Lec 38.

Fourier expansion

$$f(x) = \sum_{n=1}^{\infty} f_n \sin(nx). \qquad x \in [0, \pi]$$

a fundamental difference

$$f(x) = \sum_{n=1}^{N} f_n \sin(nx)$$

$$V = \{ f(x) \mid \sum_{n=1}^{N} f_n \sin(nx), \; f_n \in \mathbb{R} \}$$

$$\dim V = N$$

Fourier expansion $N \to \infty$.

f_n ?

$\begin{cases} ① \text{ Solve lin sys} \\ ② \text{ orthogonal basis.} \end{cases}$

$\to$ inner product.

$f(0) = f(L) = 0$

$$f_n = \frac{2}{L} \int_0^L \sin\left(\frac{n\pi}{L} x\right) f(x)\, dx. \quad n \geq 1$$

normalization

ortho. basis

$$f'(0) = f'(L) = 0$$

$$f(x) = \frac{f_0}{2} + \sum_{n=1}^{\infty} f_n \cos\left(\frac{n\pi}{L} x\right)$$

$$f_n = \frac{2}{L} \int_0^L \cos\left(\frac{n\pi}{L} x\right) f(x)\, dx \qquad n \geq 0$$

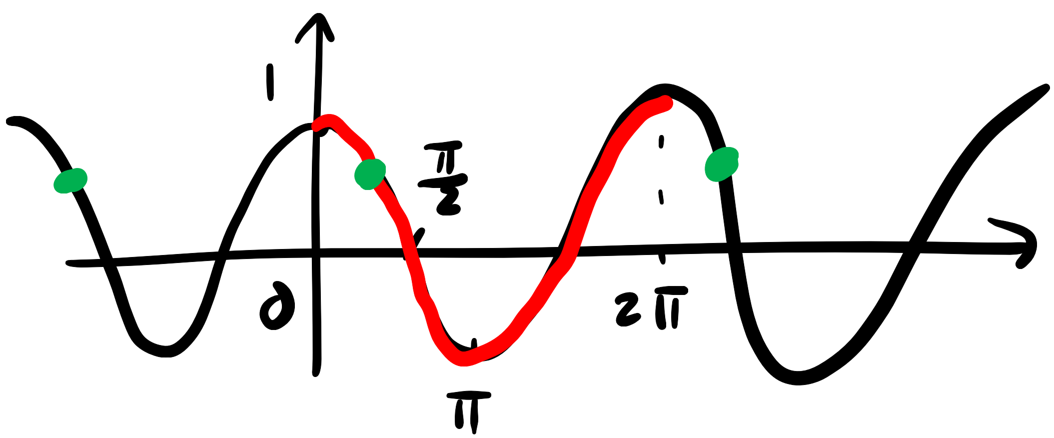
Period functions

$$f(x) = f(x+T) \quad , \text{ for any } x \in \mathbb{R}$$

period

Find smallest period.

1. $\cos(x)$



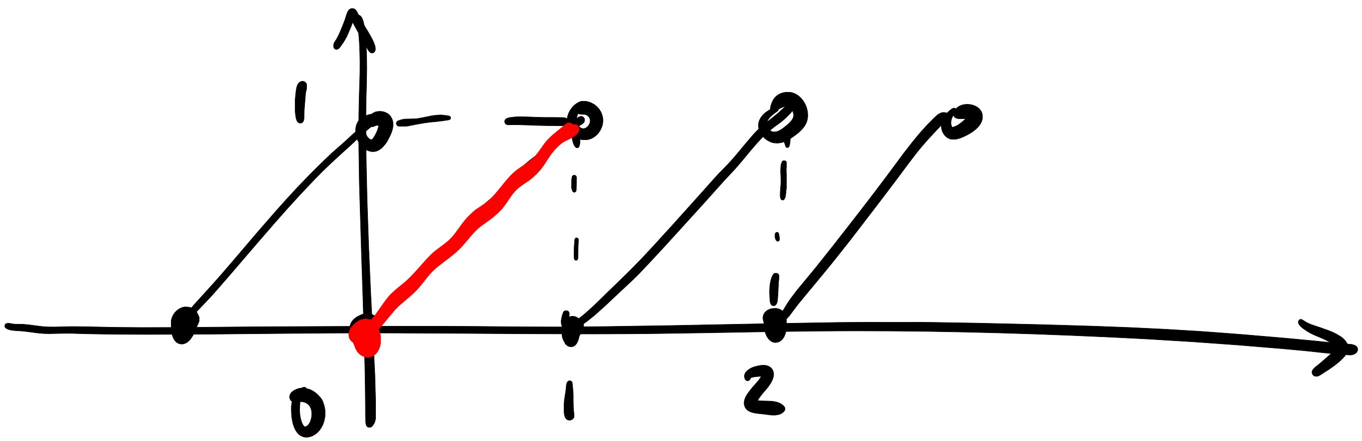
$T = 2\pi$

2. $e^{i\pi x} = \cos(\pi x) + i\sin(\pi x)$

$e^{i\pi(x+2)} = \cos(\pi x \underbrace{+2\pi}) + i\sin(\pi x \underbrace{+2\pi})$

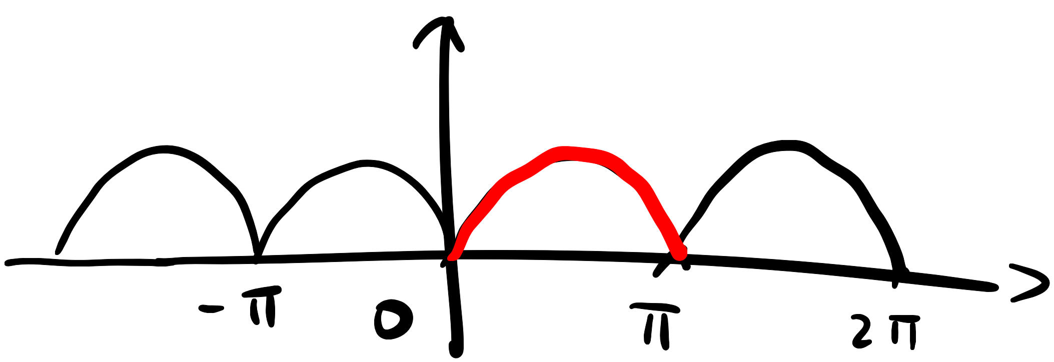
$= e^{i\pi x}$. $\quad T = 2$.

3. $f(x) = x - n, \quad n \le x < n+1, \quad n \in \mathbb{Z}$



$T = 1$

4. $f(x) = |\sin(x)|$



$T = \pi.$

Fourier analysis is expansion for periodic function.

WLOG interval $[-L, L]$

period is $T = 2L$.

$$V = \{ f(x) \mid f(x) = f(x+2L) \}.$$

$$\mathcal{B} = \{ 1, \sin\left(\frac{\pi x}{L}\right), \cos\left(\frac{\pi x}{L}\right), \sin\left(\frac{2\pi}{L} x\right),$$

$\cos\left(\frac{2\pi}{L}x\right), \dots \}$

"good" $f(x) \in V$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right)$$

$\mathcal{B}$ is an ortho. set.

a) $\displaystyle\int_{-L}^{L} \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx$

$$= 2\underbrace{\int_{0}^{L} \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx} = \begin{cases} 0, & m \neq n \\ L, & m = n \neq 0 \end{cases}$$

b) $\displaystyle\int_{-L}^{L} \cos\frac{m\pi x}{L}\cos\frac{n\pi}{L}x\,dx$

$$= 2\underbrace{\int_{0}^{L} \cos\frac{m\pi x}{L}\cos\frac{n\pi}{L}x\,dx} = \begin{cases} 0, & m \neq n \\ L, & m = n \neq 0 \\ 2L, & m = n = 0 \end{cases}$$

c) $$\int_{-L}^{L} \underbrace{\underbrace{\sin\frac{m\pi}{L}x}_{\text{odd}}\;\;\underbrace{\cos\frac{n\pi}{L}x}_{\text{even}}}_{\text{odd}}\,dx = 0$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx, \qquad n \geq 0$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx. \qquad n \geq 1$$

Connection w. sine / cosine trans.

$f(x) \quad [-L, L]$

① f is an odd function.

$$f(x) = f(x+2L)$$

$$f(x) = -f(-x)$$

$a_n = 0$ by symmetry.

$$b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L} x\right) dx$$

② f is an even function.

$$f(x) = f(x+2L)$$

$$f(x) = f(-x)$$

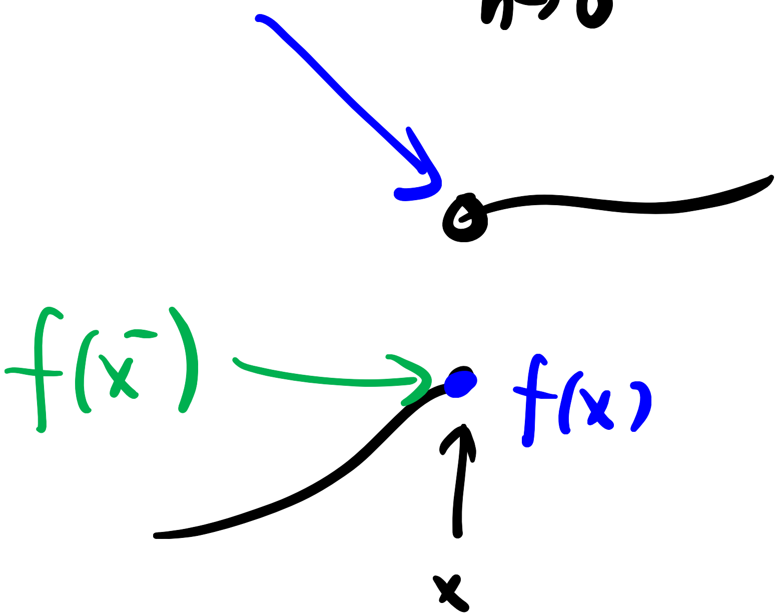
$b_n = 0$ by symmetry.

$$a_n = \frac{2}{L}\int_0^L f(x) \cos \frac{n\pi}{L} x \, dx .$$

Pointwise convergence

$f(x)$ is piecewise continuous on $[-L, L]$.

$$f(x^+) = \lim_{h \to 0^+} f(x+h)$$



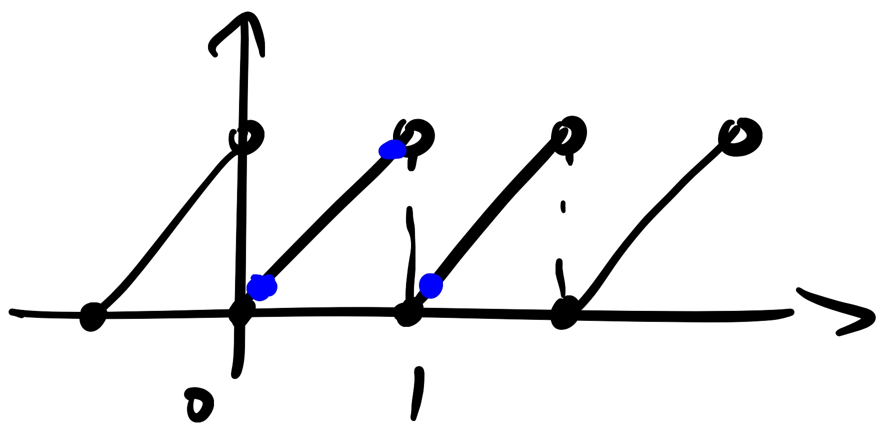
$$f(x^-) = \lim_{h \to 0^+} f(x-h)$$

<u>Thm</u>. f, f' are piecewise cont. on $[-L, L]$ for any $x \in (-L, L)$

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L} x\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L} x\right)$$

$$= \frac{1}{2}\left(f(x^+) + f(x^-)\right)$$

at $x = \pm L$, $\longrightarrow \frac{1}{2}\left[f(-L^+) + f(L^-)\right]$



if $f(x)$ is continuous on $\mathbb{R}$

$\longrightarrow f(x). \quad x \in [-L, L]$

Ex. $f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases}$

odd. $a_n = 0$

$$b_n = \frac{2}{\pi}\int_0^{\pi} \sin nx \, dx = \frac{-2}{n\pi}\int_0^{\pi} d\cos(nx)$$

$$= -\frac{2}{n\pi}\left[(-1)^n - 1\right]$$

$$= \begin{cases} 0, & n \text{ even} \\ \dfrac{4}{\pi n}, & n \text{ odd.} \end{cases}$$

$$\tilde{f}(x) = \frac{4}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right]$$