Lec 12. Warmup.

Are these vector spaces? $c \in \mathbb{R}$.

1) $V = \mathbb{R}^+ := \{x \mid x > 0\}$

$x + y := xy, \quad cx := cx$

X not closed under mult.

2) $V = \mathbb{R}^{m \times n} \equiv M_{m \times n} \equiv \{A \mid A \text{ is a } m \times n \text{ matrix}\}$

$A + B, \quad cA.$

✓

3) $V = \mathbb{R}$

$x \oplus y := \max(x, y)$ $\qquad$ $c \cdot x := cx$

✗ Let us say $a \in \mathbb{R}$ is "$\vec{0}$" $\qquad$ $x = a-1$

$x \oplus a := \max(a-1, a) = a \neq x \Rightarrow$ a is not $\vec{0} \Rightarrow \vec{0}$ doesn't exist

4) $V = \left\{ f(x) = \sum_{n=1}^{N} a_n \cos(nx) \;\middle|\; a_n \in \mathbb{R}, \; x \in \mathbb{R} \right\}.$

$(f+g)(x) := f(x) + g(x), \quad (cf)(x) := c\,f(x)$

✓

5) $V = \mathbb{Z} = \{\text{integers}\}$.

$x + y := x + y$, $cx := cx$

X . $\frac{1}{2} \cdot 1 \notin V$.

6) $V = \mathbb{Q} = \{\text{rational numbers}\}$

X $\pi \cdot 1 \notin \mathbb{Q}$

key properties in $\mathbb{R}^n$ $\longrightarrow$ ANY vector space V.

- span
- linear dependence.
- subspace.
- basis.
- linear transformation ...

Def Basis of a vector space V.

① (not too large) $\{\vec{v}_1, \cdots, \vec{v}_n\}$ lin. indep.

$x_1\vec{v}_1 + \cdots + x_n\vec{v}_n = \vec{0}$ has only trivial sol

② (not too small) $\text{span}\{\vec{v}_1, \cdots, \vec{v}_n\} = V$

take any $\vec{b} \in V$, always find $x_1, \cdots, x_n \in \mathbb{R}$

s.t. $x_1\vec{v}_1 + \cdots + x_n\vec{v}_n = \vec{b}$

Ex. $V = \mathbb{P}_3 = \{$ poly func. $f: \mathbb{R} \to \mathbb{R}$ of degree $\leq 3\}$.

Find a basis of V.

guess: $\{1, x, x^2, x^3\}$.

① solve $a_0 + a_1 x + a_2 x^2 + a_3 x^3 = 0$ for all $x \in \mathbb{R}$

By fundamental thm of algebra, any poly $\in \mathbb{P}_3$ has at most 3 roots $\Rightarrow a_0 = a_1 = a_2 = a_3 = 0$.

② $\operatorname{span}\{1, x, x^2, x^3\} = \mathbb{P}_3$ by def.

Ex. a) Is $\mathbb{R}^2$ a subspace of $\mathbb{R}^3$?

$\times$ $\quad \mathbb{R}^2 \not\subseteq \mathbb{R}^3$

b) Is $\mathbb{P}_2$ a subspace of $\mathbb{P}_3$?

$\checkmark$ $\quad \mathbb{P}_2 \subset \mathbb{P}_3$

$\dim V :=$ number of vectors of any basis of V.

Ex. $\dim \mathbb{P}_n = n+1$

$\dim V < \infty$: finite dimensional vector space.

$\dim V = \infty$: infinite " " .

Ex. $\mathbb{P} = \{$ all poly func $: \mathbb{R} \to \mathbb{R}\}$.

basis : $\{1, x, x^2, x^3, \cdots\}$.

Ex. $V = S = \{ (a_1, a_2, a_3, \cdots) \mid a_i \in \mathbb{R},$

with finite nonzero a_i's $\}$.

basis $:= \{ (1, 0, 0, \cdots), (0, 1, 0, 0, \cdots) \cdots \}$.

(rarely used in THIS course)

Linear transformation

$T : V \to W$, V, W vector spaces

$\vec{u} \mapsto T(\vec{u})$

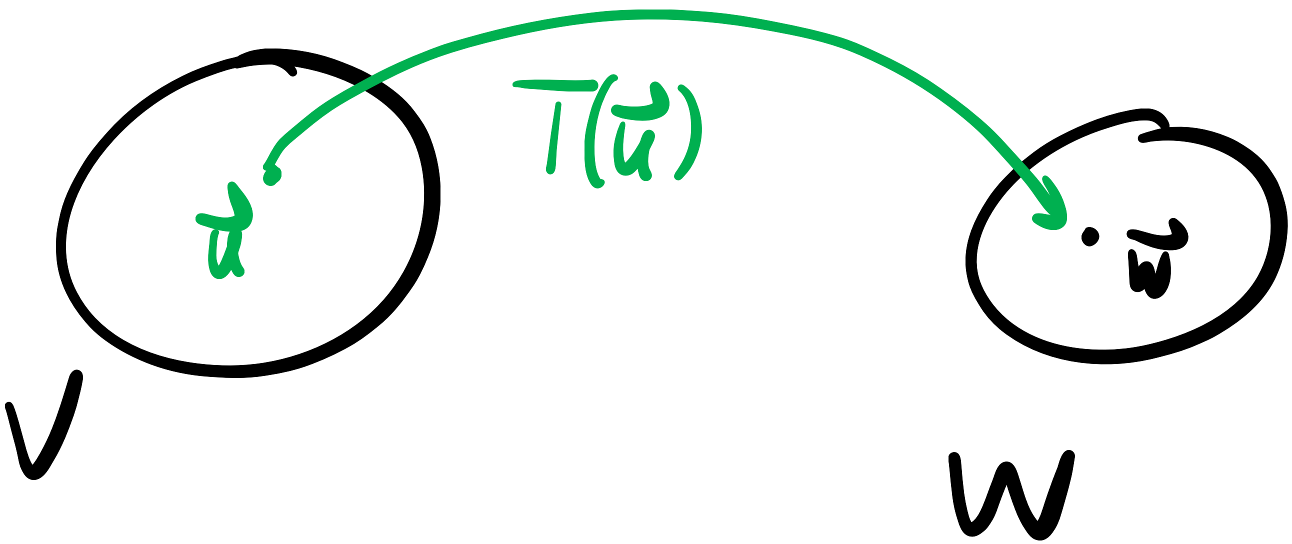
(1) $\vec{u}, \vec{v} \in V$, $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$

(2) $\vec{u} \in V$, $c \in \mathbb{R}$ $T(c\vec{u}) = cT(\vec{u})$

<u>Def</u> $\mathrm{Image}(T) = \{ \vec{w} \in W \mid \vec{w} = T(\vec{u})$ for some $\vec{u} \in V \}$

$\|$

$\mathrm{Range}(T)$



<u>Def</u> $\mathrm{rank}(T) := \dim \mathrm{Image}(T$

Def $Null(T) = \{ \vec{u} \in V \mid T(\vec{u}) = \vec{0} \}$

$\equiv$

$Kernel(T)$