Lec 11.

Warm up

$$A = \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{bmatrix}$$

$Col(A) = span\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$, $rank(A) = 2$

$Null(A) = \left\{ x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \middle| x_2 \in \mathbb{R} \right\} \equiv span\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

check rank thm.

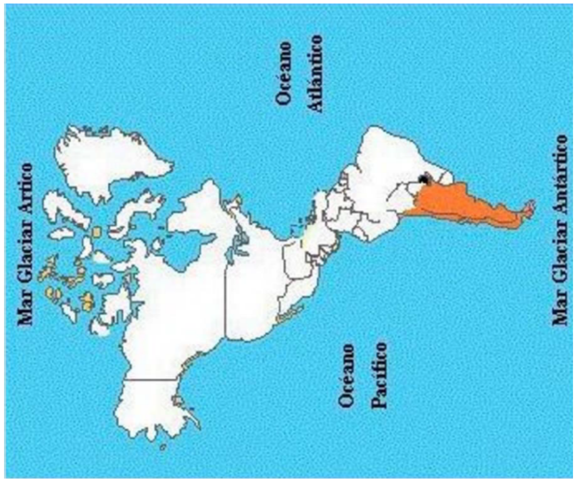
$$\begin{array}{c} \dim col(A) \\ ||| \end{array}$$

$3 = 2 + 1 \leftarrow \#cols = rank(A) + \dim Null(A)$

Vector space

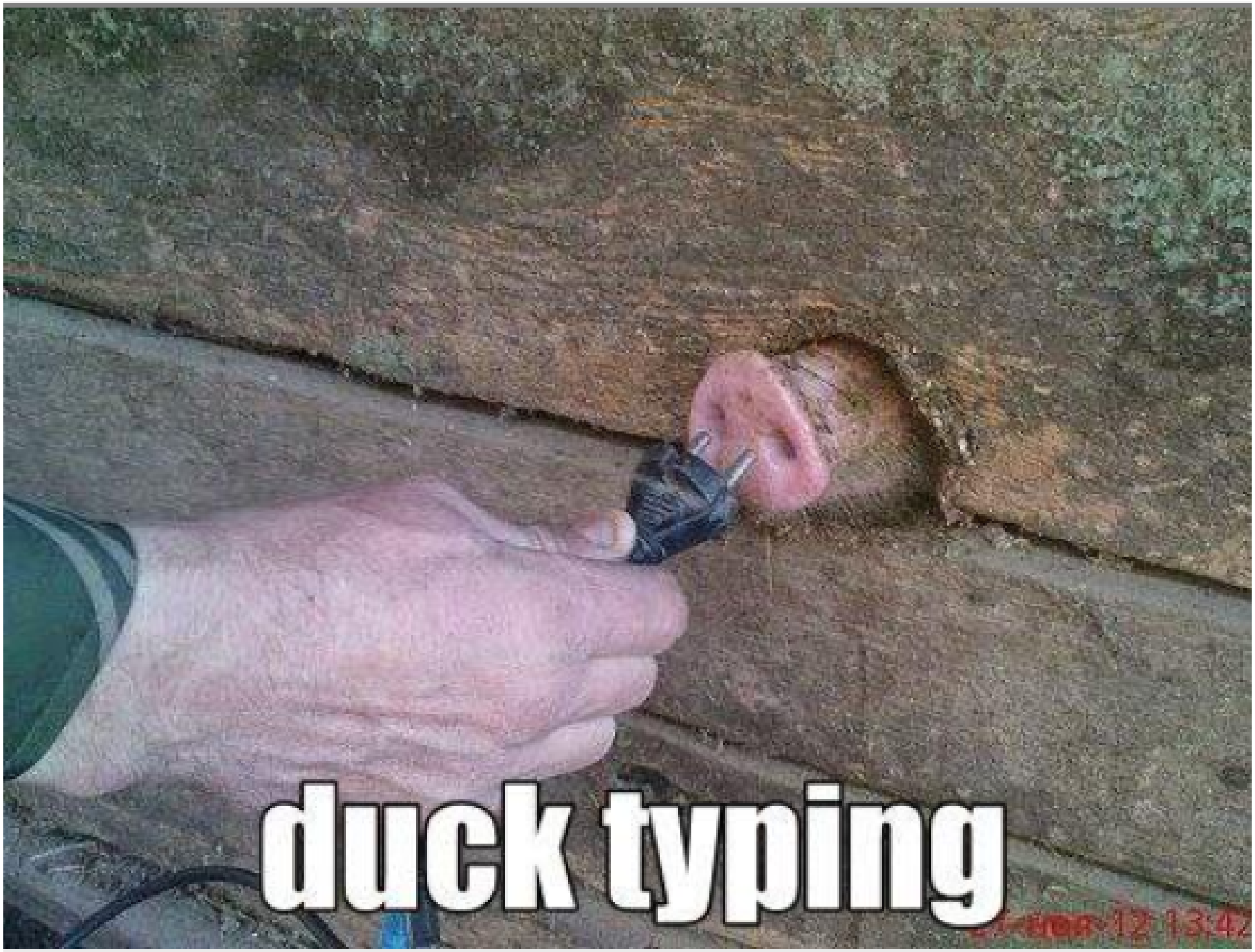
axiomatic approach

(1) looks



(2) quacks

$\Rightarrow$ duck.



A humorous and apt representation of duck typing. Source: Mastracci, 2014.*

A mathematical duck looks like...

A **vector space** is a nonempty set V of objects, called *vectors*, on which are defined two operations, called *addition* and *multiplication by scalars* (real numbers), subject to the ten axioms (or rules) listed below.[1] The axioms must hold for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and for all scalars c and d.

1. The sum of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u} + \mathbf{v}$, is in V.
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.
4. There is a **zero** vector $\mathbf{0}$ in V such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.
5. For each $\mathbf{u}$ in V, there is a vector $-\mathbf{u}$ in V such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
6. The scalar multiple of $\mathbf{u}$ by c, denoted by $c\mathbf{u}$, is in V.
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$.
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$.
10. $1\mathbf{u} = \mathbf{u}$.

A **vector space** is a nonempty set V of objects, called *vectors*, on which are defined two operations, called *addition* and *multiplication by scalars* (real numbers), subject to the ten axioms (or rules) listed below.[1] The axioms must hold for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and for all scalars c and d.

1. The sum of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u} + \mathbf{v}$, is in V. addition closed
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$. addition commutative
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$. addition associative
4. There is a **zero** vector $\mathbf{0}$ in V such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$. existence of 0
5. For each $\mathbf{u}$ in V, there is a vector $-\mathbf{u}$ in V such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$. existence negative
6. The scalar multiple of $\mathbf{u}$ by c, denoted by $c\mathbf{u}$, is in V. multiplication closed
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$. distributive (i)
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$. distributive (ii)
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$. multiplication associative
10. $1\mathbf{u} = \mathbf{u}$. multiplication by scalar 1.

Ex. 1) $V = \mathbb{R}^n$. addition, multiplication defined as before in Chap 1.

2) $V = \{ f : \mathbb{R} \to \mathbb{R} \text{ is a function} \}$.

addition $\quad (f+g)(x) := f(x) + g(x)$

scalar mult. $\quad (cf)(x) := c\, f(x)$

Ex. $V = \mathbb{P} = \{$ polynomial functions of finite degree $: \mathbb{R} \to \mathbb{R}\}$.

$$a_0 + a_1 x + \cdots + \underset{\neq 0}{a_n} x^n$$ $\to$ degree

Ex. $V = \mathbb{P}_n = \{$ polynomial $: \mathbb{R} \to \mathbb{R}$ of degree $\leq n \}$

Ex. $V = \{ \text{polynomial} : \mathbb{R} \to \mathbb{R}$ of degree $= n \}$ $(n > 0)$

NOT closed under addition:

$\vec{u} = x^n + x^{n-1}$, $\vec{v} = -x^n + x^{n-1}$

$\vec{u} + \vec{v} = 2x^{n-1} \in \mathbb{P}_{n-1}$

Ex. $V = S = \{(a_1, a_2, a_3, \cdots) \mid a_i \in \mathbb{R}\}$

$\hookrightarrow$ infinite seq.

addition: $(a_1, a_2, a_3, \cdots) + (b_1, b_2, b_3, \cdots)$

$= (a_1 + b_1, a_2 + b_2, \cdots)$

mul. $c(a_1, a_2, \cdots) = (ca_1, ca_2, \cdots)$

zero vector : $(0, 0, \cdots)$

Ex. $V =$ collection of all possible ducks differing only in width

 $+$  $=$ 

3  $=$ 

Ex. $V = \{ A \in \mathbb{R}^{n \times n} \text{ in REF} \}$

Not a vector space

$$\begin{bmatrix} \boxed{1} & 2 \\ 0 & \boxed{1} \end{bmatrix} + \begin{bmatrix} \boxed{-1} & 2 \\ 0 & \boxed{1} \end{bmatrix} = \begin{bmatrix} 0 & \boxed{4} \\ 0 & 2 \end{bmatrix}$$

should be 0 if in REF