Lec 4.

Warmup. Solve lin. sys.

$$A = \begin{bmatrix} 2 & -5 & 8 \\ -2 & -4 & 1 \\ 4 & -1 & 7 \end{bmatrix} \qquad b = \begin{bmatrix} 5 \\ -5 \\ 10 \end{bmatrix}$$

Aug. matrix

$$\left[\begin{array}{ccc|c} 2 & -5 & 8 & 5 \\ -2 & -4 & 1 & -5 \\ 4 & -1 & 7 & 10 \end{array}\right] \xrightarrow{REF} \left[\begin{array}{ccc|c} 2 & -5 & 8 & 5 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{cases} 2x_1 - 5x_2 + 8x_3 = 5 \\ \qquad\quad x_2 - x_3 = 0 \end{cases}$$

parametric form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -\frac{3}{2} \\ 1 \\ 1 \end{bmatrix}$$

special sol. sol. to hom. lin. sys.

$$[\,A \mid b\,] \iff x_1\vec{a_1} + \cdots + x_n\vec{a_n} = \vec{b}$$

$$A = [\,\vec{a_1} \cdots \vec{a_n}\,]$$

admits 2 sols. $\vec{x} = \vec{u}, \vec{v}$

$$\begin{cases} u_1\vec{a_1} + \cdots + u_n\vec{a_n} = \vec{b} \\ v_1\vec{a_1} + \cdots + v_n\vec{a_n} = \vec{b} \end{cases}$$

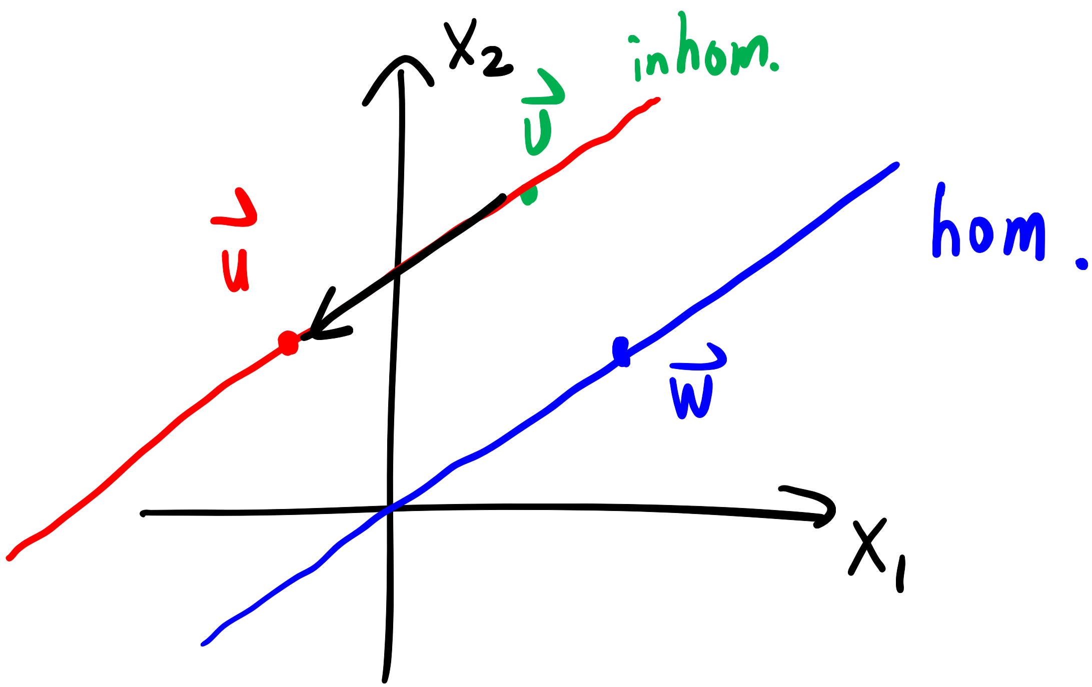
$$\Rightarrow (u_1 - v_1)\vec{a_1} + \cdots + (u_n - v_n)\vec{a_n} = \vec{0}.$$

$\Rightarrow$ $\vec{u} - \vec{v}$ is a sol. to hom. lin. sys. $[\,A \mid \vec{0}\,]$

$$\vec{u} = \vec{v} + \underset{\text{sol. hom. sys.}}{\vec{w}} := (\vec{u} - \vec{v})$$

Geometric meaning $(\mathbb{R}^2)$.

$$\left[\begin{array}{cc|c} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{array}\right] \xrightarrow{\text{sol}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



Hom. lin. sys.	General inhomogeneous
Only trivial sol.	at most one sol.
at least one non-trivial sol	if sol exists. then inf sols.

Vector eq. perspective of hom. lin. sys.

$\longrightarrow$ linear dependency.

<u>Def</u> A vector $\vec{u}$ is a <u>linear combination</u> of vectors $\vec{v}_1, \cdots, \vec{v}_k$, if

$$\vec{u} = a_1 \vec{v}_1 + \cdots + a_k \vec{v}_k, \quad a_i \in \mathbb{R}, 1 \le i \le k.$$

$\Longleftrightarrow$ lin sys. $[\vec{v}_1, \cdots, \vec{v}_k \mid \vec{u}]$ is consistent

<u>Def</u> The span of $\vec{v_1}, \dots, \vec{v_k}$ is the set of all vectors that can be written as the lin. combination of $\vec{v_1}, \dots, \vec{v_k}$

$$\operatorname{span}\{\vec{v_1}, \dots, \vec{v_k}\} = \{a_1\vec{v_1} + \dots + a_k\vec{v_k} \mid a_1, \dots, a_k \in \mathbb{R}\}$$

Ex. Is $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ in $\operatorname{span}\left\{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 6 \end{bmatrix}\right\}$

Sol. Find coefs $x_1, x_2 \in \mathbb{R}$. s.t.

$$x_1 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\Longleftrightarrow \left[\begin{array}{cc|c} 2 & -2 & 1 \\ 1 & 2 & 2 \\ 0 & 6 & 3 \end{array}\right] \quad \text{has sol?}$$

$$\xrightarrow[\text{Ex.}]{\text{REF}} \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}$$

Yes. in span.

Linear dependence $\longleftrightarrow$ redundancy.

<u>Def</u> A set of vectors $\{\vec{v}_1, \cdots, \vec{v}_k\}$ are lin. independent if

$$x_1 \vec{v}_1 + \cdots + x_k \vec{v}_k = 0$$

has only trivial sol.

Otherwise lin. dependent.

Ex. $\vec{v}_1 = \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 4 \\ -6 \\ 7 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} -2 \\ 2 \\ -4 \end{bmatrix}$

whether $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ lin. dependent ?

Ans. $\xrightarrow[\text{ref}]{\text{Exer}} \left[\begin{array}{ccc|c} \boxed{2} & 4 & -2 & 0 \\ 0 & \boxed{-2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\rightarrow$ inf sol. $\rightarrow$ lin. dep.

Sol set. $\left\{ x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \middle| \; x_3 \in \mathbb{R} \right\}$.

Plug back to vector eq.

$$x_3 \vec{v_1} + 0 \cdot \vec{v_2} + x_3 \cdot \vec{v_3} = \vec{0}, \quad x_3 \in \mathbb{R}$$

Take $x_3 = 1$. $\Rightarrow \vec{v_1} = -\vec{v_3} = \underbrace{0 \cdot \vec{v_2} + (-1) \cdot \vec{v_3}}_{\text{lin. comb.}}$

$$\operatorname{span}\{\vec{v_1}, \vec{v_2}, \vec{v_3}\} = \operatorname{span}\{\vec{v_2}, \vec{v_3}\}$$

$$= \operatorname{span}\{\vec{v_1}, \vec{v_2}\}.$$

Corner case ($k=1$)

$x_1 \vec{v}_1 = \vec{0}$ has nontrivial sol?

$$\begin{bmatrix} x_1 v_1 \\ \vdots \\ x_1 v_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

$\vec{v} \neq \vec{0} \Rightarrow x_1 = 0 \Rightarrow \{\vec{v}_1\}$ lin. indep.

$\vec{v}_1 = \vec{0} \Rightarrow x_1 \in \mathbb{R} \Rightarrow \{\vec{0}\}$ lin. dep.