Lec 36.

Partial differential equations. (PDE)

Ordinary " " (ODE)

ODE: one variable.

$y(t)$, $y(x)$. $\vec{y}(t) = \begin{bmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{bmatrix}$ ...

PDE: more than one variable

$u(x,t)$, $u(x,y,t)$, $u(x,y)$,

$$\vec{u}(x,t) = \begin{bmatrix} u_1(x,t) \\ u_2(x,t) \end{bmatrix} \cdots$$

Partial derivative.

$$\frac{\partial}{\partial t} u(x,t) = \lim_{\Delta t \to 0} \frac{u(x, t+\Delta t) - u(x,t)}{\Delta t}$$

$$\frac{\partial u}{\partial x}(x,t) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, t) - u(x,t)}{\Delta x}$$

$$\frac{\partial^2 u}{\partial x^2}(x,t) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} u(x,t) \right)$$

$$= \lim_{\Delta x \to 0} \frac{\frac{\partial u}{\partial x}(x+\Delta x, t) - \frac{\partial u}{\partial x}(x,t)}{\Delta x}$$

$$\cdots$$

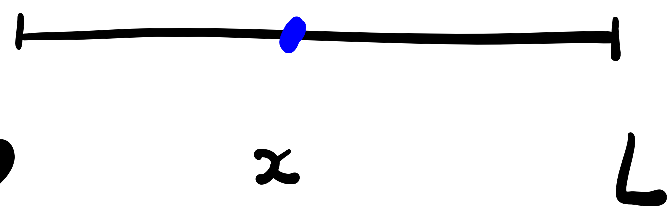
<u>Heat equation</u>

only type we care about in this class

1D. Fourier's law.

$$\frac{\partial u}{\partial t}(x,t) = \beta \frac{\partial^2 u}{\partial x^2}(x,t), \qquad \beta > 0$$

$u(x,t)$ temperature



rod

$u(x,0)$: initial temperature distribution.

what happens at the boundary ($x=0$, $x=L$)

$$u(0,t) = u(L,t) = 0$$

homogeneous.

$$\begin{cases} \dfrac{\partial u}{\partial t} = \beta \dfrac{\partial^2 u}{\partial x^2} \\ u(0,t) = u(L,t) = 0 \end{cases}$$

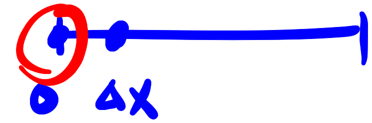
$u(x,t)$ is a sol.

$\Rightarrow c\, u(x,t)$ is a sol.

Dirichlet boundary condition.

other hom. eg.

$$\begin{cases} \dfrac{\partial u}{\partial t} = \beta \dfrac{\partial^2 u}{\partial x^2} \\[2ex] \dfrac{\partial u}{\partial x}(0,t) = \dfrac{\partial u}{\partial x}(L,t) = 0. \end{cases}$$



$$\frac{\partial u}{\partial x}(0,t)$$

heat flux

rod insulated from the enviroment

Neumann boundary condition.

Now solve

$$\begin{cases} \dfrac{\partial u}{\partial t} = \beta \dfrac{\partial^2 u}{\partial x^2} \\[2ex] u(0,t) = u(L,t) = 0. \end{cases}$$

Linear algebra perspective.

vector space.

$$V = \{ f(x) \mid f \text{ smooth}, \ f(0) = f(L) = 0 \}$$

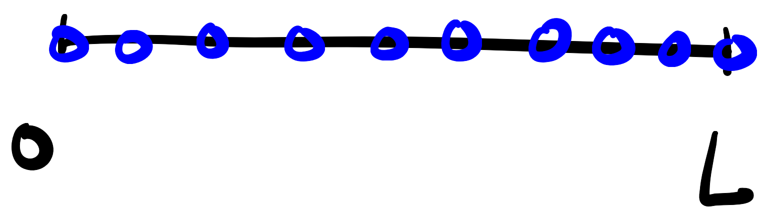
$$T : V \longrightarrow V$$

$$f(x) \longmapsto \frac{d^2}{dx^2} f(x)$$

Think $u(\cdot, t)$ as a function of x

$\uparrow$ fix

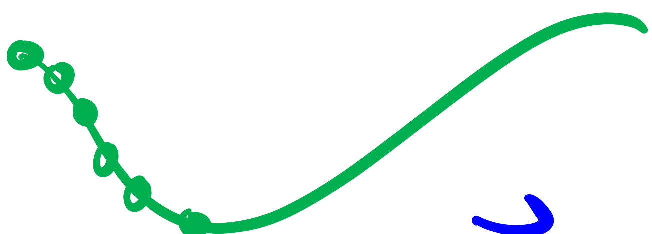
$\in V$



$u(x_i, t)$

$x_i = \frac{(i-1)}{N-1} L, \; i = 1, \cdots, N$

$x_1 = 0, \; x_N = L$



$$\vec{u}(t) = \begin{bmatrix} u(x_1, t) \\ \vdots \\ u(x_N, t) \end{bmatrix} \in \mathbb{R}^N$$

Heat

Chap 9.

" $\frac{d}{dt}\vec{u}(t) = A\,\vec{u}(t)$ "

finite dim. rep. of ∂T

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}\begin{bmatrix} u(x_1,t) \\ \vdots \\ u(x_N,t) \end{bmatrix} = \begin{bmatrix} \sum_j A_{1j}\,u(x_j,t) \\ \vdots \\ \vdots \end{bmatrix}$$

if $A\vec{v} = \lambda \vec{v}$

then $\vec{u}(t) = e^{\lambda t}\vec{v}$ is a sol.

if A diagonalizable.

$\vec{u}_1(t) = e^{\lambda_1 t}\vec{v}_1, \ldots, \vec{u}_n(t) = e^{\lambda_n t}\vec{v}_n$.

fundamental sol. set.

Method of separation of variables

inf. dim. eigenval. Problem

$\Phi(x) \in V$

$$\begin{cases} T\,\Phi(x) = \lambda\,\Phi(x). \\ \Phi(0) = \Phi(L) = 0. \end{cases}$$

$\vec{v}(x)$

$$T\,\Phi(x) = \Phi''(x).$$

$$\begin{cases} \Phi''(x) - \lambda\,\Phi(x) = 0 \\ \Phi(0) = \Phi(L) = 0 \end{cases}$$

$\leftarrow$ 1D $\underbrace{\text{ODE}}_{}$. 2nd order

$$r^2 - \lambda r = 0. \quad r_1 = \sqrt{\lambda},\ r_2 = -\sqrt{\lambda}$$

① case : $\lambda > 0$.

$$X(x) = C_1 e^{\sqrt{\lambda} x} + C_2 e^{-\sqrt{\lambda} x}$$

$$0 = X(0) = C_1 + C_2$$

$$0 = X(L) = C_1 e^{\sqrt{\lambda} L} + C_2 e^{-\sqrt{\lambda} L}$$

$$\begin{bmatrix} 1 & 1 \\ e^{\sqrt{\lambda} L} & e^{-\sqrt{\lambda} L} \end{bmatrix}$$ invertible $$\Rightarrow \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

not possible

② case : $\lambda = 0$. repeated root

$$X(x) = C_1 + C_2 x$$

$$0 = X(0) = C_1$$

$$0 = X(L) = C_1 + C_2 L$$

$$\Rightarrow \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

not possible

③ case : $\lambda < 0$.

$$X(x) = C_1 \cos\left(\sqrt{-\lambda}\, x\right) + C_2 \sin\left(\sqrt{-\lambda}\, x\right)$$

$$0 = X(0) = c_1$$

$$0 = X(L) = C_2 \sin\left(\sqrt{-\lambda}\, L\right)$$

$$\pi n \quad . \quad n \in \mathbb{N}_+$$

C_2 can be anything.

$$\sqrt{-\lambda}\, L = \pi n \implies \lambda_n = -\left(\frac{\pi n}{L}\right)^2$$

$$X_n(x) = \sin\left(\frac{\pi n}{L} x\right)$$

Check:

$$X_n''(x) = -\left(\frac{\pi n}{L}\right)^2 X_n(x) \quad \checkmark$$

$$X_n(0) = X_n(L) = 0 \quad \checkmark$$

an "educated guess" for sol of heat eq.

$$u_n(x,t) = e^{-\beta\left(\frac{\pi n}{L}\right)^2 t} X_n(x)$$

Check:

$$\frac{\partial}{\partial t} u_n(x,t) = -\beta\left(\frac{\pi n}{L}\right)^2 u_n(x,t)$$

$$= \beta \frac{\partial^2 u_n}{\partial x^2}(x,t) \quad \checkmark$$

By superposition principle

$u(x,t) = \sum_{n=1}^{\infty} c_n u_n(x,t)$ is also a sol.

Match initial data.

$$u(x,0) = \sum_{n=1}^{\infty} c_n u_n(x,0)$$

$$= \sum_{n=1}^{\infty} c_n \sin\left(\frac{\pi n}{L} x\right)$$

Fourier analysis.