Lec 19. Warmup

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad \vec{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Compute

$$\vec{y} = A^{1000} \vec{x}_0$$

Try $A\vec{x}_0 = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3\vec{x}_0$ $\longrightarrow$ scale $\vec{x}_0$ by a number.

$$A^2 \vec{x_0} = 3^2 \cdot \vec{x_0}. \Rightarrow A^{1000} \vec{x_0} = 3^{1000} \cdot \vec{x_0}$$

$\vec{x_0}$ is an <u>eigenvector</u> of A.

$$\boxed{\det(AB) = \det(A)\det(B)}$$

covered in discussion.

Cramer's rule.

How to compute $\dfrac{\det(A')}{\det(A)}$

$A, A', I' \in \mathbb{R}^{n\times n}$. A invertible $\Rightarrow \det(A) \neq 0$

A and A' differ only by 1 column

$$A = [\vec{a}_1 \ \vec{a}_2 \cdots \vec{a}_n]$$

$$A' = [\vec{a}_1 \cdots \vec{a}_{i-1} \ \vec{b} \ \vec{a}_{i+1} \cdots \vec{a}_n]$$

$\uparrow$

i-th col

$$A \cdot I' = A' \iff I' = A^{-1} \cdot A'$$

$$\det(I') = \underbrace{\det(A^{-1}) \cdot \det(A')}_{\text{product formula}}$$

Recall

$$A \cdot A^{-1} = I_n$$

$$\Rightarrow \det(A) \cdot \det(A^{-1}) = \det(I_n) = 1$$

$$\Rightarrow \det(A^{-1}) = \det(A)^{-1}$$

So what we want

$$\det(I') = \frac{\det(A')}{\det(A)}$$

$$I' = A^{-1}[\vec{a}_1 \cdots \vec{b} \cdots \vec{a}_n] \qquad \vec{x} = A^{-1}\vec{b}$$

$$= [\vec{e}_1 \cdots \vec{e}_{i-1} \; \vec{x} \; \vec{e}_{i+1} \cdots \vec{e}_n]$$

$$= \begin{bmatrix} 1 & 0 & 0 & \cdots & x_1 & \cdots & & 0 \\ & 1 & 0 & \cdot & x_2 & \cdots & & 0 \\ & & & \cdots & x_i & \cdots & & 0 \\ & & & & \cdots & & 1 & 0 \\ 0 & & \cdots & & x_n & \cdots & & 1 \end{bmatrix}$$

Cofactor expansion i-th row

$$\det(I') = x_i (-1)^{i+i} \det(I_{n-1})$$

$$= x_i$$

Cramer's rule

$$\frac{\det(A')}{\det(A)} = x_i \qquad x_i = (A^{-1}\vec{b})_i$$

ratio of 2 dets

sol of a linear sys.

Another formula for A^{-1}.

j-th col A^{-1} is the sol. of

$$A\vec{x} = \vec{e_j}$$

i-th col

$$(A^{-1})_{ij} \equiv (\vec{x})_i \underset{\text{Cramer}}{=} \frac{1}{\det(A)} \begin{vmatrix} a_{11} & \cdots & 0 & \cdots & a_{1n} \\ \vdots & & \vdots & & \\ & & 1 & & \\ & & \vdots & & \\ a_{n1} & & 0 & \cdots & a_{nn} \end{vmatrix}$$

e_j

$$= \frac{1}{\det(A)} C_{ji}$$

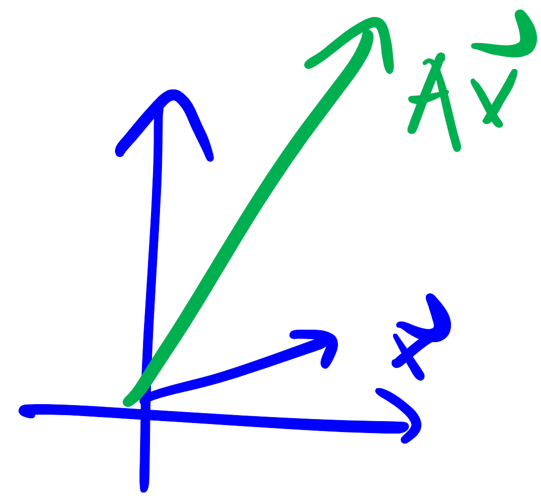
$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} C_{11} & \cdots & C_{n1} \\ \vdots & & \\ C_{1n} & \cdots & C_{nn} \end{bmatrix}$$

transpose of co-factor matrix.

Eigenvalue & Eigenvectors

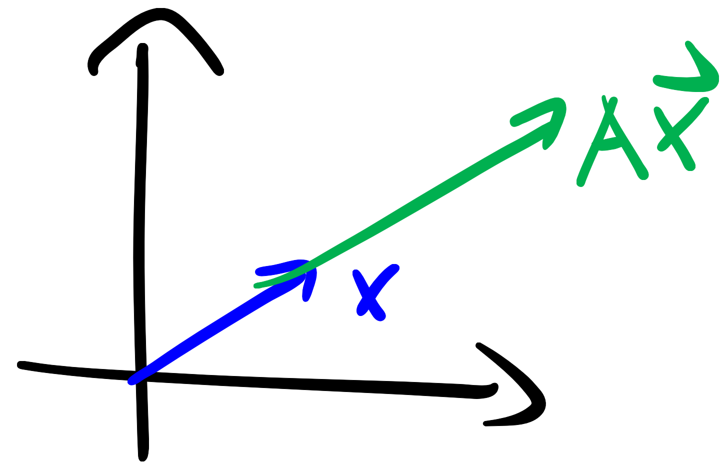
$A \in \mathbb{R}^{n\times n}$, $\quad \vec{x} \in \mathbb{R}^n$

$\vec{y} = A\vec{x}$.



Generally both length & direction of $\vec{x}$ are changed.

Find such $\vec{x}$, $A\vec{x}$ does not change direction?



$$\Longleftrightarrow \quad A\vec{x} = \lambda \vec{x}$$

eigenvalue

eigenvector.

Ex. $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

one eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ w. $\lambda = 3$.

another " " $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ w. $\lambda = -1$.

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = (-1) \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ in $\mathbb{R}^2$.

$\lambda_1 \leftarrow \vec{b_1}$ $\quad$ $\vec{b_2} \rightarrow \lambda_2$

Any vector $\vec{x} \in \mathbb{R}^2$

$$\vec{x} = c_1 \vec{b_1} + c_2 \vec{b_2}, \quad c_1, c_2 \in \mathbb{R}.$$

$$A\vec{x} = c_1 \underbrace{\left(A\vec{b_1}\right)} + c_2 \left(A\vec{b_2}\right)$$

$$= c_1 \lambda_1 \vec{b_1} + c_2 \lambda_2 \vec{b_2}$$

$$A^{1000}\vec{x} = c_1\lambda^{1000}\vec{b_1} + c_2\lambda_2^{1000}\vec{b_2}$$

$$A\vec{b_1} = \lambda_1\vec{b_1}, \quad A\vec{b_2} = \lambda_2\vec{b_2}$$

$$V = \begin{bmatrix} \vec{b_1} & \vec{b_2} \end{bmatrix} \in \mathbb{R}^{2\times 2}. \quad D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

(check) $AV = VD$

V is invertible : V^{-1} exists

$$A = VDV^{-1}$$ → diagonalization

$$\Longleftrightarrow \quad D = V^{-1}AV$$

diagonal. general