

## Lec 12. Warmup.

Are these vector spaces?  $c \in \mathbb{R}$ .

1)  $V = \mathbb{R}^+ := \{x \mid x > 0\}$

$$x+y := xy, \quad cx := cx$$

✗ not closed under mult.

2)  $V = \mathbb{R}^{m \times n} \equiv M_{m \times n} \equiv \{A \mid A \text{ is a } m \times n \text{ matrix}\}$

$$A+B, cA$$

✓

$$3) V = \mathbb{R}$$

$$x \oplus y := \max(x, y) \quad c \cdot x := cx$$

X Let us say  $a \in \mathbb{R}$  is " $\vec{0}$ "  $x = a - 1$

$$x \oplus a := \max(a - 1, a) = a \neq x \Rightarrow a \text{ is not } \vec{0} \text{ doesn't exist}$$

$$4) V = \left\{ f(x) = \sum_{n=1}^N a_n \cos(nx) \mid a_n \in \mathbb{R}, x \in \mathbb{R} \right\}.$$

$$(f + g)(x) := f(x) + g(x), \quad (cf)(x) := c f(x)$$



5)  $V = \mathbb{Z} = \{ \text{integers} \}.$

$$x+y := x+y, \quad cx := cx$$

$x \cdot \frac{1}{2} + 1 \notin V$ .

6)  $V = \mathbb{Q} = \{ \text{rational numbers} \}$

$\pi \cdot 1 \notin \mathbb{Q}$

key properties in  $\mathbb{R}^n \rightarrow$  ANY vector space  
V.

- span
- linear dependence .
- subspace .
- basis .
- linear transformation . . .

Def Basis of a vector space  $V$ .

① (not too large)  $\{\vec{v}_1, \dots, \vec{v}_n\}$  lin. indep.

$x_1 \vec{v}_1 + \dots + x_n \vec{v}_n = \vec{0}$  has only trivial sol

② (not too small)  $\text{Span}\{\vec{v}_1, \dots, \vec{v}_n\} = V$

take any  $\vec{b} \in V$ , always find  $x_1, \dots, x_n \in \mathbb{R}$

s.t.  $x_1 \vec{v}_1 + \dots + x_n \vec{v}_n = \vec{b}$

Ex.  $V = P_3 = \{ \text{poly func. } f : \mathbb{R} \rightarrow \mathbb{R} \mid f$   
 $\text{degree } \leq 3 \}$ .

Find a basis of  $V$ .

guess:  $\{1, x, x^2, x^3\}$ .

① solve  $a_0 + a_1x + a_2x^2 + a_3x^3 = 0$  for all  $x \in \mathbb{R}$

By fundamental thm of algebra, any poly  $\in P_3$  has at most 3 roots  $\Rightarrow a_0 = a_1 = a_2 = a_3 = 0$ .

②  $\text{Span}\{1, x, x^2, x^3\} = P_3$  by def.

Ex. a) Is  $\mathbb{R}^2$  a subspace of  $\mathbb{R}^3$ ?

$\times \quad \mathbb{R}^2 \notin \mathbb{R}^3$

b) Is  $P_2$  a subspace of  $P_3$ ?

$\checkmark \quad P_2 \subset P_3$

$\dim V :=$  number of vectors of  
any basis of  $V$ .

Ex.  $\dim \mathbb{P}_n = n+1$

$\dim V < \infty$  : finite dimensional vector space.

$\dim V = \infty$  ∴ infinite "

Ex.  $P = \{ \text{all poly func} : \mathbb{R} \rightarrow \mathbb{R} \}$ .

basis :  $\{1, x, x^2, x^3, \dots\}$ .

Ex.  $V = S = \{ (a_1, a_2, a_3, \dots) \mid a_i \in \mathbb{R},$   
with finite nonzeros  $a_i\}$ .

basis :=  $\{(1, 0, 0, \dots), (0, 1, 0, 0, \dots)\}$ .

(rarely used in THIS course)

# Linear transformation

$T: V \rightarrow W$ ,  $V, W$  vector spaces

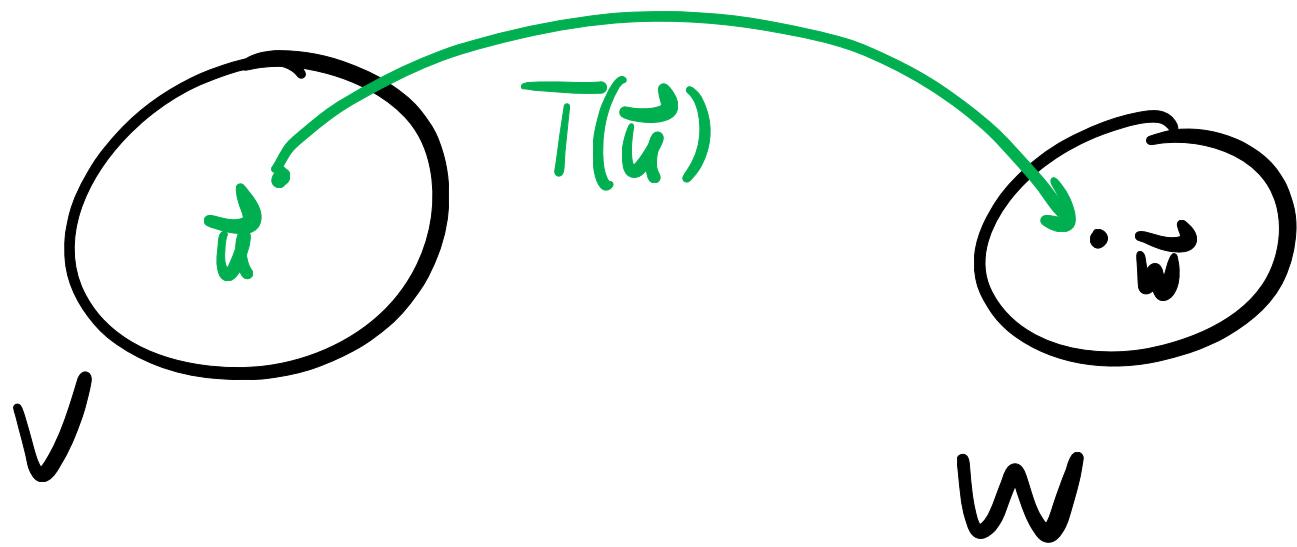
$$\vec{u} \mapsto T(\vec{u})$$

(1)  $\vec{u}, \vec{v} \in V, T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$

(2)  $\vec{u} \in V, c \in \mathbb{R} \quad T(c\vec{u}) = cT(\vec{u})$

Def  $\text{Image}(\tau) = \{ \vec{w} \in W \mid \vec{w} = \tau(\vec{u}) \text{ for some } \vec{u} \in V \}$

|||  
Range ( $\tau$ )



Def  $\text{rank}(\tau) := \dim \text{Image}(\tau)$

Def  $\text{Null}(\tau) = \{ \vec{u} \in V \mid \tau(\vec{u}) = \vec{0} \}$

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**Kernel( $\tau$ )**

