Lec 28. Warm up.

Is this an orthogonal matrix $U^TU=I \Rightarrow U^{-1}=U^T$

$\Updownarrow$

$UU^T=I$. $U\in\mathbb{R}^{n\times n}$

① $U=\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ T

② $U=\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, $\theta\in\mathbb{R}$ F. $\vec{u}_1\cdot\vec{u}_2=\sin 2\theta$

$\neq 0$. (may not be)

③ $U=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ T

$\vec{u}, \vec{v} \in \mathbb{C}^n$

$$\langle \vec{u}, \vec{v} \rangle = \sum_{i=1}^{n} \overline{u_i}\, v_i$$

$z = a + ib,\ a, b \in \mathbb{R}$

$\bar{z} = a - ib$

$|z|^2 = a^2 + b^2$

(exer) check ① - ③

④ $\langle \vec{u}, \vec{u} \rangle = \sum_{i=1}^{n} \bar{u}_i u_i = \sum_{i=1}^{n} |u_i|^2 \geq 0$

If $\langle \vec{u}, \vec{u} \rangle = 0 \Rightarrow u_i = 0 \Rightarrow \vec{u} = \vec{0}$

Inner product ✓

$$\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \in \mathbb{C}^n \qquad \overline{\vec{u}} = \begin{bmatrix} \bar{u}_1 \\ \vdots \\ \bar{u}_n \end{bmatrix}$$

$$\vec{u}^T = [u_1 \cdots u_n] \qquad \vec{u}^* \equiv \left(\overline{\vec{u}}\right)^T = [\bar{u}_1 \cdots \bar{u}_n]$$

$$\langle \vec{u}, \vec{v} \rangle = \vec{u}^* \vec{v}$$

(recall $\vec{u}, \vec{v} \in \mathbb{R}^n$,

$$\langle \vec{u}, \vec{v} \rangle = \vec{u} \cdot \vec{v} = \vec{u}^T \vec{v}$$

Orthogonal matrix . $U \in \mathbb{R}^{n \times n}$

$$U^T U = I_n \iff \langle \vec{u}_i, \vec{u}_j \rangle = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

unitary matrix

$$U^* U = I_n \iff \qquad \uparrow \qquad U \in \mathbb{C}^{n \times n}$$

$$U = [\vec{u}_1 \cdots \vec{u}_n] \; . \quad U^* = \begin{bmatrix} \vec{u}_1^* \\ \vdots \\ \vec{u}_n^* \end{bmatrix}$$

$\{\text{orthogonal matrix}\} \subset \{\text{unitary matrix}\}$.

Diagonalizability

$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ not diagonalizable.

$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $\lambda^2 = -1 \Rightarrow \lambda = \pm i$

real matrix.

eig. val. complex

Real symmetric matrix

$A \in \mathbb{R}^{n\times n}$, $A = A^T$

e.g. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\lambda^2 = 1 \Rightarrow \lambda = \pm 1$

$\lambda = 1$, $\vec{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = -1$, $\vec{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

observation

✓ ① eigenvalues are real

✓ ② $\vec{v}_1 \cdot \vec{v}_2 = 0$ → stronger than lin. independence.

$$V^T V = I_2 \qquad V = [\vec{v}_1, \vec{v}_2]$$

$$\Updownarrow$$

$$V^{-1} = V^T$$

✓ ③ $A = VDV^T$. $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

Want to show: ① - ③ generalizable to real sym. matrix.

① eigenvalues are real. $A \in \mathbb{R}^{n \times n}$

$$A\vec{v} = \lambda \vec{v}$$

at this point, I don't know

$\lambda \in \mathbb{R}$ or $\vec{v} \in \mathbb{R}^n$

So I assume

$$\lambda \in \mathbb{C}, \quad \vec{v} \in \mathbb{C}^n$$

If we can show $\lambda \in \mathbb{R}$.

$$(\underbrace{A - \lambda I}_{\text{real}})\vec{v} = 0.$$

$\Rightarrow \vec{v} \in \mathbb{R}^n$ (by elimination procedure)

$$\langle \vec{v}, A\vec{v} \rangle = \langle \vec{v}, \lambda \vec{v} \rangle$$

complex inner prod.

$$\vec{v}^* A \vec{v} = \lambda \langle \vec{v}, \vec{v} \rangle = \lambda \vec{v}^* \vec{v}$$

$$(\vec{v}^* A \vec{v})^* = (\lambda \vec{v}^* \vec{v})^*$$

complex conjugation

$$\Downarrow$$

$$\vec{v}^* \underset{\substack{\| \\ \vec{v}^* A \vec{v}}}{A^*} \vec{v} = \bar{\lambda} \vec{v}^* \vec{v}$$

$$(AB)^T = B^T A^T$$

$$(AB)^* = B^* A^*$$

$$A^* = A^T = A$$

$$\downarrow \qquad \downarrow$$

real $\qquad$ sym

$$\Rightarrow (\bar{\lambda} - \lambda) \underbrace{\vec{v}^* \vec{v}}_{\substack{\| \\ \|\vec{v}\|^2 \neq 0}} = 0$$

$$\Rightarrow \lambda = \bar{\lambda}$$

$$\lambda = a + ib \quad \Rightarrow b = 0 \Rightarrow \lambda \in \mathbb{R}. \quad \square$$

② $A\vec{v}_1 = \lambda_1 \vec{v}_1$, $\lambda_1 \neq \lambda_2 \in \mathbb{R}$

$A\vec{v}_2 = \lambda_2 \vec{v}_2$, $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^n$

$\Rightarrow \vec{v}_1 \cdot \vec{v}_2 = \vec{v}_1^T \vec{v}_2 = 0$.

Pf: $\vec{v}_2^T A \vec{v}_1 = \lambda_1 \vec{v}_2^T \vec{v}_1$

$\Downarrow$ apply trans.

$$(\vec{v}_2^T A \vec{v}_1)^T = \lambda_1 (\vec{v}_2^T \vec{v}_1)^T$$

$$\Downarrow$$

$$\vec{v}_1^{\,T} \underbrace{A\vec{v}_2} = \lambda_1 \vec{v}_1^{\,T} \underbrace{\vec{v}_2} \qquad (1)$$

$$\|$$

$$\vec{v}_1^{\,T}\left(\lambda_2 \vec{v}_2\right)$$

$$\Rightarrow \underbrace{(\lambda_1 - \lambda_2)}_{\neq 0} \vec{v}_1^{\,T} \vec{v}_2 = 0$$

$$\Rightarrow \vec{v}_1^{\,T} \vec{v}_2 = 0 \qquad \square$$

Thm $A \in \mathbb{R}^{n\times n}$. sym. If A has n distinct eigenvalues → strong assump. but not necessary.

$\lambda_1, \cdots, \lambda_n \in \mathbb{R}$. Then the eigenvectors

$\vec{v}_1, \cdots, \vec{v}_n$ form an orthogonal matrix.

$V = [\vec{v}_1, \cdots, \vec{v}_n]$ $\quad V^T V = I_n$

$$D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

$$A = VDV^T$$

$$\left[\begin{array}{c} (VDV^T)^T = VDV^T \\ A^T = A \end{array}\right]$$

State the (stronger) spectral theorem (w.o. proof. Need Math 110)

Thm. $A \in \mathbb{R}^{n\times n}$. $A^T = A$. Then

① A is always diagonalizable.

w. real eigenvalues.

② eigenvectors can always be

chosen to form an orthogonal matrix.

$$A = VDV^T \qquad D \text{ diag.}$$
$$V^TV = I_n$$

Ex. $A = A^T$. $A \in \mathbb{R}^{n\times n}$. $\vec{u}, \vec{v} \in \mathbb{R}^n$

$$\langle \vec{u}, \vec{v} \rangle := \vec{u}^T A \vec{v}$$

Inner product ?

No. $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ real sym.

(exer) check ①–③ hold.

$\langle \vec{u}, \vec{u} \rangle = \vec{u}^T A \vec{u} \geq 0$?

$= 0 \Rightarrow \vec{u} = \vec{0}$?

Use spec. decomp.

$$A = VDV^T. \quad D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

$$\langle \vec{u}, \vec{u} \rangle = (\vec{u}^T V) D (V^T \vec{u})$$

$\vec{w}$, $\vec{w}^T = \vec{u}^T V$

$$= \vec{w}^T D \vec{w}$$

$$= \sum_{i=1}^{n} (w_i)^2 \lambda_i \geq 0 \Rightarrow \lambda_i \geq 0$$

$\langle \vec{u}, \vec{u} \rangle = 0$, $\lambda_i > 0$

$\Rightarrow w_i = 0$, $i = 1, \cdots, n$

V is invertible

$\vec{w} = V^T \vec{u} = \vec{0} \Rightarrow \vec{u} = 0$.

Inner product ✓

$A \succ 0 \to A$ is positive definite

$$\Updownarrow$$

$$\lambda_i > 0$$

$$\lambda_i \geq 0$$

$$A \geq 0 \longrightarrow A \text{ is positive-semidefinite.}$$