The Spectral Dynamical Core

You are strongly encouraged to read the companion files on the non-divergent barotropic vorticity equation and the shallow water equation before proceeding to the documentation for the full spectral dynamical core. We do not repeat the discussion that can be found there concerning spherical coordinates and the spherical harmonic spectral transform method. The numerics of the shallow water model, in particular, are closely analogous to those for the full model.

Thermodynamics

The dry component of the atmosphere is assumed to have uniform composition and to be an ideal gas:

$$p_d = \rho_d R_d T.$$

The only variable constituent of the atmosphere that has a significant effect on its density is water vapor. Strictly speaking, there are a number of effects of water vapor, and condensate, on the thermodynamics and dynamics, but because water vapor mixing ratios are at most 2% in the atmosphere, and are generally much smaller, we ignore all of these in the dynamical core except for the effect of vapor on the density, often referred to as the *virtual temperature effect*.

Setting the ratio of the mass of a water molecule to the average mass of a molecule of dry air equal to $\xi = m_w/m_d = R_d/R_w \approx 0.622$, and using the ideal gas law for the water vapor as well, $p_w = \rho_w R_w T$, we have $\rho = \rho_d + \rho_w$ and $p = p_d + p_w = (\rho R_d + \rho_w(R_w - R_d))T = \rho R_d T_v$, with

$$T_v \equiv T(1 + \mu q); \quad \mu = \frac{1-\xi}{\xi} \approx 0.608 \tag{1}$$

q is the specific humidity, the ratio of the density of vapor to the total density, $\rho_w/(\rho_d + \rho_w)$. In the following, we drop the subscript on R_d.

Within this model, one can advect several tracers if desired: only one of these, specific humidity, affects the rest of the dynamical core of the model, and only through this virtual temperature effect. Other effects of water, such as on the heat capacity per unit mass, and the various effects of condensed

water, are neglected. There is also no provision for changes in the mass of the atmosphere due to precipitation: in particular, there is no mass flux through the lower boundary of the model. In its present form, the model should not be utilized for extremely warm climates in which water vapor becomes a substantial part of the atmospheric mass.

In terms of the material derivative, D/Dt, conservation of mass takes the form

$$\frac{D\rho}{Dt} = -\rho\nabla \cdot \mathbf{v} \tag{2}$$

or, in flux form,

$$\frac{\partial\rho}{\partial t} = -\nabla \cdot (\rho\mathbf{v}) \tag{3}$$

The conservation law for any tracer, including specific humidity, can be written in either flux or advective form as well:

$$\frac{D\xi}{Dt} = S \tag{4}$$

$$\frac{\partial\rho\xi}{\partial t} = -\nabla \cdot (\rho\mathbf{v}\xi) + \rho S \tag{5}$$

where S represents the sources/sinks.

We can write the first law of thermodynamics in terms of the internal energy $c_v T$

$$c_v \frac{DT}{Dt} = -p\frac{D(1/\rho)}{Dt} + Q = \frac{p}{\rho^2}\frac{D\rho}{Dt} + Q \tag{6}$$

where Q is the diabatic heating and c_v is the heat capacity at fixed volume. One can also write the first law in terms of the enthalpy $c_p T$, where c_p is the heat capacity at constant pressure,

$$c_p \frac{DT}{Dt} = \frac{1}{\rho}\frac{Dp}{Dt} + Q \tag{7}$$

or, using the equation of state

$$\frac{DT}{Dt} = \frac{\kappa T_v \omega}{p} + \frac{Q}{c_p} \tag{8}$$

where $\kappa = R/c_p$ and ω is the traditional notation for Dp/Dt. Q includes the heating due to frictional dissipation. For our ideal gas, $c_p = c_v + R$.

Equation of motion

In a spherical coordinate system rotating with angular velocity of magnitude Ω, we have for the rate of change of the zonal, meridional, and radial components of the flow (u, v, w):

$$\frac{Du}{Dt} = 2\Omega \sin(\theta) v - 2\Omega \cos(\theta) w + \frac{uv \tan(\theta)}{r} - \frac{uw}{r} - \frac{1}{\rho r \cos(\theta)} \frac{\partial p}{\partial \lambda} \tag{9}$$

$$\frac{Dv}{Dt} = -2\Omega \sin(\theta) u - \frac{u^2 \tan(\theta)}{r} - \frac{vw}{r} - \frac{1}{\rho r} \frac{\partial p}{\partial \theta} \tag{10}$$

$$\frac{Dw}{Dt} = -2\Omega \cos(\theta) u + \frac{u^2 + v^2}{r} - \frac{1}{\rho} \frac{\partial p}{\partial r} - g \tag{11}$$

where

$$\frac{D}{Dt} = \frac{u}{r \cos(\theta)} \frac{\partial}{\partial \lambda} + \frac{v}{r} \frac{\partial}{\partial \theta} + w \frac{\partial}{\partial r} \tag{12}$$

Here gz ($z \equiv r - a$, with a a constant) is the geopotential (the gravitational plus the centrifugal) potential. We ignore the equatorial bulge of the Earth, and simply set g to be a constant as well. The lower boundary of the model is at $z = h(\lambda, \theta)$, where h defined the orography of the surface.

The hydrostatic approximation is made throughout, justified by the small aspect ratio, the ratio of vertical to horizontal scale, of the flow. Generally speaking, the resulting *primitive equations* will be well-posed only if one's subgrid closure does not allow aspect ratios of order unity from developing, (i.e., scales comparable to the vertical scale height, say 10 km). One avoids this issue by never pushing the horizontal resolution to this point. The hydrostatic relation is obtained by ignoring all but the final two terms in (11) (we also replace r by z):

$$\frac{\partial p}{\partial z} = -\rho g \tag{13}$$

or, in terms of the geopotential $\Phi \equiv gz$,

$$\frac{\partial \Phi}{\partial \ln p} = -RT_v \tag{14}$$

Because we have omitted the vertical component of the Coriolis force and the metric term in the w-equation, as well as the vertical acceleration,

one must also ignore the components of the horizontal Coriolis force due to vertical motion and the metric terms involving w so as to conserve energy. In order to conserve angular momentum, one must also replace r with a constant radius a as follows:

$$\frac{Du}{Dt} = 2\Omega \sin(\theta) v + \frac{uv \tan(\theta)}{a} - \frac{1}{\rho a \cos(\theta)} \frac{\partial p}{\partial \lambda} \tag{15}$$

$$\frac{Dv}{Dt} = -2\Omega \sin(\theta) v - \frac{u^2 \tan(\theta)}{a} - \frac{1}{\rho a} \frac{\partial p}{\partial \theta} \tag{16}$$

where

$$\frac{D}{Dt} = \frac{u}{a \cos(\theta)} \frac{\partial}{\partial \lambda} + \frac{v}{a} \frac{\partial}{\partial \theta} + w \frac{\partial}{\partial z} \tag{17}$$

These are referred to as the *primitive* equations in the *traditional* approximation.

Our equation set now includes 4 prognostic equations, for (u, v, ρ, T), with the ideal gas law then determining p. Consistency with the hydrostatic equation must then determine the vertical motion. This is an awkward implicit system. The problem is simplified if one moves to pressure as a vertical coordinate. The only change to the equations of motion for $\mathbf{v} \equiv (u, v)$ is that

$$\frac{1}{\rho} \nabla \cdot p \Rightarrow \nabla \cdot \Phi \tag{18}$$

where the derivatives on the right are taken at constant pressure. (All vectors from here on are assumed to lie along the surface of the sphere). In addition,

$$\frac{D}{Dt} = \frac{u}{a \cos(\theta)} \frac{\partial}{\partial \lambda} + \frac{v}{a} \frac{\partial}{\partial \theta} + \omega \frac{\partial}{\partial p} \tag{19}$$

Conservation of mass in pressure coordinates takes the simple form

$$0 = \frac{\partial \omega}{\partial p} + \nabla \cdot \mathbf{v} \tag{20}$$

We also have, either by integrating (20) over the atmosphere, or directly from conservation of mass and the hydrostatic equation, that

$$\frac{\partial p_s}{\partial t} = -\nabla \cdot \int_0^{p_s} \mathbf{v} dp \tag{21}$$

These form a less implicit set of equations, but the lower boundary is no longer a coordinate surface. In a spectral model, in particular, there is no alternative to using a terrain following coordinate. The simplest and traditional choice is Phillips' sigma-coordinate, $\sigma \equiv p/p_s$. But rather than write down these equations and then finite-difference them in the vertical, it is actually more informative to finite difference the pressure coordinate equations first by dividing the flow into layers with interfaces defined in a more general way, as described below.

Vertical differencing

The spectral core uses the vertical differencing described by

Simmons, A. J. and D. M. Burridge, 1981: An energy and angular-momentum conserving vertical finite-difference scheme and hybrid vertical coordinates. Monthly Weather Review, 109, pp 758 - 766.

Simmons and Burridge describe a simple way of using a σ-like coordinate near the surface, gradually moving to pressure above the tops of the highest mountains. They describe how to implement such a vertical differencing scheme while conserving angular momentum and energy exactly.

Start by dividing the atmosphere into N layers, bounded by the pressures $(p_{k-1/2}; k = 1, N+1)$. The $k'th$ layer has the pressure thickness

$$\Delta p_k \equiv p_{k+1/2} - p_{k-1/2} \tag{22}$$

$p_{N+1/2}$ is the surface pressure, p_s, while $p_{1/2}$ is the pressure at the top of the model, which can be zero, but need not be. However, if it is not zero, we assume that $\nabla p_{1/2} = 0$, i.e., that the top of the model is at constant pressure. We then assume the relationship

$$p_{k-1/2} = A_{k-1/2} p_{ref} + B_{k-1/2} p_s, \;\; k = 1, N+1 \tag{23}$$

Here p_{ref} is a constant, chosen to be close to the global mean surface pressure for convenience. It is included here so as to make A dimensionless, and so that the values of A lie between 0 and 1. If $A_k = 0$ for all k, then this reduces to sigma coordinates, with the choice of $B_k's$ determining the level spacing. One requires that $A_{N+1/2} = 0$ so that we have a σ-coordinate at the ground.

If $B_k = 0$ for all $k < k_p$, then we have pressure coordinates above this level. One should always have $B_{1/2} = 0$.

A natural way of defining a hybrid coordinate (one with $A \neq 0$) is by first defining a σ-coordinate $[A_k^\sigma = 0, B_k^\sigma]$, then defining the corresponding p-coordinate, $[A_k^p = B_k^\sigma, B_k^p = 0]$, and finally making a smooth transition from one to the other

$$[A_k, B_k] = \gamma_k [A_k^\sigma, B_k^\sigma] + (1 - \gamma_k)[A_k^p, B_k^p] = B_k^\sigma [(1 - \gamma_k), \gamma_k] \tag{24}$$

where γ_k is unity near the surface and vanishes near the top of the model.

Much of the following can be generalized to other choices for the definitions of the interfaces between model layers (for example one can transition to potential temperature rather than pressure coordinates) but none of these generalizations are currently implemented.

Surface pressure tendency

The surface pressure changes due to the divergence of mass within the individual layers:

$$\frac{\partial p_s}{\partial t} = -\sum_{k=1}^{N} D_k \tag{25}$$

where

$$D_k = \nabla \cdot (\mathbf{v}_k \Delta p_k) \tag{26}$$

where Δp_k is the pressure thickness of the $k'th$ layer and $\mathbf{v}_k$ is the velocity within the $k'th$ layer.

Mass conservation for the $k'th$ layer takes the form

$$\frac{\partial (\Delta_k p)}{\partial t} = -D_k - (M_{k+1/2} - M_{k-1/2}) \tag{27}$$

where M/g is the downward mass flux through the interface. We require that $M_{1/2} = 0$. Integrating down from the top, the downward mass flux per unit area across the $k + 1/2$ interface is

$$\begin{aligned} M_{k+1/2} &= -\sum_{r=1}^{k} D_r - \frac{\partial p_{k+1/2}}{\partial t} \\ &= -\sum_{r=1}^{k} D_r - B_{k+1/2} \frac{\partial p_s}{\partial t} \end{aligned} \tag{28}$$

If the interface were a material surface, the change in the pressure on it would be given by the divergence of mass above the interface (the first term in the equation above). The difference between this pressure tendency and the actual pressure tendency on the interface gives the mass moving through the interface. Consistent with the finite-differencing of the surface pressure equation, $M_{N+1/2}$ will be identically zero.

In the σ-coordinate special case, $M = p_s\dot{\sigma}$

Vertical advection

The vertical advection in the spectral model is in advective rather than flux form, but start with the flux form for a tracer ξ,

$$\frac{\partial(\xi_k \Delta_k p)}{\partial t} = -\nabla \cdot (\mathbf{v}_k \xi_k \Delta_k p) - (M_{k+1/2}\xi_{k+1/2} - M_{k-1/2}\xi_{k-1/2}) \qquad (29)$$

This requires a specification of ξ at the interfaces. Using conservation of mass for this layer, in the finite-differenced form (27), we can also write this as

$$\frac{\partial \xi_k}{\partial t} = -\mathbf{v}_k \cdot \nabla \xi_k - \frac{1}{\Delta_k p}(M_{k+1/2}(\xi_{k+1/2} - \xi_k) + M_{k-1/2}(\xi_k - \xi_{k-1/2})) \qquad (30)$$

This is the advective form that we use.

Different vertical advection options are defined by different choices for the interface values of the tracer. A vertical advection module is shared by our B-grid point and spectral dynamical cores, which provides several choices: 2nd order centered differencing, 4'th order centered, piecewise linear finite-volume, and piecewise parabolic finite volume.

The finite volume schemes can be thought of as *remappings*, in the following sense: postulate a sub-grid scale distribution of ξ within each layer, constrained by the requirement that the mean of this distribution is the model's value of ξ in that layer. Then determine that part of the upstream grid box that would flow through the interface in question over the given timestep, assuming uniform flow with the velocity (M) defined at the interface. Average ξ over this part of the upstream grid box to define the interfacial ξ. If the assumed subgrid distribution is simply a constant, this results in the simplest upstream scheme (which is far to diffusive to be useful). Linear and parabolic distributions result in usable schemes, the art being to determine

the slope in the linear scheme, or the parameters defining the parabola in the parabolic scheme, to provide features such as monotonicity (producing no values of the tracer that are outside of the range of values present initially) with minimal dissipation. See the documentation of the vertical advection module for details.

The hydrostatic equation

Writing the hydrostatic equation for an ideal gas in the form

$$\frac{\partial \Phi}{\partial lnp} = -RT^v \tag{31}$$

where T^v is the virtual temperature (the subscript has been raised to get it out of the way), one has the simple expression for half-level values of the geopotential

$$\Phi_{k-1/2} = \Phi_{k+1/2} + RT^v_k(lnp_{k+1/2} - lnp_{k-1/2}) \tag{32}$$

which can be summed up from the ground, given the surface geopotential $\Phi_{N+1/2} \equiv \Phi_s$. However, the equation of motion require the full-level values of the geopotential. To obtain these, one must define the full-level pressures, p_k, and then compute

$$\Phi_k = \Phi_{k+1/2} + RT^v_k(lnp_{k+1/2} - lnp_k) \tag{33}$$

See below for the specification of full pressure values

Pressure gradient force

In the equations of motion, one must compute the pressure gradient force on the coordinate surface –

$$\nabla\Phi + \frac{\partial \Phi}{\partial p}\nabla p = \nabla\phi + \frac{\partial \Phi}{\partial \ln p}\nabla \ln p = \nabla\phi - RT_v \nabla \ln p \tag{34}$$

If we were to write the momentum equation in flux form, the pressure gradient term for the k'th layer contributes

$$\Delta_k p\, \nabla\Phi_k + \Delta_k p \left(\frac{\partial \Phi}{\partial p}\nabla p\right)_k \tag{35}$$

or

$$\nabla(\Delta_k p\,\Phi_k) - \Phi_k(\nabla p_{k+1/2} - \nabla p_{k-1/2}) + \Delta_k p\,\Big(\frac{\partial \Phi}{\partial p}\nabla p\Big)_k \tag{36}$$

The first term represents exchange of momentum in the horizontal within a layer; the remaining terms represent the exchange of momentum between layers. In order to conserve (angular) momentum, one needs to insure that the latter terms are expressible as a difference between the forces on the upper and lower boundaries of the layer. The simplest expression of this form would be

$$\Phi_{k+1/2}\nabla p_{k+1/2} - \Phi_{k-1/2}\nabla p_{k-1/2} \tag{37}$$

In pressure coordinates, one can think of Φ as the "pressure" whose gradient determines the pressure gradient force. Here, the pressure on the upper or lower interfaces is multiplied by the slope of the interface (∇p) to generate the horizontal component of the force. To generate this expression, we need

$$\Big(\frac{\partial \Phi}{\partial p}\nabla p\Big)_k = \frac{1}{\Delta_k p}\Big((\Phi_{k+1/2} - \Phi_k)\nabla p_{k+1/2} + (\Phi_k - \Phi_{k-1/2})\nabla p_{k-1/2}\Big) \tag{38}$$

That is,

$$RT_k(\frac{\nabla p}{p})_k = \frac{RT_k}{\Delta p_k}(lnp_k - lnp_{k-1/2})\nabla p_{k-1/2} + (lnp_{k+1/2} - lnp_k)\nabla p_{k+1/2} \tag{39}$$

For the Simmons-Burridge coordinate, $\nabla p_{k-1/2} = B_{k-1/2}\nabla p_s$.

Temperature equation

One also requires an expression for the term $\kappa T\omega/p$ in the temperature equation. We set

$$(\omega/p)_k = (\frac{D\ln p}{Dt})_k = (\frac{1}{p}\frac{\partial p}{\partial t})_k + (\mathbf{v}\cdot\frac{1}{p}\nabla p)_k + V_k \tag{40}$$

where

$$V_k \equiv \frac{1}{\Delta p_k}\Big[M_{k+1/2}(lnp_{k+1/2} - lnp_k) + M_{k-1/2}(lnp_k - lnp_{k-1/2})\Big] \tag{41}$$

using the same formulation of the vertical advection of lnp as in (30). The term involving the horizontal gradient is computed as

$$\mathbf{v}_k\cdot(\frac{1}{p}\nabla p)_k \tag{42}$$

where $(\nabla p/p)_k$ is treated just as in the momentum equations. We also have an expression for $\partial p/\partial t$ at the half-levels, from (28).

$$\frac{\partial p_{k+1/2}}{\partial t} = -\sum_{r=1}^{k} D_r - M_{k+1/2} \tag{43}$$

To conserve energy, one must weight these in the same way as in the computation of the pressure gradient

$$(\frac{1}{p}\frac{\partial p}{\partial t})_k = \frac{1}{\Delta p_k}\left[[lnp_{k+1/2} - lnp_k]\frac{\partial p_{k+1/2}}{\partial t} + [lnp_k - lnp_{k-1/2}]\frac{\partial p_{k-1/2}}{\partial t}\right] \tag{44}$$

or

$$\begin{aligned}(\frac{1}{p}\frac{\partial p}{\partial t})_k = \; & - \; \frac{1}{\Delta p_k}\left[[lnp_{k+1/2} - lnp_{k-1/2}]\sum_{r=1}^{k} D_r + [lnp_k - lnp_{k-1/2}]D_k\right] \\ & - \; V_k \end{aligned} \tag{45}$$

The term V_k involving the vertical mass flux cancels out nicely and we have

$$\begin{aligned}(\omega/p)_k = \; & - \; \frac{1}{\Delta p_k}\left[[lnp_{k+1/2} - lnp_{k-1/2}]\sum_{r=1}^{k} D_r + [lnp_k - lnp_{k-1/2}]D_k\right] \\ & + \; \mathbf{v}_k \cdot (\frac{1}{p}\nabla p)_k \end{aligned} \tag{46}$$

where the last term is computed as in (39). The required term in the temperature equation is then written as $\kappa T_k(\omega/p)_k$.

Full-level pressures

In the σ-coordinate limit ($A = 0$), the expression (39) for ∇lnp reduces to $C\nabla lnp_s$ where

$$C \equiv \frac{1}{B_{k+1/2} - B_{k-1/2}}\left[(lnp_k - lnp_{k-1/2})B_{k-1/2} + (lnp_{k+1/2} - lnp_k)B_{k+1/2}\right] \tag{47}$$

Since $\nabla lnp \equiv \nabla lnp_s$ in σ-coordinates, it would be nice if $C \equiv 1$. This will be true in the σ-coordinate special case if, more generally,

$$1 = \frac{1}{\Delta p_k}\left[(lnp_k - lnp_{k-1/2})p_{k-1/2} + (lnp_{k+1/2} - lnp_k)p_{k+1/2}\right] \tag{48}$$

or

$$lnp_k = \frac{p_{k+1/2} lnp_{k+1/2} - p_{k-1/2} lnp_{k-1/2}}{\Delta p_k} - 1 \tag{49}$$

This can also be written as

$$lnp_k = lnp_{k+1/2} - \alpha_k \tag{50}$$

where

$$\alpha_k \equiv 1 - \frac{p_{k-1/2}}{\Delta p_k}(lnp_{k+1/2} - lnp_{k-1/2}) \tag{51}$$

which is the notation used by Simmons and Burridge. One can think of (49) as a finite-differencing of the expression

$$lnp = \frac{\partial(p\, lnp)}{\partial p} - 1 \tag{52}$$

If one takes (48), multiplies by Δp_k and then computes the horizontal gradient or time derivative, one can rederive the expressions (39) and (44) for the gradient and time derivative of lnp on the full-level surfaces.

A slight complication occurs if the top of the model is at zero pressure, as would be true in the σ-coordinate special case, since $lnp_{1/2}$ is then undefined. But $\lim_{p \to 0} plnp = 0$, so the natural extension of (49) to this case is

$$lnp_1 = \frac{p_{3/2} lnp_{3/2}}{p_{3/2}} - 1 = lnp_{3/2} - 1 \tag{53}$$

or

$$p_1 = p_{3/2}/e. \tag{54}$$

When $p_{1/2} \neq 0$, (49) is used to define p_1.

For an atmosphere in which the temperature is a function of pressure only, the pressure gradient force in the horizontal equations of motion vanishes identically, and the two terms into which this force is decomposed in this coordinate should cancel exactly. An important consequence of the choice (49) for full-level pressures is that it results in exact cancellation for the special case of σ-coordinates as long as the temperature is a linear function of lnp (see Simmons and Burridge).

Vorticity/divergence form of equations of motion

Define the vertical advection operator

$$W_k(\xi) \equiv -\frac{1}{2\Delta p_k}(M_{k+1/2}(\xi_{k+1} - \xi_k) + M_{k-1/2}(\xi_k - \xi_{k-1})) \tag{55}$$

with the full-level values ξ_k determined by one's choice of advection scheme. The zonal and meridional equations of motion can then be manipulated into the form

$$\frac{\partial u_k}{\partial t} = (f + \zeta_k)v_k - \frac{1}{acos(\theta)}(\frac{\partial E_k}{\partial \lambda} + RT_k^v \frac{\partial lnp_k}{\partial \lambda}) + W_k(u) \tag{56}$$

$$\frac{\partial v_k}{\partial t} = -(f + \zeta_k)u_k - \frac{1}{a}(\frac{\partial E_k}{\partial \theta} + RT_k^v \frac{\partial lnp_k}{\partial \theta}) + W_k(v) \tag{57}$$

where $f \equiv 2\Omega \sin(\theta)$ is the vorticity of solid body rotation and ζ is the vertical component of relative vorticity,

$$\zeta \equiv \frac{1}{a\cos(\theta)}\frac{\partial v}{\partial \lambda} - \frac{1}{a\cos(\theta)}\frac{\partial u \cos(\theta)}{\partial \theta} \tag{58}$$

and

$$E_k \equiv \Phi_k + \frac{1}{2}(u_k^2 + v_k^2) \tag{59}$$

Rewrite this as

$$\frac{\partial u_k}{\partial t} = A_k - \frac{1}{acos(\theta)}\frac{\partial E_k}{\partial \lambda} \tag{60}$$

$$\frac{\partial v_k}{\partial t} = B_k - \frac{1}{a}\frac{\partial E_k}{\partial \theta} \tag{61}$$

where

$$A_k \equiv (f + \zeta_k)v_k - \frac{RT_k^v}{acos(\theta)}\frac{\partial lnp_k}{\partial \lambda} + W_k(u) \tag{62}$$

$$B_k \equiv -(f + \zeta_k)u_k - \frac{RT_k^v}{a}\frac{\partial lnp_k}{\partial \theta} + W_k(v) \tag{63}$$

We can now easily write these equations in vorticity/divergence form. With

$$D \equiv \frac{1}{a\cos(\theta)}\frac{\partial u}{\partial \lambda} + \frac{1}{a\cos(\theta)}\frac{\partial v \cos(\theta)}{\partial \theta} \tag{64}$$

and defining the vectors $\mathbf{T} \equiv (A, B)$ and $\mathbf{T}_\perp \equiv (B, -A)$, we have

$$\frac{\partial \zeta_k}{\partial t} = \nabla \cdot \mathbf{T}_{\perp k} \tag{65}$$

$$\frac{\partial D_k}{\partial t} = \nabla \cdot \mathbf{T}_k - \nabla^2 E_k \tag{66}$$

Our system of equations now consists of prognostic equations for $\zeta, D, T, \ln p_s$ and whatever tracers are included, particularly, specific humidity. The geopotential is computed by integrating up from the surface, with its surface value set to gz where z is the prescribed orography. The vertical mass fluxes M are computed by integrating down from the top of the model, where $M = 0$. The vorticity, divergence, and temperature are diffused horizontally with a ∇^{2n} operator, in which one typically chooses $n = 2$ or 4. If tracers are advected spectrally, they are also subject to this diffusion. Tracers advected horizontally by the finite-volume scheme (see below) are not subject to explicit diffusion.

Following the procedure used in nearly all global spectral models, $\ln p_s$ is utilized as a prognostic variable

$$\frac{\partial \ln p_s}{\partial t} = -\frac{1}{p_s} \sum_{k=1}^{N} \nabla \cdot (\mathbf{v}_k \Delta_k p) \tag{67}$$

The advantage of making $\ln p_s$ the spectral variable is evidently that the geopotential and pressure gradient computation works with $\ln p_s$. The spectral model is designed to work well for quadratic nonlinearities – the extra aliasing in the pressure gradients that results from using p_s as a spectral variable results in an unworkable model. Unfortunately, the use of $\ln p_s$ as the prognostic variable makes it impossible to conserve atmospheric mass and tracer mass exactly. An option is provided to adjust $\ln p_s$ every time step by an additive constant, or equivalently p_s by a multiplicative constant, so that the global mean of surface pressure is unchanged.

There is also an optional correction to conserve the global integral of the specific humidity with a globally uniform multiplicative correction every time step. And there is a hole filling algorithm that generally needs to be used when one is using the spectral advection option for specific humidity to try to avoid negative values. Hole filling is not needed when using finite volume tracer advection. Because the spectral model is currently implemented with

1D decomposition in latitude only, for efficiency hole filling is performed by looking at nearest neighbors in the latitude-height plane only, and reduces these values in a uniform ratio to fill in negative values at the central point.

Finally, an optional energy conservation correction is also provided which corrects the temperature field by adding a globally uniform constant every time step so as to conserve the sum of c_pT and the kinetic energy of the horizontal flow. (In a hydrostatic model, the vertical integral of c_pT is equal to the vertical integral of the sum of the internal energy c_vT and potential energy gz.) The dynamical core should conserve energy except for the action of the horizontal n-th harmonic diffusion and the time filter associated with the leapfrog time step.

Semi-implicit time-stepping

The code uses a standard *semi-implicit* leapfrog scheme, followed by a Robert-Asselin time filter.

The semi-implicit aspect of the algorithm allows one to take time steps that are determined by the strength of the advection, rather that the gravity wave speeds. This technique is invariably employed in spectral models and is, in fact, in large part responsible for the predominant use of the spectral transform algorithm in global atmospheric modeling. Let V represent the state of the system, and divide the operator N describing the evolution of this system into two parts, $N = N_E + N_I$:

$$\frac{\partial V}{\partial t} = N_E(V) + N_I(V) \tag{68}$$

Use a centered leapfrog scheme for N_E and an implicit scheme for N_I:

$$\frac{V^{i+1} - V^{i-1}}{2\Delta t} = N_E(V^i) + N_I(\alpha V^{i+1} + (1-\alpha)V^{i-1}) \tag{69}$$

We assume that N_I is a linear operator in V, otherwise an expensive iteration would be required to solve this implicit system. The choice of $\alpha = 0.5$, centered implicit, is the one typically used, but one can also consider a fully backward scheme ($\alpha = 1.0$), for example, if one prefers to damp as well as slow down the gravity waves.

Using the notation

$$\xi \equiv 2\alpha\Delta t, \tag{70}$$

and

$$\delta V \equiv \frac{V^{i+1} - V^{i-1}}{2\Delta t} \tag{71}$$

we find

$$\delta V = N_E V^i + N_I V^{i-1}) + \xi N_I(\delta V) \tag{72}$$

or

$$\delta V = (1 - \xi N_I)^{-1} G \tag{73}$$

where

$$G = N_E V^i + N_I(V^{i-1}) = N(V^i) + N_I(V^{i-1} - V^i) \tag{74}$$

We need first of all to define the parts of the equations that are to be part of the implicit operator N_I. This is determined by linearizing the equations for a non-rotating atmosphere about a state of rest, in which orography is ignored, surface pressure is uniform, and temperatures are a function of the vertical coordinate only.

With this simple reference state, the linear matrix to be inverted is diagonal in the spectral domain, just as for the shallow water equations. The different vertical levels are coupled to each other, however.

Our equations for each complex spectral harmonic take the form:

$$\begin{aligned} \delta T &= G_T + \xi L \delta D \\ \delta D &= G_D + \xi(R\delta T + U\delta \ln p_s) \\ \delta \ln p_s &= G_{\ln p} + \xi W \delta D \end{aligned} \tag{75}$$

where δT and δD are vectors of dimension N, as are G_T and G_D, while L and R are $N \times N$ matrices, U is $N \times 1$, and W is $1 \times N$. In this derivation, one uses the fact that perturbations in temperature and in surface pressure modify the geopotential, while perturbations in divergence and in surface pressure modify the vertical mass fluxes M

Eliminating in favor of δD, one obtains a single $N \times N$ matrix, S, to invert for each harmonic,

$$\delta D = S^{-1} G \tag{76}$$

where

$$S = 1 + \xi^2(RL + UW) \tag{77}$$

and

$$G = G_D + \xi(RG_T + UG_{\ln p}) \tag{78}$$

We compute S^{-1} at initialization for each harmonic (S depends only on the total horizontal wavenumber ℓ). On each time step, we first compute all tendencies with all terms explicit and centered in time. (This procedure has the advantage that it is easy to return to an explicit model for specialized purposes and code checkout). We then compute G, being careful to include the term in (74) that moves the terms treated implicitly from i to $i-1$. δD is then computed by matrix multiplication for each harmonic, and then δT and $\delta \ln p_s$ are generated by substituting back into 76).

Algorithm

Transforms_mod provides several high-level routines for moving from vorticity and divergence to (u, v) and back, as well as for computing the advection operator $-\mathbf{v} \cdot \nabla\xi$. *subroutine uv_grid_from_vor_div* takes as input vorticity and divergence in the spectral domain and returns (u, v) on the grid. It first computes $(cos(\theta)u, cos(\theta)v$ in the spectral domain, then transforms these to the grid and divides by $\cos(\theta)$. *subroutine vor_div_from_uv_grid* reverses this procedure. *subroutine horizontal_advection* computes $-\mathbf{v} \cdot \nabla\xi$ in the grid domain, given ξ in the spectral domain and (u, v) on the grid. It does this by first computing $cos(\theta)\nabla\xi$ in the spectral domain, transforms the components to grid space, and then divides by $\cos(\theta)$ and multiplies by the components of the velocity field in the grid domain. If one also has available the divergence, $\mathcal{D}$, on the grid, one can then compute $-\nabla \cdot (\mathbf{v}\xi) = -\mathbf{v} \cdot \nabla\xi - \xi\mathcal{D}$. One could also compute this quantity using *vor_div_from_uv_grid*. The answer would be identical.

An outline of the steps involved in integrating the spectral model is as follows:

Assume that we know the spectral vorticity, divergence, temperature, and log(surface pressure), and the grid values of vorticity divergence, temperature, surface pressure, and of u and v, at $t - \Delta t$ and t. Then

1: Compute the grid tendencies of u, v, T, and tracers due to non-dynamical processes

2: Compute the information that will be needed at the end of the time step for any global corrections to mass, energy, or water vapor.

3: Compute the full and half level values of pressure and log(pressure) from the surface pressure, as well as the virtual temperature and the geopotential

4: Compute the gradient of surface pressure by differentiating in the spectral domain and transforming to the grid.

5: Using these gradients and the grid divergence, compute the vertical mass fluxes, the log(surface pressure) tendency ($\delta \ln p_s$). and the $RT\nabla \ln p_s$ term in $(\delta u, \delta v)$, the u and v tendencies

6: add on the tendencies due to vertical advection to $\delta u, \delta v, \delta T$

7: Compute $(f + \zeta)v$ and $-(f + \zeta)u$ on the grid at time t, and add to δu and δv respectively.

8: compute the divergence and curl of $(\delta u, \delta v)$ to obtain $\delta\zeta$ and $\delta\mathcal{D}$ in the spectral domain, using *subroutine vor_div_from_uv_grid*.

9: add to δT on the grid the horizontal advection term $-\nabla \cdot (\mathbf{vT})$. using *subroutine horizontal_advection*, and then convert δT to spectral domain

10: compute the kinetic energy, add to Φ at t on the grid, transform to spectral domain, take the Laplacian, and add $-\nabla^2 E$ to the spectral divergence tendency $\delta\mathcal{D}$;

11: correct (δT, $\delta\mathcal{D}$, $\delta \ln p_s$) tendencies to take into account the semi-implicit algorithm, as described above

12: add the (nth-)harmonic damping to $\delta\zeta, \delta\mathcal{D}, \delta\mathcal{T}$ in the spectral domain, treating the damping implicitly, as in the barotropic and shallow water models

13: Use leapfrog to generate spectral $t + \Delta t$ values of $\zeta, \mathcal{D}$, $T, \ln p_s$ and apply Robert filter to the t values.

14: Generate spectral (u, v) from (ζ, $\mathcal{D}$) at $t + \Delta t$

15: compute grid ζ, $\mathcal{D}$, u, v, T, $\ln p_s$ (and p_s) at $t + \Delta t$

16: Update spectral tracers by first computing horizontal advection, adding on vertical advection, performing hole filling where needed, transferring the tendency back to the spectral domain, performing leapfrog and time filter and transforming $t + \Delta t$ values back to the grid to prepare for next step.

17: Update grid tracers by first creating a temporary updated $t + \Delta t$ field with all physical tendencies, performing horizontal finite volume advection from $t - \Delta t$ to $t + \Delta t$ on this field to create yet another updated field, then performing vertical advection on this last field, once again from $t - \Delta t$ to $t + \Delta t$, using one of the finite volume options (we do not recommend mixing finite volume horizontal advection and centered vertical advection), then applying the time filter. Grid tracers are never transformed to the spectral domain and back. The advecting velocities are taken from the centered time t.

Note that we do not bother to compute the new grid values of those fields modified by the Robert filter.

The finite volume advection scheme follows closely that described in

Lin, S.-J. and R. B. Rood, 1996: Multidimensional flux-form semi-Lagrangian transport schemes. Monthly Weather Review, 124, pp. 2046-2069.

For backwards compatibility, there is also the option of finally passing the new grid temperatures and humidities to a second physics routine , for an *adjustment* step, after which these adjusted temperatures and humidities are returned to the spectral domain. In this case, the way in which the time filter is imposed is modified, as one cannot compute the finalized time t fields from $t - \Delta t$ and $t + \Delta t$ until the latter has been finalized. The standard leapfrog and time filter take the form

- $\xi(t + \Delta t) = \xi(t - \Delta t) + 2\Delta t\, d\xi/dt$
- $\xi(t) = \xi(t) + r(\xi(t + \Delta t) + \xi(t - \Delta t) - 2\xi(t))$

To avoid adding more memory (the values at $t+\Delta t$ and $t-\Delta t$ share the same memory), this is rearranged as follows when an adjustment step is utilized

- $\xi(t) = \xi(t) + r(\xi(t - \Delta t) - 2\xi(t))$

- $\xi(t + \Delta t) = \xi(t - \Delta t) + 2\Delta t \, d\xi/dt$
- do the physics adjustment to finalize $\xi(t + \Delta t)$
- then finalize $\xi(t) = \xi(t) + r\xi(t + \Delta t)$

Held-Suarez forcing

When extracted from the FMS repository as an isolated dynamical core, rather than as a component of a comprehensive model, the spectral model is packaged to run with the simple forcing defined in

Held, I. M., and M. J. Suarez, 1994: A proposal for the intercomparison of the dynamical cores of atmospheric general circulation models. Bulletin of the American Meteorological Society, 75(10), 1825-1830.

This consists of a simple relaxation of temperature to a specified *radiative equilibrium* and a linear damping of momentum near the surface. This forcing has been designed to generate an atmospheric climate that resembles that observed. A meaningful climate can be obtained from this model by integrating for roughly 1000 days. But there are a variety of important ways in which the resulting model differs from reality, many of which relate to the fact that the model is dry. This simple physics package is meant to provide a way of testing dynamical cores and addressing certain fundamental questions about atmospheric circulation that can be approached in an idealized dry setting. See http://www.gfdl.noaa.gov/∼gth/netscape/1994/ih9401.pdf