

Lec 31. Inhomogeneous 2nd order linear ODEs.

$$y'' + by' + cy = t^l e^{rt}, \quad l \in \mathbb{N}, \quad r \in \mathbb{C}.$$

$$T: y \mapsto y'' + by' + cy$$

Observe

$$(e^{rt})' = r e^{rt}, \quad (e^{rt})'' = r^2 e^{rt}$$

$$(t^a e^{rt})' = a t^{a-1} e^{rt} + r t^a e^{rt}$$

$$\in \text{Span} \{ t^{a-1} e^{rt}, t^a e^{rt} \}.$$

Consider a subspace  $W \subset C^\infty(\mathbb{R})$ .

$$W := \text{span} \{ e^{rt} \cdot t^m, e^{rt} t^{m-1}, \dots, e^{rt} \}$$

$$\dim W = m+1, \quad \text{intuitively } "W = e^{rt} \cdot P_m"$$

$$\text{Our hope: } t^l e^{rt} \in \text{Image}(T) \rightarrow m \geq l$$

$$T: W \rightarrow W.$$

$$\begin{aligned} \text{Then } y(t) &= \underline{e^{rt}} p(t), \quad p(t) \in P_m \\ &= e^{rt} (a_0 + a_1 t + \dots + a_m t^m) \end{aligned}$$

$a_0, \dots, a_m$  are unknown.

## Method of undetermined coefficients

$$\begin{aligned} T(y) &= (e^{rt} p(t))'' + b(e^{rt} p(t))' + c e^{rt} p(t) \\ &= (r e^{rt} p(t) + e^{rt} p'(t))' + b(r e^{rt} p(t) + e^{rt} p'(t)) \\ &\quad + c e^{rt} p(t) \\ &= (r^2 e^{rt} p(t) + 2r e^{rt} p'(t) + e^{rt} p''(t)) \\ &\quad + b(r e^{rt} p(t) + e^{rt} p'(t)) + c e^{rt} p(t) \end{aligned}$$

$$\begin{aligned}
 &= e^{rt} \left\{ \underbrace{(r^2 + br + c)}_{\text{red wavy}} p(t) \rightarrow \in \mathbb{P}_m \right. \\
 &\quad + \underbrace{(2r + b)}_{\text{red wavy}} p'(t) \quad \in \mathbb{P}_{m-1} \\
 &\quad \left. + p''(t) \quad \in \mathbb{P}_{m-2} \right\}
 \end{aligned}$$

$$= e^{rt} t^l$$

Case 1:  $r^2 + br + c \neq 0$ ,  $m = l$ .  
 aux. eq.  $\rightarrow$  not a root

Case 2 :  $r^2 + br + c = 0$

$$m = l + 1$$

$$2r + b \neq 0$$

Observe  $2r + b = (r^2 + br + c)'$

$\Rightarrow r$  is a single root of aux. eq.

Case 3 :  $r^2 + br + c = 0$

$$m = l + 2.$$

$$2r + b = 0$$

$r$  is a double root of aux. eq.

Ex.  $y'' + 2y' + y = t^2$ .

Step 1:  $t^2 = t^2 \cdot e^{0t}$ ,  $r=0$ .

Step 2: plug in  $r=0$  into

$$r^2 + 2r + 1 = 1 \neq 0.$$

$\Rightarrow r=0$  is not a root

should choose  $m=2$ ,

$$W = \text{span} \{ \underbrace{t^2, t, 1}_B \}.$$

Step 3. Solve this as a lin. sys. problem.

$$[t^2]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$T: W \rightarrow W$$

$$[T]_B = \begin{bmatrix} [T(\vec{b}_1)]_B & [T(\vec{b}_2)]_B & [T(\vec{b}_3)]_B \end{bmatrix}$$

$$T(\vec{b}_1) = T(t^2) = 2 + 4t + t^2$$

$$[T(\vec{b}_1)]_B = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$$

Similarly

$$[T(\vec{b}_2)]_B = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \quad [T(\vec{b}_3)]_B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Solve

$$\left[ \begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 4 & 1 & 0 & 1 \\ 2 & 2 & 1 & 0 \end{array} \right] \rightarrow [\vec{y}]_B = \begin{bmatrix} 1 \\ -4 \\ 6 \end{bmatrix}$$

**A** Solution:  $y_p(t) = t^2 - 4t + 6$



All solutions

$$y(t) = C_1 e^{-t} + C_2 t e^{-t}$$

$$+ y_p(t)$$

fixed by  
initial condition.

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Recall lin sys.

$$A\vec{x} = \vec{b}, \quad \textcircled{1} \text{ Find } \vec{x}_p, \text{ s.t. } A\vec{x}_p = \vec{b}$$

$$\textcircled{2} \text{ sol set} = \{ \vec{x}_p + \vec{w} \mid A\vec{w} = \vec{0} \}.$$

Ex .  $y'' + 3y' + 2y = e^{-2t}$

Step 1:  $e^{-2t} = t^l e^{rt}$  with  $l=0, r=-2$

Step 2 :  $r = -2$  is a single root  
of aux eq.  $m = l + 1 = 1$ .

$$W = \text{span} \{ \underbrace{te^{-2t}, e^{-2t}}_B \}.$$

Step 3:

$$T(\vec{b}_1) = (te^{-2t})'' + 3(te^{-2t})' + 2(te^{-2t})$$

$$= -e^{-2t}$$

$$T(\vec{b}_2) = 0$$

$$[T(\vec{b}_1)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \quad [T(\vec{b}_2)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$[e^{-2t}]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\text{Solve } \left[ \begin{array}{cc|c} 0 & 0 & 0 \\ -1 & 0 & 1 \end{array} \right] \Rightarrow [\vec{y}_p]_B = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$\vec{y}_p(t) = -t e^{-2t}$$

$$\text{Solution set } \vec{y}(t) = C_1 e^{-2t} + C_2 e^{-t} + t e^{-2t}$$

Ex.  $y'' - 2y' + y = e^t$

Step 1.  $e^t = t^l e^{rt}$  ,  $l=0, r=1$ .

Step 2.  $r=1$  is a double root  
of aux. eq.

$$m = l + 2 = 2$$

$$W = \text{span} \{ \underbrace{t^2 e^t, t e^t, e^t}_B \}.$$

Step 3.

$$T(\vec{b}_1) = 2e^t \quad . \quad [T(\vec{b}_1)]_B = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

$$T(\vec{b}_2) = T(\vec{b}_3) = 0.$$

$$[e^t]_B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{Solve } \begin{bmatrix} 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ \underline{2} & 0 & 0 & | & 1 \end{bmatrix} \rightarrow [\vec{y}_p]_B = \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \end{bmatrix}$$

$$y_p(t) = \frac{1}{2} t^2 e^t.$$

$$\text{All sol: } y(t) = c_1 e^t + c_2 t e^t \\ + \frac{1}{2} t^2 e^t$$







