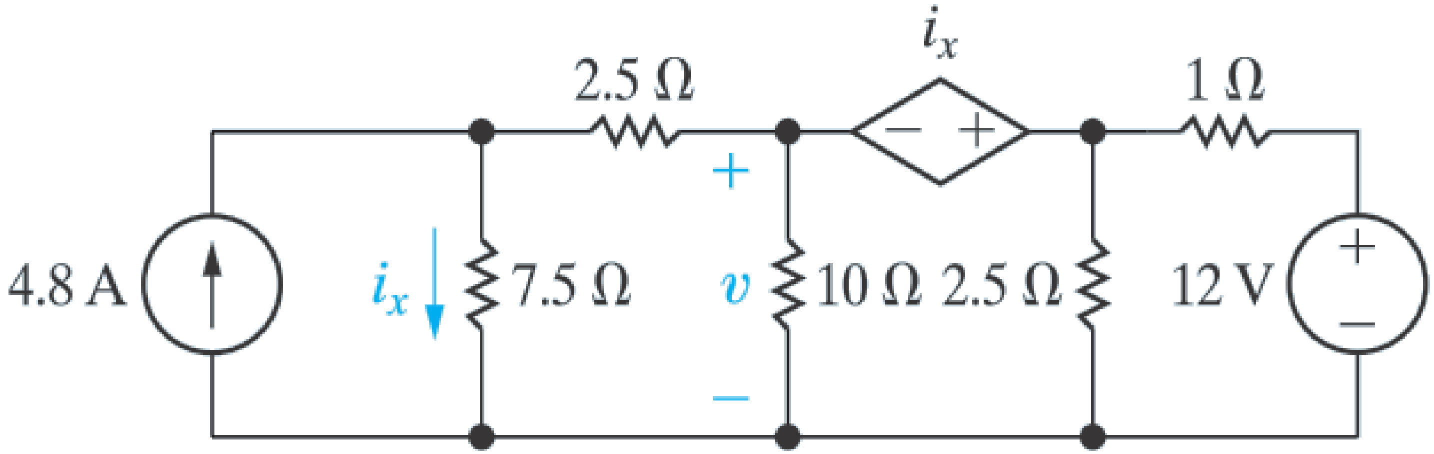
- Set v_1, v_2, and v_3 as unknowns; set a reference node
- What's the relation between v_1 and v_3?
- How to express v_x?

Step 2

- Apply KCL to the 3rd node and the supernode

Ex. 2.6

Example of Nodal Analysis

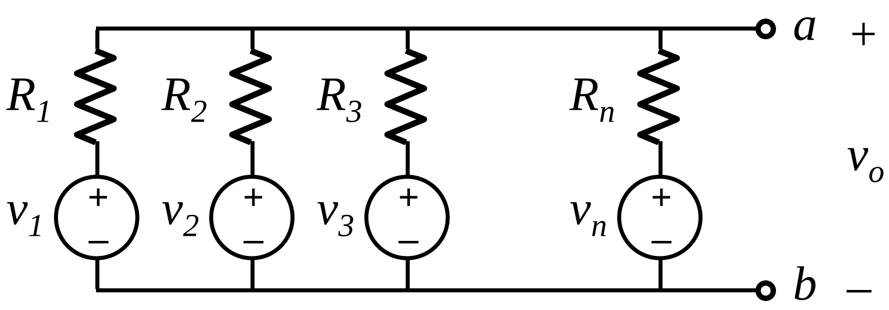


1. Find v
2. Find the power consumed (or supplied) of the dependent voltage source

J. Nilsson and S. Riedel, *Electric Circuits*, 9th Edition, Pearson, NJ, 2011

Millman's Theorem

A.K.A. "Sharing bus method"



- By using node-voltage method (choose b as the reference node):

$$\frac{v_o - v_1}{R_1} + \frac{v_o - v_2}{R_2} + \ldots + \frac{v_o - v_n}{R_n} = 0$$

$$\Rightarrow \quad v_o = \frac{\dfrac{v_1}{R_1} + \dfrac{v_2}{R_2} + \ldots + \dfrac{v_n}{R_n}}{\dfrac{1}{R_1} + \dfrac{1}{R_2} + \ldots + \dfrac{1}{R_n}}$$

- If $R_1 = R_2 = \ldots = R_n = R$, $v_o = \dfrac{1}{n}\sum_{k=1}^{n} v_k$

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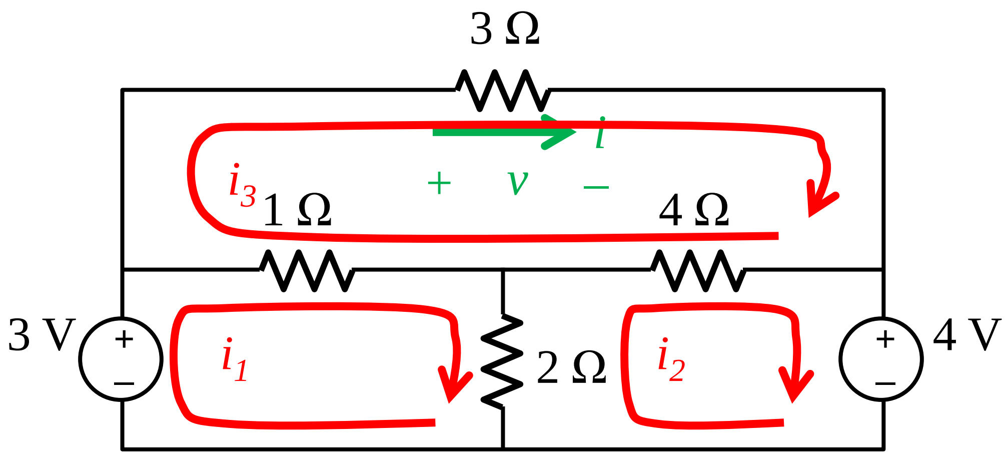
2.2 Mesh Analysis (Mesh-Current Method)

The Idea of Mesh Analysis



Only valid for planar circuit

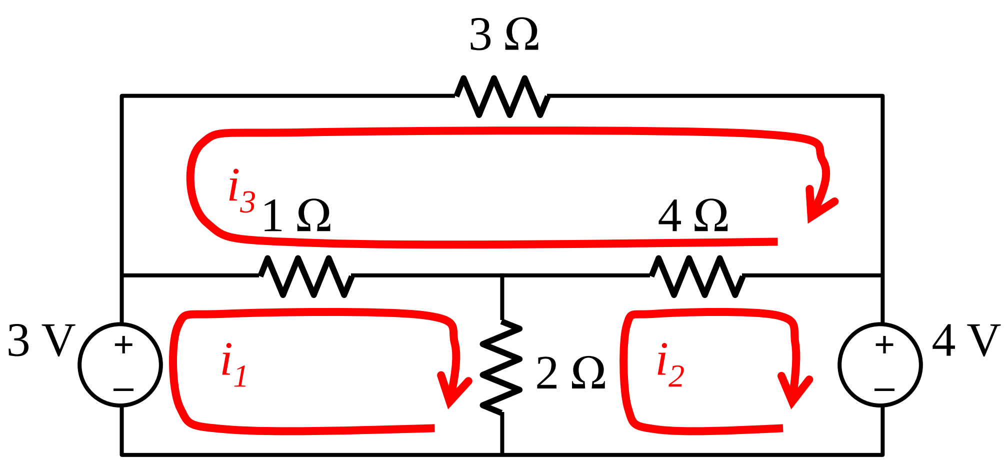
Previous method: use **branch** voltages as the variable

- But the number of branch is much larger than the number of mesh

New method: use **Mesh** currents as the variable

- Each branch current can be obtained from mesh currents
- The corresponding branch voltage can be calculated by its component model
- 12 branch variables → 3 mesh variables

Characteristics of Mesh-Current Method



The current which runs through 1 Ω:

$i_1 - i_3$

The current which runs through 2 Ω:

$i_1 - i_2$

The current which runs through 3 V:

i_1

KVL on loop 1

$(-3) + 1(i_1 - i_3) + 2(i_1 - i_2)=0$

1. **KCL is automatically satisfied in every node (why?)**
2. We only need to find the independent KVL equations
3. For a planar circuit consists of B branches and N nodes, one can have $B-(N-1)$ independent meshes
4. The direction of mesh current should be kept as the same orientation

Five Cases in the Following Discussion

Only containing voltage sources

1. Only with independent voltage sources: basic case
2. With dependent voltage sources: expressing the controlling parameter in terms of mesh currents

Containing current sources

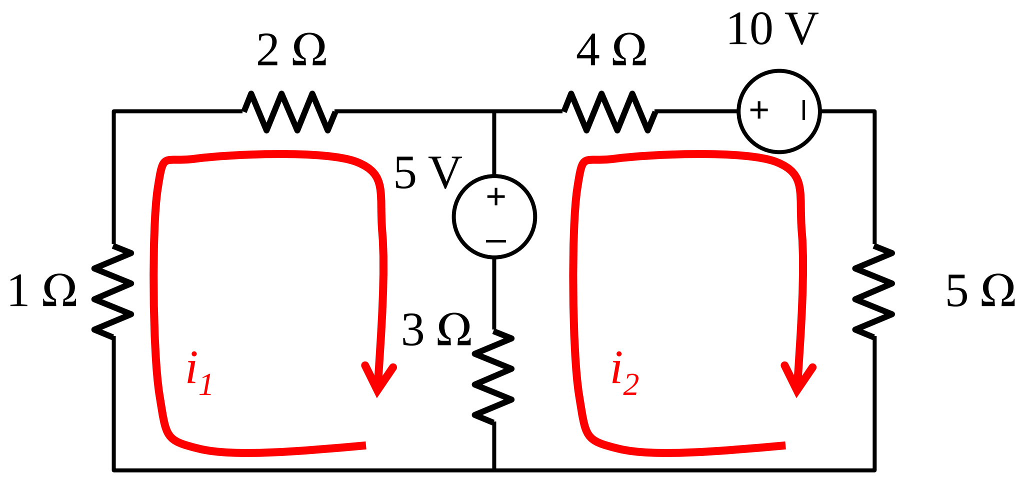
3. Independent current sources exist in single mesh: the easiest case
4. Independent current sources exist between two adjacent meshes: by supermesh concept
5. With dependent current sources: expressing the controlling parameters in terms of the mesh currents

Ex. 2.7 With Independent Voltage Source

CASE 1

$b = 7, n = 6$

$b - (n - 1) = 2$



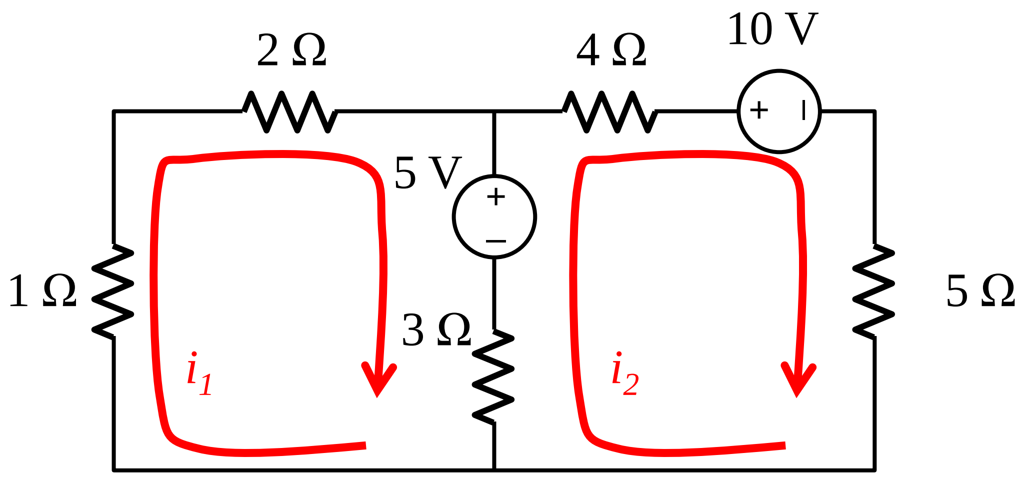
- The number of branch: $b = 7$
- The number of node: $n = 6$
- So, there are $b - (n - 1) = 7 - (6 - 1) = 2$ meshes

Step 1

- Find 2 meshes and select i_1 and i_2 as mesh currents
- Choose the mesh currents as clockwise direction

Ex. 2.7 With Independent Voltage Source

CASE 1



Step 2

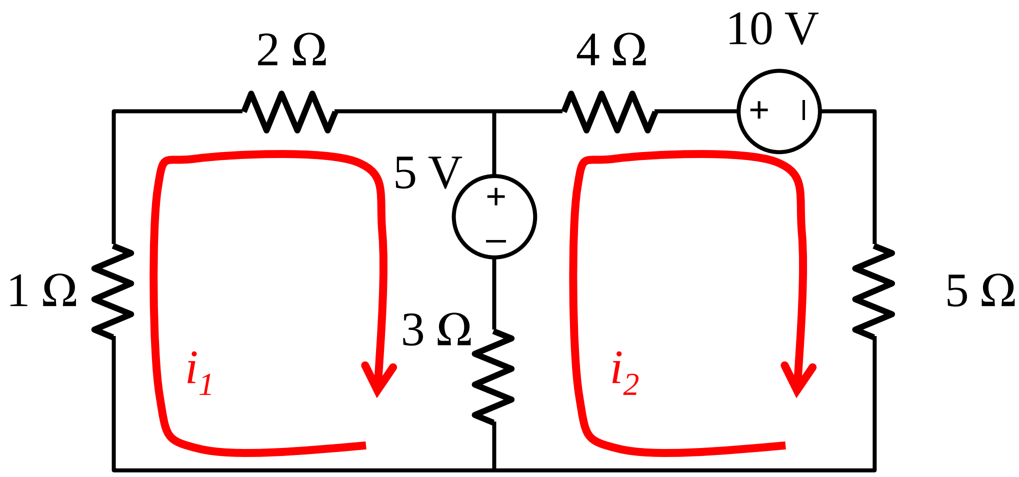
- Apply KVL to 2 meshes and use the component model (Ohm's law) to express the branch voltages in terms of mesh currents

$$\sum_{drop} v_k = 0 \quad \Rightarrow$$

$\forall$ mesh

Ex. 2.7 With Independent Voltage Source

CASE 1



Step 3 (not necessary)

- Matrix expression

$$\begin{bmatrix} 1+2+3 & -3 \\ -3 & 3+4+5 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} -5 \\ 5-10 \end{bmatrix}$$

$$Ri = V$$

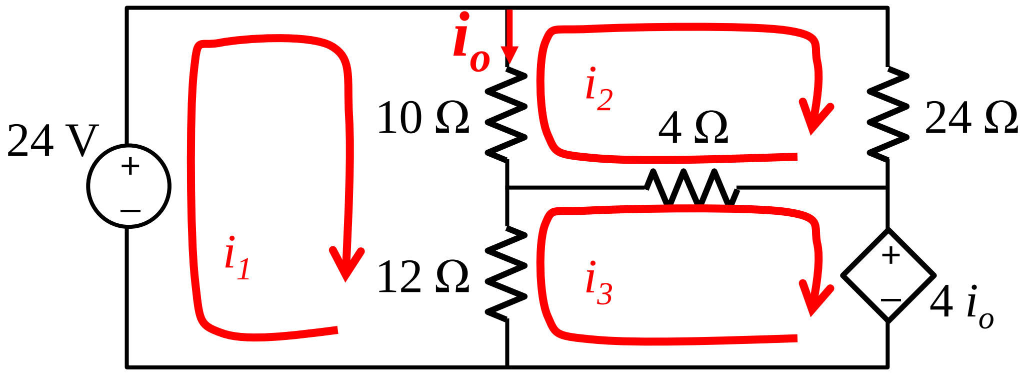
R	Resistance matrix
R_{kk}	Sum of the resistances in mesh k
R_{kj}	Sum of the resistance between meshes i and k and the algebraic sign depends on the relative direction of meshes i and k, plus (minus) sign for same (opposite) direction
V_k	Sum of independent voltage ***rises*** at the k^{th} mesh
i_k	Unknown; mesh current at the k^{th} mesh

With Dependent Voltage Source

CASE 2

$b = 6, n = 4$

$b - (n - 1) = 3$



Step 1

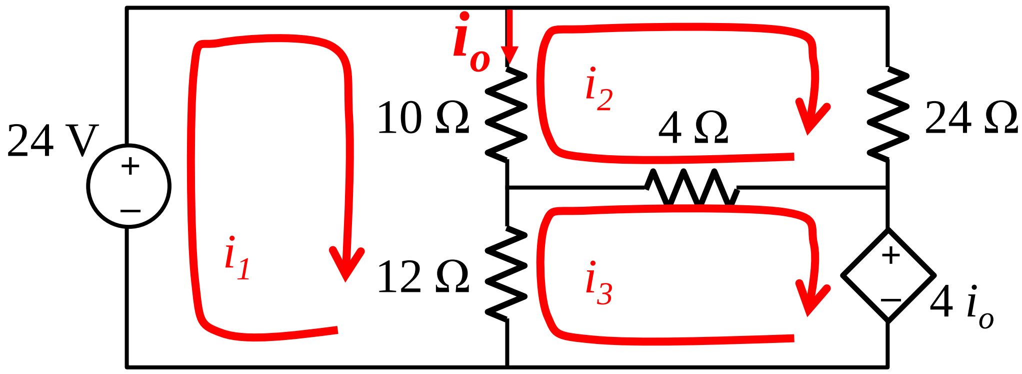
- Assign mesh currents: i_1, i_2,and i_3
- Express i_0 in terms of the mesh currents (i_1, i_2,and i_3)

Step 2

- Apply KVL to 3 meshes:

With Dependent Voltage Source

CASE 2



Step 3

- Matrix expression (not necessary)

$$\begin{bmatrix} 10+12 & -10 & -12 \\ -10 & 10+4+24 & -4 \\ -12+4 & -4-4 & 4+12 \end{bmatrix}\begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 24 \\ 0 \\ 0 \end{bmatrix} \qquad 4\,i_o = 4\,(i_1 - i_2)$$

i_1 = 2.25 A, i_2 = 0.75 A, i_3 = 1.5 A

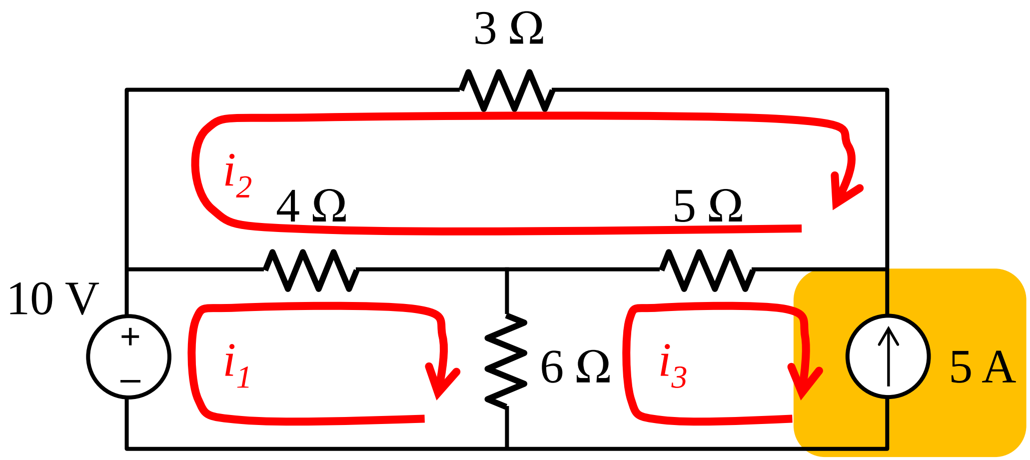
How about the current, voltage and power through the 10-Ω resistor and the CCVS?

With Current Source Existed in Only One Mesh

CASE 3

$b = 6, n = 4$

$b - (n - 1) = 3$



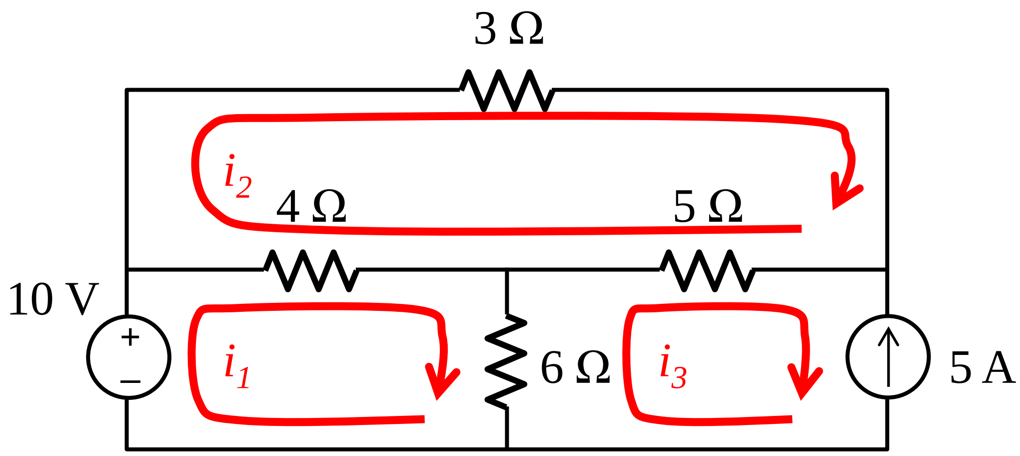
Step 1

- Assign mesh currents: i_1, i_2,and i_3
- How to express i_3?

Step 2

- Apply KVL to 2 meshes:

Ex. 2.9 With Current Source Existed in Only One Mesh



- Rearrange the equations:

Step 3

- Matrix expression (not necessary)

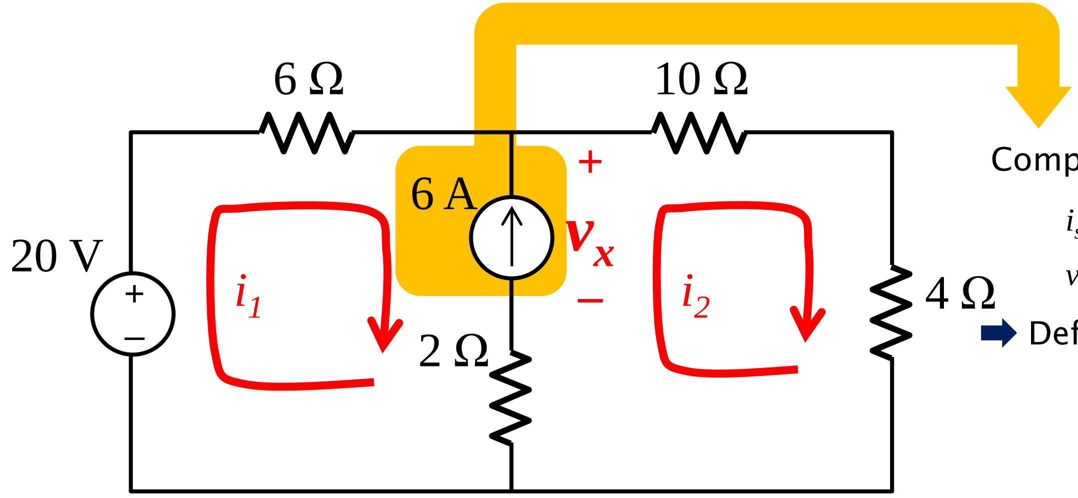
$$\begin{bmatrix} 4+6 & -4 \\ -4 & 3+5+4 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 10-30 \\ -25 \end{bmatrix}$$

Ex. 2.10 With Current Source Existed Between Two Meshes

CASE 4

$b = 6, n = 5$

$b - (n - 1) = 2$



Component model of (↑) :

i_s = 6 A

v_s = ?

➡ Define an extra unknown v_x

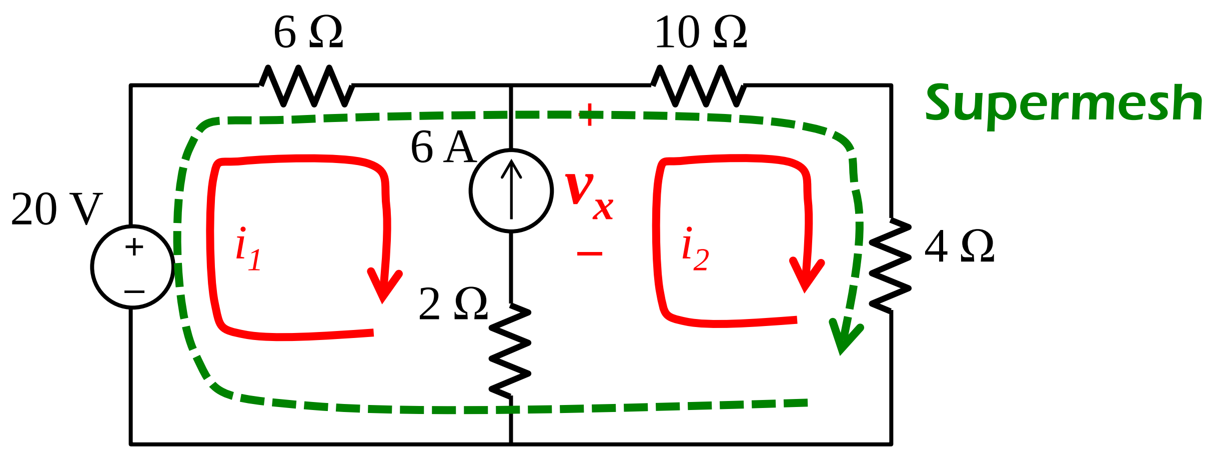
Step 1

- Assign mesh currents: i_1 and i_2
- What's the relation between i_1 and i_2?

Step 2

- Apply KVL to 2 meshes:

With Current Source Existed Between Two Meshes



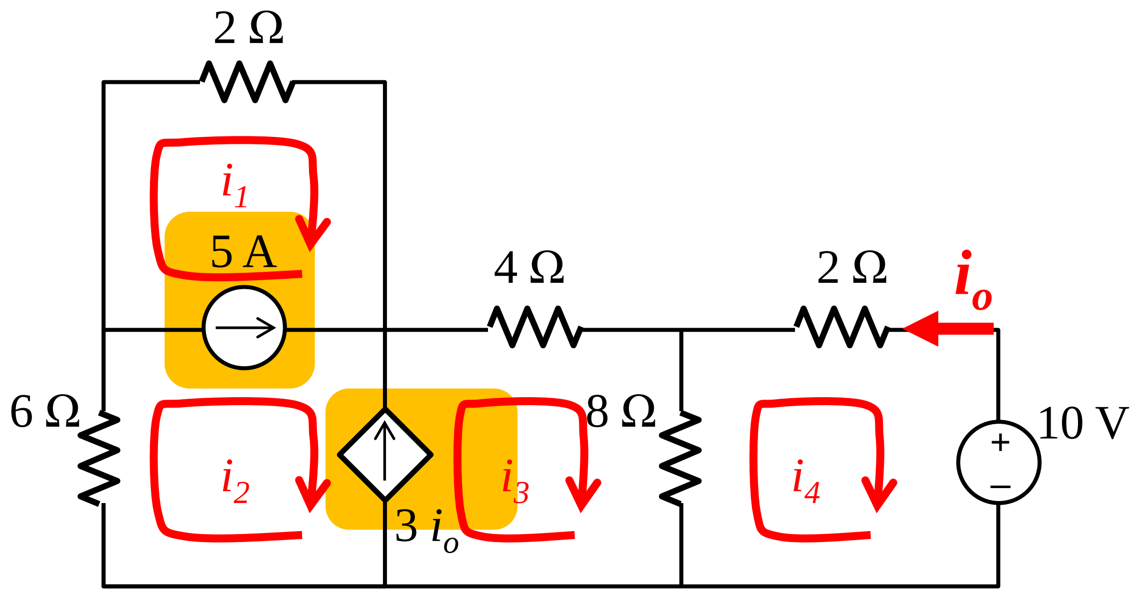
- And we already know that $i_2 - i_1 = 6$ A

With Dependent Current Source

CASE 5

$b = 8, n = 5$

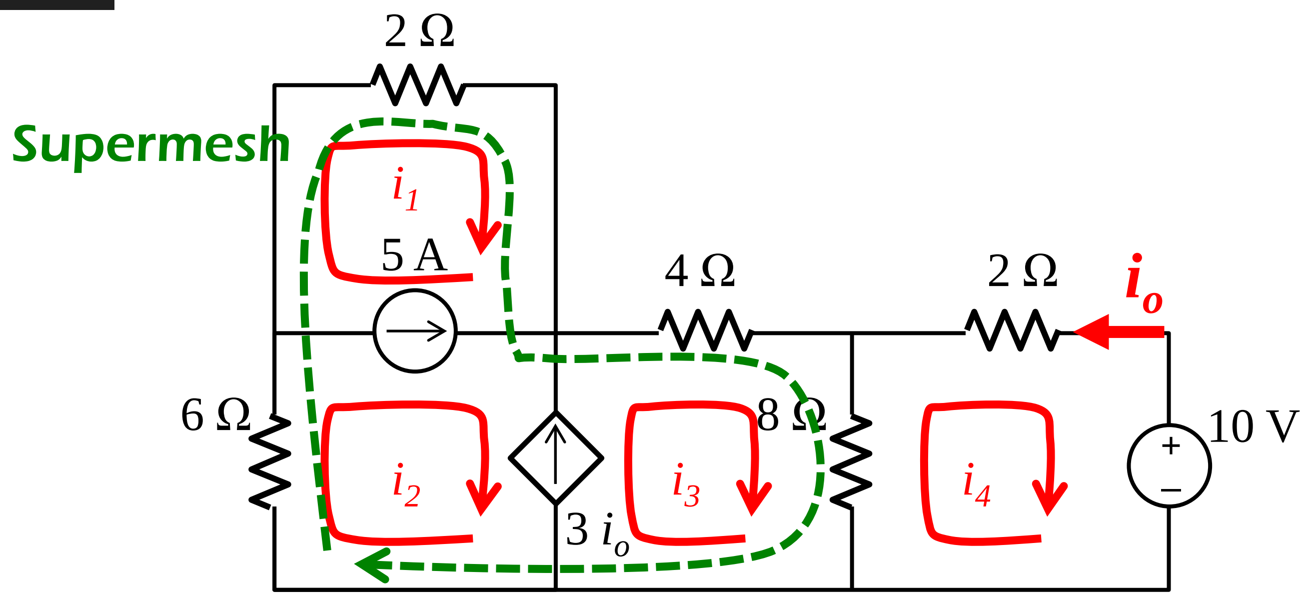
$b - (n - 1) = 4$



Step 1

- Assign mesh currents: i_1, i_2, i_3 and i_4

Ex. 2.11 With Dependent Current Source



Step 2

- We can find that both the 5A independent current source and the $3i_o$ dependent current source lie between two meshes
- Apply the supermesh concept to the 1st, 2nd, and 3rd mesh
- Only two KVL equations are required

Nodal Analysis vs. Mesh Analysis

Advantage of nodal analysis and mesh analysis:

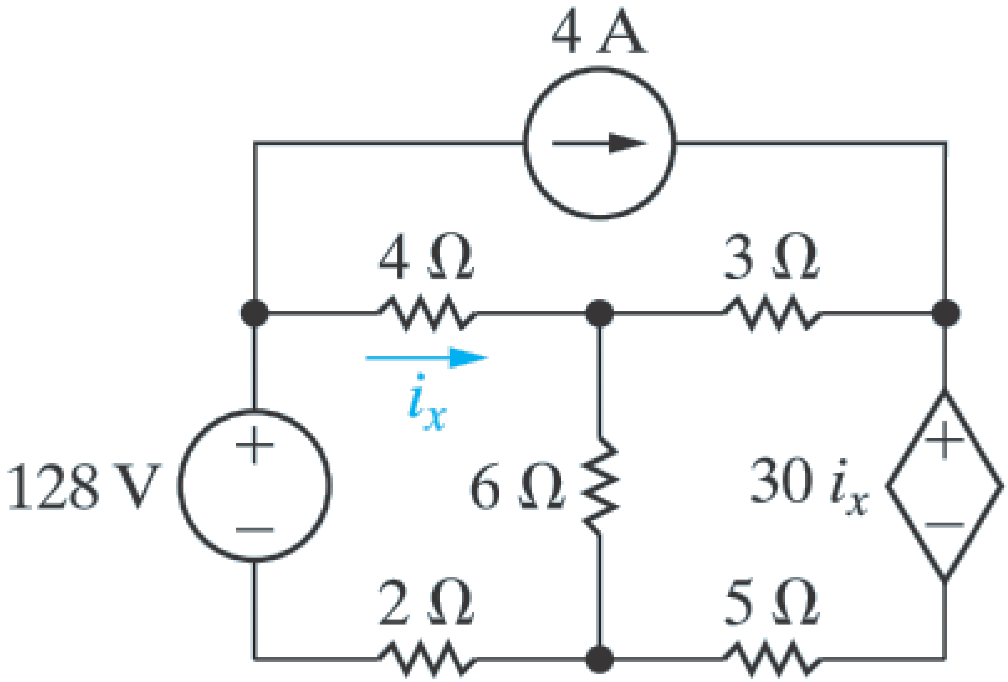
- Both methods provide a systematic way of analyzing a complex circuit
- The number of unknowns are much fewer than those of the 2B method

Nodal analysis		Mesh analysis
Node voltage	Unknown	Mesh current
$n-1$	Number of unknowns	$b-(n-1)$
$n-1-m_v$	Number of unknowns (including sources)	$b-(n-1)-m_i$

(m_v: number of voltage sources, m_i: number of current sources)

- Which method do we prefer? It depends on the solution required!

Ex. 2.12 Nodal Analysis & Mesh Analysis



1. Find the power supplied of the 4 A independent current source (by nodal analysis and mesh analysis, respectively)

Contents

2.3 Superposition Theorem

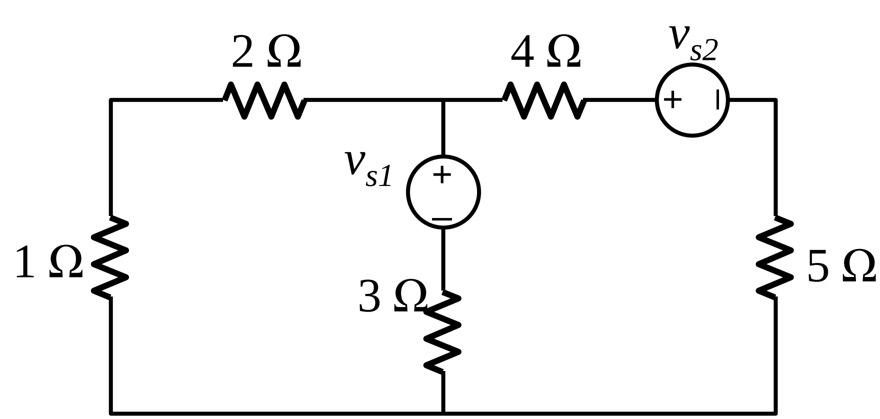
Superposition Theorem in Circuits

- The circuit theorem studied in this semester belongs to the area of "linear systems"
- So, we can apply some theorems of linearity to solve circuit problems

Two theorems are introduced in this section:

1. **Homogeneous theorem**
2. **Superposition theorem**

- Actually, these two theorems are not critical for dc independent sources
- However, they are important for the scenario that two independent sources are not of the same form, such as $v_{s1} = v_1 e^{-2t}$ and $v_{s2} = v_2 \cos\omega t$



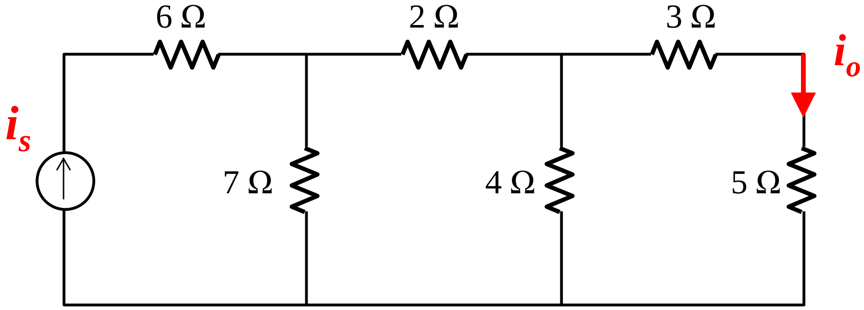
Linear Systems

A circuit can be considered as a **linear system**:

x → $f(\quad)$ → y

Input → The value of i_s

Output → The current on the interested load (i_o)



Properties of linear system (between output y and input x):

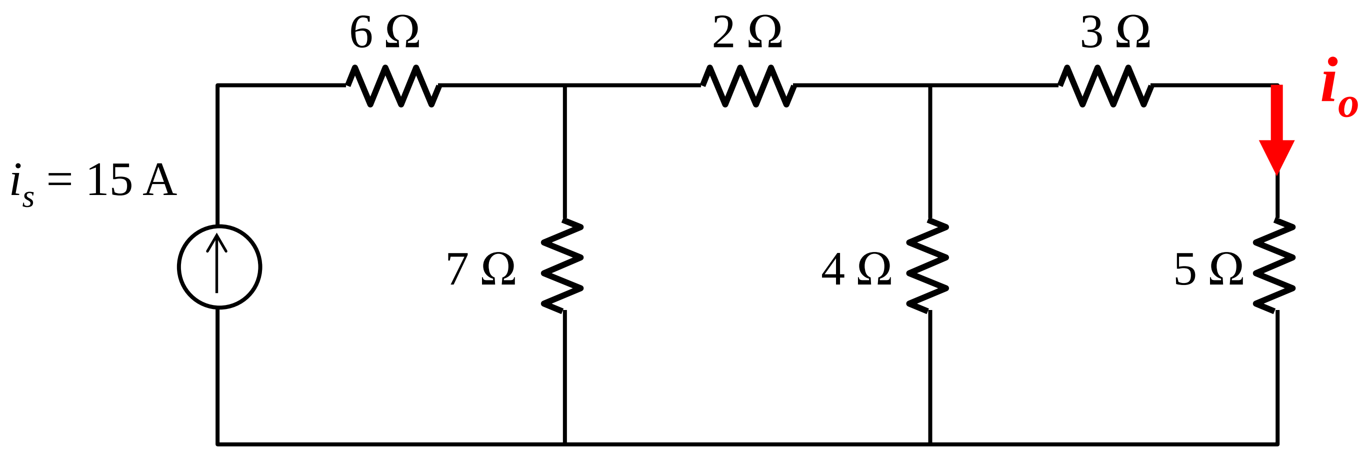
- Homogeneous property

  If $f(x_1) = y_1$, then $f(mx_1) = my_1$

- Additive property

  If $f(x_1) = y_1$, $f(x_2) = y_2$, then $f(x_1 + x_2) = y_1 + y_2$

Ex. 2.13 Homogeneous Property in Circuit

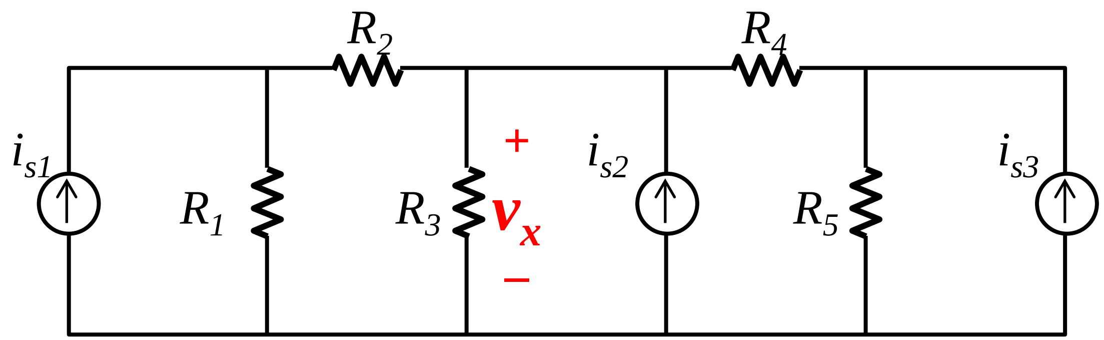


1. Find i_o by assuming $i_o = 1$ A and use linear properties to find its actual value

Input: The value of i_s ⟶ $f(\quad)$ ⟶ *Output:* The current on the interested load i_o

Assumption	5	⟵	1
Real value	15	⟶	3

Additive Property in Circuit (1/2)



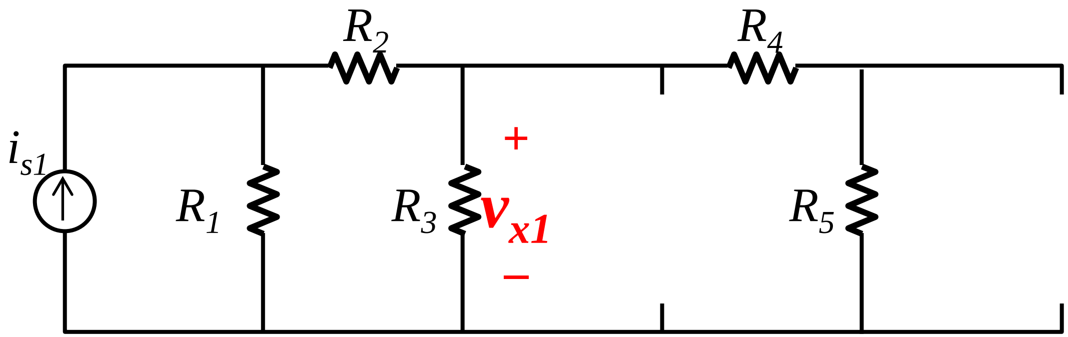
- For a linear circuit consisting of n input source (u_1, u_2, u_3,…, u_n), then the output response can be calculated as the sum of its components

$$y = f(u_1) + f(u_2) + ... + f(u_n)$$

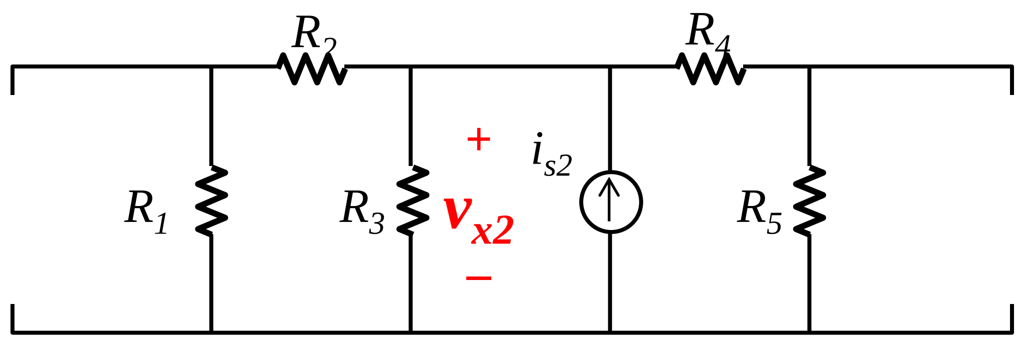
- Input sources: i_{s1}, i_{s2}, and i_{s3}
- Interesting output response: v_x
- We can activate one source at a time and sum the resultant output responses to determine the final result

Additive Property in Circuit (2/2)

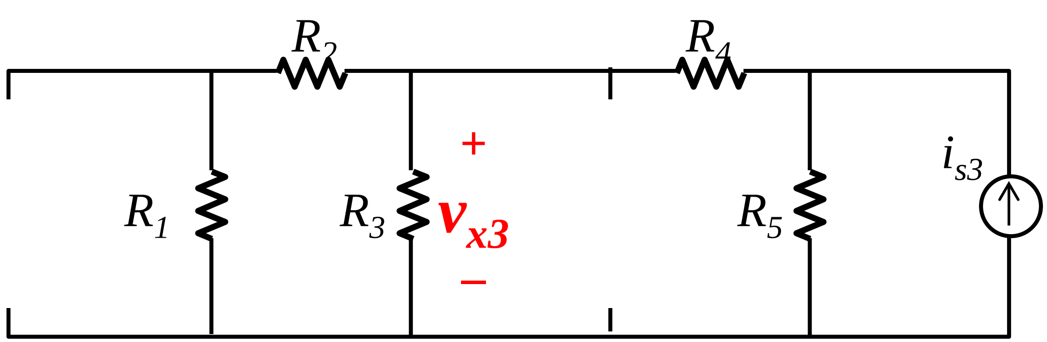
Source 1:



Source 2:

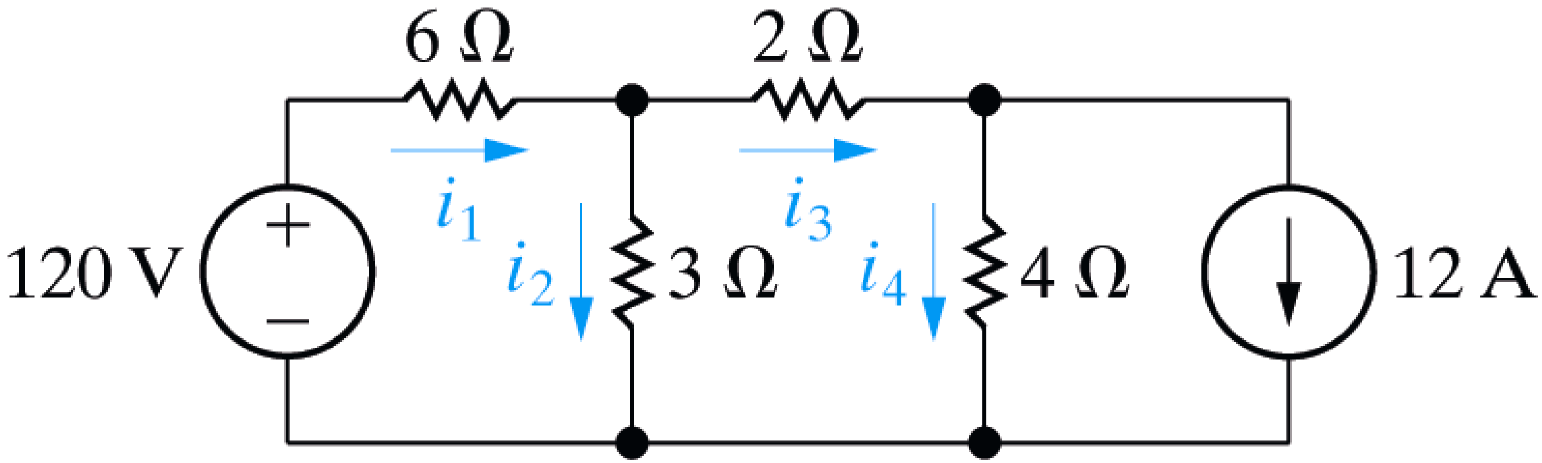


Source 3:



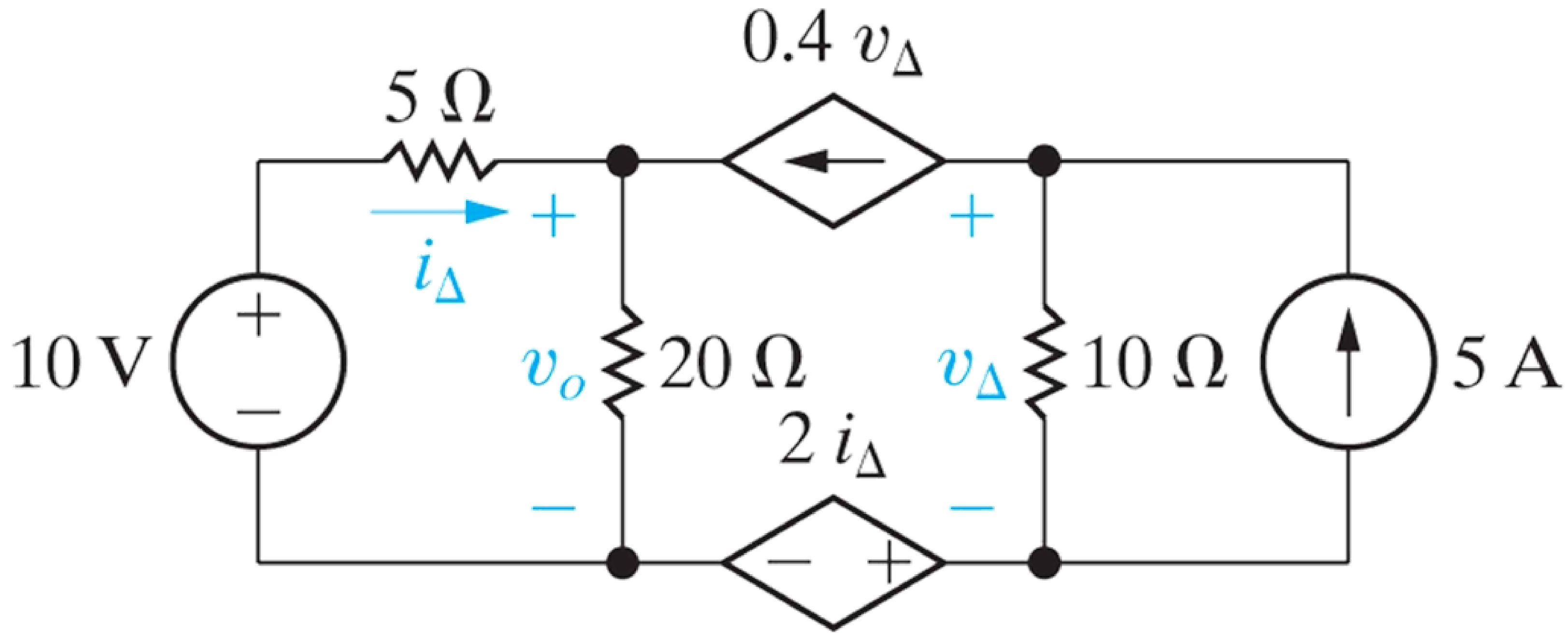
The final response $v_x = v_{x1} + v_{x2} + v_{x3}$

Ex. 2.14 Superposition Theorem



1. Calculate i_1, i_2, i_3, and i_4 by superposition theorems

Ex. 2.15 Superposition Theorem



1. Use the principle of superposition to find v_o and the associated power delivered to it

(Note that dependent sources cannot be deactivate!!)

J. Nilsson and S. Riedel, *Electric Circuits*, 9th Edition, Pearson, NJ, 2011

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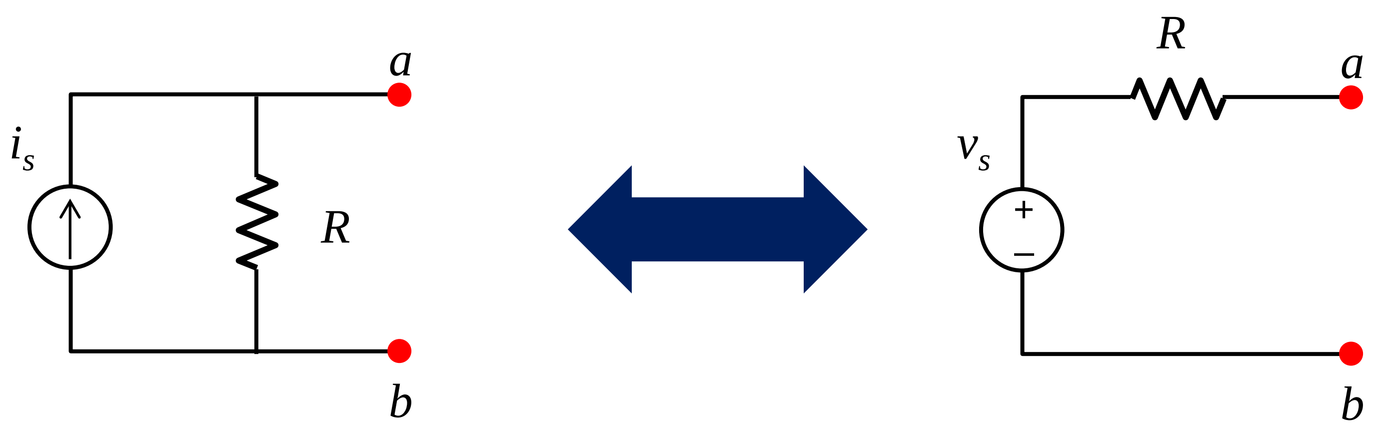
2.4 Source Transformation

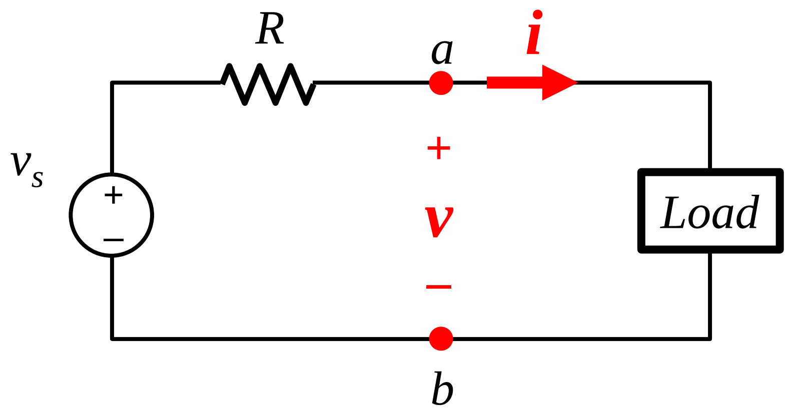
Source Transformation



- It allows a voltage source in series with a resistor to be replaced by a current source in parallel with the same resistor
- Vice versa

Voltage Source → Current Source

Case 1:

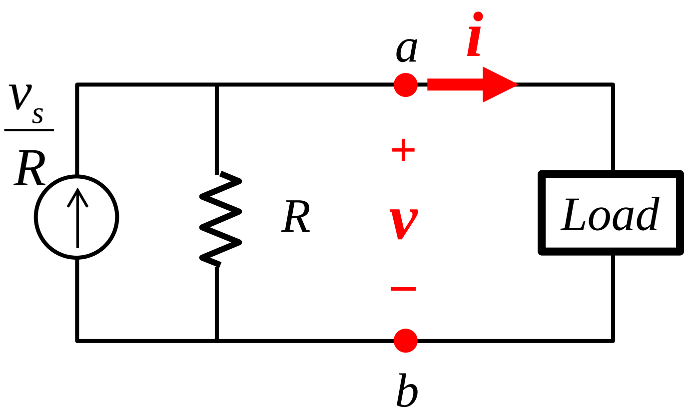


- The current run through the load:

$$i = \frac{v_s - v}{R} = \boxed{\frac{v_s}{R}} - \frac{v}{R}$$

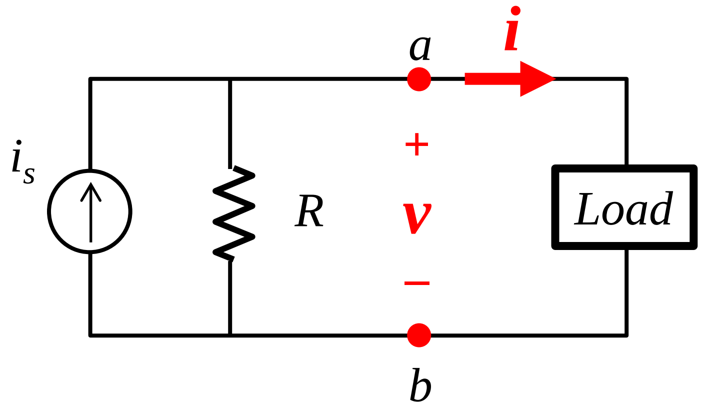
Let this term be $\boxed{i_s}$, so it is equivalent to:

$$i_s = \frac{v_s}{R}$$



Current Source → Voltage Source

Case 2:

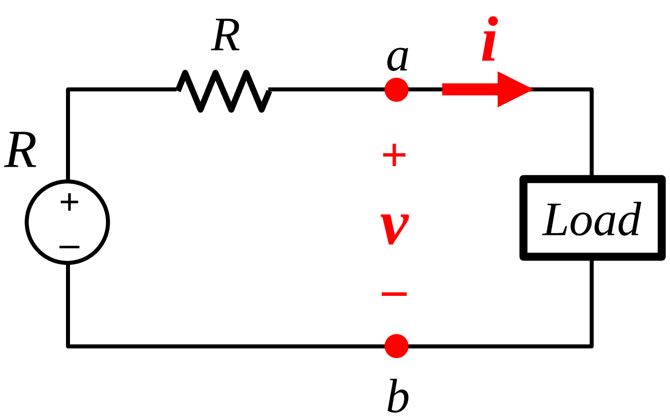


- The voltage across the load:

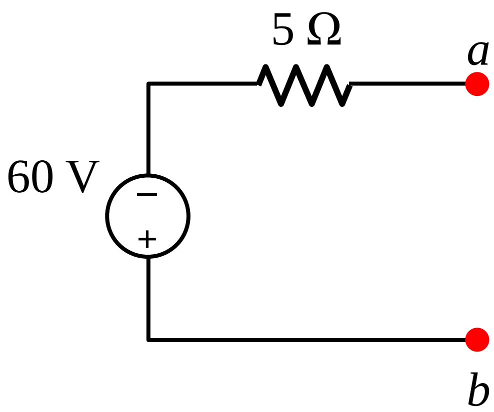
$$v = (i_s - i)R = \boxed{i_s R} - iR$$

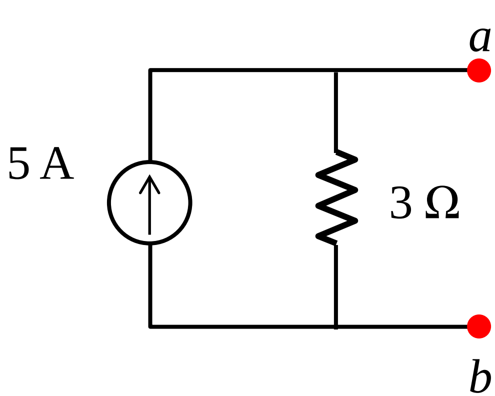
Let this term be $\boxed{V_S}$, so it is equivalent to:

$$v_s = i_s R$$



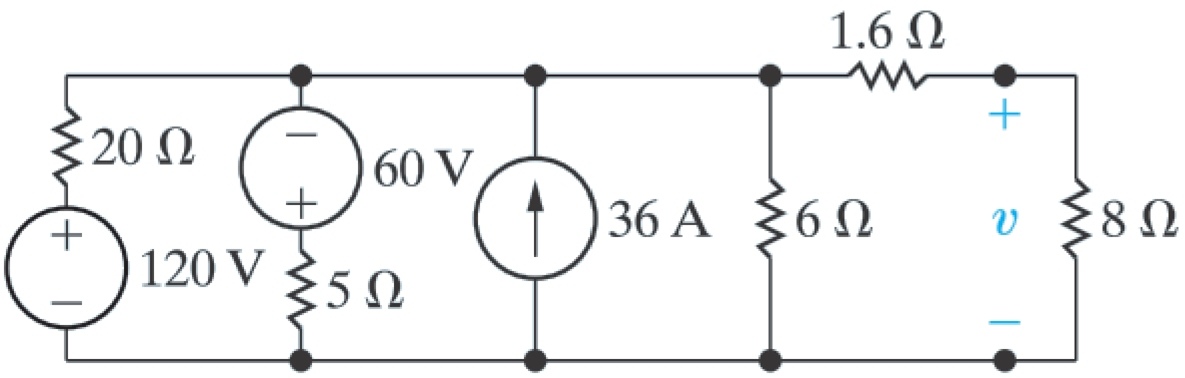
Ex. 2.16 Source Transformation





- Both the source and resistor representations are equivalent for the load connected at *a*-*b* terminals

Ex. 2.17 Source Transformation



1. Use source transformations to find the voltage *v*
2. Find the power developed by the 120-V voltage source

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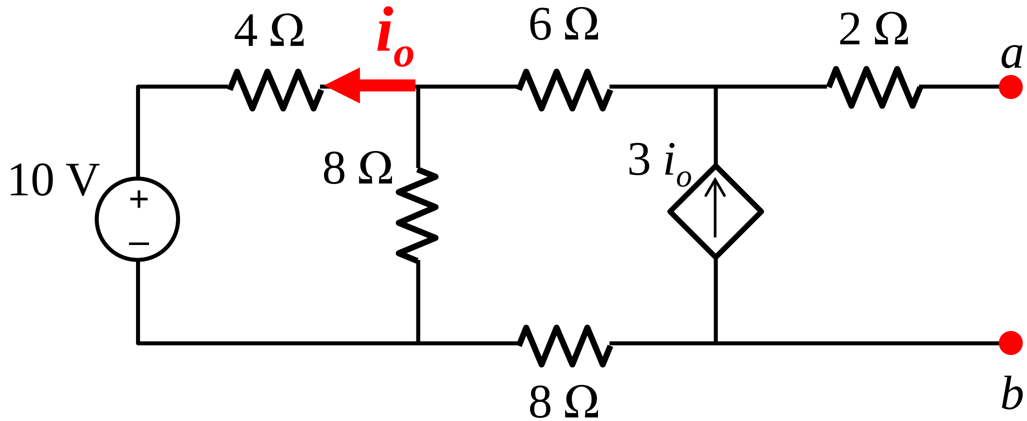
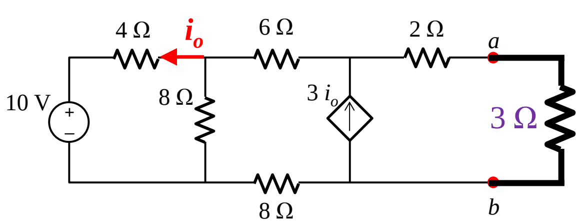
2.5 Thévenin and Norton Equivalents

Objective of Thévenin Equivalent Circuit (1/3)

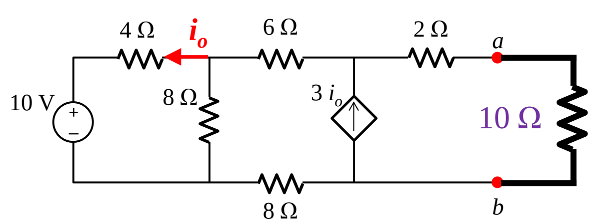


- Usually, we have a given circuit
- We connect a load to the a-b terminals; the load derives power from that given circuit
- The objective is to calculate the **voltage, current, or power on the load**

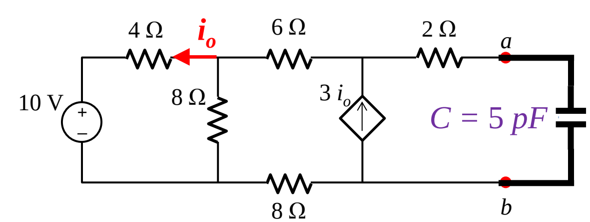
$R_L = 3\ \Omega$



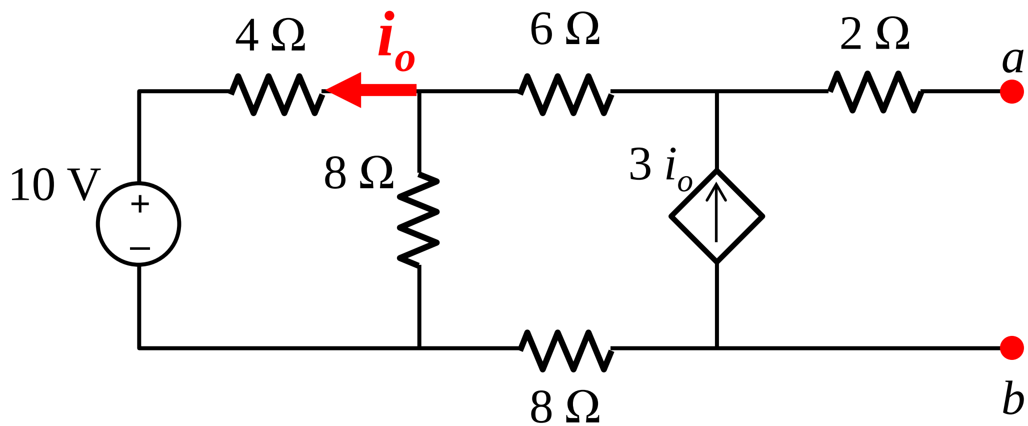
$R_L = 10\ \Omega$



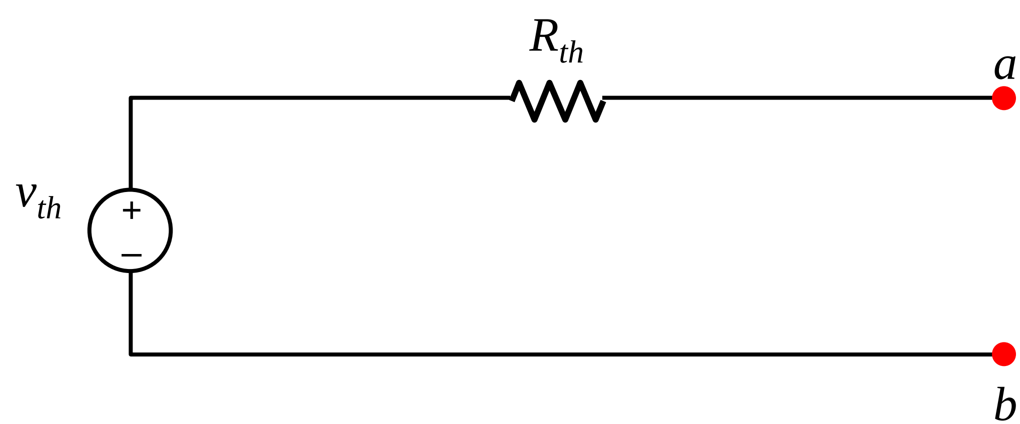
$C_L = 5\ \text{pF}$



Objective of Thévenin Equivalent Circuit (2/3)

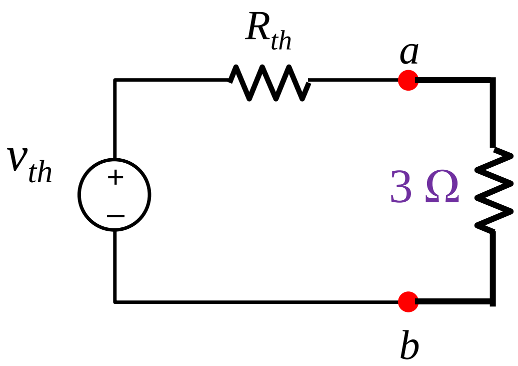


- Instead of calculating v_L and i_L one at a time, the given two-terminal circuit can be replaced by:

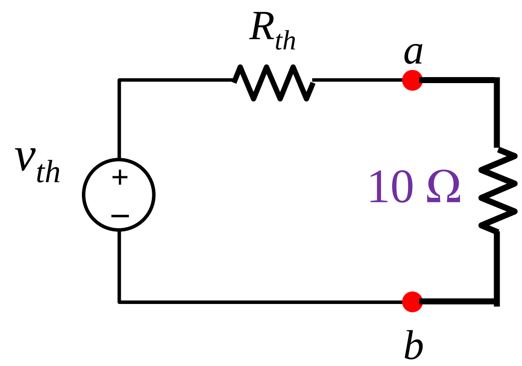


Objective of Thévenin Equivalent Circuit (3/3)

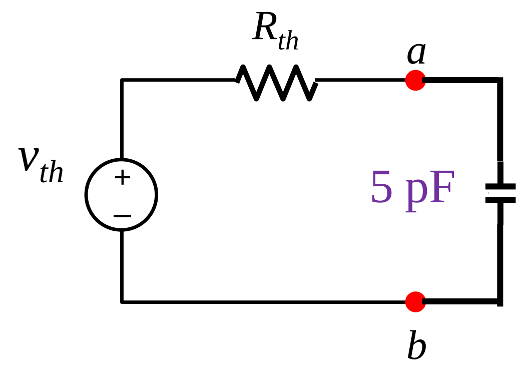
$R_L = 3\ \Omega$



$R_L = 10\ \Omega$



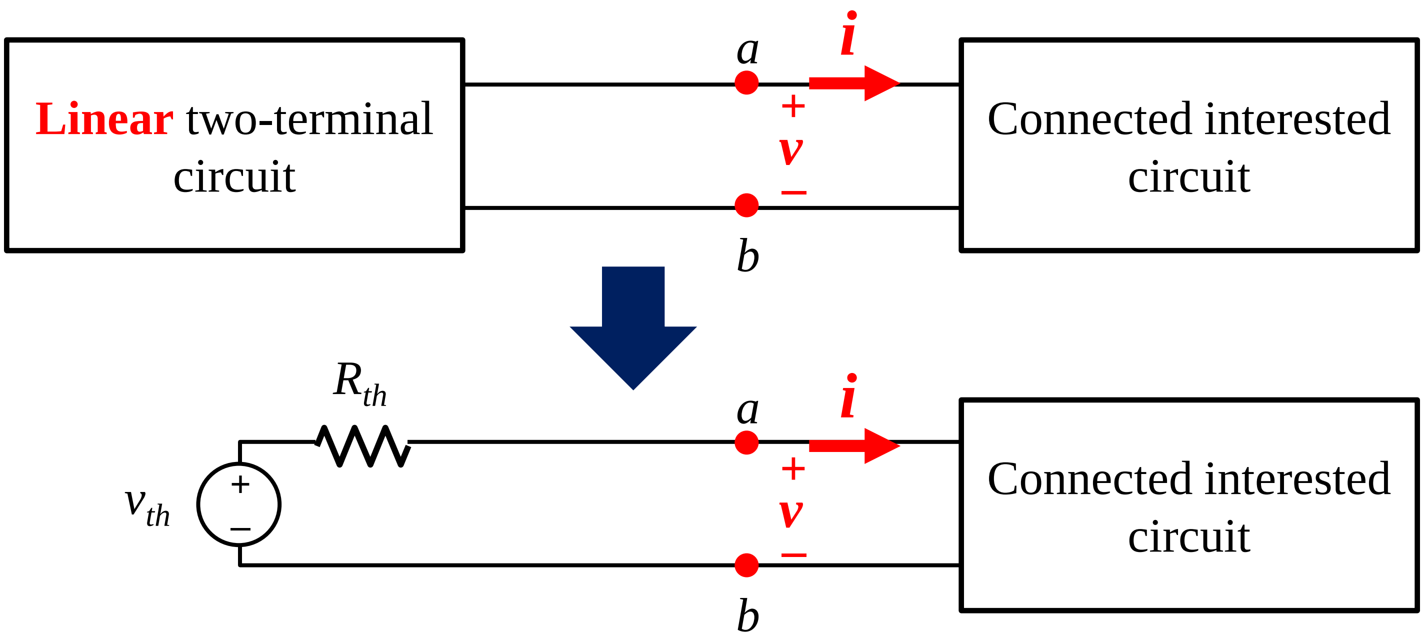
$C_L = 5\ \text{pF}$



- To compute the voltage and current on the load becomes much easier
- **Voltage divider will do!**
- We don't have to reanalyze the entire circuit
- So, the most imperative task would be:
  - How to find v_{th}?
  - How to find R_{th}?

Thévenin Theorem

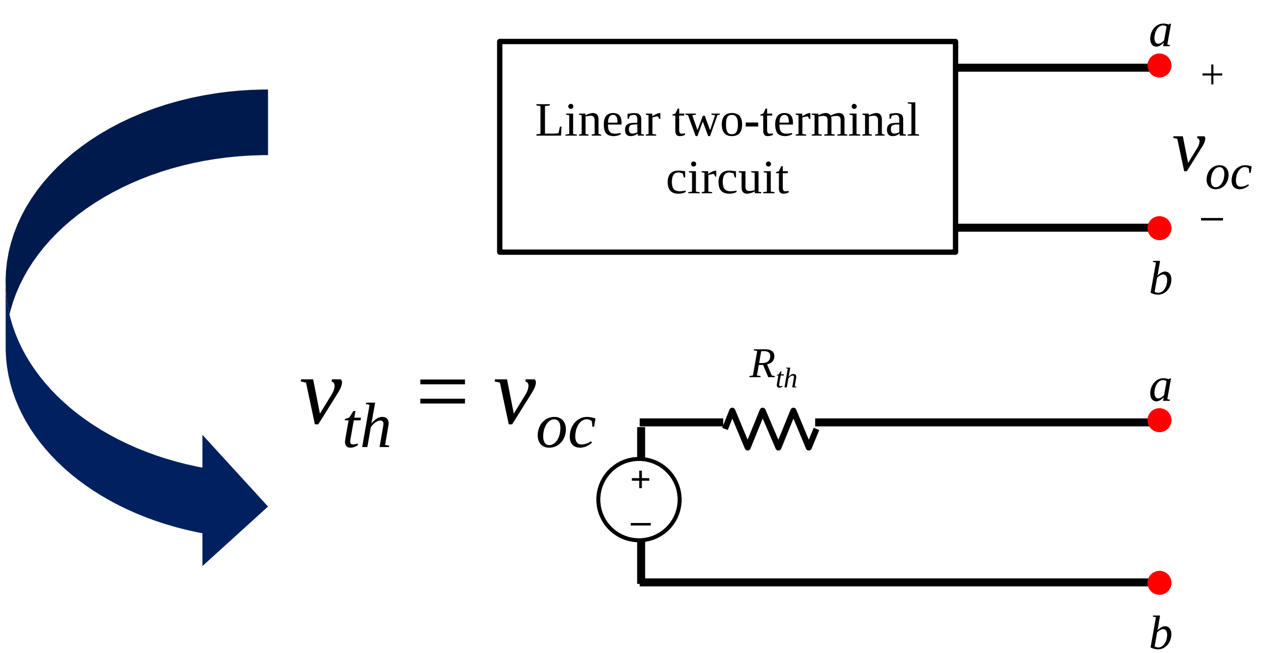
- A linear two-terminal circuit can be replaced by an equivalent circuit consisting of a voltage source v_{th} in series with a resistor R_{th} where



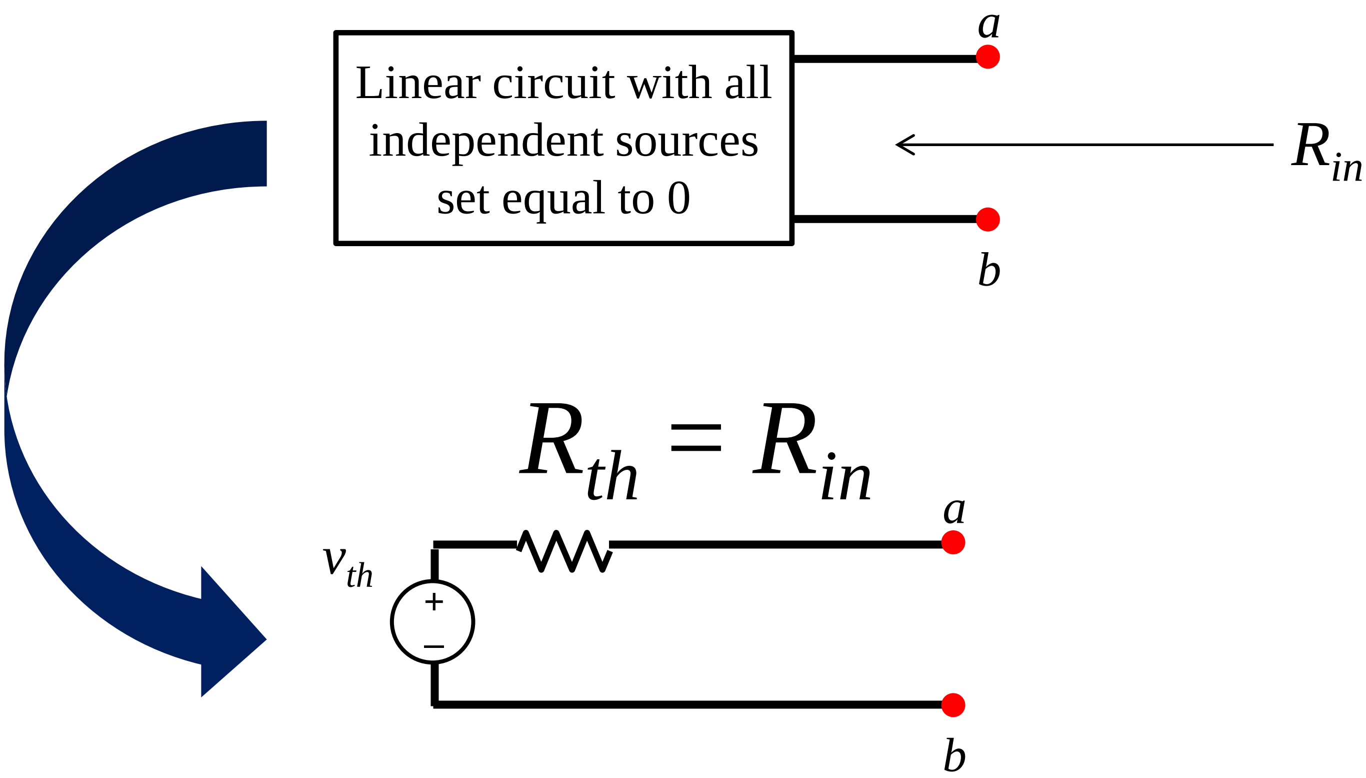
- v_{th} : The open-circuit voltage at the terminals
- R_{th} :The input resistance at the terminals when the ***independent*** sources are turned off

v_{th}: New Voltage Source

- Equivalent circuit: The same voltage-current relation at the terminals
- The equivalence is only valid for the viewpoint of the load
- However, if you concern about the voltage-current relation at some components of the source circuit, you can't find such information by using the equivalent circuit

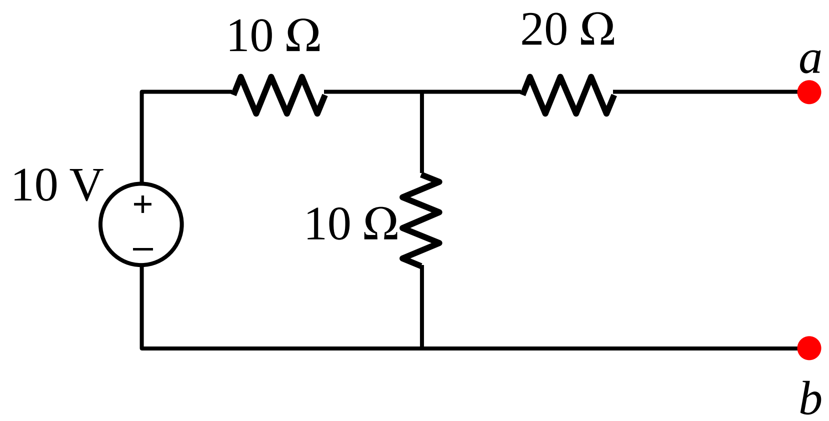


R_{th}: New Internal Resistance (Case 1)



- If dependent sources are included in the original circuit, we have to use another method for finding R_{th}

Ex. 2.18 If the Original Circuit Only Has Independent Sources (1/2)

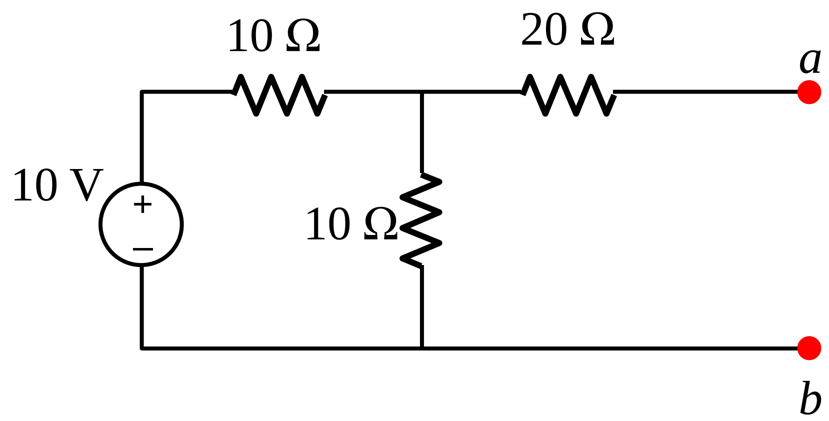


1. Derive the Thévenin equivalent circuit

① v_{th} : By voltage divider principle:

② R_{th} :

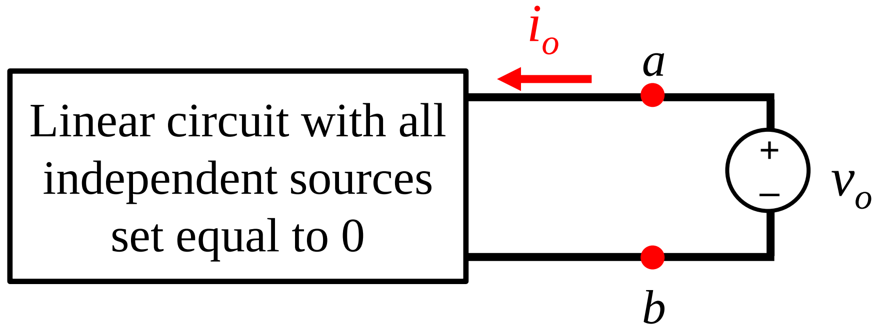
Ex. 2.18 If the Original Circuit Only Has Independent Sources (2/2)



Thévenin equivalent circuit:

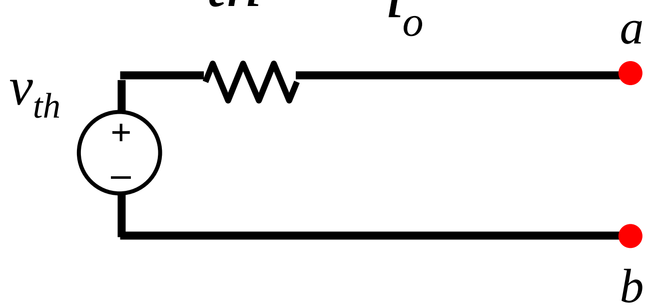
- Verification: If $R_L = 25\ \Omega$:
  - Original circuit: $i_L = 0.1$ A, $v_L = 2.5$ V
  - Thévenin equivalent circuit: $i_L = 0.1$ A, $v_L = 2.5$ V

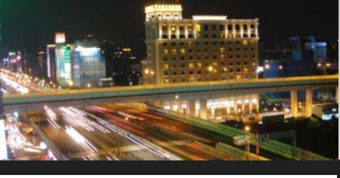
R_{th}: If the Original Network Has Dependent Sources (Case 2) (1/2)



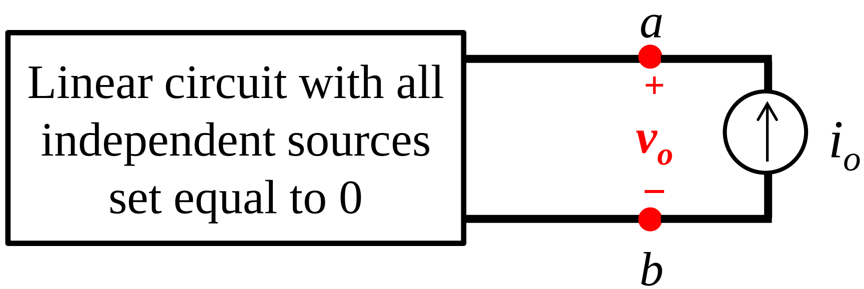
- Arbitrarily apply a testing voltage source v_o
- And then find the associated i_o

$$R_{th} = \frac{v_o}{i_o}$$



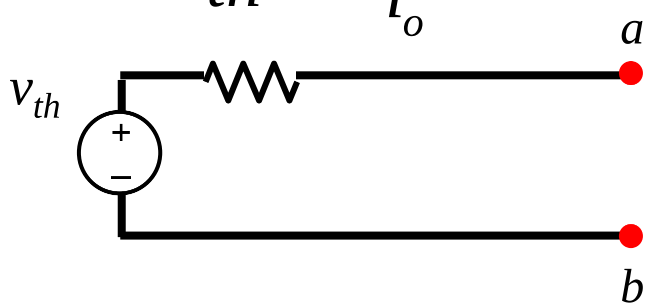


R_{th}: If the Original Network Has Dependent Sources (Case 2) (2/2)

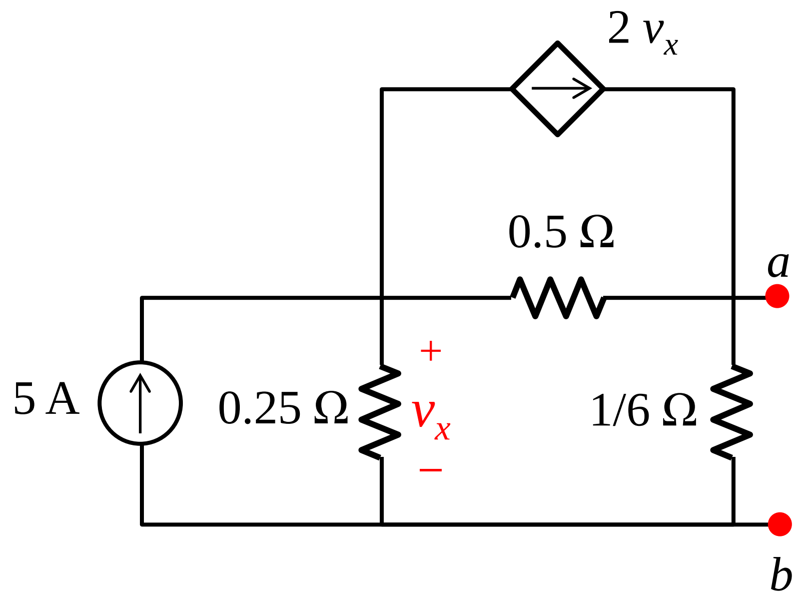


- Arbitrarily apply a testing current source i_o
- And then find the associated v_o

$$R_{th} = \frac{v_o}{i_o}$$



Ex. 2.19 If the Original Circuit Has Dependent Sources

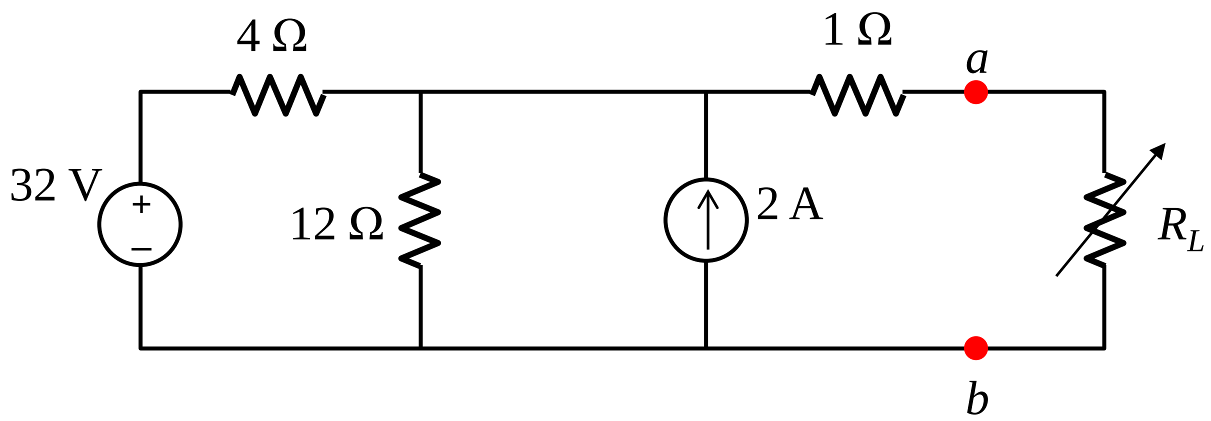


1. Derive the Thévenin equivalent circuit

① v_{th} : By nodal analysis

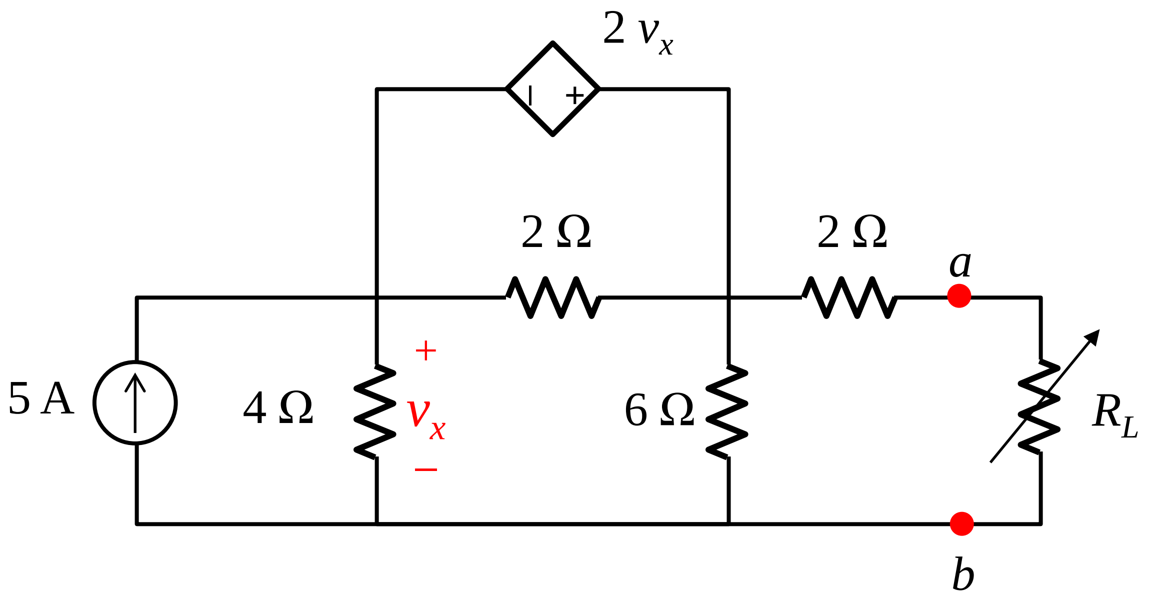
② R_{th} :

Ex. 2.20 Application of Thévenin Equivalent Circuit



1. Find the current through the adjustable resistor where R_L = 6, 16, and 36 Ω, respectively

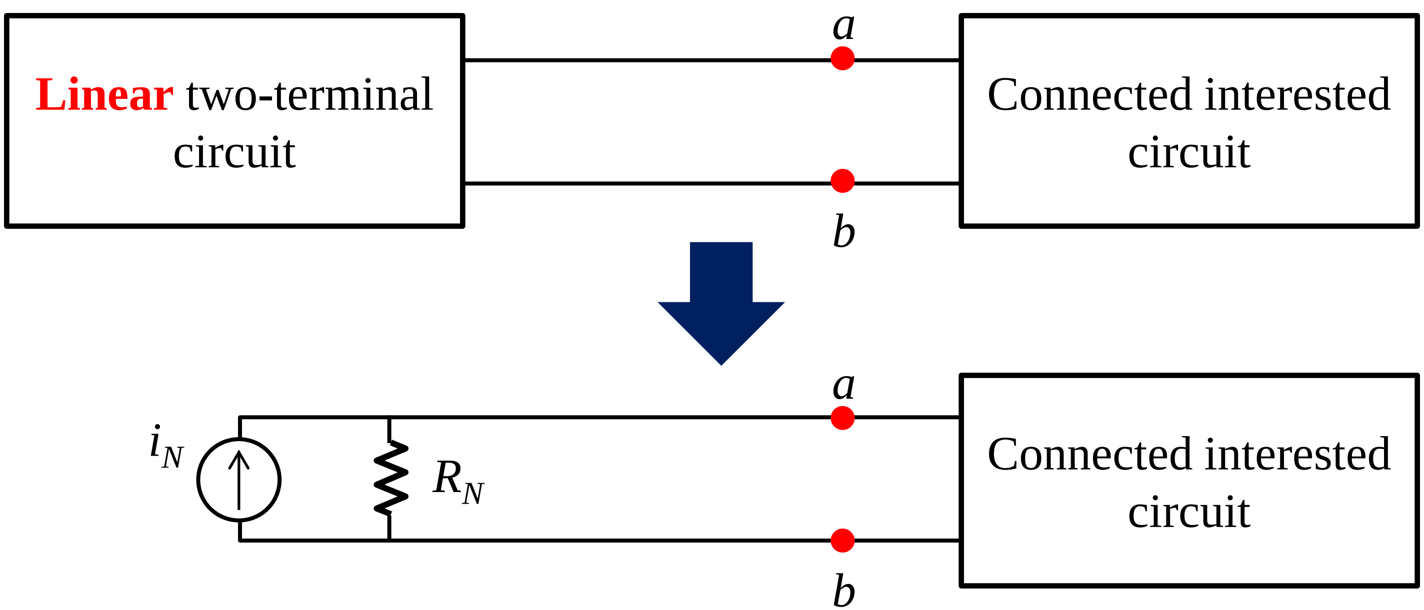
Ex. 2.21 A More Complicated Example



1. Find the consumed power on the adjustable resistor where R_L = 6, 16, and 36 Ω, respectively
2. What value of R_L can derive maximum power from the original circuit?

Norton Equivalent Circuit

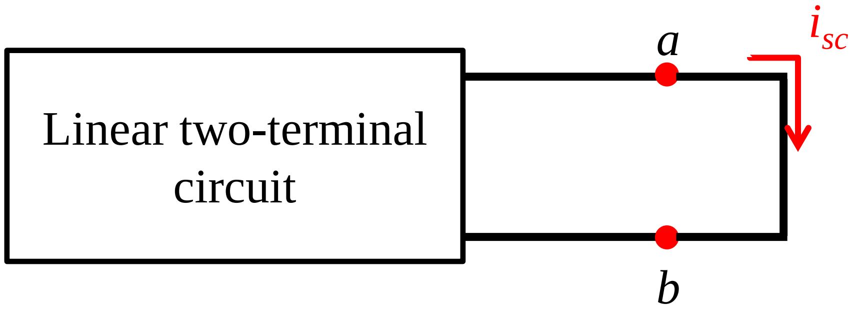
- A linear two-terminal circuit can be replaced by an equivalent circuit consisting of a current source i_N in parallel with a resistor R_N where



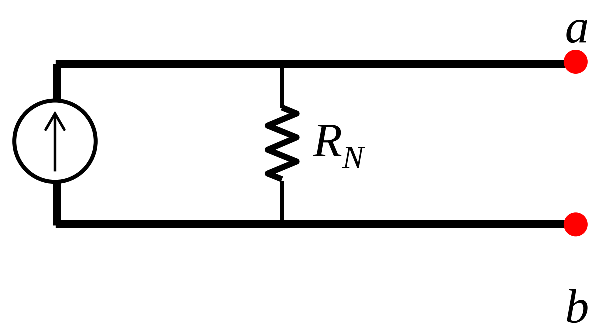
- i_N : The short-circuit current through the terminals
- R_N :The input resistance at the terminals when the independent sources are turned off

Computation of i_N and R_N

- i_N : The short-circuit current run through the terminals



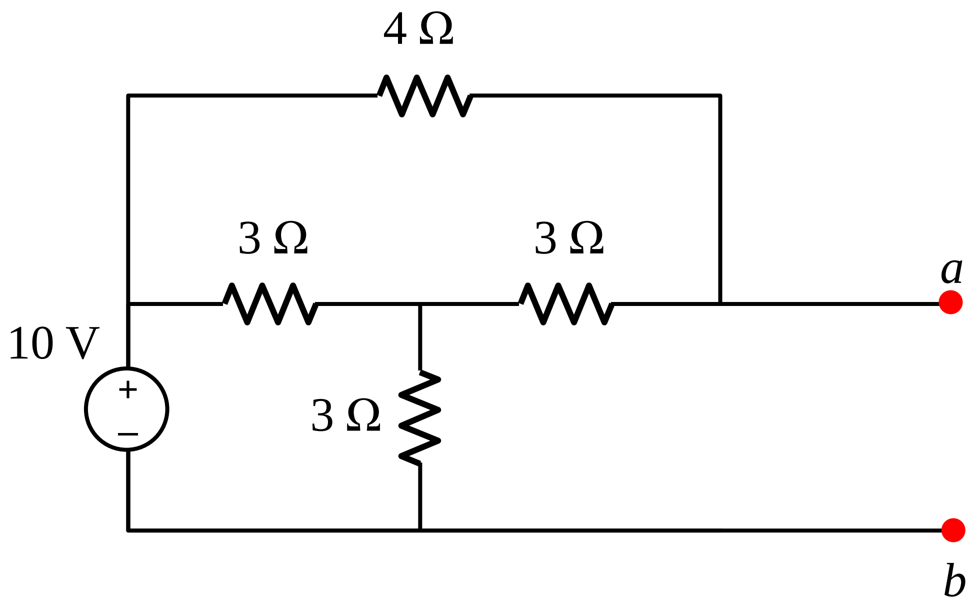
$$i_N = i_{sc}$$



- R_N : The computation of R_N is the same as that of R_{th}

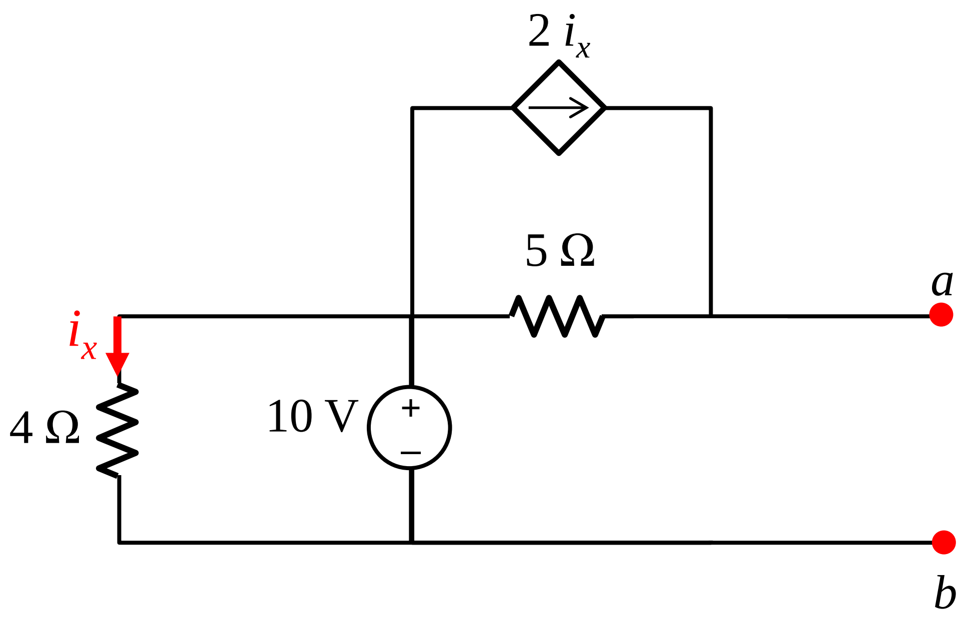
Norton equivalent circuit is the **source transformation** of Thévenin equivalent circuit

Ex. 2.22 Norton Equivalent Circuit



1. Derive the Norton equivalent circuit

Ex. 2.23 Norton Equivalent Circuit



1. Derive the Norton equivalent circuit

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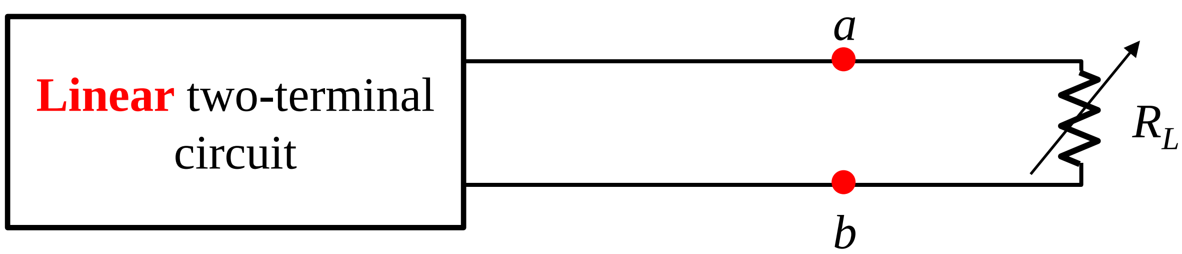
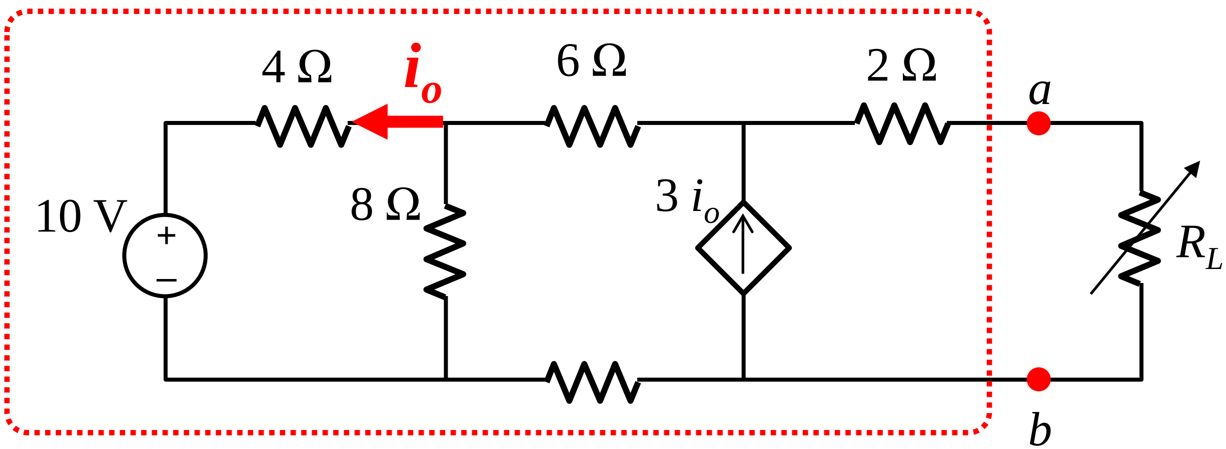
2.6 Maximum Power Transfer Theorem

Objective of This Section



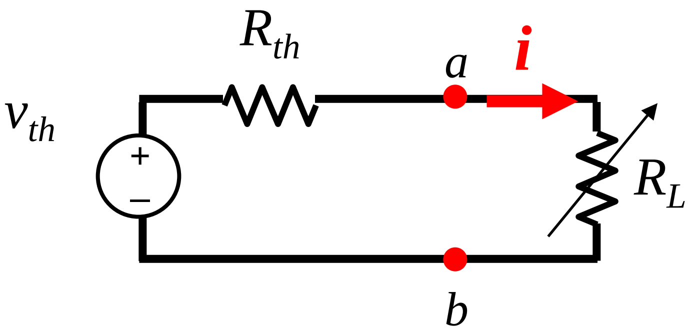
Find the value of R_L that permits maximum power delivery to it

- The linear two-terminal circuit can be arbitrarily assigned, such as



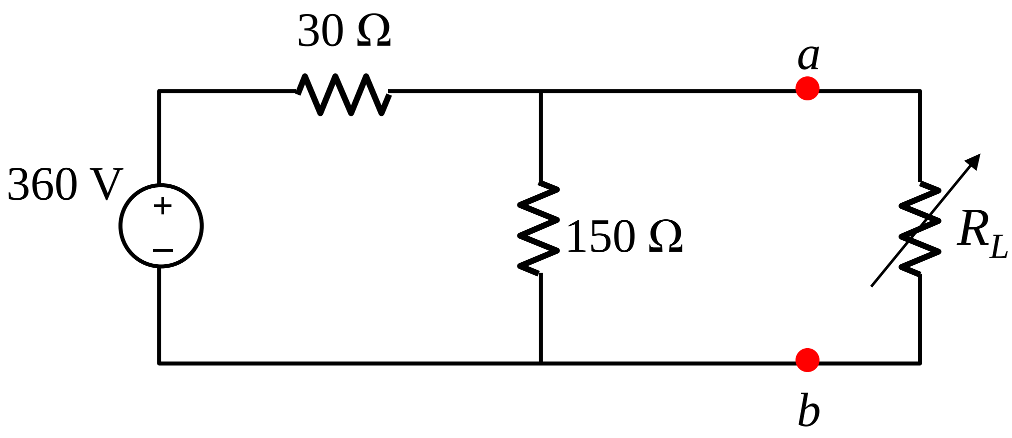
Solution to This Problem

- To simplify the problem, we replace the two-terminal circuit with its Thévenin equivalent circuit



- The power on R_L: $P = i^2 R_L = \left(\dfrac{v_{th}}{R_{th} + R_L} \right)^2 R_L$
- In order to have the maximum power transfer, let $\dfrac{dP}{dR_L} = 0$
- Solution: $when\ R_L = R_{th},\ P_{\max} = \left(\dfrac{v_{th}}{2R_L} \right)^2 R_L = \dfrac{v_{th}^2}{4R_L}$

Ex. 2.24 Maximum Power Transfer



1. Find the value of R_L that results in maximum power transferred to R_L
2. Calculate the maximum power P_{max} that can be delivered to R_L
3. When R_L is adjusted for maximum power transfer, what percentage of the power delivered by the 360 V source reaches R_L?