

Lec 21. Warm up

True / False :

Is this matrix diagonalizable?

(1) $A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$ $\begin{vmatrix} 1-\lambda & 0 \\ 1 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(2-\lambda) = 0$
 $\Rightarrow \lambda_1 = 1, \lambda_2 = 2$ ✓
check eig. vec.

(2) $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $\begin{vmatrix} 1-\lambda & 0 \\ 0 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda(1-\lambda) = 0$
 $\Rightarrow \lambda_1 = 0, \lambda_2 = 1.$ ✓
 $a=2$ $\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

(3) $A+B = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$ $\begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}$ is not diagonalizable X

True / False

There exists an $n \times n$ matrix ω .

$(n+1)$ eigenvalues.

False.

$$0 = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & & \ddots & \\ a_{n1} & \cdots & a_{nn}-\lambda & \end{vmatrix} \rightarrow P_n(\lambda)$$

λ must be a root of n -deg. poly.

\Rightarrow at most n eigenvalues.

Thm. $A \in \mathbb{R}^{n \times n}$ has n distinct eigenvalues

$\lambda_1, \dots, \lambda_n$. Then A is diagonalizable.

Pf : For λ_i , $1 \leq i \leq n$.

$A - \lambda_i I_n$ is not invertible

$\Rightarrow \dim \text{Null}(A - \lambda_i I) \neq 0$.

\vec{v}_i a eigenvector.

$\{\vec{v}_1, \dots, \vec{v}_n\}$ n lin. indep. vectors

 (exer)

forms a basis of \mathbb{R}^n

$\rightarrow V = [\vec{v}_1, \dots, \vec{v}_n]$ diagonalize A.

Cor. $A = \begin{bmatrix} \lambda_1 & * \\ 0 & \ddots \\ & & \lambda_n \end{bmatrix}$

← only a sufficient condition

$\lambda_1, \dots, \lambda_n$ are distinct.

$\Rightarrow A$ diagonalizable. (think I_n)

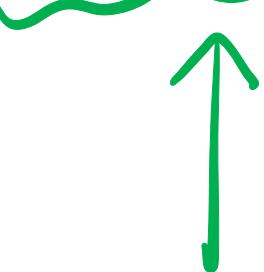
Fact : $A \in \mathbb{R}^{n \times n}$ λ is an eigenvalue.

$$1 \leq \dim \text{Null}(A - \lambda I) \leq \text{multiplicity}$$



$A - \lambda I$ is
singular

geometric
multiplicity of
eigenvalue λ .



of λ in
 $|A - \lambda I| = 0$.



algebraic
multiplicity of
eigenvalue λ

Ex. 4×4 matrix.

$$\begin{bmatrix} \lambda & & & 0 \\ & \lambda & & \\ 0 & & \lambda & \\ & & & \lambda \end{bmatrix} = \lambda \cdot I_4$$

$$\dim \text{Null}(A - \lambda I) = 4$$

$$\begin{bmatrix} \lambda & 1 & & \\ 0 & \lambda & & \\ \vdots & & & \\ 0 & & & \lambda \end{bmatrix} \quad \begin{bmatrix} & & & 0 \\ & & & \\ & & & \\ 0 & & & \lambda \end{bmatrix}$$

$$\dim \text{Null}(A - \lambda I) = 3$$

$$\left[\begin{array}{cc|cc} \lambda & 1 & 0 & \\ 0 & \lambda & & \\ \hline 0 & & \lambda & 1 \\ 0 & & 0 & \lambda \end{array} \right] \text{ or } \left[\begin{array}{ccc|c} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ \hline 0 & 0 & 0 & \lambda \end{array} \right]$$

$$\dim \text{Null}(A - \lambda I) = 2$$

$$\begin{bmatrix} \lambda & 1 & 0 & \\ & \lambda & 1 & \\ 0 & & \lambda & 1 \\ & & & \lambda \end{bmatrix}$$

$$\dim \text{Null}(A - \lambda I) = 1$$

Ex. $A - \lambda I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

dim of sol set of $(A - \lambda I) \vec{x} = 0$.

$$\Rightarrow \dim \text{Null}(A - \lambda I) = 3$$

Ex. $A \in \mathbb{R}^{6 \times 6}$

$$|A - \lambda I| = (-\lambda)^3 (2 - \lambda) \lambda^2$$

Possible dim of eigenspace



span of all eigenvects?

$\lambda =$ 0 1 2

(algebraic)
multiplicity 2 3 1

$$1 \leq \dim \text{Null}(A) \leq 2. \quad 1 \leq \dim \text{Null}(A - I) \leq 3$$

$$1 \leq \dim \text{Null}(A - 2I)$$

`dim span {all eigenvectors}`

`3 , 4 , 5 , 6`

(exer). create an actual matrix
for each case above.

What does diagonalization mean?

$$A \begin{bmatrix} \vec{v}_1, \dots, \vec{v}_n \end{bmatrix} = \begin{bmatrix} \vec{v}_1, \dots, \vec{v}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$



$$AV = VD \Leftrightarrow D = V^{-1}AV.$$

View from lin. trans.
matrix representation.

$$T : \mathbb{R}^n \rightarrow \mathbb{R}^n$$

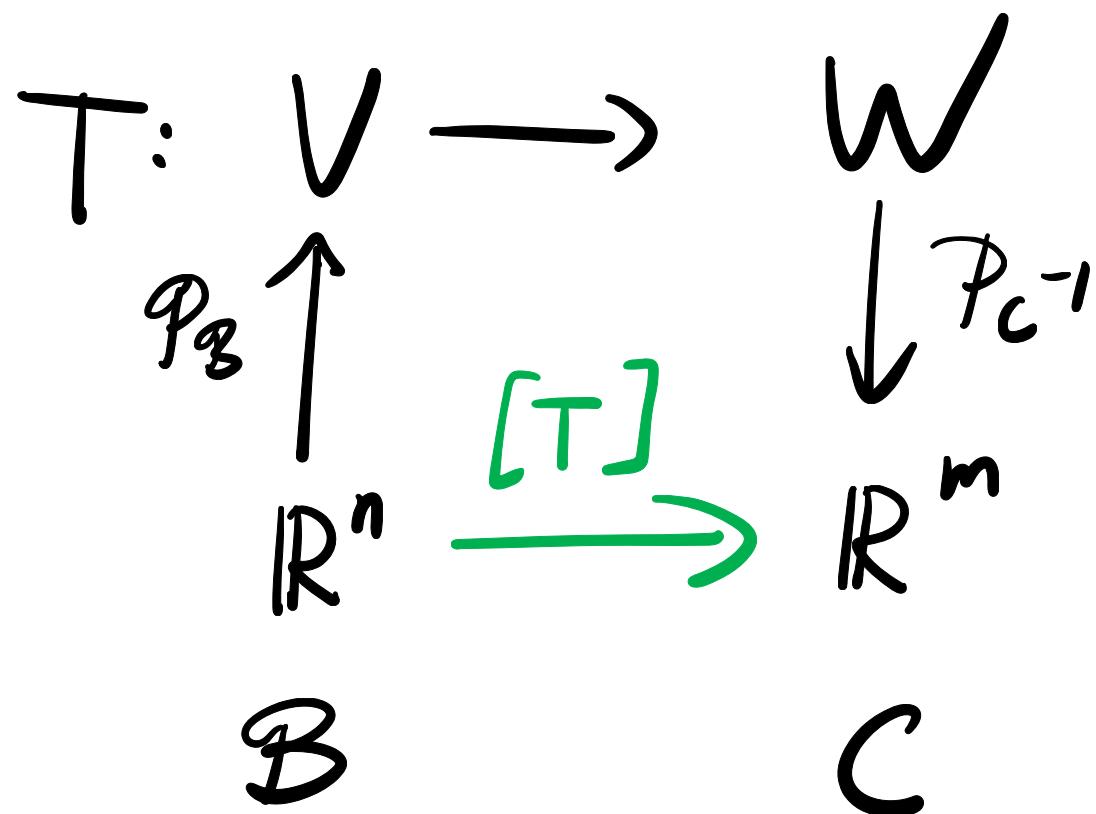
Standard $\leftarrow \vec{x} \mapsto A\vec{x}$

basis E

What if choose $B = \{\vec{v}_1, \dots, \vec{v}_n\}$

$$P_{\mathcal{B}} = [\vec{v}_1, \dots, \vec{v}_n] \equiv V$$

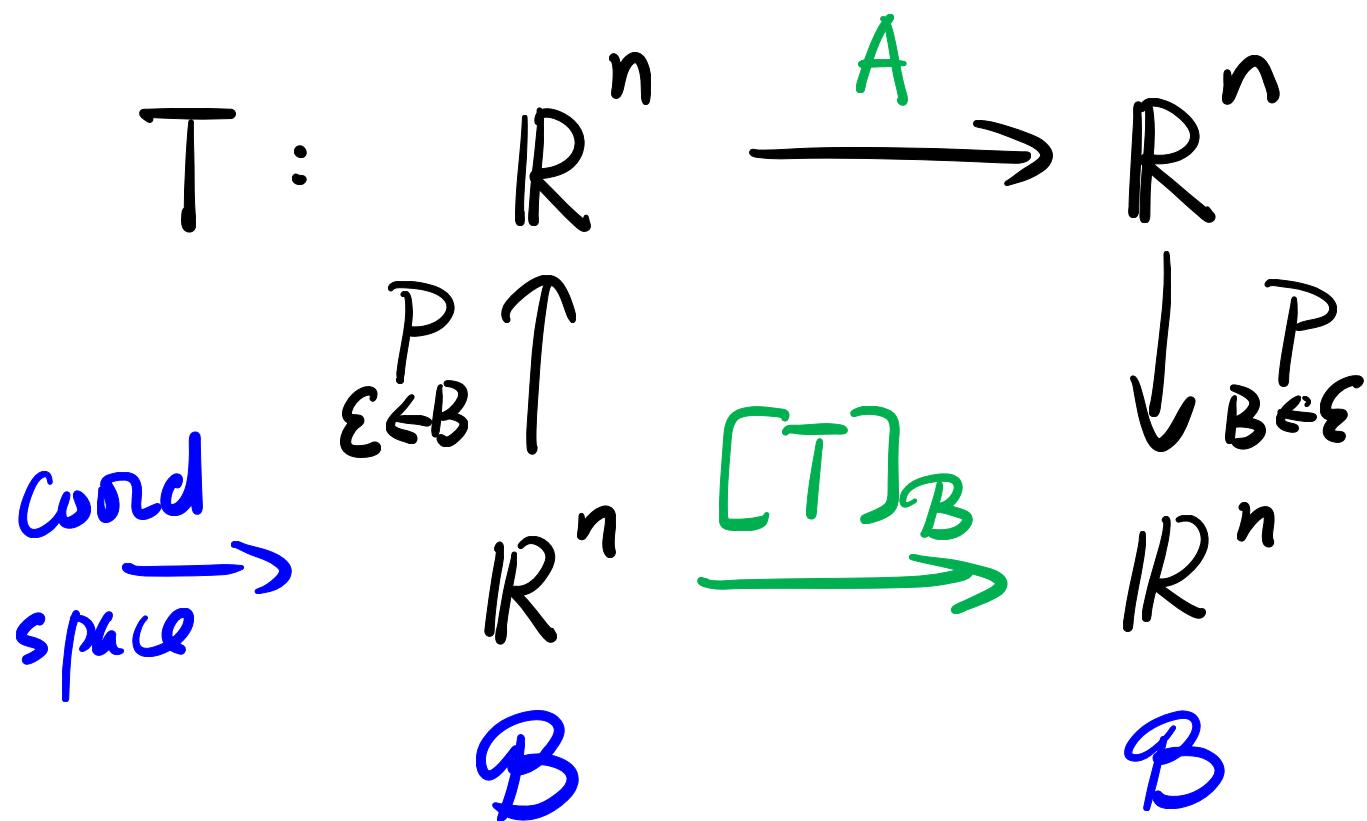
General matrix representation



$$V = W = \mathbb{R}^n$$

$$\mathcal{B} = C = \{\vec{v}_1, \dots, \vec{v}_n\}$$

Simplified as follows



$$[\mathcal{T}]_B = \underset{B \leftarrow \mathcal{E}}{\mathcal{P}} A \underset{\mathcal{E} \leftarrow B}{\mathcal{P}}$$

$$= V^{-1} A V = \mathcal{D} \rightarrow \text{diag.}$$

Diagonalization:

In the eigen-basis, the matrix representation $[\mathcal{T}]_B$ is diagonal.

