Dynamics Derivation for Cart Inverted Pendulum

Tianpeng Zhang

August 1, 2020

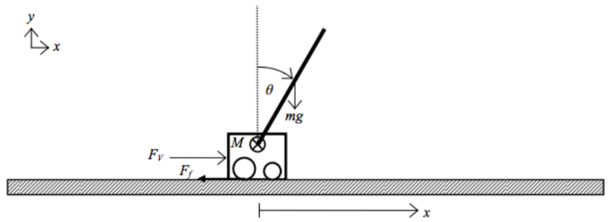

Figure 1: System Diagram for cart inverted pendulum. We assume the rod on the cart is a rigid straight rod with uniform mass distribution. The positive direction of rod rotation is chosen as pointing towards negative z direction as shown in the diagram.

As a correction to the old lab notes, the equation of motion for the system should be

$$\begin{aligned}(m_c + m_p)\ddot{x} + c\dot{x} &= F + m_p r(\sin\theta\ \dot{\theta}^2 - \cos\theta\ \ddot{\theta}) \\ (J + m_p r^2)\ddot{\theta} + \gamma\dot{\theta} &= m_p r g \sin\theta - m_p r \ddot{x}\cos\theta\end{aligned} \tag{1}$$

Denote $M = m_c + m_p, R = m_p r, K = J + m_p r^2$, the equations above simplifies as

$$\begin{aligned}M\ddot{x} + c\dot{x} &= F + R(\sin\theta\ \dot{\theta}^2 - \cos\theta\ \ddot{\theta}) \\ K\ddot{\theta} + \gamma\dot{\theta} &= Rg\sin\theta - R\ddot{x}\cos\theta\end{aligned} \tag{2}$$

And we have

$$\ddot{x} = \frac{Rg\sin\theta - K\ddot{\theta} - \gamma\dot{\theta}}{R\cos\theta} \tag{3}$$

Substitute this back to (2), get $\ddot{\theta}$ as

$$\ddot{\theta} = \frac{M\gamma\dot{\theta} - MgR\sin\theta + (F - c\dot{x} + R\sin\theta\ \dot{\theta}^2)R\cos\theta}{-MK + R^2\cos^2\theta} \tag{4}$$

With $\ddot{\theta}$ we can use (3) to calculate $\ddot{x}$ already. Nonetheless, we also give the explicit formula for $\ddot{x}$ as below

$$\ddot{x} = \frac{R^2 g \sin\theta \cos\theta - K(F - c\dot{x} + R\sin\theta\ \dot{\theta}^2) - \gamma R\dot{\theta}\cos\theta}{-MK + R^2\cos^2\theta} \tag{5}$$