Lec 31. Inhomogeneous 2nd order linear ODEs.

$$y'' + by' + cy = t^l e^{rt}, \quad l \in \mathbb{N}, \quad r \in \mathbb{C}.$$

$$T: \quad y \mapsto y'' + by' + cy$$

Observe

$$(e^{rt})' = re^{rt}, \quad (e^{rt})'' = r^2 e^{rt}$$

$$(t^a e^{rt})' = a t^{a-1} e^{rt} + r t^a e^{rt}$$

$$\in \operatorname{Span}\{t^{a-1} e^{rt}, t^a e^{rt}\}.$$

Consider a subspace $W \subset C^{\infty}(\mathbb{R})$.

$$W := \operatorname{span}\{e^{rt} \cdot t^{m}, e^{rt} t^{m-1}, \cdots, e^{rt}\}$$

$\dim W = m+1$, intuitively "$W = e^{rt} \cdot \mathbb{P}_m$"

Our hope: $t^{l} e^{rt} \in \operatorname{Image}(T) \rightarrow m \geq l$

$$T : W \longrightarrow W.$$

Then $y(t) = e^{rt} p(t)$, $p(t) \in \mathbb{P}_m$

$$= e^{rt} (a_0 + a_1 t + \cdots + a_m t^{m})$$

$a_0, \cdots, a_m$ are unknown.

Method of undetermined coefficients

$$T(y) = (e^{rt}p(t))'' + b(e^{rt}p(t))' + c e^{rt}p(t)$$

$$= (re^{rt}p(t) + e^{rt}p'(t))' + b(re^{rt}p(t) + e^{rt}p'(t))$$

$$+ c e^{rt}p(t)$$

$$= (r^2e^{rt}p(t) + 2re^{rt}p'(t) + e^{rt}p''(t))$$

$$+ b(re^{rt}p(t) + e^{rt}p'(t)) + c e^{rt}p(t)$$

$$= e^{rt} \Big\{ (\underbrace{r^2 + br + c}) \, p(t) \longrightarrow \in \mathbb{P}_m$$

$$+ (\underbrace{2r + b}) \, p'(t) \qquad \in \mathbb{P}_{m-1}$$

$$+ \quad p''(t) \qquad \in \mathbb{P}_{m-2}$$

$$\Big\}$$

$$= e^{rt} \; t^{l}$$

Case 1: $r^2 + br + c \neq 0$, $m = l$.

aux. eq. $\hookrightarrow$ not a root

Case 2 : $r^2+br+c=0$ $\qquad m=l+1$

$2r+b \neq 0$

observe $2r+b=(r^2+br+c)'$

$\Rightarrow$ r is a single root of aux. eq.

$(r-r_1)(r-r_2)$
$=r^2+br+c$, $r_1 \neq r_2$

$2r_1+b=2r_1-(r_1+r_2)$
$=r_1-r_2 \neq 0$

Case 3: $r^2+br+c=0$ $\qquad m=l+2$.

$2r+b=0$

r is a double root of aux. eq.

double root: $(r-r_1)^2=r^2+br+c$. $b=-2r_1 \Rightarrow 2r_1+b=0$

Ex. $y'' + 2y' + y = t^2$.

Step 1: $t^2 = t^2 \cdot e^{0t}$, $r = 0$.

Step 2: plug in $r=0$ into

$$r^2 + 2r + 1 = 1 \neq 0.$$

$\Rightarrow$ $r=0$ is not a root

should choose $m = 2$.

$$W = \operatorname{span}\{\underbrace{t^2, t, 1}_{B}\}.$$

Step 3. Solve this as a lin. sys. problem.

$$[t^2]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$T: W \to W$$

$$[T]_B = \left[[T(\vec{b}_1)]_B \quad [T(\vec{b}_2)]_B \quad [T(\vec{b}_3)]_B \right]$$

$$T(\vec{b}_1) = T(t^2) = 2 + 4t + t^2$$

$$[T(\vec{b}_1)]_B = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$$

$$T(t^2) = (t^2)'' + 2(t^2)' + t^2$$
$$= 2 + 4t + t^2$$

Similarly

$$[T(\vec{b_2})]_B = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \quad [T(\vec{b_3})]_B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Solve

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 4 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 \end{array}\right] \longrightarrow [\vec{y}]_B = \begin{bmatrix} 1 \\ -4 \\ 6 \end{bmatrix}$$

A Solution: $y_p(t) = t^2 - 4t + 6$

All solutions

$$y(t) = c_1 e^{-t} + c_2 t e^{-t}$$

$$+ y_p(t)$$

fixed by initial condition.

Recall lin sys.

$A\vec{x} = \vec{b}$, ① Find $\vec{x}_p$, s.t. $A\vec{x}_p = \vec{b}$

② sol set $= \{ \vec{x}_p + \vec{w} \mid A\vec{w} = \vec{0} \}$.

Ex. $y'' + 3y' + 2y = e^{-2t}$

Step 1: $e^{-2t} = t^{l} e^{rt}$ with $l = 0$, $r = -2$

Step 2: $r = -2$ is a single root of aux eq. $m = l + 1 = 1$.

$$W = \operatorname{span}\underbrace{\{te^{-2t}, e^{-2t}\}}_{B}.$$

step 3:

$$T(\vec{b_1}) = (te^{-2t})'' + 3(te^{-2t})' + 2(te^{-2t})$$

$$= -e^{-2t}$$

$$T(\vec{b_2}) = 0$$

$$[T(\vec{b_1})]_B = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \qquad [T(\vec{b_2})]_B = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$[e^{-2t}]_B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Solve $\left[\begin{array}{cc|c} 0 & 0 & 0 \\ -1 & 0 & 1 \end{array}\right] \Rightarrow [\vec{y}_p]_B = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$$\vec{y}_p(t) = -\ te^{-2t}$$

Sol set $\vec{y}(t) = c_1 e^{-2t} + c_2 e^{-t} + \ te^{-2t}$

Ex. $y'' - 2y' + y = e^t$

Step 1. $e^t = t^l e^{rt}$, $l = 0$, $r = 1$.

Step 2. $r = 1$ is a double root of aux. eq.

$m = l + 2 = 2$

$$W = \operatorname{span}\underbrace{\{t^2 e^t, te^t, e^t\}}_{B}.$$

Step 3.

$$T(\vec{b}_1) = 2e^t \quad . \quad [T(\vec{b}_1)]_B = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

$$T(\vec{b}_2) = T(\vec{b}_3) = 0 .$$

$$[e^t]_B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Solve $\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 \end{array}\right] \to [\vec{y}_p]_B = \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \end{bmatrix}$

$$y_p(t) = \frac{1}{2} t^2 e^t .$$

All sol: $y(t) = c_1 e^t + c_2 t e^t$

$$+ \frac{1}{2} t^2 e^t$$