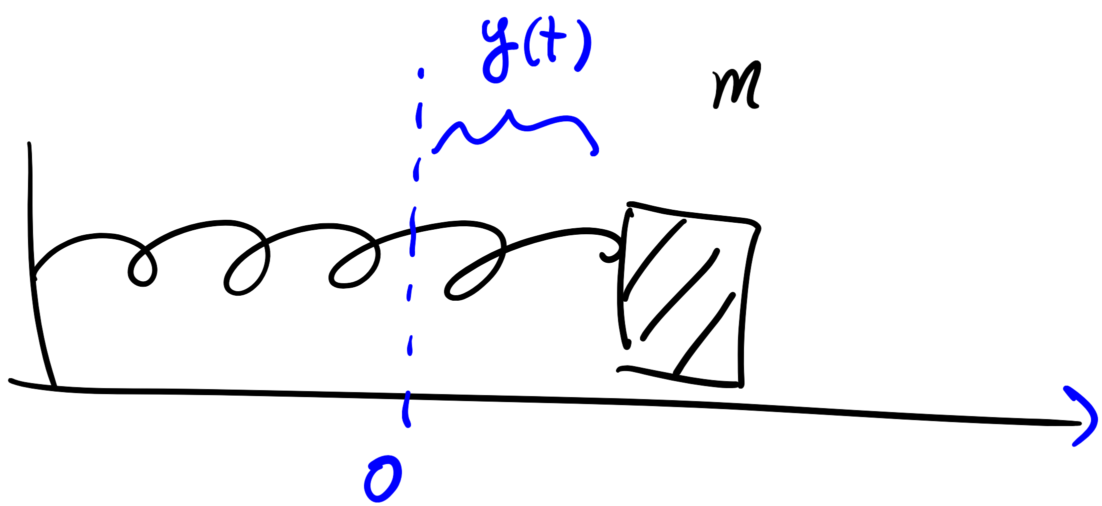
Lec 29. Homogeneous 2nd order ODEs.

Physical system. (spring mass system)



Newton's law

$$m\ddot{y}(t) = \text{FORCE} = F_{spring} + F_{friction}$$

$$F_{spring} = -c\, y(t) \quad : \text{Hooke's law}$$

$$F_{friction} = -b\, \dot{y}(t)$$

$$m\, \ddot{y}(t) = -b\, \dot{y}(t) - c\, y(t)$$

$$\Rightarrow \boxed{m\, \ddot{y}(t) + b\, \dot{y}(t) + c\, y(t) = 0}.$$

Hom. 2nd order ODE.

$\neq 0$. inhomgeneous.

Initial condition

$$\begin{cases} y(0) = y_0 \\ \dot{y}(0) = y_0' \end{cases}$$

Spring simulator (check course page)

$m = 1$.

stiffness : $c = 1$

damping : $b \rightarrow$ vary

$$\begin{cases} y(0) = -2. \\ \dot{y}(0) = 0 \end{cases}$$

b^2, $4mc = 4$

Def. A homogeneous (RHS $=0$) second order ($y, \dot{y}, \ddot{y}$) ordinary differential equation is an ($y(t)$) eq. of the form

$$a\ddot{y}(t) + b\dot{y}(t) + c = 0$$

If we are given

$$\begin{cases} y(0) = y_0 \\ \dot{y}(0) = y_0' \end{cases}$$

this is called an initial value problem (IVP).

Strategy:

① Find general sol. for all initial values

② Find specific sol. satisfying initial cond.

Connect to Lin. Alg.

$V = C^{\infty}(\mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{R} \text{ is infinitely differentiable} \}$

$T : V \to V$

$y \mapsto a\ddot{y} + b\dot{y} + c$

$T(y) = a\ddot{y} + b\dot{y} + c$

General sol $= \underbrace{\mathrm{Nul}(T)}_{\dim = 2}$.

$$a\ddot{y}(t) + b\dot{y}(t) + c\,y(t) = 0$$

auxiliary eq. $\quad r \in \mathbb{C}$

$$ar^2 + br + c = 0$$ Find roots

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2}$$

① $b^2 - 4ac > 0$: 2 distinct real roots

② $b^2 - 4ac = 0$. 1 real root of multiplicity 2

③ $b^2 - 4ac < 0$ 2 distinct complex roots

(conjugate to each other)

Case 1. Ex. $y'' - 3y' - 4y = 0$

aux. eq. $r^2 - 3r - 4 = 0$

$$r = \frac{3 \pm \sqrt{9+16}}{2}$$

$r_1 = 4, \quad r_2 = -1.$

Make an educated guess

$$y_1(t) = e^{4t}, \quad y_2(t) = e^{-t}$$

$$y_1' = 4e^{4t} \qquad y_1'' = 16e^{4t}$$

$$y_1'' - 3y_1' - 4y_1 = e^{4t}(16 - 12 - 4) = 0$$

$$y_2'' - 3y_2' - 4y_2 = e^{-t} \cdot 0 = 0.$$

$y_1(t)$, $y_2(t)$ lin. indep. in V

i.e., If $c_1 y_1(t) + c_2 y_2(t) = c_1 e^{4t} + c_2 e^{-t} = 0$

$$\Rightarrow \underbrace{c_1 e^{5t}}_{\text{not a constant}} = \underbrace{-c_2}_{\text{constant}}$$

unless $c_1 = 0 \Rightarrow c_2 = 0$.

$$Nul(T) = \text{span}\{\underbrace{e^{4t}, e^{-t}}_{\to \text{basis}}\}$$

$$= \{c_1 e^{4t} + c_2 e^{-t} \mid c_1, c_2 \in \mathbb{R} \text{ or } \mathbb{C}\}$$

why?

$$y(t) = e^{rt} \quad \text{for unknown } r \in \mathbb{C}$$

$$\dot{y}(t) = r e^{rt}, \quad \ddot{y}(t) = r^2 e^{rt}$$

$$a r^2 e^{rt} + b r e^{rt} + c \underbrace{e^{rt}}_{\neq 0} = 0$$

$$\Longleftrightarrow \quad ar^2 + br + c = 0.$$

aux. eq!

<u>Ex</u>. $y'' - 2y' + y = 0$.

aux. eq. $r^2 - 2r + 1 = 0$. $\Rightarrow$ $r = 1$

(w. multiplicity 2)

$y_1(t) = e^t$

$y_2(t) = te^t$

$y_2'(t) = e^t + te^t = (1+t)e^t$

$$y_2''(t) = e^t + (1+t)e^t = (2+t)e^t$$

$$y_2'' - 2y_2' + y_2 = e^t\left[(2+t) - 2(1+t) + t\right] = 0.$$

$$\{e^t, te^t\} \text{ lin. indep.}$$

$$\mathrm{Nul}(T) = \mathrm{span}\{e^t, te^t\}.$$