Lec 23. Warm up

True / False.

(a) The sum of 2 eig. vectors of $A \in \mathbb{R}^{n\times n}$

is still an eig. vector of A

False. If $Av_1 = \lambda v_1$, $Av_2 = \lambda v_2$

$A(v_1+v_2) = \lambda(v_1+v_2)$.

If different eigenvalue.

$Av_1 = \lambda_1 v_1$, $Av_2 = \lambda_2 v_2$. $\Rightarrow A(v_1+v_2) = \lambda_1 v_1 + \lambda_2 v_2$

$\neq \lambda'(v_1+v_2)$

(b) $A, B \in \mathbb{R}^{n\times n}$ are similar, then they have the same set of eigenvectors.

False. $A = V^{-1} B V$

$\Rightarrow$ A, B share the same set of eig. values.

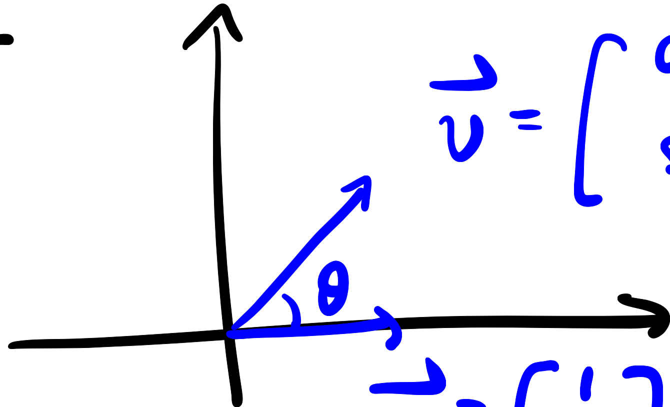
Just think. $A \in \mathbb{R}^{2\times 2}$.

$$AV = V \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

B. eigvec. $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Geomerty length. angle in $\mathbb{R}^n$

$\mathbb{R}^2$



$$\vec{v} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$$\vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\|\vec{v}\| := \sqrt{v_1^2 + v_2^2} = \sqrt{\cos^2\theta + \sin^2\theta} = 1.$$

length

$$u_1 v_1 + u_2 v_2 = \cos\theta$$

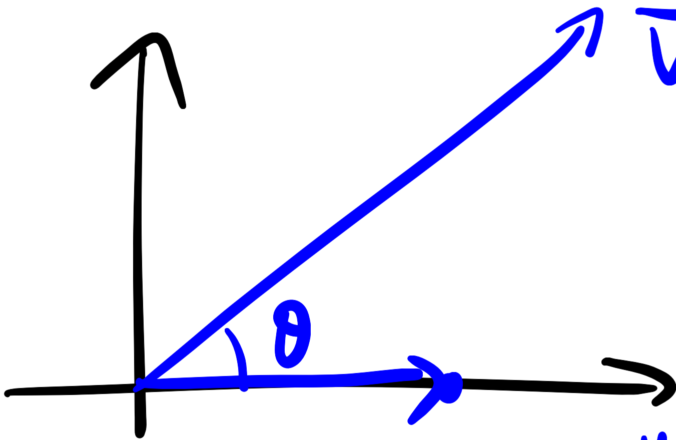
Def $\vec{u}, \vec{v} \in \mathbb{R}^2$, the inner product (dot product) of $\vec{u}, \vec{v}$ is denoted by $\left(\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right)$

$$\vec{u} \cdot \vec{v} := u_1 v_1 + u_2 v_2 .$$

The length of $\vec{u}$ is

$$\|\vec{u}\| := \sqrt{u_1^2 + u_2^2} \equiv \sqrt{\vec{u} \cdot \vec{u}}$$

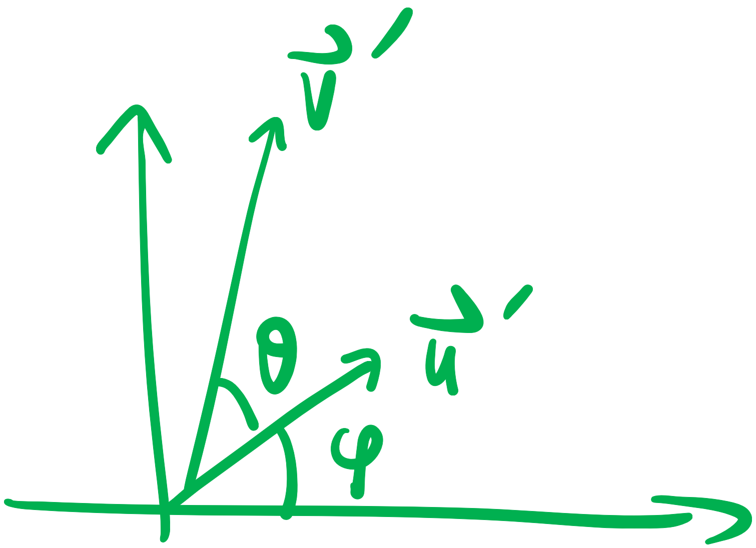
Note: $\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v} = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$



$$\vec{v} = \begin{bmatrix} \|\vec{v}\| \cos\theta \\ \|\vec{v}\| \sin\theta \end{bmatrix}$$

$$\vec{u} = \begin{bmatrix} \|\vec{u}\| \\ 0 \end{bmatrix}$$

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| \cdot \cos \underbrace{\theta}_{\text{angle}}$$



whether

$$\vec{u}' \cdot \vec{v}' = \|\vec{u}\| \, \|\vec{v}\| \cos\theta \, ?$$

Rotation operation is a matrix transformation.

$$\vec{u}' = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \vec{u}$$

$$\vec{u}' \cdot \vec{v}' = \underbrace{(\vec{u}')^T \ \vec{v}'}_{\text{matrix-matrix mult.}}$$

$$\vec{u}'^T = \vec{u}^T \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix}$$

$$= \vec{u}^T \left(\begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \right) \vec{v}$$

$$= \vec{u}^T \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \vec{v}$$

$$= \vec{u}^T \vec{v} = \vec{u} \cdot \vec{v}$$

$$\boxed{\vec{u}' \cdot \vec{v}' = \vec{u} \cdot \vec{v}}$$

after rotation before rotation

$\mathbb{R}^2 \longrightarrow \mathbb{R}^n$

Use previous algebraic definition.

<u>Def</u>. $\vec{u}, \vec{v} \in \mathbb{R}^n$, dot product / inner product / Euclidean inner product is

$$\vec{u} \cdot \vec{v} = u_1 v_1 + \cdots + u_n v_n$$

The length of $\vec{u}$ is

$$\|\vec{u}\| = \sqrt{\vec{u}\cdot\vec{u}}$$

Again

$$\vec{u}\cdot\vec{v} \equiv \vec{u}^{T}\vec{v} = [u_1 \cdots u_n]\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

Properties

1) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

2) $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$

3) $c \in \mathbb{R}$ $\quad (c\vec{u}) \cdot \vec{v} = c(\vec{u} \cdot \vec{v})$

4) $\vec{u} \cdot \vec{u} \geq 0$, $\quad \vec{u} \cdot \vec{u} = 0 \Leftrightarrow \vec{u} = \vec{0}$

length is 0 $\Leftrightarrow$ zero vector.

Def Unit vector $\vec{u}$ if $\|\vec{u}\| = 1$

For any nonzero vector $\vec{u}$

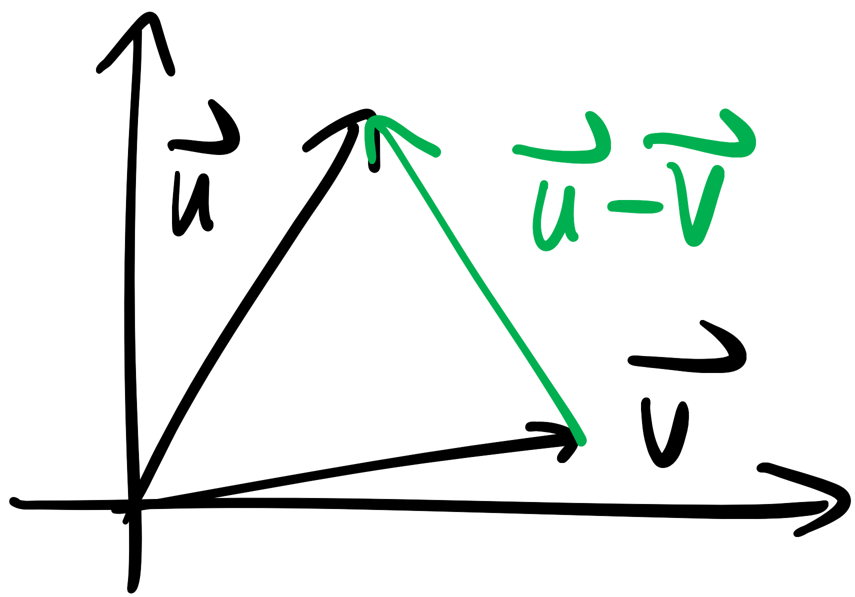
Normalization

$$\vec{v} = \left(\frac{1}{\|\vec{u}\|}\right)\vec{u} \quad \text{is a unit vector}$$

$$\vec{v}\cdot\vec{v} = \frac{1}{\|\vec{u}\|^2}\,\vec{u}\cdot\vec{u} = 1 \,.$$

<u>Def</u> Distance between $\vec{u}, \vec{v} \in \mathbb{R}^n$

$$d(\vec{u}, \vec{v}) := \|\vec{u} - \vec{v}\|$$



$$\|\vec{u}-\vec{v}\|^2 = (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v})$$

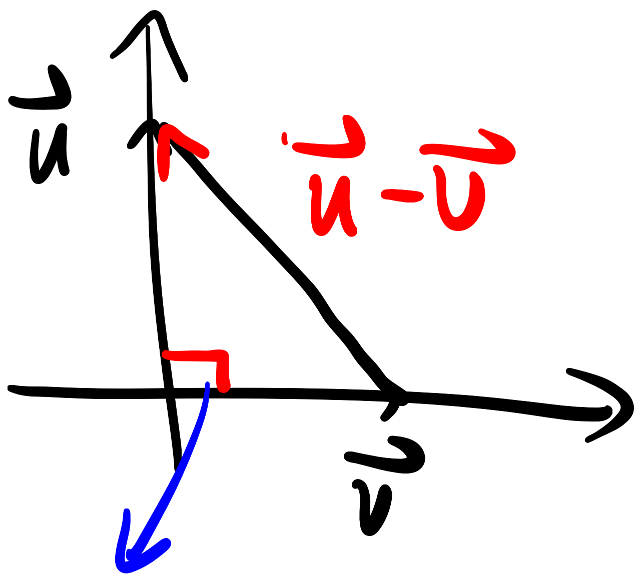
$$= \vec{u}\cdot\vec{u} - \vec{u}\cdot\vec{v} - \vec{v}\cdot\vec{u} + \vec{v}\cdot\vec{v}$$

$$= \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\,\vec{u}\cdot\vec{v}$$

Important special case

$\vec{u}\cdot\vec{v} = 0$.

$$\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$$



$\theta = \frac{\pi}{2}$

Generalization of Pythagorean thm to $\mathbb{R}^n$.

Recall

$$\vec{u} \cdot \vec{v} =: \|\vec{u}\| \, \|\vec{v}\| \underbrace{\cos\theta}$$

define angle between $\vec{u}, \vec{v} \in \mathbb{R}^n$

If $\vec{u} \cdot \vec{v} = 0 \Rightarrow \cos\theta = 0$.

$\theta \in [0, \pi]$, $\theta = \frac{\pi}{2}$

$\vec{u}, \vec{v}$ are orthogonal vectors. $\vec{u} \perp \vec{v}$

In Chap 4.

$\mathbb{R}^{2 \text{ or } 3} \longrightarrow \mathbb{R}^n \longrightarrow V$ 

In Chap 6.

equip vector space V by an additional structure (inner product) to describe geometry.

V

$+$

(regular) vector space

inner product

$=$ 

Inner product space

Def. $\{\vec{v_1}, \cdots, \vec{v_K}\}$ orthogonal set

$\vec{v_i} \in \mathbb{R}^n$, if $\vec{v_i} \perp \vec{v_j}$ for all

$1 \le i \ne j \le K$ $\quad \vec{v_i} \cdot \vec{v_j} = 0$

Orthonormal set if

1) Orthogonal set

2) all $\|\vec{v_i}\| = 1$.