Lec 18. Warm up

$$A = \begin{bmatrix} 3 & 0 \\ 6 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix}$$

$$\det(A) = 3 \cdot 1 - 0 = 3$$

$$\det(B) = 2 \cdot 2 - 2 = 2$$

$$\det(AB) = \det\left(\begin{bmatrix} 6 & 3 \\ 14 & 8 \end{bmatrix}\right) = 6 \cdot 8 - 3 \cdot 14 = 6$$

$$\boxed{\det(AB) = \det(A) \cdot \det(B)}$$

Co-factor expansion (of the determinant)

$A \in \mathbb{R}^{n\times n}$. Cofactor matrix $C \in \mathbb{R}^{n\times n}$

$$[C]_{ij} := (-1)^{i+j} \underbrace{\det(A_{ij})}_{M_{ij}\,:\,\text{(first) minor}}$$

$$A_{ij} = \begin{array}{cc} & j \\ i & \begin{bmatrix} a_{11} & \vdots & a_{1n} \\ \cdots & a_{ij} & - \\ a_{n1} & \vdots & a_{nn} \end{bmatrix} \end{array}$$

$$C = \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & & \vdots \\ c_{n1} & \cdots & c_{nn} \end{bmatrix}$$

use of cofactor

$$\det(A) = a_{11} C_{11} + a_{12} C_{12} + \cdots + a_{1n} C_{1n}$$

definition

<u>Take as a fact</u>

Thm. (Cofactor expansion) $A \in \mathbb{R}^{n \times n}$

(1) expand any row. $\forall\, 1 \le i \le n$

$$\det(A) = a_{i1} C_{i1} + \cdots + a_{in} C_{in}$$

(2) expand any column $\forall 1 \le j \le n$

$$\det(A) = a_{1j} C_{1j} + \cdots + a_{nj} C_{nj}$$

A lot to assume. but life gets easier after this...

Ex. $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ -1 & 0 & -2 \end{bmatrix}$ expand w.r.t. 3rd row

$$|A| = (-1)\cdot(-1)^{3+1}\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} + (-2)(-1)^{3+3}\begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix}$$

$$= (-1)\cdot(-1) + (-2)\cdot 3 = -5.$$

Ex. $A = \begin{bmatrix} 3 & -7 & 8 & 9 & -6 \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 4 & -1 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix}$ expand over 5th row

$$|A| = (-2)\cdot(-1)^{5+4} \begin{vmatrix} 3 & -7 & 8 & -6 \\ 0 & 2 & -5 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}$$

upper triangular

$$= (-2)\cdot(-1)\ 3\cdot 2\cdot 1\cdot(-1)$$

$$= -12.$$

Ex.
$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & \cdots & a_{2n} \\ 0 & & \ddots & \vdots \\ & & & a_{nn} \end{vmatrix}$$

expand w.r.t. 1st col

$$a_{11} \cdot (-1)^{1+1} \cdot \begin{vmatrix} a_{22} & \cdots & a_{2n} \\ & \ddots & \\ & & a_{nn} \end{vmatrix}$$

$$= a_{11} a_{22} \begin{vmatrix} a_{33} & \cdots & a_{3n} \\ & \ddots & \\ & & a_{nn} \end{vmatrix} = \cdots = a_{11} \cdots a_{nn}$$

Ex. $\begin{vmatrix} a_{11} & & 0 \\ a_{21} & \ddots & \\ \vdots & & \ddots & \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$ expand w.r.t 1st row $a_{11} \cdots a_{nn}$

Thm. $\det(A) = \det(A^T)$

Pf: $\det(A) \underset{\text{1st row}}{\overset{\text{cofactor}}{=\!=\!=}} a_{11}C_{11} + \cdots + a_{1n}C_{1n}$

$$\Big\| \begin{array}{l}\text{cofactor}\\ \text{1st col}\end{array}$$

$\det(A^T)$

Effect of elem. row op. on det(A)

<u>Thm</u>. $\det(A') = -\det(A)$. if A' swaps 2 row / 2 cols of A.

$$\begin{vmatrix} a_{21} & \cdots & a_{2n} \\ a_{11} & \cdots & a_{1n} \\ a_{31} & \cdots & a_{3n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} \overset{\text{cofactor}}{\underset{\text{2nd row}}{=\!=}} a_{11}\cdot(-1)^{2+1}\det(A_{11}) + \cdots + a_{1n}(-1)^{2+n}\det(A_{1n})$$

$$\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} \underset{\text{1st row}}{\overset{\text{cofactor}}{=\!=}} a_{11} \cdot \det(A_{11}) + \cdots + a_{1n} (-1)^{1+n} \det(A_{1n})$$

Corollary If A has 2 rows/cols that are the same.

$$\det(A) = 0.$$

Pf: swap THESE 2 rows/cols

$$\det(A) = -\det(A) \Rightarrow \det(A) = 0.$$

Add scaled row to another row.

$$\begin{vmatrix} a_{11}+\lambda a_{21} & \cdots & a_{1n}+\lambda a_{2n} \\ a_{21} & & a_{2n} \\ \vdots & & \vdots \\ a_{n1} & & a_{nn} \end{vmatrix}$$

$$\underset{\text{1st row}}{\overset{\text{cofactor}}{=}} (a_{11}+\lambda a_{21}) C_{11} + \cdots + (a_{1n}+\lambda a_{2n}) C_{1n}$$

$$= (a_{11} C_{11} + \cdots + a_{1n} C_{1n}) \rightarrow \det(A)$$

$$+ \lambda (a_{21} C_{11} + \cdots + a_{2n} C_{1n}) = \begin{vmatrix} a_{21} & \cdots & a_{2n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = 0$$

scale a row by λ.

$$\begin{vmatrix} \lambda a_{11} & \cdots & \lambda a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = \lambda \det(A).$$

Summarize.

$$A \xrightarrow[\text{elem. row. op.}]{\text{REF}} A'$$

at each step. value of $|A|$ MAY change. but it DOES NOT change its nullity.

REF

$$A' = \begin{bmatrix} \boxed{\lambda_1} & * & * & & * \\ 0 & \boxed{\lambda_2} & * & & * \\ 0 & 0 & \ddots & & \\ 0 & 0 & & \ddots & * \\ 0 & 0 & & & \boxed{\lambda_n} \\ 0 & 0 & & & \end{bmatrix}$$

λ_i could be zero or nonzero

upper triangular.

$\det(A') = \lambda_1 \cdots \lambda_n \neq 0$

$\lambda_1, \cdots, \lambda_n \neq 0$

↪ are pivots → invertible

$\det(A') = 0$. some $\lambda_i = 0$.

→ not invertible.

$\boxed{\text{Thm. } \det(A) \neq 0 \iff A \text{ is invertible}}$