Lec 21. Warm up

True/False:

Is this matrix diagonalizable?

(1) $A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$ $\qquad \begin{vmatrix} 1-\lambda & 0 \\ 1 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (\lambda-1)(\lambda-2)=0$

$\Rightarrow \lambda_1 = 1, \lambda_2 = 2$

check eig.vec.

✓

(2) $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $\qquad \begin{vmatrix} 1-\lambda & 0 \\ 0 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda(1-\lambda)=0$

$\Rightarrow \lambda_1 = 0, \lambda_2 = 1.$

$\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

✓

(3) $A+B = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$ $\qquad \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}$ is not diagonalizable

$a=2$

✗

True / False

There exists an $n \times n$ matrix w. $(n+1)$ eigenvalues.

False.

$$0 = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ & & \ddots & \\ a_{n1} & \cdots & & a_{nn}-\lambda \end{vmatrix} \to P_n(\lambda)$$

λ must be a root of n-deg. poly.

$\Rightarrow$ at most n eigenvalues.

Thm. $A \in \mathbb{R}^{n \times n}$ has n distinct eigenvalues $\lambda_1, \cdots, \lambda_n$. Then A is diagonalizable.

Pf: For λ_i, $1 \le i \le n$.

$A - \lambda_i I_n$ is not invertible

$\Rightarrow \dim \mathrm{Null}(A - \lambda_i I) \neq 0.$

$\vec{v}_i$ a eigenvector.

$\{\vec{v}_1, \cdots, \vec{v}_n\}$ n $\underbrace{\text{lin. indep.}}_{\text{(exer)}}$ vectors forms a basis of $\mathbb{R}^n$

$\to V = [\vec{v}_1, \cdots, \vec{v}_n]$ diagonalize A.

<u>Cor</u>. $A = \begin{bmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ ← only a sufficient condition (think I_n)

$\lambda_1, \cdots, \lambda_n$ are distinct.

$\Rightarrow A$ diagonalizable.

Fact: $A \in \mathbb{R}^{n \times n}$ λ is an eigenvalue.

$$1 \leq \underbrace{\dim \text{Null}(A - \lambda I)}_{\text{geometric multiplicity of eigenvalue } \lambda.} \leq \text{multiplicity of } \lambda \text{ in } |A - \lambda I| = 0.$$

$A - \lambda I$ is singular

geometric multiplicity of eigenvalue λ.

algebraic multiplicity of eigenvalue λ

Ex. 4×4 matrix.

$$\begin{bmatrix} \lambda & & & 0 \\ & \lambda & & \\ & & \lambda & \\ 0 & & & \lambda \end{bmatrix} = \lambda \cdot I_4$$

$$\dim \operatorname{Null}(A - \lambda I) = 4$$

$$\begin{bmatrix} \boxed{\begin{matrix} \lambda & 1 \\ 0 & \lambda \end{matrix}} & 0 \\ 0 & \begin{matrix} \lambda & 0 \\ 0 & \lambda \end{matrix} \end{bmatrix}$$

$$\dim \operatorname{Null}(A - \lambda I) = 3$$

$$\begin{bmatrix} \boxed{\begin{matrix} \lambda & 1 \\ 0 & \lambda \end{matrix}} & 0 \\ 0 & \boxed{\begin{matrix} \lambda & 1 \\ 0 & \lambda \end{matrix}} \end{bmatrix} \text{ or } \begin{bmatrix} \boxed{\begin{matrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{matrix}} & 0 \\ 0 & \lambda \end{bmatrix}$$

$$\dim \operatorname{Null}(A - \lambda I) = 2$$

$$\begin{bmatrix} \lambda & 1 & & 0 \\ & \lambda & 1 & \\ & & \lambda & 1 \\ 0 & & & \lambda \end{bmatrix}$$

$$\dim \operatorname{Null}(A - \lambda I) = 1$$

Ex. $A - \lambda I = \begin{bmatrix} \begin{matrix} 0 & \boxed{1} \\ 0 & 0 \end{matrix} & O \\ O & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \end{bmatrix}$

dim of sol set of $(A-\lambda I)\vec{x} = 0$.

$\Rightarrow$ dim Null $(A-\lambda I) = 3$

Ex. $A \in \mathbb{R}^{6\times 6}$

$$|A-\lambda I| = (1-\lambda)^3 (2-\lambda)\lambda^2$$

possible dim of $\underbrace{\text{eigenspace}}$

span of all eigenvecs?

$\lambda =$ 0 1 2

(algebraic) multiplicity 2 3 1

$1 \leq \dim Null(A) \leq 2$. $1 \leq \dim Null(A-I) \leq 3$

$1 = \dim Null(A-2I)$

$\dim \operatorname{span}\{\text{all eigenvectors}\}$

$3 \quad , \quad 4 \quad , \quad 5, \quad 6$

(exer). create an actual matrix for each case above.

What does diagonalization mean?

$$A\underbrace{[\vec{v}_1, \cdots, \vec{v}_n]}_{V} = [\vec{v}_1, \cdots, \vec{v}_n]\underbrace{\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}}_{D}$$

$$AV = VD \iff D = V^{-1}AV.$$

View from lin. trans.

matrix representation.

$$T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

$$\vec{x} \longmapsto A\vec{x}$$

standard basis $\mathcal{E}$

what if choose $\mathcal{B} = \{\vec{v}_1, \cdots, \vec{v}_n\}$

$$P_{\mathcal{E} \leftarrow \mathcal{B}} = \left[\vec{v}_1, \dots, \vec{v}_n \right] \equiv V$$

General matrix representation

$$\begin{tikzcd}
T: V \arrow[r] & W \arrow[d, "P_{\mathcal{C}}^{-1}"] \\
\mathbb{R}^n \arrow[u, "P_{\mathcal{B}}"] \arrow[r, "{[T]}"] & \mathbb{R}^m \\
\mathcal{B} & \mathcal{C}
\end{tikzcd}$$

$$V = W = \mathbb{R}^n$$

$$\mathcal{B} = \mathcal{C} = \{\vec{v}_1, \dots, \vec{v}_n\}$$

Simplified as follows

$$
\begin{tikzcd}
T: & \mathbb{R}^n \arrow[r, "A"] & \mathbb{R}^n \arrow[d, "P_{\mathcal{B}\leftarrow\mathcal{E}}"] \\
\text{coord space} \arrow[r] & \mathbb{R}^n \arrow[u, "P_{\mathcal{E}\leftarrow\mathcal{B}}"] \arrow[r, "{[T]_{\mathcal{B}}}"] & \mathbb{R}^n \\
& \mathcal{B} & \mathcal{B}
\end{tikzcd}
$$

$$[T]_B = \underset{B \leftarrow \mathcal{E}}{P} \; A \; \underset{\mathcal{E} \leftarrow B}{P}$$

$$= V^{-1} A V = D \rightarrow \text{diag.}$$

Diagonalization:

In the eigen-basis, the matrix representation $[T]_B$ is diagonal.