Lec 22. Warmup

True / False :

$A \in \mathbb{R}^{n \times n}$ diagonalizable. Then A^T diagonalizable.

True .

$$A = VDV^{-1}, \quad V \in \mathbb{R}^{n \times n} \text{ invertible}$$

$$D \in \mathbb{R}^{n \times n} \text{ diagonal.}$$

$$A^T = \underbrace{(V^{-1})^T}_{U} D \, V^T \qquad A^T = UDU^{-1}$$

$$U^{-1} = V^T$$

A^T, A have the same set of eigenvalues (counting multiplicities)

(Recall $(AB)^T = B^T A^T$).

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matrix representation.

$$
\begin{tikzcd}
T: \mathbb{R}^n \arrow[r, "A"] & \mathbb{R}^n \arrow[d, "P_{\mathcal{B}\leftarrow\mathcal{E}}"] & \mathcal{E}\text{-basis} \\
\mathbb{R}^n \arrow[u, "P_{\mathcal{E}\leftarrow\mathcal{B}}"] \arrow[r, "{[T]_{\mathcal{B}}}"] & \mathbb{R}^n & \mathcal{B}\text{-basis}
\end{tikzcd}
$$

$$\mathcal{B} = \{\vec{v}_1, \cdots, \vec{v}_n\}$$

$$[T]_{\mathcal{B}} = \begin{bmatrix} [T\vec{v}_1]_{\mathcal{B}} & \cdots & [T\vec{v}_n]_{\mathcal{B}} \end{bmatrix}$$

Now if $T\vec{v}_i := A\vec{v}_i = \lambda_i \vec{v}_i$

$$[T]_{\mathcal{B}} = \begin{bmatrix} \lambda_1 [\vec{v}_1]_{\mathcal{B}} & \cdots & \lambda_n [\vec{v}_n]_{\mathcal{B}} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 \vec{e}_1 & \cdots & \lambda_n \vec{e}_n \end{bmatrix}$$

$$\equiv \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} = D \rightarrow \text{diagonal matrix.}$$

Ex. $T: \mathbb{P}_2 \to \mathbb{P}_2$. $T(p) = (x+1)\frac{dp}{dx}(x)$

Is there a basis of $\mathbb{P}_2$ s.t

$[T]_{\mathcal{B}}$ is a diagonal matrix?

Sol. standard basis $\mathcal{E} = \{1, x, x^2\}$.

$$[T]_{\mathcal{E}} = \left[\, [T(1)]_{\mathcal{E}} \;\; [T(x)]_{\mathcal{E}} \;\; [T(x^2)]_{\mathcal{E}} \,\right]$$

$T(1) = 0$, $T(x) = 1+x$.

$$T(x^2) = 2x(x+1) = 2x + 2x^2$$

$$[T]_{\mathcal{E}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \leftarrow A$$

Now diagonalize A.

① eigen values.

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & 1-\lambda & 2 \\ 0 & 0 & 2-\lambda \end{vmatrix} = -\lambda(1-\lambda)(2-\lambda)$$

$\lambda_1 = 0, \quad \lambda_2 = 1, \quad \lambda_3 = 2.$

3 distinct eigenvalues $\Rightarrow$ diagonalizable

② eigenvectors. (exer)

$V = [\vec{v}_1, \ \vec{v}_2, \ \vec{v}_3]$

$AV = VD$. $\quad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Find basis $\mathcal{B}$ (in $\mathbb{P}_2$)

$\mathcal{B} = \{ \vec{b}_1, \vec{b}_2, \vec{b}_3 \}$

polynomials.

$$\vec{b}_i = 1 \cdot (\vec{v}_i)_1 + x \cdot (\vec{v}_i)_2 + x^2 \cdot (\vec{v}_i)_3$$

(exer) check: $[T]_{\mathcal{B}}$ is indeed diagonal.

<u>Def</u> $A, B \in \mathbb{R}^{n \times n}$. A is similar to B if there exists an invertible

matrix V s.t.

$$A = V^{-1} B V$$

If A diagonalizable. $\Rightarrow$ A, D are similar.

<u>Thm</u>. $A, B \in \mathbb{R}^{n \times n}$ are similar.

then $$|A - \lambda I| = |B - \lambda I| . \quad \lambda \in \mathbb{C}$$

Therefore A, B share the same set of eig. values. (counting multiplicities)

Pf: $A = V^{-1} B V$ $\qquad (\lambda I = \lambda V^{-1} V)$

$$A - \lambda I = V^{-1} B V - V^{-1} \cdot (\lambda I) V$$

$$= V^{-1} (B - \lambda I) V$$

$$\Rightarrow |A - \lambda I| = |V^{-1}| \cdot |B - \lambda I| \cdot |V|$$

$$= |V^{-1} \cdot V| \cdot |B - \lambda I|$$

$$= |B - \lambda I| . \qquad \square .$$

Ex. $|A - \lambda I| = |B - \lambda I|$ for any λ

$\overset{?}{\Rightarrow}$ A is similar to B ?

False. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

not diagonalizable.

$$|A-\lambda I| = (1-\lambda)^2 = |B-\lambda I|$$

Now we show A cannot be similar to B.

Assume this is not the case, i.e. there exists $V \in \mathbb{R}^{2\times 2}$ s.t.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = V^{-1} B V.$$

Then B is diagonalizable. contradiction. $\square$.

More generally. A diagonalizable

B not "

$\Rightarrow$ A. B cannot be similar.

Ex. $A = \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix}$. $B = \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix}$

similar ?

$$|A-\lambda I| = \begin{vmatrix} -2-\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda(2+\lambda)+1$$

$$= (\lambda+1)^2$$

$$|B-\lambda I| = \begin{vmatrix} 3-\lambda & 1 \\ -2 & -\lambda \end{vmatrix} = \lambda(\lambda-3)+2$$

$$= (\lambda-1)(\lambda-2)$$

$\neq$

not similar.

Thm. $A, B, C \in \mathbb{R}^{n \times n}$.

A similar to B

B " " C

$\Rightarrow A$ " " C.

$$\underbrace{A \sim B \sim C}_{\sim}$$

Pf: there exist $V, W \in \mathbb{R}^{n \times n}$ invertible

$$A = V^{-1} B V, \quad B = W^{-1} C W$$

$$\Rightarrow A = \underbrace{V^{-1} W^{-1}} C\, W V$$

$$= (\underbrace{WV}_{\text{invertible.}})^{-1} C\, (WV) \qquad \square$$

(optional reading. Book 5.6.

applications to discrete dyn.

systems).