Lec 32.

Ex. $y'' + 3y' + 2y = 20e^{3t}$

Try $y_p(t) = C e^{3t}$

$y_p'(t) = 3Ce^{3t}$

$y_p''(t) = 9Ce^{3t}$

$\rightarrow e^{3t}(9C + 9C + 2C) = 20e^{3t}$

$\Rightarrow c = 1. \quad y_p(t) = e^{3t}$

$$y(t) = c_1 e^{-t} + c_2 e^{-2t} + e^{3t}$$

Ex. $\underbrace{y'' + 3y' + 2y}_{T(y)} = e^{-2t}$

Try $y(t) = c\, e^{-2t}$

$T(e^{-2t}) = 0.$ RHS $\in Null(T)$

"ϵ - perturbation"

$$y_\epsilon'' + 3y_\epsilon' + 2y_\epsilon = e^{-(2+\epsilon)t} \notin Null(T)$$

$$y_\epsilon(t) = c_\epsilon e^{-(2+\epsilon)t}$$

$$c_\epsilon \cancel{e^{-(2+\epsilon)t}} \Big[\underbrace{(2+\epsilon)^2 - 3(2+\epsilon) + 2}_{\epsilon + \epsilon^2} \Big] = \cancel{e^{-(2+\epsilon)t}}$$

$$c_\epsilon = \frac{1}{\epsilon + \epsilon^2}$$

$$y_\epsilon(t) = \frac{1}{\epsilon+\epsilon^2}\left[e^{-(2+\epsilon)t} - \underbrace{e^{-2t}}_{\in \, Null(T)}\right]$$

$$\lim_{\epsilon\to 0^+} y_\epsilon(t) = -t e^{-2t}$$

$$e^{-(2+\epsilon)t} = e^{-2t}\cdot e^{-\epsilon t} = e^{-2t}(1-\epsilon t + O(\epsilon^2))$$

Make an "educated guess"

$$y_p(t) = C t e^{-2t}$$

Check:

$$y_p'(t) = c\left(e^{-2t} - 2te^{-2t}\right)$$

$$y_p''(t) = c\left(-4e^{-2t} + 4te^{-2t}\right)$$

$$y_p'' + 3y_p' + 2y_p = ce^{-2t}\left(-4 + 3 \qquad\right)$$

$$+ cte^{-2t}\left(4 - 6 + 2 \quad\right)$$

$$= -ce^{-2t}$$

$$\Rightarrow c = -1. \quad \text{or} \quad \boxed{y_p(t) = -te^{-2t}}$$

Ex. $y'' - 2y' + y = e^t$

aux. eq. $(r-1)^2 = 0 \Rightarrow r_1 = r_2 = 1$

2 repeated roots.

$T(e^t) = 0$, $T(te^t) = 0$

$y_p(t) = c\, t^2 e^t$

$y_p'(t) = c\left(2t e^t + t^2 e^t\right)$

$y_p''(t) = c\left(2e^t + 4te^t + t^2 e^t\right)$

$$y_p'' - 2y_p' + y_p = ce^t\left(2 - 0 + 0 \qquad\right)$$

$$+ c\cancel{e^t \cdot t}\left(4 - 4 + 0 \qquad\right)$$

$$+ c\cancel{e^t t^2}\left(1 - 2 + 1 \qquad\right)$$

$$= 2ce^t = e^t$$

$$\Rightarrow c = \frac{1}{2}. \quad y_p(t) = \frac{1}{2}t^2 e^t$$

Thm $y'' + by' + cy = e^{rt}$ $r^2 + br + c = 0$

r	$y_p(t)$	W
not a root	ce^{rt}	$\text{span}\{e^{rt}\}$
single root	$c\,te^{rt}$	$\text{span}\{te^{rt}, e^{rt}\}$
double root	ct^2e^{rt}	$\text{span}\{t^2e^{rt}, te^{rt}, e^{rt}\}$

Linear algebra perspective.

$T: V \longrightarrow V$ inf-dim.

$W \subset V$. $\dim W < \infty$.

$$
\begin{tikzcd}
T: W \arrow[r] & W \arrow[d] \\
\mathcal{B} \quad \mathbb{R}^m \arrow[u] \arrow[r, "{[T]_{\mathcal{B}}}"] & \mathbb{R}^m
\end{tikzcd}
$$

single root. $W = \text{span}\{te^{rt}, e^{rt}\}$.

$$[T]_{\mathcal{B}} = \Big[\, [T(te^{rt})]_{\mathcal{B}} \quad \underset{\underset{0}{\parallel}}{[T(e^{rt})]_{\mathcal{B}}} \,\Big]$$

$$= \begin{bmatrix} 0 & 0 \\ T_{2,1} & 0 \end{bmatrix} \in \mathbb{R}^{2\times 2}$$

$$[e^{rt}]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

particular.

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ T_{2,1} & 0 & 1 \end{array}\right] \xrightarrow{\text{sol}} [y_p]_{\mathcal{B}} = \begin{bmatrix} \frac{1}{T_{2,1}} \\ 0 \end{bmatrix}$$

$$+ c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

e^{rt}. Null space

double root

$$W = \operatorname{span}\{ t^2 e^{rt}, \underbrace{t e^{rt}, e^{rt}}_{\substack{\pitchfork \\ Null(T)}} \}$$

$$[T]_{\mathcal{B}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ T_{3,1} & 0 & 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ T_{3,1} & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{sol}} [y_p]_{\mathcal{B}} = \begin{bmatrix} \frac{1}{T_{3,1}} \\ 0 \\ 0 \end{bmatrix}$$

Superposition principle
(a.k.a. linear combination)

Ex. $y'' + by' + cy = c_1 f(t) + c_2 g(t)$

$f(t) = t^3, \quad g(t) = e^t$

Sol. $\begin{cases} y_1'' + by_1' + cy_1 = f(t) \\ y_2'' + by_2' + cy_2 = g(t) \end{cases}$

$$y(t) = c_1 y_1(t) + c_2 y_2(t).$$

$$Ty = c_1 T(y_1) + c_2 T(y_2) \longrightarrow \text{linearity}$$

$$= c_1 f(t) + c_2 g(t) \qquad \square$$

Ex. $y'' = t e^t$

method of undeter. coeff.

"educated guess"

$$y(t) = (A_1 t + A_0) e^t$$

$$W = \text{span}\{te^t, e^t\}.$$

$$[T]_{\mathcal{B}} = \left[\ [T(te^t)]_{\mathcal{B}} \quad [T(e^t)]_{\mathcal{B}}\ \right]$$

$$= \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$T(te^t) = (te^t)'' = (e^t + te^t)' = 2e^t + te^t$$

$$T(e^t) = e^t$$

$$[te^t]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 2 & 1 & 0 \end{array}\right] \xrightarrow{\text{sol}} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$y_p(t) = te^t - 2e^t$$

More general form.

$$y'' + by' + cy = t^l e^{rt}$$

"educated guess"

$$y_p(t) = (A_m t^m + A_{m-1} t^{m-1} + \cdots + A_0) e^{rt}$$

Q: ① what is m? → what is W?

② once W is obtained. → linear alg.

how to find A_i's.

$$[T]_B [\vec{y}_p]_B = [\vec{b}]_B$$

$$W = \operatorname{span}\{t^m e^{rt}, t^{m-1} e^{rt}, \cdots, e^{rt}\}.$$