Lec 27. Real symmetric matrix

singular value decomposition.

$A \in \mathbb{R}^{n\times n}$. $A = A^T$.

Thm. (Spectral decomp). A real symmetric

① A is always diagonalizable.

all eigenvalues are <u>real</u>.

② eigenvectors can always be chosen to be a real orthogonal matrix.

$$AV = VD \iff A = VD\underline{V^{-1}} \iff A = VDV^T$$

$$D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \qquad V^T V = I_n$$

$$\Updownarrow$$

$\lambda_i \in \mathbb{R}$. $\qquad V^{-1} = V^T$

A real symmetric. is <u>orthogonally</u> diagonalizable.

We will show. ① λ is real.

If so then eigenvector $\vec{v}$ satisfies

$$(\underset{\mathbb{R}^{n\times n}}{A} - \underset{\mathbb{R}}{\lambda} I)\vec{v} = \vec{0}$$

$\vec{v}$ → real vector.

Pf: $A\vec{v} = \lambda\vec{v}$ $\quad \vec{v} \in \mathbb{C}^n, \lambda \in \mathbb{C}$

$$\vec{v}^*\vec{v} = \sum_{i=1}^{n} \bar{v}_i v_i = \sum_{i=1}^{n} |v_i|^2 \geq 0$$

$$\Rightarrow \quad \vec{v}^* A \vec{v} = \lambda \underbrace{\vec{v}^* \vec{v}}_{>0} \qquad (1)$$

Apply Hermitian conjugate to both sides of (1)

$$(\vec{v}^* A \vec{v})^* = \bar{\lambda}\, \vec{v}^* \vec{v}$$

$$\parallel$$

$$\vec{v}^* A^* \vec{v}$$

$$\parallel$$

$$\vec{v}^* A \vec{v} = \lambda \vec{v}^* \vec{v}$$

$$A^* = \bar{A}^T \underset{\text{real}}{=} A^T \underset{\text{symmetric}}{=} A$$

$$\Rightarrow \quad \lambda = \bar{\lambda} \quad \Rightarrow \quad \lambda \in \mathbb{R}.$$

Ex. In Chap 5.

not diagonalizable

$$\begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} \quad \begin{bmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{bmatrix} \quad \cdots$$

not symmetric.

eigenvalues not real.

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ +\sin\theta & \cos\theta \end{bmatrix}$$

not symmetric

Ex. $A = A^T$. $A \in \mathbb{R}^{n\times n}$. $\vec{u}, \vec{v} \in \mathbb{R}^n$.

$$\langle \vec{u}, \vec{v} \rangle = \vec{u}^T A \vec{v}$$

For which A, $\langle \vec{u}, \vec{v} \rangle$ defines an inner prod?

check: ① $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$

$$= \vec{v}^T A \vec{u} = (\vec{v}^T A \vec{u})^T = \vec{u}^T A^T \vec{v}$$

$$= \vec{u}^T A \vec{v}$$

$\downarrow$

sym.

② $\langle \vec{u} + \vec{v}, \vec{w} \rangle = (\vec{u} + \vec{v})^T A \vec{w} = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$

③ $\forall c \in \mathbb{R}$, $\langle c\vec{u}, \vec{v} \rangle = c \langle \vec{u}, \vec{v} \rangle$.

④ $\langle \vec{u}, A\vec{u} \rangle = \vec{u}^T A \vec{u} \geq 0$. $\vec{u}^T A \vec{u} = 0 \Leftrightarrow \vec{u} = \vec{0}$.

Obviously wrong when

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Use spectral decomp. thm.

$$A = VDV^T. \quad V^TV = I_n$$

$$D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \quad \lambda_i \in \mathbb{R}$$

$$\vec{u}^T A \vec{v} = \vec{u}^T V D V^T \vec{v}$$

$$= \left(V^T \vec{u}\right)^T D \left(V^T \vec{v}\right)$$

change of basis.

Define $\vec{w} = V^T \vec{u} \in \mathbb{R}^n$

$$\vec{u}^T A \vec{u} = \vec{w}^T D \vec{w} = \underbrace{\sum_{i=1}^{n} w_i^2 \lambda_i}$$

Inner prod $\iff$ $\lambda_i > 0$. for all $1 \le i \le n$.

$\underbrace{A^T = A}$, $A \succ 0$ reads "A is positive definite".

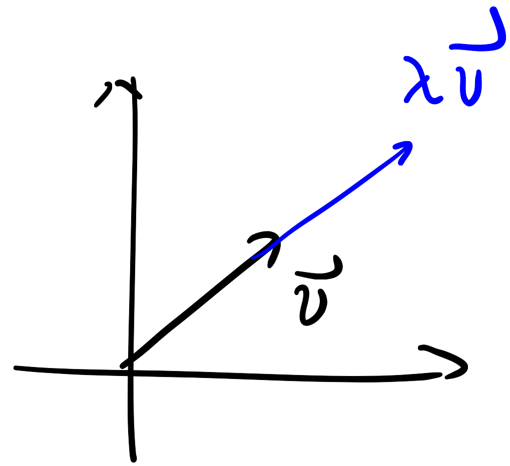
means $\lambda_i > 0$.

Singular value decomposition (SVD)

In Chap 5.

$A \in \mathbb{R}^{n\times n}$. eigenvector.

$$A\vec{v} = \lambda\vec{v}$$



Drawback: ① A may not be diagonalizable.

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

② $A \in \mathbb{R}^{m\times n}$ $(m \neq n)$ cannot even talk about diagonalizability

Ex. $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \in \mathbb{R}^{5\times 2}$

$\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ $\quad \vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\vec{u}\,\vec{v}^T = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} = A.$$

Ex. $A = 10^6 \left\{ \underbrace{\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & & \ddots & 1 \\ 1 & & & 1 \end{bmatrix}}_{10^5} \right.$.

$\vec{u} = 10^6 \left\{ \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \right.$ $\qquad$ $\vec{v} = 10^5 \left\{ \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \right.$

$$10^6 + 10^5 = 1.1 \times 10^6 \lll 10^6 \times 10^5 = 10^{11}.$$

$$\vec{u}\,\vec{v}^T = A.$$

In general. SVD seeks the following decomposition. $A \in \mathbb{R}^{m \times n}$ (assume $m \geq n$)

$$A = \sum_{k=1}^{n} \sigma_k \vec{u}_k \vec{v}_k^T$$

singular value.

$\sigma_k \geq 0$. $\vec{u}_k \in \mathbb{R}^m$, $\vec{v}_k \in \mathbb{R}^n$.

$$U = [\vec{u}_1, \cdots, \vec{u}_n] \in \mathbb{R}^{m \times n}$$

$$V = [\vec{v}_1, \cdots, \vec{v}_n] \in \mathbb{R}^{n \times n}.$$

$U^T U = I_n$. $V^T V = I_n$.

When is this useful?

Most useful if most $\sigma_k \approx 0$.

$$A \approx \sum_{k=1}^{K} \sigma_k \vec{u}_k \vec{v}_k^T \in \mathbb{R}^{m\times n}$$

$K \ll n$.

data compression.