Lec 17. Warm up

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad a \neq 0$$

what is the condition for A to be invertible?

Row reduce.
$$\begin{bmatrix} \boxed{a} & b \\ 0 & \boxed{d - \frac{bc}{a}} \end{bmatrix}$$

$$\Rightarrow d - \frac{bc}{a} \neq 0 \Rightarrow ad - bc \neq 0.$$

Determinant. Use a single number to decide whether a square matrix is invertible.

Used in many other contexts.

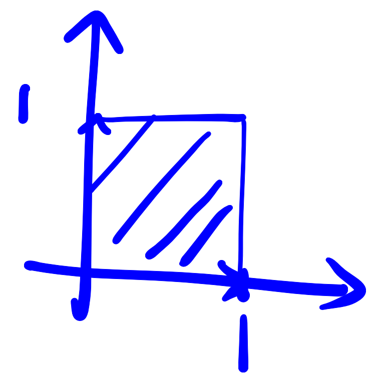
1) change of var. in multi.var. calculus (Math 53)

2) statistics.

3) quantum mechanics. (anti-symmetric)

4) Chap 5. eigenvalues.

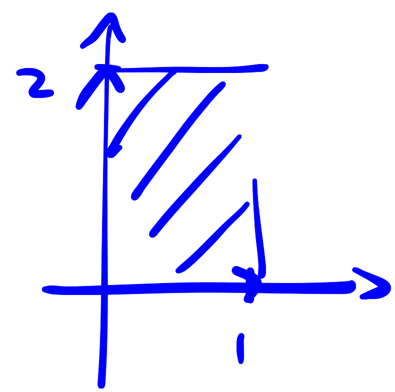
Ex. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$



area = 1

$\det(A) = 1$

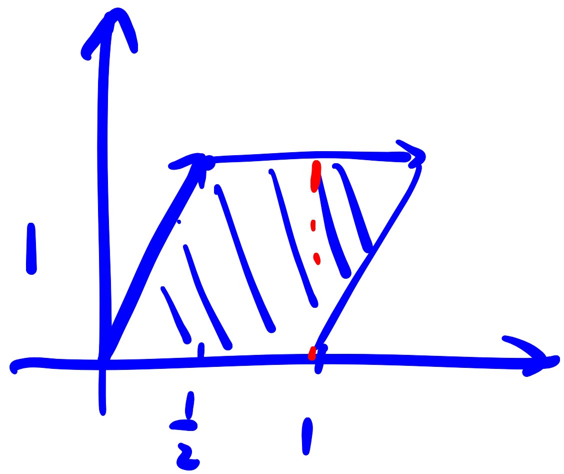
Ex. $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$



$\det(A) = \text{area} = 2$

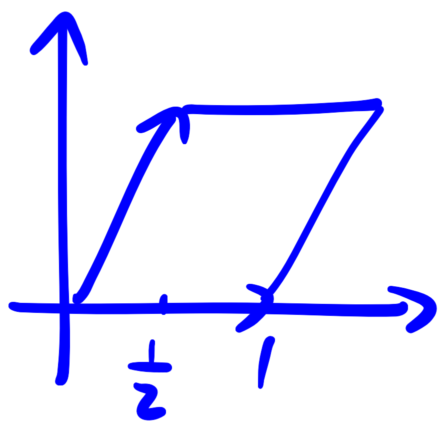
$A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$. $\det = \text{area} = \lambda_1 \lambda_2$

Ex. $A = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix}$



$\det(A) = 1 = \text{area}$.

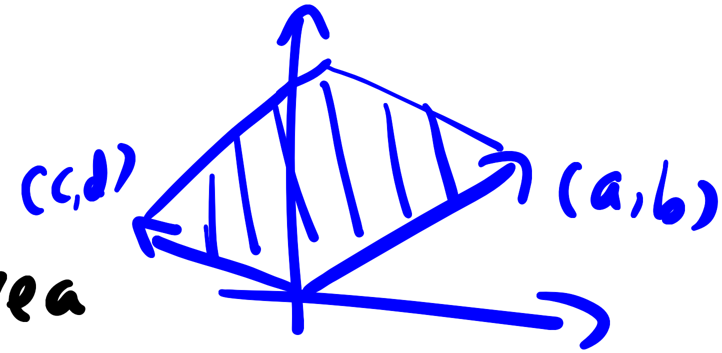
Ex. $A = \begin{bmatrix} \frac{1}{2} & 1 \\ 1 & 0 \end{bmatrix}$



$\det(A) = -1 =$ signed area.

(exer). $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\det(A) = ad - bc =$ signed area



summarize $\det(A)$ v.s. elem. row op. (in $\mathbb{R}^2$)

① add scalar multiple of a row to another row

$\det(A)$ does not change

② exchange 2 rows.

$\det(A) \times (-1)$

③ scale any row by $\lambda \in \mathbb{R}$ $(\lambda \neq 0)$

$\det(A) \times (\lambda)$

$$A \xrightarrow{\text{elem. row. op.}} A' \text{ (RREF)}$$

RREF	$\det(A')$
$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	0
$\begin{bmatrix} 1 & * \\ 0 & 0 \end{bmatrix}$	0
$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$	0
$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	1

$\det(A') =$ Some nonzero number $\times$ $\det(A)$

$\Rightarrow \det(A) \neq 0 \Rightarrow \det(A') \neq 0$.

$\Downarrow$

invertible.

General matrix in $\mathbb{R}^n$.

Inductive definition.

start from $n=1$.

$A = [a_{11}]$ $\det(A) := a_{11}$

assume we know how to define

$\det(A)$ for any matrix $A \in \mathbb{R}^{(n-1)\times(n-1)}$

Now given $A \in \mathbb{R}^{n \times n}$

$$
A = \begin{array}{cc}
 & \text{col } j \\
 & \downarrow \\
\text{row } i \leftarrow &
\begin{bmatrix}
a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\
\vdots & & \vdots & & \vdots \\
a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\
\vdots & & & & \\
a_{n1} & \cdots & a_{nj} & \cdots & a_{nn}
\end{bmatrix}
\end{array}
$$

confusing notation

$$
A_{ij} := \begin{bmatrix}
a_{11} & \cdots & \cancel{a_{1j}} & \cdots & a_{1n} \\
\vdots & & \cancel{\vdots} & & \vdots \\
\cancel{a_{i1}} & \cancel{\cdots} & \cancel{a_{ij}} & \cancel{\cdots} & \cancel{a_{in}} \\
\vdots & & \cancel{} & & \\
a_{n1} & \cdots & \cancel{a_{nj}} & \cdots & a_{nn}
\end{bmatrix}
$$

$\mathbb{R}^{(n-1)\times(n-1)}$

Define

$$\det(A) := a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + a_{13}\det(A_{13}) - a_{14}\det(A_{14}) + \cdots + (-1)^{n+1} a_{1n}\det(A_{1n})$$

n terms.

$$\begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & & & \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix}$$

Ex. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\det(A) = a \cdot \underset{\det(A_{11})}{\underset{\shortparallel}{d}} - b \cdot \underset{\det(A_{12})}{\underset{\shortparallel}{c}}$$

Ex. $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ -1 & 0 & -2 \end{bmatrix}$

$\det(A)$

$\parallel$

$$|A| = 1 \cdot \begin{vmatrix} 3 & 1 \\ 0 & -2 \end{vmatrix} - 2 \begin{vmatrix} 0 & 1 \\ -1 & -2 \end{vmatrix}$$

$$+ 1 \cdot \begin{vmatrix} 0 & 3 \\ -1 & 0 \end{vmatrix}$$

$$= 1 \cdot (-6) - 2 \cdot 1 + 1 \cdot 3 = -5.$$

Ex. $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$

(exer)

$$\det(A) = a_1 b_2 c_3 + a_3 b_1 c_2 + a_2 b_3 c_1$$

$$- a_3 b_2 c_1 - a_2 b_1 c_3 - a_1 b_3 c_2$$