Lec 33.

$$y'' + by' + cy = \underbrace{t^{l} e^{rt}}_{}$$

↳ most general form.

Revisit. $r = 0$, $l = 0$

$$y'' + by' + cy = 1 \quad \leftarrow e^{rt} \text{ w. } r = 0$$

① 0 is not a root

$$y_p = \tilde{C} \Rightarrow c\tilde{C} = 1 \Rightarrow \tilde{C} = \frac{1}{c}$$

② 0 is a single root $\Rightarrow c=0$

$$y_p = \tilde{C}\, t$$

$$\tilde{C}\, b = 1 \Rightarrow y_p = \frac{1}{b} t$$

③ 0 is a double root $\Rightarrow b = c = 0$

$$y_p = \tilde{C}\, t^2$$

$$2\tilde{C} = 1 \Rightarrow y_p(t) = \frac{1}{2} t^2$$

General ℓ, $r=0$ polynomial.

$$y'' + by' + cy = t^{\ell}$$

① 0 is not a root of $r^2 + br + c = 0$.

$W = P_{\ell}$

$$y_p(t) = A_{\ell} t^{\ell} + \cdots + A_0.$$

② 0 is a single root

$$y_p(t) = t\,(A_{\ell} t^{\ell} + \cdots + A_0)$$

$W = P_{\ell+1}$

③ 0 is a double root

$$y_p(t) = t^2 (A_\ell t^\ell + \cdots + A_0) \qquad W = \mathbb{P}_{\ell+2}$$

Most general case

$$y'' + by' + cy = t^\ell e^{rt}.$$

$$W = \text{“} e^{rt}\,\mathbb{P}_m \text{”} := \operatorname{span}\{e^{rt}t^m, \dots, e^{rt}\}$$

$$\dim(W) = m+1$$

$$y_p(t) = e^{rt} p(t), \quad p(t) \in \mathbb{P}_m$$

$$T(y) = \left(e^{rt}p(t)\right)'' + b\left(e^{rt}p(t)\right)' + ce^{rt}p(t)$$

$$= \left(re^{rt}p(t) + e^{rt}p'(t)\right)'$$

$$+ b\left(re^{rt}p(t) + e^{rt}p'(t)\right)$$

$$+ ce^{rt}p(t)$$

$$= \left(r^2e^{rt}p(t) + 2re^{rt}p'(t) + e^{rt}p''(t)\right)$$
$$+ b\left(re^{rt}p(t) + e^{rt}p'(t)\right)$$

$$+ ce^{rt}p(t)$$

$$= \underbrace{(r^2+br+c)}_{\text{aux eq.}}\cancel{e^{rt}}\,p(t) \in \mathbb{P}_m$$

$$+ (2r+b)\cancel{e^{rt}}\,p'(t) \in \mathbb{P}_{m-1}$$

$$+ \cancel{e^{rt}}\,p''(t) = \cancel{e^{rt}}\,t^{\ell}$$

$$\in \mathbb{P}_{m-2}$$

① r is not a root $r^2+br+c \neq 0$

$W =$ "$e^{rt}\mathbb{P}_\ell$" is enough.

$$y_p(t) = (A_\ell t^\ell + \cdots + A_0)\, e^{rt}$$

② r is a single root

$r^2 + br + c = 0$. $2r + b \neq 0$.

$$W = \text{"} e^{rt} \mathbb{P}_{\ell+1} \text{"}$$

$$y_p(t) = t\,(A_\ell t^\ell + \cdots + A_0)\, e^{rt}$$

Note $e^{rt} \in Null(T)$

③ r is a double rout

$r^2+br+c=0$. $2r+b=0$.

$W =$ " $e^{rt}\mathbb{P}_{l+2}$ "

$$y_p(t) = t^2(A_l t^l + \cdots + A_0)\,e^{rt}$$

$e^{rt}, te^{rt} \in Null(T)$

Ex. $y'' = t\cos t$

$$\cos t = \frac{1}{2}\left(e^{it} + e^{-it}\right)$$

$$y'' = \frac{t}{2}e^{it} + \frac{t}{2}e^{-it}$$

superposition principle.

(1) $y'' = \frac{t}{2}e^{it}$. $\quad r = i \quad$ not a root

$$y_p^{(1)}(t) = (\tilde{A}_1 t + \tilde{A}_0) e^{it}. \qquad \tilde{A}_1, \tilde{A}_0 \in \mathbb{C}$$

(2) $y'' = \frac{t}{2} e^{-it}$. $\quad r = -i \quad$ not a root

$$y_p^{(2)}(t) = (\tilde{B}_1 t + \tilde{B}_0) e^{-it} \qquad \tilde{B}_1, \tilde{B}_0 \in \mathbb{C}$$

Rewrite

$$A_1, A_0, B_1, B_0 \in \mathbb{R}$$

$$y_p(t) = y_p^{(1)} + y_p^{(2)}$$

$$\overset{\mathbb{R}}{=} (A_1 t + A_0) \cos t + (B_1 t + B_0) \sin t \qquad (*)$$

$$y_p' = A_1 \cos t + (A_1 t + A_0)(-\sin t)$$

$$+ B_1 \sin t + (B_1 t + B_0) \cos t$$

$$y_p'' = -2A_1 \sin t - (A_1 t + A_0) \cos t$$

$$+ 2B_1 \cos t - (B_1 t + B_0) \sin t$$

$$= t \cos t$$

$$\begin{cases} -2A_1 - B_0 = 0 \\ B_1 = 0 \\ -A_1 = 1 \\ -A_0 + 2B_1 = 0 \end{cases} \Rightarrow \begin{cases} A_0 = 0 \\ A_1 = -1 \\ B_1 = 0 \\ B_0 = 2 \end{cases}$$

$$y_p(t) = -t\cos t + 2\sin t$$

Alternatively

$$T: W \longrightarrow W.$$

$$W = \text{span}\{t\cos t,\ t\sin t,\ \cos t,\ \sin t\}$$

$$\left[\begin{array}{c|c} [T]_{\mathcal{B}} & \begin{matrix}1\\0\\0\\0\end{matrix}\end{array}\right] \longrightarrow [y_p]_{\mathcal{B}}$$

So far.

$$y(t) = y_p(t) + \underset{\sim}{C_1}\, y_1(t) + \underset{\sim}{C_2}\, y_2(t)$$

$\mathrm{Null}(T)$

$$\begin{cases} y'' + 2y' - 3y = e^t \cos t \\ y(0) = 1 \\ y'(0) = 1 \end{cases}$$

Step 1.

$$e^t \cos t = \frac{1}{2}\left(e^{(1+i)t} + e^{(1-i)t}\right)$$

r

$r^2+2r-3=0 \Rightarrow r_1=-3,\ r_2=+1.$

$1\pm i$ not a root.

$W = \text{span}\{e^t\cos t,\ e^t\sin t\}$

$T[e^t\cos t] = -4e^t\sin t - e^t\cos t$

$T[e^t\sin t] = 4e^t\cos t - e^t\sin t$

$$[T]_{\mathcal{B}} = \begin{bmatrix} -1 & 4 \\ -4 & -1 \end{bmatrix}$$

Solve $\left[\begin{array}{cc|c} -1 & 4 & 1 \\ -4 & -1 & 0 \end{array}\right] \longrightarrow [y_p]_{\mathcal{B}} = \begin{bmatrix} -\frac{1}{17} \\ \frac{4}{17} \end{bmatrix}$

$$y_p(t) = -\frac{1}{17} e^t \cos t + \frac{4}{17} e^t \sin t$$

Step 2.

general sol.

$$y(t) = y_p(t) + C_1 y_1(t) + C_2 y_2(t)$$

$$y_1(t) = e^{-3t} \quad y_2(t) = e^t$$

Step 3. Find c_1, c_2 via initial vals.

$$y(0) = \boxed{y_p(0)} + c_1 + c_2 = 1$$

$$y'(0) = \boxed{y_p'(0)} - 3c_1 + c_2 = 1$$

$$\Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{17} \\ 1 \end{bmatrix}$$

$$y(t) = \frac{1}{17}e^{-3t} + e^{t} - \frac{1}{17}e^{t}\cos t + \frac{4}{17}e^{t}\sin t \qquad \square .$$