Lec 37. Warm up

Solve the following heat eq.

$$\begin{cases} \frac{\partial u}{\partial t} = 8 \frac{\partial^2 u}{\partial x^2} \\ u(0,t) = u(\pi,t) = 0 \\ u(x,0) = 6\sin x - 8\sin(3x) \end{cases}$$

Sol:

$$\begin{cases} X''(x) = \lambda X(x) \\ X(0) = X(\pi) = 0 \end{cases}$$

$$\Rightarrow \lambda_n = -n^2, \quad X_n(x) = \sin(nx), \quad n \in \mathbb{N}_+$$

general sol. $u(x,t) = \sum_{n=1}^{\infty} c_n e^{-8n^2 t} \sin(nx)$

match initial data

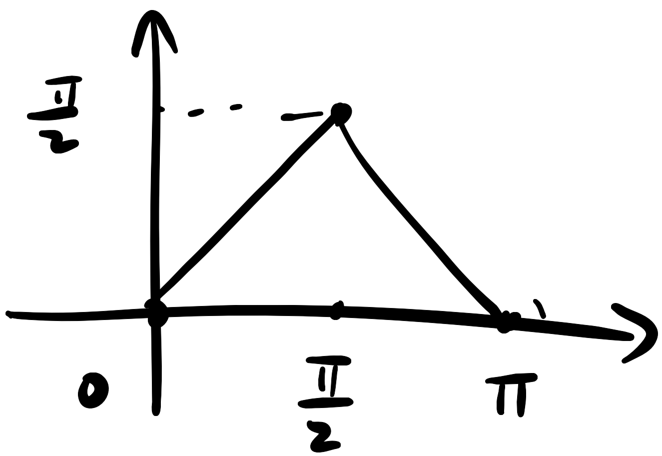
$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin(nx) = 6 \sin x - 8 \sin 3x$$

$$c_n = \begin{cases} 6, & n=1 \\ -8, & n=3 \\ 0, & \text{otherwise} \end{cases}$$

$$u(x,t) = 6 e^{-8t} \sin x - 8 e^{-72t} \sin 3x$$

$\square$.

Ex. $f(x) = \begin{cases} x, & 0 \le x < \frac{\pi}{2} \\ \pi - x, & \frac{\pi}{2} \le x \le \pi \end{cases}$




"hat function"

does not look like

$$\sum_{n=1}^{\infty} f_n \sin(nx)$$

Fourier expansion

Recall chap 6.

Vector space (finite dim) $V$

inner product $\langle \vec{u}, \vec{v} \rangle$

Basis $\mathcal{B} = \{\vec{b}_1, \cdots, \vec{b}_n\}$

$\vec{x} \in V$. Coordinate $[\vec{x}]_{\mathcal{B}}$

Method 1. Solve lin eq. $\to$ row reduction does NOT generalize to inf-dim setting

Method 2. $\mathcal{B}$ is an orthogonal basis

$$\left([\vec{x}]_{\mathcal{B}}\right)_i = \frac{\langle \vec{b}_i , \vec{x} \rangle}{\langle \vec{b}_i , \vec{b}_i \rangle}$$

at least formally. it generalizes to $\infty$ dim setting

→ Fourier analysis on $[0, L]$

$$\underset{\substack{\Uparrow \\ V}}{f(x)} = \sum_{n=1}^{\infty} \underbrace{\sin\left(\frac{n\pi x}{L}\right)}_{\vec{b}_n \in V} \underbrace{f_n}_{([f]_{\mathcal{B}})_n}$$

$$\mathcal{B} = \left\{ \sin\frac{\pi}{L}x,\ \sin\frac{2\pi}{L}x,\ \cdots \right\}.$$

Inner product

$$\langle f, g \rangle = \int_0^L f(x)\, g(x)\, dx$$

Check orthogonality

$$\int_0^L \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)dx = \langle \vec{b_n}, \vec{b_m} \rangle$$

$$= \frac{1}{2}\int_0^L \cos\left(\frac{n-m}{L}\pi x\right) - \cos\left(\frac{n+m}{L}\pi x\right)dx$$

$$\overset{\text{exer}}{=} \begin{cases} \frac{L}{2}, & m=n \\ 0, & m \neq n. \end{cases} \longrightarrow \text{orthogonal.}$$

$$\|\vec{b_n}\|^2 = \left\|\sin\left(\frac{n\pi}{L}x\right)\right\|^2 = \frac{L}{2}$$

$$f_n = \frac{\langle \sin \frac{n\pi}{L} x,\ f \rangle}{\langle \sin \frac{n\pi}{L} x,\ \sin \frac{n\pi}{L} x \rangle} = \frac{2}{L} \int_0^L \sin\left(\frac{n\pi}{L} x\right) f(x)\, dx$$

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Apply to "hat function"

$$f_n = \frac{2}{\pi} \int_0^{\pi} \sin(nx)\, f(x)\, dx$$

$$= \frac{2}{\pi} \left( \int_0^{\frac{\pi}{2}} \sin(nx)\, x\, dx + \int_{\frac{\pi}{2}}^{\pi} \sin(nx)(\pi - x)\, dx \right)$$

exer

$$\begin{cases} 0, & n \text{ even} \\ \dfrac{4}{\pi n^2}(-1)^{\frac{n-1}{2}}, & n \text{ odd} \end{cases}$$

$$f(x) = \frac{4}{\pi}\left(\sin x - \frac{1}{9}\sin 3x + \frac{1}{25}\sin 5x - \cdots\right)$$

not exact representation

but a finite term approximation.

How to quantify convergence?

<u>Thm</u> (Pointwise convergence)

Any function $f: [0, L] \to \mathbb{R}$.

$f(0) = f(L) = 0$. $f'(x)$ piecewise continuous

$$f(x) = \sum_{n=1}^{\infty} f_n \sin\left(\frac{n\pi}{L} x\right)$$

holds for every point $x \in [0, L]$.

$$\begin{cases} \dfrac{\partial u}{\partial t} = \beta \dfrac{\partial^2 u}{\partial x^2} \\ \dfrac{\partial u}{\partial x}(0,t) = \dfrac{\partial u}{\partial t}(L,t) = 0 . \\ u(x,0) = f(x) \end{cases}$$

$\longrightarrow$ Neumann b.c.

$$\begin{cases} X''(x) = \lambda X(x) \\ X'(0) = X'(L) = 0 \end{cases}$$

① $\lambda > 0$ $\quad c_1 e^{\sqrt{\lambda} x} + c_2 e^{-\sqrt{\lambda} x}$

② $\lambda = 0$. $\quad c_1 + c_2 x$ . $\quad \lambda = 0$. $X(x) = 1$.

③ $\lambda < 0.$ $c_1 \cos(\sqrt{-\lambda}\, x) + c_2 \sin(\sqrt{-\lambda}\, x)$

$$\lambda_n = -\left(\frac{n\pi}{L}\right)^2 \qquad n \in \mathbb{N}$$

$$X_n(x) = \cos\left(\frac{n\pi}{L} x\right).$$

$n=0.$ $\lambda_0 = 0.$ $X_0(x) = 1$

general sol.

$$u(x,t) = \sum_{n=0}^{\infty} c_n e^{-\beta\left(\frac{n\pi}{L}\right)^2 t} \cos\left(\frac{n\pi}{L} x\right)$$

$$u(x,0) = \sum_{n=0}^{\infty} c_n \cos\left(\frac{n\pi}{L} x\right) = f(x)$$

How to obtain $c_n$?

$$\mathcal{B} = \left\{ 1,\ \cos\frac{\pi}{L}x,\ \cos\frac{2\pi}{L}x, \cdots \right\}$$

Is $\mathcal{B}$ an orthogonal basis?

$$\int_0^L \cos\frac{n\pi}{L}x \cdot \cos\frac{m\pi}{L}x \, dx = \langle \vec{b}_n, \vec{b}_m \rangle$$

$$= \frac{1}{2}\int_0^L \left\{ \cos\left(\frac{n+m}{L}\pi x\right) + \cos\left(\frac{n-m}{L}\pi x\right) \right\} dx$$

$$= \begin{cases} \frac{L}{2}, & n = m \neq 0 \\ L, & n = m = 0 \\ 0, & n \neq m \end{cases}$$

$\leftarrow$ new

orthogonal!

$$c_0 = \frac{1}{L} \int_0^L f(x)\, dx$$

$$c_n = \frac{2}{L} \int_0^L \cos \frac{n\pi}{L} x \cdot f(x)\, dx, \quad n \geq 1$$

Long time behavior. $t \to \infty$

Dirichlet $u(0,t) = u(L,t) = 0$.

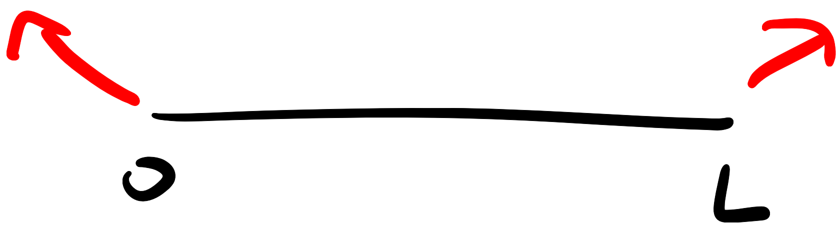

$t \to \infty$ $u(x,t) \approx 0$ $x \in [0, L]$

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\beta\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L} x\right)$$

$\downarrow$ exp. decay

0 w.r.t. $t$.

Neumann. $\dfrac{\partial u}{\partial x}(0,t) = \dfrac{\partial u}{\partial x}(L,t) = 0$




insulated from environment

$$u(x,t) = C_0 + \sum_{n=1}^{\infty} C_n e^{-\beta\left(\frac{n\pi}{L}\right)^2 t} \cos\left(\frac{n\pi}{L}x\right)$$

$$\xrightarrow{t \to \infty} C_0 = \underbrace{\frac{1}{L}\int_0^L f(x)\,dx}_{\text{average temperature.}}$$

initial temperature