Lec 30. Warm up

Find all sols to

$$y''(t) = 12t^2$$

Guess

$y(t) = t^4$. $y'(t) = 4t^3$, $y''(t) = 12t^2$

$y(t) = 1$, $y'(t) = 0$ $\Rightarrow$ $y(t) = t^4 + c$, $c \in \mathbb{R}$

$y(t) = t$, $y'(t) = 1$, $y''(t) = 0$ $\Rightarrow y(t) = t^4 + c_1 t + c_2$, $c_1, c_2 \in \mathbb{R}$

Draw an analogy w. lin. sys.

$A\vec{x}=b$.

① a particular sol. $\vec{x}_p$

② hom. eq.

$A\vec{x}=\vec{0}$ sol set W ← subspace.

All sol. $\vec{x}_p+\vec{w}$, $\vec{w}\in W$

$y''(t)$ $\quad$ V: vector space of inf. differtiable functions on $\mathbb{R}$

$$T: V \longrightarrow V$$

$$y(t) \longmapsto y''(t)$$

$$T\vec{y} = \vec{b} \longleftarrow 12t^2$$

$$T(a_1\vec{y}_1 + a_2\vec{y}_2) = (a_1 y_1(t) + a_2 y_2(t))''$$

$$= a_1 T(\vec{y}_1) + a_2 T(\vec{y}_2)$$

particular sol. $\vec{y}_p$: t^4

hom. eq.

$$T\vec{y} = \vec{0} \iff y''(t) = 0$$

$$y'(t) = c_1 \Rightarrow \underbrace{y(t) = c_1 t + c_2}_{\text{all sols. to hom. eq.}}$$

$\Rightarrow$ sol set: $t^4 + (c_1 t + c_2 \quad)$

Much Simpler.

$$y'' = 12t^2 \xrightarrow{\text{integrate}} \text{Calculus 1A.}$$

<u>Ex</u>. $y'(t) = \lambda y(t)$, $\lambda \in \mathbb{R}$.

Sol: $$\left(e^{-\lambda t} y(t)\right)' = -\lambda e^{-\lambda t} y(t) + e^{-\lambda t} y'(t)$$

$$= e^{-\lambda t} \big(\underbrace{y'(t) - \lambda y(t)}_{\shortparallel \atop 0} \big)$$

$$e^{-\lambda t} y(t) = C$$

$$\Rightarrow y(t) = C e^{\lambda t}$$

$$T : V \longrightarrow V$$

$$y(t) \longmapsto y'(t)$$

$T\vec{y} = \lambda \vec{y}$: eigenvalue problem.

$A \in \mathbb{R}^{n \times n}$: # eigenvalues $\leq n$

$\forall \lambda \in \mathbb{R}$ (in fact, $\mathbb{C}$) is an eig. val.

$e^{\lambda t}$: eigenvector

T: defined on an inf dim. space.

All first order lin. diff. eq. w. constant coef. are solved.

Second order diff. w. const coef.

$$y'' + by' + cy = 0.$$

$$T: V \to V$$

$$y(t) \mapsto y'' + by' + cy$$

$\mathrm{Null}(T)$

$\underline{Ex}$. Solve $y'' - y = 0$.

aux. eq. $r^2 - 1 = 0 \Rightarrow r_1 = 1,\ r_2 = -1.$

$= (r-1)(r+1)$

$$T(\vec{y}) = \underbrace{\left(\frac{d^2}{dt^2} - I\right)}\, y(t)$$

$\rightarrow$ factorization on the operator level

$$= \underbrace{\left(\frac{d}{dt} - I\right)\left(\frac{d}{dt} + I\right) y(t)} \rightarrow \star$$

Check: $\star = \left(\frac{d}{dt} - I\right)\left[y'(t) + y(t)\right]$

$= y''(t) + y'(t) - y'(t) - y(t)$

$= y''(t) - y(t)$

$\frac{d}{dt}$, I they commute.

$\frac{d}{dt} \cdot I \cdot \vec{y} = I \cdot \frac{d}{dt} \vec{y}$

Replace $\frac{d}{dt} \longrightarrow r$

$$T(\vec{y}) = \left(\frac{d}{dt} - I\right)\underbrace{\left(\frac{d}{dt} + I\right) y(t)}_{0}$$

$$y'(t) + y(t) = 0 \Rightarrow y(t) = c_1 e^{-t}$$

$$T(\vec{y}) = \left(\frac{d}{dt} + I\right)\underbrace{\left(\frac{d}{dt} - I\right) y(t)}_{0}$$

$$y'(t) - y(t) = 0 \Rightarrow y(t) = C_2 e^{t}$$

$$\Rightarrow y(t) = c_1 e^{-t} + c_2 e^{+t}$$

<u>Ex</u>. $\begin{cases} y'' - y = 0 \\ \boxed{\begin{matrix} y(0) = 1 \\ y'(0) = 0 \end{matrix}} \end{cases}$ Initial value problem (IVP).

initial condition

For general sol.

$$y(t) = C_1 e^{-t} + C_2 e^{t}$$

Plug in initial value

$$y(0) = C_1 + C_2 = 1$$

$$y'(0) = -C_1 + C_2 = 0$$

$$\Rightarrow y(t) = \frac{1}{2}(e^{t} + e^{-t})$$

Ex. Solve $y'' - 2y' + y = 0$

aux. eq. $r^2 - 2r + 1 = 0 \Rightarrow (r-1)^2 = 0$

$$T(\vec{y}) = \left(\frac{d^2}{dt^2} - 2\frac{d}{dt} + I\right) y(t)$$

$$T(\vec{y}) = \left(\frac{d}{dt} - I\right)^2 y(t)$$

$$\left(\frac{d}{dt} - I\right) y(t) = 0 \Rightarrow y(t) = c_1 e^t$$

Have run of first order diff. eqs !

Claim $y_2(t) = t e^t$

$$y_2'(t) = (te^t)' = e^t + te^t$$

$$y_2'(t) - y_2(t) = e^t$$

$$T(y_2) = \left(\frac{d}{dt} - I\right)\left[\left(\frac{d}{dt} - I\right)\right] y_2(t)$$

$$= \left(\frac{d}{dt} - I\right) e^t$$

$$= 0$$

$$y_2(t) = c_2 \, t \, e^t$$

Sol set. $y(t) = c_1 e^t + c_2 \underset{\sim}{t} \cdot e^t$

An interpretation

$$y'' - 2y' + (1+\epsilon)(1-\epsilon)\, y = 0.$$

ε: small number. $\varepsilon \to 0^+$

aux. eq. $r^2 - 2r + (1+\epsilon)(1-\epsilon) = 0$

$\Rightarrow \quad r_1 = 1+\epsilon, \quad r_2 = 1-\epsilon$

$$\Rightarrow \begin{cases} y_1^{\varepsilon}(t) = e^{(1+\epsilon)t} \\ y_2^{\varepsilon}(t) = e^{(1-\epsilon)t} \end{cases}$$

$$\tilde{y}_2^{\varepsilon}(t) := \frac{y_1^{\varepsilon}(t) - y_2^{\varepsilon}(t)}{2\varepsilon} \quad \leftarrow \text{is a sol.}$$

$$= \frac{e^{(1+\varepsilon)t} - e^{(1-\varepsilon t)}}{2\varepsilon}$$

$$\xrightarrow{\varepsilon \to 0} t e^{t}$$