Lec 23. Projection.

least squares problem

$$\text{``} A\vec{x}=\vec{b}\text{''}$$

$$\left[\,A\,\right]\vec{x}=\vec{b}$$
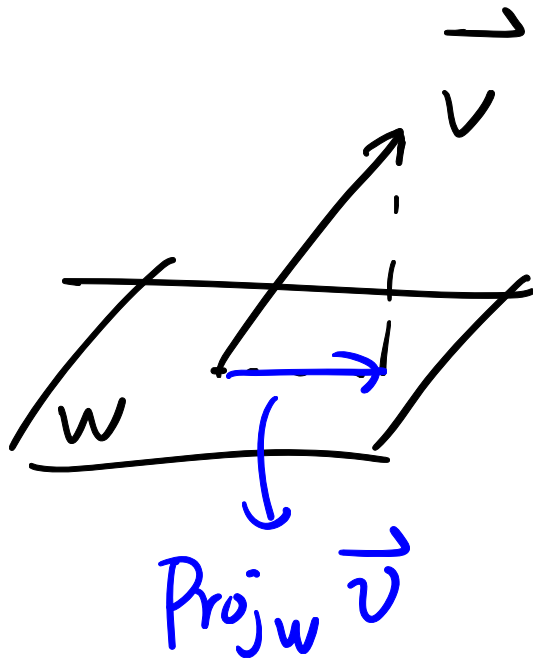
Col(A)

Ex. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad \vec{b} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$

$\underbrace{\phantom{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}}}$
orthogonal set

Compute $\text{Proj}_{\text{Col}(A)} \vec{b}$

$$= \frac{\vec{a_1} \cdot \vec{b}}{\vec{a_1} \cdot \vec{a_1}} \, \vec{a_1} + \frac{\vec{a_2} \cdot \vec{b}}{\vec{a_2} \cdot \vec{a_2}} \, \vec{a_2}$$

$$= \frac{-1}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \frac{0}{1} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= -\frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$U \in \mathbb{R}^{n \times k}.$$

$$U = \begin{bmatrix} \vec{u_1} & \cdots & \vec{u_k} \end{bmatrix} \quad U \text{ has } \textcolor{blue}{\text{orthonormal columns}}$$

if $\quad \vec{u_i} \cdot \vec{u_j} = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j. \end{cases}$

i.e., $\{ \vec{u_1}, \cdots, \vec{u_k} \}$ is an orthonormal set.

$$\underbrace{\left(\textcolor{blue}{U^T U} \right)}_{\underset{\mathbb{R}^{k \times k}}{\uparrow}}{}_{ij} = \vec{u_i}^T \vec{u_j} = \textcolor{blue}{\vec{u_i} \cdot \vec{u_j}}$$

$$\overbrace{\underbrace{U^T U = I_K}}.$$

Special case: $\boxed{k = n}$.

$$U^T U = I_n \quad . \quad U \in \mathbb{R}^{n \times n}.$$

$\{\vec{u}_1, \cdots, \vec{u}_n\}$ is **ONB** of $\mathbb{R}^n$.

$$U^{-1} = U^T .$$

special when $k = n$.

$$n \left[\begin{array}{c} k \\ U \end{array} \right] \quad : \qquad \boxed{U^T} \; \boxed{U} = \boxed{I_k}$$

$$\boxed{U} \; \boxed{U^T}$$

$\underrightarrow{\quad} \to$ projection.

$$n \left[\begin{array}{c} n \\ U \end{array} \right] \qquad U U^T = U U^{-1} = I_n .$$

$U \in \mathbb{R}^{n \times n}$. ONB.

orthogonal matrix

① orthonormal "

Confusing notation!

Connect to projection.

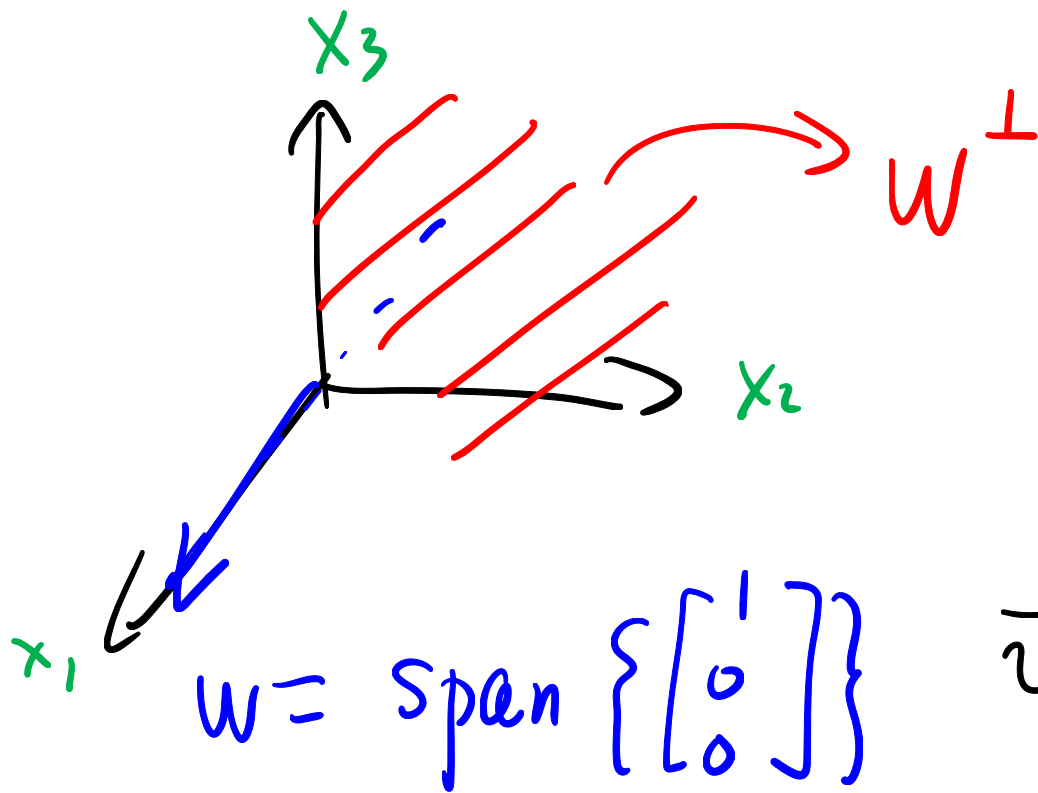$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

$U = [\vec{u_1}, \vec{u_2}] = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 \\ \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$

$$\text{Proj}_{\text{Col}(A)} \vec{b} = \vec{u_1}(\vec{u_1} \cdot \vec{b}) + \vec{u_2}(\vec{u_2} \cdot \vec{b})$$

$$= (\vec{u_1}\vec{u_1}^T + \vec{u_2}\vec{u_2}^T)\vec{b}$$

$$= U U^T \vec{b}$$

$$\left(U U^T = [\vec{u_1} \ \vec{u_2}] \begin{bmatrix} \vec{u_1}^T \\ \vec{u_2}^T \end{bmatrix} = \vec{u_1}\vec{u_1}^T + \vec{u_2}\vec{u_2}^T \right)$$

Def $W \subseteq \mathbb{R}^n$ is a subspace.

$$W^{\perp} = \{ \vec{v} \in \mathbb{R}^n \mid \vec{v} \perp \vec{w} \text{ for all } \vec{w} \in W \}$$

$W = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

$\vec{v} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 0 \iff v_1 = 0.$

Thm. 1) $W^\perp$ is a subspace

2) $(W^\perp)^\perp = W$

Thm. (Uniqueness of projection)

$W \subseteq \mathbb{R}^n$ is a subspace. any vector $\vec{v} \in \mathbb{R}^n$ has a unique decomposition

$$\vec{v} = \vec{w} + \vec{z}, \quad \vec{w} \in W, \quad \vec{z} \in W^\perp.$$

$$\left(\vec{w} = \text{Proj}_W \vec{v} \right)$$

Pf: (of uniqueness).

$$\vec{v} = \vec{w} + \vec{z} = \vec{w}\,' + \vec{z}\,'$$

$$\vec{w}, \vec{w}\,' \in W$$

$$\vec{z}, \vec{z}\,' \in W^{\perp}$$

$$\Rightarrow \quad \vec{w} - \vec{w}\,' = \vec{z}\,' - \vec{z}$$

$$(\vec{w} - \vec{w}\,') \cdot (\vec{w} - \vec{w}\,') = 0$$

$$\underbrace{\phantom{(\vec{w} - \vec{w}\,') \cdot (\vec{w} - \vec{w}\,')}}_{\|\vec{w} - \vec{w}\,'\|^2}$$

$$\Rightarrow \vec{w} = \vec{w}\,', \quad \vec{z} = \vec{z}\,'. \text{ Therefore}$$

uniqueness

$\square$

Ex. $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ Find $\text{Col}(A)^{\perp}$

$\text{Col}(A)^{\perp} = \left\{ \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \;\middle|\; \vec{a_1} \cdot \vec{x} = 0, \; \vec{a_2} \cdot \vec{x} = 0 \right\}$

$\begin{aligned} x_1 + x_2 &= 0 \\ 2x_1 + 2x_2 &= 0 \end{aligned} \implies \begin{bmatrix} 1 & 1 & | & 0 \\ 2 & 2 & | & 0 \end{bmatrix}$

$\implies$ sol set is $\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} c \;\middle|\; c \in \mathbb{R} \right\}$,

$= \text{span} \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$.

More generally.

$$A = \begin{bmatrix} \vec{a_1} & \cdots & \vec{a_k} \end{bmatrix} \quad Col(A)^\perp$$

$$\begin{cases} \vec{a_1} \cdot \vec{x} = 0 \\ \vdots \\ \vec{a_k} \cdot \vec{x} = 0 \end{cases} \Rightarrow \begin{cases} \vec{a_1}^T \vec{x} = 0 \\ \vdots \\ \vec{a_k}^T \vec{x} = 0 \end{cases} \Rightarrow A^T \vec{x} = \vec{0}$$

$$Col(A)^\perp = Nul(A^T)$$

Least squares
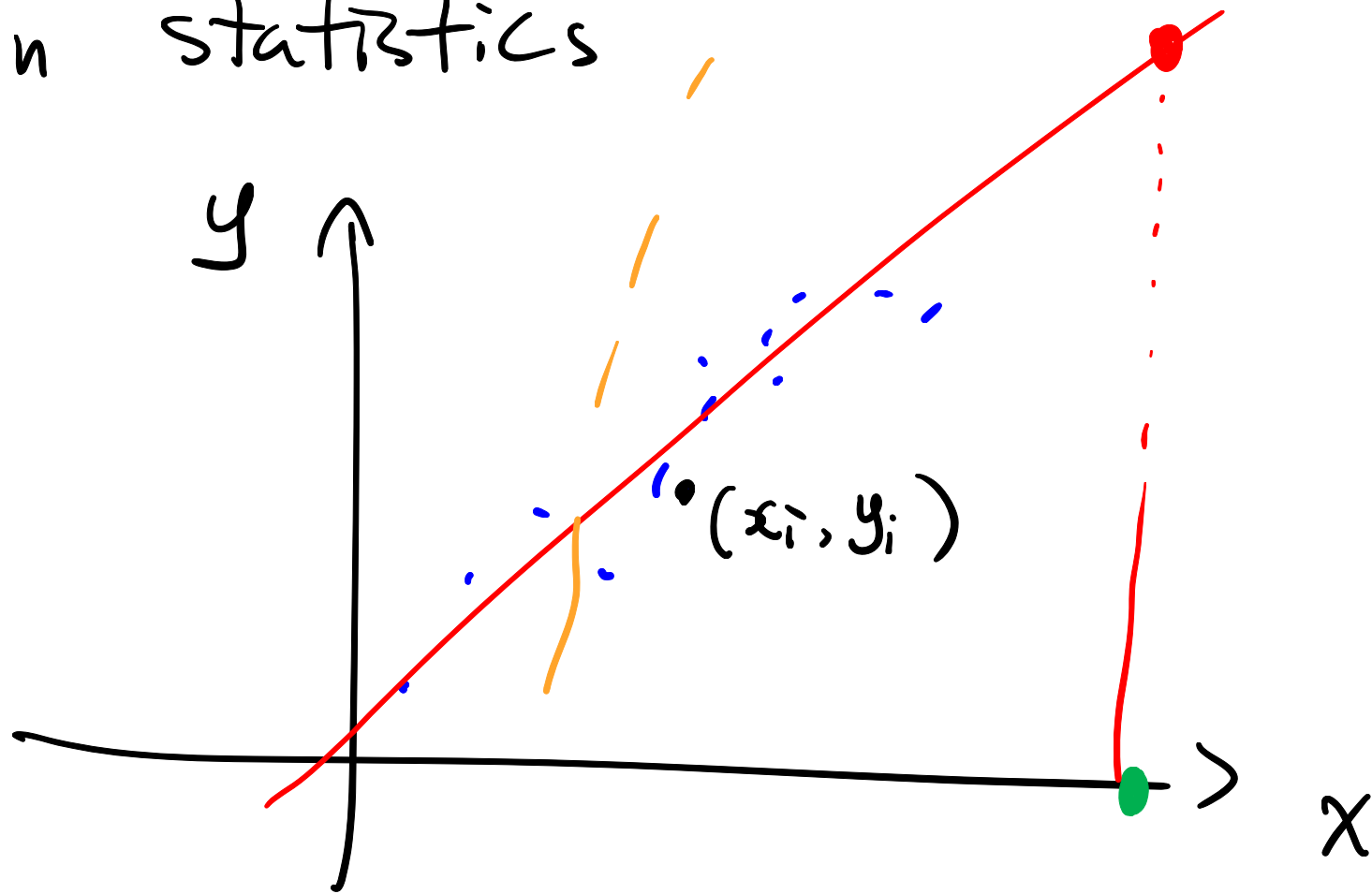
$$A\vec{x} = \vec{b} \qquad \text{has no sol.}$$

$$A \in \mathbb{R}^{m \times n}, \quad m > n$$

$$m \overbrace{\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}}^{n} \begin{bmatrix} x \end{bmatrix} = m \begin{bmatrix} b \end{bmatrix}$$

In statistics

y

(x_i, y_i)
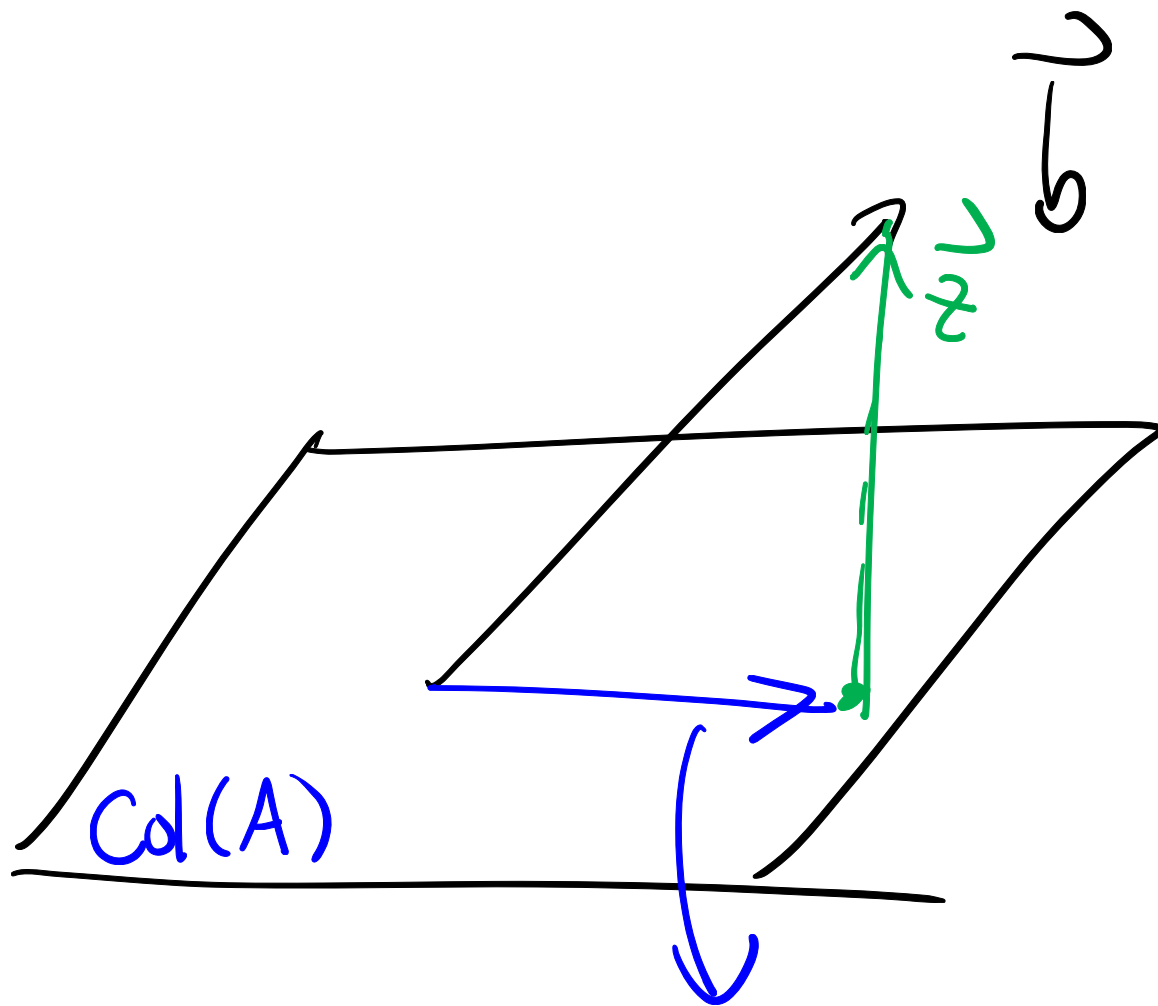
x

observations (x_i, y_i)

$$a x_i + b = y_i, \quad i = 1, \dots, m$$

unknowns are a, b.

$$\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_m & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

$$A \vec{x} = \vec{b}$$

$$\Longleftrightarrow \quad \min_{\vec{x}} \| A\vec{x} - \vec{b} \|^2$$

$\longrightarrow$ residual

$$\text{Proj}_{\text{Col}(A)}\vec{b} = A\vec{x}$$

minimal distance $\quad \| \vec{z} \| = \| \vec{b} - \text{Proj}_{\text{Col}(A)}\vec{b} \|.$