Dynamics Derivation for Cart Inverted Pendulum

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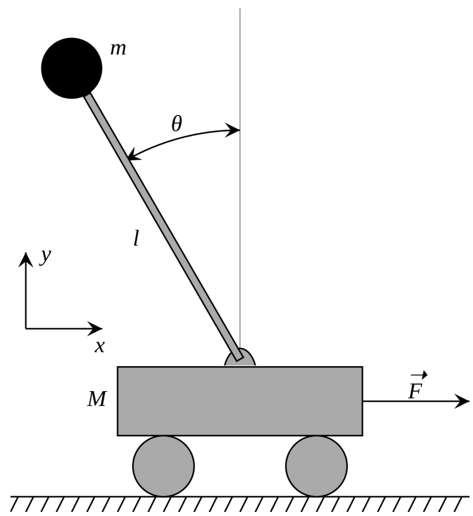

Figure 1: Forces and Coordinate System for Cart Inverted Pendulum

See Figure 1 for the forces acted on the system. We assume the rod connected the cart M and the weight m is massless, and ignore any drag/friction in the system. The most important external forces are gravity on the weight m and our control input force F.

See this Wikipedia Page for the derivation of the following equation of motions:

$$\begin{aligned} &(M+m)\ddot{x} - ml\ddot{\theta}\cos\theta + ml\dot{\theta}^2\sin\theta = F \\ &l\ddot{\theta} - g\sin\theta = \ddot{x}\cos\theta \end{aligned} \tag{1}$$

Notice from the second equation in (1),

$$\ddot{x} = (l\ddot{\theta} - g\sin\theta)/\cos\theta \tag{2}$$

Substitute (2) into the first equation in (1), can get $\ddot{\theta}$ as

$$\ddot{\theta} = [(M+m)g\sin\theta - ml\dot{\theta}^2\sin\theta\cos\theta + F\cos\theta]/(M+m\sin^2\theta)l \tag{3}$$

Notice $\ddot{\theta}$ does not depend on $x, \dot{x}$, therefore the $(\theta, \dot{\theta})$ subsystem it can be treated as an isolated system.

And substitute (3) into (2) we get $\ddot{x}$ as

$$\ddot{x} = (mg\sin\theta\cos\theta - ml\dot{\theta}^2\sin\theta + F)/(M+m\sin^2\theta) \tag{4}$$

Let's denote the system state vector as s, which contains four elements: $s = (x, \dot{x}, \theta, \dot{\theta})$. Let us denote $u = F$ as the control input. The state transition function for s is then

$$\dot{s} = \begin{bmatrix} s_2 \\ (mg\sin s_3\cos s_3 - mls_4^2\sin s_3 + u)/(M+m\sin^2 s_3) \\ s_4 \\ [(M+m)g\sin s_3 - mls_4^2\sin s_3\cos s_3 + u\cos s_3]/(M+m\sin^2 s_3)l \end{bmatrix} \tag{5}$$

Again, notice the (s_3, s_4) subsystem can be treated as an isolated system.