ML Programming assignment VIII

Name : 林欣妤
Student number : 314652023

October 26, 2025

1 Stochastic Differential Equations (SDE)

1.1 Definition

A **stochastic differential equation (SDE)** describes a process $x_t \in \mathbb{R}^d$ whose infinitesimal change is influenced by both deterministic and random terms:

$$dx_t = \underbrace{f(x_t, t)\, dt}_{\text{drift}} + \underbrace{G(x_t, t)\, dW_t}_{\text{diffusion}}, \quad x(0) = x_0, \tag{1.1}$$

where

- x_t is the stochastic process (unknown we want to solve for),
- $f \in \mathbb{R}^d$ is the drift vector,
- $G \in \mathbb{R}^{d\times d}$ is the diffusion matrix,
- W_t is a standard Brownian motion (Wiener process).

The integral form of the SDE (Itô integral) is:

$$x_t = x_0 + \int_0^t f(x_s, s)\, ds + \int_0^t G(x_s, s)\, dW_s. \tag{1.2}$$

A stochastic process that solves an SDE is called an **Itô process**.
Drift and diffusion must satisfy certain regularity conditions such as Lipschitz continuity.

1.2 Special Cases

- Pure drift: $dx_t = f(x_t, t)\, dt \to$ deterministic process
- Pure diffusion: $dx_t = G(x_t, t)\, dW_t \to$ Brownian-like random motion

1.3 Stochastic Process

A stochastic process is a collection of random variables $\{x_t\}_{t\in T}$ indexed by a set T (discrete or continuous), defined on a probability space, and taking values in $\mathbb{R}^d$.

- For each fixed t, $x_t(\omega)$ is a random variable
- For each $\omega \in \Omega$, $t \mapsto x_t(\omega)$ is called a **path** (realization)

1.4 Wiener Process (Brownian Motion)

$W_t \in \mathbb{R}^d$ is a continuous stochastic process satisfying:

- Initial condition: $W_0 = 0$
- Stationary Gaussian increments: $\Delta W_t = W(t + \Delta t) - W(t) \sim \mathcal{N}(0, \Delta t I)$
- Independent increments
- Continuous paths: with probability 1, $t \mapsto W(t)$ is continuous

Properties: $E[W(t)] = 0$, $\mathrm{Var}[W(t)] = t$, and almost surely nowhere differentiable.
White noise can be interpreted as the (formal) derivative of Brownian motion: $h(t) = \frac{dW}{dt}$.

1.5 White Noise

A white noise process $h(t) \in \mathbb{R}^d$ satisfies:

- $h(t)$ and $h(t')$ are independent if $t \neq t'$
- Gaussian process with zero mean and Dirac delta correlation:

$$E[h(t)] = 0, \quad E[h(t)h^T(s)] = \delta(t - s)I$$

Properties: almost everywhere discontinuous, unbounded, can take arbitrarily large values.
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1.6 Interpretation of Stochastic Integral

$$\int_0^t G(x_s, s)\, dW_s = \lim_{n\to\infty} \sum_{k=0}^{n-1} G(x_{t_k}, t_k)\,[W(t_{k+1}) - W(t_k)]\,,$$

with $0 = t_0 < t_1 < \cdots < t_n = t$, and $W(t_{k+1}) - W(t_k) \sim \mathcal{N}(0, (t_{k+1} - t_k)I)$.
Remark: $\int_0^t dW_s = W_t - W_0 = W_t \sim \mathcal{N}(0, tI)$.

1.7 Euler—Maruyama Method / Euler—Maruyama 方法

A forward-Euler-like scheme is used to numerically approximate the process:
Partition $[0, T]$ into N equal subintervals of width $\Delta t = T/N$, $t_k = k\Delta t$, $k = 0, \ldots, N$.
Set $X_0 = x_0$. Update rule:

$$X_{n+1} = X_n + f(X_n, t_n)\Delta t + G(X_n, t_n)\Delta W(t_n), \tag{1.3}$$

where $\Delta W(t_n) = W(t_{n+1}) - W(t_n)$, or equivalently

$$X_{n+1} = X_n + f(X_n, t_n)\Delta t + G(X_n, t_n)\Delta t Z(t_n), \tag{1.4}$$

with $Z(t_i)$ i.i.d. $\sim \mathcal{N}(0, 1)$.

References

[1] Chat-GPT (Apply GPT to revise and correct the English content of the report)