Lec 14. Warmup

$V = \mathbb{R}^{2\times 2}$

$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $A_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ $A_3 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$A_4 = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$. Are $\{A_1, \cdots, A_4\}$ lin. indep.?

method 1.

$x_1 A_1 + \cdots + x_4 A_4 = 0$ has nontrivial sol?

(exer for details)

method 2. Find a basis for V

$e_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $e_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $e_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$$A_1 = 1 \cdot e_1 - e_4 = [e_1\ e_2\ e_3\ e_4] \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}}_{\to \text{ coord.}}$$

similarly for A_2, A_3, A_4

$$[A_1\ A_2\ A_3\ A_4] = [e_1\ e_2\ e_3\ e_4] \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

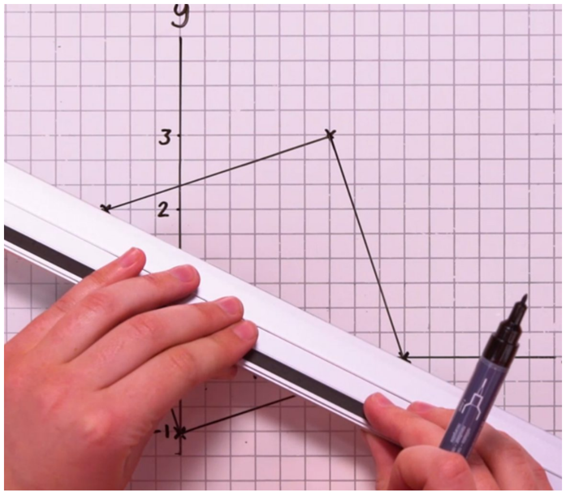
$$0 = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

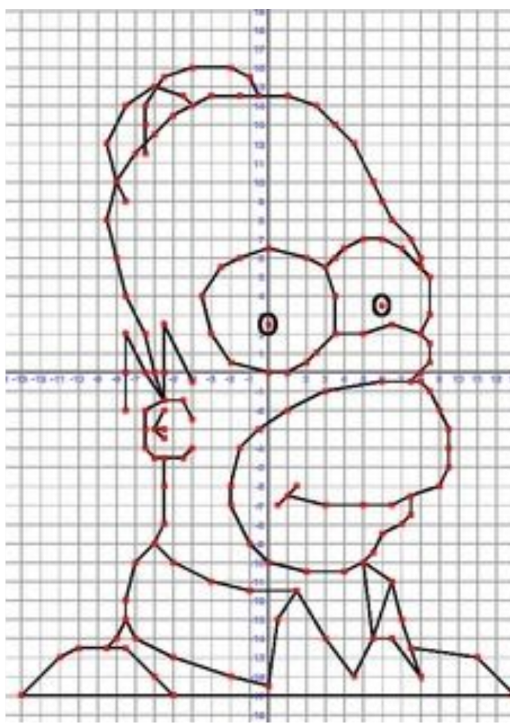
$$= \underbrace{\begin{bmatrix} e_1 & e_2 & e_3 & e_4 \end{bmatrix}}_{\text{lin indep.}} \underbrace{\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{= 0}$$

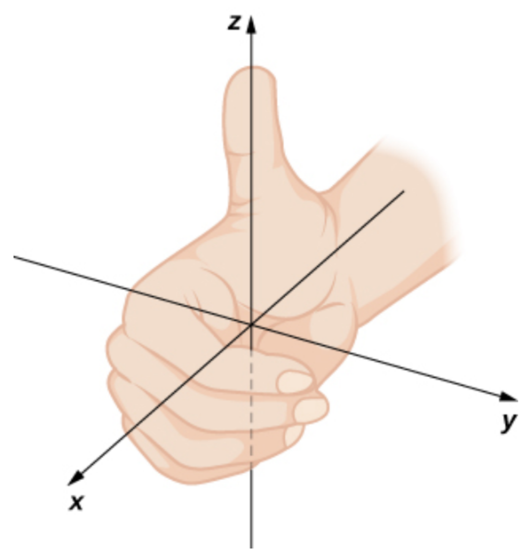
$$\xrightarrow{\text{REF}} \begin{bmatrix} \boxed{1} & 0 & 0 & -1 \\ 0 & \boxed{1} & 0 & 1 \\ 0 & 0 & \boxed{1} & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

free var. $\Rightarrow$ lin. dep.

What is coordinate







To go beyond 2d/3d Euclidean space

Conceptualize

Ex. $V =$ collection of all possible ducks differing only in width

$\cdots$

$\vec{v}_1$ $\quad$ $\vec{v}_2$ $\quad$ $\vec{v}_3$

Pick one duck $\vec{b} \in \vec{V}$, $\vec{b} \neq \vec{0}$

$\vec{b}$

For all other $\vec{v} \in V$, $\vec{v} = a\vec{b}$, $a \in \mathbb{R}$

 $=$ 1.8 

$$\vec{v} = \underbrace{1.8}_{\text{coord.}} \vec{b}$$

Lin trans:

$$P_B : \mathbb{R} \longrightarrow V \qquad B = \{\vec{b}\}.$$
$$a \mapsto a\vec{b}$$

① one-to-one

② onto

$\Big\} \longrightarrow$ " P_B^{-1} " exists.

$$P_B^{-1} : V \to \mathbb{R}$$

$$\vec{v} \mapsto \underbrace{[\vec{v}]_B}_{\text{coordinate of } \vec{v} \text{ w.r.t. } B} \in \mathbb{R}$$

Coordinate of $\vec{v}$ w.r.t. B.

$\equiv$ B-coordinate of $\vec{v}$

Ex. V = collection of all possible ducks differing only by height & width

. . . .

Check.

$B = \{\vec{b_1}, \vec{b_2}\}$ is a basis for V

$\vec{b_1}$

$\vec{b_2}$

$$P_B : \mathbb{R}^2 \to V$$

$$P_B\left(\begin{bmatrix} h \\ w \end{bmatrix}\right) = h\vec{b}_1 + w\vec{b}_2$$

$$P_B^{-1} : V \to \mathbb{R}^2$$

$$\vec{v} \mapsto \begin{bmatrix} h \\ w \end{bmatrix} := [\vec{v}]_B$$

coord is nothing but a special lin. trans.

where W is always $\mathbb{R}^n$. $n = \dim V$

Ex. $V = \mathbb{P}_2$ Find. coord of

$f(x) = x^2 + 3x + 2$ w.r.t. basis.

1) $\mathcal{B}_1 = \{1, x, x^2\}$,

2) $\mathcal{B}_2 = \{x, 1, 1 - x^2\} := \{\vec{b_1}, \vec{b_2}, \vec{b_3}\}$

1) $[f]_{\mathcal{B}_1} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$

2) want $f = a_1 \vec{b_1} + a_2 \vec{b_2} + a_3 \vec{b_3}$

$$[\vec{b_1}]_{\mathcal{B}_1} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad [\vec{b_2}]_{\mathcal{B}_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad [\vec{b_3}]_{\mathcal{B}_1} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & x & x^2 \end{bmatrix} [f]_{\mathcal{B}_1} = \begin{bmatrix} 1 & x & x^2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 0 & 1 & 1 & 2 \\ 1 & 0 & 0 & 3 \\ 0 & 0 & -1 & 1 \end{array}\right] \xrightarrow{\text{Sol}} [f]_{\mathcal{B}_2}.$$

Ex. Find coord $\vec{v} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ $V = \mathbb{R}^2$

w.r.t. basis

$B = \left\{ \vec{b}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$.

$$\rightarrow \left[\begin{array}{cc|c} 1 & 1 & -3 \\ 1 & -1 & 2 \end{array}\right] \xrightarrow{\text{sol}} [\vec{v}]_B.$$

generalization of matrix inverse.

<u>Def</u> Lin. trans. $T: V \to W$ is called an isomorphism if there is lin. trans. called

$T^{-1}: W \to V$ s.t.

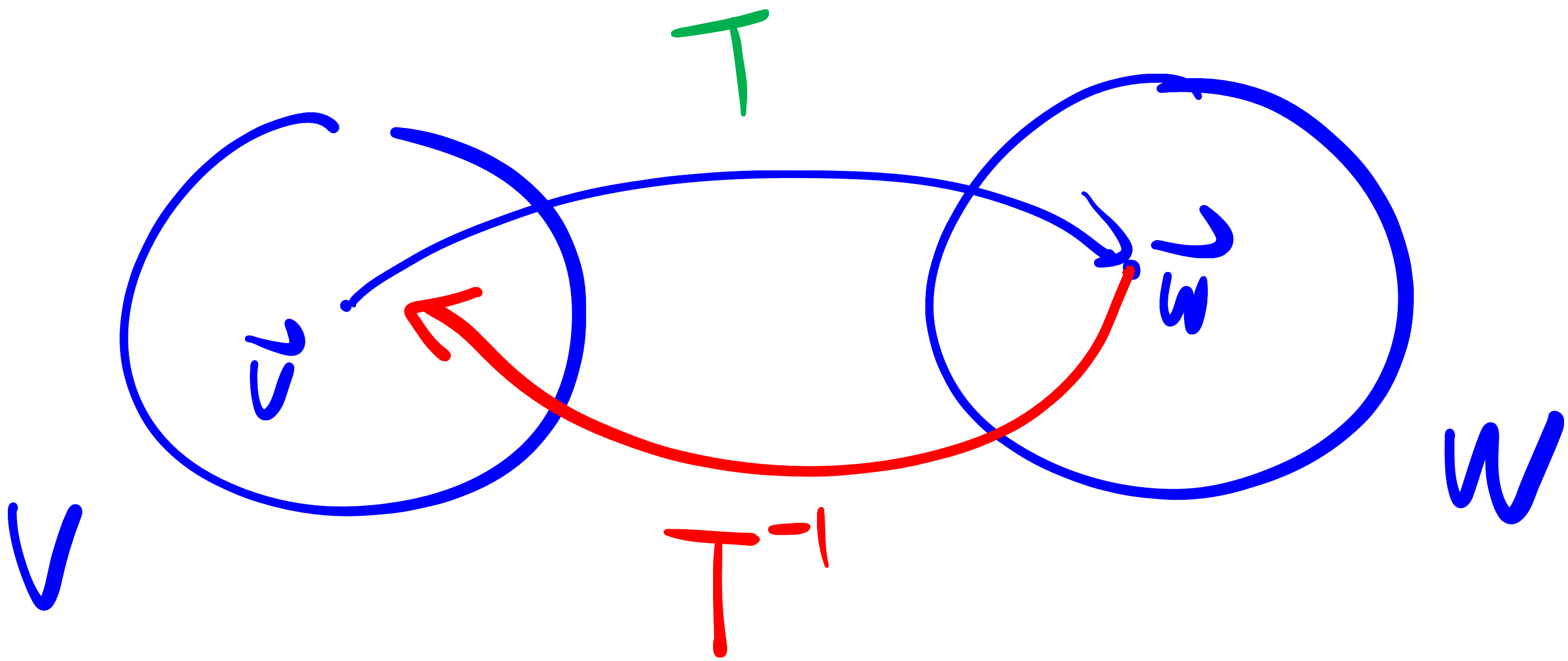
$T(T^{-1}(\vec{w})) = \vec{w}$, for any $\vec{w} \in W$

$T^{-1}(T(\vec{v})) = \vec{v}$, " " $\vec{v} \in V$

iso morphism $\Longleftrightarrow$ bijective.

iso $\downarrow$ equal

morphism $\downarrow$ shape



Thm: Finite dim vector space V has a basis $\mathcal{B} = [\vec{b}_1, \dots, \vec{b}_n]$

Then

$$P_{\mathcal{B}}\left(\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}\right) = a_1 \vec{b}_1 + \dots + a_n \vec{b}_n$$

is an isomorphism. and $\underbrace{P_{\mathcal{B}}^{-1}}$ exists.

define coord.

pf: exer.

Ex. Is T an isomorphism?

$T: \mathbb{P}_2 \to \mathbb{R}^3$,

$$p(x) \mapsto \begin{bmatrix} p(0) \\ p(1) \\ p(2) \end{bmatrix}$$

$$p(x) = a_0 + a_1 x + a_2 x^2$$

$$T(p) = \begin{bmatrix} a_0 \\ a_0 + a_1 + a_2 \\ a_0 + 2a_1 + 4a_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

$\downarrow$ REF

every row/col has pivot ✓

“We share a philosophy about linear algebra: we think basis-free, we write basis-free , but when the chips are down we close the office door and compute with matrices like fury."

Kaplansky, Irving

“Don’t forget to call it a ‘procedure’—makes it less scary.”