Математический анализ

«Производная неявной функции. Экстремум функции нескольких переменных»

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МОСКОВСКИЙ АВИАЦИОННЫЙ ИНСТИТУТ
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Производная неявной функции

Пусть функция $z(x_1, \ldots, x_n)$ задана неявно через соотношение

$$F(x_1, \ldots, x_n, z) = 0.$$

Тогда если все условия теоремы о неявной функции выполнены, то

$$\frac{\partial z}{\partial x_i} = -\frac{F'_{x_i}}{F'_z}, \quad i = \overline{1, n}.$$

3372. Вычислить y', y'', если $\ln\sqrt{x^2+y^2} = \operatorname{arctg}\frac{y}{x}$

$$F(x,y) = \frac{1}{2}\ln(x^2+y^2) - \operatorname{arctg}\frac{y}{x},$$

$$F'_x = \frac{2x}{2(x^2+y^2)} - \frac{1}{\left(\frac{y}{x}\right)^2+1}\left(-\frac{y}{x^2}\right) = \frac{x+y}{x^2+y^2},$$

$$F'_y = \frac{2y}{2(x^2+y^2)} - \frac{1}{\left(\frac{y}{x}\right)^2+1}\frac{1}{x} = \frac{y-x}{x^2+y^2},$$

$$y' = \frac{x+y}{x-y}.$$

$$y'' = \frac{\partial y'}{\partial x} + \frac{\partial y'}{\partial y}\frac{dy}{dx} = \frac{(x-y)-(x+y)}{(x-y)^2} + \frac{(x-y)+(x+y)}{(x-y)^2}\cdot\frac{x+y}{x-y} =$$

$$= \frac{2(x^2+y^2)}{(x-y)^3}.$$

3389. Вычислить $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial x \partial y}$, $\frac{\partial^2 z}{\partial y^2}$ в точке $(1, -2, 1)$, если $x^2 + 2y^2 + 3z^2 + xy - z - 9 = 0$

$$F'_x = 2x + y, \quad F'_y = 4y + x, \quad F'_z = 6z - 1.$$

$$\frac{\partial z}{\partial x} = -\frac{2x + y}{6z - 1}, \quad \frac{\partial z}{\partial y} = -\frac{x + 4y}{6z - 1}.$$

$$\frac{\partial^2 z}{\partial x^2} = -\frac{2}{6z - 1} + \frac{6(2x + y)}{(6z - 1)^2} \cdot \left(-\frac{2x + y}{6z - 1}\right),$$

$$\frac{\partial^2 z}{\partial y^2} = -\frac{4}{6z - 1} + \frac{6(x + 4y)}{(6z - 1)^2} \cdot \left(-\frac{x + 4y}{6z - 1}\right),$$

$$\frac{\partial^2 z}{\partial x \partial y} = -\frac{1}{6z - 1} + \frac{6(2x + y)}{(6z - 1)^2} \cdot \left(-\frac{x + 4y}{6z - 1}\right).$$

$$\left.\frac{\partial^2 z}{\partial x^2}\right|_{(1,-2,1)} = -\frac{2}{5}, \quad \left.\frac{\partial^2 z}{\partial y^2}\right|_{(1,-2,1)} = -\frac{394}{125}, \quad \left.\frac{\partial^2 z}{\partial x \partial y}\right|_{(1,-2,1)} = -\frac{1}{5}.$$

3410. Найти dz и d^2z при $u = v = 0$, если $x = e^{u+v}$, $y = e^{u-v}$, $z = uv$

$$dz = udv + vdu, \quad dx = e^{u+v}dv + e^{u+v}du, \quad dy = -e^{u-v}dv + e^{u-v}du.$$

$$\begin{cases} e^{-(u+v)}dx = du + dv, \\ e^{v-u}dy = du - dv. \end{cases} \quad \begin{cases} du = \frac{1}{2}\left(e^{-(u+v)}dx + e^{v-u}dy\right), \\ dv = \frac{1}{2}\left(e^{-(u+v)}dx - e^{v-u}dy\right), \end{cases}$$

$$dz = \frac{1}{2}(u+v)e^{-(u+v)}dx + \frac{1}{2}(v-u)e^{v-u}dy,$$

$$d^2z = d(dz) = dudv + dvdu = 2dudv = \frac{2}{4}e^{-2(u+v)}dx^2 - \frac{2}{4}e^{2(v-u)}dy^2.$$

При $u = v = 0$

$$dz = 0, \quad d^2z = \frac{1}{2}dx^2 - \frac{1}{2}dy^2.$$

3585. Разложить функцию $z = x^y$ в окрестности точки $(1,1)$ по формуле Тейлора до членов второго порядка включительно

$$\frac{\partial z}{\partial x} = y \cdot x^{y-1}, \quad \left.\frac{\partial z}{\partial x}\right|_{(1,1)} = 1,$$

$$\frac{\partial z}{\partial y} = \ln(x) \cdot x^{y}, \quad \left.\frac{\partial z}{\partial y}\right|_{(1,1)} = 0,$$

$$\frac{\partial^2 z}{\partial x^2} = y(y-1)x^{y-2}, \quad \left.\frac{\partial^2 z}{\partial x^2}\right|_{(1,1)} = 0,$$

$$\frac{\partial^2 z}{\partial y^2} = \ln^2(x)x^{y}, \quad \left.\frac{\partial^2 z}{\partial y^2}\right|_{(1,1)} = 0,$$

$$\frac{\partial^2 z}{\partial x \partial y} = x^{y-1} + y\ln(x) \cdot x^{y-1}, \quad \left.\frac{\partial^2 z}{\partial x \partial y}\right|_{(1,1)} = 1,$$

$$z(x,y) = z(1,1) + \left.\frac{\partial z}{\partial x}\right|_{(1,1)}(x-1) + \left.\frac{\partial z}{\partial y}\right|_{(1,1)}(y-1) +$$

$$+\frac{1}{2!}\left(\left.\frac{\partial^2 z}{\partial x^2}\right|_{(1,1)}(x-1)^2 + \left.\frac{\partial^2 z}{\partial y^2}\right|_{(1,1)}(y-1)^2 + 2\left.\frac{\partial^2 z}{\partial x \partial y}\right|_{(1,1)}(x-1)(y-1)\right) + \ldots =$$

$$= 1 + (x-1) + (x-1)(y-1) + o(x^2+y^2).$$

3625. Исследовать на экстремум функцию $z = x^2y^3(6-x-y)$

$$\begin{cases} \dfrac{\partial z}{\partial x} = 2xy^3(6-x-y) - x^2y^3 = 0, \\ \dfrac{\partial z}{\partial y} = 3x^2y^2(6-x-y) - x^2y^3 = 0, \end{cases} \quad \begin{array}{l} (2xy^3 - 3x^2y^2)(6-x-y) = 0, \\ xy^2(2y-3x)(6-x-y) = 0, \end{array}$$

$$I\colon 6-x-y=0, \quad x^2y^3 = 0, \quad \begin{array}{l} x=0,\ y=6; \\ x=6,\ y=0. \end{array}$$

$$II\colon y = \frac{3}{2}x, \quad \frac{27}{4}x^4\left(6-\frac{5}{2}x\right) - \frac{27}{8}x^5 = 0, \quad x^4(2-x) = 0, \quad \begin{array}{l} x=0,\ y=0; \\ x=2,\ y=3. \end{array}$$

$$III\colon x=0,\ y\in\mathbb{R}; \quad y=0,\ x\in\mathbb{R}.$$

$$d^2z = (2y^3(6-x-y) - 2xy^3 - 2xy^3)dx^2 +$$
$$+2(6xy^2(6-x-y) - 2xy^3 - 3x^2y^2)dxdy +$$
$$+(6x^2y(6-x-y) - 3x^2y^2 - 2x^2y^2)dy^2 =$$
$$= 2y^3(6-3x-y)dx^2 + 2xy^2(36-9x-8y)dxdy + 6x^2y(6-x-2y).$$

$$x=0,\ d^2z = 2y^3(6-y)dx^2,\ y\in(0;6) \Rightarrow \min,\ y\in(-\infty;0)\cup(6;+\infty) \Rightarrow \max;$$

$$y=0, \quad d^2z = 0;$$

$$x=2,\ y=3, \quad d^2z = -162dx^2 - 216dxdy - 144dy^2 < 0 \ \Rightarrow\ \max.$$

$$-162 < 0, \quad (-162)\cdot(-144) - (-108)^2 > 0, \quad \text{(Критерий Сильвестра)}$$

3642. Исследовать на экстремум функцию $u = x^2 + y^2 + z^2 + 2x + 4y - 6z$

$$\begin{cases} \dfrac{\partial u}{\partial x} = 2x + 2 = 0, \\ \dfrac{\partial u}{\partial y} = 2y + 4 = 0, \\ \dfrac{\partial u}{\partial z} = 2z - 6 = 0, \end{cases} \quad \begin{pmatrix} x^* \\ y^* \\ z^* \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \\ 6 \end{pmatrix}$$

Построим матрицу Гессе

$$H = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} > 0,$$

$$\Delta_1 = |2| > 0, \quad \Delta_2 = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} > 0, \quad \Delta_3 = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} > 0.$$

Таким образом (x^*, y^*, z^*) – точка минимума.

3671.1. Вычислить экстремум функции $z = x^2 + 12xy + 2y^2$, если $4x^2 + y^2 = 25$

$$L(x, y, \lambda) = x^2 + 12xy + 2y^2 + \lambda(4x^2 + y^2 - 25),$$

$$\begin{cases} \dfrac{\partial L}{\partial x} = 2x + 12y + 8\lambda x = 0, \\ \dfrac{\partial L}{\partial y} = 12x + 4y + 2\lambda y = 0, \\ 4x^2 + y^2 = 25, \end{cases} \quad \begin{array}{l} x = \dfrac{-6y}{4\lambda + 1}, \\ x = \dfrac{(\lambda+2)y}{-6}, \\ 4x^2 + y^2 = 25, \end{array} \quad \begin{array}{l} \dfrac{-6}{4\lambda+1} = \dfrac{(\lambda+2)}{-6}, \\ 4\lambda^2 + 9\lambda + 2 = 34, \\ 4\lambda^2 + 9\lambda - 36 = 0, \\ D = 81 + 16 \cdot 34 = 625, \end{array}$$

$$\lambda = \frac{9 \pm 25}{8} = 2; -\frac{17}{4}.$$

$$\min\colon \lambda = 2, \quad x = -\frac{2}{3}y, \quad \frac{16}{9}y^2 + y^2 = 25, \quad y = \pm 3, \quad x = \mp 2.$$

$$\max\colon \lambda = -\frac{17}{4}, \quad x = -\frac{3}{8}y, \quad \frac{36}{64}y^2 + y^2 = 25, \quad y = \pm 4, \quad x = \mp\frac{3}{2}.$$

$$d^2L = (2 + 8\lambda)dx^2 + 24dxdy + (4 + 2\lambda)dy^2,$$

$$8xdx + 2ydy = 0, \implies dx = -\frac{y}{4x}dy = \frac{-3}{2(\lambda+2)}dy,$$

$$d^2L = (2 + 8\lambda)\frac{9}{4(\lambda+2)^2}dy^2 - \frac{72}{2(\lambda+2)}dy^2 + 2(\lambda+2)dy^2.$$

Домашнее задание

Теорема о неявной функции: 3372, 3380, 3387, 3389, 3397, 3404-3407.2, 3410, 3415, 3581, 3585.
Экстремум: 3621-3628, 3642, 3651, 3654, 3657.1, 3659, 3663.