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108 группа
Вариант – 23

① $\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$ – кососимметрическая матрица

$$\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} = a\underbrace{\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{e_1} + b\underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}}_{e_2} + c\underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}}_{e_3} \Rightarrow \text{размерность } 3.$$

Ответ: $e_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $e_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$ $e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$

$\dim V = 3.$

② $A = Lin(x^2+x;\ x+1)$ $B = Lin(2x^2+1;\ x^2+x+1)$ $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}^T$ $B = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}^T$

$$(A\,|\,B) = \left(\begin{array}{cc|cc} 1 & 0 & 2 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & 2 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & 2 & 1 \\ 0 & 0 & -3 & -1 \\ 0 & 1 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -3 & -1 \end{array}\right)$$

$x = \alpha_1 \cdot (x^2+x) + \alpha_2(x+1) = \beta_1(2x^2+1) + \beta_2(x^2+x+1)$

$\alpha_1 = 2\beta_1 + \beta_2$
$\alpha_1 + \alpha_2 = \beta_2$
$\alpha_2 = \beta_1 + \beta_2$

a_1, a_2 – базисные

$\alpha_1 = 2\beta_1 + \beta_2$
$\alpha_2 = \beta_1 + \beta_2$
$-3\beta_1 - \beta_2 = 0.$
$\beta_2 = -3\beta_1$
$\alpha_1 = -\beta_1$ $\alpha_2 = -2\beta_1$
$\beta_1 = 0 \Rightarrow \varphi = \cancel{(0\ 0\,|\,0\ 0)}$
$\beta_1 = 1 \Rightarrow \varphi = (-1\ -2\,|\,1\ -3)^T$
$x_1 = -1(x^2+x) - 2(x+1) =$
$= 1(2x^2+1) - 3(x^2+x+1)$
$A \cap B = Lin(x_1)$
$\dim(A \cap B) = 1.$

Ответ:

$A \cap B = Lin(-x^2 - 3x - 2)$

③ Угол между вектором $x \in (1\ 0\ 1\ 1)^T \in \mathbb{R}$ и лин. оболоч. векторов $a = (1\ 0\ 0\ 1)^T$ и $b = (0\ 1\ 1\ 1)^T$.

$$|h| = \sqrt{\frac{\det G(a,b,x)}{\det G(a,b)}}$$

$(a,a) = 1\cdot1 + 0\cdot0 + 0\cdot0 + 1\cdot1 = 2$
$(b,b) = 0\cdot0 + 1\cdot1 + 1\cdot1 + 1\cdot1 = 3$
$(b,x) = 0\cdot1 + 1\cdot0 + 1\cdot1 + 1\cdot1 = 2$
$(a,b) = 1\cdot0 + 0\cdot1 + 0\cdot1 + 1\cdot1 = 1.$
$(a,x) = 1\cdot1 + 0\cdot0 + 0\cdot1 + 1\cdot1 = 2$
$(x,x) = 1\cdot1 + 0\cdot0 + 1\cdot1 + 1\cdot1 = 3.$

$$\det G(a,b,x) = \begin{vmatrix} (a,a) & (a,b) & (a,x) \\ (a,b) & (b,b) & (b,x) \\ (a,x) & (b,x) & (x,x) \end{vmatrix} = \begin{vmatrix} 2 & 1 & 2 \\ 1 & 3 & 2 \\ 2 & 2 & 3 \end{vmatrix} = 3\cdot3\cdot2 + 2\cdot2\cdot1 + 1\cdot2\cdot2 -$$

$$- 2\cdot3\cdot2 - 2\cdot2\cdot2 - 1\cdot1\cdot3 =$$

$$= 18 + 4 + 4 - 12 - 8 - 3 =$$

$$= 26 - 15 - 8 = 9 - 8 = 3.$$

$$\det G(a,b) = \begin{vmatrix} (a,a) & (a,b) \\ (a,b) & (b,b) \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 6 - 1 = 5$$

$$|h| = \sqrt{\frac{3}{5}} = \frac{\sqrt{3}}{\sqrt{5}} \Rightarrow \sin\varphi = \frac{|h|}{|x|} = \frac{\frac{\sqrt{3}}{\sqrt{5}}}{\sqrt{3}} = \frac{\sqrt{3}}{\sqrt{3}\sqrt{5}} \Rightarrow$$

Ответ: $\boxed{\varphi = \arcsin\left(\frac{1}{\sqrt{5}}\right)}$

4) $A(x\bar{i} + y\bar{j} + z\bar{k}) = (x+y+z)\bar{i} + (x+y+z)\bar{j} + (2x+2y+2z)\bar{k}$

$$B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \Rightarrow \begin{cases} x+y+z=0 \\ x+y+z=0 \\ 2x+2y+2z=0 \end{cases}$$

$\operatorname{Ker} B$ – пустое ядро
$d = 0$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \operatorname{rg} B = \dim(\operatorname{Im} B) = 1$$

$\operatorname{Im} B = (x\bar{i}, y\bar{j}, z\bar{k})$

Нет нулевого столбца $\Rightarrow$ инъективное
Не сюръективное $\Rightarrow$ не обратимое $\det B = 0$.

Ответ:

а) $B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix}$

в) инъективное
не сюръективное
не обратимое

б) $\operatorname{Ker} A = \{0\}$, $d = 0$, $r = 1$, $\operatorname{Im} B = (x\bar{i}, y\bar{j}, z\bar{k})$

5. $q(x) = 5x_1^2 + x_2^2 + 2x_3^2 + 4x_1x_3$

$$A = \begin{pmatrix} 5 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 2 \end{pmatrix} \qquad \begin{vmatrix} 5-\lambda & 0 & 2 \\ 0 & 1-\lambda & 0 \\ 2 & 0 & 2-\lambda \end{vmatrix} = 0$$

$-\lambda^3 + 8\lambda^2 - 13\lambda + 6 =$

$= -(\lambda-1)(\lambda^2-7\lambda-6) = -(\lambda-1)(\lambda-1)(\lambda-6)$

$\lambda_1 = 1 \quad \lambda_2 = 6.$

$n_1 = 2 \quad n_2 = 1$

$$A - \lambda_1 E = \begin{pmatrix} 5-1 & 0 & 2 \\ 0 & 1-1 & 0 \\ 2 & 0 & 2-1 \end{pmatrix} \sim \begin{pmatrix} 4 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$2x_1 + x_3 = 0$

$x_3 = -2x_1$

$x_1 = (1 \ 0 \ -2)^T \quad x_2 = (0 \ 1 \ 0)^T$

$|x_1| = \sqrt{1+4} = \sqrt{5} \quad |x_2| = \sqrt{1} = 1$

$s_1 = \left(\frac{1}{\sqrt{5}} \ 0 \ \frac{-2}{\sqrt{5}}\right)^T.$

$s_2 = (0 \ 1 \ 0)^T$

$$A - \lambda_2 E = \begin{pmatrix} 5-6 & 0 & 2 \\ 0 & 1-6 & 0 \\ 2 & 0 & 2-6 \end{pmatrix} \sim \begin{pmatrix} -1 & 0 & 2 \\ 0 & -5 & 0 \\ 2 & 0 & -4 \end{pmatrix} \sim \begin{pmatrix} -1 & 0 & 2 \\ 0 & -5 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$-x_1 + 2x_3 = 0 \quad x_1 = 2x_3$

$-5x_2 = 0$

$x_3 = (2 \ 0 \ 1)^T.$

$|x_3| = \sqrt{4+1} = \sqrt{5}$

$s_3 = \left(\frac{2}{\sqrt{5}} \ 0 \ \frac{1}{\sqrt{5}}\right)^T$

$(S) = (S_1 \ S_2 \ S_3)$

$$S = \begin{pmatrix} \frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ 0 & 1 & 0 \\ -\frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \end{pmatrix}$$

$$A_{(S)} = S^TAS = \begin{pmatrix} \frac{1}{\sqrt{5}} & 0 & -\frac{2}{\sqrt{5}} \\ 0 & 1 & 0 \\ \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} 5 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ 0 & 1 & 0 \\ -\frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{\sqrt{5}}{5} & 0 & \frac{-2\sqrt{5}}{5} \\ 0 & 1 & 0 \\ \frac{12\sqrt{5}}{5} & 0 & \frac{6\sqrt{5}}{5} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ 0 & 1 & 0 \\ -\frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{pmatrix} = diag(1,1,6)$$

$$\boxed{\tilde{q}(Sy) = y_1^2 + y_2^2 + 6y_3^2}$$

$\bar{s}_1 = \frac{1}{\sqrt{5}}\bar{i} + \left(-\frac{2}{\sqrt{5}}\bar{k}\right) \quad \bar{s}_2 = \bar{j} \quad \bar{s}_3 = \frac{2}{\sqrt{5}}\bar{i} + \frac{1}{\sqrt{5}}\bar{k}$

⑥ $p(\lambda) = (\lambda-2)^2(\lambda-3)^2$ $\lambda_1 = 2$ $\lambda_2 = 3$

$n_1 = 2$ $n_2 = 2$

$$\left(\begin{array}{cc|cc} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ \hline 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{array}\right)$$

$\cancel{\lambda_1 = 2 \ n_1 = 2}$: