TAIPEI TECH

國立臺北科技大學

電路學 Circuit Theory

Lecture 4

Sinusoidal Steady-State Analysis

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Contents

Lecture 4: Sinusoidal Steady-State Analysis

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4.1 The Phasor

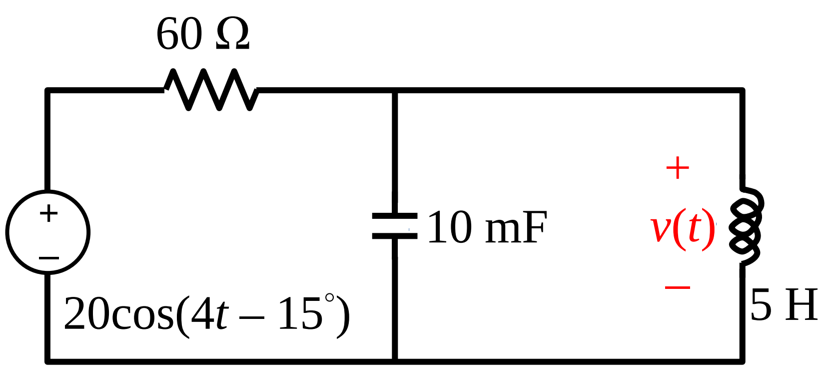
Introduction to This Lecture

Lecture 1–3

- We consider DC sources
- Transient solution + steady-state solution
- Such as:
  $v_S = 12$ V & $i_S = 200$ mA

Lecture 4

- Now we deal with AC sources
- Steady-state solution
- Such as:
  $v_S = 2\cos(60t + 30°)$ V
  $i_S = I_m\cos(\omega t + \varphi)$ A



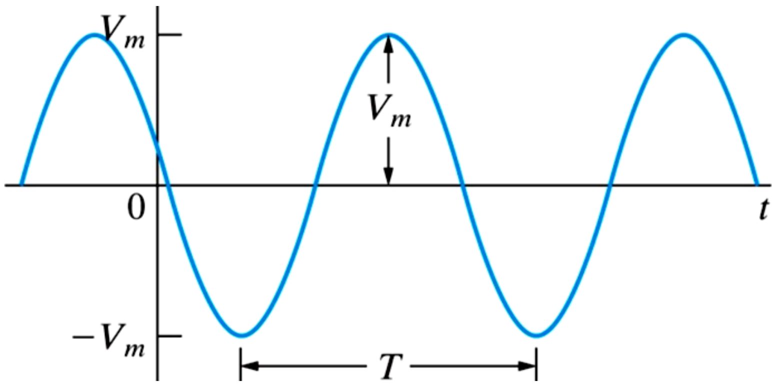
- Give you a circuit, and ask you find the current/voltage/power on a component
- How to solve it? The AC version of
  - Node analysis
  - Mesh analysis
  - Thévenin equivalent circuit

Definition of AC Circuit

AC circuits:

- Circuits driven by sinusoidal current or voltage sources

$$v(t) = V_m \cos(\omega t + \theta)$$



V_m	The amplitude of the sinusoid
ω	The angular frequency, unit: rad/s
θ	Phase angle, unit: rad
f	frequency, unit: Hz $\omega = 2\pi f$
T	Period, unit: s. $v(t + T) = v(t)$ $T = 2\pi/\omega$

Characteristics:

- It's the dominant form of signal in *Communication*, *Electromagnetics*, and *Electric Power Industries*
- Through Fourier analysis, any practical periodic signal can be represented by a sum of sinusoids
- AC circuits can be easily handled by the phasor

Some Manipulation of Sinusoidal Sources

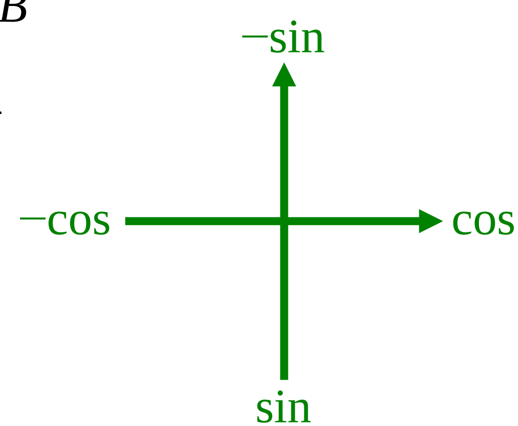
1. Changing sine into cosine:

- Cosine is the standard reference of AC signals/circuits
- $v(t) = V_m \sin(\omega t + \theta) \quad \rightarrow \quad v(t) = V_m \cos(\omega t + \theta - 90^\circ)$
  $v(t) = -V_m \sin(\omega t + \theta) \quad \rightarrow \quad v(t) = V_m \cos(\omega t + \theta + 90^\circ)$
  $v(t) = -V_m \cos(\omega t + \theta) \quad \rightarrow \quad v(t) = V_m \cos(\omega t + \theta + 180^\circ)$
- $V_m > 0$; only the cosine function is involved

2. The superposition of sine and cosine:

- $A\cos(\omega t) + B\sin(\omega t) = C\cos(\omega t - \theta)$, where $C = \sqrt{A^2 + B^2}$

$$\theta = \tan^{-1}\frac{B}{A}$$



3. Another form of sinusoidal function:

$$e^{j\theta} = \cos\theta + j\sin\theta$$

Why is the Phasor So Important?

Prerequisites of using the phasor domain:

- It's useful when you only care about the steady-state solution
- You must have the capability to move back and forth between the polar and rectangular forms of complex number

Benefits of using the phasor representation:

- For a linear circuit with sinusoidal sources, the current and voltage at any branch will also have sinusoidal forms
- It greatly simplifies the complexity of computation:
  - The addition of complex number
  - It turns the differential/integral equations into algebraic equations

Time domain | Phasor domain

① $\xrightarrow{P\left(v(t)\right)}$ ②

② $\downarrow$ ③ Solve the problem!

④ $\xleftarrow{}$ ③ $P^{-1}(\mathbf{V}) = \mathrm{Re}\left[\mathbf{V}e^{j\omega t}\right]$

A New Approach for Solving AC Circuits: Phasor

Phasor: a complex number that represents the amplitude and phase of a sinusoid

Time domain

$$v(t) = V_m \cos(\omega t + \theta)$$

$$\xrightarrow{P(v(t))}$$

Phasor domain (or frequency domain)

$\mathbf{V}$ (Complex form)

$V_m \angle \theta$ (Polar form)

$V_m \cos\theta + jV_m \sin\theta$ (Rectangular form)

$V_m e^{j\theta}$ (Exponential form)

$$\xleftarrow{P^{-1}(\mathbf{V}) = \mathrm{Re}\left[\mathbf{V} e^{j\omega t}\right]}$$

- Real value
- Voltage and current are the functions of t

- Complex
- Voltage and current are NOT the functions of t

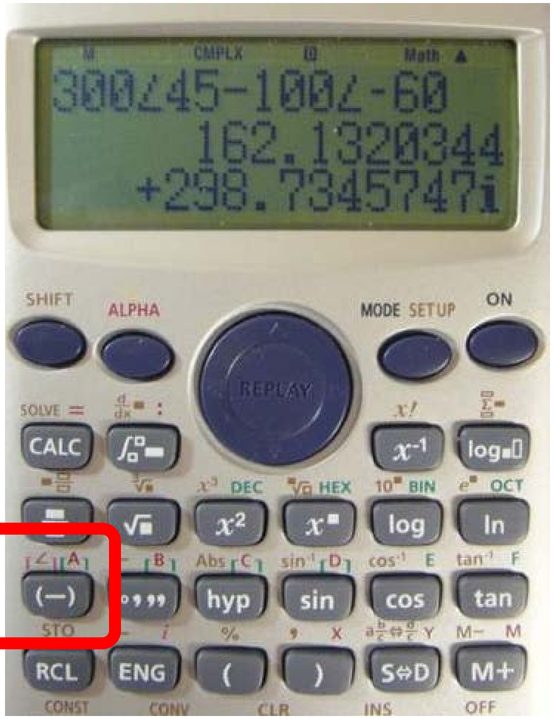
EX 4.1 Phasor Transform

1. Find the phasor transform of each trigonometric function:

   $v(t) = 170\cos(377t - 40^{\circ})$ V
   $i(t) = 10\sin(1000t + 20^{\circ})$ A
   $v(t) = 300\cos(20000\,\pi t + 45^{\circ}) - 100\sin(20000\,\pi t + 30^{\circ})$ mV

2. Find the time-domain expression associated to each phasor:

   $\mathbf{V} = 18.6\angle{-54^{\circ}}$ V
   $\mathbf{I} = (20\angle 45^{\circ} - 50\angle{-30^{\circ}})$ mA
   $\mathbf{V} = (20 + j80 - 30\angle 15^{\circ})$ V



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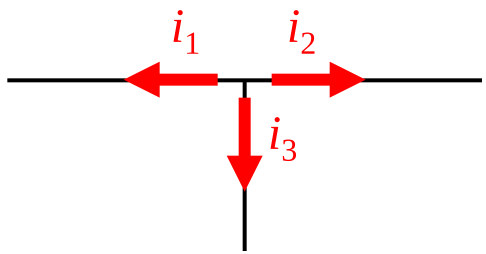
4.2 Circuit Theorems in the Phasor Domain

KCL in Phasor Domain



KCL in Lecture 1:

$$\sum_{K=1}^{n} i_K(t) = 0 \qquad \text{(for any node)}$$

Now every branch has sinusoidal current:

$$i_K(t) = I_{mK}\cos(\omega t + \theta_K)$$

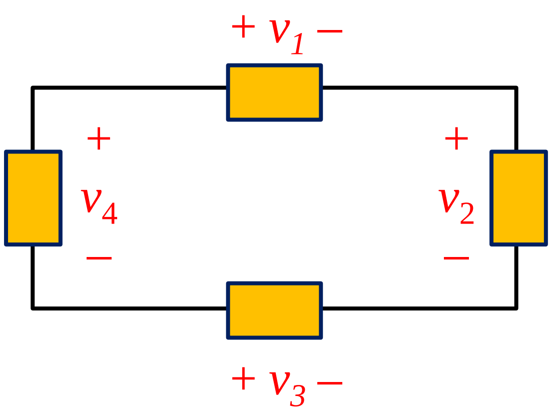
$$\therefore \sum_{K=1}^{n} I_{mK}\cos(\omega t + \theta_K) = 0$$

Take phasor transform on both sides:

$$\sum_{K=1}^{n} I_{mK}\angle\theta_K = 0 + j0$$

$$\text{or } \mathbf{I_1} + \mathbf{I_2} + \ldots + \mathbf{I_n} = 0$$

KVL in Phasor Domain



KVL in Lecture 1:

$$\sum_{K=1}^{n} v_K(t) = 0 \qquad \text{(for any loop)}$$

Now every branch has sinusoidal voltage:

$$v_K(t) = V_{mK}\cos(\omega t + \phi_K)$$

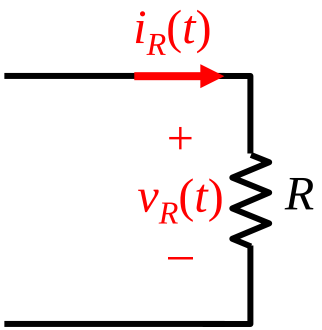
$$\therefore \sum_{K=1}^{n} V_{mK}\cos(\omega t + \phi_K) = 0$$

Take phasor transform on both sides:

$$\sum_{K=1}^{n} V_{mK}\angle\phi_K = 0 + j0$$

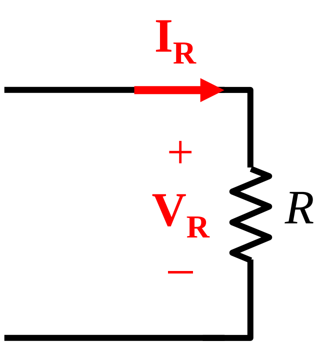
$$\text{or } \mathbf{V_1} + \mathbf{V_2} + \ldots + \mathbf{V_n} = 0$$

Resistor Model in Phasor Domain



R: resistance
G: conductance

Phasor domain model



Component model in Lecture 1:

$$v_R(t) = Ri_R(t)$$

Suppose the sinusoidal current is given:

Given $i_R(t) = I_m \cos(\omega t + \theta)$

Then $v_R(t) = RI_m \cos(\omega t + \theta)$

Take phasor transform on both sides:

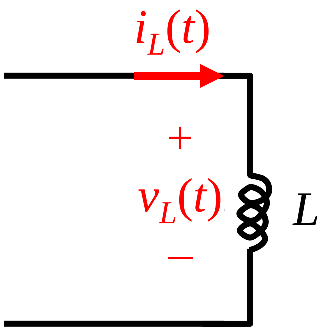
Given $\mathbf{I_R} = I_m \angle \theta$

Then $\mathrm{P}(v_R(t)) = RI_m \angle \theta = R\mathbf{I_R} = \mathbf{V_R}$

Similarly, given $\mathbf{V_R}$, one can find

$$\mathbf{I_R} = \frac{1}{R}\mathbf{V_R} = G\mathbf{V_R}$$

Inductor Model in Phasor Domain



Component model in Lecture 3:

$$v_L(t) = L\frac{di_L(t)}{dt}$$

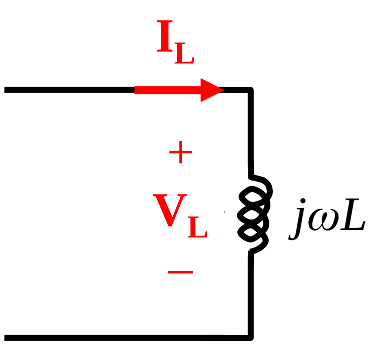
Suppose the sinusoidal current is given:

$$\text{Given } i_L(t) = I_m \cos(\omega t + \theta)$$

$$\text{Then } v_L(t) = L\frac{di_L(t)}{dt} = -\omega L I_m \sin(\omega t + \theta)$$

$$= \omega L I_m \cos(\omega t + \theta + 90°)$$

Phasor domain model

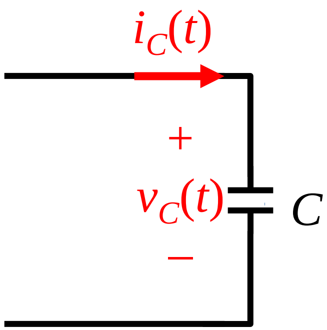


Take phasor transform on both sides:

$$\mathrm{P}(v_L(t)) = \omega L I_m e^{j(\theta+90°)} = \omega L I_m e^{j\theta} e^{j90°} = j\omega L I_m e^{j\theta} = j\omega L \mathbf{I_L}$$

$\Rightarrow$ $\mathbf{V_L} = j\omega L \mathbf{I_L}$ Similarly, given $\mathbf{V_L}$: $\mathbf{I_L} = \frac{1}{j\omega L}\mathbf{V_L}$

Capacitor Model in Phasor Domain



Component model in Lecture 3:

$$i_C(t) = C\frac{dv_C(t)}{dt}$$

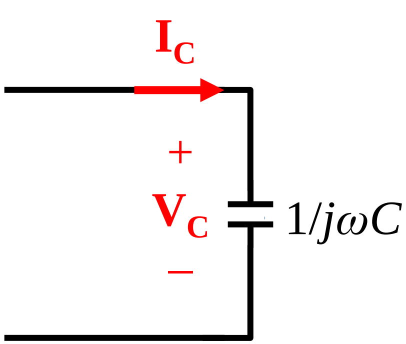
Suppose the sinusoidal voltage is given:

$$\text{Given } v_C(t) = V_m \cos(\omega t + \theta)$$

$$\text{Then } i_C(t) = C\frac{dv_C(t)}{dt} = -\omega C V_m \sin(\omega t + \theta)$$

$$= \omega C V_m \cos(\omega t + \theta + 90°)$$

Phasor domain model



Take phasor transform on both sides:

$$\mathrm{P}(i_C(t)) = \omega C V_m e^{j(\theta+90°)} = \omega C V_m e^{j\theta} e^{j90°} = j\omega C V_m e^{j\theta} = j\omega C \mathbf{V_C}$$

$\Rightarrow$ $\mathbf{I_C} = j\omega C \mathbf{V_C}$ Similarly, given $\mathbf{I_C}$: $\mathbf{V_C} = \dfrac{1}{j\omega C}\mathbf{I_C}$

Summary of the Component Models

Element	Time domain	Phasor domain
R	$v = Ri$ $i = Gv$	$\mathbf{V} = R\mathbf{I}$ $\mathbf{I} = G\mathbf{V}$
L	$v = L\dfrac{di}{dt}$ $i = i(0) + \dfrac{1}{L}\int_0^t v(\tau)\,d\tau$	$\mathbf{V} = j\omega L\mathbf{I}$ $\mathbf{I} = \dfrac{1}{j\omega L}\mathbf{V}$
C	$i = C\dfrac{dv}{dt}$ $v = v(0) + \dfrac{1}{C}\int_0^t i(\tau)\,d\tau$	$\mathbf{V} = \dfrac{1}{j\omega C}\mathbf{I} = -j\dfrac{1}{\omega C}\mathbf{I}$ $\mathbf{I} = j\omega C\mathbf{V}$

The Concept of "Impedance"

- From the above phasor-domain models of R, L, and C, the ratio of the phasor voltage to the phasor current:

$$\frac{\mathbf{V}}{\mathbf{I}} = \begin{cases} R, & \text{for resistor} \\ j\omega L, & \text{for inductor} \\ \dfrac{1}{j\omega C} = -j\dfrac{1}{\omega C}, & \text{for capacitor} \end{cases}$$

➡ Define $\dfrac{\mathbf{V}}{\mathbf{I}} \triangleq Z$ $\longrightarrow$ Impedance (Unit: Ω)

- Impedance is a complex quantity; it's not a phasor
- It can be further expanded into $R + jX$. R: resistance; X: reactance

The Concept of "Admittance"

Definition:

$$Y = \frac{1}{Z} \triangleq \frac{\mathbf{I}}{\mathbf{V}} = G + jB$$

- Y: admittance, unit: S
- G: conductance, unit: S
- B: susceptance, unit: S

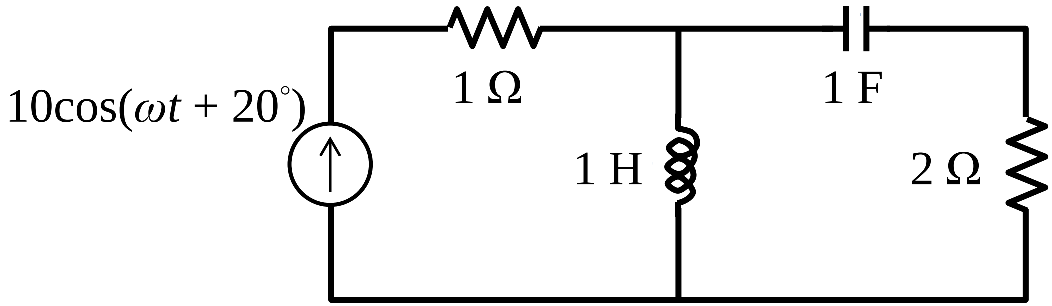
Summary:

$$\mathbf{V} = Z\mathbf{I}, \ \mathbf{I} = Y\mathbf{V}$$

Suitable for resistors, inductors, and capacitors

→ complex Ohm's law

EX 4.2 Circuitry in Frequency Domain



1. Redraw the above circuit in frequency domain (phasor domain with $\omega = 3$)

Remarks

For a resistor with resistance R:

$$G = \frac{1}{R}$$

For a component with impedance $Z = R + jX$:

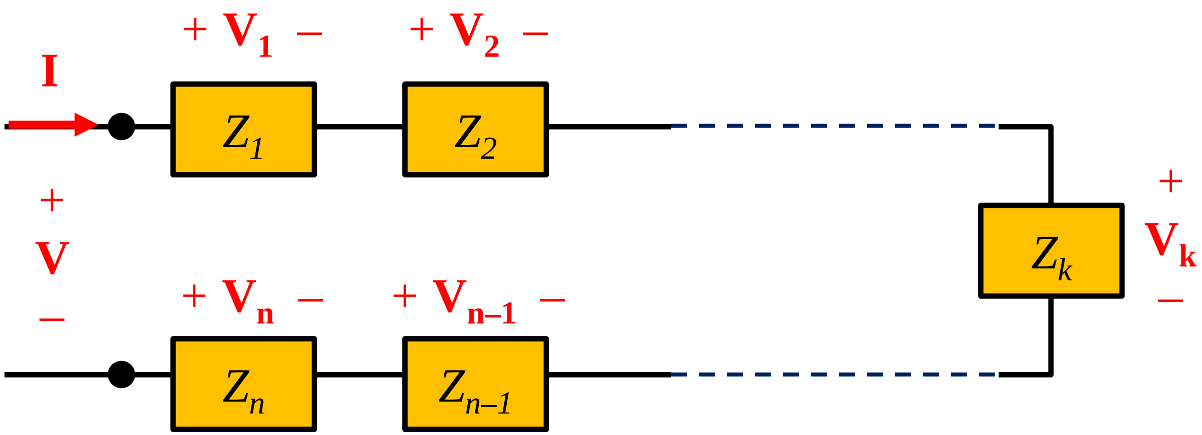
$$\because Y = \frac{1}{Z} = \frac{1}{R + jX} = \frac{R}{R^2 + X^2} - j\frac{X}{R^2 + X^2} = G + jB$$

$$\therefore G = \frac{R}{R^2 + X^2}$$

$$B = \frac{-X}{R^2 + X^2}$$

$\Rightarrow$ $G \neq \frac{1}{R}$ (unless $X = 0$)

Series Connection

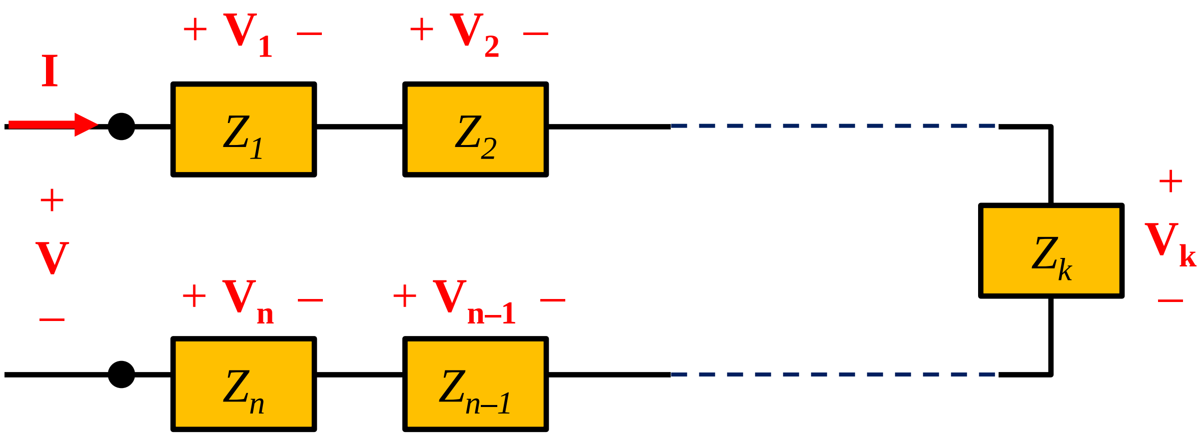


- Equivalent impedance:

$$\text{By KVL: } \mathbf{V} = \sum_{k=1}^{n} \mathbf{V}_k$$

$$Z_{eq} \triangleq \frac{\mathbf{V}}{\mathbf{I}} = \frac{\sum_{k=1}^{n} \mathbf{V}_k}{\mathbf{I}} = \frac{\mathbf{V}_1 + \mathbf{V}_2 + \ldots + \mathbf{V}_n}{\mathbf{I}} = \sum_{k=1}^{n} Z_k$$

Voltage Division Principle

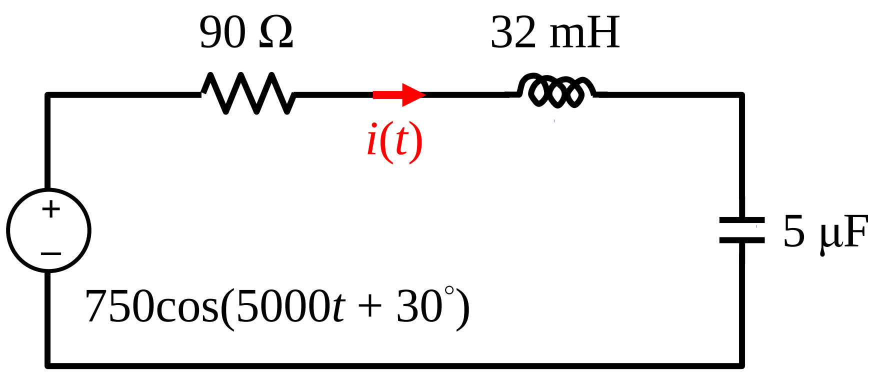


- The voltage on each component $\mathbf{V}_k$:

$$\mathbf{V}_k = \frac{Z_k}{\sum_{k=1}^{n} Z_k}\mathbf{V}$$

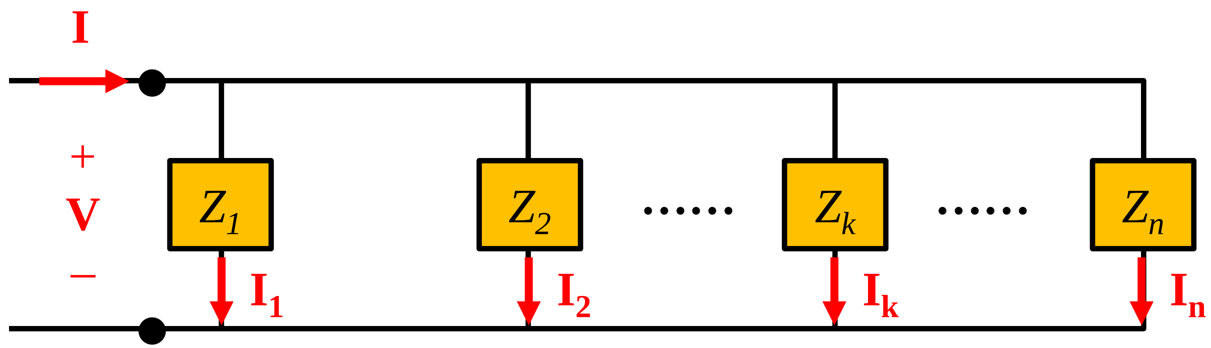
EX 4.3

Combining Impedances in Series



1. Calculate the steady-state current $i(t)$ by the phasor approach

Parallel Connection

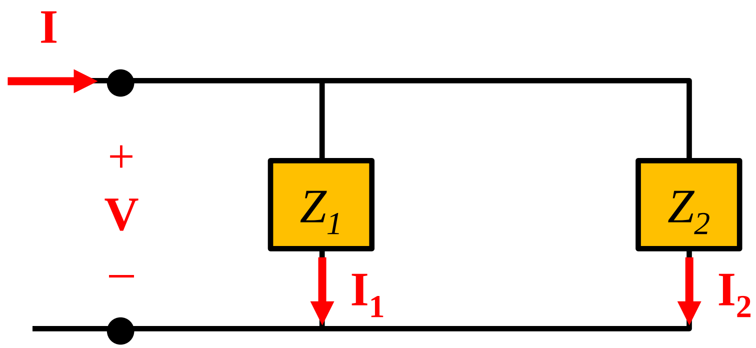


- Equivalent impedance (or equivalent admittance):

$$\text{By KCL: } \mathbf{I} = \sum_{k=1}^{n} \mathbf{I}_k$$

$$\frac{1}{Z_{eq}} \triangleq \frac{\mathbf{I}}{\mathbf{V}} = \frac{\sum_{k=1}^{n} \mathbf{I}_k}{\mathbf{V}} = \sum_{k=1}^{n} \frac{1}{Z_k} \qquad \text{or} \qquad Y_{eq} = \sum_{k=1}^{n} Y_k$$

Current Division Principle



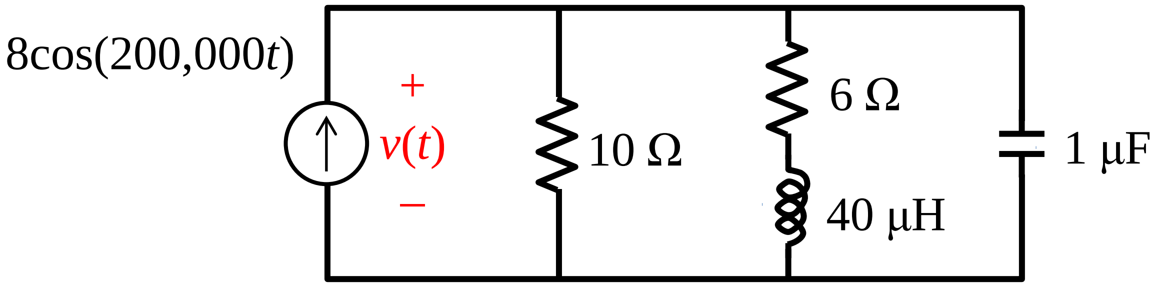
- The current through each component:

$$\mathbf{I}_1 = \frac{Z_2}{Z_1 + Z_2}\mathbf{I}$$

$$\mathbf{I}_2 = \frac{Z_1}{Z_1 + Z_2}\mathbf{I}$$

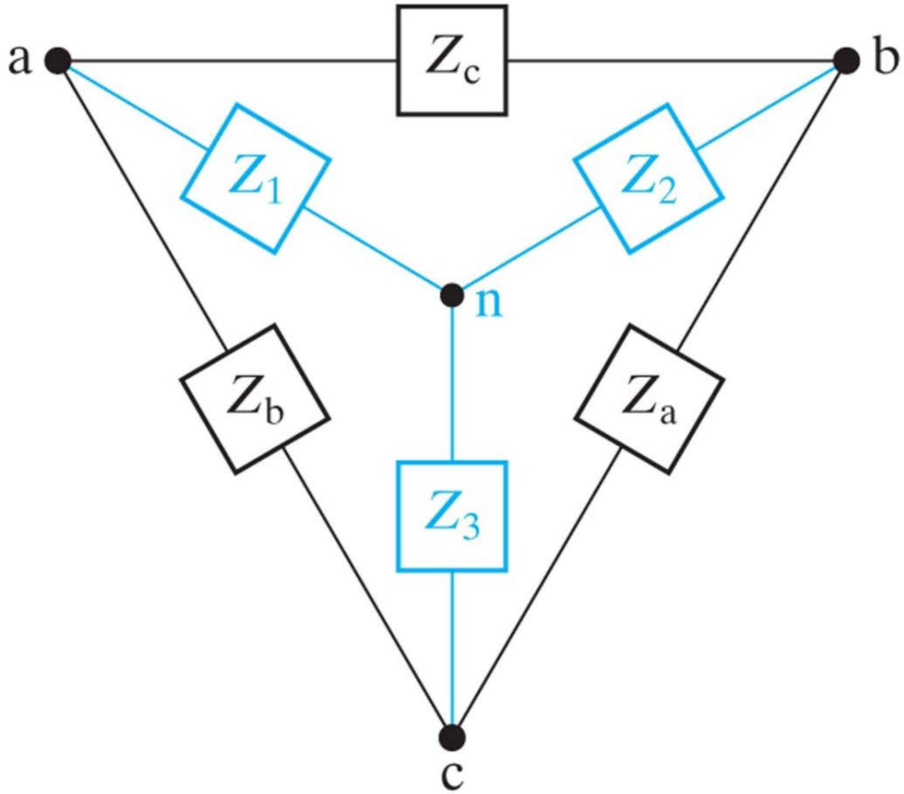
(The equivalent impedance can be written as $Z_{eq} = \dfrac{Z_1 Z_2}{Z_1 + Z_2}$)

EX 4.4 Combining Impedances in Parallel



1. Calculate the steady-state voltage $v(t)$ by the phasor approach
2. Find the current run through each branch

Δ-Y Transformation



Δ-Y:

$$Z_1 = \frac{Z_b Z_c}{Z_a + Z_b + Z_c}$$

$$Z_2 = \frac{Z_c Z_a}{Z_a + Z_b + Z_c}$$

$$Z_3 = \frac{Z_a Z_b}{Z_a + Z_b + Z_c}$$

Y-Δ:

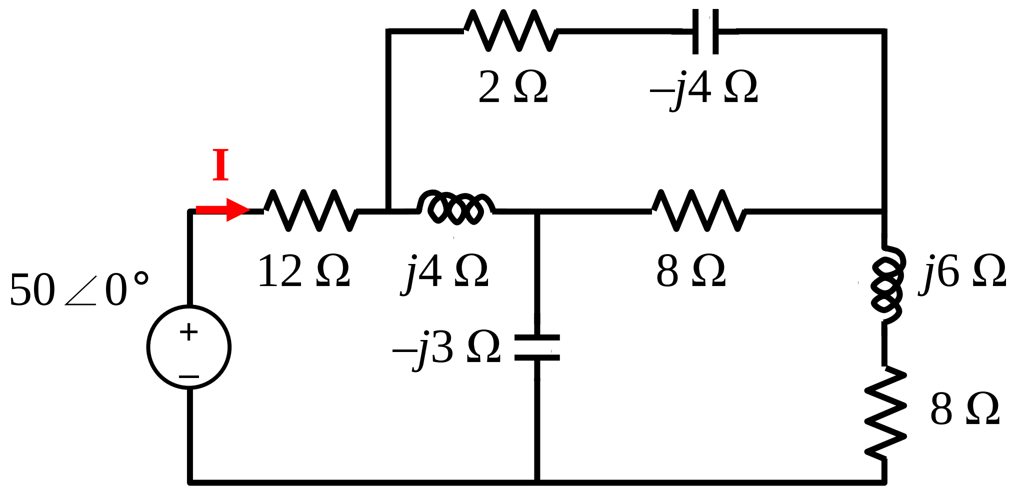
$$Z_a = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1}{Z_1}$$

$$Z_b = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1}{Z_2}$$

$$Z_c = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1}{Z_3}$$

J. Nilsson and S. Riedel, *Electric Circuits*, 9th Edition, Pearson, NJ, 2011

Δ-Y Transformation



1. Find **I** in the above circuit

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4.3 Advanced Techniques for Phasor Analysis

How to Solve More Complex Circuits?

Lecture 2

Resistive networks

Lecture 4

AC circuits

The same methods in Lecture 2 can be extended to this lecture:

- Nodal analysis
- Mesh analysis
- Superposition theorem
- Source transformation
- Thévenin equivalent circuits
- Norton equivalent circuits

Solution Steps

Steps to analysis of AC circuits:

Step 1: Transform the circuit to the phasor domain (or frequency domain)

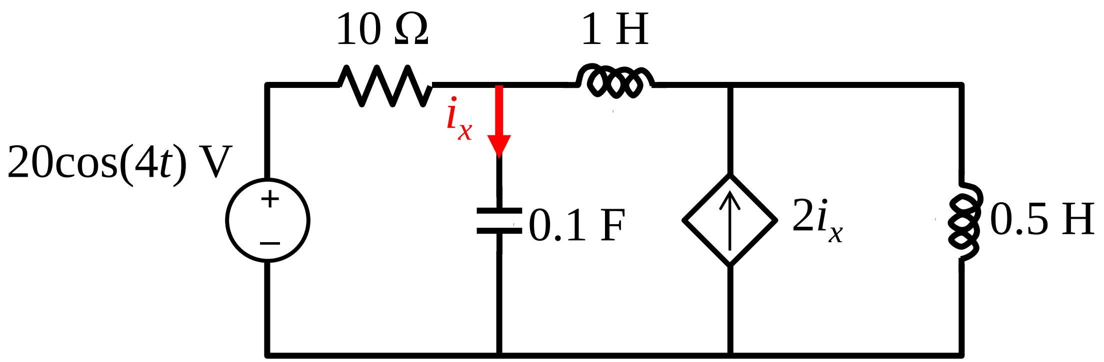
Step 2: Solve the **complex** algebraic equations

Step 3: Transform the answer back to the time domain

Assumptions for using the phasor approach:

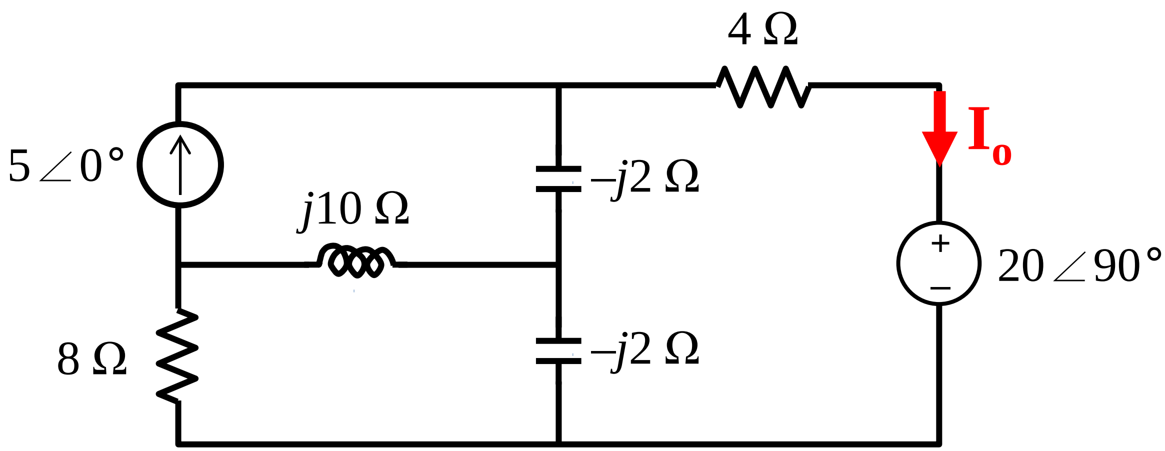
1. We only need the steady-state solutions
2. Only one frequency existed in the circuit
3. The voltage source and current source must be of sinusoidal form
4. All of the elements in the circuit are linear components

EX 4.6 Nodal Analysis



1. Find the steady-state $i_x(t)$ in the above circuit

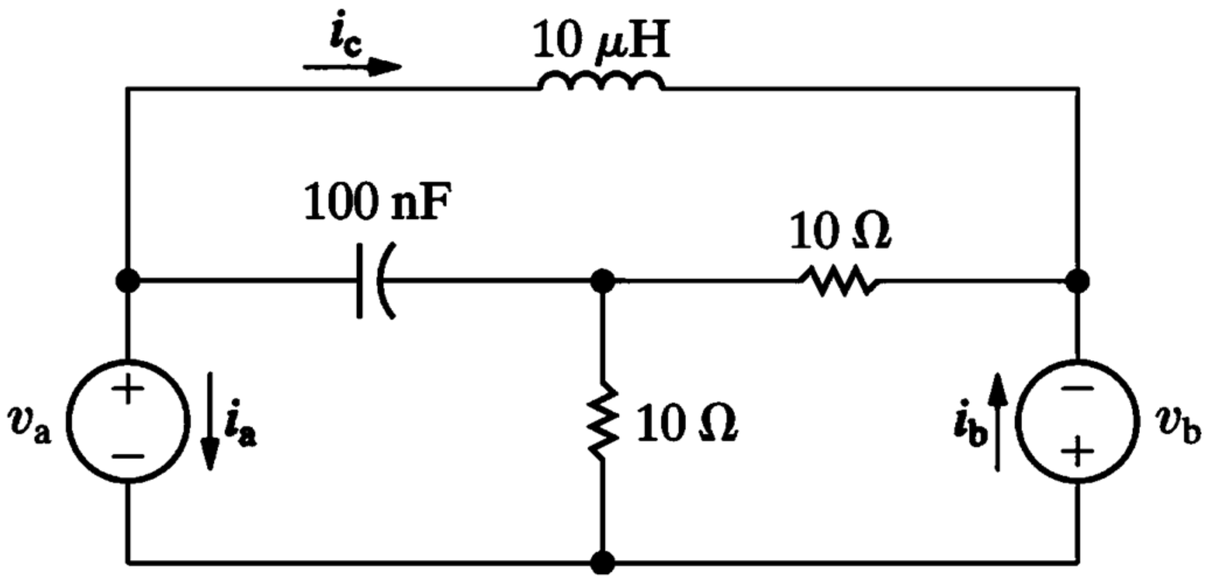
EX 4.7 Mesh Analysis



- The circuit has been already transformed to the phasor domain

1. Find $\mathbf{I_o}$

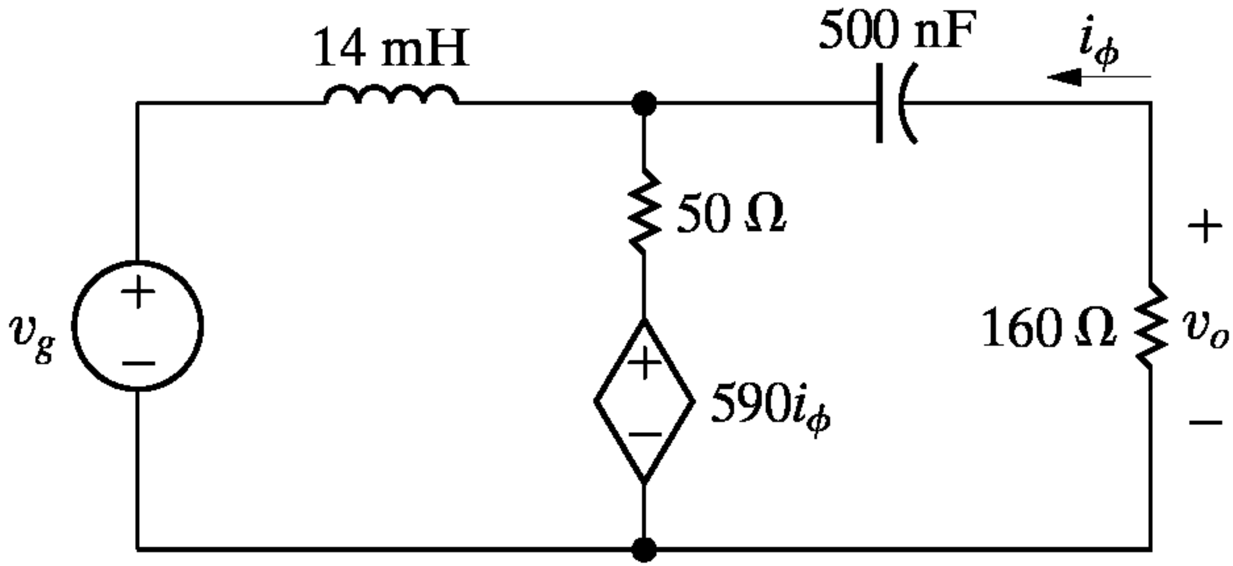
Solving AC Circuit



- $v_a = 50\sin(10^6 t)$, $v_b = 25\cos(10^6 t + 90°)$

  1. Find the steady-state expressions for the branch currents i_a, i_b, and i_c

EX 4.9 Solving AC Circuit



- $v_g = 72\cos(5000t)$

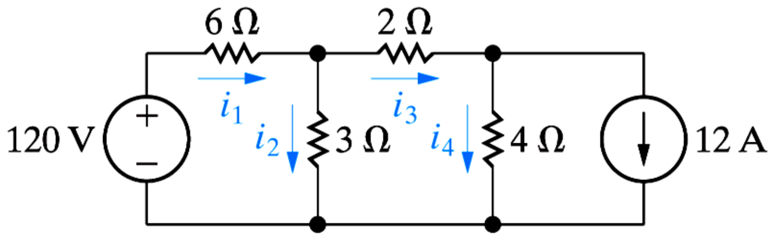
1. Find the steady-state expressions for v_o

J. Nilsson and S. Riedel, *Electric Circuits*, 9th Edition, Pearson, NJ, 2011

Superposition Theorem

The superposition theorem in Lecture 2:

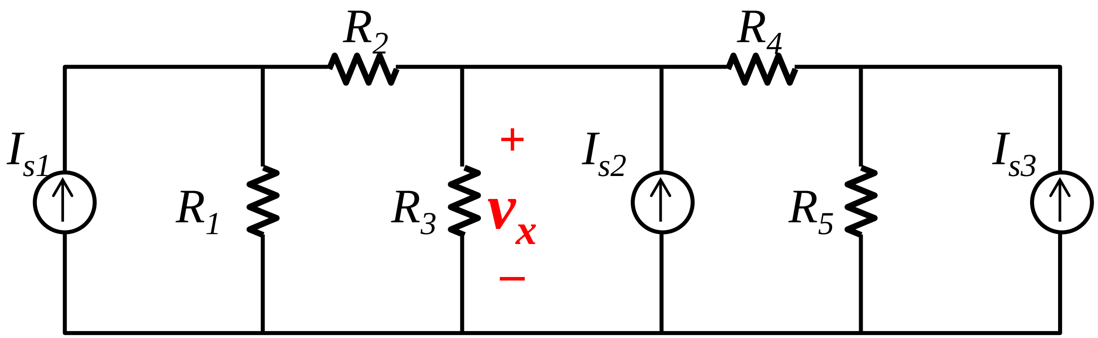
- For a linear circuit consisting of n input sources, We can activate one source at a time and sum the resultant output responses to determine the final result
- But, if the circuit contains only DC sources, the superposition theorem is of little help



The superposition theorem in Lecture 4:

- The superposition theorem becomes a very important technique NOW
- When the AC sources have **DIFFERENT frequencies**, we can activate one source at a time, and the total response is obtained by adding the individual responses **<u>in the time domain</u>**

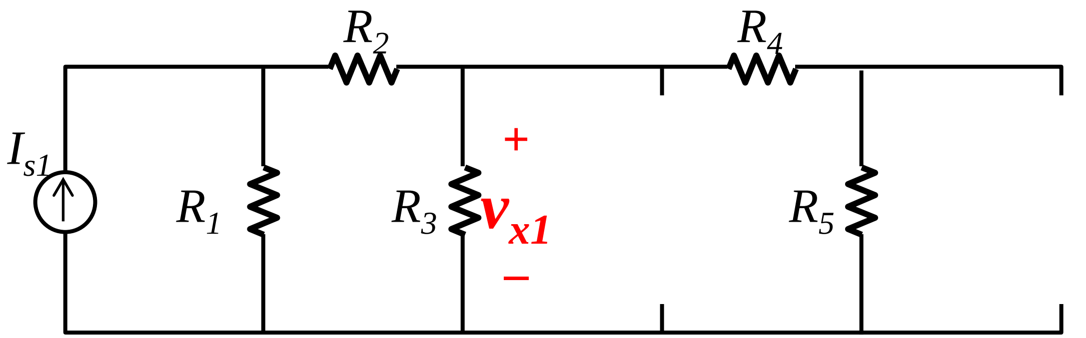
Example of Superposition Theorem (1/2)



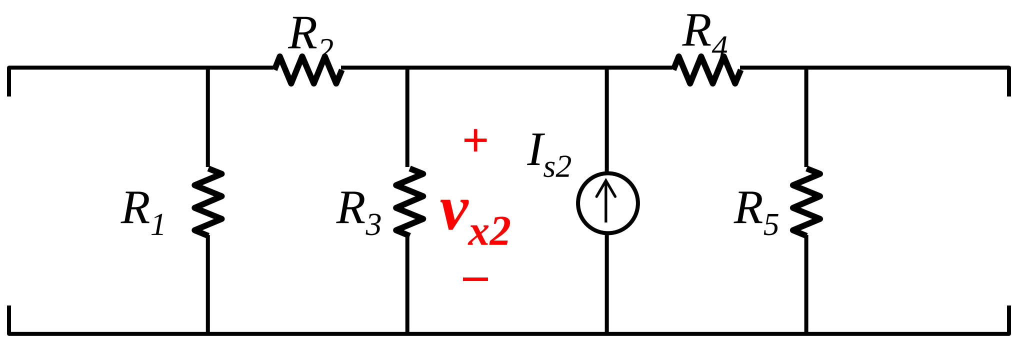
How to find v_x by superposition theorem?

Example of Superposition Theorem (2/2)

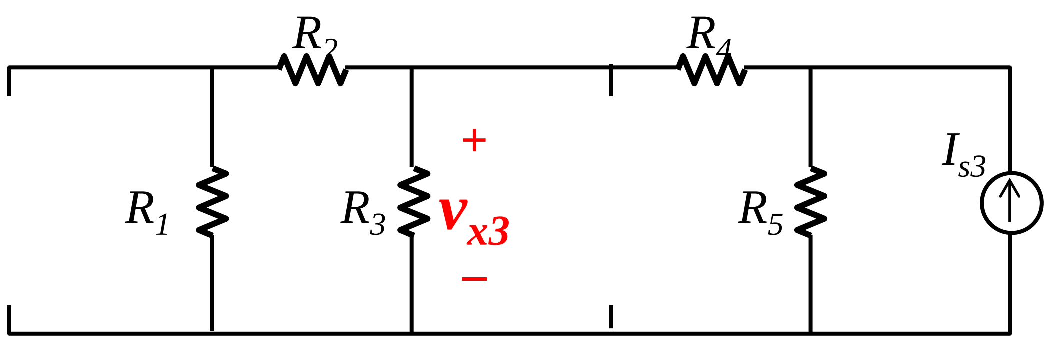
Source 1:



Source 2:

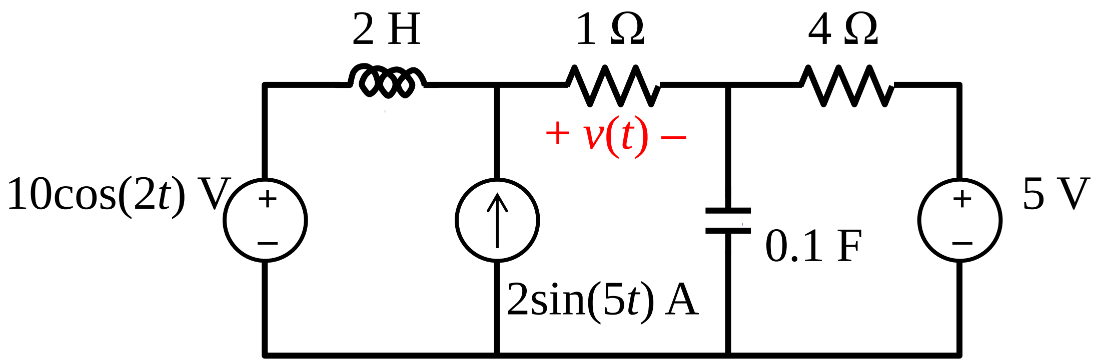


Source 3:



The final response $v_x = v_{x1} + v_{x2} + v_{x3}$

EX 4.10 Superposition Theorem (1/6)



- The linear AC circuit contains 3 source frequencies
- ω: 2, 5, and 0

1. Find the steady-state $v(t)$

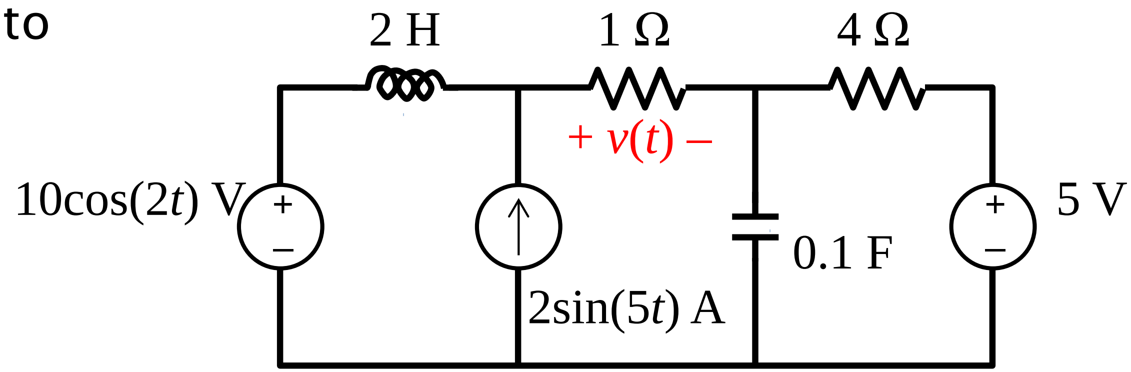
Now we have 3 frequencies, the response should be regarded as:

$$\omega_1 = 0,\ \ \omega_2 = 2,\ \ \omega_3 = 5$$

$$\therefore v(t) = v_1(t) + v_2(t) + v_3(t)$$

EX 4.10 Superposition Theorem (2/6)

Original circuit:



Step 1: Transform the circuit to the phasor domain (frequency domain)

1. $v_1(t)$: ($\omega_1 = 0$)

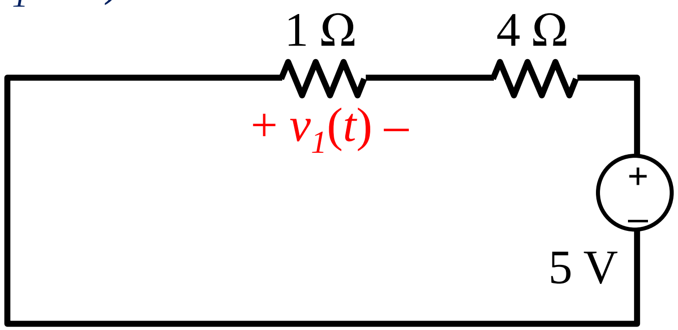
2. $v_2(t)$: ($\omega_2 = 2$)

3. $v_3(t)$: ($\omega_3 = 5$)

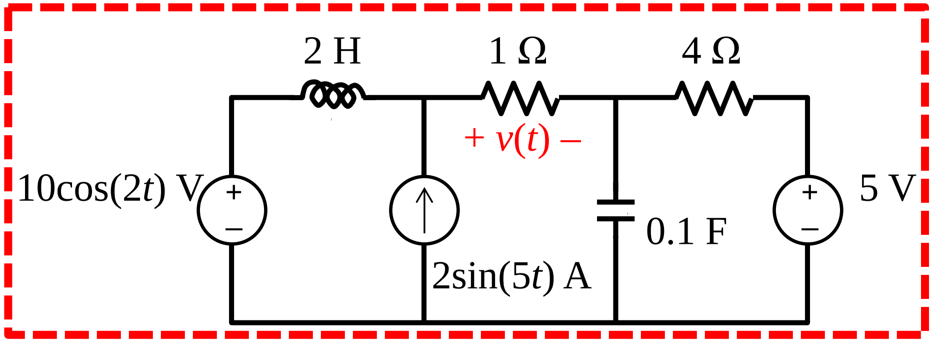
$$v(t) = v_1(t) + v_2(t) + v_3(t)$$

Superposition Theorem (3/6)

1. $v_1(t)$: ($\omega_1 = 0$)



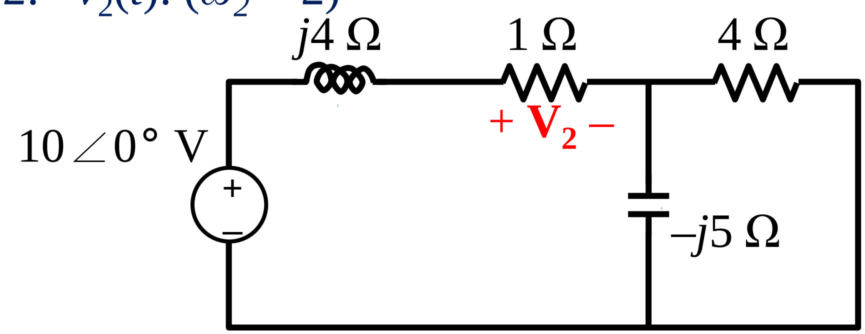
(Original circuit)



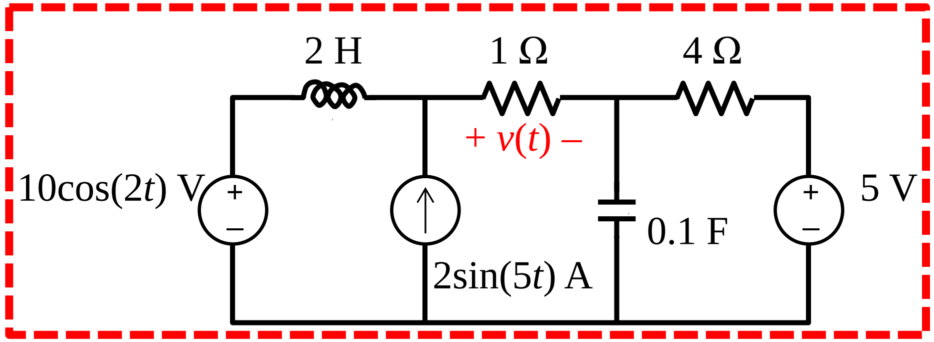
Step 2-1: Solve the circuit

Superposition Theorem (4/6)

2. $v_2(t)$: ($\omega_2 = 2$)



(Original circuit)

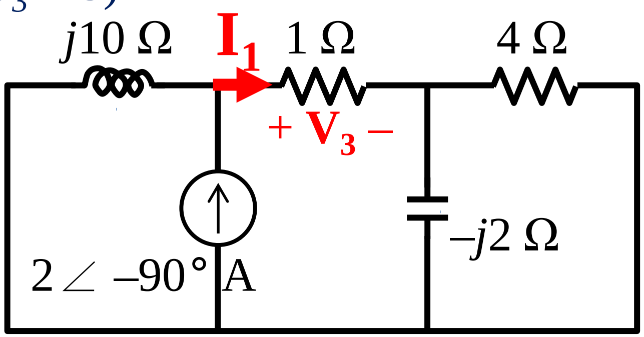


<u>Step 2-2</u>: Solve the circuit

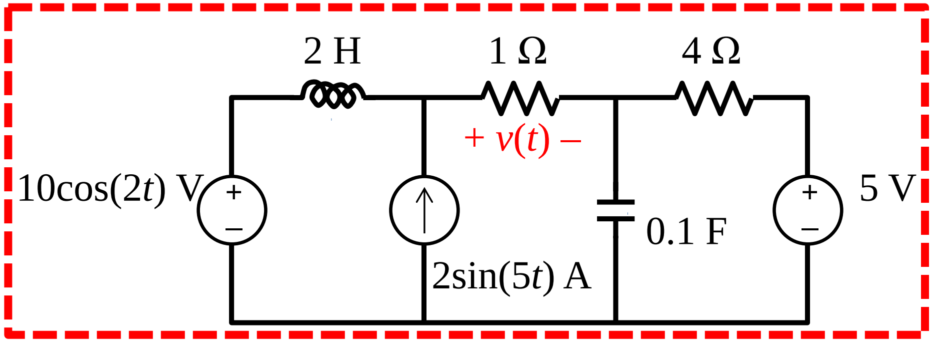
<u>Step 3-2</u>: Transform the answer back to the time domain

Superposition Theorem (5/6)

3. $v_3(t)$: ($\omega_3 = 5$)



(Original circuit)



Step 2-3: Solve the circuit

Step 3-3: Transform the answer back to the time domain

EX 4.10 Superposition Theorem (6/6)

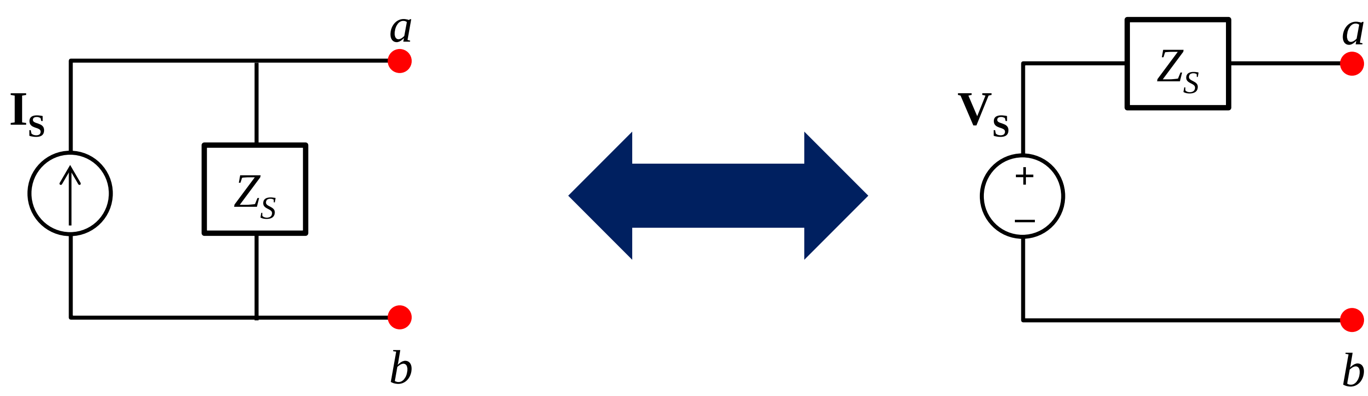
Step 4: Sum up the individual response in time domain

$$v_1(t) = -5 \times \frac{1}{1+4} = -1 \text{ V}$$

$$v_2(t) = 2.5\cos(2t - 30.8°) \text{ V}$$

$$v_3(t) = 2.33\cos(5t - 80°) \text{ V}$$

Source Transformation



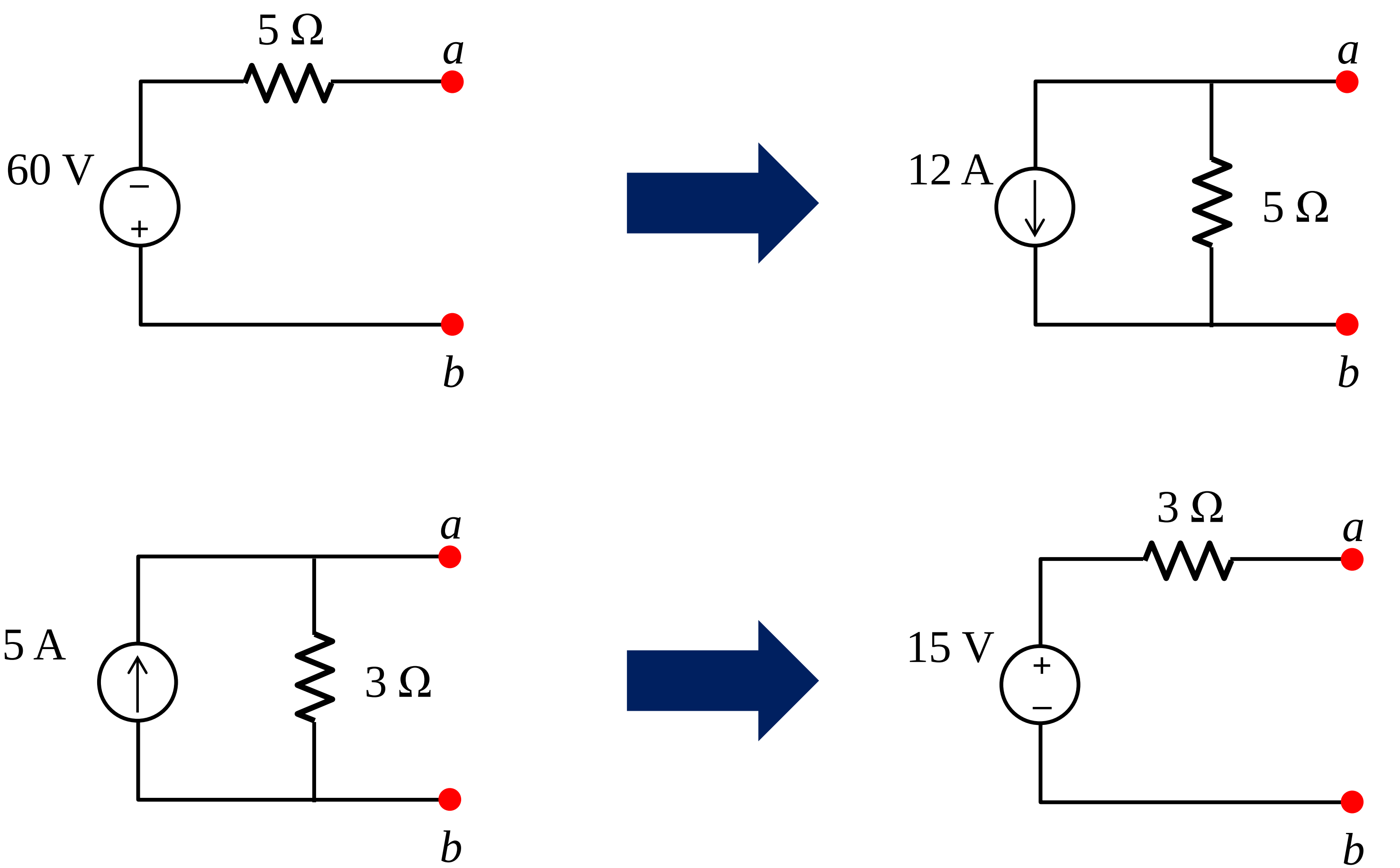
- If the current source ($\mathbf{I}_S$) is given, then it can be transformed to a voltage source with

$$\mathbf{V}_S = Z_S \mathbf{I}_S$$

- If the voltage source ($\mathbf{V}_S$) is given, then it can be transformed to a current source with

$$\mathbf{I}_S = \frac{\mathbf{V}_S}{Z_S}$$

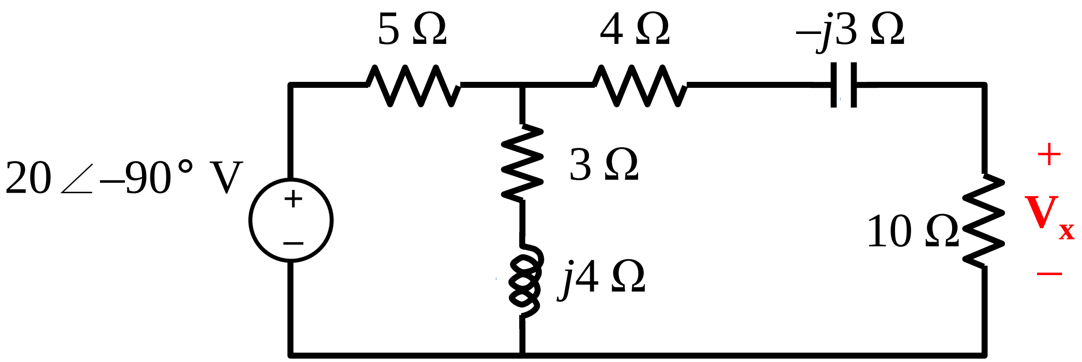
Example of Source Transformation



- Both the source and resistor representations are equivalent for the load connected at *a*-*b* terminals

EX 4.11

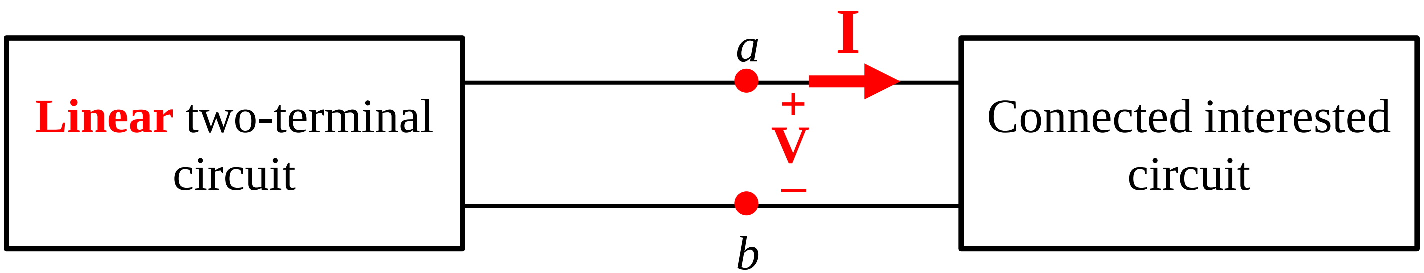
Source Transformation

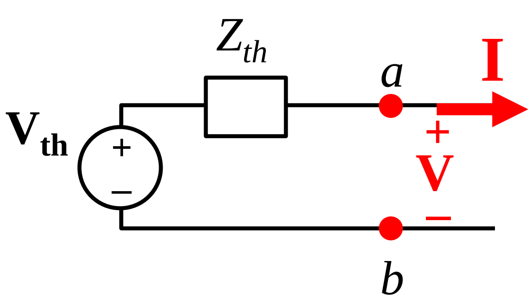
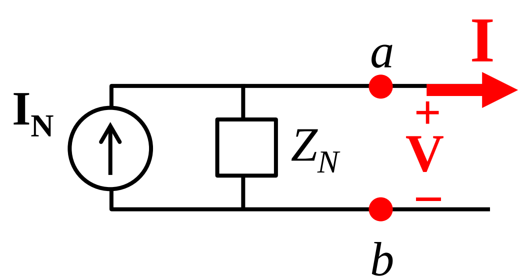


1. Calculate $\mathbf{V_x}$

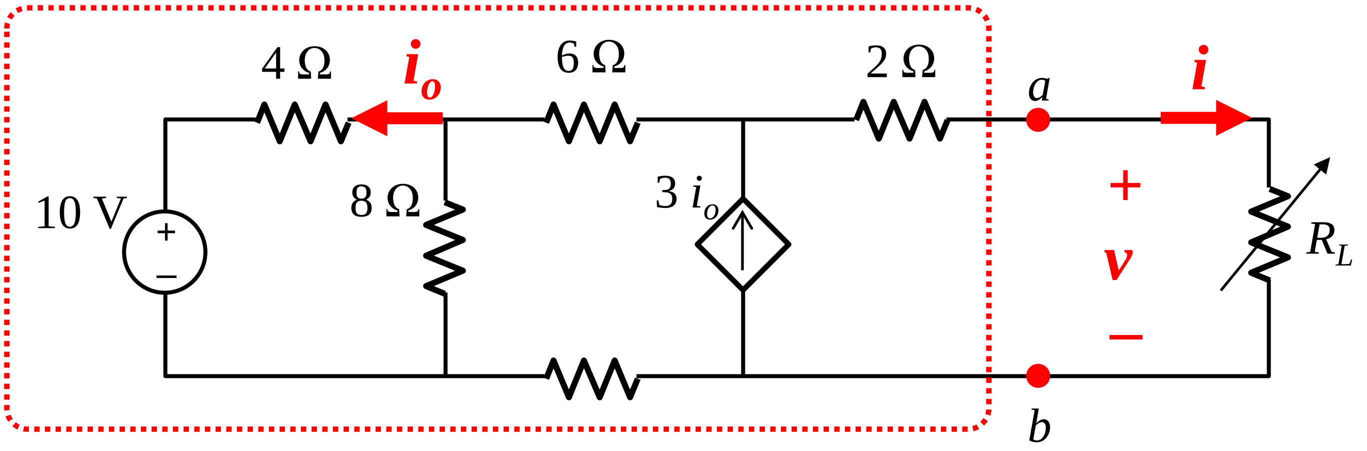
Thévenin-Norton Equivalent Circuits

A linear two-terminal circuit under sinusoidal steady-state condition:

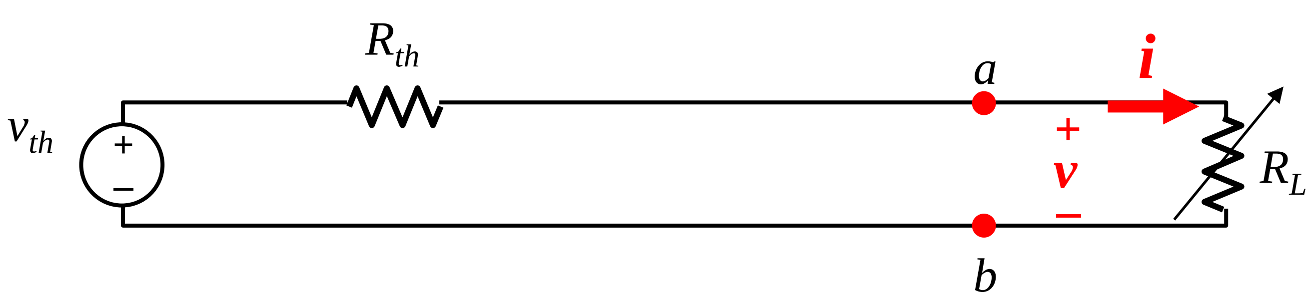


<table>
<tr><th>Thévenin equivalents</th><th>Norton equivalents</th></tr>
<tr><td>
</td><td>
</td></tr>
</table>

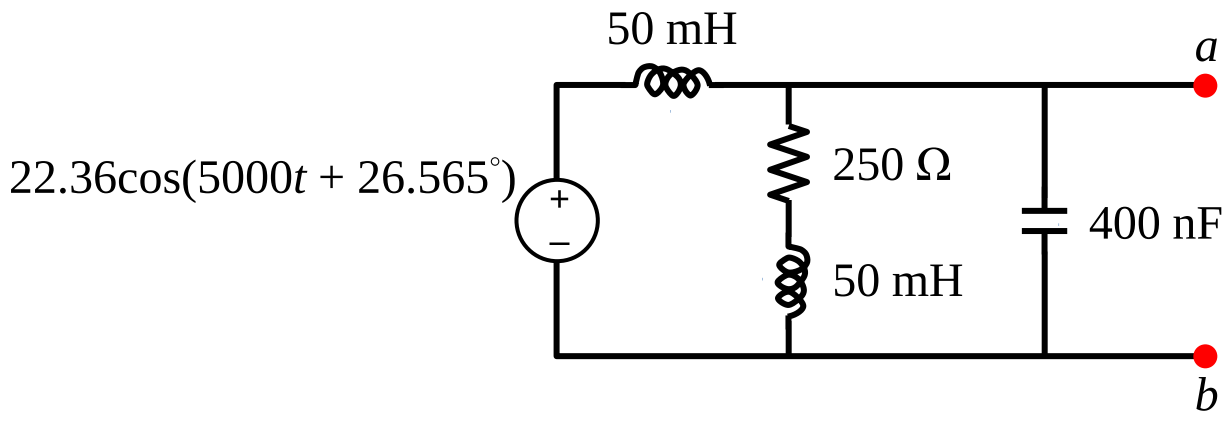
Example of Thévenin Equivalent Circuit



- v_{th} : The open-circuit voltage at the terminals
- R_{th} :The input resistance at the terminals when the ***independent*** sources are turned off

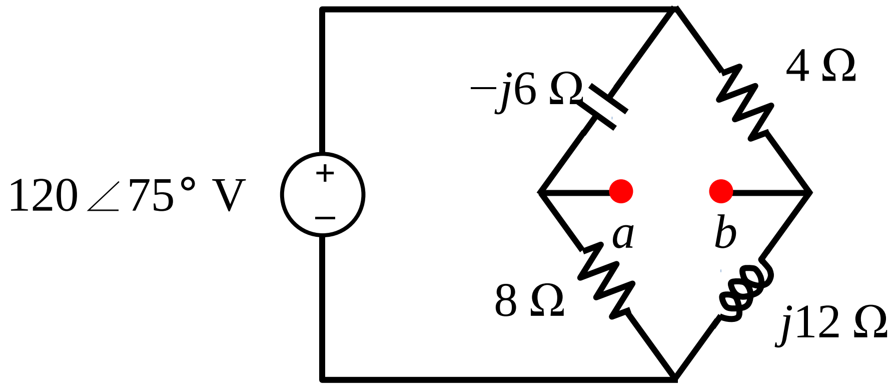


EX 4.12 Thévenin Equivalent Circuits



1. Find the Thévenin equivalent circuit with respect to the terminals *a*, *b*

EX 4.13 Thévenin Equivalent Circuits



1. Find the Thévenin equivalent circuit with respect to the terminals a, b

Contents

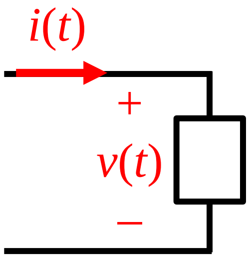
4.4 Sinusoidal Steady-State Power Calculation

Introduction to AC Power

The topic of the following discussion: AC power

- Nearly all electric energy is supplied in the form of sinusoidal voltages and currents
- Prior to this lecture, we have only one definition for power calculation:

	Power calculation
Resistive circuits	$p = v \times i$
Circuits with L & C	$p(t) = v(t) \times i(t)$



- But AC power has 6 definitions:
  1. Instantaneous power
  2. Average power (namely, real power)
  3. Reactive power (namely, imaginary power)
  4. Complex power
  5. Apparent power
  6. Power factor

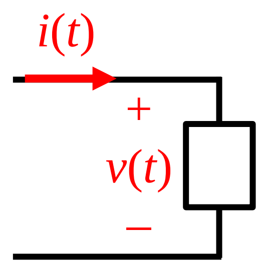
Instantaneous Power

Instantaneous power

$$p(t) = v(t) \times i(t)$$

(Watt)

Assume passive sign convention



$p(t) > 0$: The power is dissipated

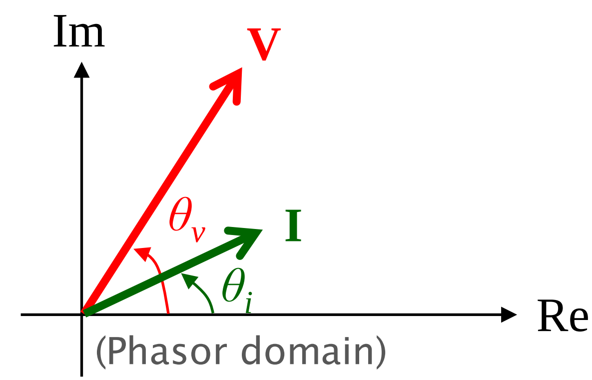
$p(t) < 0$: The power is pulled out

In AC circuits, $v(t)$ and $i(t)$ are expressed as:

$$v'(t) = V_m \cos(\omega t + \theta_v)$$

$$i'(t) = I_m \cos(\omega t + \theta_i)$$

(Time domain)

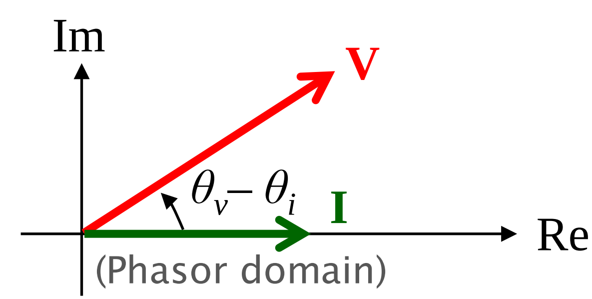


It's equivalent to:

$$v(t) = V_m \cos(\omega t + \theta_v - \theta_i)$$

$$i(t) = I_m \cos(\omega t)$$

(Time domain)



Derivation of Instantaneous Power

By definition, the instantaneous power is:

$$p(t) = V_m I_m \cos(\omega t + \theta_v - \theta_i)\cos(\omega t)$$

On the other hand, from the property of trigonometric function:

$$\cos\alpha\cos\beta = \frac{1}{2}\cos(\alpha-\beta) + \frac{1}{2}\cos(\alpha+\beta)$$

$$\Rightarrow\quad p(t) = \frac{V_m I_m}{2}\cos(\theta_v - \theta_i) + \boxed{\frac{V_m I_m}{2}\cos(2\omega t + \theta_v - \theta_i)}$$

Again, using the property of trigonometric function:

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\Rightarrow\quad p(t) = \frac{V_m I_m}{2}\cos(\theta_v - \theta_i) + \frac{V_m I_m}{2}\cos(\theta_v - \theta_i)\cos 2\omega t - \frac{V_m I_m}{2}\sin(\theta_v - \theta_i)\sin 2\omega t$$

This expression leads to the definition of average power and reactive power

Average Power & Reactive Power

$$p(t)=\underbrace{\frac{V_m I_m}{2}\cos(\theta_v-\theta_i)}_{P}+\underbrace{\frac{V_m I_m}{2}\cos(\theta_v-\theta_i)}_{P}\cos 2\omega t-\underbrace{\frac{V_m I_m}{2}\sin(\theta_v-\theta_i)}_{Q}\sin 2\omega t$$

We "**define**" them as: P P Q

$$\Rightarrow\quad p(t)=P+P\cos 2\omega t-Q\sin 2\omega t$$

We define the following quantities:

Average power

$$P=\frac{V_m I_m}{2}\cos(\theta_v-\theta_i)$$

(Watt)

Reactive power

$$Q=\frac{V_m I_m}{2}\sin(\theta_v-\theta_i)$$

(Var)

Physical Meaning of P

Why is P called "average" power?

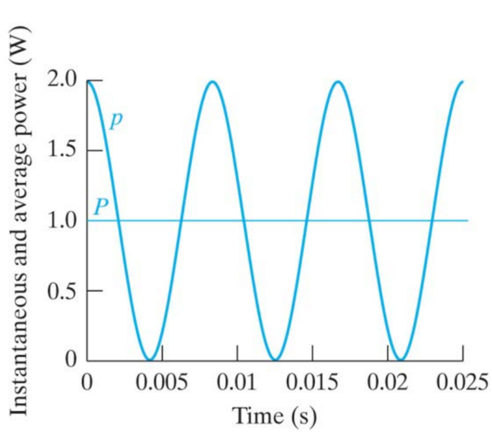
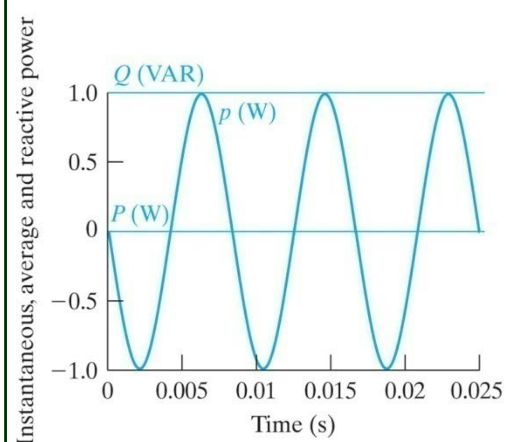
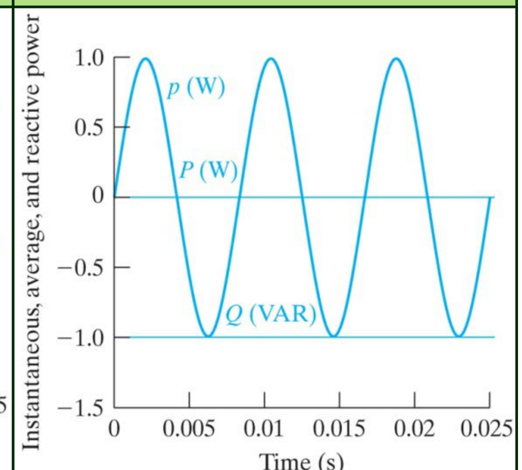
$$\because\ p(t) = P + P\cos 2\omega t - Q\sin 2\omega t$$

$$\therefore\ \frac{1}{T}\int_{t_0}^{t_0+T} p(\tau)\,d\tau = P$$

- T is the period of the sinusoidal function
- The average of the instantaneous power over one period is exactly P itself

Physical Meaning of Q

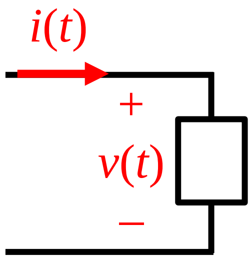
Why is Q called "reactive" power?

	Purely resistive circuit	Purely inductive circuit	Purely capacitive circuit
$\theta_v - \theta_i$	0	90°	–90°
$p(t)$	$P + P\cos 2\omega t$	$-Q\sin 2\omega t$	$Q\sin 2\omega t$
P	1	0	0
Q	0	1	–1
Figure			

- Reactive power: Inductors and capacitors are reactive element

J. Nilsson and S. Riedel, *Electric Circuits*, 9th Edition, Pearson, NJ, 2011

EX 4.14 Average Power & Reactive Power



1. Calculate the average power and the reactive power at the terminals of the network if

$$v(t) = 100\cos(\omega t + 15^\circ)$$
$$i(t) = 4\sin(\omega t - 15^\circ)$$

2. State whether the network is absorbing or delivering average power
3. State whether the network inside the box is absorbing or supplying magnetizing vars

Representation in Phasor Domain

Time domain

$$v(t) = V_m \cos(\omega t + \theta_v)$$

$$i(t) = I_m \cos(\omega t + \theta_i)$$

Phasor domain

$$\mathbf{V} = V_m \angle \theta_v$$

$$\mathbf{I} = I_m \angle \theta_i$$

$$P(v(t))$$

$$P^{-1}(\mathbf{V}) = \mathrm{Re}\left[\mathbf{V} e^{j\omega t}\right]$$

- Average power and reactive power represented by its phasor:

$$P = \frac{V_m I_m}{2} \cos(\theta_v - \theta_i) = \frac{1}{2} \mathrm{Re}\left[\mathbf{V}\mathbf{I}^*\right]$$

$$Q = \frac{V_m I_m}{2} \sin(\theta_v - \theta_i) = \frac{1}{2} \mathrm{Im}\left[\mathbf{V}\mathbf{I}^*\right]$$

Are the Formulations Familiar with Our Background?

$$P = \frac{V_m I_m}{2}\cos(\theta_v - \theta_i) = \frac{1}{2}\mathrm{Re}\left[\mathbf{VI}^*\right]$$

$$Q = \frac{V_m I_m}{2}\sin(\theta_v - \theta_i) = \frac{1}{2}\mathrm{Im}\left[\mathbf{VI}^*\right]$$

That is, the instantaneous power in time domain

- We are more familiar with the form: $p = v \times i$ instead of $P = V_m \times I_m^*/2$
- Can we create effective terms of voltage and current, such as

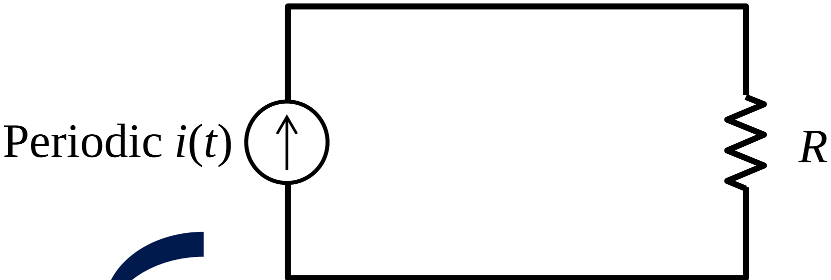
V_{eff}, I_{eff}

and make $P = V_{eff} \times I_{eff}^*$?

I_{eff} of a periodic current $i(t)$: $I_{eff} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} i^2(\tau)d\tau}$

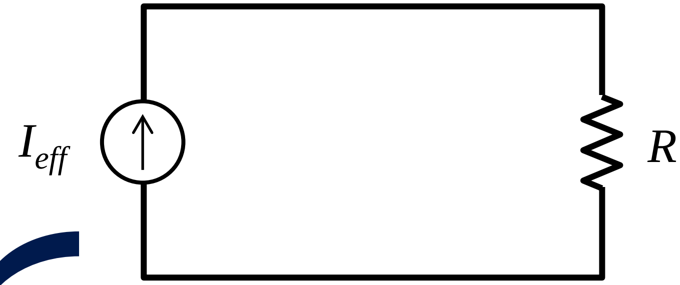
The equivalent **DC** current that delivers the same average power to a resistor R as the periodic current

The Meaning of I_{eff}



- Periodic current source
- Not necessary being a sinusoidal source

$$P_1 = \frac{1}{T}\int_{t_0}^{t_0+T} i^2(\tau) \times R\, d\tau$$



- DC source

$$P_2 = I_{eff}^2 \times R$$

By definition, $P_1 = P_2$, so:

$$I_{eff} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} i^2(\tau)\, d\tau}$$

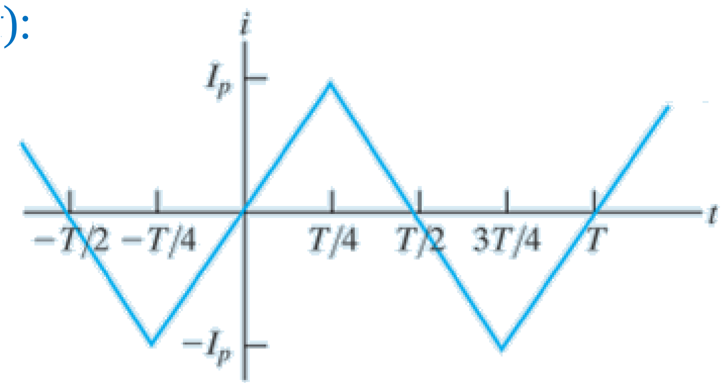
The effective value of a periodic signal is its root mean square (RMS) value!

EX 4.15 Calculations of RMS Value

1. If $i(t) = I_m \cos(\omega t + \theta_i)$

   Find its RMS current

2. If $i(t)$:



   Find its RMS current

Special Case: AC Circuits

The major concern in this lecture: $i(t) = I_m\cos(\omega t + \theta_i)$

$$\Rightarrow \quad I_{rms} = \sqrt{\frac{I_m^2}{T}\int_0^T \cos\left(\omega t + \theta_i\right)^2 d\tau} = \frac{I_m}{\sqrt{2}}$$

- We define a new type of phasor, called "**rms phasor**" of $i(t)$, as

$$\mathbf{I}_{rms} = I_{rms}\angle\theta_i = \frac{I_m}{\sqrt{2}}\angle\theta_i$$

- Average power can be rewritten as:

$$P = \frac{V_m I_m}{2}\cos(\theta_v - \theta_i) = \frac{V_m}{\sqrt{2}}\frac{I_m}{\sqrt{2}}\cos(\theta_v - \theta_i) = V_{rms}I_{rms}\cos(\theta_v - \theta_i)$$

- Reactive power can be rewritten as:

$$Q = V_{rms}I_{rms}\sin(\theta_v - \theta_i)$$

EX 4.16 Finding Average Power by the RMS Voltage

- A sinusoidal voltage having a maximum amplitude of 625 V is applied to the terminals of a 50 Ω resistor

    1. Find the average power delivered to the resistor

Complex Power of Phasor Domain

- If we express them in terms of RMS voltage phasor & RMS current phasor:

$$P = \frac{1}{2}\mathrm{Re}\left[\mathbf{V}\mathbf{I}^*\right] = \mathrm{Re}\left[\frac{\mathbf{V}}{\sqrt{2}}\frac{\mathbf{I}^*}{\sqrt{2}}\right] = \mathrm{Re}\left[\mathbf{V}_{rms}\mathbf{I}^*_{rms}\right]$$

→ So P is also called “real power”

$$Q = \frac{1}{2}\mathrm{Im}\left[\mathbf{V}\mathbf{I}^*\right] = \mathrm{Im}\left[\frac{\mathbf{V}}{\sqrt{2}}\frac{\mathbf{I}^*}{\sqrt{2}}\right] = \mathrm{Im}\left[\mathbf{V}_{rms}\mathbf{I}^*_{rms}\right]$$

→ So Q is also called “imaginary power”

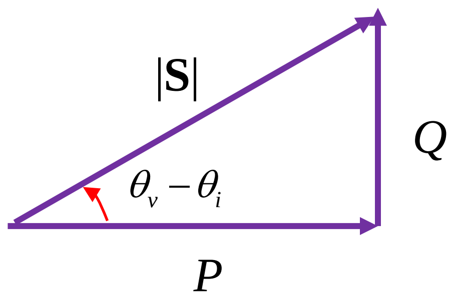
- So, we define the complex power:

$$\mathbf{S} = P + jQ = \mathbf{V}_{rms}\mathbf{I}^*_{rms}$$

Complex power

$$\mathbf{S} = \mathbf{V}_{rms}\mathbf{I}^*_{rms} = P + jQ \quad \text{(VA)}$$

Power Triangle (1/2)



$$P = \frac{V_m I_m}{2}\cos(\theta_v - \theta_i) = \frac{1}{2}\mathrm{Re}\left[\mathbf{V}\mathbf{I}^*\right] = \mathrm{Re}\left[\mathbf{V}_{rms}\mathbf{I}^*_{rms}\right]$$

$$Q = \frac{V_m I_m}{2}\sin(\theta_v - \theta_i) = \frac{1}{2}\mathrm{Im}\left[\mathbf{V}\mathbf{I}^*\right] = \mathrm{Im}\left[\mathbf{V}_{rms}\mathbf{I}^*_{rms}\right]$$

- Definition of "apparent power (unit: VA)":

Apparent power = $|\mathbf{S}|$, or

$$|\mathbf{S}| = \sqrt{P^2 + Q^2}$$

- The apparent power represents the volt-amp capacity required to supply the average power

- Definition of "power factor (pf)":

$$pf = \frac{P}{|\mathbf{S}|} = \cos(\theta_v - \theta_i)$$

- Clearly, $pf \le 1$
- What does $pf = 1$ mean?

Apparent power

$$|\mathbf{S}| = \sqrt{P^2 + Q^2}$$

(VA)

Power Triangle (2/2)

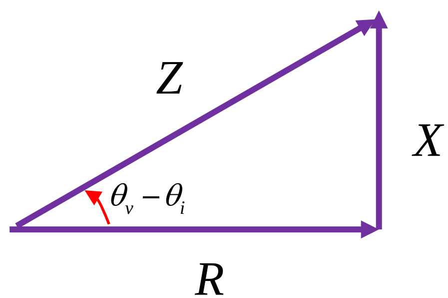
- From complex Ohm's law:

$$\mathbf{V}_{rms} = Z\mathbf{I}_{rms} = (R + jX)\mathbf{I}_{rms}$$

- Therefore, complex power can be written as:

$$\mathbf{S} = \mathbf{V}_{rms}\mathbf{I}^*_{rms} = Z\mathbf{I}_{rms} \times \mathbf{I}^*_{rms} = Z\left|\mathbf{I}_{rms}\right|^2 = R\left|\mathbf{I}_{rms}\right|^2 + jX\left|\mathbf{I}_{rms}\right|^2 = P + jQ$$

$\Rightarrow$ $R \propto P$ and $X \propto Q$



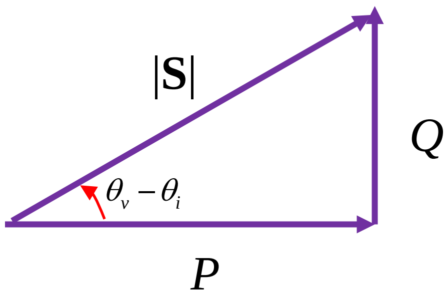
Note:

$$pf = \frac{P}{|\mathbf{S}|} = \frac{R}{|Z|}$$

- Q is a measure of the energy exchange between the source and the reactive part of the load
- Reactive power represents a lossless interchange between the load and the source

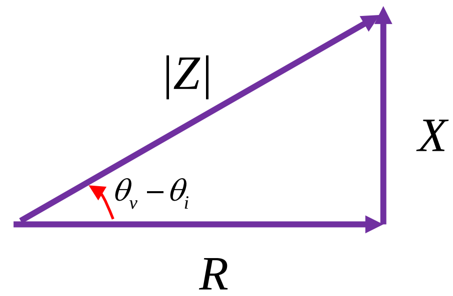
Meaning of Power Factor

From the perspective of power:



$$P = |\mathbf{S}|\cos(\theta_v - \theta_i)$$

From the perspective of impedance:



$$Z = \frac{\mathbf{V}}{\mathbf{I}} = \frac{V_m}{I_m}\angle\theta_v - \theta_i$$

Lagging pf and leading pf:

	Load	*pf*
$Q = 0$	Resistive load	$pf = 1$
$Q > 0$	Inductive load	Lagging *pf*
$Q < 0$	Capacitive load	Leading *pf*

Current lags voltage

Current leads voltage

EX 4.17 Calculating Power for Arbitrary Load

- The load voltage and current are given as follow:

$$v(t) = 60\cos(\omega t - 10°)\ \text{V}$$
$$i(t) = 1.5\cos(\omega t + 50°)\ \text{A}$$

- Calculate the following quantities:

1. Complex power
2. Apparent power
3. Average power
4. Reactive power
5. Power factor
6. Impedance Z_L

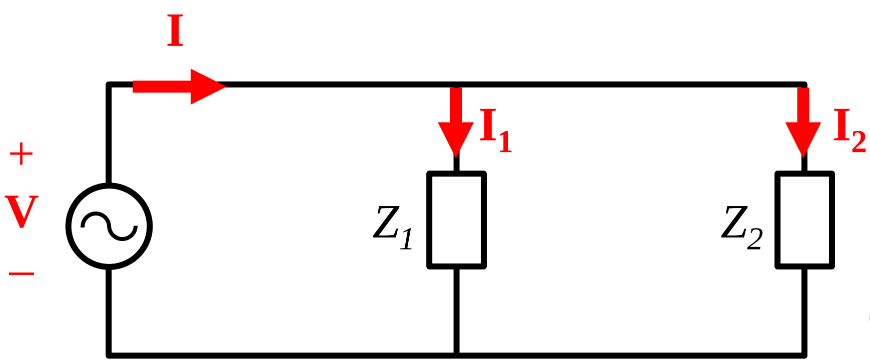
EX 4.18 Calculating Complex Power

- An electrical load operates at $|\mathbf{V}_{rms}| = V_{rms} = 240$ V
- The load absorbs an average power of 8 kW at a lagging power factor of 0.8

1. Calculate the complex power of the load
2. Calculate the impedance of the load

Conservation of Complex Power (1/2)

Parallel connection:



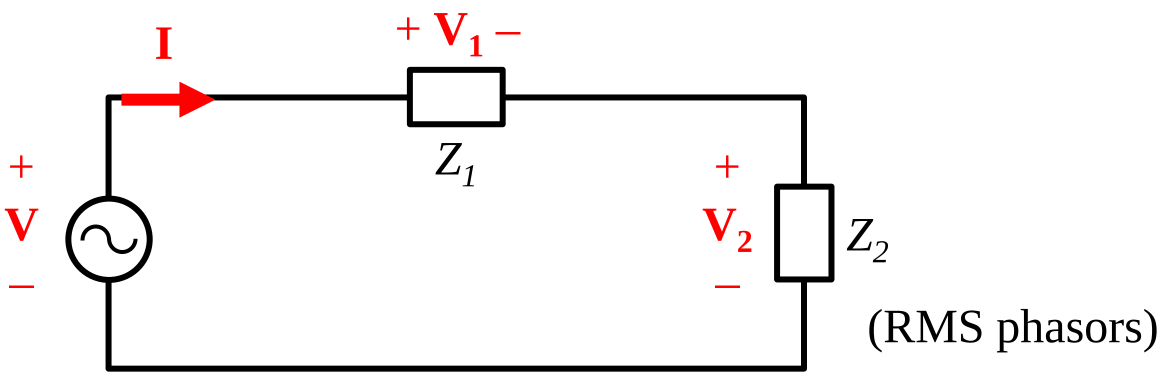
(RMS phasors)

- The complex power supplied by the source:

- The complex power of the source equals the respective sum of the complex powers of the individual loads

Conservation of Complex Power (2/2)

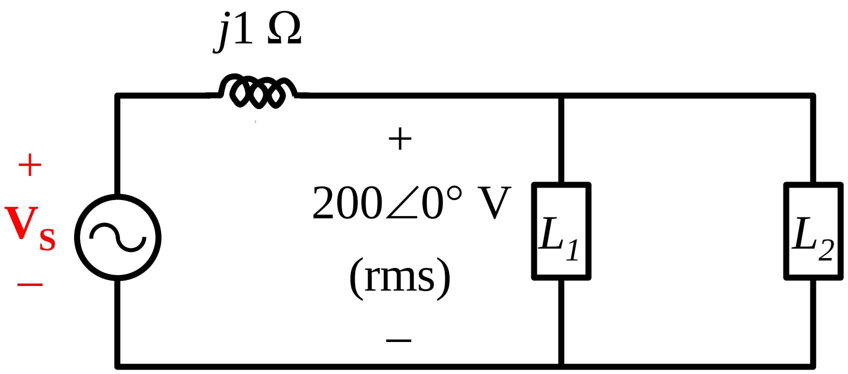
Series connection:



- The complex power supplied by the source:

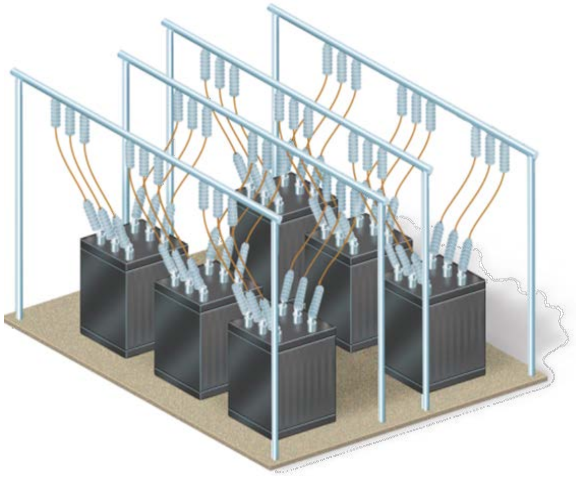
- The complex power of the source equals the respective sum of the complex powers of the individual loads

Power Conservation



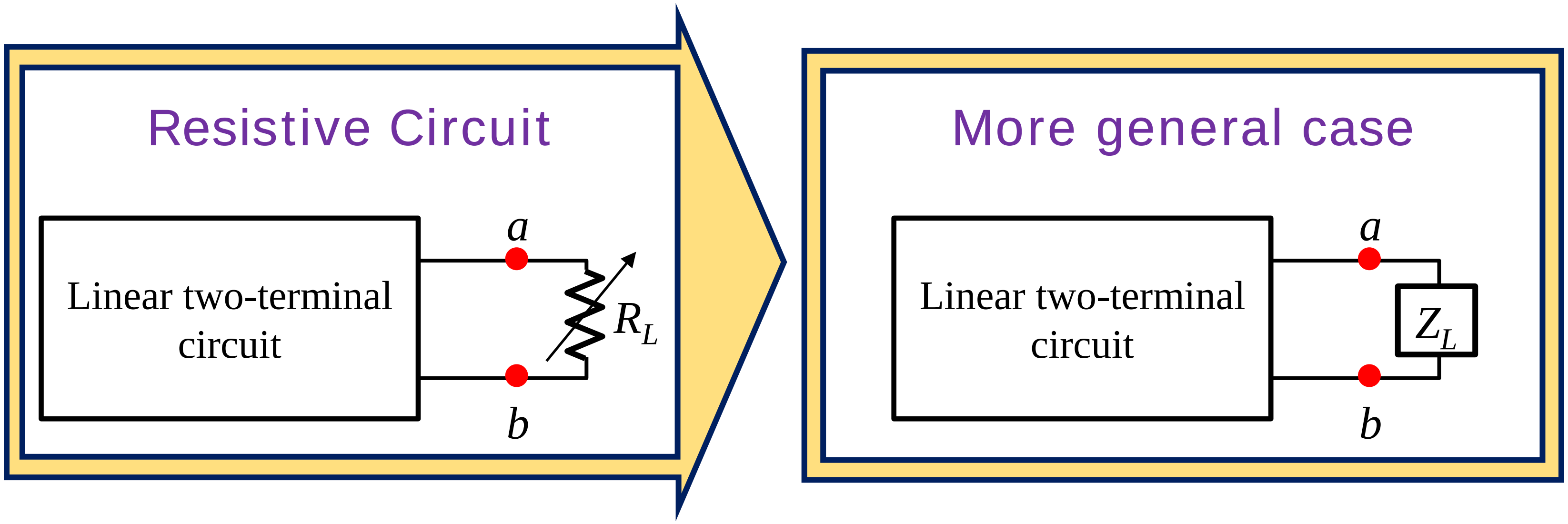
- Load L_1 is absorbing 15 kVA at 0.6 *pf* lagging
- Load L_2 is absorbing 6 kVA at 0.8 *pf* leading

1. Find the phasor voltage $\mathbf{V}_S$ (rms) in this circuit
   (Express $\mathbf{V}_S$ in polar form)

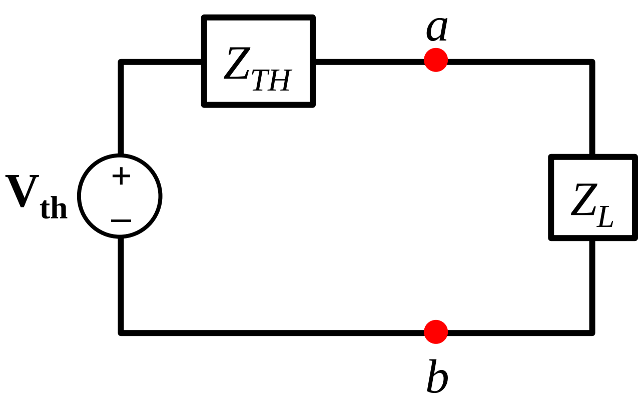


J. Nilsson and S. Riedel, *Electric Circuits*, 9th Edition, Pearson, NJ, 2011

Maximum Power Transfer in AC Circuits

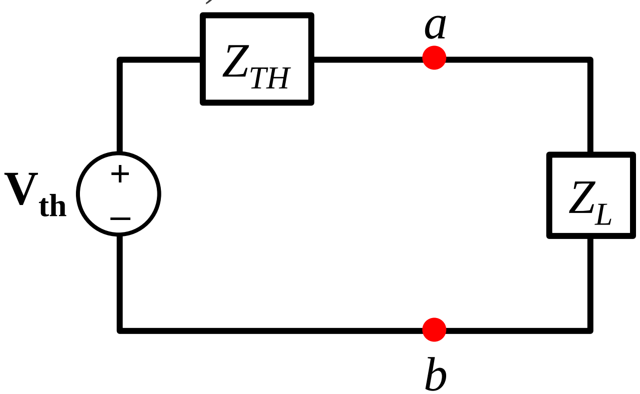


- Find the value of Z_L that permits maximum power delivery to it
- How to solve it? From its Thévenin equivalent circuit:



The Condition for Maximum Power Transfer (1/2)

(Phasor domain)



- Let

$$Z_{TH} = R_{TH} + jX_{TH}$$
$$Z_L = R_L + jX_L$$

$$\Rightarrow \quad \mathbf{I} = \frac{\mathbf{V_{TH}}}{Z_{TH} + Z_L} = \frac{\mathbf{V_{TH}}}{(R_{TH} + R_L) + j(X_{TH} + X_L)}$$

$$P = \frac{1}{2} \times |\mathbf{I}|^2 \times R_L = \frac{1}{2} \times \frac{|\mathbf{V_{TH}}|^2 \times R_L}{(R_{TH} + R_L)^2 + (X_{TH} + X_L)^2}$$

(P: Average power)

- To have maximum output P:

$$\frac{\partial P}{\partial R_L} = 0 \text{ and } \frac{\partial P}{\partial X_L} = 0$$

(1) (2)

The Condition for Maximum Power Transfer (2/2)

(1) $$\frac{\partial P}{\partial R_L} = \frac{-\left|\mathbf{V_{TH}}\right|^2 [(R_L + R_{TH})^2 + (X_L + X_{TH})^2 - 2R_L(R_L + R_{TH})]}{[(R_L + R_{TH})^2 + (X_L + X_{TH})^2]^2} = 0$$

$$\rightarrow (R_L + R_{TH})^2 + (X_L + X_{TH})^2 - 2R_L(R_L + R_{TH}) = 0$$

$$\rightarrow R_L^2 + 2R_L R_{TH} + R_{TH}^2 + (X_L + X_{TH})^2 - 2R_L^2 - 2R_L X_{TH} = 0$$

$$\rightarrow R_{TH}^2 + (X_L + X_{TH})^2 - R_L^2 = 0 \Rightarrow \boxed{R_L = \sqrt{R_{TH}^2 + (X_L + X_{TH})^2}}$$

(2) $$\frac{\partial P}{\partial X_L} = \frac{-\left|V_{TH}\right|^2 2R_L(X_L + X_{TH})}{[(R_L + R_{TH})^2 + (X_L + X_{TH})^2]^2} = 0 \Rightarrow \boxed{X_L = -X_{TH}}$$

(Amplitude phasor)

To have maximum output P:

$$Z_L = Z_{TH}^*$$

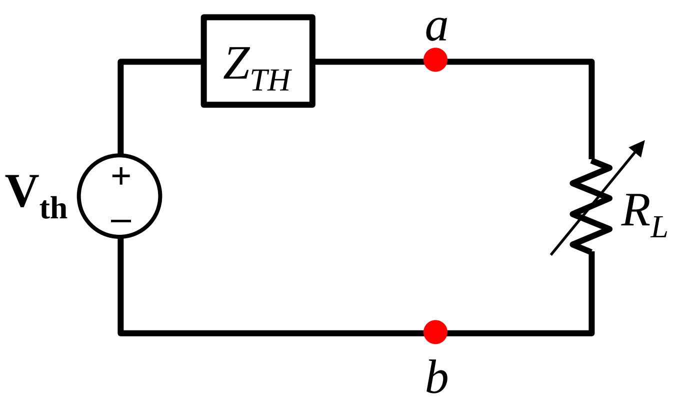
$$P_{\max} = \frac{1}{2} \times \left|\mathbf{I}\right|^2 \times R_L$$

$$= \frac{1}{2} \times \frac{\left|\mathbf{V}_{TH}\right|^2}{2R_{TH}} \times R_{TH} = \frac{\left|\mathbf{V}_{TH}\right|^2}{8R_{TH}}$$

Special Case: If the Load Is Purely Real

The load is not a complex number; it's purely real:

$$Z_{TH} = R_{TH} + jX_{TH}$$



- What's the condition for maximum power transfer?

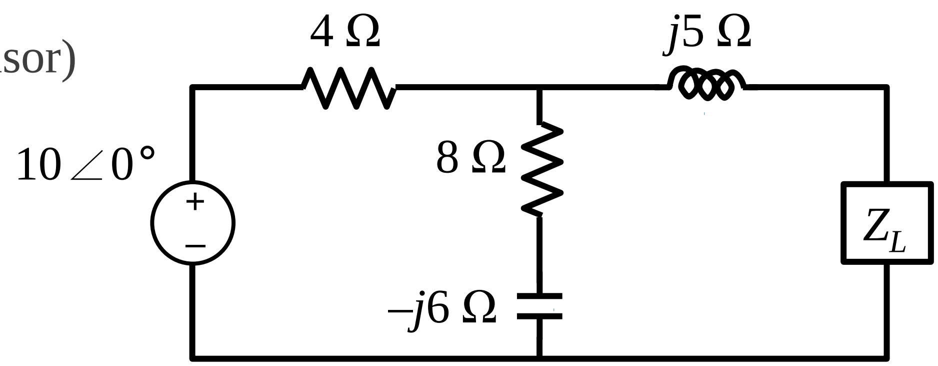
1. $R_L = R_{TH}$

   or

2. $R_L = |Z_{TH}|$

EX 4.20 Case 1: Z_L Is an Arbitrary Complex Number

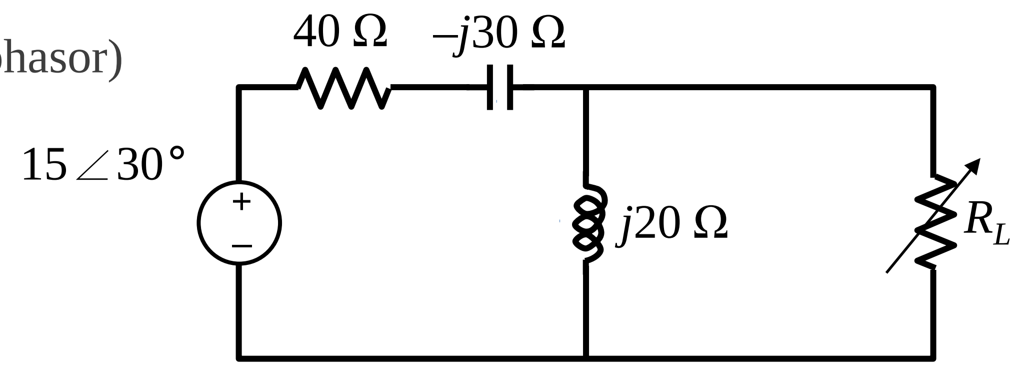
(Amplitude phasor)



1. Determine Z_L to get maximum average power output P_{Z_L}
2. What is the value of P_{Z_L}?

EX 4.21 Case 2: Z_L Is an Arbitrary Real Number

(Amplitude phasor)



1. Find R_L such that it will absorb maximum average power
2. What is the value of this power?

Contents

4.5 Balanced Three-Phase Circuits

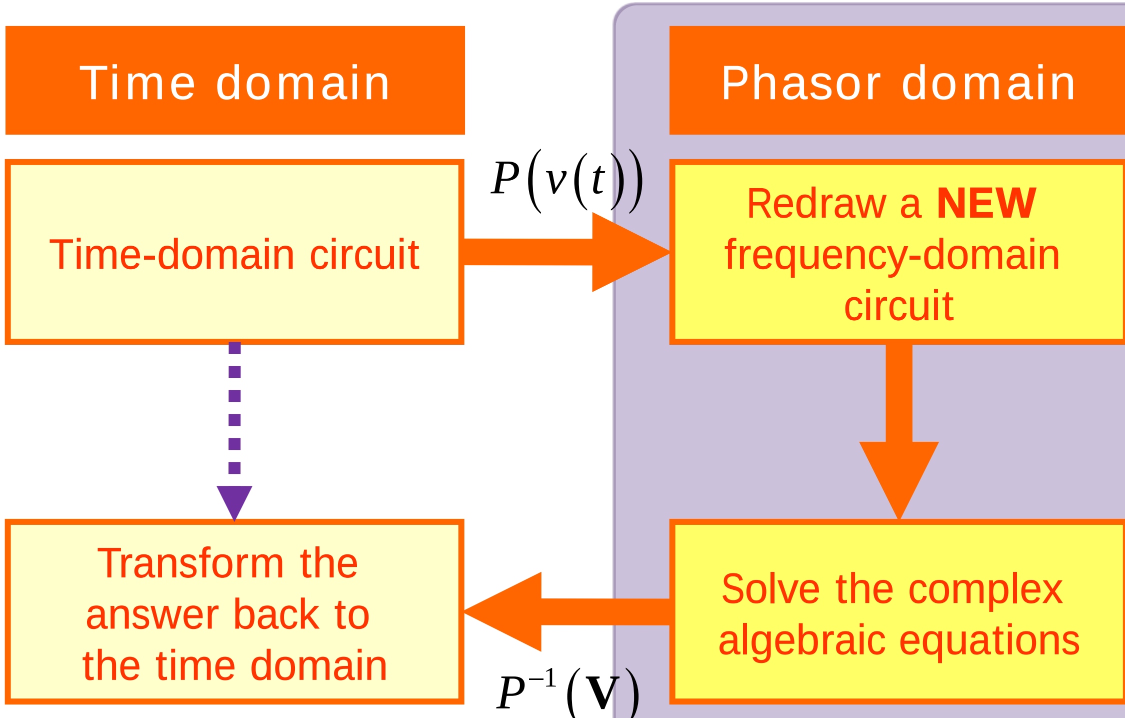
Review of the Phasor Approach

- Phasor transform of $v(t) = V_m \cos(\omega t + \theta) \xrightarrow{P(v(t))} V_m \angle \theta$ (polar form)

  $\mathbf{V}$ (complex form)

  $V_m e^{j\theta}$ (Exponential form)

- Inverse phasor transform: $P^{-1}(\mathbf{V}) = \mathrm{Re}\left[\mathbf{V}e^{j\omega t}\right]$



1. KCL for a node:

$$\sum_{K=1}^{n} I_{mK} \angle \theta_K = 0 + j0$$

or $\mathbf{I_1} + \mathbf{I_2} + \ldots + \mathbf{I_n} = \mathbf{0}$

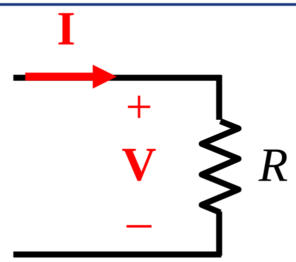
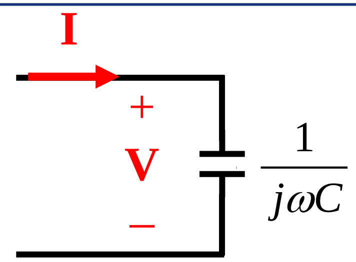
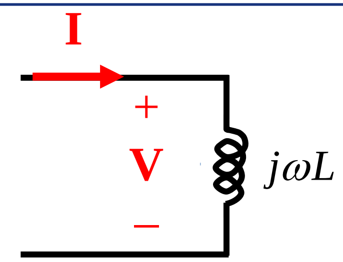
2. KVL for a loop:

$$\sum_{K=1}^{n} V_{mK} \angle \phi_K = 0 + j0$$

or $\mathbf{V_1} + \mathbf{V_2} + \ldots + \mathbf{V_n} = \mathbf{0}$

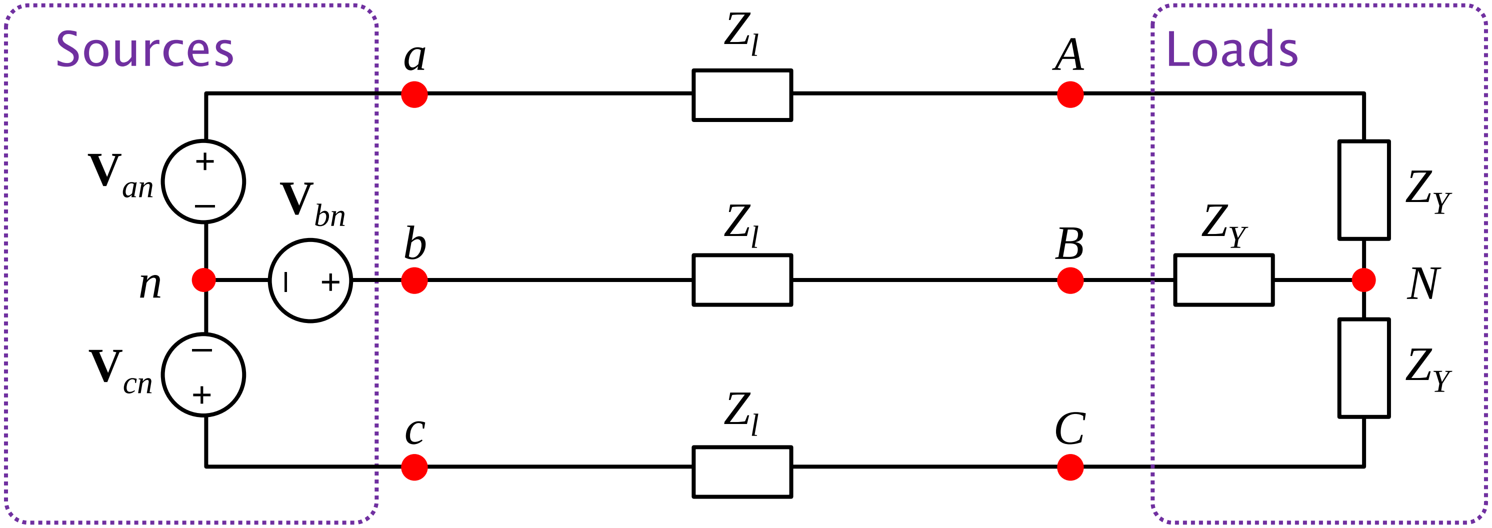
3. Component models

Component Models in the Phasor Domain

	 	 	
Given $\mathbf{I}$, express $\mathbf{V}$	$\mathbf{V} = R\mathbf{I}$	$\mathbf{V} = \frac{1}{j\omega C}\mathbf{I} = -j\frac{1}{\omega C}\mathbf{I}$	$\mathbf{V} = j\omega L\mathbf{I}$
Given $\mathbf{V}$, express $\mathbf{I}$	$\mathbf{I} = \frac{1}{R}\mathbf{V}$	$\mathbf{I} = j\omega C\mathbf{V}$	$\mathbf{I} = \frac{1}{j\omega L}\mathbf{V}$
Impedance Z	$Z \triangleq \frac{\mathbf{V}}{\mathbf{I}} = R$	$Z \triangleq \frac{\mathbf{V}}{\mathbf{I}} = \frac{1}{j\omega C}$	$Z \triangleq \frac{\mathbf{V}}{\mathbf{I}} = j\omega L$

- Impedance is a complex quantity made by our definition; it's not a phasor
- It can be further expanded into $R + jX$; R: Resistance, X: Reactance

An Overview of Balanced Three-Phase Systems



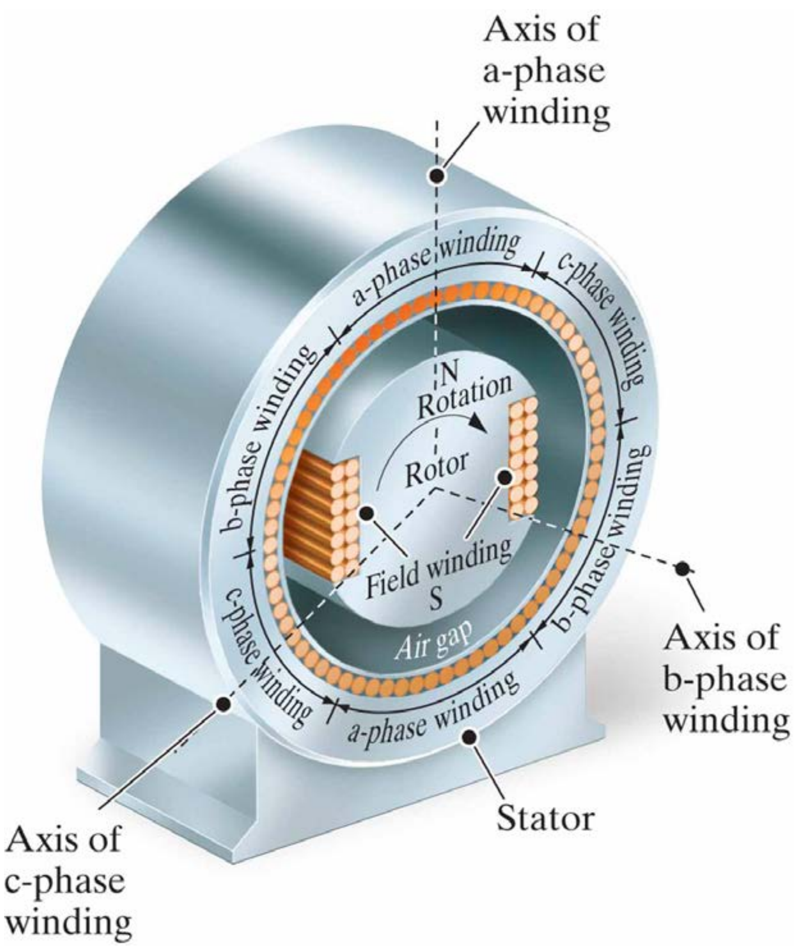
(Phasor domain)

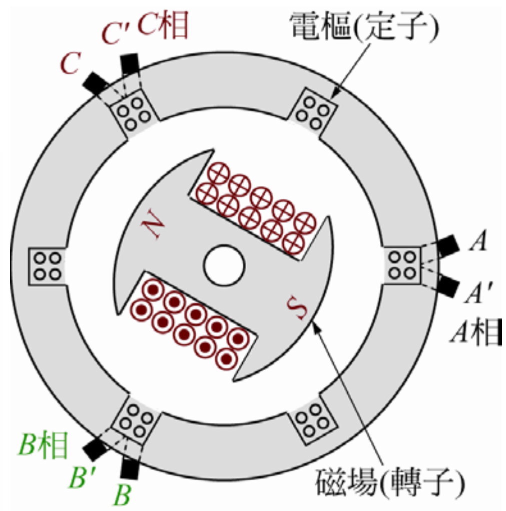
A balance three-phase system contains:

- Three voltage sources
  - Equal amplitude
  - Phases are different by 120°
- Three loads with equal value
- Three distributed lines

Three-Phase Voltage Sources (1/2)

Three-phase generator:





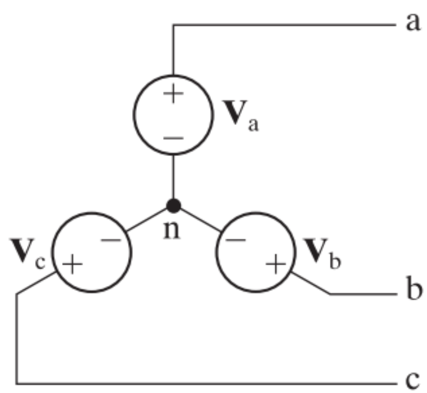
The induced voltage:

$$E = -N\frac{d\phi}{dt}$$

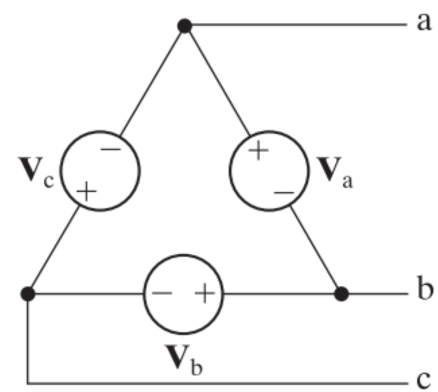
(N: number of wires;
φ: magnetic flux)

Mathematical representation:

Y-connection:



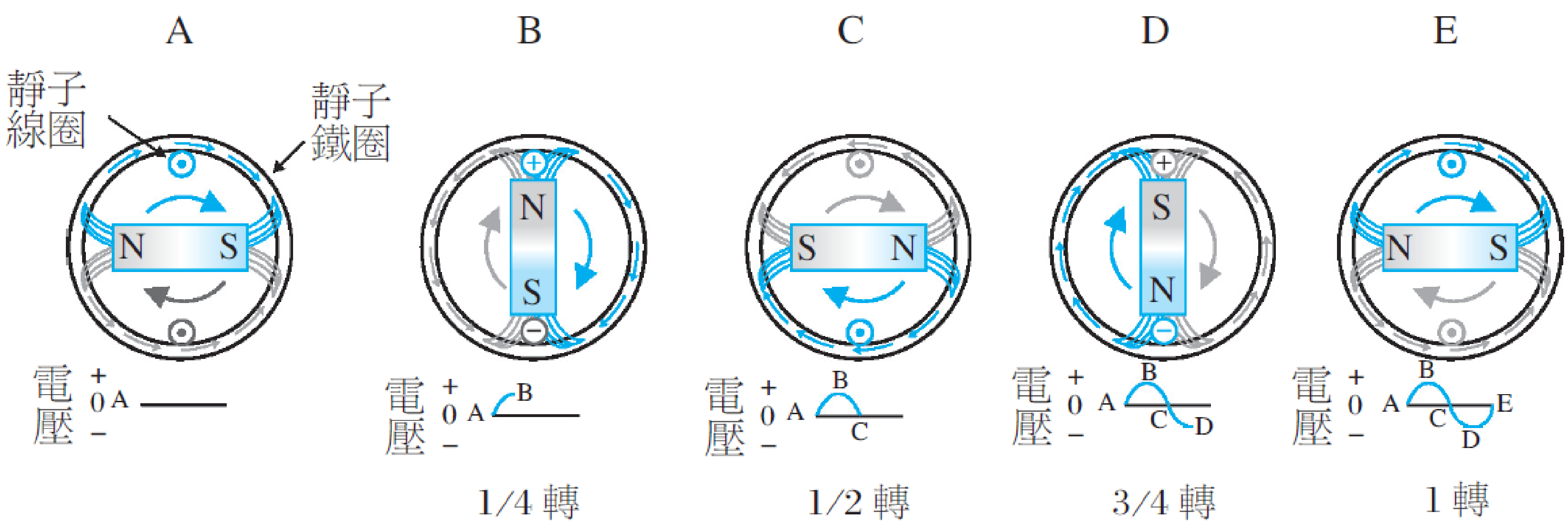
Δ-connection:



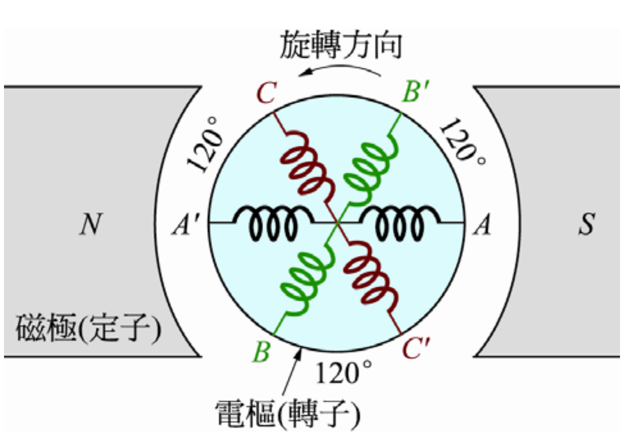
J. Nilsson and S. Riedel, *Electric Circuits*, 9th Edition, Pearson, NJ, 2011

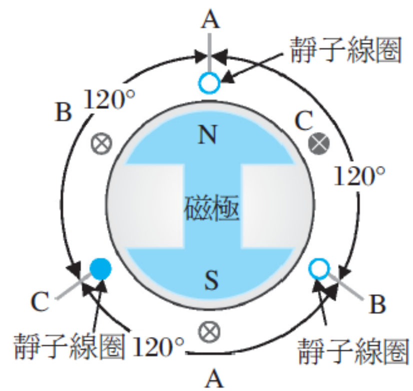
Three-Phase Voltage Sources (2/2)

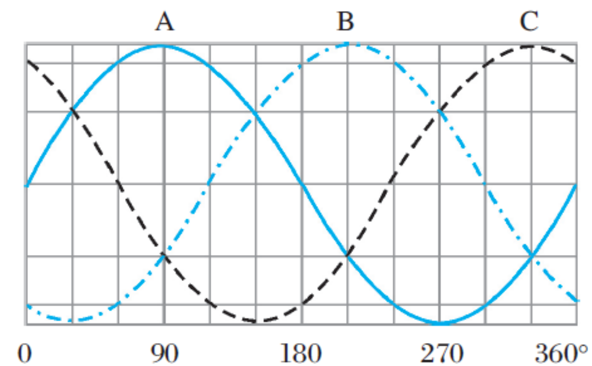
Creating one-phase voltage:



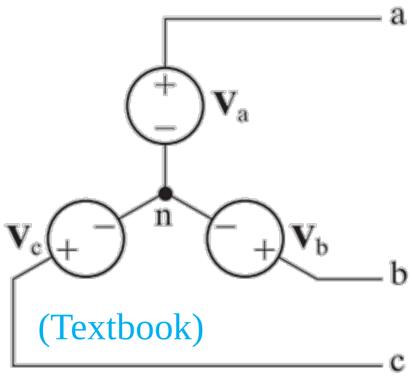
Creating three-phase voltages:



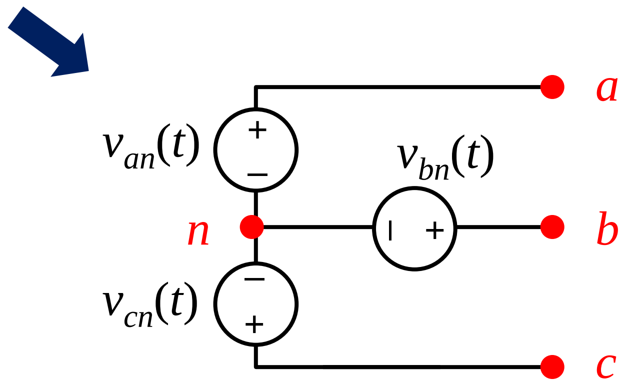




Y-Connection (1/3)



(Textbook)



Positive sequence

- Phase voltage in time domain:

$$v_{an}(t) = V_p \cos\omega t$$

$$v_{bn}(t) = V_p \cos(\omega t - 120^\circ)$$

$$v_{cn}(t) = V_p \cos(\omega t - 240^\circ)$$

$$= V_p \cos(\omega t + 120^\circ)$$

Negative sequence

- Phase voltage in time domain:

$$v_{an}(t) = V_p \cos\omega t$$

$$v_{bn}(t) = V_p \cos(\omega t + 120^\circ)$$

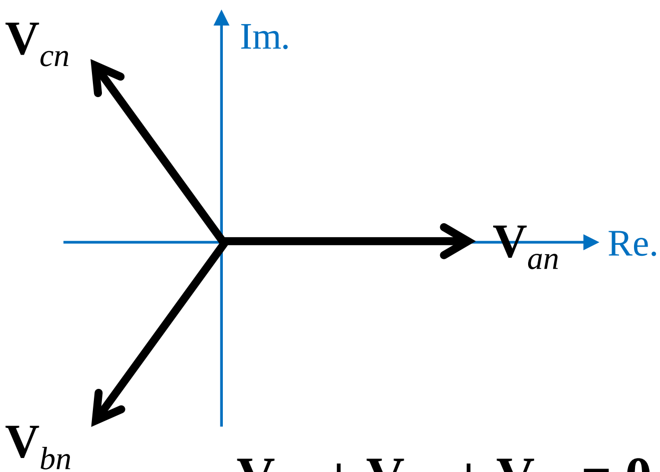
$$v_{cn}(t) = V_p \cos(\omega t - 120^\circ)$$

Y-Connection (2/3)

Positive sequence

- Phase voltage in phasor domain:

$$\mathbf{V}_{an} = V_p \angle 0^\circ$$
$$\mathbf{V}_{bn} = V_p \angle -120^\circ$$
$$\mathbf{V}_{cn} = V_p \angle 120^\circ$$

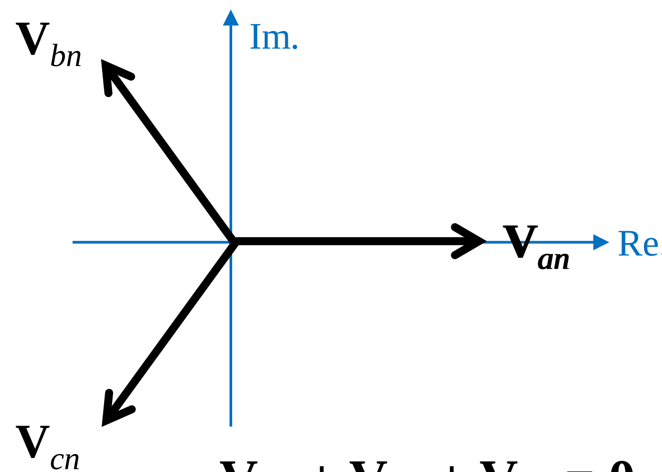


$$\mathbf{V}_{an} + \mathbf{V}_{bn} + \mathbf{V}_{cn} = \mathbf{0}$$

Negative sequence

- Phase voltage in phasor domain:

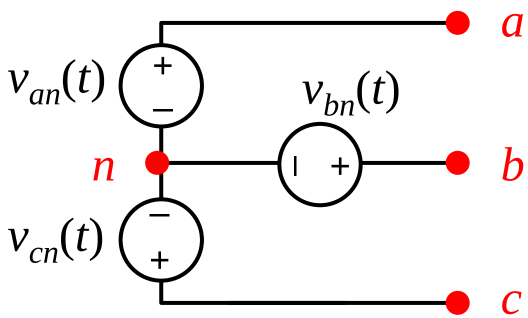
$$\mathbf{V}_{an} = V_p \angle 0^\circ$$
$$\mathbf{V}_{bn} = V_p \angle 120^\circ$$
$$\mathbf{V}_{cn} = V_p \angle -120^\circ$$



$$\mathbf{V}_{an} + \mathbf{V}_{bn} + \mathbf{V}_{cn} = \mathbf{0}$$

Y-Connection (3/3)

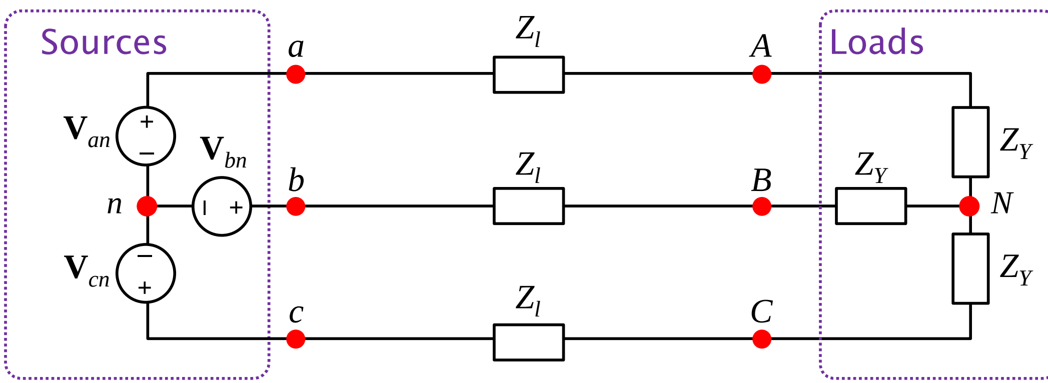
4 important physical quantities:



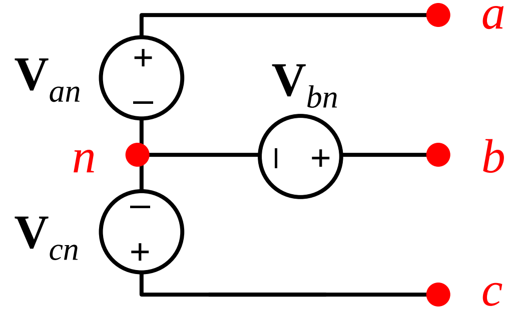
- Phase voltage:
  - Time domain: $v_{an}(t)$, $v_{bn}(t)$, $v_{cn}(t)$
  - Phasor domain: $\mathbf{V}_{an}$, $\mathbf{V}_{bn}$, $\mathbf{V}_{cn}$
- Line voltage (line-to-line voltage):
  - Time domain: $v_{ab}(t)$, $v_{bc}(t)$, $v_{ca}(t)$
  - Phasor domain: $\mathbf{V}_{ab}$, $\mathbf{V}_{bc}$, $\mathbf{V}_{ca}$
- The relation between phase voltage and line voltage:

$$v_{ab}(t) = v_{an}(t) - v_{bn}(t)$$
$$v_{bc}(t) = v_{bn}(t) - v_{cn}(t)$$
$$v_{ca}(t) = v_{cn}(t) - v_{an}(t)$$

- Phase current = line current



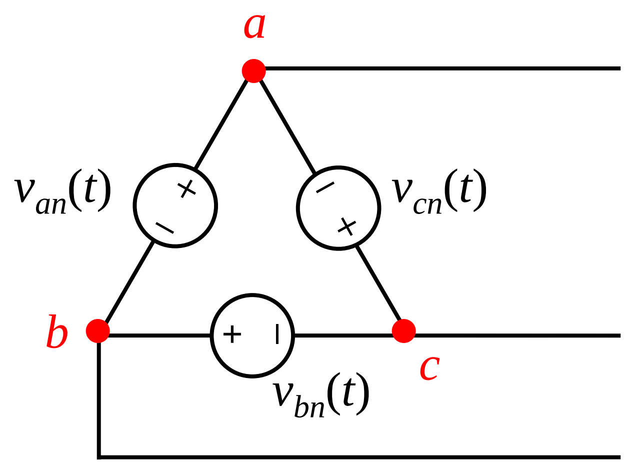
EX 4.22 Line Current vs. Phase Current



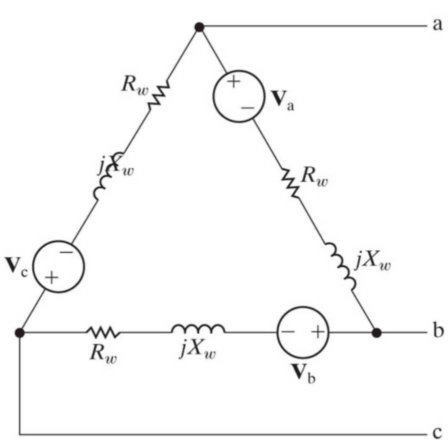
- The balanced three-phase system is of positive sequence
- Now, $\mathbf{V}_{an} = 100\angle 0^\circ$

1. Find $\mathbf{V}_{ab}$, $\mathbf{V}_{bc}$, and $\mathbf{V}_{ca}$

Δ-Connection



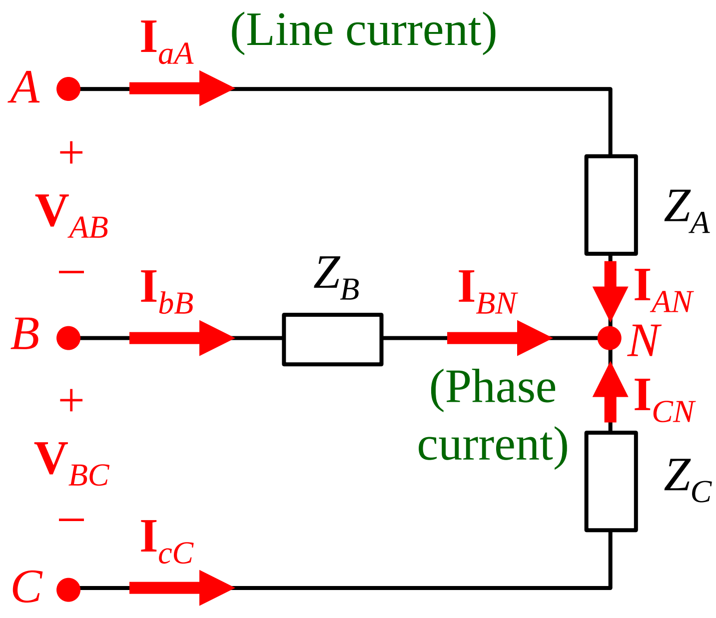
- There is a circulation current for the Δ-Connection systems
- The circulating current cause power loss



- Moreover, the usage life of the generator decreases
- <u>We only consider the Y-connection in this course</u>

Practical scenario: $\mathbf{V}_{an} + \mathbf{V}_{bn} + \mathbf{V}_{cn} \neq \mathbf{0}$

Y-Connected Load (1/2)



In “balanced” three phase systems:

- The three load impedances are equal:

$$Z_A = Z_B = Z_C = Z_Y$$

- Line current = phase current

$$\mathbf{I}_{aA} = \mathbf{I}_{AN}$$

$$\mathbf{I}_{bB} = \mathbf{I}_{BN}$$

$$\mathbf{I}_{cC} = \mathbf{I}_{CN}$$

Considering positive sequence:

$$V_P = |\mathbf{V}_{AN}| = |\mathbf{V}_{BN}| = |\mathbf{V}_{CN}|$$

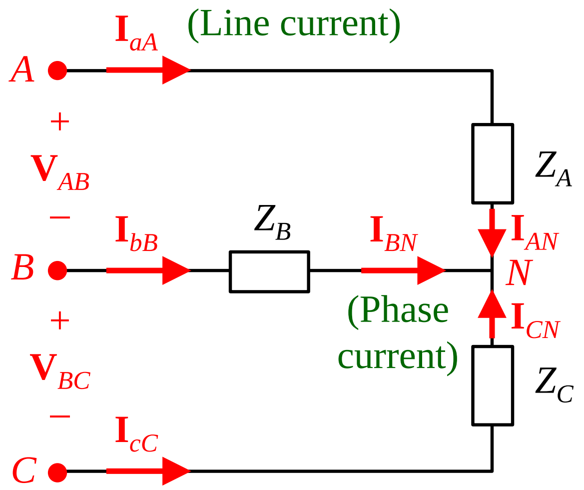
- Line voltage (line-to-line voltage):

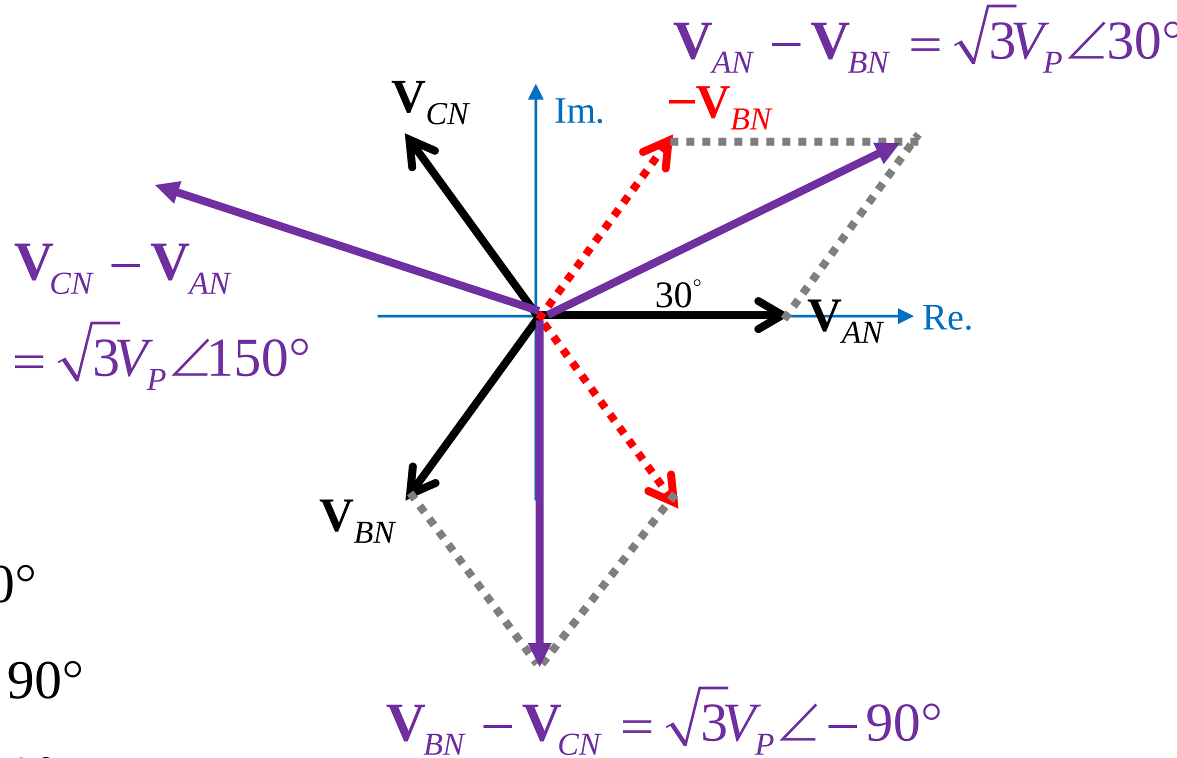
$$\mathbf{V}_{AB} = \mathbf{V}_{AN} - \mathbf{V}_{BN} = \sqrt{3}V_P\angle 30°,\ \mathbf{V}_{BC} = \mathbf{V}_{BN} - \mathbf{V}_{CN} = \sqrt{3}V_P\angle -90°,$$

$$\mathbf{V}_{CA} = \mathbf{V}_{CN} - \mathbf{V}_{AN} = \sqrt{3}V_P\angle 150°$$

Y-Connected Load (2/2)

Assuming positive sequence:





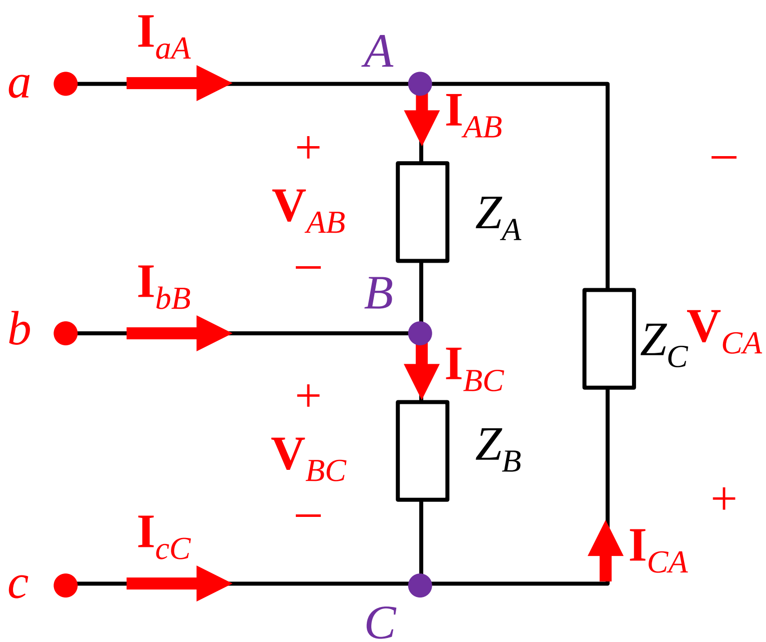
$$\mathbf{V}_{AB} = \mathbf{V}_{AN} - \mathbf{V}_{BN} = \sqrt{3}V_P\angle 30°$$

$$\mathbf{V}_{BC} = \mathbf{V}_{BN} - \mathbf{V}_{CN} = \sqrt{3}V_P\angle -90°$$

$$\mathbf{V}_{CA} = \mathbf{V}_{CN} - \mathbf{V}_{AN} = \sqrt{3}V_P\angle 150°$$

- If the phase voltage is obtained, so is the line voltage
- If the solution of the phase a is obtained, so are that of the phase b and phase c

Δ-Connected Load (1/2)



In "balanced" three phase systems:

- The three load impedances are equal:

$$Z_A = Z_B = Z_C = Z_\Delta$$

- Line voltage = phase voltage

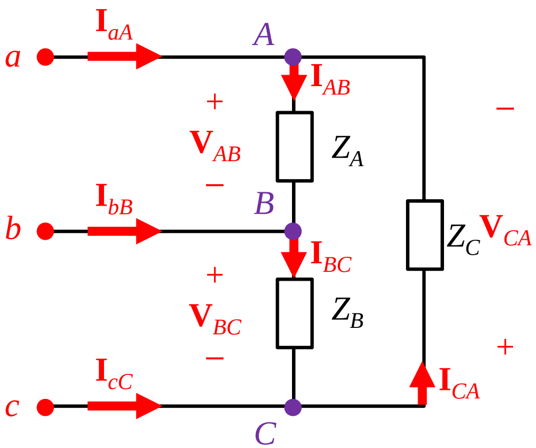
$$\mathbf{V}_{ab} = \mathbf{V}_{AB}$$
$$\mathbf{V}_{bc} = \mathbf{V}_{BC}$$
$$\mathbf{V}_{ca} = \mathbf{V}_{CA}$$

Considering positive sequence:

- For **given** phase voltages $\mathbf{V}_{AB} = V_P\angle 0^\circ$, $\mathbf{V}_{BC} = V_P\angle -120^\circ$, $\mathbf{V}_{CA} = V_P\angle 120^\circ$:

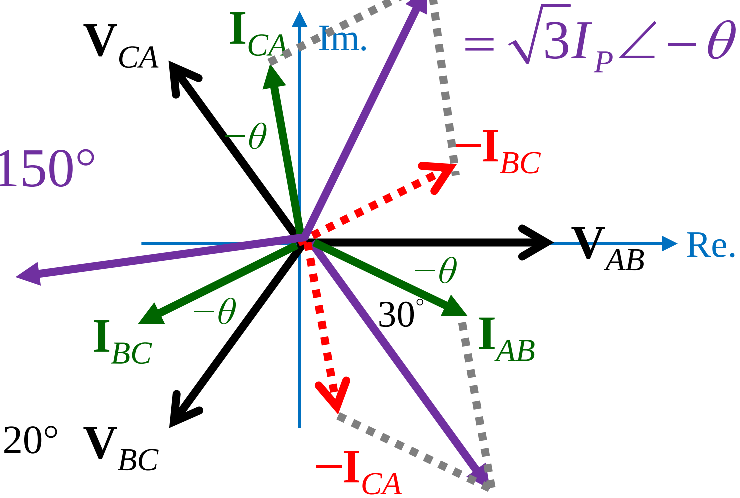
$$\mathbf{I}_{AB} = \frac{\mathbf{V}_{AB}}{Z_\Delta} = I_P\angle -\theta,\ \mathbf{I}_{BC} = \frac{\mathbf{V}_{BC}}{Z_\Delta} = I_P\angle -\theta - 120^\circ,\ \mathbf{I}_{CA} = \frac{\mathbf{V}_{CA}}{Z_\Delta} = I_P\angle -\theta + 120^\circ$$

Δ-Connected Load (2/2)



$$\mathbf{I}_{bB} = \mathbf{I}_{BC} - \mathbf{I}_{AB} = \sqrt{3} I_P \angle -\theta - 150^\circ$$

$$\mathbf{I}_{cC} = \mathbf{I}_{CA} - \mathbf{I}_{BC} = \sqrt{3} I_P \angle -\theta + 90^\circ$$



$$\mathbf{I}_{AB} = \frac{\mathbf{V}_{AB}}{Z_\Delta} = I_P \angle -\theta$$

$$\mathbf{I}_{BC} = \frac{\mathbf{V}_{BC}}{Z_\Delta} = I_P \angle -\theta - 120^\circ$$

$$\mathbf{I}_{CA} = \frac{\mathbf{V}_{CA}}{Z_\Delta} = I_P \angle -\theta + 120^\circ$$

$$\mathbf{I}_{aA} = \mathbf{I}_{AB} - \mathbf{I}_{CA} = \sqrt{3} I_P \angle -\theta - 30^\circ$$

$$I_P = |\mathbf{I}_{AN}| = |\mathbf{I}_{BN}| = |\mathbf{I}_{CN}|$$

- Line current:

$$\mathbf{I}_{aA} = \mathbf{I}_{AB} - \mathbf{I}_{CA} = \sqrt{3} I_P \angle -\theta - 30^\circ$$

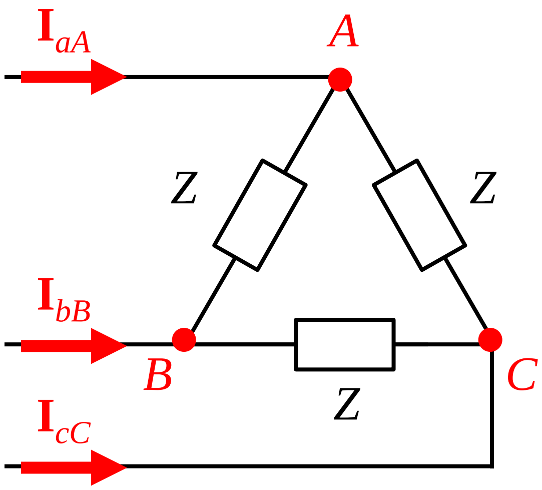
$$\mathbf{I}_{bB} = \mathbf{I}_{BC} - \mathbf{I}_{AB} = \sqrt{3} I_P \angle -\theta - 150^\circ$$

$$\mathbf{I}_{cC} = \mathbf{I}_{CA} - \mathbf{I}_{BC} = \sqrt{3} I_P \angle -\theta + 90^\circ$$

Summary

	Positive sequence (P.S. or a b c)	Negative sequence (N.S. or a c b)
Y-connected	$\angle\mathbf{V}_{PA} \to \angle\mathbf{V}_{PB} \to \angle\mathbf{V}_{PC} \to 120°$ V: $\begin{cases} \dfrac{\lvert\mathbf{V}_{aA}\rvert}{\lvert\mathbf{V}_{AN}\rvert} = \sqrt{3}, \\ \angle\mathbf{V}_{aA} - \angle\mathbf{V}_{AN} = 30° \end{cases}$ I: $\{\ \mathbf{I}_{aA} = \mathbf{I}_{AN}$	$\angle\mathbf{V}_{PA} \to \angle\mathbf{V}_{PB} \to \angle\mathbf{V}_{PC} \to -120°$ V: $\begin{cases} \dfrac{\lvert\mathbf{V}_{aA}\rvert}{\lvert\mathbf{V}_{AN}\rvert} = \sqrt{3}, \\ \angle\mathbf{V}_{aA} - \angle\mathbf{V}_{AN} = -30° \end{cases}$ I: $\{\ \mathbf{I}_{aA} = \mathbf{I}_{AN}$
Δ-connected	$\angle\mathbf{V}_{PA} \to \angle\mathbf{V}_{PB} \to \angle\mathbf{V}_{PC} \to 120°$ V: $\{\ \mathbf{V}_{aA} = \mathbf{V}_{AN}$ I: $\begin{cases} \dfrac{\lvert\mathbf{I}_{aA}\rvert}{\lvert\mathbf{I}_{AN}\rvert} = \sqrt{3}, \\ \angle\mathbf{I}_{aA} - \angle\mathbf{I}_{AN} = -30° \end{cases}$	$\angle\mathbf{V}_{PA} \to \angle\mathbf{V}_{PB} \to \angle\mathbf{V}_{PC} \to -120°$ V: $\{\ \mathbf{V}_{aA} = \mathbf{V}_{AN}$ I: $\begin{cases} \dfrac{\lvert\mathbf{I}_{aA}\rvert}{\lvert\mathbf{I}_{AN}\rvert} = \sqrt{3}, \\ \angle\mathbf{I}_{aA} - \angle\mathbf{I}_{AN} = 30° \end{cases}$

EX 4.23 Line Current vs. Phase Current



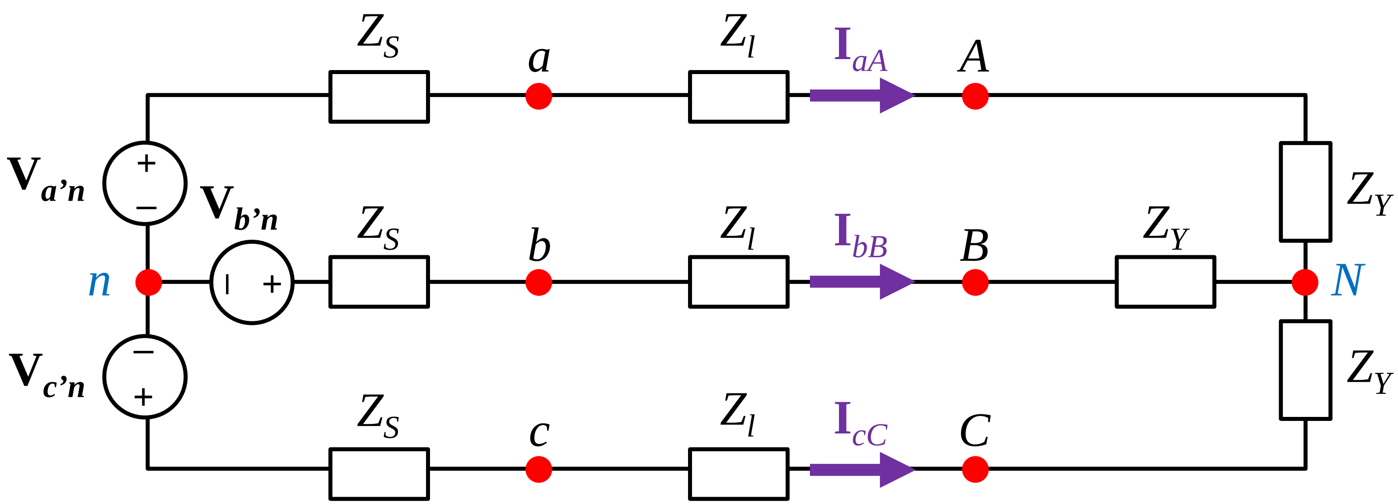
- The balanced three-phase system is of positive sequence
- Now, $\mathbf{V}_{AB} = 200\angle 30^\circ$ and $Z = 10\angle 10^\circ$

  1. Find $\mathbf{I}_{aA}$, $\mathbf{I}_{bB}$, and $\mathbf{I}_{cC}$

Contents

4.6 Analysis of the Y-Y and Y-△ Circuits

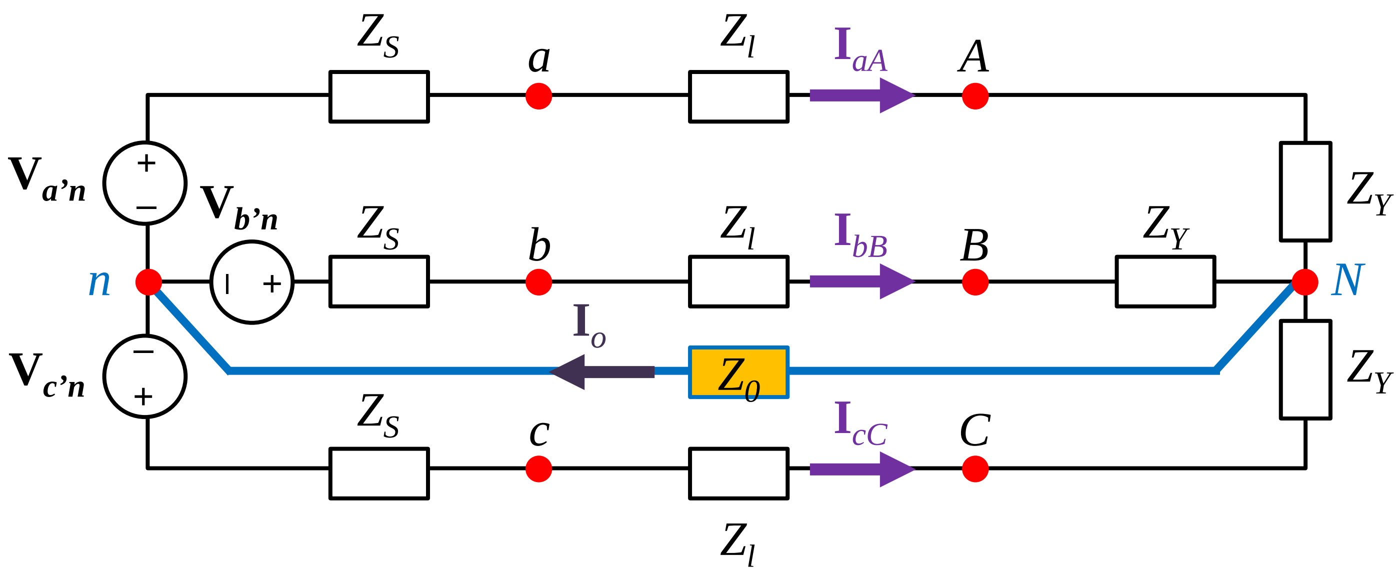
Y-Y Circuits



A three-phase Y-Y balanced system:

- Balanced source conditions
- Balanced distribution lines (Z_l)
- Balanced load conditions (Z_Y)

If We Purposely Add a Neutral Wire (1/2)



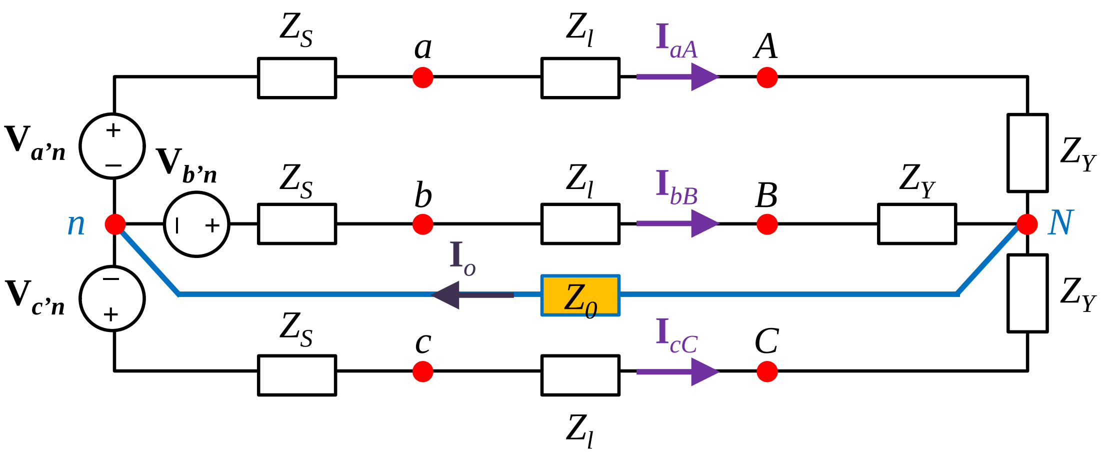
Nn line with impedance Z_0: Let the voltage across the wire is $\mathbf{V}_{Nn}$

$$\mathbf{I}_{aA} = \frac{\mathbf{V}_{a'n} - \mathbf{V}_{Nn}}{Z_S + Z_l + Z_Y}$$

$$\mathbf{I}_{bB} = \frac{\mathbf{V}_{b'n} - \mathbf{V}_{Nn}}{Z_S + Z_l + Z_Y} \qquad \mathbf{I}_o = \frac{\mathbf{V}_{Nn}}{Z_o}$$

$$\mathbf{I}_{cC} = \frac{\mathbf{V}_{c'n} - \mathbf{V}_{Nn}}{Z_S + Z_l + Z_Y}$$

If We Purposely Add a Neutral Wire (2/2)



$$\mathbf{I}_{aA} = \frac{\mathbf{V}_{a'n} - \mathbf{V}_{Nn}}{Z_S + Z_l + Z_Y}$$

$$\mathbf{I}_{bB} = \frac{\mathbf{V}_{b'n} - \mathbf{V}_{Nn}}{Z_S + Z_l + Z_Y} \qquad \mathbf{I}_o = \frac{\mathbf{V}_{Nn}}{Z_o}$$

$$\mathbf{I}_{cC} = \frac{\mathbf{V}_{c'n} - \mathbf{V}_{Nn}}{Z_S + Z_l + Z_Y}$$

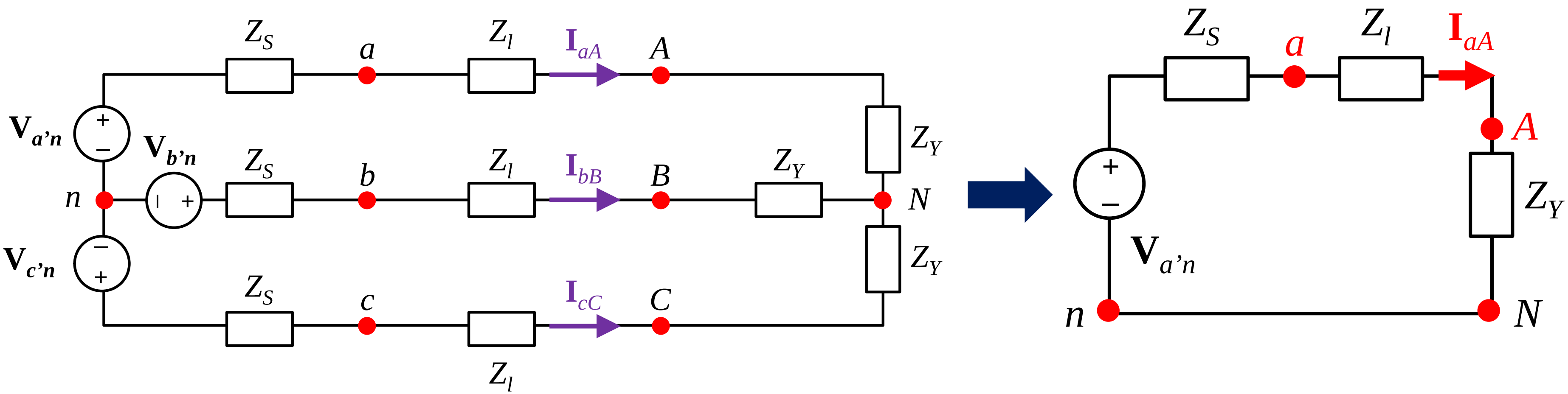
- From KCL at node *n*: $\mathbf{I}_{aA} + \mathbf{I}_{bB} + \mathbf{I}_{cC} - \mathbf{I}_o = 0$

$$\Rightarrow \frac{\overbrace{\mathbf{V}_{a'n} + \mathbf{V}_{b'n} + \mathbf{V}_{c'n}}^{=0}}{Z_S + Z_l + Z_Y} - \mathbf{V}_{Nn} \underbrace{\left(\frac{3}{Z_S + Z_l + Z_Y} + \frac{1}{Z_o} \right)}_{\neq 0} = 0$$

$$\Rightarrow \mathbf{V}_{Nn} = 0$$

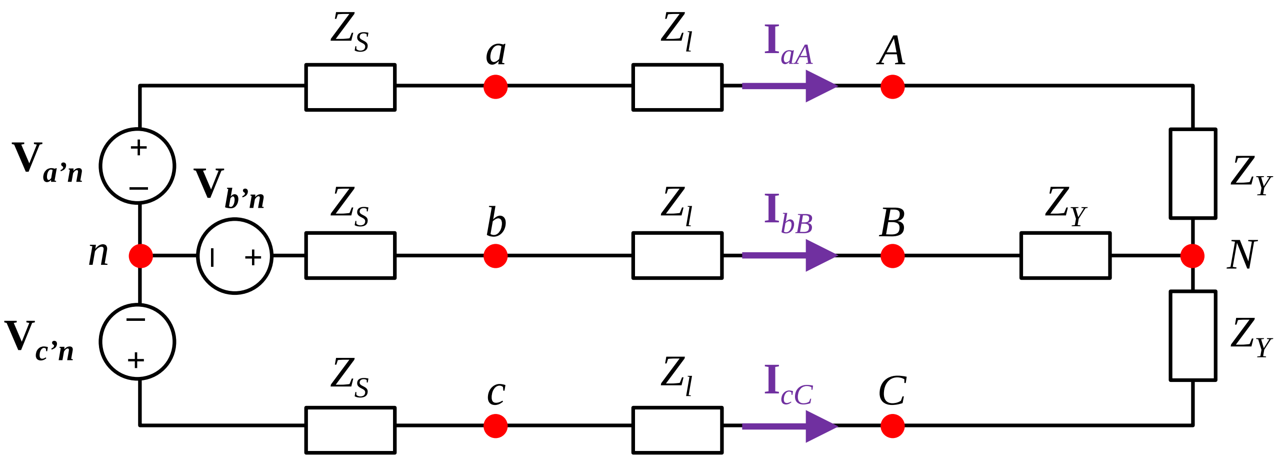
- The neutral wire can be replaced by a shorted circuit, despite the fact that Z_0 is arbitrarily chosen!

A Short Cut for Balanced 3φ Systems



- The three-phase circuit can be decomposed into 3 single-phase equivalent circuits
- We solve a-phase circuit, calculating its line current ($\mathbf{I}_{aA}$) and phase voltage ($\mathbf{V}_{AN}$)
- *If the phase voltage is obtained, so is the line voltage*
- *If the solution of the phase a is obtained, so are that of the phase b and phase c*

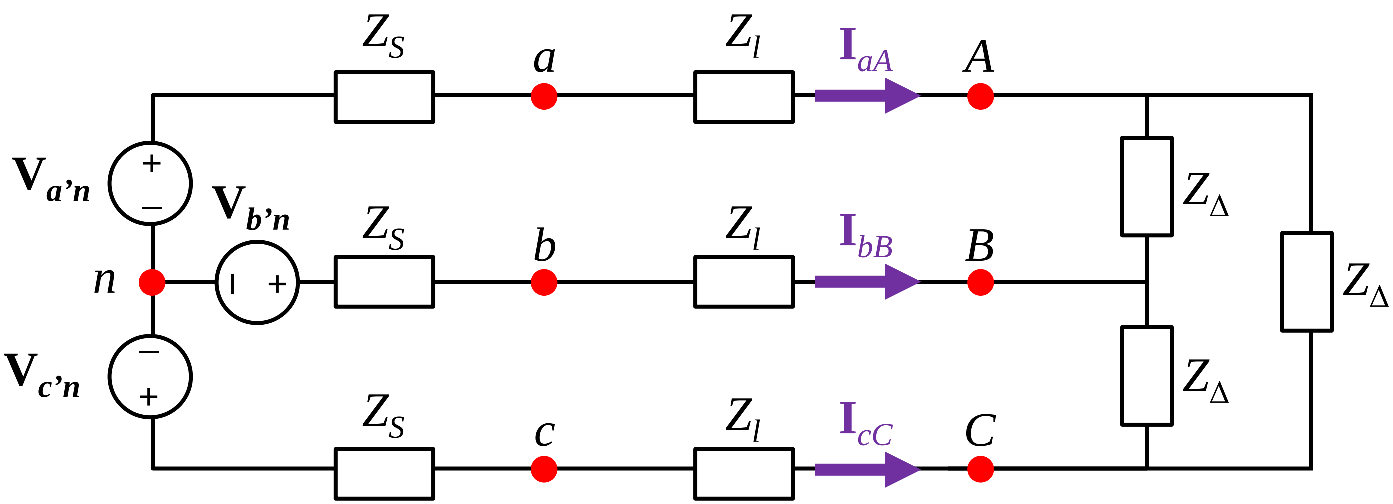
EX 4.24 Analyzing a Y-Y Circuit



- We already know that the three-phase system is of positive sequence, $\mathbf{V}_{a'n} = 120\angle 0^\circ$, $Z_S = 0.2 + j0.5\ \Omega$, $Z_Y = 39 + j28\ \Omega$, and $Z_l = 0.8 + j1.5\ \Omega$
- The a-phase internal voltage of the generator is chosen as the reference phasor

1. Construct the a-phase equivalent circuit of the system
2. Calculate the line currents $\mathbf{I}_{aA}$, $\mathbf{I}_{bB}$, and $\mathbf{I}_{cC}$
3. Calculate the phase voltages at the load $\mathbf{V}_{AN}$, $\mathbf{V}_{BN}$, and $\mathbf{V}_{CN}$
4. Calculate the line voltages at the load $\mathbf{V}_{AB}$, $\mathbf{V}_{BC}$, and $\mathbf{V}_{CA}$
5. Calculate the phase voltages at the source $\mathbf{V}_{an}$, $\mathbf{V}_{bn}$, and $\mathbf{V}_{cn}$
6. Calculate the line voltages at the source $\mathbf{V}_{ab}$, $\mathbf{V}_{bc}$, and $\mathbf{V}_{ca}$

Y-Δ Circuits

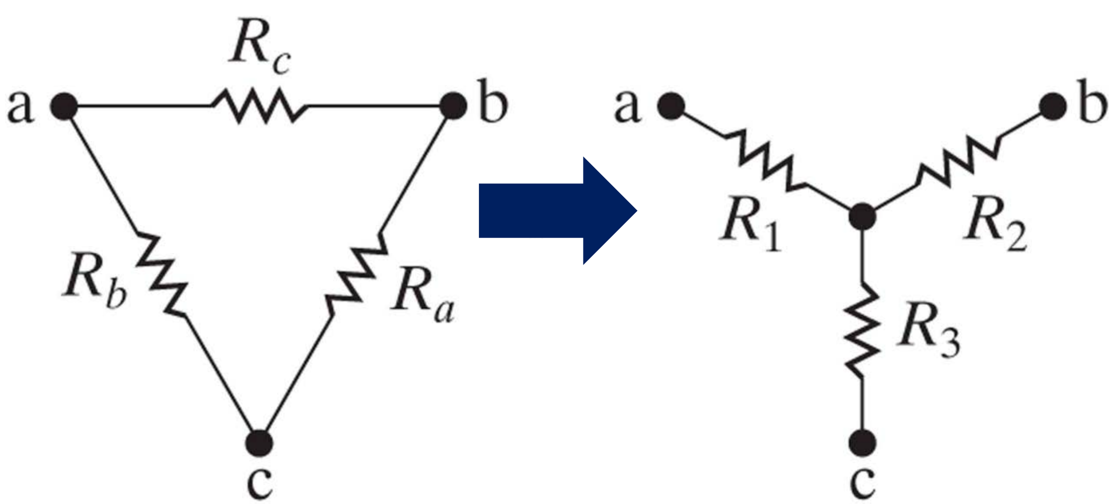


A three-phase Y-Δ balanced system:

- Now the loads are Δ-Connected, while the sources are Y-connected
- We can't add an N-n neutral line, so the three-phase system can't be decomposed into 3 single one-phase circuits

Δ-Y Transformation

Δ-Y transformation



$$R_1 = \frac{R_b R_c}{R_a + R_b + R_c}$$

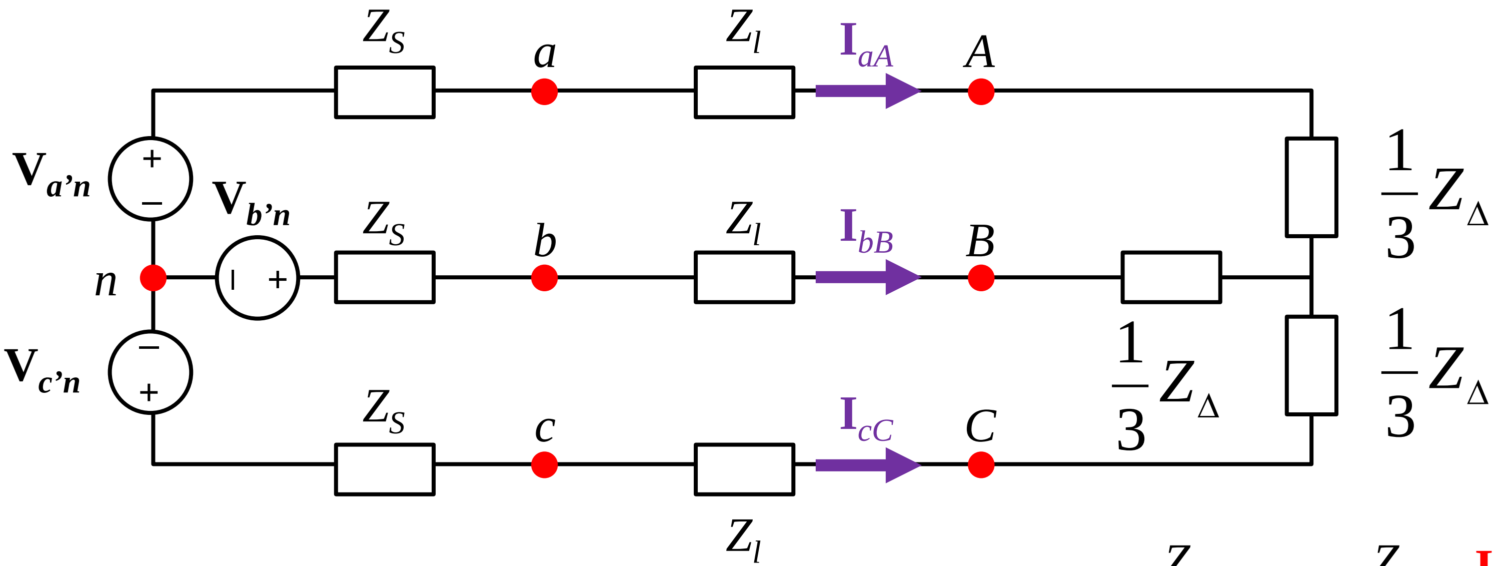
$$R_2 = \frac{R_c R_a}{R_a + R_b + R_c}$$

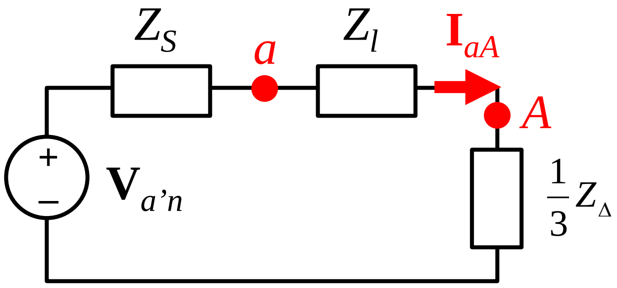
$$R_3 = \frac{R_a R_b}{R_a + R_b + R_c}$$

- Special case: If $R_a = R_b = R_c = R$

$$\Rightarrow R_1 = R_2 = R_3 = R/3$$

Transforming Y-Δ to Y-Y

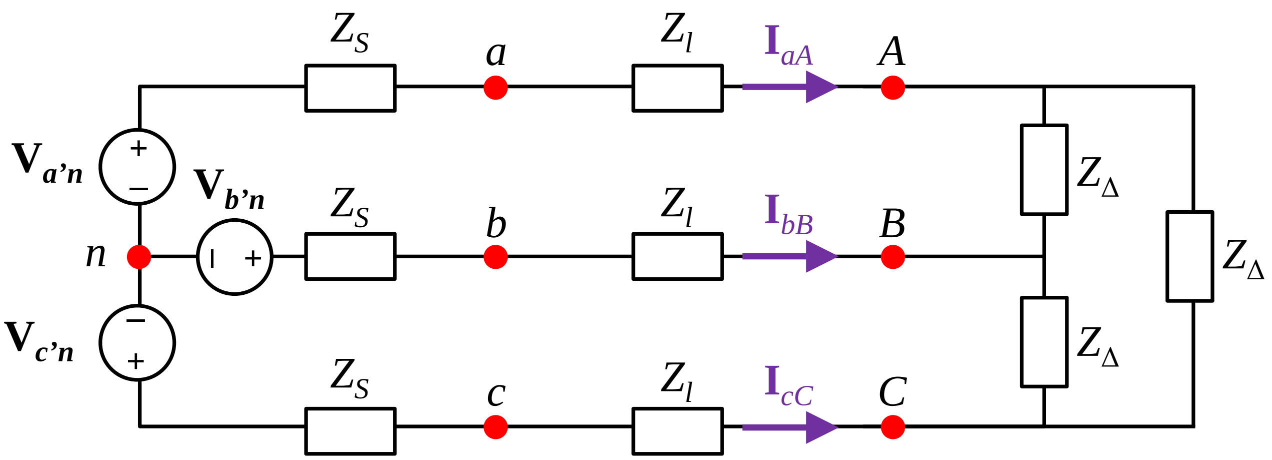




Solution step:

1. Redraw a single-phase circuit for phase a
2. Find the line current $\mathbf{I}_{aA}$
3. For other phases: line quantities can be obtained directly form the information of phase sequence

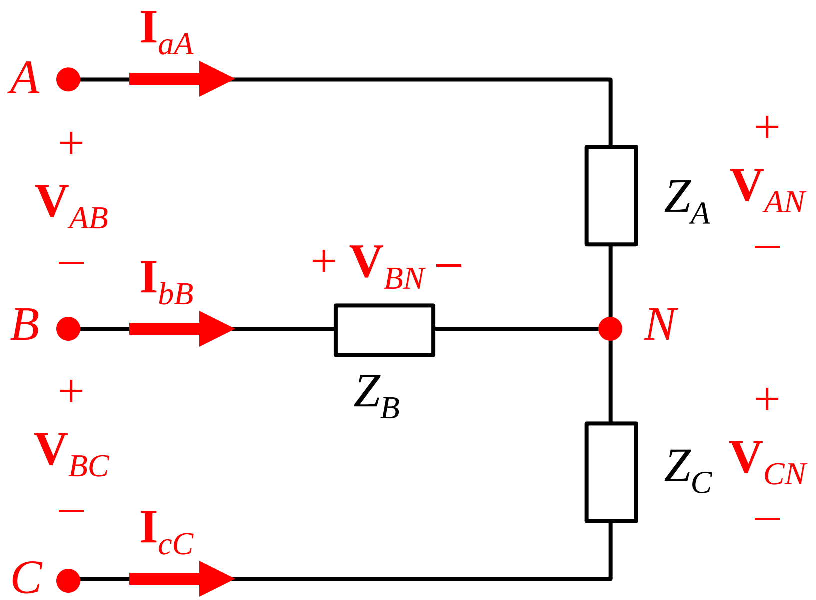
EX 4.25 Analyzing a Y-Δ Circuit



- We already know that the three-phase system is of positive sequence, $\mathbf{V}_{a'n} = 120\angle 0^\circ$, $Z_S = 0.2 + j0.5\ \Omega$, $Z_\Delta = 118.5 + j85.8\ \Omega$, and $Z_l = 0.3 + j0.9\ \Omega$
- The a-phase internal voltage of the generator is chosen as the reference phasor

1. Construct a single-phase equivalent circuit of the system
2. Calculate the line currents $\mathbf{I}_{aA}$, $\mathbf{I}_{bB}$, and $\mathbf{I}_{cC}$
3. Calculate the phase currents of the load
4. Calculate the phase voltages at the load terminals
5. Calculate the line voltages at the source terminals

A Balanced Y Load: Instantaneous Power (1/2)



1. Instantaneous power:

- Assuming the system has positive sequence
- Let the loads are $Z_A = Z_B = Z_C = |Z|\angle\theta$
- Let the phase voltages be:

$$v_{AN} = \sqrt{2}V_P \cos\omega t$$

$$v_{BN} = \sqrt{2}V_P \cos\left(\omega t - 120^\circ\right)$$

$$v_{CN} = \sqrt{2}V_P \cos\left(\omega t + 120^\circ\right)$$

RMS phasor

$$i_{AN} = \sqrt{2}I_P \cos\left(\omega t - \theta\right)$$

$$i_{BN} = \sqrt{2}I_P \cos\left(\omega t - \theta - 120^\circ\right)$$

$$i_{CN} = \sqrt{2}I_P \cos\left(\omega t - \theta + 120^\circ\right)$$

$$where\ I_P = \frac{V_P}{|Z|}$$

A Balanced Y Load: Instantaneous Power (2/2)

- The formula of instantaneous power:

$$
\begin{aligned}
p(t) &= p_A + p_B + p_C \\
&= v_{AN} i_{AN} + v_{BN} i_{BN} + v_{CN} i_{CN} \\
&= 2V_P I_P \cos\omega t \cos(\omega t - \theta) \\
&\quad + 2V_P I_P \cos(\omega t - 120°) \cos(\omega t - \theta - 120°) \\
&\quad + 2V_P I_P \cos(\omega t + 120°) \cos(\omega t - \theta + 120°)
\end{aligned}
$$

- By using $\cos A \cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$

$\Rightarrow$ $p(t) = 3V_P I_P \cos\theta$

- In a balanced three-phase circuit, the power is invariant with time

A Balanced Y Load: Complex Power

2. Complex power: $\mathbf{S} = P + jQ = \mathbf{V}_{rms}\mathbf{I}_{rms}^{*}$

- Complex power for each phase:

$$\mathbf{S}_A = P_A + jQ_A = \left(V_P\angle 0^\circ\right)\left(I_P\angle -\theta\right)^* = V_P I_P \angle\theta$$

$$\mathbf{S}_B = P_B + jQ_B = \left(V_P\angle -120^\circ\right)\left(I_P\angle -\theta - 120^\circ\right)^* = V_P I_P \angle\theta$$

$$\mathbf{S}_C = P_C + jQ_C = \left(V_P\angle 120^\circ\right)\left(I_P\angle -\theta + 120^\circ\right)^* = V_P I_P \angle\theta$$

- The complex power for each phase is the same as those of each other
- Total complex power for the three phases:

$$\mathbf{S}_{3\phi} = \mathbf{S}_A + \mathbf{S}_B + \mathbf{S}_C = 3V_P I_P \angle\theta$$

- Total complex power for the three phases in terms of line voltages and line currents:

$$\mathbf{S}_{3\phi} = \sqrt{3}V_L I_L \angle\theta$$

A Balanced Y Load: Average Power

3. Average power:

$$\mathbf{S}_{3\phi} = \mathbf{S}_A + \mathbf{S}_B + \mathbf{S}_C = 3V_P I_P \angle\theta$$

- The average power of the three phases is the real part of the complex power:

$$P_{3\phi} = 3V_P I_P \cos\theta$$

- In terms of the line voltage and line current:
  - Line current I_L = Phase current I_P
  - Line voltage magnitude $V_L = \sqrt{3}V_P$

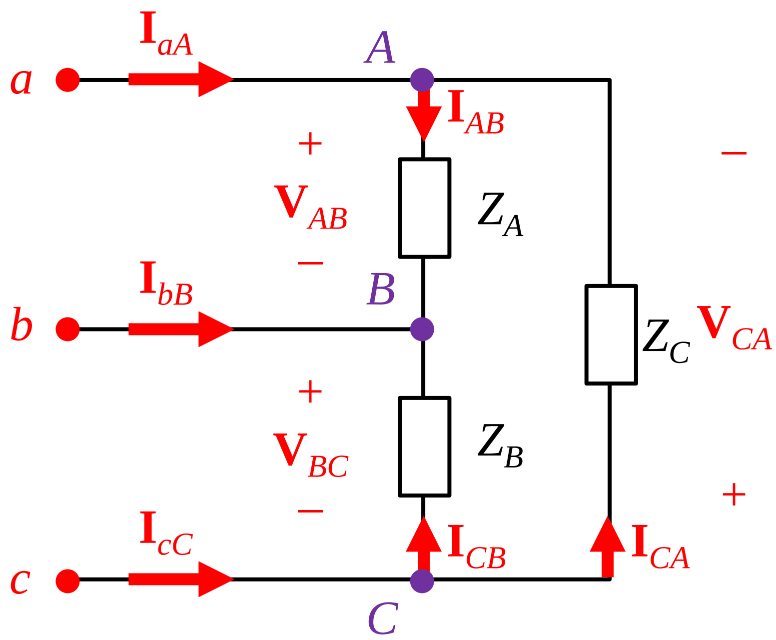
$$\Rightarrow \quad P_{3\phi} = \sqrt{3}V_L I_L \cos\theta$$

RMS phasor

4. Reactive power:

$$Q_{3\phi} = \sqrt{3}V_L I_L \sin\theta$$

A Balanced Δ Load: Instantaneous Power (1/2)



$$i_{AB} = \sqrt{2} I_P \cos\left(\omega t - \theta\right)$$

$$i_{BC} = \sqrt{2} I_P \cos\left(\omega t - \theta - 120^\circ\right)$$

$$i_{CA} = \sqrt{2} I_P \cos\left(\omega t - \theta + 120^\circ\right)$$

$$where\ I_P = \frac{V_P}{|Z|}$$

1. Instantaneous power:

- Assuming the system has positive sequence
- Let the loads are $Z_A = Z_B = Z_C = |Z| \angle \theta$
- Let the phase voltages be:

$$v_{AB} = \sqrt{2} V_P \cos \omega t$$

$$v_{BC} = \sqrt{2} V_P \cos\left(\omega t - 120^\circ\right)$$

$$v_{CA} = \sqrt{2} V_P \cos\left(\omega t + 120^\circ\right)$$

RMS phasor

A Balanced Δ Load: Instantaneous Power (2/2)

- The formula of instantaneous power:

$$
\begin{aligned}
p(t) &= p_A + p_B + p_C \\
&= v_{AB} i_{AB} + v_{BC} i_{BC} + v_{CA} i_{CA} \\
&= 2V_P I_P \cos \omega t \cos(\omega t - \theta) \\
&+ 2V_P I_P \cos(\omega t - 120^\circ)\cos(\omega t - \theta - 120^\circ) \\
&+ 2V_P I_P \cos(\omega t + 120^\circ)\cos(\omega t - \theta + 120^\circ)
\end{aligned}
$$

- By using $\cos A \cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$

$$\Rightarrow \quad p(t) = 3V_P I_P \cos\theta$$

- In a balanced three-phase circuit, **no matter what the connection manner is**, the instantaneous power is invariant with time

A Balanced Δ Load: Complex Power

2. Complex power: $\mathbf{S} = P + jQ = \mathbf{V}_{rms}\mathbf{I}^*_{rms}$

- Complex power for each phase:

$$\mathbf{S}_A = P_A + jQ_A = \left(V_P\angle 0°\right)\left(I_P\angle -\theta\right)^* = V_P I_P \angle\theta$$

$$\mathbf{S}_B = P_B + jQ_B = \left(V_P\angle -120°\right)\left(I_P\angle -\theta - 120°\right)^* = V_P I_P \angle\theta$$

$$\mathbf{S}_C = P_C + jQ_C = \left(V_P\angle 120°\right)\left(I_P\angle -\theta + 120°\right)^* = V_P I_P \angle\theta$$

- The complex power for each phase is the same as those of each other
- Total complex power for the three phases:

$$\mathbf{S}_{3\phi} = \mathbf{S}_A + \mathbf{S}_B + \mathbf{S}_C = 3V_P I_P \angle\theta$$

- Total complex power for the three phases in terms of line voltages and line currents:

$$\mathbf{S}_{3\phi} = \sqrt{3} V_L I_L \angle\theta$$

Again, the expression of complex power for two connection manners are identical

A Balanced Δ Load: Average Power

3. Average power:

$$\mathbf{S}_{3\phi} = \mathbf{S}_A + \mathbf{S}_B + \mathbf{S}_C = 3V_P I_P \angle\theta$$

- The average power of the three phases is the real part of the complex power:

$$P_{3\phi} = 3V_P I_P \cos\theta$$

- In terms of the line voltage and line current:

  - Line current magnitude $I_L = \sqrt{3} I_P$
  - Line voltage V_L = Phase voltage V_P

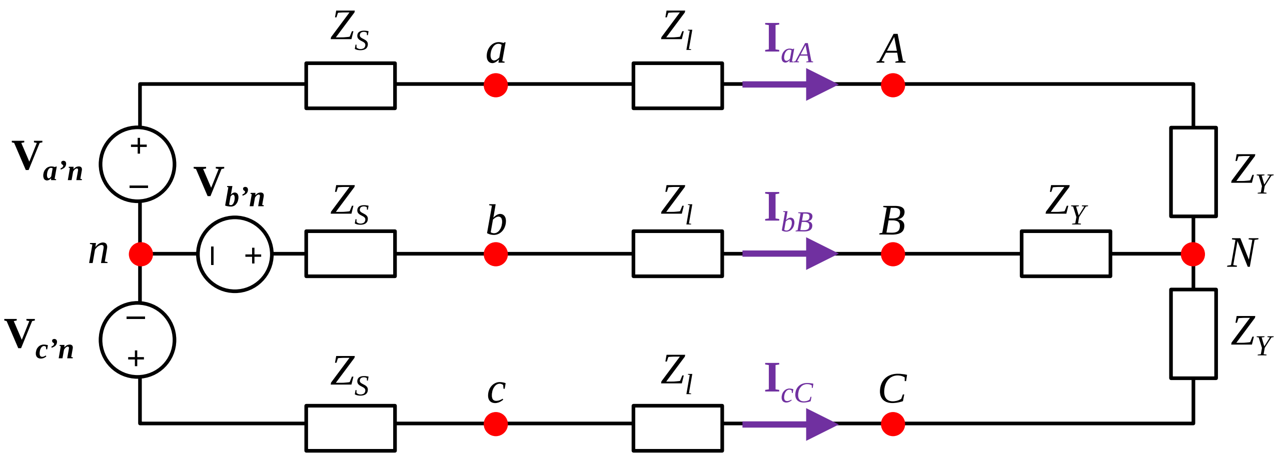
$$\Rightarrow P_{3\phi} = \sqrt{3} V_L I_L \cos\theta$$

RMS phasor

4. Reactive power:

$$Q_{3\phi} = \sqrt{3} V_L I_L \sin\theta$$

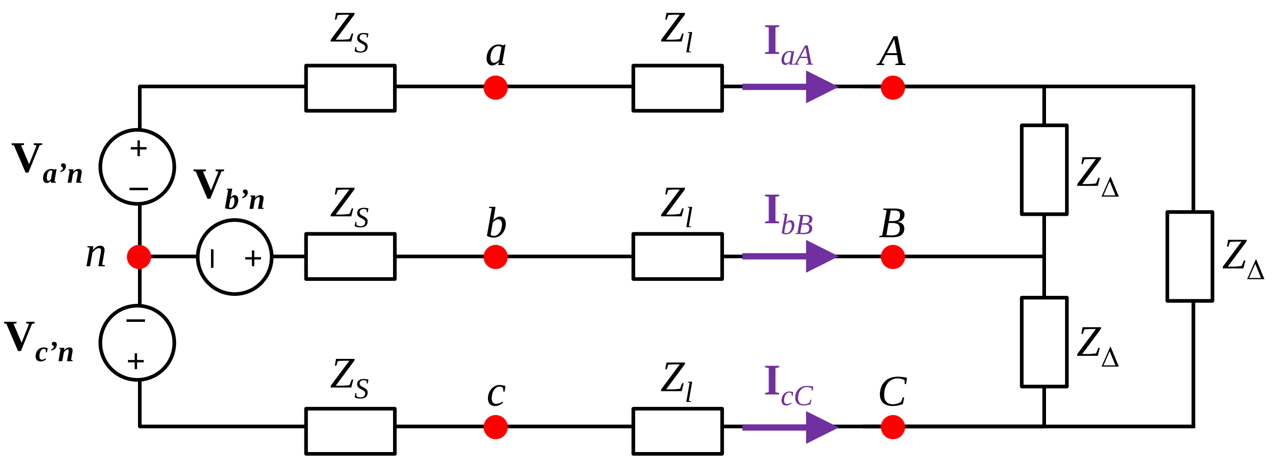
EX 4.26 Calculating Power in the Y-Y Circuit (Ex 4.24)



- We already know that the three-phase system is of positive sequence, $\mathbf{V}_{a'n} = 120\angle 0^\circ$, $Z_S = 0.2 + j0.5\ \Omega$, $Z_Y = 39 + j28\ \Omega$, and $Z_l = 0.8 + j1.5\ \Omega$
- The a-phase internal voltage of the generator is chosen as the reference phasor

  1. Calculate the average power per phase delivered to the Y-connected load
  2. Calculate the total average power delivered to the load
  3. Calculate the total average power lost in the line
  4. Calculate the total average power lost in the generator
  5. Calculate the total number of magnetizing vars absorbed by the load

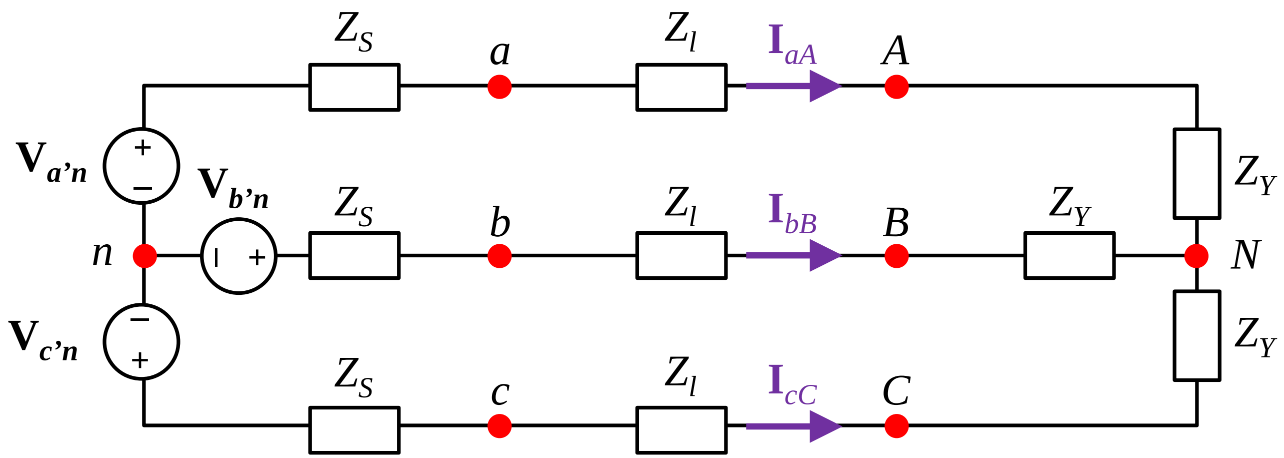
EX 4.27 Calculating Power in the Y-Δ Circuit (Ex 4.25)



- We already know that the three-phase system is of positive sequence, $\mathbf{V}_{a'n} = 120\angle 0^\circ$, $Z_S = 0.2 + j0.5\ \Omega$, $Z_\Delta = 118.5 + j85.8\ \Omega$, and $Z_l = 0.3 + j0.9\ \Omega$
- The a-phase internal voltage of the generator is chosen as the reference phasor

1. Calculate the total complex power delivered to the Δ-connected load
2. What percentage of the average power at the sending end of the line is delivered to the load?

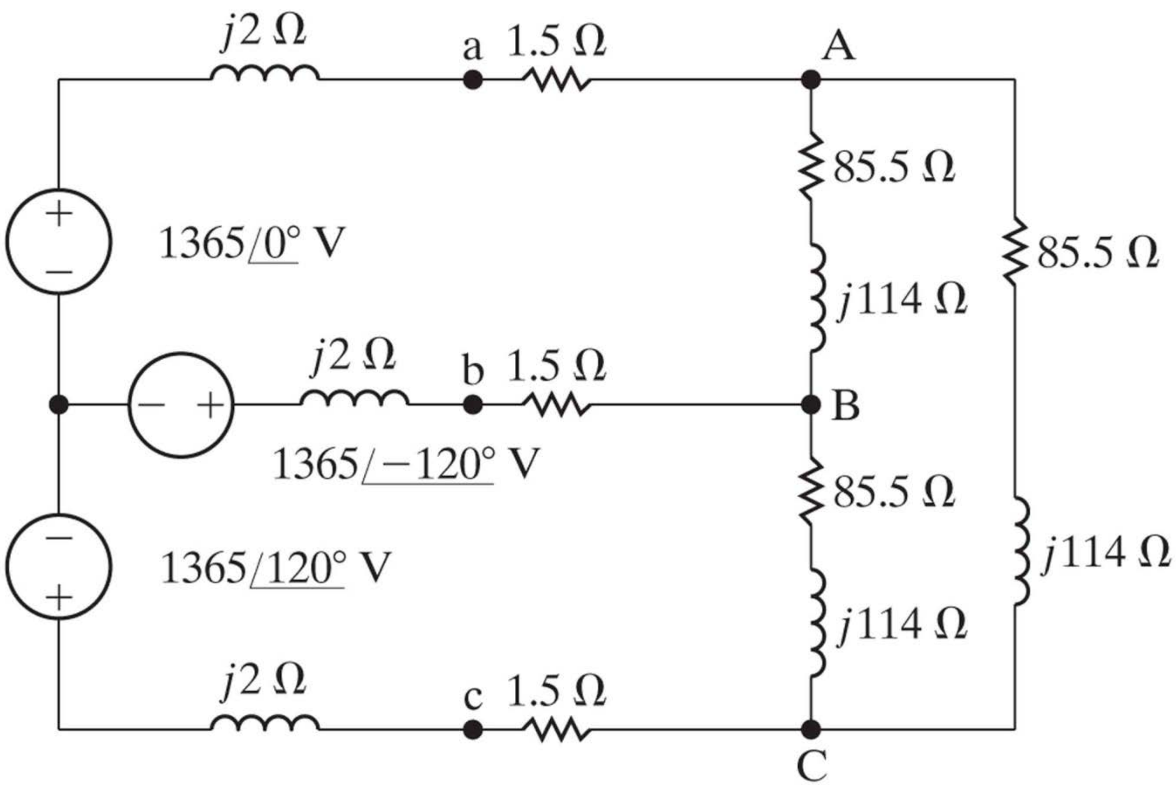
EX 4.28 Calculating Power with an Unspecified Load



- A balanced three-phase load Z_Y requires 480 kW at a lagging power factor of 0.8
- The load is fed from a line having an impedance of $Z_l = 0.005 + j0.025\ \Omega$
- The line voltage at the terminals of the load is $600\angle 30^\circ$ V (RMS phasor)

1. Construct a single-phase equivalent circuit of the system
2. Calculate the magnitude of the line current
3. Calculate the magnitude of the line voltage at the sending end of the line
4. Calculate the complex power at the sending end of the line

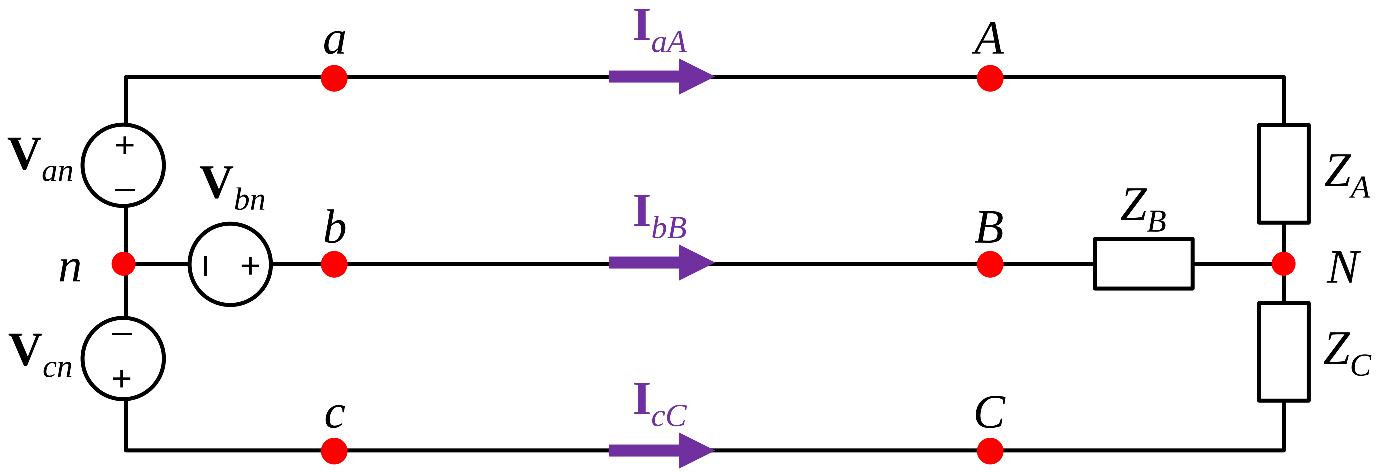
EX 4.29 Calculation of Line Current and Power



1. Find the RMS magnitude and the phase angle of $\mathbf{I}_{CA}$
2. What percentage of the average power delivered by the three-phase source is dissipated in the three-phase load

J. Nilsson and S. Riedel, *Electric Circuits*, 9th Edition, Pearson, NJ, 2011

EX 4.30 An Unbalanced Y-Y Circuit



- We already know that the three-phase system is of positive sequence with $\mathbf{V}_{an} = 110\angle 0^\circ$ (rms)
- The load impedances are $Z_A = 50 + j80\ \Omega$, $Z_B = j50\ \Omega$, and $Z_C = 100 + j25\ \Omega$

1. Calculate the line currents $\mathbf{I}_{aA}$, $\mathbf{I}_{bB}$, and $\mathbf{I}_{cC}$
2. Determine the complex power of the load for each phase
3. Calculate the total complex power delivered to the three-phase load