MLProgramming assignment III

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1 Background

Neural networks with the tanh activation can be used to approximate many functions. Since the **Weierstrass Approximation Theorem** states that polynomials can approximate any continuous function, it is sufficient to show that neural networks can approximate polynomials. The lemmas below show how shallow tanh networks can approximate both odd and even monomials, which are the building blocks of polynomials.

2 Lemma 3.1 (Odd Powers)

Lemma 3.1 states that for any odd monomial y^p (such as $y, y^3, y^5, \dots$), there exists a shallow tanh neural network that approximates it arbitrarily well on $[-M, M]$. The key reason is that tanh itself is an odd function, making it naturally suited for constructing odd powers. The proof uses finite difference operators and combinations of shifted tanh functions to cancel out lower-order terms, leaving only the desired odd power.

Example. To approximate y^3, one can combine $\tanh(y + h)$ and $\tanh(y - h)$. By subtracting and scaling properly, the linear terms cancel, leaving something that looks like y^3.

3 Lemma 3.2 (Even Powers)

Lemma 3.2 extends the approximation to even monomials ($y^2, y^4, y^6, \dots$). Since tanh is odd, even powers cannot be obtained directly. The trick is to use an algebraic identity that expresses even powers in terms of odd powers and lower-order even powers. By recursion, once odd powers are approximated (via Lemma 3.1), we can build accurate approximations of even powers.

4 Overall Understanding

Lemma 3.1 + Lemma 3.2 together show that shallow tanh neural networks can approximate **any low-degree polynomial** with arbitrarily small error. Since polynomials are building blocks for continuous functions (Weierstrass theorem), this implies that shallow tanh networks can approximate any continuous function on a bounded interval.

References

[1] T. D. Ryck and S. Lanthaler and S. Mishra, *On the approximation of functions by tanh neural networks*, Neural Networks, 143 (2021), pp. 732–750.

[2] Chat gpt (Apply GPT to revise and correct the English content of the report, and learning.)