Linear Programming Fundamentals

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Outline

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Linear Programming (LP): A Historic and Vital Tool in OR

- 1950: LP Discovery - A Landmark Moment in OR for Linear Programming.
- Linear Programming's Impact: Saving Money in Diverse Industries, such as education, forestry, petroleum, …

- In a survey of Fortune 500 firms, 85% of the respondents said they had used LP

Prototype example

You run a company that produces two products: **chairs and tables.**

- You have 500 hours of labor, 400 m^2 of wood, and 300 m^2 of fabric available.
- Each chair requires 3 hours of labor, 4 m^2 of wood, and 2 sm^2 of fabric.
- Each table requires 4 hours of labor, 6 m^2 of wood, and 3 m^2 of fabric.
- You sell each chair for DZD 100 and each table for DZD 150.

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- Each table requires 4 hours of labor, 6 m^2 of wood, and 3 m^2 of fabric.
- You sell each chair for DZD 100 and each table for DZD 150.

How many chairs and tables should you produce in order to maximize your profit?

Prototype example

- **Decision variables ?**
- **Objective function ?**
- **Constraints ?**

Prototype example

- **Decision variables:**
  - Let x_1 be the number of chairs to produce.
  - Let x_2 be the number of tables to produce.
- **Objective function:**
  - Maximize $100x_1 + 150x_2$
- **Constraints:**
  - $3x1 + 4x2 \leq 500$ (labor constraint)
  - $4x_1 + 6x_2 \leq 400$ (wood constraint)
  - $2x1 + 3x2 \leq 300$ (fabric constraint)
  - $x1, x2 \geq 0$ (non-negativity constraint)

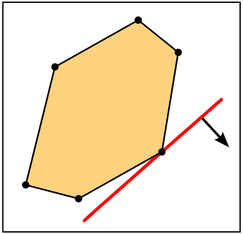
Prototype example

$$
\begin{aligned}
\textbf{Maximize} \quad & 100x_1 + 150x_2 \\
\textbf{Subject to} \quad & 3x_1 + 4x_2 \leq 500 \\
& 4x_1 + 6x_2 \leq 400 \\
& 2x_1 + 3x_2 \leq 300 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

What is the solution approach for this problem on a two-dimensional plane?

What is Linear Programming (LP) ?

An optimization model is a Linear Program (or LP) if it has ***continuous variables****, a* ***single linear objective function****, and* ***all constraints are linear equalities or inequalities.***



What is Linear Programming (LP) ?

- **Decisions variables** : that we seek to determine
  - $x_1, x_2, ..., x_n$
- **Linear objetive function:** (goal) that we need to optimize (maximize or minimize)
  - $z = c_1x_1 + c_2x_2 + ... + c_nx_n$
  - $z = \sum_{i=1}^{n} c_ix_i$
- **Linear constraints:** that the solution must satisfy
  - $a_{1j}x_1 + a_{2j}x_2 + ... + a_{nj}x_n \leq b_j$ for $j = 1...m$

What is Linear Programming (LP)?

$$
\begin{aligned}
\textbf{Maximize} \quad & z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\
\textbf{Subject to} \quad & a_{11}x_1 + a_{21}x_2 + \cdots + a_{n1}x_n \leq b_1, \\
& a_{12}x_1 + a_{22}x_2 + \cdots + a_{n2}x_n \leq b_2, \\
& \quad \vdots \\
& a_{1m}x_1 + a_{2m}x_2 + \cdots + a_{nm}x_n \leq b_m, \\
& x_i \geq 0 \quad (1 \leq i \leq n)
\end{aligned}
$$

- Here, x_i represents the decision variables, c_i, a_{ij} and and b_j are fixed constants extracted from the studied problem. $x_i \geq 0$ represents non-negativity constraint.

What is Linear Programming (LP)?

- Any LP problem can be converted to the previous form 11.
- An equality ($=$) can be represented by two inequalities ($\leq$ and $\geq$)
- An inequality $\geq$ can be converted to $\leq$ by multiplying by -1.
- A negative variable x_i can be replaced with $x_i = -x_i^{'}$ and $x_i^{'} \geq 0$.
- An unrestricted variables (i.e., it can take both positive and negative values) can be replaced by two variables $x_i = x_i^{+} - x_i^{-}$ and $x_i^{+}, x_i^{-} \geq 0$.

LP Applications: Resource Allocation

- LP commonly used for allocating resources to activities with limited resources
- The objective is to **choose levels of activities** that achieve the **best overall measure of performance**.
- E.g. Optimizing the scheduling of personnel and equipment in a hospital or allocating budgets across different departments in a company
- Organizations can make the most efficient use of their resources and achieve better performance outcomes using LP.

Resource	Resource Usage per Unit of Activity: Activity 1	2	. . .	n	Amount of Resource Available
1	a_{11}	a_{12}	. . .	a_{1n}	b_1
2	a_{21}	a_{22}	. . .	a_{2n}	b_2
.					.
.	. . .	. . .	. . .	. . .	.
.					.
m	a_{m1}	a_{m2}	. . .	a_{mn}	b_m
Contribution to Z per unit of activity	c_1	c_2	. . .	c_n	

Example: Resource Allocation

- **Context:** manufacturing company has two factories and three product lines.
- **The company's objective:** maximize the profit.
- **Constraint 1:** each factory has a limited capacity for producing each product line.
- **Constraint 2:** limited demand for each product line.

Example: Resource Allocation

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Maximize $15x_1 + 20x_2 + 10x_3$

Subject to:

$2x_1 + 3x_2 + x_3 \leq 30$ (Factory 1 capacity constraint)

$x_1 + 2x_2 + 4x_3 \leq 40$ (Factory 2 capacity constraint)

$x_1 + x_2 + x_3 \leq 20$ (Product 1 demand constraint)

$2x_2 + 3x_3 \leq 25$ (Product 2 demand constraint)

$x_1 + 2x_3 \leq 15$ (Product 3 demand constraint)

$x_1, x_2, x_3 \geq 0$ (Non-negativity constraint)

LP Applications

1. Production planning:
   - Determine optimal mix of products to produce
   - Determine quantity of each product to manufacture
   - Utilize resources such as labor, materials, and machines efficiently
2. Transportation and distribution:
   - Minimize transportation costs
   - Satisfy demand and supply constraints
3. Portfolio optimization:
   - Select best mix of assets
   - Maximize returns while minimizing risks

LP Applications

4. Marketing optimization:
   - Optimize advertising campaigns and promotions
   - Determine best media channels, advertising frequency, and promotion offers
   - Maximize sales and profits
5. Energy and environmental management:
   - Optimize use of energy resources
   - Reduce environmental impacts
   - Determine optimal mix of energy sources and production technologies
6. Other applications ...

LP assumptions

- LP assumptions are implicit in the mathematical formulation.
- In order to use LP from a modeling perspective, it is necessary for the assumptions about the data to be fulfilled..

Understanding LP assumptions can help evaluate how well linear programming applies to a problem.

LP assumptions: Proportionality

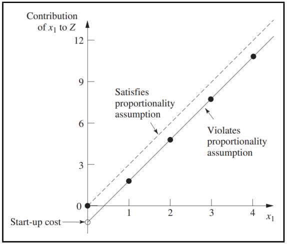
- The contribution of each variable to the objective function z and functional constraints is proportional to x_j.
- This is represented by the terms $c_j x_j$ in the objective function and $a_i j x_j$ in the functional constraint.

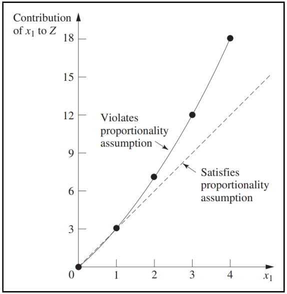
In a LP model, only variables with an exponent of 1 are allowed in the terms of any function, including the objective function and constraints.

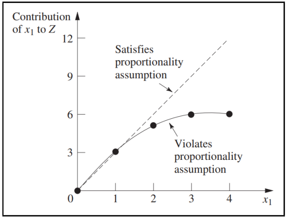
LP assumptions: Proportionality

x_1	Proportionality Satisfied	Proportionality Violated		
		Case 1	Case 2	Case 3
0	0	0	0	0
1	3	2	3	3
2	6	5	7	5
3	9	8	12	6
4	12	11	18	6

LP assumptions: Proportionality







LP assumptions: Additivity

- Proportionality assumption in LP model restricts exponents to 1.
- It does not forbid cross-product terms like $x_i x_j$.
- Additivity assumption in LP model rules out cross-product terms.

Every function in the model can be represented as the sum of the separate contributions of the individual activities.

LP assumptions: Additivity

	Value of the objective function z		
		Additivity Violated	
(x_1, x_2)	Additivity Satisfied	Case 1	Case 2
(1,0)	3	3	3
(0,1)	5	5	5
(1,1)	8	9	7

LP assumptions: Divisibility

- The divisibility assumption allows decision variables to have values that are not limited to integers.
- Integer Programming can be used if the propblem requires integer decision variables.

LP allows non-integer decision variable values, implying the possibility to run at fractional levels, which can result in more efficient solutions in some scenarios.

LP assumptions: Certainty

- The certainty assumption in LP models assumes that all the LP coefficients are **known fixed constants**.
- In practical applications, satisfying the certainty assumption in can be challenging as parameter values are often **estimated or predicted values**.
- Therefore, **sensitivity analysis** is crucial to identify sensitive parameters that may affect the optimal solution.

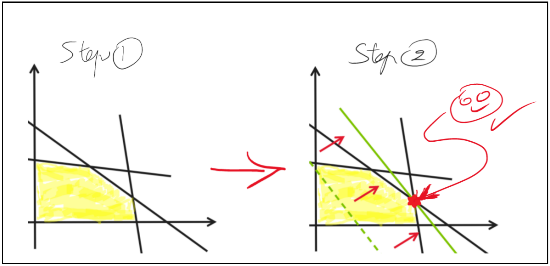
LP assumptions: Conclusion

- Mathematical models are just approximations and simplifying assumptions.
- Reasonably high correlation between the model's prediction and the actual problem is sufficient for useful analysis.
- If any of the assumptions are violated significantly, alternative models are available.
- LP outperforms other complex methods in terms of its effectiveness and efficiency.

Graphical solution of LP

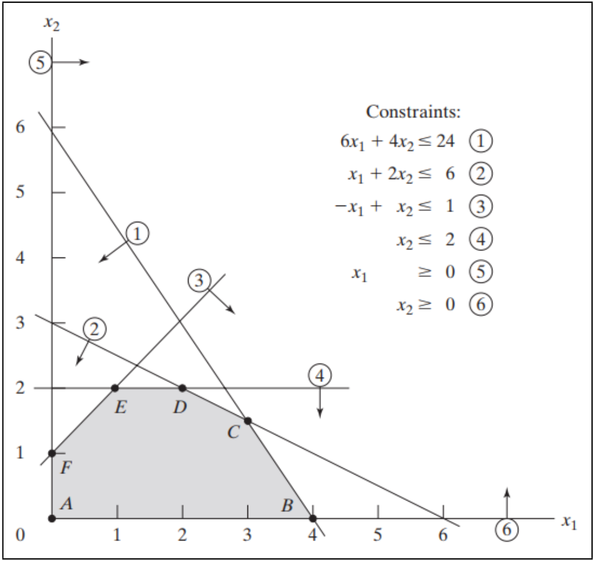
The graphical solution includes two steps:

1. Determination of the feasible solution space (feasible region).
2. Determination of the optimum solution among all points of the feasible region.



Determination of the feasible region

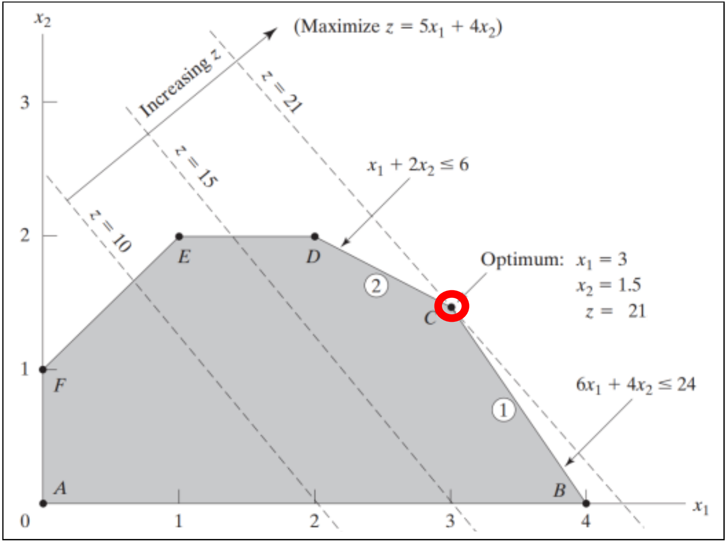
1. **Feasible region:** set of possible solutions that meet the problem constraints.
2. **Graph inequalities:** represent constraints on a coordinate plane.
3. **Shade region:** satisfy all constraints to determine feasible solution space.



Constraints:
$6x_1 + 4x_2 \le 24$ (1)
$x_1 + 2x_2 \le 6$ (2)
$-x_1 + x_2 \le 1$ (3)
$x_2 \le 2$ (4)
$x_1 \ge 0$ (5)
$x_2 \ge 0$ (6)
x_1
x_2
A
B
C
D
E
F
0
1
2
3
4
5
6

Determination of the optimum solution

1. **Optimum solution:** determine best solution from feasible solution space.
2. **Graph the objective function:** Fix z and draw the line called **isoprofit/ iso-cost**
3. **Shift the line:** shift the iso-profit line in a parallel manner from its original position in a direction that increases of z.


(Maximize $z = 5x_1 + 4x_2$)
Increasing z
$z = 21$
$z = 15$
$z = 10$
$x_1 + 2x_2 \leq 6$
Optimum: $x_1 = 3$
$x_2 = 1.5$
$z = 21$
$6x_1 + 4x_2 \leq 24$
x_2
x_1
A
B
C
D
E
F
1
2
0
1
2
3
4
1
2
3

Why study the graphical solution of LP ?

- In practice, LPs can have hundreds or even thousands of variables and constraints.
- Two-variable LP may seem limited, but the graphical solution offers a key insight.
- The optimal solution of an LP is always at a **corner point** of the feasible region.

Key result

The graphical solution of a two-variable LP problem provides a fundamental insight into LP optimization, which is that **the optimal solution is always associated with a corner point of the feasible region.**

Why Corner Points help to solve LP ?

- This result limits the search for the optimal solution from an **infinite number of feasible points** to a **finite number of corner points.**
- Simplex method is based on this powerful result to solve complex LP problems.
- In two dimension case, the upper bound of the number of corner points is

Number of corner points $\leq \binom{n}{2} = \frac{n(n-1)}{2}$: where *n* is the number of constraints.

Key result

Intelligently navigating the corner points of a LP problem can simplify solving very large LP problems using optimization algorithms.

Convex Sets, Extreme Points, and LP

Convex set

A set of points S is a convex set if the line segment joining any pair of points in S is wholly contained in S.

S is convex $\Rightarrow \forall A, B \in S, \beta \in [0,1], \beta A + (1-\beta)B \in S$

Extreme Point (Corner point)

For any convex set S, a point P in S is an extreme point if each line segment that lies completely in S and contains the point P has P as an endpoint of the line segment.

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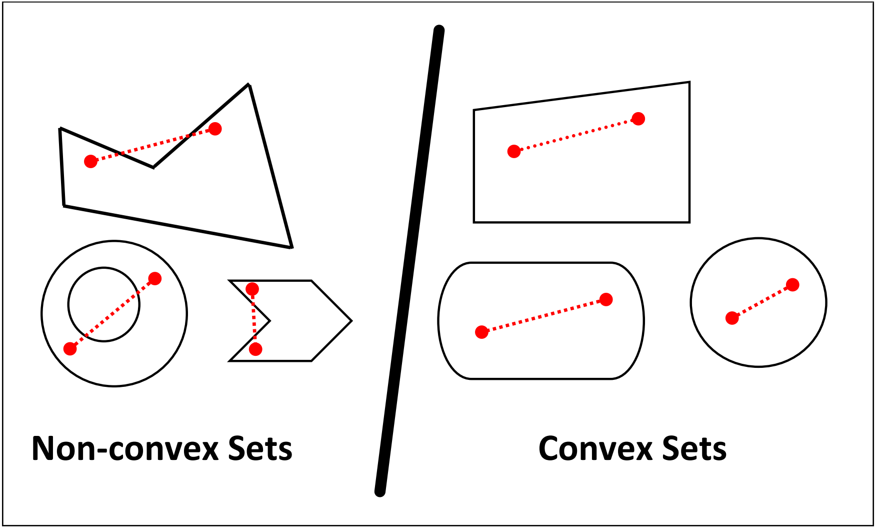

Non-convex Sets
Convex Sets

Convex Sets, Extreme Points, and LP

- Feasible region **cannot comprise several disconnected feasible regions.**
- Convexity guarantees **the absence of local optima.**
- **No local optimum** that could trap the solver.

Key result

Feasible region convexity is a critical property and it assures that if the LP problem has an optimal solution, then **at least one corner point of the feasible region is a global optimum**.

LP Special Cases

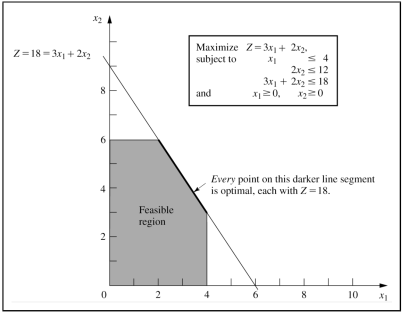
1. **Alternative or multiple optimal solutions:** LPs have an infinite number of optimal solutions.

2. **Infeasible LPs:** LPs have no feasible solutions.

3. **Unbounded LPs:** There are points in the feasible region with arbitrarily large (in a max problem) z-values.

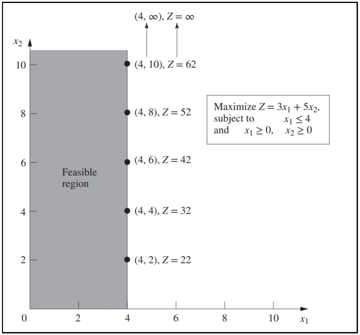
Alternative or multiple optimal solutions

- The isoprofit intersects an **entire line segment** corresponding to the constraint.
- The decision maker can use a **secondary criterion** to choose between optimal solutions.



Unbounded LPs

- Objective function can **be arbitrarily large.**
- Unbounded optimal solution **should not occur in a correctly formulated LP.**
- **Modify the problem formulation** to remove the unboundedness to make the problem solvable.



Infeasible LPs

- **Feasible region to be empty** (contain no points), resulting in an infeasible LP.
- Because the optimal solution to an LP is the best point in the feasible region, an **infeasible LP has no optimal solution.**
- **Over-constraining the problem**, which makes the constraints are too tight to allow any feasible solution.
- **Introduction of incompatible constraints**, such as when one constraint requires a variable to be positive while another requires it to be negative.
- **Modify the problem formulation** to remove the contradictory constraint or constraints and make the problem feasible.

Summary

- LP problem has three parts:
  1. **Decision variables:** variable that we can control and aim to optimize by finding the best values.
  2. **Objective function:** a linear function of decision variables that we want to maximize or minimize.
  3. **Constraints:** linear inequalities or equalities that restrict the values of decision variables.
- Optimal solution to the LP is a point in the feasible region that maximizes / minimizes the objective function.
- Verifying LP assumptions with data assesses the feasibility of using LP.

Summary

- The feasible region for any LP is a convex set, and an optimal solution to an LP is an extreme point of the feasible region.

- To solve a max LP with two decision variables graphically, follow these steps:
  1. **Step 1:** Graph the feasible region.
  2. **Step 2:** Draw an isoprofit line.
  3. **Step 3:** Move parallel to the isoprofit line in the direction of increasing z. The last point in the feasible region that contacts an isoprofit line is an optimal solution to the LP.

- The optimal solution to a max LP with two decision variables is the extreme point of the feasible region that lies on the highest isoprofit line.

Summary

Four cases that can occur when solving an LP:

1. **Case 1:** The LP has a unique solution.
   - Only one point in the feasible region that maximizes or minimizes the objective function.
2. **Case 2:** The LP has more than one optimal solution.
   - Only one point in the feasible region that maximizes or minimizes the objective function.
   - The feasible region has multiple points that optimize the objective function.
   - Graphically, the isoprofit line last hits an entire line segment before leaving the feasible region.

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Summary

5. **Case 3:** The LP is infeasible.
   - No points in the feasible region.
   - The LP has no feasible solution.
6. **Case 4:** The LP is unbounded.
   - There are points in the feasible region with arbitrarily large objective function values.
   - Graphically, when we move parallel to an isoprofit line in the direction of increasing z, we never lose contact with the LP's feasible region.

References

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