Lec. 27 Warmup

$$U = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \longrightarrow \text{rotation matrix in } \mathbb{R}^2$$

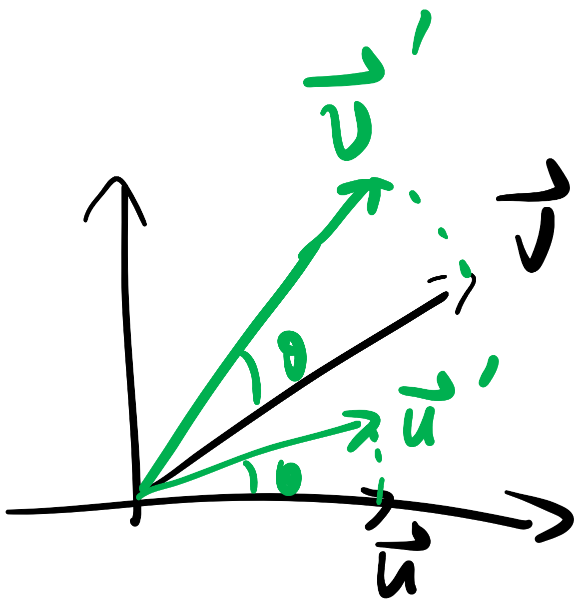
$\forall \vec{u}, \vec{v} \in \mathbb{R}^2$

$$\vec{u} \cdot \vec{v} = (U\vec{u}) \cdot (U\vec{v}) \ ? \quad \checkmark$$

$$RHS = (U\vec{u})^T (U\vec{v}) = \vec{u}^T (U^T U) \vec{v}$$

$$\left(U^T U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 \right)$$

$= \vec{u} \cdot \vec{v} \qquad \forall \ \vec{u}, \vec{v} \in \mathbb{R}^2$



U orthogonal matrix

$U^T U = I_N$

$\Rightarrow$ preserves inner prod.

angle.

Inner product space $\mathbb{R}^N$

(General) inner product space.

Def An inner product of a vector space V is a function that maps a pair of vectors $\vec{u}, \vec{v} \in V$ to a real (complex) number, denoted by

$$\langle \vec{u}, \vec{v} \rangle$$ satisfying

① $\langle \vec{u}, \vec{v} \rangle = \overline{\langle \vec{v}, \vec{u} \rangle}$

② $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$

③ $\langle c\vec{u}, \vec{v} \rangle = \overline{c} \langle \vec{u}, \vec{v} \rangle \quad c \in \mathbb{C}$

$\boxed{④}$ $\langle \vec{u}, \vec{u} \rangle \geq 0, \quad \langle \vec{u}, \vec{u} \rangle = 0$

$\Leftrightarrow \vec{u} = \vec{0}$

complex conjugation "$\bar{c}$" is for complex case.

$$③' \quad \langle \vec{u}, c\vec{v} \rangle \overset{①}{=} \overline{\langle c\vec{v}, \vec{u} \rangle}$$

$$\overset{③}{=} \overline{\bar{c} \langle \vec{v}, \vec{u} \rangle}$$

$$\overset{①}{=} c \langle \vec{u}, \vec{v} \rangle$$

Ex. $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, $\vec{u}, \vec{v} \in \mathbb{R}^2$

$\langle \vec{u}, \vec{v} \rangle = \vec{u}^T A \vec{v}$, $a, b \in \mathbb{R}$

is it an inner prod on $\mathbb{R}^2$?

False.

say $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\forall \vec{u} \in \mathbb{R}^2$

$\langle \vec{u}, \vec{u} \rangle \equiv 0 \not\Rightarrow \vec{u} = \vec{0}$ violates 4 same problem, $a, b > 0$

True.

① - ③ check by yourself.

④ $\vec{u} \in \mathbb{R}^2$, $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

$$\vec{u}^T A \vec{u} = a u_1^2 + b u_2^2 = 0$$

$$\Rightarrow \begin{cases} a u_1^2 = 0 \\ b u_2^2 = 0 \end{cases} \Rightarrow u_1 = u_2 = 0 \Rightarrow \vec{u} = \vec{0} \checkmark$$

Ex. $V = \mathbb{P}_n$, $f, g \in V$

$$\langle f, g \rangle = \int_0^1 f(x)\, g(x)\, dx .$$

① - ③ ✓

Directly check ④. $f \in V$

$$\langle f, f \rangle = \int_0^1 f^2(x)\, dx = 0$$

$\Rightarrow f \equiv 0$. $f = \vec{0}$ ✓

Ex. Same setup.

$$\langle f, g \rangle = \int_0^1 f'(x)\, g'(x)\, dx$$

False

$$f(x) \equiv 1$$

$$\langle f, f \rangle = 0 \not\Rightarrow f = \vec{0}$$

Ex. Same setup.

$$\langle f, g \rangle = \left(\int_0^1 f'(x)\, g'(x)\, dx \right) + f(0)\, g(0)$$

$f \in V$

$$0 = \langle f, f \rangle = \int_0^1 (f'(x))^2 dx + f^2(0)$$

$$\Rightarrow \begin{cases} \int_0^1 (f'(x))^2 dx = 0 \Rightarrow \underline{f'(x) \equiv 0} \\ f(0) = 0 \end{cases}$$

$$f(x) = \int_0^x f'(y)\,dy + f(0) \equiv 0$$

$$\Rightarrow f = \vec{0}$$

$$L^2([0,1]) = \left\{ f \;\middle|\; \int_0^1 f^2(x)\,dx < \infty \right\}$$

$$H^1([0,1]) = \left\{ f \;\middle|\; \int_0^1 (f'(x))^2\,dx + \int_0^1 f^2(x)\,dx < \infty \right\}$$

Everything in Chap 6

$\mathbb{R}^n \longrightarrow V$

① length . angle

$\sqrt{\langle \vec{u}, \vec{u} \rangle} \equiv \|\vec{u}\|$ $\cos\theta = \dfrac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \|\vec{v}\|}$

orthogonality. $\langle \vec{u}, \vec{v} \rangle = 0$ $\vec{u} \perp \vec{v}$

② Projection. Gram-Schmidt

③ Best approx. Least-squares

Ex. $P_1(x) = 1$, $P_2(x) = x$, $P_3(x) = \frac{3}{2}x^2$

Orthogonalize (G-S) w.r.t.

inner prod.

$$\langle f, g \rangle = \int_{-1}^{1} f(x)\, g(x)\, dx$$

Apply G-S.

$$q_1(x) = p_1(x) = 1$$

$$q_2(x) = p_2(x) - \frac{\langle q_1, p_2 \rangle}{\langle q_1, q_1 \rangle} q_1(x) = x$$

$$\langle q_1, p_2 \rangle = \int_{-1}^{1} 1 \cdot x\, dx = 0$$

$$q_3(x) = p_3(x) - \frac{\langle q_1, p_3 \rangle}{\langle q_1, q_1 \rangle} q_1(x)$$

$$- \frac{\langle q_2, p_3 \rangle}{\langle q_2, q_2 \rangle} q_2(x)$$

$$\langle q_1, p_3 \rangle = \int_{-1}^{1} 1 \cdot \frac{3}{2} x^2 \, dx = \int_0^1 3x^2 \, dx$$

$$= x^3 \Big|_0^1 = 1$$

$$\langle q_1, q_1 \rangle = \int_{-1}^{1} 1 \cdot 1 \, dx = 2$$

$$\langle q_2, q_3 \rangle = \int_{-1}^{1} x \cdot \frac{3}{2} x^2 \, dx = 0$$

$$q_3(x) = \frac{3}{2} x^2 - \frac{1}{2}$$

$\{q_1, q_2, q_3\}$ is orthogonal w.r.t. inner prod.

First 3 Legendre polynomial

$\{1, x, x^2, \ldots, x^n\}$

Apply G-S. Then up to normalization constants

w.r.t. L^2 inner product.

$\Rightarrow$ Legendre poly.