Lec 35. Warm up

Convert $y^{(3)}(t) + y^{(2)}(t) = f(t)$ into first order eq. arrays

$$\vec{y}(t) = \begin{bmatrix} y(t) \\ y'(t) \\ y''(t) \end{bmatrix}$$

$$\frac{d}{dt}\vec{y}(t) = \begin{bmatrix} y'(t) \\ y''(t) \\ f(t) - y''(t) \end{bmatrix} = \overbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}}^{A} \begin{bmatrix} y(t) \\ y'(t) \\ y''(t) \end{bmatrix} + \overbrace{\begin{bmatrix} 0 \\ 0 \\ f(t) \end{bmatrix}}^{\vec{f}(t)}$$

$$\frac{d}{dt}\vec{y}(t) = A\vec{y}(t) + \vec{f}(t)$$ normal form

General high order eq.

$$y^{(n)}(t) + a_1 y^{(n-1)}(t) + \cdots + a_n y(t) = f(t).$$

$$\vec{y}(t) = \begin{bmatrix} y(t) \\ y'(t) \\ \vdots \\ y^{(n-1)}(t) \end{bmatrix}$$

indep. of eq.

$$\frac{d}{dt}\vec{y}(t) = \begin{bmatrix} y'(t) \\ y''(t) \\ \vdots \\ y^{(n)}(t) \end{bmatrix} = \begin{array}{c} \\ n-1 \\ \\ \\ \end{array} \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ 0 & - & - & - & 1 \\ \hline -a_n & -a_{n-1} & & & -a_1 \end{bmatrix} \vec{y}(t) + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \hline f(t) \end{bmatrix}$$

IVP.

$$\begin{cases} \frac{d}{dt}\vec{y}(t) = A\vec{y}(t) + \vec{f}(t) \\ \vec{y}(t_0) = Y_0 \in \mathbb{R}^n \end{cases}$$

$\leftarrow$ n-th order diff. eq.

f(t) is "sufficiently nice"

$\vec{y}(t)$ unique sol.

Hom. eq. $\frac{d}{dt}\vec{y}(t) = A\vec{y}(t)$.

① n lin. indep. sol.

$$Y(t) = [\vec{y}_1(t), \cdots, \vec{y}_n(t)]$$ fundamental sol. set.

$$\begin{cases} \frac{d}{dt} Y(t) = A\, Y(t) \\ Y(0) = I_n \end{cases}$$

$$\begin{cases} \frac{d\vec{y}(t)}{dt} = A \vec{y}(t) \\ \vec{y}(t) = Y_0 \end{cases}$$

$$\boxed{\vec{y}(t) = Y(t) Y_0}$$

why?

① satisfies eq. ✓

② satisfies initial data

③ unique

② Wronskian lemma.

2nd order.

$$\begin{array}{ccc} & \vec{y}_1(t) & \vec{y}_2(t) \\ W[y_1(t), y_2(t)] = & \left| \begin{matrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{matrix} \right. & \left. \begin{matrix} \\ \end{matrix} \right| \end{array}$$

$$= |Y(t)|$$

$$W(Y(t_0)) \neq 0 \iff W(Y(t)) \neq 0 \quad \forall t$$

$$A\vec{u} = \lambda \vec{u}$$

$$\vec{y}(t) = c(t)\,\vec{u} \quad . \qquad c(t) \in \mathbb{R}$$

must be a sol for some $c(t)$.

$$\frac{d}{dt}\vec{y}(t) = c'(t)\,\vec{u} = A\left(c(t)\vec{u}\right) = c(t)\lambda\,\vec{u}$$

$$\Rightarrow \left(\underbrace{c'(t) - c(t)\lambda}_{=0}\right)\vec{u} = 0$$

$$\Rightarrow \quad c(t) = c_0 e^{\lambda t} \quad \Rightarrow \quad \vec{y}(t) = c_0 e^{\lambda t}\vec{u}$$

Thm. A diagonalizable

$\{\vec{u}_1, \cdots, \vec{u}_n\}$ lin. indep. cor. $\{\lambda_1, \cdots, \lambda_n\}$

(some eig. vals may be the same)

Then $\{e^{\lambda_1 t}\vec{u}_1, \cdots, e^{\lambda_n t}\vec{u}_n\}$ is a

basis for sol set of hom. eq.

$$\frac{d}{dt}\vec{y}(t) = A\vec{y}(t)$$

Ex. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

$$\begin{bmatrix} y'(t) \\ y''(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}$$

$\Longleftrightarrow \quad y''(t) = y(t) \qquad r^2 - 1 = 0 \;.\; r = \pm 1.$

$y_1(t) = e^{t}. \qquad y_2(t) = e^{-t}.$

New method. diagonalize A.

$|A - \lambda I| = \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1.$

$\lambda_1 = 1. \quad \vec{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = -1, \quad \vec{u}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$y_1(t)$ $\qquad\qquad\qquad\qquad\qquad\qquad$ $y_2(t)$

$\vec{y}_1(t) = e^{t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \qquad \vec{y}_2(t) = e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$y_1'(t)$ $\qquad\qquad\qquad\qquad\qquad\qquad$ $y_2'(t)$

High order diff. $\Longrightarrow$ $\frac{d}{dt}\vec{y}(t) = A\vec{y}(t)$

$\not\Longleftarrow$

Ex. $\frac{d}{dt}\vec{y}(t) = A\vec{y}(t) \qquad A = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & -3 \\ 1 & -2-\lambda \end{vmatrix} = \lambda^2 - 4 + 3 = 0$$

$$\Rightarrow \lambda = \pm 1.$$

$$\lambda_1 = 1 \Rightarrow \vec{u}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\lambda_2 = -1 \Rightarrow \vec{u}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\vec{y}(t) = c_1 e^{t} \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Think :

Fact 1. $y'' + by' + cy = 0. \Rightarrow r^2 + br + c = 0$

$\Rightarrow r_1 = r_2 = \lambda$

$y_1(t) = e^{\lambda t}$. $y_2(t) = t e^{\lambda t}$

Fact 2. $\frac{d}{dt}\vec{y}(t) = \underbrace{\begin{bmatrix} 0 & 1 \\ -c & -b \end{bmatrix}}_{\downarrow} \vec{y}(t)$

not a diagonalizable matrix

$$\begin{cases} \dfrac{d}{dt}\vec{y}(t) = A\vec{y}(t). \\ \vec{y}(0) = Y_0. \end{cases}$$

closed form sol. for all $A \in \mathbb{R}^{n\times n}$.

$$\vec{y}(t) = e^{At} Y_0$$

$\uparrow$

matrix exponential function.

Cool. Read 9.8 (not required).

Inhom. first order eq. variation of parameters.

$$\frac{d}{dt}\vec{y}(t) = A\vec{y}(t) + \vec{f}(t)$$

$$\vec{f}(t) = e^{rt} t^{l} \vec{g} \qquad \vec{g} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$y_p(t) = e^{rt}(A_m t^m + \cdots + A_0) \qquad m \geq l$$

$$\begin{cases} \dfrac{d\,\underset{\sim}{Y}(t)}{dt} = A\,Y(t) \\ \underset{\sim}{Y}(0) = I_n \end{cases}$$

Assume $Y(t)$ is given

$$\vec{y}_p(t) = Y(t)\,\underbrace{\vec{c}(t)}_{}$$

const vector if hom. eq. and $\vec{c}(t) \equiv \vec{c}$ initial data

$$\frac{d}{dt}\vec{y}_p(t) = \frac{d}{dt}Y(t)\,\vec{c}(t) + Y(t)\,\dot{\vec{c}}(t)$$

$$= A\,\cancel{\vec{y}_p}(t) + Y(t)\,\dot{\vec{c}}(t)$$

$$= A\,\cancel{\vec{y}_p}(t) + \vec{f}(t)$$

$$\Rightarrow \quad \dot{\vec{c}}(t) = Y^{-1}(t)\,\vec{f}(t)$$

integrate

$$\Rightarrow \quad \vec{c}(t) = \vec{c} + \int_0^t Y^{-1}(s)\,\vec{f}(s)\,ds$$

$$\vec{y}_p(t) = Y(t)\,\vec{c} + Y(t)\int_0^t Y^{-1}(s)\,\vec{f}(s)\,ds$$