Lec. 31 Warm up.

Ex. Solve $y'' - 4y' + 5y = 0$

aux. eq. $r^2 - 4r + 5 = 0$

$$r = \frac{4 \pm \sqrt{16-20}}{2} = 2 \pm i$$

$$\left(\frac{d}{dt} - r_1 I\right)\left(\frac{d}{dt} - r_2 I\right) y = 0$$

$$\begin{cases} y_1(t) = e^{r_1 t} \\ y_2(t) = e^{r_2 t} \end{cases}$$

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

$$e^{(2+i)t} = e^{2t} e^{it}$$

Euler formula

$$e^{it} = \cos t + i \sin t$$

$$e^{(\alpha + i\beta)t} = e^{\alpha t} (\cos \beta t + i \sin \beta t)$$

$\alpha, \beta \in \mathbb{R}$

Rewrite general sol

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}, \quad c_1, c_2 \in \mathbb{C}$$

$r_1 = 2+i$

$r_2 = 2-i$

$$= c_1 e^{2t}(\cos t + i \sin t)$$
$$+ c_2 e^{2t}(\cos t - i \sin t)$$

$$= \underbrace{(c_1 + c_2)}_{D_1} e^{2t}\cos t + \underbrace{(i c_1 - i c_2)}_{D_2} e^{2t}\sin t$$

real

If the initial condition

$$y(0),\ y'(0) \in \mathbb{R} \Rightarrow D_1, D_2 \in \mathbb{R}$$

Ex. $\begin{cases} y'' - 4y' + 5y = 0 \\ y(0) = 3 \\ y'(0) = -4 \end{cases}$

Method I. complex basis

$$y(t) = c_1 e^{(2+i)t} + c_2 e^{(2-i)t}$$

$$\begin{cases} y(0) = c_1 + c_2 = 3 \\ y'(0) = (2+i)c_1 + (2-i)c_2 = -4 \end{cases}$$

$$\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} + 5i \\ \frac{3}{2} - 5i \end{bmatrix}$$

$$y(t) = 3e^{2t}\cos t - 10e^{2t}\sin t \qquad \underline{\text{real sol}}$$

Method II. real basis

$$y(t) = D_1 e^{2t}\cos t + D_2 e^{2t}\sin t$$

$$y(0) = D_1 = 3$$

$$y'(0) = 2D_1 + D_2 = -4$$

$$\Rightarrow \begin{cases} D_1 = 3 \\ D_2 = -10 \end{cases}$$

same sol. but much simpler

Recommended approach.

Review. $y'' + by' + cy = 0$. $b, c \in \mathbb{R}$

aux. $r^2 + br + c = 0 \Rightarrow r_{1,2} = \dfrac{-b \pm \sqrt{b^2 - 4c}}{2}$

① $b^2 > 4c$, r_1, r_2 2 distinct real roots

$$e^{r_1 t}, \; e^{r_2 t}$$

② $b^2 = 4c$, r_1, r_2 2 repeated real roots

$$e^{r_1 t}, t e^{r_1 t}$$

③ $b^2 < 4c$, r_1, r_2 2 complex roots conjugate of each other

$r_1 = \overline{r_2} = \alpha + i\beta$, $\alpha, \beta \in \mathbb{R}$

(a) $e^{r_1 t}, e^{r_2 t}$

(b) $e^{\alpha t} \cos \beta t, e^{\alpha t} \sin \beta t$

Linear dependence

V : vector space of functions
infinitely differentiable.

$\vec{y}_1, \vec{y}_2 \in V$. lin. indep.

$c_1 \vec{y}_1 + c_2 \vec{y}_2 = 0 \Rightarrow c_1 = c_2 = 0$.

$\Leftrightarrow$ there does not exist α s.t.

$$\vec{y}_2 = \alpha \vec{y}_1$$

$y_1(t)$, $y_2(t)$ depend on t

<u>Def</u> (Wronskian) $y_1(t), y_2(t) \in V$

$$W[y_1, y_2](t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} \in V$$

Use of Wronskian: IVP.

$y_1(t), y_2(t)$ 2 lin. indep. sol. to

$$\begin{cases} y'' + by' + cy = 0 \\ y(t_0) = A \\ y'(t_0) = B \end{cases}$$

$$y(t) = C_1 y_1(t) + C_2 y_2(t)$$

$$\Rightarrow \begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} A \\ B \end{bmatrix}$$

Solve for any A, B

$\Longleftrightarrow W[y_1, y_2](t_0) \neq 0$

Next Q: relation between

$W[y_1, y_2](t_1)$, $W[y_1, y_2](t_2)$

$t_1 \neq t_2$.

Ex. $y_1(t) = e^{(2+i)t}$, $y_2(t) = e^{(2-i)t}$

$$W[y_1, y_2](t) = \begin{vmatrix} e^{(2+i)t} & e^{(2-i)t} \\ (2+i)e^{(2+i)t} & (2-i)e^{(2-i)t} \end{vmatrix}$$

$$= (2-i)e^{4t} - (2+i)e^{4t}$$

$$= -2i e^{4t} \neq 0$$

for ANY $t \in \mathbb{R}$.

Lemma (Wronskian).

$y_1(t)$, $y_2(t)$ lin. indep. sol.

$$y'' + by' + c = 0.$$

$\Longleftrightarrow$ $\begin{bmatrix} y_1(t_0) \\ y'(t_0) \end{bmatrix}$, $\begin{bmatrix} y_1(t_0) \\ y_2'(t_0) \end{bmatrix}$ are lin. indep. column vectors at some point $t_0 \in \mathbb{R}$. $W[y_1, y_2](t_0) \neq 0$.

$\Longleftrightarrow \quad W[y_1, y_2](t) \neq 0. \quad \forall t \in \mathbb{R}.$

(HW).

Inhom. eq.

Ex. $y''(t) + y(t) = t$. Find sol.

$y(t) = t$ is a sol.

$y''(t) + y(t) = 0.$

$r^2+1=0$. $\quad r=\pm i$

$$y(t) = c_1 \cos t + c_2 \sin t + t$$

$$y'' + by' + cy = g(t)$$

I. $g(t) = t^{l}$, $l \in \mathbb{N}$.

"educated guess"

$$y_p(t) = A_m t^m + A_{m-1} t^{m-1} + \cdots + A_1 t + A_0$$

$$\left.\begin{array}{l} A_0, \cdots, A_m \\ \text{relation } l, m \end{array}\right\} \text{to be determined.}$$

Method of undetermined coeffients

Linear algebra.

$y_p \in V$: inf-dim space.

$\downarrow$

finite-dim representation.

$$\mathcal{B} = \{t^m, \cdots, t, 1\} \subset \mathbb{P}_m$$

$W = \mathbb{P}_m$. $\dim W = m+1$.

$$T : \frac{d^2}{dt^2} + b\frac{d}{dt} + cI$$

$V \longrightarrow V$ $\qquad \dim V = \infty$

Restrict T to $W \longrightarrow W$

$$\begin{tikzcd} T: W \arrow[r] & W \arrow[d, "P_{\mathcal{B}}^{-1}"] \\ \mathcal{B}: \mathbb{R}^{m+1} \arrow[u, "P_{\mathcal{B}}"] \arrow[r, "{[T]_{\mathcal{B}}}"] & \mathbb{R}^{m+1} \end{tikzcd}$$

finite dim!

basis: $\mathcal{B}$

$$T\vec{y} = \vec{b} \leftarrow t^{\ell}$$

$$[T]_{\mathcal{B}}[\vec{y}]_{\mathcal{B}} = [\vec{b}]_{\mathcal{B}}$$

Solve finite dim lin. sys. Row reduction!

$\underline{Ex}$. $y'' + 2y' + y = t^2$

Pick $W = \mathbb{P}_2$ $\lambda = 2$, $m = 2$.

$\mathcal{B} = \{t^2, t, 1\}$. $[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix}$

$\downarrow$

lower triangular

$T(t^2) = 2 + 4t + t^2$

$T(t) = 2 + t$

$T(1) = 1$

$$[t^2]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 4 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 \end{array}\right] \longrightarrow [\vec{y}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -4 \\ 6 \end{bmatrix}$$

particular sol $\vec{y}_p = t^2 - 4t + 6$ (exer)

$$y'' + 2y' + y = 0 \quad . \quad r = -1$$

$$y_1(t) = e^{-t} \, , \quad y_2(t) = e^{-t} \cdot t$$

All sols: $y(t) = c_1 e^{-t} + c_2 t e^{-t}$

$+ (t^2 - 4t + 6 \qquad)$