

## Lec 23. Warm up

True / False.

- (a) The sum of 2 eig. vectors of  $A \in \mathbb{R}^{n \times n}$   
is still an eig. vector of A

False. If  $A\mathbf{v}_1 = \lambda_1 \mathbf{v}_1$ ,  $A\mathbf{v}_2 = \lambda_2 \mathbf{v}_2$

$$A(\mathbf{v}_1 + \mathbf{v}_2) = \lambda (\mathbf{v}_1 + \mathbf{v}_2).$$

If different eigenvalue.

$$A\mathbf{v}_1 = \lambda_1 \mathbf{v}_1, A\mathbf{v}_2 = \lambda_2 \mathbf{v}_2. \Rightarrow A(\mathbf{v}_1 + \mathbf{v}_2) = \lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 \\ \pm \lambda' (\mathbf{v}_1 + \mathbf{v}_2)$$

(b)  $A, B \in \mathbb{R}^{n \times n}$  are similar, then  
they have the same set of eigenvectors.

False.  $A = V^{-1} B V$

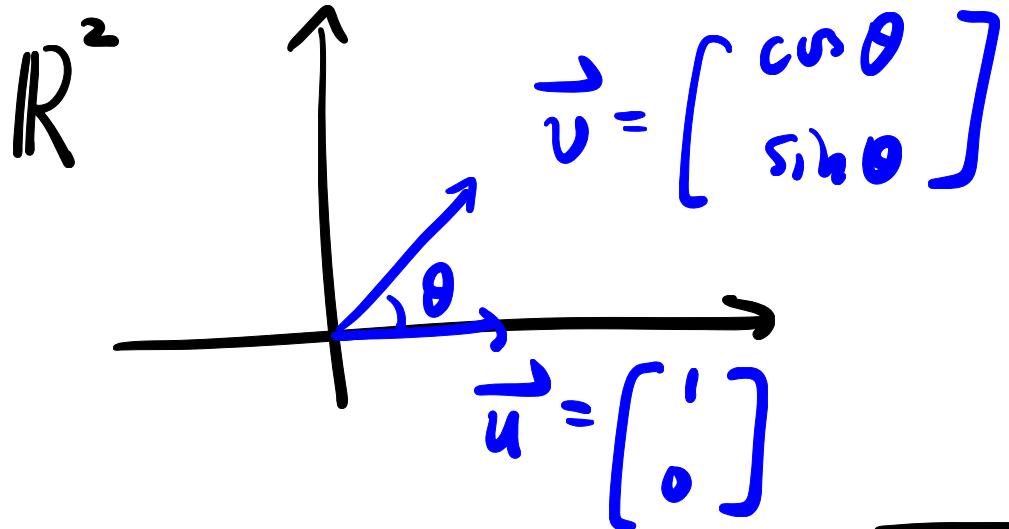
$\Rightarrow A, B$  share the same set of  
eig. values.

Just think.  $A \in \mathbb{R}^{2 \times 2}$ .

$$AV = V \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

B. eigvec.  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

# Geometry length. angle in $\mathbb{R}^n$



$$\|\vec{v}\| := \sqrt{v_1^2 + v_2^2} = \sqrt{\cos^2 \theta + \sin^2 \theta} = 1.$$

length  $u_1 v_1 + u_2 v_2 = \cos \theta$

Def  $\vec{u}, \vec{v} \in \mathbb{R}^2$ , the inner product  
(dot product)

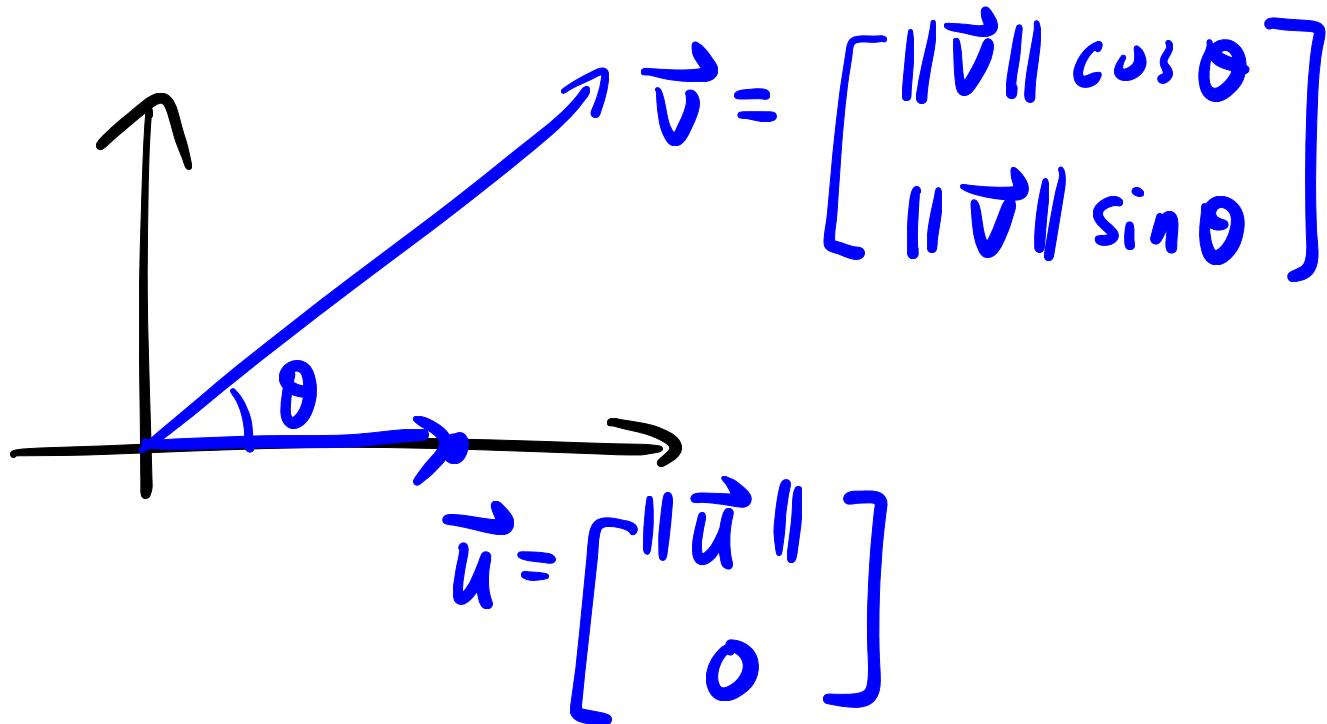
of  $\vec{u}, \vec{v}$  is denoted by ( $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ )

$$\vec{u} \cdot \vec{v} := u_1 v_1 + u_2 v_2.$$

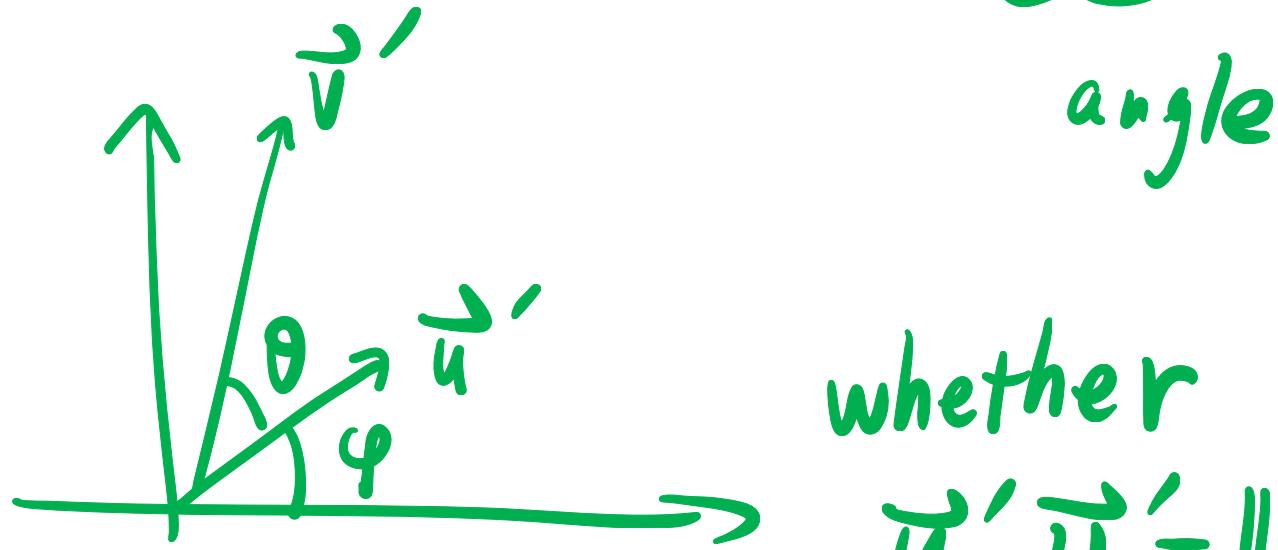
The length of  $\vec{u}$  is

$$\|\vec{u}\| := \sqrt{u_1^2 + u_2^2} \equiv \sqrt{\vec{u} \cdot \vec{u}}$$

Note:  $\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v} = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$



$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| \cdot \cos \underline{\theta}$$



Rotation operation is a matrix transformation.

$$\vec{u}' = \begin{bmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{bmatrix} \vec{u}$$

$$\vec{u}' \cdot \vec{v}' = (\vec{u}')^T \vec{v}'$$

matrix-matrix  
mult.

$$\vec{u}'^T = \vec{u}^T \begin{bmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{bmatrix}$$

$$= \vec{u}^T \begin{pmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{pmatrix} \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix} \vec{v}$$

$$= \vec{u}^T \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \vec{v}$$

$$= \vec{u}^T \vec{v} = \vec{u} \cdot \vec{v}$$

$$\vec{u}' \cdot \vec{v}' = \vec{u} \cdot \vec{v}$$

↑  
after rotation      ↑  
before rotation

$$\mathbb{R}^2 \rightarrow \mathbb{R}^n$$

Use previous algebraic definition.

Def.  $\vec{u}, \vec{v} \in \mathbb{R}^n$ , dot product /  
inner product / Euclidean inner product  
is

$$\vec{u} \cdot \vec{v} = u_1 v_1 + \cdots + u_n v_n$$

The length of  $\vec{u}$  is

$$\|\vec{u}\| = \sqrt{\vec{u} \cdot \vec{u}}$$

Again

$$\vec{u} \cdot \vec{v} \equiv \vec{u}^T \vec{v} = [u_1 \cdots u_n] \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

# Properties

$$1) \vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$$

$$2) (\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$$

$$3) c \in \mathbb{R} \quad (c\vec{u}) \cdot \vec{v} = c(\vec{u} \cdot \vec{v})$$

$$4) \vec{u} \cdot \vec{u} \geq 0, \quad \vec{u} \cdot \vec{u} = 0 \Leftrightarrow \vec{u} = \vec{0}$$

length is 0  $\Leftrightarrow$  zero vector.

Def Unit vector  $\vec{u}$  : if  $\|\vec{u}\| = 1$

For any **nonzero** vector  $\vec{u}$

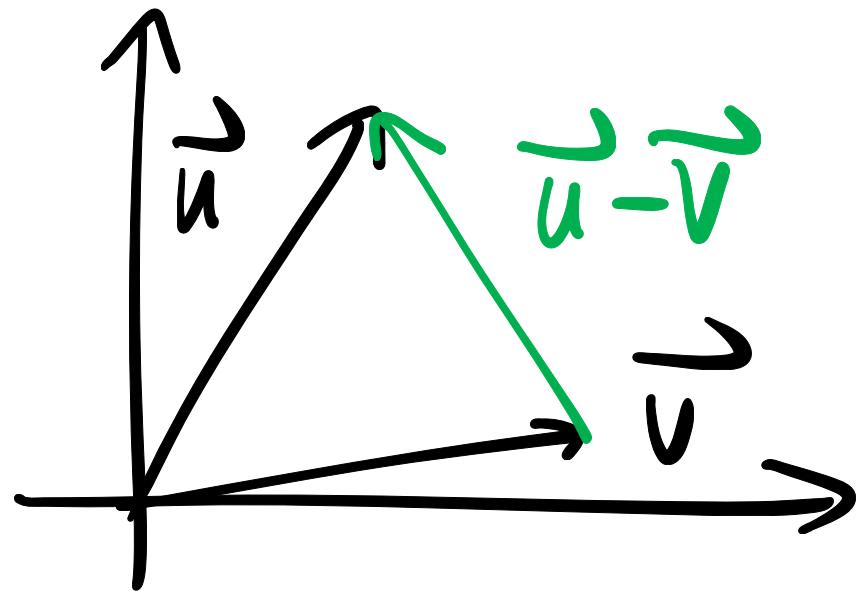
Normalization

$$\vec{v} = \left( \frac{1}{\|\vec{u}\|} \right) \vec{u} \quad \text{is a unit vector}$$

$$\vec{v} \cdot \vec{v} = \frac{1}{\|\vec{u}\|^2} \vec{u} \cdot \vec{u} = 1 .$$

Def Distance between  $\vec{u}, \vec{v} \in \mathbb{R}^n$

$$d(\vec{u}, \vec{v}) := \|\vec{u} - \vec{v}\|$$

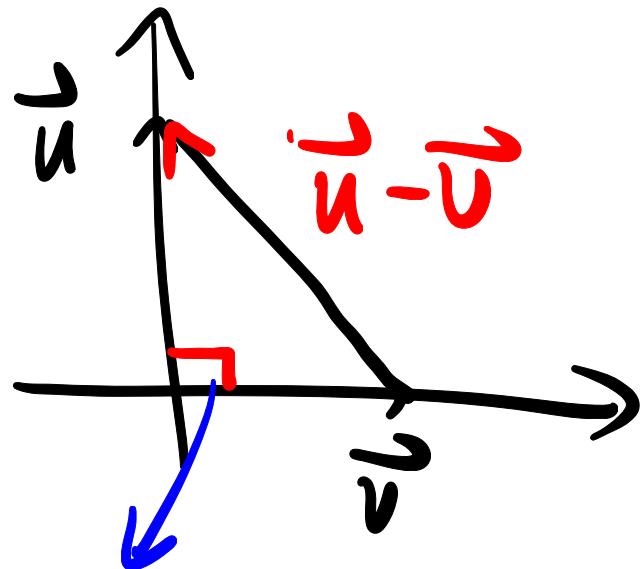


$$\begin{aligned}
 \|\vec{u} - \vec{v}\|^2 &= (\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v}) \\
 &= \vec{u} \cdot \vec{u} - \vec{u} \cdot \vec{v} - \vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{v} \\
 &= \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2 \vec{u} \cdot \vec{v}
 \end{aligned}$$

Important special case

$$\vec{u} \cdot \vec{v} = 0.$$

$$\|\vec{u} - \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$$



$$\theta = \frac{\pi}{2}$$

Recall

Generalization of  
Pythagorean theorem  
to  $\mathbb{R}^n$ .

$$\vec{u} \cdot \vec{v} =: \|\vec{u}\| \|\vec{v}\| \underbrace{\cos \theta}_{\uparrow}$$

define angle  
between  $\vec{u}, \vec{v} \in \mathbb{R}^n$

If  $\vec{u} \cdot \vec{v} = 0 \Rightarrow \cos \theta = 0.$

$$\theta \in [0, \pi], \quad \theta = \frac{\pi}{2}$$

$\vec{u}, \vec{v}$  are orthogonal vectors.  $\vec{u} \perp \vec{v}$

In Chap 4.

$$\mathbb{R}^{2 \text{ or } 3} \rightarrow \mathbb{R}^n \rightarrow V$$



In Chap 6.

equip vector space  $V$  by  
an additional structure (inner product)  
to describe geometry.

v



+



(regular) vector  
space

inner product

—

—



Inner product space

Def. .  $\{\vec{v}_1, \dots, \vec{v}_k\}$  orthogonal set

$\vec{v}_i \in \mathbb{R}^n$ , if  $\vec{v}_i \perp \vec{v}_j$  for all

$$1 \leq i \neq j \leq k \quad \vec{v}_i \cdot \vec{v}_j = 0$$

Orthonormal set if

- 1) Orthogonal set
- 2) all  $\|\vec{v}_i\| = 1$ .

