# Chapter 2

## Modeling in the Frequency Domain

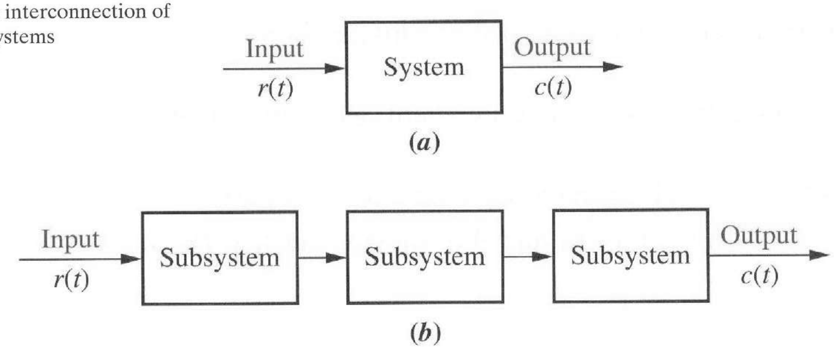
## Introduction

- We will discuss two methods:
  - (1) Transfer functions in the frequency domain
  - (1) State equations in the time domain.



## Block diagram representation

**FIGURE 2.1** **a.** Block diagram representation of a system; **b.** block diagram representation of an interconnection of subsystems




Note: The input, $r(t)$, stands for *reference input*.
The output, $c(t)$, stands for *controlled variable*.



## Laplace Transform Review

The Laplace transform is defined as

$$\mathscr{L}[f(t)] = F(s) = \int_{0_-}^{\infty} f(t)e^{-st}dt \qquad (2.1)$$

The inverse Laplace transform is defined as

$$\mathscr{L}^{-1}[F(s))] = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}ds = f(t)u(t) \qquad (2.2)$$

where

$$u(t) = 1 \quad t > 0$$
$$u(t) = 0 \quad t < 0$$

# Laplace transform table

**TABLE 2.1** Laplace transform table

| Item no. | $f(t)$ | $F(s)$ |
|---|---|---|
| 1. | $\delta(t)$ | $1$ |
| 2. | $u(t)$ | $\frac{1}{s}$ |
| 3. | $tu(t)$ | $\frac{1}{s^2}$ |
| 4. | $t^n u(t)$ | $\frac{n!}{s^n + 1}$ |
| 5. | $e^{-at}u(t)$ | $\frac{1}{s+a}$ |
| 6. | $\sin \omega t u(t)$ | $\frac{\omega}{s^2+\omega^2}$ |
| 7. | $\cos \omega t u(t)$ | $\frac{s}{s^2+\omega^2}$ |

Table 2.1


Important Table!



# Example 2.1

**PROBLEM:** Find the Laplace transform of $f(t) = Ae^{-at}u(t)$.

**SOLUTION:** Since the time function does not contain an impulse function, we can replace the lower limit of Eq. (2.1) with 0. Hence,

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt = \int_0^\infty Ae^{-at}e^{-st}\,dt = A\int_0^\infty e^{-(s+a)t}\,dt$$

$$= -\frac{A}{s+a}e^{-(s+a)t}\Big|_{t=0}^{\infty} = \frac{A}{s+a} \qquad (2.3)$$



**TABLE 2.2** Laplace transform theorems

| Item no. | Theorem | | Name |
|---|---|---|---|
| 1. | $\mathscr{L}[f(t)] = F(s)$ | $= \int_{0-}^{\infty} f(t)e^{-st}dt$ | Definition |
| 2. | $\mathscr{L}[kf(t)]$ | $= kF(s)$ | Linearity theorem |
| 3. | $\mathscr{L}[f_1(t)+f_2(t)]$ | $= F_1(s)+F_2(s)$ | Linearity theorem |
| 4. | $\mathscr{L}[e^{-at}f(t)]$ | $= F(s+a)$ | Frequency shift theorem |
| 5. | $\mathscr{L}[f(t-T)]$ | $= e^{-sT}F(s)$ | Time shift theorem |
| 6. | $\mathscr{L}[f(at)]$ | $= \frac{1}{a}F\left(\frac{s}{a}\right)$ | Scaling theorem |
| 7. | $\mathscr{L}\left[\frac{df}{dt}\right]$ | $= sF(s) - f(0-)$ | Differentiation theorem |
| 8. | $\mathscr{L}\left[\frac{d^2f}{dt^2}\right]$ | $= s^2F(s) - sf(0-) - f'(0-)$ | Differentiation theorem |
| 9. | $\mathscr{L}\left[\frac{d^nf}{dt^n}\right]$ | $= s^nF(s) - \sum_{k=1}^{n} s^{n-k}f^{k-1}(0-)$ | Differentiation theorem |
| 10. | $\mathscr{L}\left[\int_{0-}^{t} f(\tau)d\tau\right]$ | $= \frac{F(s)}{s}$ | Integration theorem |
| 11. | $f(\infty)$ | $= \lim_{s\to 0} sF(s)$ | Final value theorem[1] |
| 12. | $f(0+)$ | $= \lim_{s\to\infty} sF(s)$ | Initial value theorem[2] |

[1]For this theorem to yield correct finite results, all roots of the denominator of $F(s)$ must have negative real parts, and no more than one can be at the origin.
[2]For this theorem to be valid, $f(t)$ must be continuous or have a step discontinuity at $t = 0$ (that is, no impulses or their derivatives at $t = 0$).

Important Table!



# Example 2.2

## Inverse Laplace Transform

**PROBLEM:** Find the inverse Laplace transform of $F_1(s) = 1/(s+3)^2$.

**SOLUTION:** For this example we make use of the frequency shift theorem, Item 4 of Table 2.2, and the Laplace transform of $f(t) = tu(t)$, Item 3 of Table 2.1. If the inverse transform of $F(s) = 1/s^2$ is $tu(t)$, the inverse transform of $F(s+a) = 1/(s+a)^2$ is $e^{-at}tu(t)$. Hence, $f_1(t) = e^{-3t}tu(t)$.

# Partial-Fraction Expansion

- To find the inverse Laplace transform of a complicated function, we can convert the function to sum of simpler terms for which we know the Laplace transform of each term.
- If $F_1(s)=N(s)/D(s)$, where the order of $N(s)$ where the order of $N(s)$ is less than the order of $D(s)$, then the partial-fraction expansion can be made.
- If the order of $N(s)$ is greater than or equal to the order of $D(s)$, $N(s)$ must be divided by $D(s)$ successively until the result has a remainder whose numerator is of order less than its denominator.



# Example

$$F_1(s) = \frac{s^3+2s^2+6s+7}{s^2+s+5} \quad (2.4)$$

$$F_1(s) = s+1+\frac{2}{s^2+s+5} \quad (2.5)$$

Inverse Laplace transform

$$f_1(t) = \frac{d\delta(t)}{dt} + \delta(t) + \mathscr{L}^{-1}\left[\boxed{\frac{2}{s^2+s+5}}\right] \quad (2.6)$$

Partial-fraction expansion



# Case 1

- Roots of the denominator of $F(s)$ are real and distinct

$$F(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{(s+p_1)(s+p_2)\cdots(s+p_m)\cdots(s+p_n)}$$
$$= \frac{K_1}{(s+p_1)} + \frac{K_2}{(s+p_2)} + \cdots + \frac{K_m}{(s+p_m)} + \cdots + \frac{K_n}{(s+p_n)} \quad (2.11)$$

$$(s+p_m)F(s) = \frac{(s+p_m)N(s)}{(s+p_1)(s+p_2)\cdots(s+p_m)\cdots(s+p_n)}$$
$$= (s+p_m)\frac{K_1}{(s+p_1)} + (s+p_m)\frac{K_2}{(s+p_2)} + \cdots + K_m + \cdots$$
$$+ (s+p_m)\frac{K_n}{(s+p_n)} \quad (2.12)$$

$$\boxed{\left.\frac{\cancel{(s+p_m)}N(s)}{(s+p_1)(s+p_2)\cdots\cancel{(s+p_m)}\cdots(s+p_n)}\right|_{s\to -p_m} = K_m \quad (2.13)}$$

Example in page 30



# Example 2.3

**Laplace Transform Solution of a Differential Equation**

**PROBLEM:** Given the following differential equation, solve for $y(t)$ if all initial conditions are zero. Use the Laplace transform.

$$\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + 32y = 32u(t) \quad (2.14)$$

**SOLUTION:** Substitute the corresponding $F(s)$ for each term in Eq. (2.14), using Item 2 in Table 2.1, Items 7 and 8 in Table 2.2, and the initial conditions of $y(t)$ and $dy(t)/dt$ given by $y(0-)=0$ and $\dot{y}(0-)=0$, respectively. Hence, the Laplace transform of Eq. (2.14) is

$$s^2Y(s) + 12sY(s) + 32Y(s) = \frac{32}{s} \quad (2.15)$$

Solving for the response, $Y(s)$, yields

$$Y(s) = \frac{32}{s(s^2+12s+32)} = \frac{32}{s(s+4)(s+8)} \quad (2.16)$$

# Example 2.3

To solve for $y(t)$, we notice that Eq. (2.16) does not match any of the terms in Table 2.1. Thus, we form the partial-fraction expansion of the right-hand term and match each of the resulting terms with $F(s)$ in Table 2.1. Therefore,

$$Y(s) = \frac{32}{s(s+4)(s+8)} = \frac{K_1}{s} + \frac{K_2}{(s+4)} + \frac{K_3}{(s+8)} \qquad (2.17)$$

where, from Eq. (2.13),

$$K_1 = \left.\frac{32}{(s+4)(s+8)}\right|_{s\to 0} = 1 \qquad (2.18a)$$

$$K_2 = \left.\frac{32}{s(s+8)}\right|_{s\to -4} = -2 \qquad (2.18b)$$

$$K_3 = \left.\frac{32}{s(s+4)}\right|_{s\to -8} = 1 \qquad (2.18c)$$

Hence,

$$Y(s) = \frac{1}{s} - \frac{2}{(s+4)} + \frac{1}{(s+8)} \qquad (2.19)$$




# Example 2.3

Since each of the three component parts of Eq. (2.19) is represented as an $F(s)$ in Table 2.1, $y(t)$ is the sum of the inverse Laplace transforms of each term. Hence,

$$y(t) = (1 - 2e^{-4t} + e^{-8t})u(t) \qquad (2.20)$$

Students who are using MATLAB should now run ch2p1 through ch2p8 in Appendix B. This is your first MATLAB exercise. You will learn how to use MATLAB to (1) represent polynomials, (2) find roots of polynomials, (3) multiply polynomials, and (4) find partial-fraction expansions. Finally, Example 2.3 will be solved using MATLAB.




# Case 2

- Roots of the denominator of *F(s)* are real and repeated

$$F(s) = \frac{N(s)}{D(s)}$$

$$= \frac{N(s)}{(s+p_1)^r(s+p_2)\cdots(s+p_n)}$$

$$= \frac{K_1}{(s+p_1)^r} + \frac{K_2}{(s+p_1)^{r-1}} + \cdots + \frac{K_r}{(s+p_1)}$$

$$+ \frac{K_{r+1}}{(s+p_2)} + \cdots + \frac{K_n}{(s+p_n)} \qquad (2.27)$$

$$\boxed{K_i = \left.\frac{1}{(i-1)!}\frac{d^{i-1}F_1(s)}{ds^{i-1}}\right|_{s\to -p_1} \quad i = 1, 2, \ldots, r; \quad 0! = 1 \quad (2.29)}$$

$$F_1(s) = (s+p_1)^r F(s)$$

$$= \frac{(s+p_1)^r N(s)}{(s+p_1)^r(s+p_2)\cdots(s+p_n)}$$

$$= K_1 + (s+p_1)K_2 + (s+p_1)^2 K_3 + \cdots + (s+p_1)^{r-1}K_r$$

$$+ \frac{K_{r+1}(s+p_1)^r}{(s+p_2)} + \cdots + \frac{K_n(s+p_1)^r}{(s+p_n)} \qquad (2.28)$$

Example in page 32




# Case 3

- **Roots of the denominator of *F(s)* are complex or imaginary**

$$F(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{(s+p_1)(s^2+as+b)\cdots}$$

$$= \frac{K_1}{(s+p_1)} + \boxed{\frac{(K_2 s + K_3)}{(s^2+as+b)}} + \cdots \qquad (2.42)$$

$$\Downarrow$$

$$\boxed{\frac{A(s+a)+B\omega}{(s+a)^2+\omega^2}}$$

$$\Downarrow$$

$$\mathscr{L}[Ae^{-at}\cos\omega t + Be^{-at}\sin\omega t] = \frac{A(s+a)+B\omega}{(s+a)^2+\omega^2} \qquad (2.36)$$

$$\mathscr{L}[Ae^{-at}\cos\omega t] = \frac{A(s+a)}{(s+a)^2+\omega^2} \qquad (2.34)$$

$$\mathscr{L}[Be^{-at}\sin\omega t] = \frac{B\omega}{(s+a)^2+\omega^2} \qquad (2.35)$$

Example in page 33

# The Transfer Function

- Let us begin by writing a general nth-order, linear, time-invariant differential equation

$$a_n \frac{d^n c(t)}{dt^n} + a_{n-1} \frac{d^{n-1} c(t)}{dt^{n-1}} + \cdots + a_0 c(t) = b_m \frac{d^m r(t)}{dt^m} + b_{m-1} \frac{d^{m-1} r(t)}{dt^{m-1}} + \cdots + b_0 r(t) \quad (2.50)$$

- Taking the Laplace transform of both sides

$$a_n s^n C(s) + a_{n-1} s^{n-1} C(s) + \cdots + a_0 C(s) + \text{initial condition terms involving } c(t)$$
$$= b_m s^m R(s) + b_{m-1} s^{m-1} R(s) + \cdots + b_0 R(s) + \text{initial condition terms involving } r(t) \quad (2.51)$$

- Assume that *all initial conditions are zero*

$$(a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0) C(s) = (b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0) R(s) \quad (2.52)$$



# The Transfer Function

- The ratio of the output transform, $C(s)$, divided by the input transform, $R(s)$:

$$\boxed{\frac{C(s)}{R(s)} = G(s) = \frac{(b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0)}{(a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0)}} \quad (2.53)$$

- We call this ratio, $G(s)$, the *transfer function* and evaluate it with *zero initial conditions*.

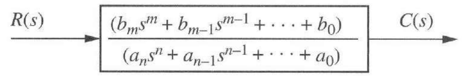



**FIGURE 2.2** Block diagram of a transfer function



# Example 2.4

### Transfer Function for a Differential Equation

**PROBLEM:** Find the transfer function represented by

$$\frac{dc(t)}{dt} + 2c(t) = r(t) \quad (2.55)$$

**SOLUTION:** Taking the Laplace transform of both sides, assuming zero initial conditions, we have

$$sC(s) + 2C(s) = R(s) \quad (2.56)$$

The transfer function, $G(s)$, is

$$G(s) = \frac{C(s)}{R(s)} = \frac{1}{s+2} \quad (2.57)$$



# Example 2.5

### System Response from the Transfer Function

**PROBLEM:** Use the result of Example 2.4 to find the response, $c(t)$ to an input, $r(t) = u(t)$, a unit step, assuming zero initial conditions.

**SOLUTION:** To solve the problem, we use Eq. (2.54), where $G(s) = 1/(s+2)$ as found in Example 2.4. Since $r(t) = u(t)$, $R(s) = 1/s$, from Table 2.1. Since the initial conditions are zero,

$$C(s) = R(s)G(s) = \frac{1}{s(s+2)} \quad (2.58)$$

Expanding by partial fractions, we get

$$C(s) = \frac{1/2}{s} - \frac{1/2}{s+2} \quad (2.59)$$

Finally, taking the inverse Laplace transform of each term yields

$$c(t) = \frac{1}{2} - \frac{1}{2}e^{-2t} \quad (2.60)$$

# Electrical Network Transfer Functions

**TABLE 2.3** Voltage-current, voltage-charge, and impedance relationships for capacitors, resistors, and inductors

| Component | Voltage-current | Current-voltage | Voltage-charge | Impedance $Z(s) = V(s)/I(s)$ | Admittance $Y(s) = I(s)/V(s)$ |
|---|---|---|---|---|---|
| Capacitor | $v(t) = \frac{1}{C}\int_0^t i(\tau)d\tau$ | $i(t) = C\frac{dv(t)}{dt}$ | $v(t) = \frac{1}{C}q(t)$ | $\frac{1}{Cs}$ | $Cs$ |
| Resistor | $v(t) = Ri(t)$ | $i(t) = \frac{1}{R}v(t)$ | $v(t) = R\frac{dq(t)}{dt}$ | $R$ | $\frac{1}{R} = G$ |
| Inductor | $v(t) = L\frac{di(t)}{dt}$ | $i(t) = \frac{1}{L}\int_0^t v(\tau)d\tau$ | $v(t) = L\frac{d^2q(t)}{dt^2}$ | $Ls$ | $\frac{1}{Ls}$ |

Note: The following set of symbols and units is used throughout this book: $v(t)$ – V (volts), $i(t)$ – A (amps), $q(t)$ – Q (coulombs), $C$ – F (farads), $R$ – Ω (ohms), $G$ – Ω (mhos), $L$ – $H$ (henries).



# Example 2.6

**Transfer Function—Single Loop via the Differential Equation**

**PROBLEM:** Find the transfer function relating the capacitor voltage, $V_C(s)$, to the input voltage, $V(s)$ in Figure 2.3.

**SOLUTION:** In any problem, the designer must first decide what the input and output should be. In this network, several variables could have been chosen to be the output—for example, the inductor voltage, the capacitor voltage, the resistor voltage, or the current. The problem statement, however, is clear in this case: We are to treat the capacitor voltage as the output and the applied voltage as the input.

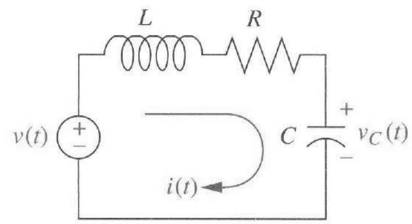



**FIGURE 2.3** *RLC* network



For the capacitor,

$$\boxed{V(s) = \frac{1}{Cs}I(s)} \quad (2.67)$$

For the resistor,

$$\boxed{V(s) = RI(s)} \quad (2.68)$$

For the inductor,

$$\boxed{V(s) = LsI(s)} \quad (2.69)$$

Now define the following transfer function:

$$\boxed{\frac{V(s)}{I(s)} = Z(s)} \quad (2.70)$$



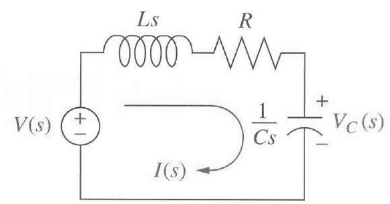



**FIGURE 2.5** Laplace-transformed network

$$\left(Ls + R + \frac{1}{Cs}\right)I(s) = V(s)$$

1. Redraw the original network showing all time variables, such as $v(t)$, $i(t)$, and $v_C(t)$, as Laplace transforms $V(s)$, $I(s)$, and $V_C(s)$, respectively.
2. Replace the component values with their impedance values. This replacement is similar to the case of dc circuits, where we represent resistors with their resistance values.

# Example 2.7

### Transfer Function—Single Loop via Transform Methods

**PROBLEM:** Repeat Example 2.6 using mesh analysis and transform methods without writing a differential equation.

**SOLUTION:** Using Figure 2.5 and writing a mesh equation using the impedances as we would use resistor values in a purely resistive circuit, we obtain

$$\left(Ls + R + \frac{1}{Cs}\right)I(s) = V(s) \tag{2.73}$$

Solving for $I(s)/V(s)$,

$$\frac{I(s)}{V(s)} = \frac{1}{Ls + R + \frac{1}{Cs}} \tag{2.74}$$

But the voltage across the capacitor, $V_C(s)$, is the product of the current and the impedance of the capacitor. Thus,

$$V_C(s) = I(s)\frac{1}{Cs} \tag{2.75}$$

Solving Eq. (2.75) for $I(s)$, substituting $I(s)$ into Eq. (2.74), and simplifying yields the same result as Eq. (2.66).



# Example 2.8

### Transfer Function—Single Node via Transform Methods

**PROBLEM:** Repeat Example 2.6 using nodal analysis and without writing a differential equation.

**SOLUTION:** The transfer function can be obtained by summing currents flowing out of the node whose voltage is $V_C(s)$ in Figure 2.5. We assume that currents leaving the node are positive and currents entering the node are negative. The currents consist of the current through the capacitor and the current flowing through the series resistor and inductor. From Eq. (2.70), each $I(s) = V(s)/Z(s)$. Hence,

$$\frac{V_C(s)}{I/Cs} + \frac{V_C(s) - V(s)}{R + Ls} = 0 \tag{2.76}$$

where $V_C(s)/(1/Cs)$ is the current flowing out of the node through the capacitor, and $[V_C(s) - V(s)]/(R + Ls)$ is the current flowing out of the node through the series resistor and inductor. Solving Eq. (2.76) for the transfer function, $V_C(s)/V(s)$, we arrive at the same result as Eq. (2.66).



# Example 2.9

### Transfer Function—Single Loop via Voltage Division

**PROBLEM:** Repeat Example 2.6 using voltage division and the transformed circuit.

**SOLUTION:** The voltage across the capacitor is some proportion of the input voltage, namely the impedance of the capacitor divided by the sum of the impedances. Thus,

$$V_C(s) = \frac{1/Cs}{\left(Ls + R + \frac{1}{Cs}\right)}V(s) \tag{2.77}$$

Solving for the transfer function, $V_C(s)/V(s)$, yields the same result as Eq. (2.66).

Review Examples 2.6 through 2.9. Which method do you think is easiest for this circuit?



# Complex Circuits via Mesh Analysis

1. Replace passive element values with their impedances.
2. Replace all sources and time variables with their Laplace transform.
3. Assume a transform current and a current direction in each mesh.
4. Write Kirchhoff's voltage law around each mesh.
5. Solve the simultaneous equations for the output.
6. Form the transfer function.

Let us look at an example.

**Kirchhoff's voltage law (KVL)**
The directed sum of the electrical potential differences (voltage) around any closed network is zero

# Example 2.10

- PROBLEM: Given the network of Figure 2.6(a), find the transfer function, $I_2(s)/V(s)$.

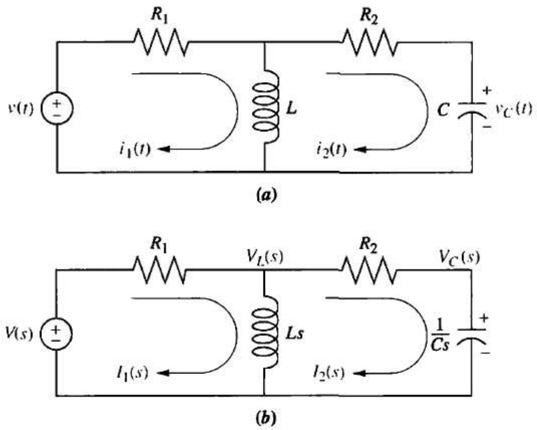



**FIGURE 2.6** **a.** Two-loop electrical network; **b.** transformed two-loop electrical network;




# Example 2.11

**Transfer Function—Multiple Nodes**

**PROBLEM:** Find the transfer function, $V_C(s)/V(s)$, for the circuit in Figure 2.6($b$). Use nodal analysis.

**SOLUTION:** For this problem, we sum currents at the nodes rather than sum voltages around the meshes. From Figure 2.6($b$) the sum of currents flowing from the nodes marked $V_L(s)$ and $V_C(s)$ are, respectively,

$$\frac{V_L(s)-V(s)}{R_1}+\frac{V_L(s)}{Ls}+\frac{V_L(s)-V_C(s)}{R_2}=0 \tag{2.85a}$$

$$CsV_C(s)+\frac{V_C(s)-V_L(s)}{R_2}=0 \tag{2.85b}$$

Rearranging and expressing the resistances as conductances,[5] $G_1=1/R_1$ and $G_2=1/R_2$, we obtain,

$$\left(G_1+G_2+\frac{1}{Ls}\right)V_L(s) \qquad -G_2V_C(s)=V(s)G_1 \tag{2.86a}$$




$$-G_2V_L(s)+(G_2+Cs)V_C(s)=0 \tag{2.86b}$$

Solving for the transfer function, $V_C(s)/V(s)$, yields

$$\frac{V_C(s)}{V(s)}=\frac{\dfrac{G_1G_2}{C}s}{(G_1+G_2)s^2+\dfrac{G_1G_2L+C}{LC}s+\dfrac{G_2}{LC}} \tag{2.87}$$

as shown in Figure 2.7.

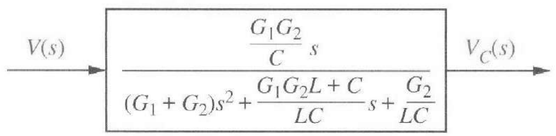



**FIGURE 2.7** Block diagram of the network of Figure 2.6




# Complex Circuits via Nodal Analysis

In order to handle multiple-node electrical networks, we can perform the following steps:

1. Replace passive element values with their admittances.
2. Replace all sources and time variables with their Laplace transform.
3. Replace transformed voltage sources with transformed current sources.
4. Write Kirchhoff's current law at each node.
5. Solve the simultaneous equations for the output.
6. Form the transfer function.

Let us look at an example.

$$\boxed{Y(s)=\frac{1}{Z(s)}=\frac{I(s)}{V(s)}}$$

**Kirchhoff's current law (KCL)**
At any node (junction) in an electrical circuit, the sum of currents flowing into that node is equal to the sum of currents flowing out of that node

## Electrical Network Transfer Functions

**TABLE 2.3** Voltage-current, voltage-charge, and impedance relationships for capacitors, resistors, and inductors

| Component | Voltage-current | Current-voltage | Voltage-charge | Impedance $Z(s) = V(s)/I(s)$ | Admittance $Y(s) = I(s)/V(s)$ |
|---|---|---|---|---|---|
| Capacitor | $v(t) = \frac{1}{C}\int_0^t i(\tau)d\tau$ | $i(t) = C\frac{dv(t)}{dt}$ | $v(t) = \frac{1}{C}q(t)$ | $\frac{1}{Cs}$ | $Cs$ |
| Resistor | $v(t) = Ri(t)$ | $i(t) = \frac{1}{R}v(t)$ | $v(t) = R\frac{dq(t)}{dt}$ | $R$ | $\frac{1}{R} = G$ |
| Inductor | $v(t) = L\frac{di(t)}{dt}$ | $i(t) = \frac{1}{L}\int_0^t v(\tau)d\tau$ | $v(t) = L\frac{d^2q(t)}{dt^2}$ | $Ls$ | $\frac{1}{Ls}$ |

Note: The following set of symbols and units is used throughout this book: $v(t)$ – V (volts), $i(t)$ – A (amps), $q(t)$ – Q (coulombs), $C$ – F (farads), $R$ – Ω (ohms), $G$ – Ω (mhos), $L$ – $H$ (henries).

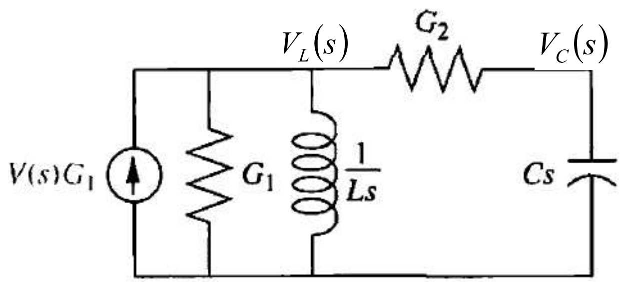


## Example 2.12

### Transfer Function—Multiple Nodes with Current Sources

**PROBLEM:** For the network of Figure 2.6, find the transfer function, $V_C(s)/V(s)$, using nodal analysis and a transformed circuit with current sources.

$$G_1 = 1/R_1,\ G_2 = 1/R_2,$$

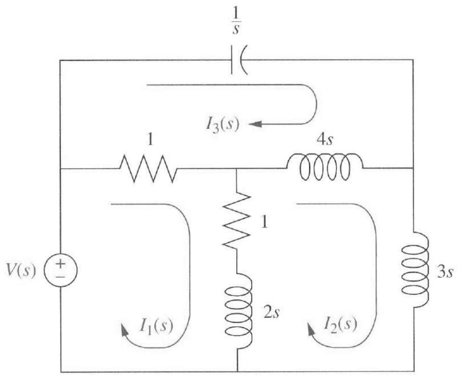





## Example 2.13

### Mesh Equations via Inspection

**PROBLEM:** Write, but do not solve, the mesh equations for the network shown in Figure 2.9.






Substituting the values from Figure 2.9 into Eqs. (2.91) through (2.93) yields

$$+(2s+2)I_1(s) - (2s+1)I_2(s) - I_3(s) = \mathrm{V}(s) \quad (2.94a)$$

$$-(2s+1)I_1(s) + (9s+1)I_2(s) - 4sI_3(s) = 0 \quad (2.94b)$$

$$-I_1(s) - 4sI_2(s) + \left(4s + 1 + \tfrac{1}{s}\right)I_3(s) = 0 \quad (2.94c)$$

which can be solved simultaneously for any desired transfer function, for example, $I_3(s)/V(s)$.

**TryIt 2.8**

Use the following MATLAB and Symbolic Math Toolbox statements to help you solve for the electrical currents in Eq. (2.94).

```
syms  s I1 I2 I3 V
A=[(2*s+2)  -(2*s+1)...
  -1
  -(2*s+1)  (9*s+1)...
  -4*s
  -1  -4*s...
  (4*s+1+1/s)];
B=[I1;I2;I3];
C=[V;0;0];
B=inv(A)*C;
pretty(B)
```

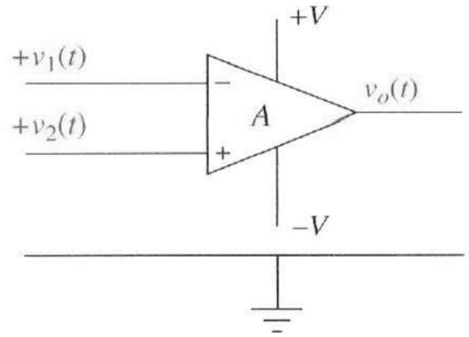


# Operational Amplifiers

1. Differential input, $V_2(t) - v_1(t)$
2. High input impedance, $Z_i = \infty$ (ideal)
3. Low output impedance, $Z_o = 0$ (ideal)
4. High constant gain amplification, $A = \infty$ (ideal)

The output, $v_o(t)$, is given by

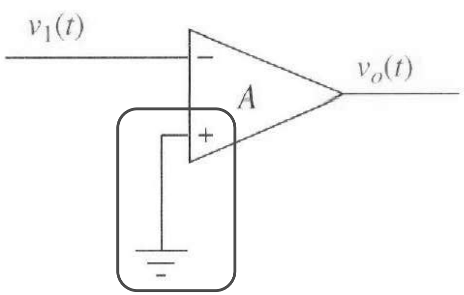
$$v_o(t) = A(v_2(t) - v_1(t))$$






# Inverting Operational Amplifier

$$v_o(t) = -Av_1(t)$$




(*b*)

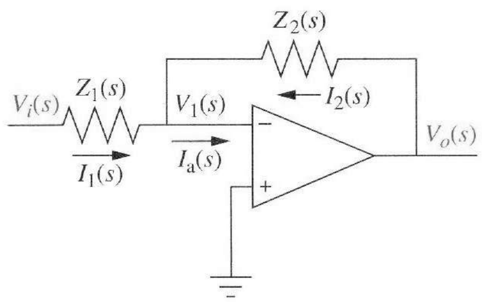


# Inverting Operational Amplifier

$I_a(s) = 0$ and $I_1(s) = -I_2(s)$

$v_1(t) \approx 0$

$$\boxed{\frac{V_o(s)}{V_i(s)} = -\frac{Z_2(s)}{Z_1(s)}} \quad (2.97)$$




(*c*)

# Example 2.14

## Transfer Function—Inverting Operational Amplifier Circuit

**PROBLEM:** Find the transfer function, $V_o(s)/V_i(s)$, for the circuit given in Figure 2.11.

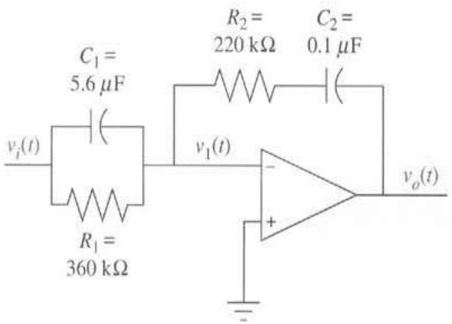



**FIGURE 2.11** Inverting operational amplifier circuit for Example 2.14



# Noninverting Operational Amplifier

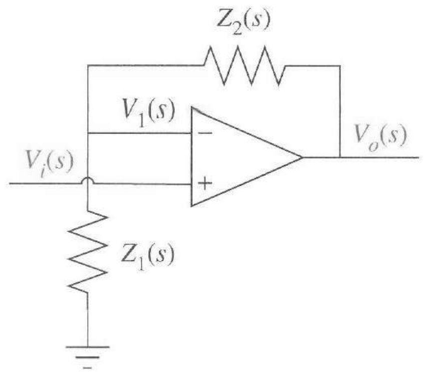



**FIGURE 2.12** General noninverting operational amplifier circuit

$$\boxed{\frac{V_o(s)}{V_i(s)} = \frac{Z_1(s) + Z_2(s)}{Z_1(s)}}$$



# Example 2.15

## Transfer Function—Noninverting Operational Amplifier Circuit

**PROBLEM:** Find the transfer function, $V_o(s)/V_i(s)$, for the circuit given in Figure 2.13.

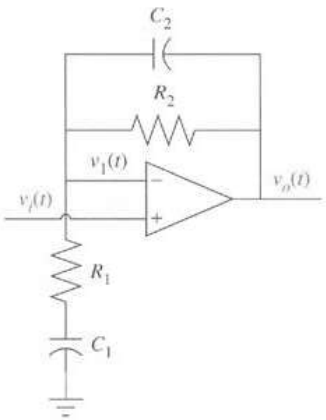
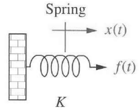
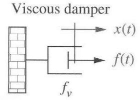
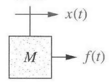



**FIGURE 2.13** Noninverting operational amplifier circuit for Example 2.15



# Translational Mechanical System Transfer Functions

**TABLE 2.4** Force-velocity, force-displacement, and impedance translational relationships for springs, viscous dampers, and mass

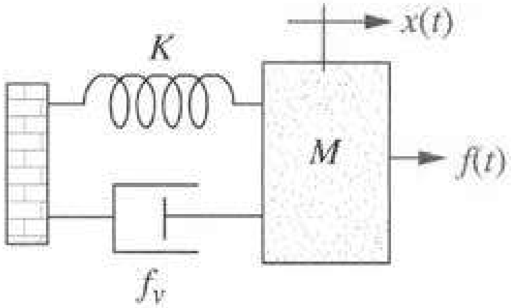
| Component | Force-velocity | Force-displacement | Impedence $Z_M(s) = F(s)/X(s)$ |
|---|---|---|---|
| Spring  | $f(t) = K\int_0^t v(\tau)d\tau$ | $f(t) = Kx(t)$ | $K$ |
| Viscous damper  | $f(t) = f_v v(t)$ | $f(t) = f_v \frac{dx(t)}{dt}$ | $f_v s$ |
| Mass  | $f(t) = M\frac{dv(t)}{dt}$ | $f(t) = M\frac{d^2x(t)}{dt^2}$ | $Ms^2$ |

Note: The following set of symbols and units is used throughout this book: $f(t)$ = N (newtons), $x(t)$ = m (meters), $v(t)$ = m/s (meters/second), $K$ = N/m (newtons/meter), $f_v$ = N-s/m(newton-seconds/meter), $M$ = kg (kilograms = newton-seconds$^2$/meter).

# Example 2.16

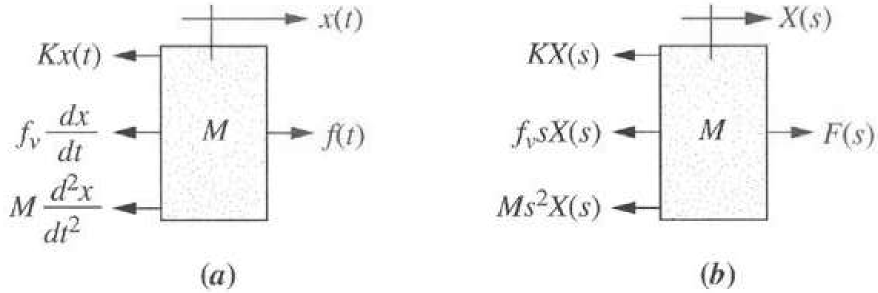
## Transfer Function—One Equation of Motion

**PROBLEM:** Find the transfer function, $X(s)/F(s)$, for the system of Figure 2.15($a$).



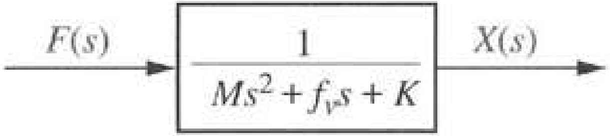

($a$)








$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{Ms^2 + f_v s + K}$$

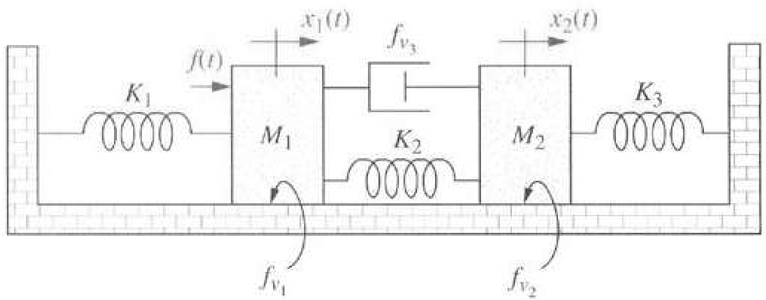





# Example 2.17

## Transfer Function—Two Degrees of Freedom

**PROBLEM:** Find the transfer function, $X_2(s)/F(s)$, for the system of Figure 2.17($a$).



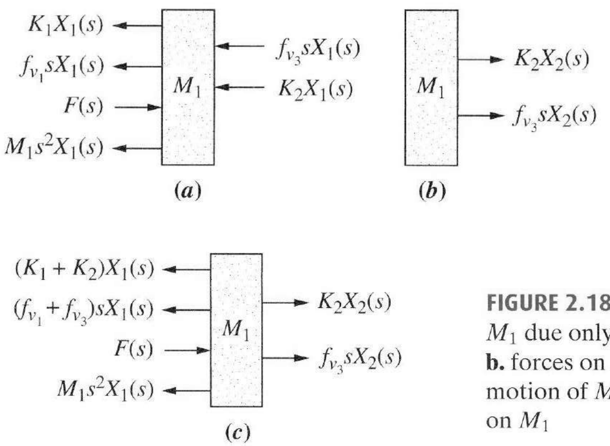



**FIGURE 2.18** **a.** Forces on $M_1$ due only to motion of $M_1$; **b.** forces on $M_1$ due only to motion of $M_2$; **c.** all forces on $M_1$



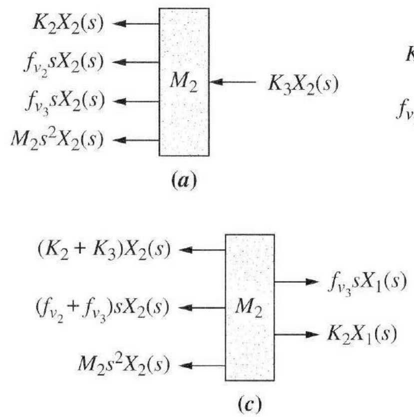



**FIGURE 2.19** **a.** Forces on $M_2$ due only to motion of $M_2$; **b.** forces on $M_2$ due only to motion of $M_1$; **c.** all forces on $M_2$



The Laplace transform of the equations of motion can now be written from Figures 2.18(*c*) and 2.19(*c*) as

$$[M_1s^2(f_{v_1}+f_{v_3})s+(K_1+K_2)]X_1(s)-(f_{v_3}s+K_2)X_2(s)=F(s) \qquad (2.118a)$$

$$-(f_{v_3}s+K_2)X_1(s)+[M_2s^2+(f_{v_2}+f_{v_3})s+(K_2+K_3)]X_2(s)=0 \qquad (2.118b)$$

From this, the transfer function, $X_2(s)/F(s)$, is

$$\frac{X_2(s)}{F(s)}=G(s)=\frac{(f_{v_3}s+K_2)}{\Delta} \qquad (2.119)$$

as shown in Figure 2.17(*b*) where

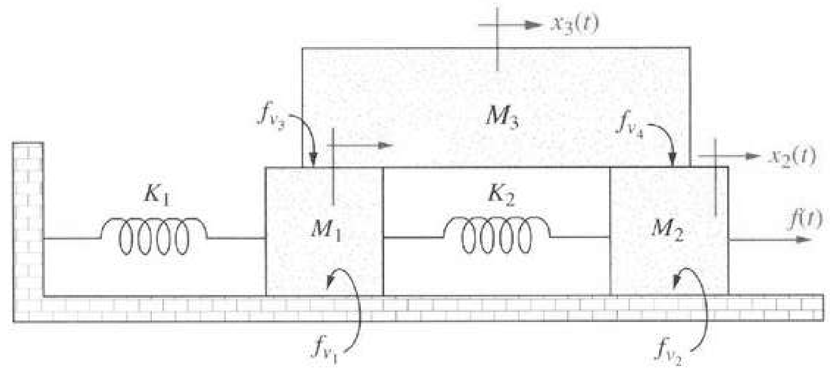
$$\Delta=\begin{vmatrix}[M_1s^2+(f_{v_1}+f_{v_3})s+(K_1+K_2)] & -(f_{v_3}s+K_2)\\ -(f_{v_3}s+K_2) & [M_2s^2+(f_{v_2}+f_{v_3})s+(K_2+K_3)]\end{vmatrix}$$



# Example 2.18

## Equations of Motion by Inspection

**PROBLEM:** Write, but do not solve, the equations of motion for the mechanical network of Figure 2.20.



$M_1$ has two springs, two viscous dampers, and mass associated with its motion. There is one spring between $M_1$ and $M_2$ and one viscous damper between $M_1$ and $M_3$. Thus, using Eq. (2.121),

$$[M_1s^2 + (f_{v_1} + f_{v_3})s + (K_1 + K_2)]X_1(s) - K_2X_2(s) - f_{v_3}sX_3(s) = 0 \qquad (2.124)$$

Similarly, using Eq. (2.122) for $M_2$,

$$-K_2X_1(s) + [M_2s^2 + (f_{v_2} + f_{v_4})s + K_2]X_2(s) - f_{v_4}sX_3(s) = F(s) \qquad (2.125)$$

and using Eq. (2.123) for $M_3$,

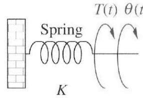
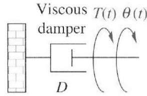
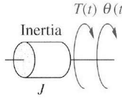
$$-f_{v_3}sX_1(s) - f_{v_4}sX_2(s) + [M_3s^2 + (f_{v_3} + f_{v_4})s]X_3(s) = 0 \qquad (2.126)$$

Equations (2.124) through (2.126) are the equations of motion. We can solve them for any displacement, $X_1(s)$, $X_2(s)$, or $X_3(s)$, or transfer function.



# Rotational Mechanical System

**TABLE 2.5** Torque-angular velocity, torque-angular displacement, and impedance rotational relationships for springs, viscous dampers, and inertia

| Component | Torque-angular velocity | Torque-angular displacement | Impedence $Z_M(s) = T(s)/\theta(s)$ |
|---|---|---|---|
|   | $T(t) = K\int_0^t \omega(\tau)d\tau$ | $T(t) = K\theta(t)$ | $K$ |
|   | $T(t) = D\omega(t)$ | $T(t) = D\frac{d\theta(t)}{dt}$ | $Ds$ |
|   | $T(t) = J\frac{d\omega(t)}{dt}$ | $T(t) = J\frac{d^2\theta(t)}{dt^2}$ | $Js^2$ |

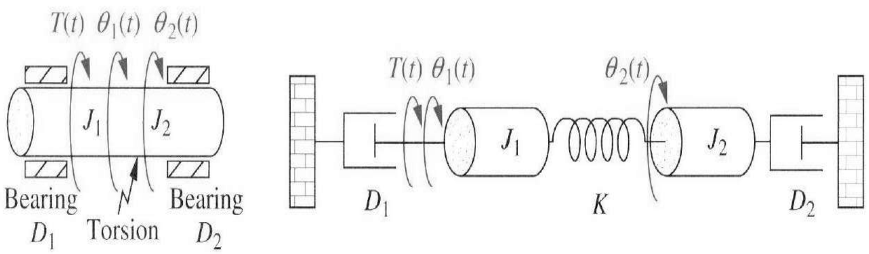
Note: The following set of symbols and units is used throughout this book: $T(t)$ – N-m (newton-meters), $\theta(t)$ – rad(radians), $\omega(t)$ – rad/s(radians/second), $K$ – N-m/rad(newton- meters/radian), $D$ – N-m-s/rad (newton- meters-seconds/radian). $J$ – kg-m$^2$(kilograms-meters$^2$ – newton-meters-seconds$^2$/radian).



# Example 2.19

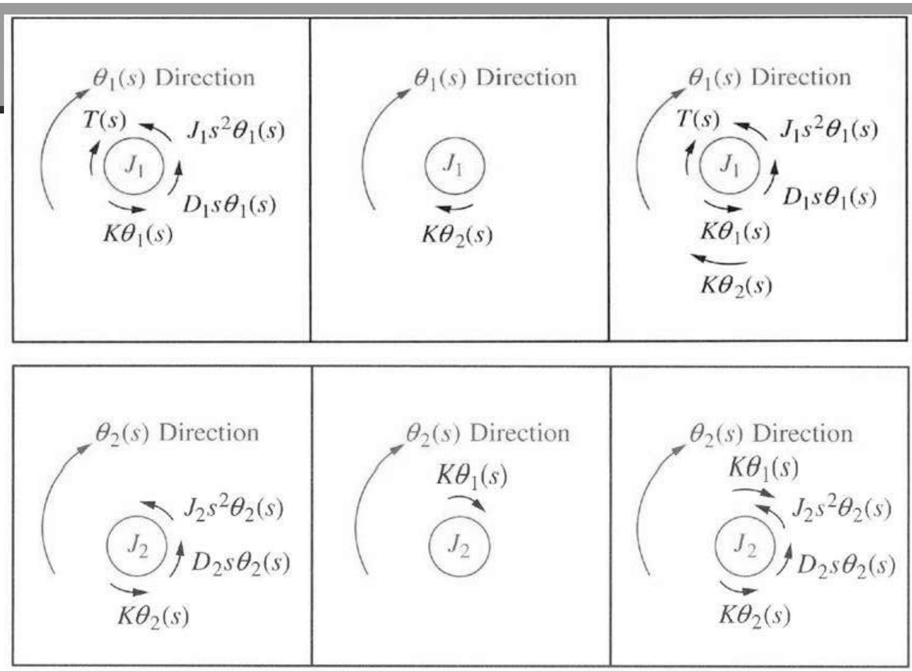
**Transfer Function—Two Equations of Motion**

**PROBLEM:** Find the transfer function, $\theta_2(s)/T(s)$, for the rotational system shown in Figure 2.22($a$). The rod is supported by bearings at either end and is undergoing torsion. A torque is applied at the left, and the displacement is measured at the right.




($a$) ($b$)





Summing torques respectively from Figures 2.23($c$) and 2.24($c$) we obtain the equations of motion,

$$(J_1s^2 + D_1s + K)\theta_1(s) \qquad - K\theta_2(s) = T(s) \qquad (2.127a)$$

$$-K\theta_1(s) + (J_2s^2 + D_2s + K)\theta_2(s) = 0 \qquad (2.127b)$$

from which the required transfer function is found to be

$$\frac{\theta_2(s)}{T(s)} = \frac{K}{\Delta} \qquad (2.128)$$

as shown in Figure 2.22($c$), where

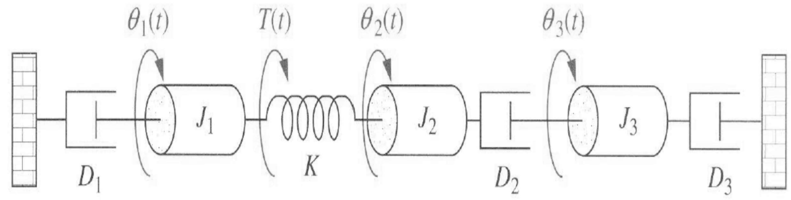
$$\Delta = \begin{vmatrix} (J_1s^2 + D_1s + K) & -K \\ -K & (J_2s^2 + D_2s + K) \end{vmatrix}$$



# Example 2.20

**Equations of Motion By Inspection**

**PROBLEM:** Write, but do not solve, the Laplace transform of the equations of motion for the system shown in Figure 2.25.




$$\begin{aligned}
(J_1s^2 + D_1s + K)\theta_1(s) \quad & -K\theta_2(s) & -0\theta_3(s) &= T(s) \\
-K\theta_1(s) \quad & +(J_2s^2 + D_2s + K)\theta_2(s) & -D_2s\theta_3(s) &= 0 \\
-0\theta_1(s) \quad & -D_2s\theta_2(s) & +(J_3s^2 + D_3s + D_2s)\theta_3(s) &= 0
\end{aligned}$$



# Transfer Functions for Systems with Gears

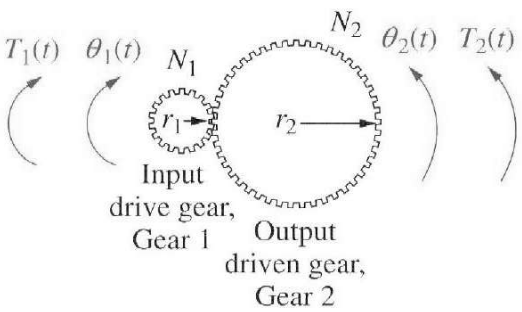
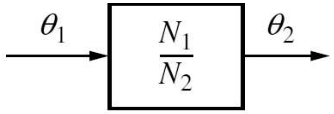
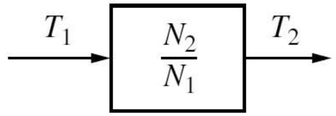



**FIGURE 2.27** A gear system

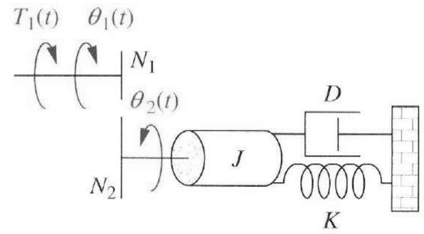
$$\boxed{\frac{\theta_2}{\theta_1} = \frac{r_1}{r_2} = \frac{N_1}{N_2}}$$



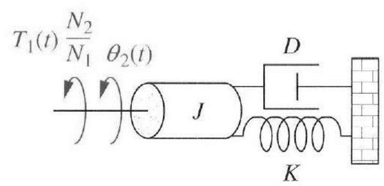

$$\boxed{\frac{T_2}{T_1} = \frac{\theta_1}{\theta_2} = \frac{N_2}{N_1}}$$



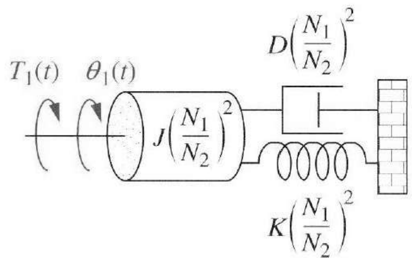



($a$)




($b$)




($c$)

$$\boxed{\left[J\left(\frac{N_1}{N_2}\right)^2 s^2 + D\left(\frac{N_1}{N_2}\right)^2 s + K\left(\frac{N_1}{N_2}\right)^2\right]\theta_1(s) = T_1(s)}$$

- Rotational mechanical impedances can be reflected through gear trains by multiplying the mechanical impedance the ratio

$$\left(\frac{\text{Number of teeth of gear on } \textit{destination} \text{ shaft}}{\text{Number of teeth of gear on } \textit{source} \text{ shaft}}\right)^2$$

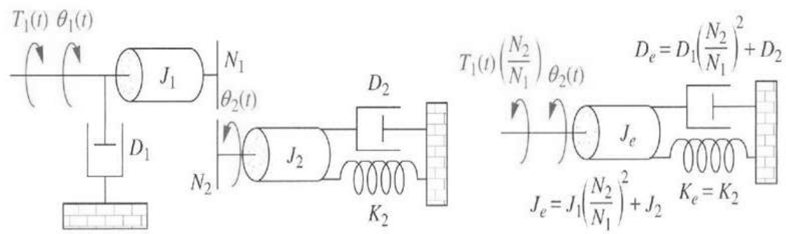
where the impedance to be reflected is attached to the source shaft and is being reflected to the destination shaft.

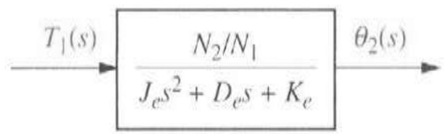


# Example 2.21

**Transfer Function—System with Lossless Gears**

**PROBLEM:** Find the transfer function, $\theta_2(s)/T_1(s)$, for the system of Figure 2.30($a$).






**SOLUTION:** It may be tempting at this point to search for two simultaneous equations corresponding to each inertia. The inertias, however, do not undergo linearly independent motion, since they are tied together by the gears. Thus, there is only one degree of freedom and hence one equation of motion.

Let us first reflect the impedances ($J_1$ and $D_1$) and torque ($T_1$) on the input shaft to the output as shown in Figure 2.30($b$), where the impedances are reflected by $(N_2/N_1)^2$ and the torque is reflected by $(N_2/N_1)$. The equation of motion can now be written as

$$(J_e s^2 + D_e s + K_e)\theta_2(s) = T_1(s)\frac{N_2}{N_1} \tag{2.139}$$

where

$$J_e = J_1\left(\frac{N_2}{N_1}\right)^2 + J_2; \quad D_e = D_1\left(\frac{N_2}{N_1}\right)^2 + D_2; \quad K_e = K_2$$

Solving for $\theta_2(s)/T_1(s)$, the transfer function is found to be

$$G(s) = \frac{\theta_2(s)}{T_1(s)} = \frac{N_2/N_1}{J_e s^2 + D_e s + K_e} \tag{2.140}$$

as shown in Figure 2.30($c$).



# Gear Train

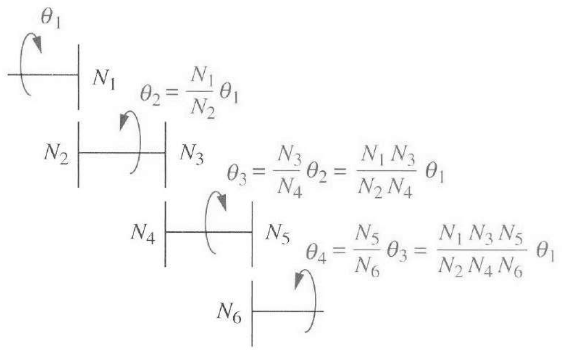


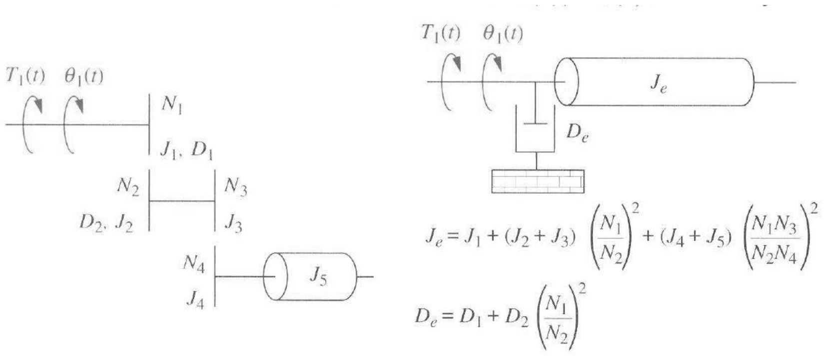

**FIGURE 2.31** Gear train

$$\theta_4 = \frac{N_1 N_3 N_5}{N_2 N_4 N_6}\theta_1$$

# Example 2.22

**Transfer Function—Gears with Loss**

**PROBLEM:** Find the transfer function, $\theta_1(s)/T_1(s)$, for the system of Figure 2.32($a$).




($a$) ($b$)



**SOLUTION:** This system, which uses a gear train, does not have lossless gears. All of the gears have inertia, and for some shafts there is viscous friction. To solve the problem, we want to reflect all of the impedances to the input shaft, $\theta_1$. The gear ratio is not the same for all impedances. For example, $D_2$ is reflected only through one gear ratio as $D_2(N_1/N_2)^2$, whereas $J_4$ plus $J_5$ is reflected through two gear ratios as $(J_4 + J_5)[(N_3/N_4)(N_1/N_2)]^2$. The result of reflecting all impedances to $\theta_1$ is shown in Figure 2.32($b$), from which the equation of motion is

$$(J_e s^2 + D_e s)\theta_1(s) = T_1(s) \tag{2.142}$$

where

$$J_e = J_1 + (J_2 + J_3)\left(\frac{N_1}{N_2}\right)^2 + (J_4 + J_5)\left(\frac{N_1 N_3}{N_2 N_4}\right)^2$$



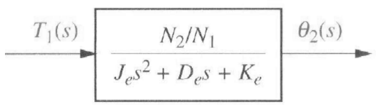
and

$$D_e = D_1 + D_2\left(\frac{N_1}{N_2}\right)^2$$

From Eq. (2.142), the transfer function is

$$G(s) = \frac{\theta_1(s)}{T_1(s)} = \frac{1}{J_e s^2 + D_e s} \tag{2.143}$$

as shown in Figure 2.32($c$).




($c$)



# Electromechanical System Transfer function

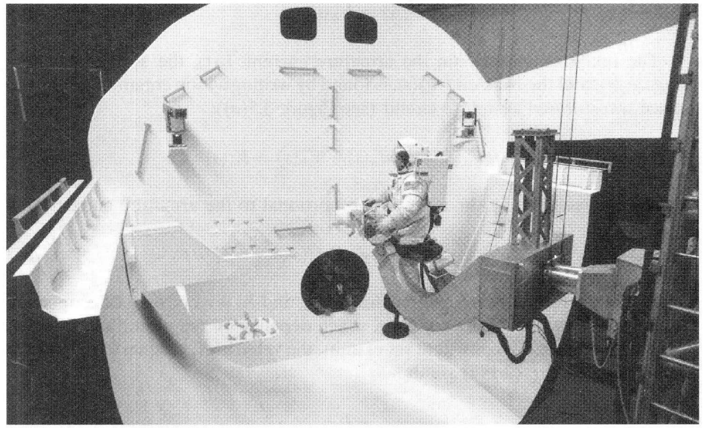
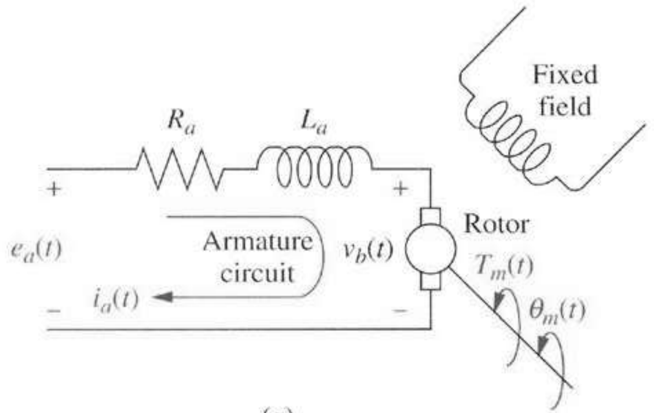


**FIGURE 2.34** NASA flight simulator robot arm with electromechanical control system components.

# Electromechanical Component: Motor

Electrical Input (Voltage) → Mechanical output (displacement)




(a)



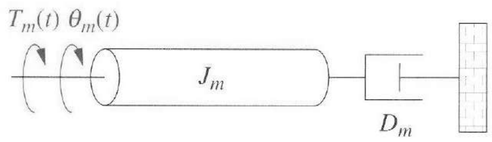
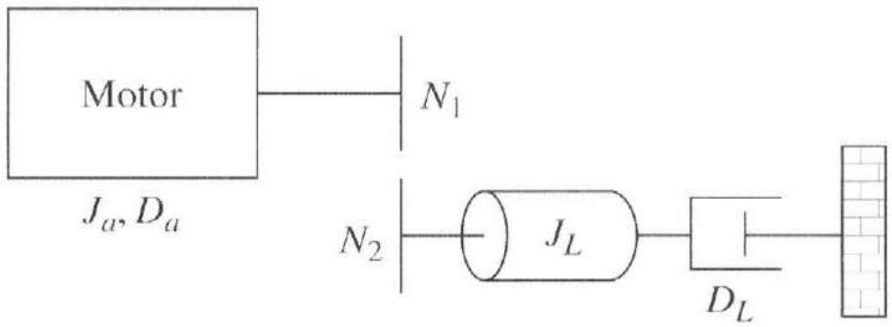



**FIGURE 2.36** Typical equivalent mechanical loading on a motor

$$\frac{\theta_m(s)}{E_a(s)} = \frac{K_t/(R_aJ_m)}{s\left[s + \frac{1}{J_m}\left(D_m + \frac{K_tK_b}{R_a}\right)\right]} \qquad (2.153)^{13}$$






$$J_m = J_a + J_L\left(\frac{N_1}{N_2}\right)^2;\ D_m = D_a + D_L\left(\frac{N_1}{N_2}\right)^2$$



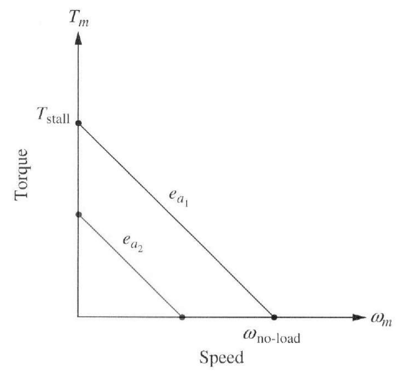



**FIGURE 2.38** Torque-speed curves with an armature voltage, $e_a$, as a parameter

$$T_m = -\frac{K_bK_t}{R_a}\omega_m + \frac{K_t}{R_a}e_a \qquad (2.159)$$

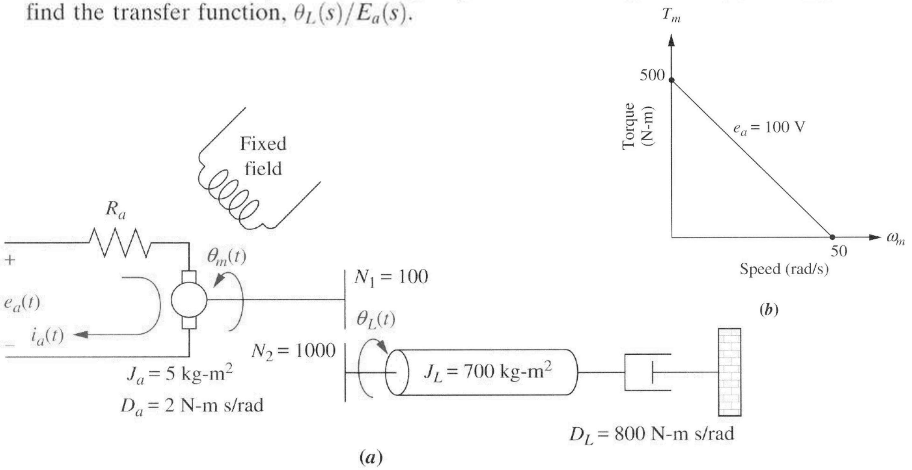
$$\frac{K_t}{R_a} = \frac{T_{stall}}{e_a} \qquad (2.162)$$

$$K_b = \frac{e_a}{\omega_{no\text{-}load}} \qquad (2.163)$$

# Example 2.23

**Transfer Function—DC Motor and Load**

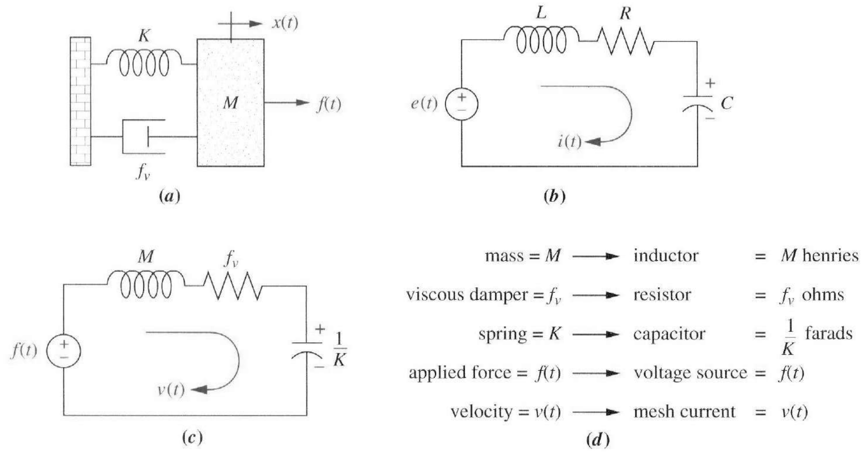
**PROBLEM:** Given the system and torque-speed curve of Figure 2.39($a$) and ($b$), find the transfer function, $\theta_L(s)/E_a(s)$.






# Electrical Circuit Analogs

## Series Analog






# Example 2.24

**Converting a Mechanical System to a Series Analog**

**PROBLEM:** Draw a series analog for the mechanical system of Figure 2.17($a$).

**SOLUTION:** Equations (2.118) are analogous to electrical mesh equations after conversion to velocity. Thus,

$$\left[M_1 s + (f_{v_1} + f_{v_3}) + \frac{(K_1 + K_2)}{s}\right]V_1(s) - \left(f_{v_3} + \frac{K_2}{s}\right)V_2(s) = F(s) \qquad (2.176a)$$

$$-\left(f_{v_3} + \frac{K_2}{s}\right)V_1(s) + \left[M_2 s + (f_{v_2} + f_{v_3}) + \frac{(K_2 + K_3)}{s}\right]V_2(s) = 0 \qquad (2.176b)$$

Coefficients represent sums of electrical impedance. Mechanical impedances associated with $M_1$ form the first mesh, where impedances between the two masses are common to the two loops. Impedances associated with $M_2$ form the second mesh. The result is shown in Figure 2.42, where $v_1(t)$ and $v_2(t)$ are the velocities of $M_1$ and $M_2$, respectively.



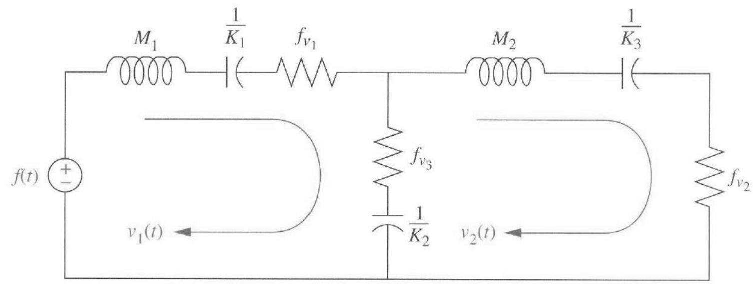
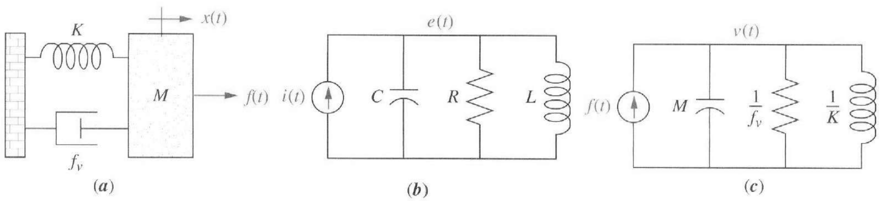



**FIGURE 2.42** Series analog of mechanical system of Figure 2.17($a$)

# Parallel Analog

$$\left(Cs+\frac{1}{R}+\frac{1}{Ls}\right)E(s)=I(s) \qquad (2.177)$$




| | | | |
|---|---|---|---|
| mass = $M$ | ⟶ | capacitor | = $M$ farads |
| viscous damper = $f_v$ | ⟶ | resistor | = $\frac{1}{f_v}$ ohms |
| spring = $K$ | ⟶ | inductor | = $\frac{1}{K}$ henries |
| applied force = $f(t)$ | ⟶ | current source | = $f(t)$ |
| velocity = $v(t)$ | ⟶ | node voltage | = $v(t)$ |

(d)

**FIGURE 2.43** Development of parallel analog: **a.** mechanical system; **b.** desired electrical representation; **c.** parallel analog; **d.** parameters for parallel analog



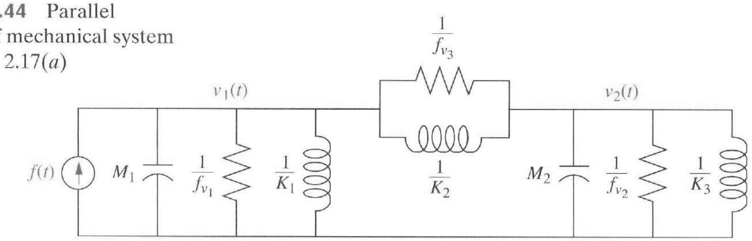
# Example 2.25

**Converting a Mechanical System to a Parallel Analog**

**PROBLEM:** Draw a parallel analog for the mechanical system of Figure 2.17($a$).

**SOLUTION:** Equation (2.176) is also analogous to electrical node equations. Coefficients represent sums of electrical admittances. Admittances associated with $M_1$ form the elements connected to the first node, where mechanical admittances between the two masses are common to the two nodes. Mechanical admittances associated with $M_2$ form the elements connected to the second node. The result is shown in Figure 2.44, where $v_1(t)$ and $v_2(t)$ are the velocities of $M_1$ and $M_2$, respectively.

**FIGURE 2.44** Parallel analog of mechanical system of Figure 2.17($a$)






# Nonlinearities

- A ***linear system*** possesses two properties.

  ***Superposition***:

  $r_1(t) \longrightarrow c_1(t)$

  $r_2(t) \longrightarrow c_2(t)$

  ⇒ $r_1(t)+r_2(t) \longrightarrow c_1(t)+c_2(t)$

  ***Homogeneity***:

  $r_1(t) \longrightarrow c_1(t)$ ⇒ $Ar_1(t) \longrightarrow Ac_1(t)$



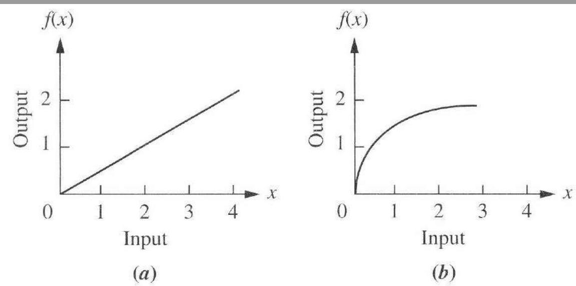
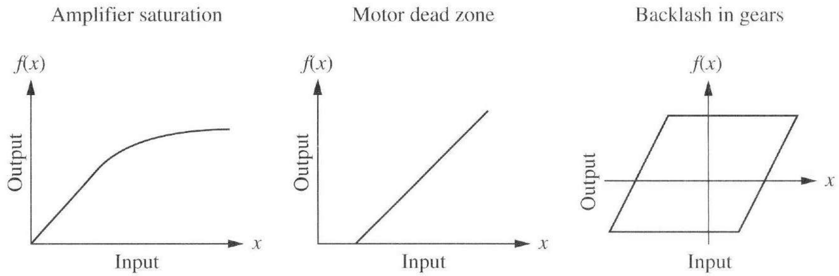



**FIGURE 2.45** **a.** Linear system; **b.** nonlinear system



# Linearization

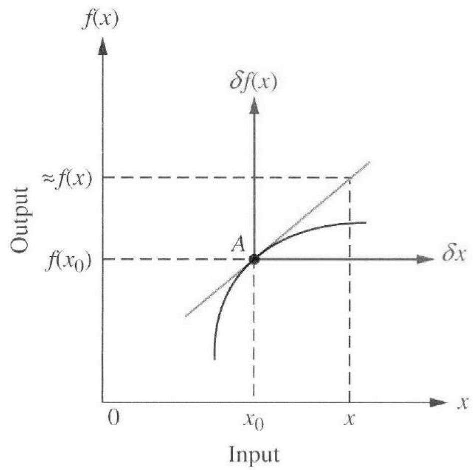



**FIGURE 2.47** Linearization about point $A$

$$[f(x) - f(x_0)] \approx m_a(x - x_0)$$

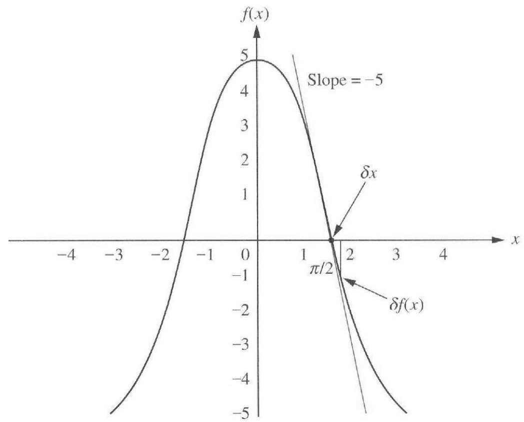
$$\delta f(x) \approx m_a \delta x$$



# Example 2.26

## Linearizing a Function

**PROBLEM:** Linearize $f(x) = 5\cos x$ about $x = \pi/2$.




$$f(x) = -5\,\delta x$$



# Example 2.27

## Linearizing a Differential Equation

**PROBLEM:** Linearize Eq. (2.184) for small excursions about $x = \pi/4$.

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + \cos x = 0 \qquad (2.184)$$

**SOLUTION:** The presence of the term $\cos x$ makes this equation nonlinear. Since we want to linearize the equation about $x = \pi/4$, we let $x = \delta x + \pi/4$, where $\delta x$ is the small excursion about $\pi/4$, and substitute $x$ into Eq. (2.184):

$$\frac{d^2\left(\delta x + \frac{\pi}{4}\right)}{dt^2} + 2\frac{d\left(\delta x + \frac{\pi}{4}\right)}{dt} + \cos\left(\delta x + \frac{\pi}{4}\right) = 0 \qquad (2.185)$$

But

$$\frac{d^2\left(\delta x + \frac{\pi}{4}\right)}{dt^2} = \frac{d^2\delta x}{dt^2} \qquad (2.186)$$

and

$$\frac{d\left(\delta x + \frac{\pi}{4}\right)}{dt} = \frac{d\delta x}{dt} \qquad (2.187)$$



Finally, the term $\cos(\delta x + (\pi/4))$ can be linearized with the truncated Taylor series. Substituting $f(x) = \cos(\delta x + (\pi/4))$, $f(x_0) = f(\pi/4) = \cos(\pi/4)$, and $(x - x_0) = \delta x$ into Eq. (2.182) yields

$$\cos\left(\delta x + \frac{\pi}{4}\right) - \cos\left(\frac{\pi}{4}\right) = \left.\frac{d\cos x}{dx}\right|_{x=\frac{\pi}{4}} \delta x = -\sin\left(\frac{\pi}{4}\right)\delta x \qquad (2.188)$$

Solving Eq. (2.188) for $\cos(\delta x + (\pi/4))$, we get

$$\cos\left(\delta x + \frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right)\delta x = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\delta x \qquad (2.189)$$

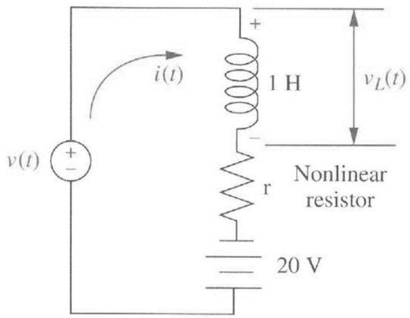
Substituting Eqs. (2.186), (2.187), and (2.189) into Eq. (2.185) yields the following linearized differential equation:

$$\frac{d^2\delta x}{dt^2} + 2\frac{d\delta x}{dt} - \frac{\sqrt{2}}{2}\delta x = -\frac{\sqrt{2}}{2} \qquad (2.190)$$

# Example 2.28

**Transfer Function—Nonlinear Electrical Network**

**PROBLEM:** Find the transfer function, $V_L(s)/V(s)$, for the electrical network shown in Figure 2.49, which contains a nonlinear resistor whose voltage-current relationship is defined by $i_r = 2e^{0.1v_r}$, where $i_r$ and $v_r$ are the resistor current and voltage, respectively. Also, $v(t)$ in Figure 2.49 is a small-signal source.






# Homework

- Skill-Assessment Exercise 2.2, 2.11, 2.13.