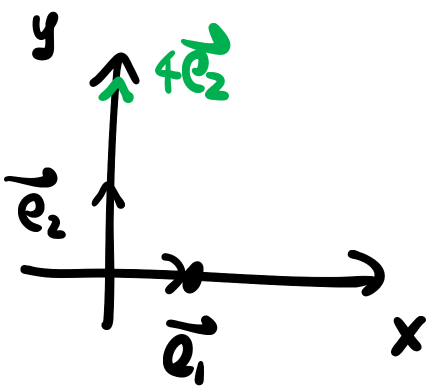
Lec 20. Warmup.

Ex. $A = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$ Find eigenvalues & eigenvectors.

geometrically



$A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$A\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix} \rightarrow 4 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

$\lambda_1 = 1$ $\qquad$ $\lambda_2 = 4$

Try $A = \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix}$

$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $A\vec{v}_1 = \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} = 2 \cdot \vec{v}_1$

w. $\lambda_1 = 2$

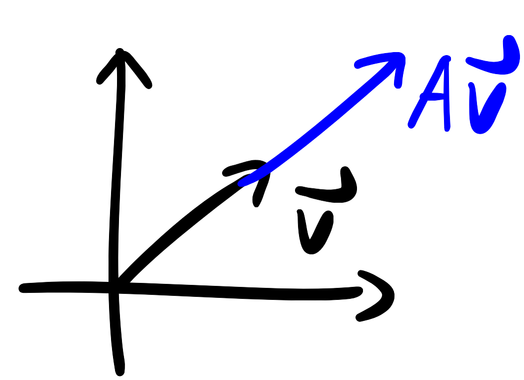
$\vec{v}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $A\vec{v}_2 = \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix} = -2 \cdot \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

w. $\lambda_2 = -2$

How to find eigenvalues &

eigenvectors systematically?

Def. $\underline{A\vec{v} = \lambda \vec{v}}$



$\iff (A-\lambda I)\vec{v} = 0.$ $\quad \vec{v} \neq \vec{0}$

$\iff A-\lambda I$ is not invertible

$\iff \det(A-\lambda I) = 0$

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ \vdots & \ddots & & \vdots \\ a_{n1} & & \cdots & a_{nn}-\lambda \end{vmatrix} = 0 \longrightarrow \text{polynomial of } \lambda \text{ of degree } n.$$

cofactor

$$\underbrace{(a_{11}-\lambda)}_{\text{degree } 1} \underbrace{C_{11}(\lambda)}_{\text{of degree } n-1} + a_{12}\, C_{12}(\lambda) + \cdots + a_{1n}\, C_{1n}(\lambda) = 0$$

$P_n(\lambda) = 0 \Rightarrow$ Find all its roots

$\Rightarrow$ eigenvalues.

Ex. $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, $\quad A - \lambda I_2 = \begin{bmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix}$

$$|A-\lambda I_2| = (1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3$$

$$= (\lambda-3)(\lambda+1) = 0$$

$$\Rightarrow \quad \lambda_1 = 3 \quad , \quad \lambda_2 = -1.$$

Find eigenvectors.

$\vec{v}$ is the non-trivial sol(s) of

$$(A - \lambda I)\,\vec{v} = \vec{0}$$

is known

$\lambda_1 = 3.$ $\quad (A - 3I)\vec{v} = \vec{0}.$

$$\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \vec{v} = \vec{0} \Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\lambda_2 = -1.$ $\quad (A + I)\vec{v} = \vec{0}$

$$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \vec{v} = \vec{0} \Rightarrow \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Q1: Is the number of eigenvalues ALWAYS equal to # of eigenvectors? ✗

Ex. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 = 0$$

$\Rightarrow \lambda = 1$ (multiplicity 2)

$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ both eig. vectors.

Q2: Is the # of eig. vals. (counting multiplicity) ALWAYS equal to # of eig. vec.?

$\times$

Ex. $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

$$|A - \lambda I| = \begin{bmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{bmatrix} = (2-\lambda)^2 = 0$$

$\Rightarrow \lambda = 2$ (multiplicity 2)

$$\begin{bmatrix} 0 & \boxed{1} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \longrightarrow$ one eigenvector.

Ex. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

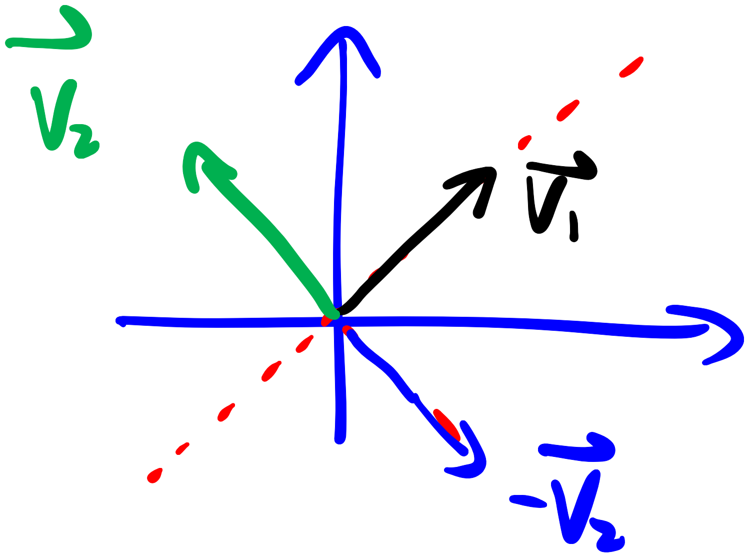
$$0 = \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 \Rightarrow \lambda_1 = 1, \ \lambda_2 = -1.$$

For $\lambda_1 = 1$.

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \vec{v}_1 = \vec{0} \Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

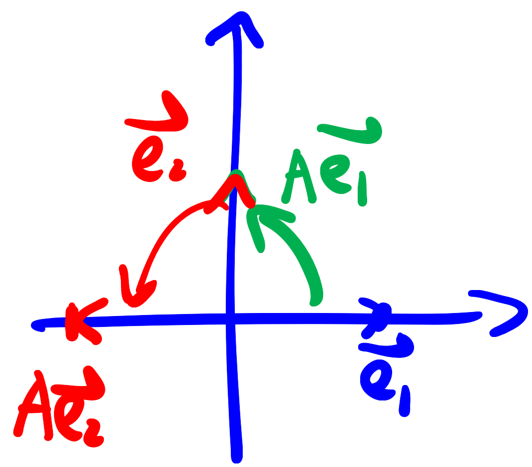
For $\lambda_2 = -1$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \vec{v}_2 = \vec{0} \Rightarrow \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$



reflection

Ex. $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$



rotation

$$|A-\lambda I| = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0$$

$$\Rightarrow \lambda_1 = +i \ , \ \lambda_2 = -i \qquad i^2 = -1$$

Complex eigenvalues for real matrix

$$\lambda_1 = i, \quad \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \vec{V_1} = \vec{0} \Rightarrow \vec{V_1} = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

$$\lambda_2 = -i, \quad \begin{bmatrix} i & 1 \\ 1 & i \end{bmatrix} \vec{V_2} = \vec{0} \Rightarrow \vec{V_2} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

Def $A \in \mathbb{R}^{n \times n}$. If we can find

n lin. indep. vectors $\vec{v}_1, \cdots, \vec{v}_n$ s.t.

$$A\vec{v}_i = \lambda_i \vec{v}_i \quad , i = 1, \cdots n$$

Then A is diagonalizable.

$$\left(AV = VD , \quad D = V^{-1}AV \right)$$

Previously. $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

is NOT diagonalizable

Q3: Is there arbitrarily large non-diagonalizable matrix? ✓

Thm. For any n. there exists $A \in \mathbb{R}^{n \times n}$ not diagonalizable.

<u>Pf</u>: constructive. any $\lambda \in \mathbb{R}$

$$A_n = \begin{bmatrix} \lambda & 1 & & 0 \\ & \lambda & 1 & \\ & & \ddots & \ddots \\ & 0 & & \ddots & 1 \\ & & & & \lambda \end{bmatrix}$$

Ex. $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$, $\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$, $\begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix}$ ...

Claim: A is not diagonalizable

$$|A-\lambda' I| = \begin{vmatrix} \lambda-\lambda' & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda-\lambda' \end{vmatrix} = (\lambda-\lambda')^n = 0$$

$\Rightarrow \lambda' = \lambda$ (of multiplicity n).

$$\begin{bmatrix} 0 & \boxed{1} & & \\ & 0 & \boxed{1} & \\ & & \ddots & \ddots \\ & & & \boxed{1} \\ & & & 0 \end{bmatrix} \vec{v} = \vec{0}$$

free

dim Null = 1

$$\vec{v} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

1 eigenvector.