Lec 34. Warm up

Write down the form of a particular sol to

(1) $y'' + 2y' - 3y = 10e^{-3t}$

aux. $r^2 + 2r - 3 = 0 \Rightarrow r_1 = -3,\ r_2 = 1$

$$y_p(t) = A_0 t e^{-3t}$$

(2) $y'' + 2y' - 3y = 2t^2 e^t$

$$y_p(t) = t\left(A_2 t^2 + A_1 t + A_0\right)e^t$$

(3) $y'' + 2y' - 3y = e^t \cos t$

$$e^t \cos t = \frac{1}{2}\left[e^{(1+i)t} + e^{(1-i)t}\right]$$

$1 \pm i$ is not a root

$$y_p(t) = A_0 e^{(1+i)t} + B_0 e^{(1-i)t}, \quad A_0, B_0 \in \mathbb{C}$$

$$= C_0 e^t \cos t + D_0 e^t \sin t, \quad C_0, D_0 \in \mathbb{R}$$

(4) $y'' + y = t \cos t$

aux. $r^2 + 1 = 0 \Rightarrow r = \pm i$

$$t \cos t = \frac{1}{2} t e^{it} + \frac{1}{2} t e^{-it}$$

$$y_p(t) = t(C_1 t + C_0)\cos t + t(D_1 t + D_0)\sin t, \quad C_0, C_1, D_0, D_1 \in \mathbb{R}.$$

High order diff eq.

Same as second order.

Ex. Solve IVP.

$$\begin{cases} y''' - y' = t + \sin(2t) \\ y(0) = 2 \\ y'(0) = 0 \\ y''(0) = 0 \end{cases}$$

Step 1. hom. eq.

$$y''' - y' = 0 \qquad T(y) = 0$$

aux. eq. $r^3 - r = 0 \Rightarrow r_1 = 0,\ r_2 = 1,\ r_3 = -1$

$$T(y) = \left(\frac{d}{dt} - r_1 I\right)\left(\frac{d}{dt} - r_2 I\right)\left(\frac{d}{dt} - r_3 I\right)$$

basis of $Null(T) = \{1, e^t, e^{-t}\}$.

Step 2. Find particular sol.

Use superposition principle

a) $y''' - y' = t$

guess work. $t = te^{0 \cdot t}$ 0. single root

$$y_p(t) = t(A_1 t + A_0)$$

$$y_p'(t) = 2A_1 t + A_0$$

$$y_p''(t) = 2A_1$$

$$y_p'''(t) = 0$$

$$\Rightarrow -2A_1 t - A_0 = t$$

$$\Rightarrow \begin{cases} A_0 = 0 \\ A_1 = -\frac{1}{2} \end{cases}$$

$$y_p(t) = -\frac{1}{2}t^2$$

b) $y''' - y' = \sin 2t$

$$\sin 2t = \frac{1}{2i}\left(e^{2it} - e^{-2it}\right)$$

$$\cos 2t = \frac{1}{2}\left(e^{2it} + e^{-2it}\right)$$

$\pm 2i$ not a root of $r^3 - r = 0$.

$y_p(t) = A_0 \cos 2t + B_0 \sin 2t$, $A_0, B_0 \in \mathbb{R}$

$y_p'(t) = -2A_0 \sin 2t + 2B_0 \cos 2t$

$$y_p''(t) = -4A_0 \cos 2t - 4B_0 \sin 2t$$

$$y_p'''(t) = 8A_0 \sin 2t - 8B_0 \cos 2t$$

$$T(y_p) = 10A_0 \sin 2t - 10B_0 \cos 2t$$

$$= \sin 2t$$

$$\Rightarrow \begin{cases} A_0 = \frac{1}{10} \\ B_0 = 0 \end{cases}$$

$$y_p(t) = \frac{1}{10} \cos 2t$$

Step 3. Plug in initial value.

$$y(t) = -\frac{1}{2}t^2 + \frac{1}{10}\cos 2t + C_1 + C_2 e^{t} + C_3 e^{-t}$$

$$y(0) = \frac{1}{10} + C_1 + C_2 + C_3 = 2$$

$$y'(0) = \qquad C_2 - C_3 = 0$$

$$y''(0) = -\frac{1}{2} - \frac{2}{5} + C_2 + C_3 = 0$$

$$\Rightarrow \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{7}{10} \\ \frac{7}{10} \end{bmatrix} \qquad \square$$

n-th order diff eq. w. const coef.

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y^{(1)} + a_n y = 0.$$

aux. eq.

$$r^n + a_1 r^{n-1} + \cdots + a_n = 0$$

Fundamental thm of algebra.

n roots (counting multiplicity) in $\mathbb{C}$

① r_0 is a simple root. $e^{r_0 t}$

② $r_1 = \alpha + i\beta$
$r_2 = \alpha - i\beta$
complex conjugate pairs.

$$\text{span}\{e^{(\alpha+i\beta)t}, e^{(\alpha-i\beta)t}\}$$

$$= \text{span}\{e^{\alpha t}\cos\beta t, e^{\alpha t}\sin\beta t\}.$$

③ r_0 repeated root of multiplicity k.

$$r^n + a_1 r^{n-1} + \cdots + a_n = (r - r_0)^k g(r)$$

$g(r_0) \neq 0$.

$$\text{span}\{e^{r_0 t}, te^{r_0 t}, \ldots, t^{k-1} e^{rt}\}.$$

Why? Recall 2nd order.

$$\text{span}\{e^{r_0 t}, e^{(r_0+\epsilon)t}\}$$

$$= \mathrm{span}\left\{e^{r_0 t}, \frac{e^{(r_0+\varepsilon)t} - e^{r_0 t}}{\varepsilon}\right\}$$

$$\xrightarrow{\varepsilon \to 0} \mathrm{span}\left\{e^{r_0 t}, t e^{r_0 t}\right\}$$

Now 3rd order

$$\mathrm{span}\left\{e^{r_0 t}, e^{(r_0+\varepsilon)t}, e^{(r_0+2\varepsilon)t}\right\}.$$

$$= \mathrm{span}\left\{e^{r_0 t}, \frac{1}{\varepsilon}\left(e^{(r_0+\varepsilon)t} - e^{r_0 t}\right), \frac{1}{2\varepsilon}\left(e^{(r_0+2\varepsilon)t} - e^{r_0 t}\right)\right\}$$

$$\xrightarrow{\varepsilon \to 0} \mathrm{span}\{e^{r_0 t},\ \underbrace{te^{r_0 t},\ te^{r_0 t}}_{\text{lin. dep. sol.}}\}$$

Fix:

$$\frac{e^{(r_0+2\varepsilon)t} - 2e^{(r_0+\varepsilon)t} + e^{r_0 t}}{\varepsilon^2}$$

$$\longrightarrow \frac{(\cancel{1}+\cancel{\varepsilon t}+\frac{1}{2}\varepsilon^2 t^2+\cancel{1}-\cancel{\varepsilon t}+\frac{1}{2}\varepsilon^2 t^2-\cancel{2})}{\varepsilon^2}\, e^{(r_0+\varepsilon)t} + O(\varepsilon)$$

$$\xrightarrow{\varepsilon \to 0} t^2 e^{r_0 t}$$

$$\text{span}\{e^{r_0 t}, te^{r_0 t}, t^2 e^{r_0 t}\}$$

Ex. Solve $y^{(4)} - y^{(2)} = 0$

aux. $r^4 - r^2 = 0 \Rightarrow r^2(r+1)(r-1) = 0$

$r_1 = 0$ mult. 2

$r_2 = 1$ " 1

$r_3 = -1$ " 1

$$y(t) = c_1 + c_2 t + c_3 e^{t} + c_4 e^{-t}$$

All high order diff. eq.

ARE first order vector diff. eq!

Ex. $y^{(4)} - y^{(2)} = 0$.

Define $\vec{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ y_4(t) \end{bmatrix} \equiv \begin{bmatrix} y(t) \\ y'(t) \\ y''(t) \\ y'''(t) \end{bmatrix}$

$$\frac{d}{dt}\vec{y}(t) = \begin{bmatrix} y'(t) \\ y''(t) \\ y'''(t) \\ y^{(4)}(t) \end{bmatrix}$$

$$\overset{\text{matrix-vectr}}{\underset{\text{mult}}{=}} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \underbrace{0 \quad 1 \quad 0 \quad 0}_{\text{eq.}} \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ y_4(t) \end{bmatrix} = A\,\vec{y}(t)$$

1st order eq. normal form