Lec 39.

Inhom. heat. eq.

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}, & x \in [0,1] \\ u(0,t) = 1, \; u(1,t) = 2 \\ u(x,0) = f(x). \end{cases}$$

$A\vec{x} = \vec{b}$. $\quad \underbrace{\vec{x} = \vec{x}_p + \vec{w}}$, $A\vec{w} = 0$

$$u(x,t) = u_p(x,t) + w(x,t)$$

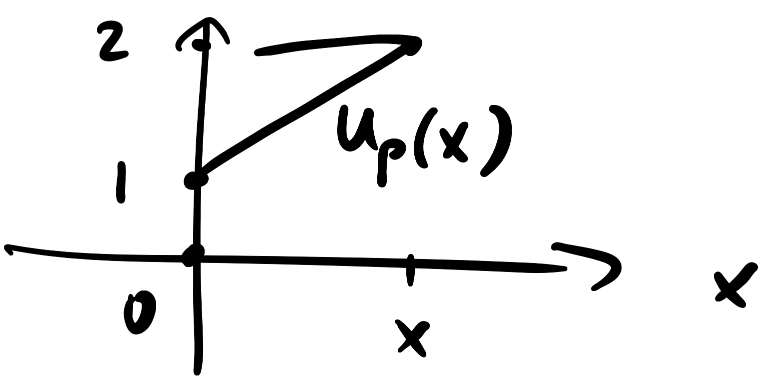
hom. heat.

$$u_p(x,t) = u_p(x) \longrightarrow \text{independent of } t$$

$$u(x,t) = u_p(x) + w(x,t)$$

$$\begin{cases} 0 = \frac{\partial^2}{\partial x^2} u_p(x) \equiv u_p''(x) \\ u_p(0) = 1 \ , \ u_p(1) = 2 \end{cases} \longrightarrow \text{ODE}$$

$$u_p(x) = C_1 + C_2 x \Rightarrow u_p(x) = 1 + x$$



$$\frac{\partial u}{\partial t} = \frac{\partial w}{\partial t}$$

$$\frac{\partial^2 u}{\partial x^2} = \cancel{\frac{\partial^2 u_p}{\partial x^2}} + \frac{\partial^2 w}{\partial x^2}$$

$$0 = \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \underbrace{\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2}}_{\text{satisfy heat eq.}}$$

$$u(0,t) = u_p(0) + w(0,t)$$

$$\Rightarrow w(0,t) = 0 = \underbrace{w(1,t)}_{\text{hom.}}$$

$$u(x,0) = u_p(x) + w(x,0)$$

$$\Rightarrow w(x,0) = f(x) - u_p(x)$$

$$\begin{cases} \dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} \\ w(0,t) = w(1,t) = 0 \\ w(x,0) = f(x) - u_p(x) \end{cases}$$

$f(x) - u_p(x) \longrightarrow$ Fourier expansion

$$\lim_{t \to \infty} u(x,t) = u_p(x)$$

$u_p(x) \longleftarrow$ steady state

Review

What have we learned?

Part I. Lin. Alg.

$$T: V \longrightarrow W \qquad \dim V = n$$

$$\dim W = m$$

$$n, m < \infty$$

$$T\vec{x} = \vec{b}$$

$\vec{x} \in V$, $\vec{b} \in W$. sol set?

① at least one sol. $\Leftrightarrow \vec{b} \in \text{Image}(T)$

$$\text{Null}(T) = \{ \vec{x} \mid T\vec{x} = \vec{0} \}.$$

$$\text{Sol set} = \{ \vec{x_p} + \vec{w} \mid \vec{w} \in \text{Null}(T),\ T\vec{x_p} = \vec{b} \}$$

particular sol.

② at most one sol. $\Leftrightarrow \text{Null}(T) = \{\vec{0}\}$

③ unique sol $\Leftrightarrow \text{Null}(T) = \{\vec{0}\}$ & $\vec{b} \in \text{Image}(T)$

$T\vec{x} = \vec{b}$ always has unique sol?

$\mathrm{Null}(T) = \{\vec{0}\}$ & $W = \mathrm{Image}(T)$

one-to-one onto

$\dim V = \dim W$

more generally rank theorem.

This is a bit abstract!

Computation $\longrightarrow$ need a basis

$$\mathcal{B} = \{\vec{b}_1, \cdots, \vec{b}_n\} \subset V$$

$$\mathcal{C} = \{\vec{c}_1, \cdots, \vec{c}_m\} \subset W$$

$$
\begin{tikzcd}
T: \quad V \arrow[r] & W \arrow[d, "P_C^{-1}"] \\
\mathbb{R}^n \arrow[u, "P_B"] \arrow[r, "{[T]_B^C}"] & \mathbb{R}^m
\end{tikzcd}
$$

B-coord $\qquad\qquad$ C-coord.

$$[T] \in \mathbb{R}^{m \times n}$$

$$\parallel$$

$$P_C^{-1}\, T\, P_B$$

Change of coord.

$$I : \begin{tikzcd} V \arrow[r] & V \arrow[d, "P_C^{-1}"] \\ \mathbb{R}^n \arrow[u, "P_B"] \arrow[r, "{[I]}"] & \mathbb{R}^n \end{tikzcd}$$

$\mathcal{B}$ $\qquad\qquad\qquad\qquad$ $\mathcal{C}$

$$[I] = \underset{B \to C}{P} = P_C^{-1} P_B \in \mathbb{R}^{n \times n}$$

Sol. $T\vec{x} = \vec{b}$

$$\Updownarrow$$

$$[T][\vec{x}]_{\mathcal{B}} = [\vec{b}]_{\mathcal{C}}$$ sys. lin. eq.

$$\left[\, [T] \;\vdots\; [\vec{b}]_{\mathcal{C}} \,\right] \xrightarrow{\text{REF}} \text{check pivots}$$

$$\xrightarrow{\text{sol}} [\vec{x}]_{\mathcal{B}}$$

$$\vec{x} = P_B([\vec{x}]_B)$$

$$= ([\vec{x}]_B)_1 \vec{b_1} + \cdots + ([\vec{x}]_B)_n \vec{b_n}$$

T is isomorphism

$\Leftrightarrow$ $[T]$ is invertible. $m = n$

$\Leftrightarrow$ $\det([T]) \neq 0$.

Part II. Lin. Alg.

Special lin. transform.

$$T: V \longrightarrow V. \quad \dim V = n < \infty$$

$$\mathcal{B} = \{\vec{b_1}, \cdots, \vec{b_n}\}$$

$$\begin{tikzcd} T: & V \arrow[r] & V \arrow[d, "P_{\mathcal{B}}^{-1}"] \\ \mathcal{B} & \mathbb{R}^n \arrow[u, "P_{\mathcal{B}}"] \arrow[r, "{[T]_{\mathcal{B}}}"] & \mathbb{R}^n & \mathcal{B} \end{tikzcd}$$

$$[T]_{\mathcal{B}} = P_{\mathcal{B}}^{-1} \, T \, P_{\mathcal{B}} \in \mathbb{R}^{n \times n}$$

eigenvalue / diagonalization.

Find $\mathcal{B}$ $\Longrightarrow$ $[T]_{\mathcal{B}} = \begin{bmatrix} \ddots & 0 \\ 0 & \ddots \end{bmatrix}$

$$T\vec{x} = \lambda\vec{x}$$

$$\iff [T]_{\mathcal{B}}[\vec{x}]_{\mathcal{B}} = \lambda[\vec{x}]_{\mathcal{B}}$$

$$A\vec{v} = \lambda\vec{v} \quad \text{or} \quad (A-\lambda I)\vec{v} = 0.$$

$$\vec{v} \neq \vec{0}$$

$$\det(A-\lambda I) = 0$$

1) Find λ

2) Solve $(A-\lambda I)\vec{v} = 0$

A diagonalizable.

n. lin. indep. eig. vecs.

1) All eigenvalues are distinct

2) $A = A^T$ $\begin{cases} \text{a. eigvals real} \\ \text{b. orthogonally diagonalizable} \end{cases}$

$$A = VDV^{-1} \xrightarrow{V^TV = I} A = VDV^T$$

singular value decomp. (SVD)

$A \in \mathbb{R}^{m \times n}$ (WLOG. $m \geq n$

cols of A lin. indep)

$$A = U \Sigma V^T = \tilde{U} \tilde{\Sigma} V^T.$$

"thin" SVD

diagonal, $\tilde{\Sigma} \geq 0$

$$\begin{array}{cc} & n \\ m & \begin{bmatrix} \quad & \quad \\ & \end{bmatrix}\end{array} = \begin{array}{cc} & m \\ m & \begin{bmatrix} \underbrace{\tilde{U}}_{n} & \vdots & * \end{bmatrix}\end{array} \begin{array}{cc} & n \\ m & \begin{bmatrix} \tilde{\Sigma} \\ \hline 0 \end{bmatrix}\end{array} \begin{array}{cc} & n \\ n & \begin{bmatrix} V^T \end{bmatrix}\end{array}$$

$$A^T A = V D V^T = V \tilde{\Sigma}^2 V^T$$

sym

Geometry. (inner prod).

$\langle \vec{u}, \vec{u} \rangle \geq 0$. $\langle \vec{u}, \vec{u} \rangle = 0 \Rightarrow \vec{u} = \vec{0}$

Orthogonality:

$\langle \vec{u}, \vec{v} \rangle = 0$

$\langle \vec{u}+\vec{v}, \vec{u}+\vec{v} \rangle = \langle \vec{u}, \vec{u} \rangle + \langle \vec{v}, \vec{v} \rangle$

$\Leftrightarrow \|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$

Orthogonal basis : simplify
find coord.

$P_B^{-1} : V \to \mathbb{R}^n$ → inner prod.

What if $A\vec{x} = \vec{b}$ has no sol?

Sol in the least-squares sense

Concept : projection. best approx. thm.

Implementation:

$$A^T A \hat{x} = A^T \vec{b}$$

least-squares sol.

How to generate ortho basis?

Gram-Schmidt $\xrightarrow{\text{order}}$ QR.

projection.

Part III. ODEs & PDEs.

V : function space. $\dim V = \infty$

T : differential op.

$T\vec{y}(t) = \vec{0}$. hom. ODE

aux. eq. $\longrightarrow$ factorize T

$\longrightarrow$ first order eq.

repeated root $\longrightarrow$ understand using limiting process.

$T\vec{y}(t) = f(t)$. $f(t) = t^m e^{rt}$

$r \in \mathbb{C}$

$T : W \to W$ $\dim W < \infty$

$f \in W$

$[T]_{\mathcal{B}} [\vec{x}]_{\mathcal{B}} = [\vec{f}]_{\mathcal{B}}$

method of undet. coeffs.

PDE.

$T: V \to V$ boundary.

Separation of variables

$$\frac{d}{dt}\vec{u}(t) = A\vec{u}(t)$$

$$\begin{cases} \mathrm{X}''(x) = \lambda \mathrm{X}(x) \\ \text{boundary} \end{cases} \to \text{ODE.}$$

Need to match initial cond.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right)$$

Fourier expansion. how to compute a_n, b_n? $\longrightarrow$ chap 6

inner prod.


So Long,and Thanks
for All the Fish!