

Lec 14. Warmup

$$V = \mathbb{R}^{2 \times 2}$$

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad A_3 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}. \quad \text{Are } \{A_1, \dots, A_4\} \text{ lin. indep. ?}$$

method 1.

$$x_1 A_1 + \dots + x_4 A_4 = 0 \quad \text{has nontrivial sol?}$$

(exer for details)

method 2. Find a basis for V

$$e_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, e_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A_1 = 1 \cdot e_1 - e_4 = [e_1 \ e_2 \ e_3 \ e_4] \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

Similarly for A_2, A_3, A_4

coord.

$$[A_1 \ A_2 \ A_3 \ A_4] = [e_1 \ e_2 \ e_3 \ e_4] \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

$$0 = [A_1 \ A_2 \ A_3 \ A_4] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$= [e_1 \ e_2 \ e_3 \ e_4] \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

↓
lin. indep.

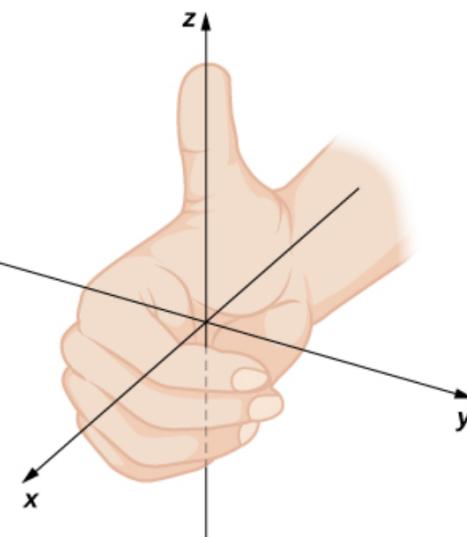
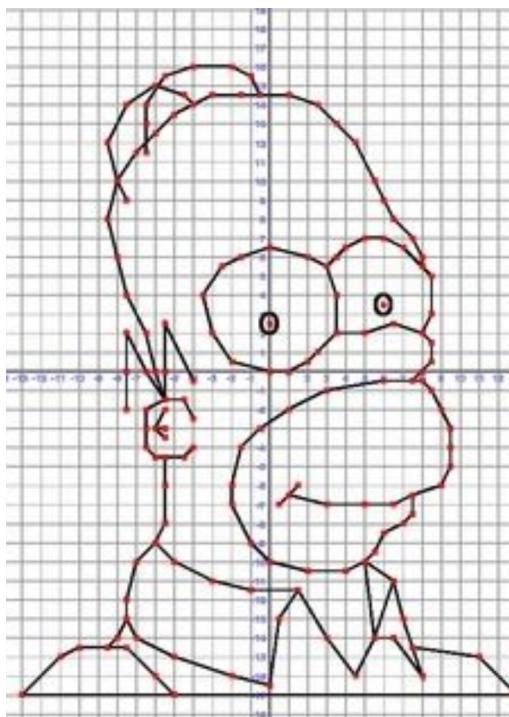
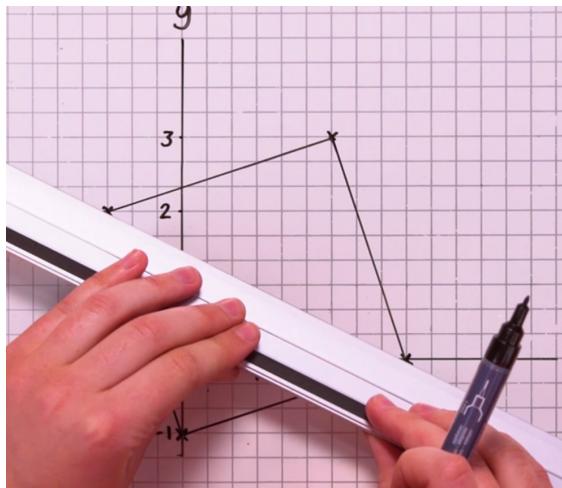
REF

"

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

free var. \Rightarrow lin. dep.

What is coordinate



To go beyond 2d/3d Euclidean space

Conceptualize

Ex. $V =$ collection of all possible ducks
differing only in width

 \vec{v}_1  \vec{v}_2  \vec{v}_3

...

Pick one duck $\vec{b} \in \vec{V}$, $\vec{b} \neq \vec{0}$

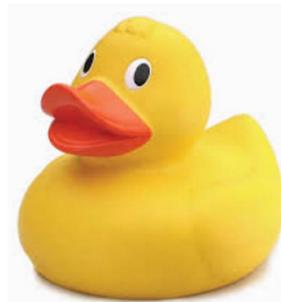


\vec{b}

For all other $\vec{v} \in V$, $\vec{v} = a\vec{b}$, $a \in \mathbb{R}$



$$= 1.8$$



$$\vec{v} = \underbrace{1.8}_{\text{coord.}} \vec{b}$$

coord.

Lin trans:

$$P_B : \mathbb{R} \rightarrow V$$
$$a \mapsto a\vec{b}$$
$$B = \{\vec{b}\}.$$

① one -to- one } $\rightarrow "P_B^{-1}"$ exists.
 ② onto

$$P_B^{-1} : V \rightarrow \mathbb{R}$$

$$\vec{v} \mapsto [\vec{r}]_B \in \mathbb{R}$$



Coordinate of \vec{v} w.r.t. B .

\equiv B - coordinate of \vec{v}

Ex. V = collection of all possible ducks
differing only by height & width



...



Check .

$B = \{\vec{b}_1, \vec{b}_2\}$ is a basis for V



\vec{b}_1



\vec{b}_2

$$P_B : \mathbb{R}^n \rightarrow V$$

$$P_B \begin{pmatrix} h \\ w \end{pmatrix} = h \vec{b}_1 + w \vec{b}_2$$

$$P_B^{-1} : V \rightarrow \mathbb{R}^n$$

$$\vec{v} \mapsto \begin{pmatrix} h \\ w \end{pmatrix} := [\vec{v}]_B$$

Coord is nothing but a special lin. trans.

where W is always \mathbb{R}^n . $n = \dim V$

Ex. $V = \mathbb{P}_2$ Find. coord of

$$f(x) = x^2 + 3x + 2 \quad \text{w.r.t. basis.}$$

1) $\mathcal{B}_1 = \{1, x, x^2\},$

2) $\mathcal{B}_2 = \{x, 1, 1-x^2\} := \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$

1) $[f]_{\mathcal{B}_1} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$

2) want $f = a_1 \vec{b}_1 + a_2 \vec{b}_2 + a_3 \vec{b}_3$

$$[\vec{b}_1]_{\mathcal{B}_1} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad [\vec{b}_2]_{\mathcal{B}_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad [\vec{b}_3]_{\mathcal{B}_1} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$[I \times X^2] [f]_{\mathcal{B}_1} = [I \times X^2] \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 1 & 1 & 1 & 2 \\ 1 & 0 & 0 & 1 & 3 \\ 0 & 0 & -1 & 1 & 1 \end{bmatrix} \xrightarrow{\text{sol}} [f]_{\mathcal{B}_2}.$$

Ex. Find coord $\vec{v} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ $V = \mathbb{R}^2$

w.r.t. basis

$$\mathcal{B} = \left\{ \vec{b}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}.$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 & -3 \\ 1 & -1 & 1 & 2 \end{bmatrix} \xrightarrow{\text{sol}} [\vec{v}]_{\mathcal{B}}.$$

generalization of matrix inverse.

Def Lin. trans. $T: V \rightarrow W$ is

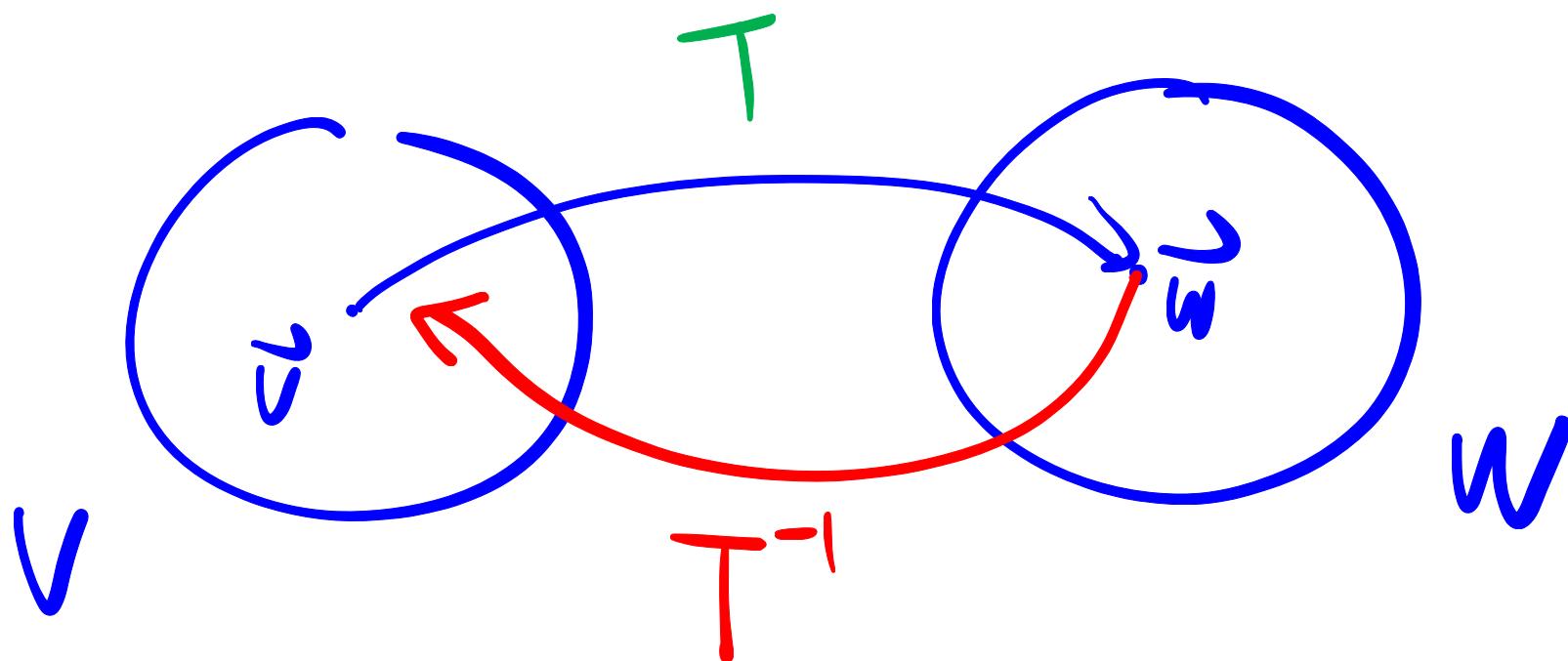
called an **isomorphism** if there
is lin. trans. called

$T^{-1}: W \rightarrow V$ s.t.

$T(T^{-1}(\vec{w})) = \vec{w}$, for any $\vec{w} \in W$

$T^{-1}(T(\vec{v})) = \vec{v}$, " " $\vec{v} \in V$

iso morphism \hookrightarrow bijective.
↓
equal Shape



Thm : Finite dim vector space V

has a basis $B = [\vec{b}_1, \dots, \vec{b}_n]$

Then

$$P_B \left(\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \right) = a_1 \vec{b}_1 + \dots + a_n \vec{b}_n$$

is an isomorphism. and $\underbrace{P_B^{-1}}_{\downarrow}$ exists.

Pf: exer.

define coord.

Ex. Is T an isomorphism?

$$T: P_2 \rightarrow \mathbb{R}^3,$$

$$P(x) \mapsto \begin{bmatrix} P(0) \\ P(1) \\ P(2) \end{bmatrix}$$

$$P(x) = a_0 + a_1 x + a_2 x^2$$

$$T(P) = \begin{bmatrix} a_0 \\ a_0 + a_1 + a_2 \\ a_0 + 2a_1 + 4a_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

↓ REF

every row/col has pivot ✓