Lec 7.

Warm up $r, \theta, \varphi \in \mathbb{R}$. $r > 0$.

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} r\cos\varphi \\ r\sin\varphi \end{bmatrix}$$

Compute the matrix transformation

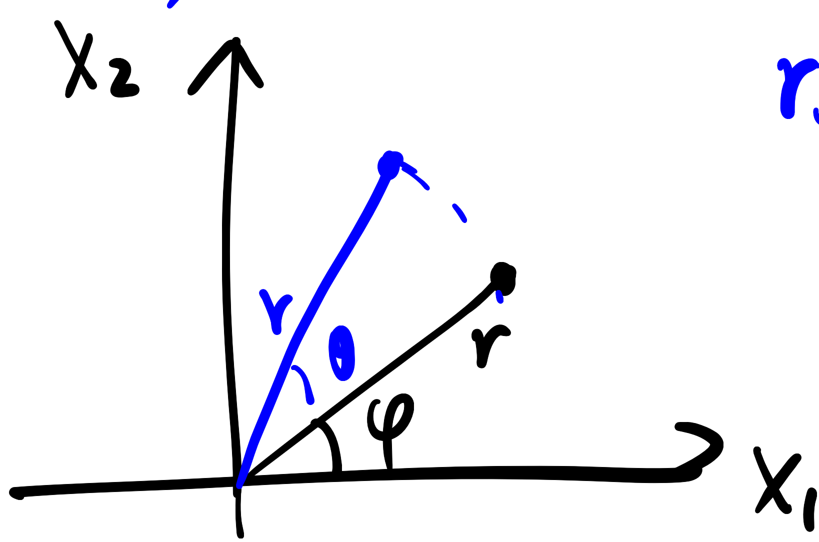
$$\vec{x} \mapsto A\vec{x}$$

$$A\vec{x} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} r\cos\varphi + \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} r\sin\varphi$$

$$= \begin{bmatrix} r(\cos\theta\cos\varphi - \sin\theta\sin\varphi) \\ r(\sin\theta\cos\varphi + \cos\theta\sin\varphi) \end{bmatrix}$$

$$= \begin{bmatrix} r\cos(\theta+\varphi) \\ r\sin(\theta+\varphi) \end{bmatrix}$$

Geometrically $\vec{x} \mapsto A\vec{x}$ is a rotation.



Def A transformation T is called a lin. trans. if $\vec{u}, \vec{v} \in \mathbb{R}^n$, $c \in \mathbb{R}$

(1) $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$

(2) $T(c\vec{u}) = c\,T(\vec{u})$

matrix trans. is a special case of lin. trans.

Not all trans. are linear.

Ex. $T: \mathbb{R}^2 \to \mathbb{R}^2$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} x_1^2 \\ x_2^2 \end{bmatrix}$$

$$T\left(\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} + \begin{bmatrix} u_2 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} (u_1+u_2)^2 \\ (v_1+v_2)^2 \end{bmatrix}$$

$$\neq \begin{bmatrix} u_1^2 \\ v_1^2 \end{bmatrix} + \begin{bmatrix} u_2^2 \\ v_2^2 \end{bmatrix}$$

Ex. translation

$$T: \mathbb{R}^2 \to \mathbb{R}^2$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} x_1 + c_1 \\ x_2 + c_2 \end{bmatrix}$$

$$T\left(a\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} ax_1 + c_1 \\ ax_2 + c_2 \end{bmatrix}$$

$$? \parallel$$

$$aT\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} ax_1 + ac_1 \\ ax_2 + ac_2 \end{bmatrix}$$

$ac_1 = c_1$

$ac_2 = c_2$

$\Rightarrow a \neq 0$ then

$c_1 = c_2 = 0.$

In particular $a=0$

$$T(\vec{0}) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \pm \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

<u>Thm</u> $T : \mathbb{R}^n \to \mathbb{R}^m$ is lin. trans.

$$\underset{\mathbb{R}^n}{T(\underset{\cap}{\vec{0}})} = \underset{\mathbb{R}^m}{\underset{\cap}{\vec{0}}}$$

Pf: Take any $\vec{u} \in \mathbb{R}^n$

$$T(0 \cdot \vec{u}) = T(\vec{0})$$

$$\shortparallel$$

$$0 \cdot T(\vec{u}) = \vec{0}$$

Any lin. trans. is a matrix trans.

Ex. $T: \mathbb{R}^2 \to \mathbb{R}^2$. $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$T(\vec{e}_1) = \begin{bmatrix} 2 \\ 5 \end{bmatrix} \qquad T(\vec{e}_2) = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \vec{e}_1 + x_2 \vec{e}_2$$

Use linearity

$$T(\vec{x}) = x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2)$$

$$= \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) \end{bmatrix} \vec{x}$$

$$= \begin{bmatrix} 2 & -1 \\ 5 & 6 \end{bmatrix} \vec{x} \rightarrow \text{mat. trans.}$$

<u>Thm</u>. $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ lin. trans.

then there exists a matrix A of size $m \times n$ s.t. for any $\vec{x} \in \mathbb{R}^n$

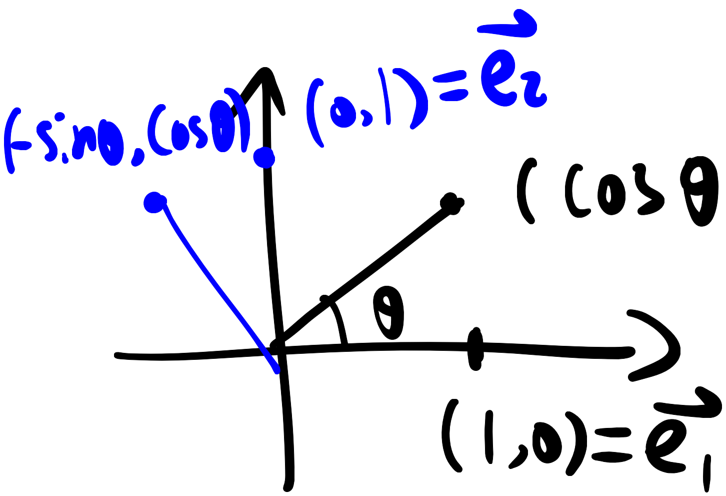
$$T(\vec{x}) = A\vec{x}$$

Pf: $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \cdots, \vec{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$

$$A = \begin{bmatrix} T(\vec{e}_1) & \cdots & T(\vec{e}_n) \end{bmatrix}$$

Standard matrix of T

Now back to warm up



$$T(\vec{e_1}) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$$T(\vec{e_2}) = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Def : $T : \mathbb{R}^n \to \mathbb{R}^m$ lin trans.

is called onto (a.k.a. surjective)

if for each $\vec{b} \in \mathbb{R}^m$ there is

at least one $\vec{x} \in \mathbb{R}^n$ s.t.

$$T(\vec{x}) = \vec{b}$$

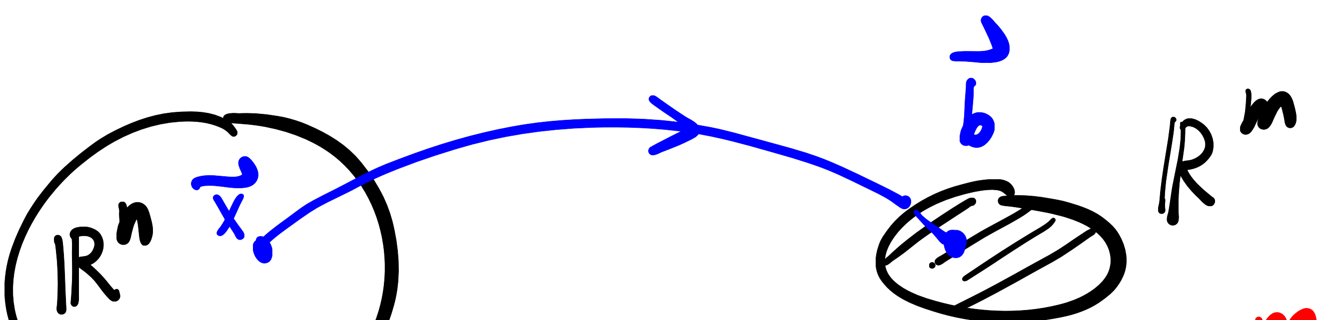
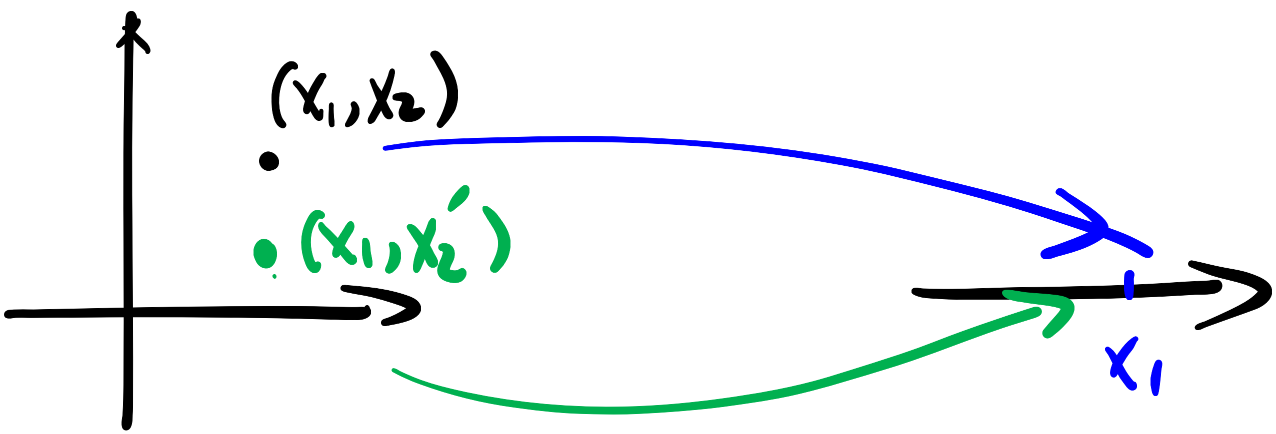


Image $(T) = \mathbb{R}^m$

Ex. $T : \mathbb{R}^2 \to \mathbb{R}$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto x_1$$



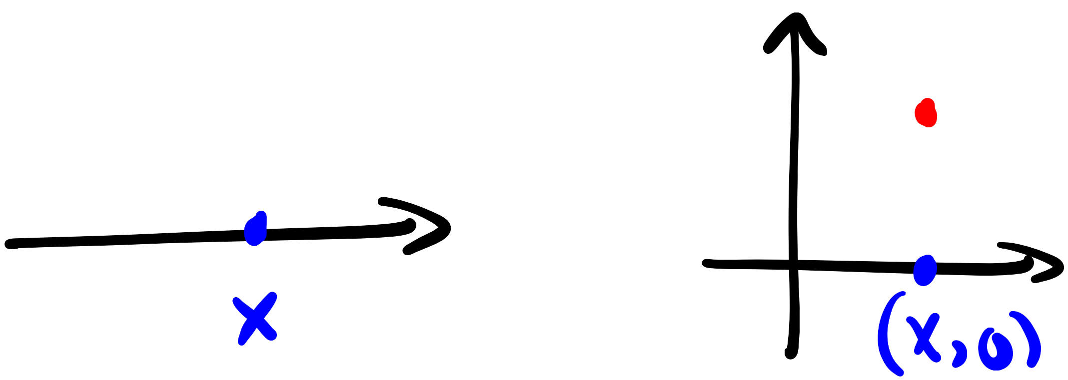
<u>Def</u> $T : \mathbb{R}^n \to \mathbb{R}^m$ is called one-to-one (a.k.a. injective) if for each $\vec{b} \in \mathbb{R}^m$

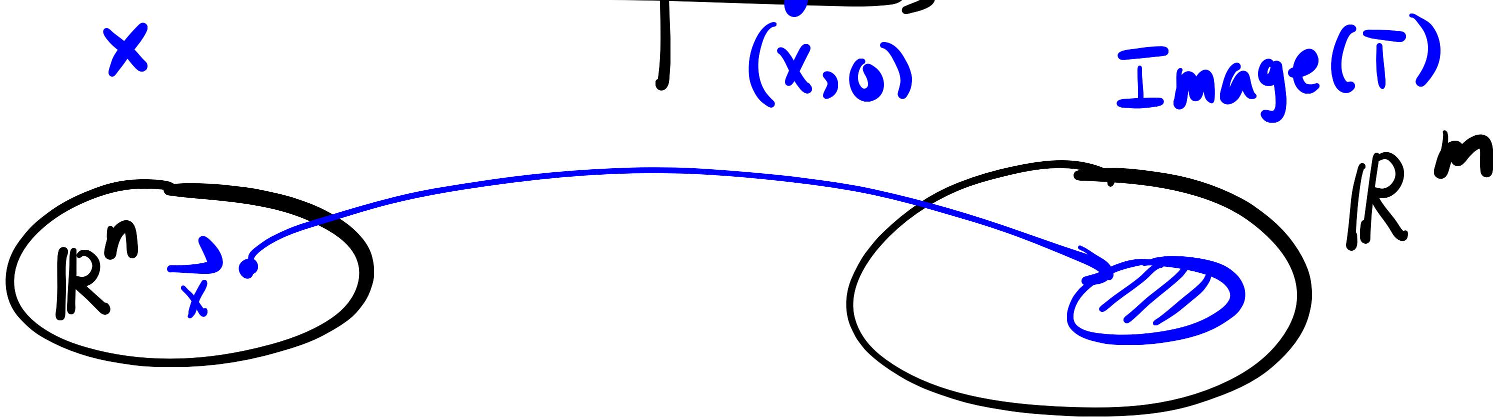
there is at most one $\vec{x} \in \mathbb{R}^n$ s.t.

$$T(\vec{x}) = \vec{b}$$

Ex. $T : \mathbb{R} \to \mathbb{R}^2$

$$x \mapsto \begin{bmatrix} x \\ 0 \end{bmatrix}$$





Def $T : \mathbb{R}^n \to \mathbb{R}^m$ lin. trans.

both injective & surjective is called bijective.

For any $\vec{b} \in \mathbb{R}^m$, there is a unique $\vec{x} \in \mathbb{R}^n$ s.t. $T(\vec{x}) = \vec{b}$

bijective? $m = n$