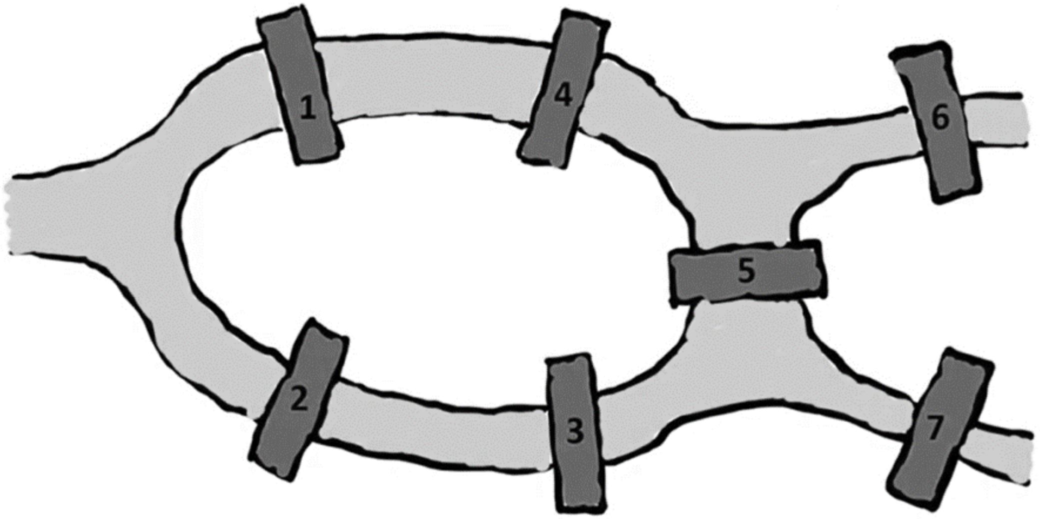
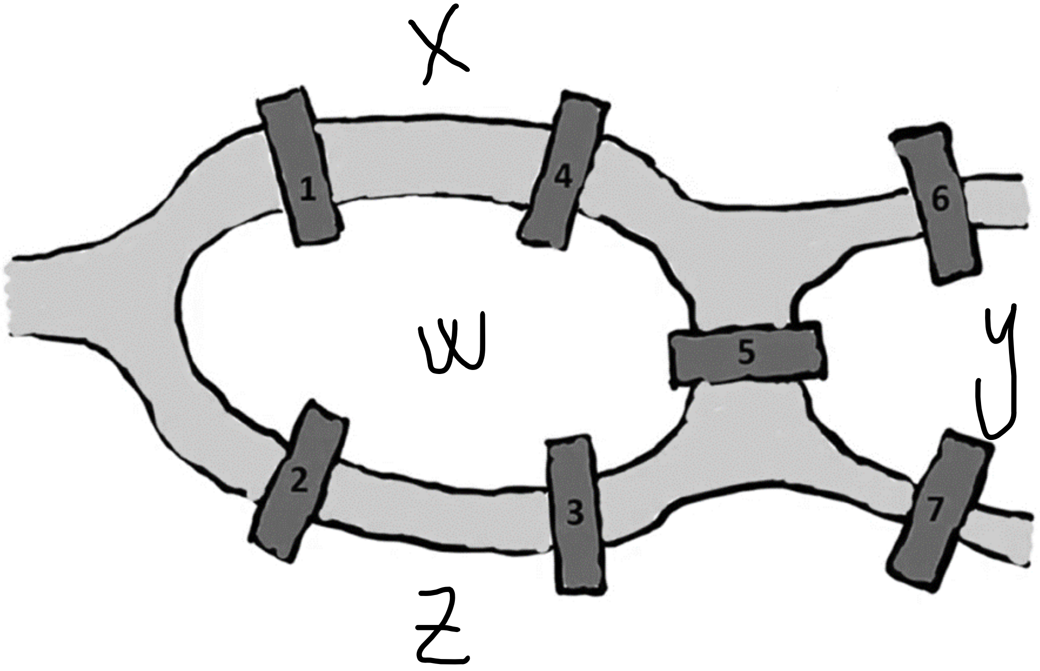
Introduction to graph theory

Mohammed Brahimi

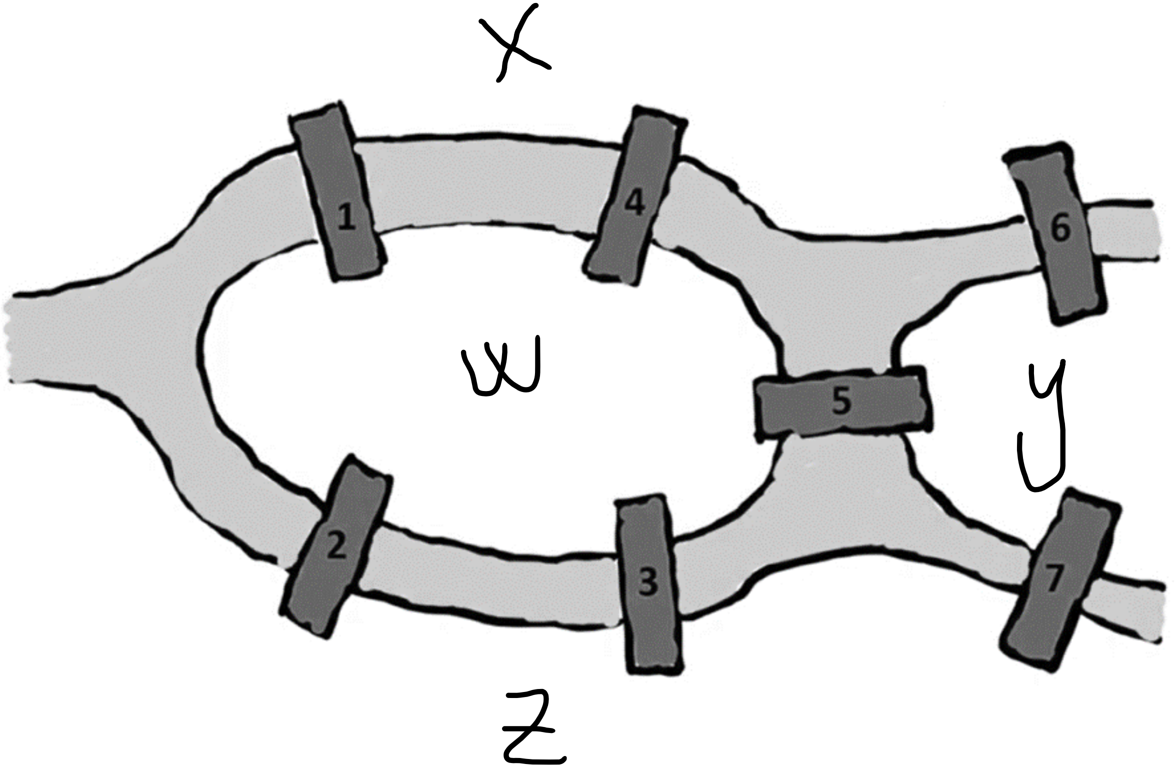


Konigsberg Bridge Problem



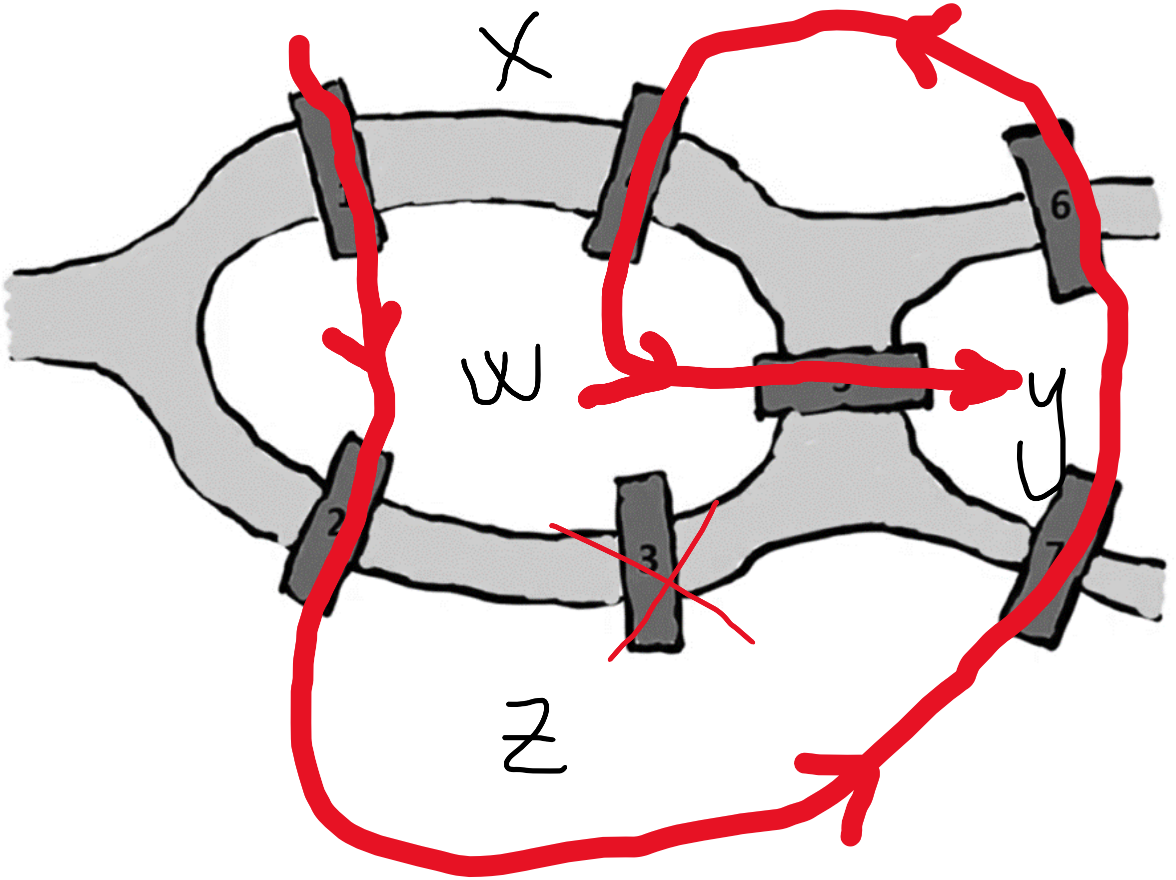
- Finding a **path crosses each of the seven bridges** in the city **exactly once** and **return to the starting point**
- Mathematicians had been discussing the problem for years, but no solution was found

Konigsberg Bridge Problem



Find a path that **crosses each of the seven bridges** in the city of Konigsberg **exactly once**

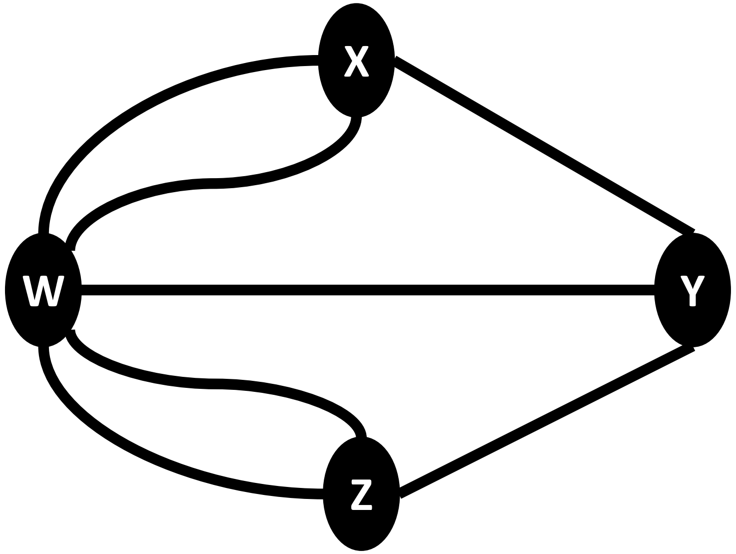
Konigsberg Bridge Problem



Bridge 3 not crossed

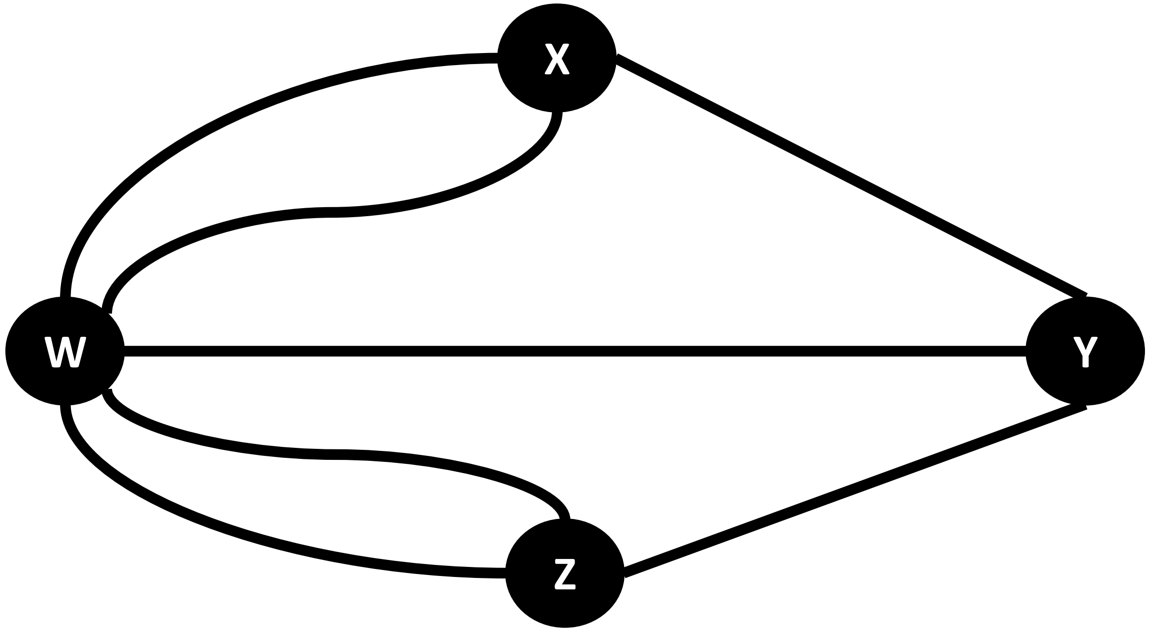
Euler Solution

- In 1741, Euler, a young mathematician, analysed the challenge
- Reduced it to a network of points (land masses) connected by lines (bridges)



To cross all bridges once, each land mass needs an even number of bridges

Konigsberg Bridge Problem



This necessary condition did not hold in Konigsberg, as some land masses were connected to an odd number of bridges.

The Birth of Graph Theory

- Euler **abstracted away the physical details of the Problem** to focus on its **fundamental network properties**.

- By doing so, he was able to prove that the desired path did not exist based on the network's topological properties.

- Euler's approach laid the foundations for graph theory

- Graph theory is now used to study networks in various fields, such as social and computer networks.

Graph theory applications

- Laying cables to make every telephone reachable.
- Finding the fastest route from one location to another.
- Scheduling a sports league into the minimum number of weeks.
- Design computer chips with non-crossing wires.
- Color maps using four colors.
- Imagine other applications …

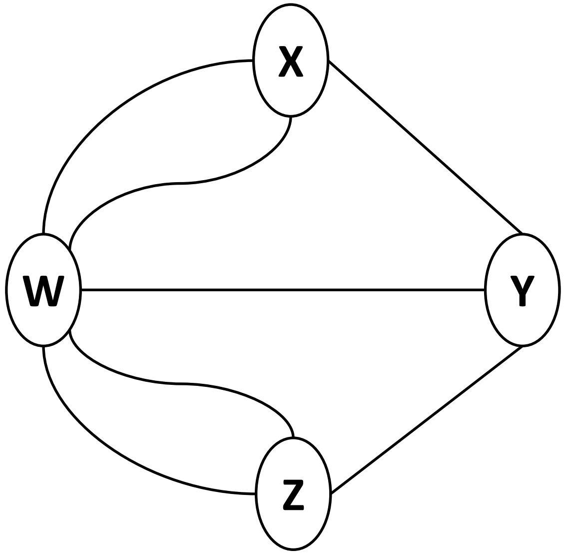
The theory of graphs is a mathematical tool that provides a way to analyze and solve problems involving networks.

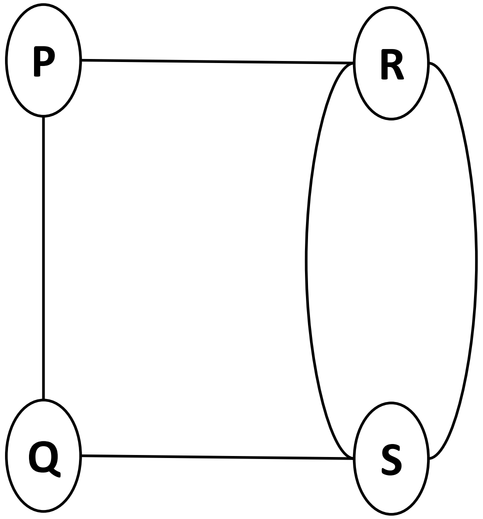
What is a graph ?

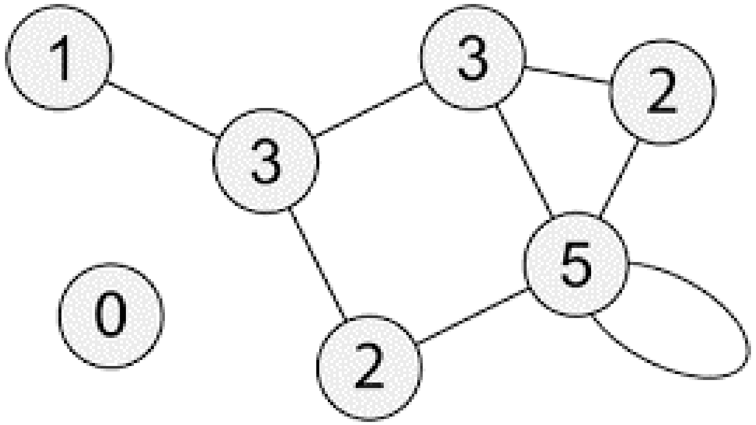
- **Graph** is a triple consisting of a:
  - **V(G)**: Vertex set
  - **E(G)**: Edge set
  - **Relation** that associates with each edge two endpoints vertices (not necessarily distinct).

- To draw a graph, we place each vertex at a point and connect them with curves to represent the edges.

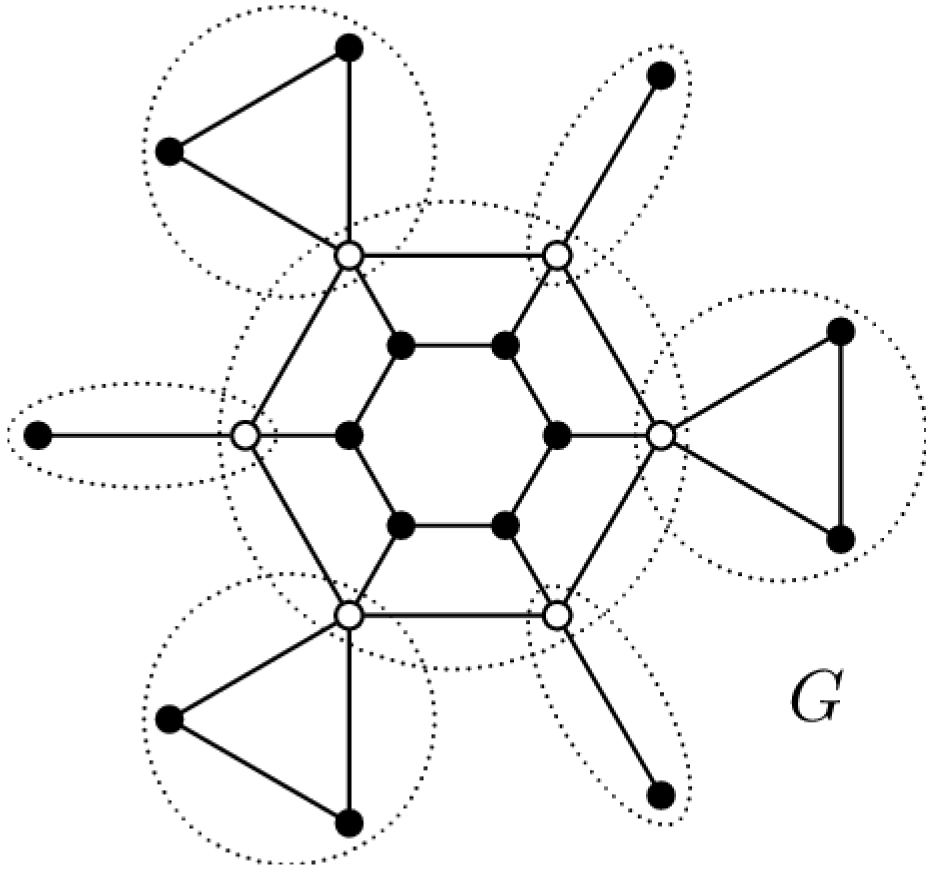
What is a graph?

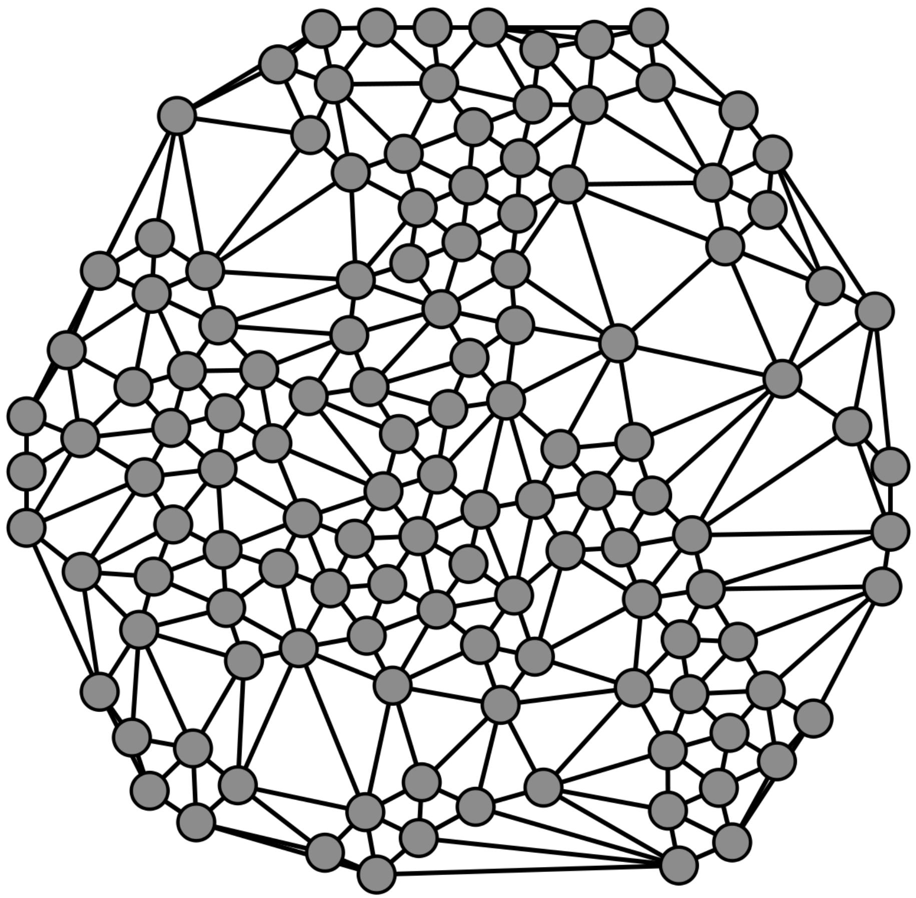






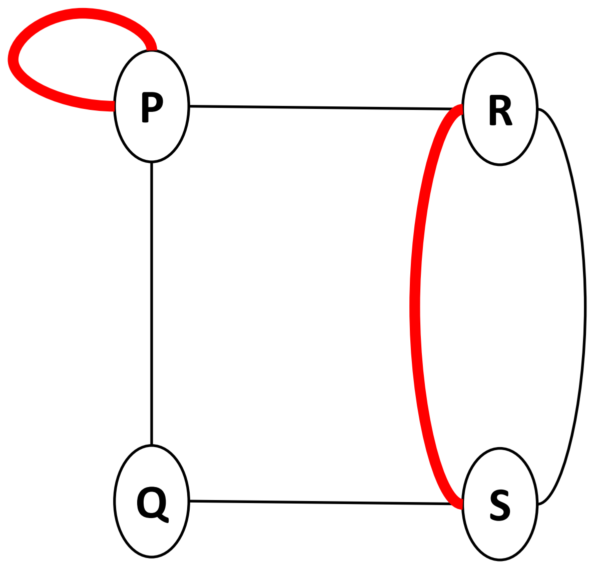
What is a graph?

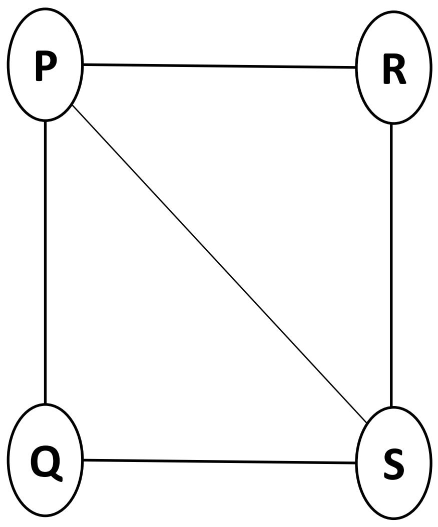




Simple Graph vs General Graph

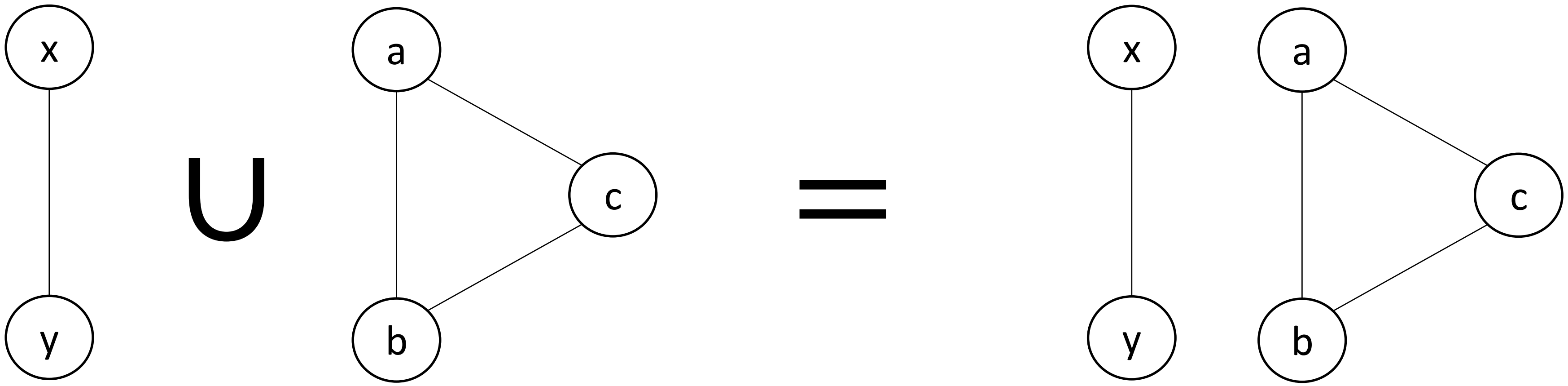
- **Simple Graph** is an graph that does not have multiple edges or self-loops.
- **General Graph** (multigraph) can have multiple edges between two vertices and also allows self-loops.





Union of two graphs

- Two graphs, G_1 and G_2, can be combined to create a larger graph G called their union.

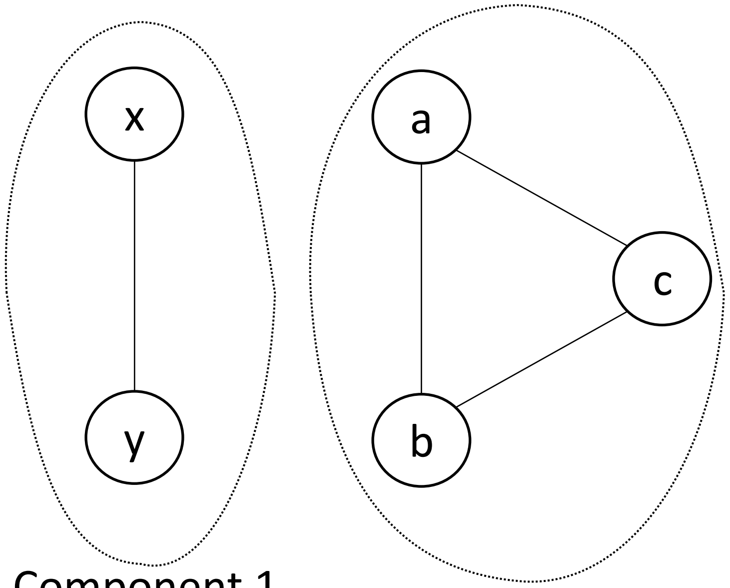


- $V(G) = V(G_1) \cup \mathrm{V}(\mathrm{G}_2) : V(G_1) \cap \mathrm{V}(\mathrm{G}_2) = \emptyset$

- $E(G) = E(G_1) \cup E(G_2)$

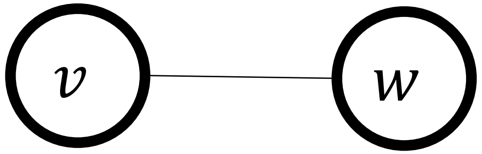
Connected vs disconnected graphs

- A graph is **connected** if it **cannot be expressed as a union of graphs**.
- **Disconnected graph** can be expressed as the union of connected graphs, each of which is called a **component**.

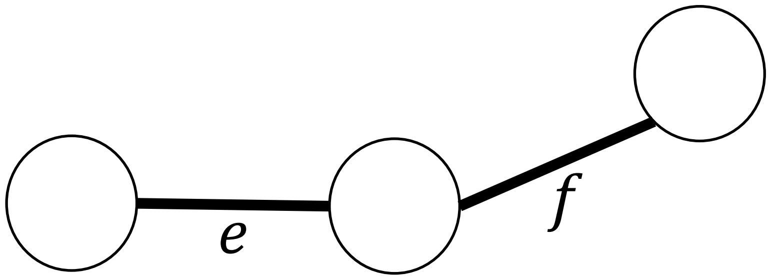


Component 1 Component 2

Adjacency



Adjacent vertices

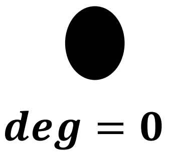


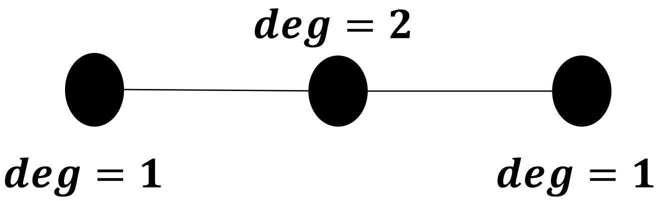
Adjacent edges

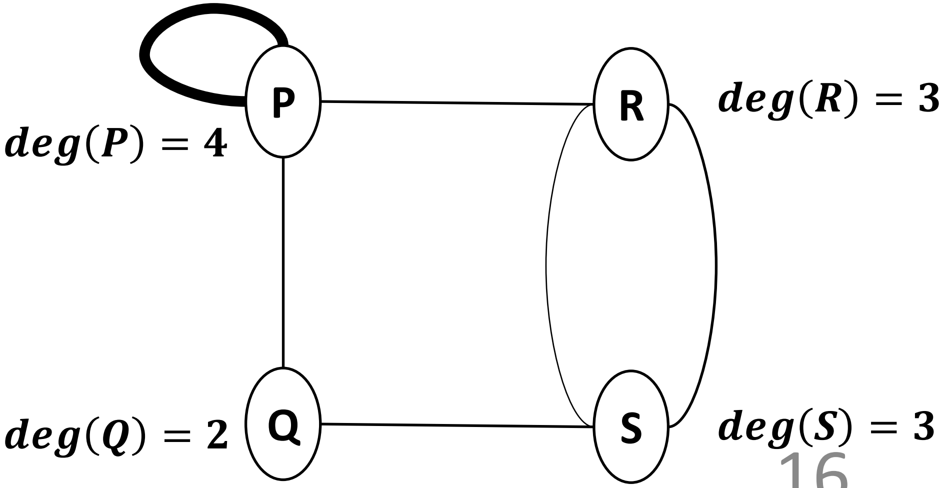
- Two vertices $\boldsymbol{v}$ and $\boldsymbol{w}$ are **adjacent** if there is an edge vw joining them.
- $\boldsymbol{v}$ and $\boldsymbol{w}$ are said to be **incident** with edge vw.
- Two distinct edges $\boldsymbol{e}$ and $\boldsymbol{f}$ are **adjacent** if they have a common vertex.

Degree of vertices

- The degree $\boldsymbol{deg(v)}$ of a vertex v is the number of edges incident with v.
  - A vertex with degree 0 is an isolated vertex
  - A vertex with degree 1 is an end-vertex.
- We usually make the convention
  - A loop at v contributes 2 (rather than 1) to $\deg(v)$.







Handshaking lemma

For any undirected graph, the total sum of degrees of all vertices is an even number, which is equal to twice the number of edges in the graph.

$$\sum_{v \in V(G)} \deg(v) = 2 \times |E(G)|$$

Proof

Each edge contributes exactly 2 to the sum. It is thus an even number.

Handshaking lemma

For any undirected graph, the total sum of degrees of all vertices is an even number, which is equal to twice the number of edges in the graph.

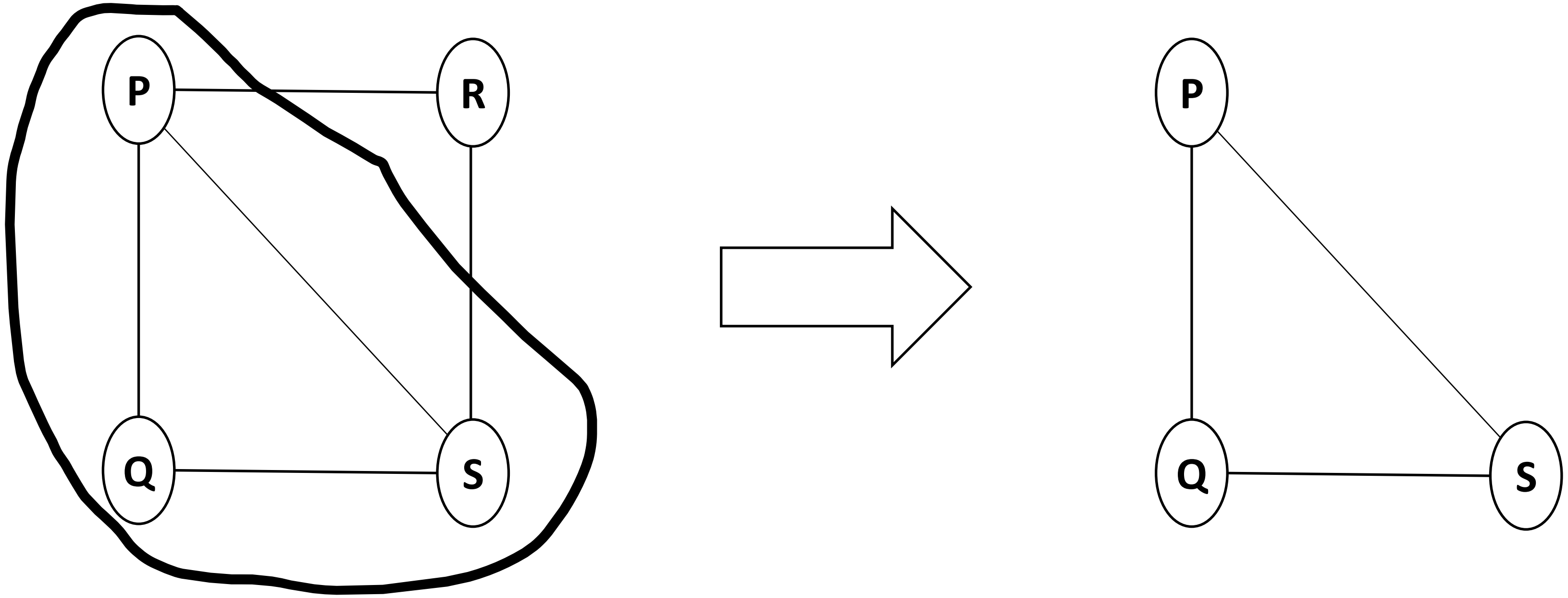
$$\sum_{v \in V(G)} \deg(v) = 2 \times |E(G)|$$

Corollary

In any undirected graph, the number of vertices of odd degree is even.

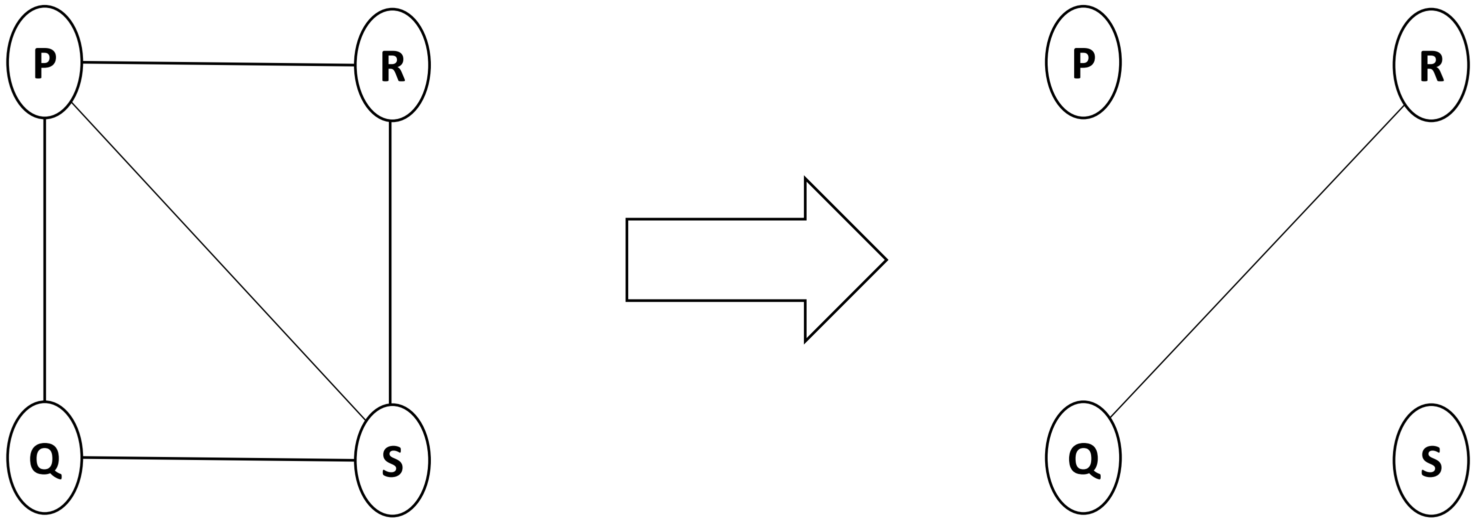
Subgraphs

- A graph H is a subgraph of a graph G if
  - Each of its vertices belongs to $V(G)$.
  - Each of its edges belongs to $E(G)$.
- Subgraphs can be obtained by deleting edges and vertices.



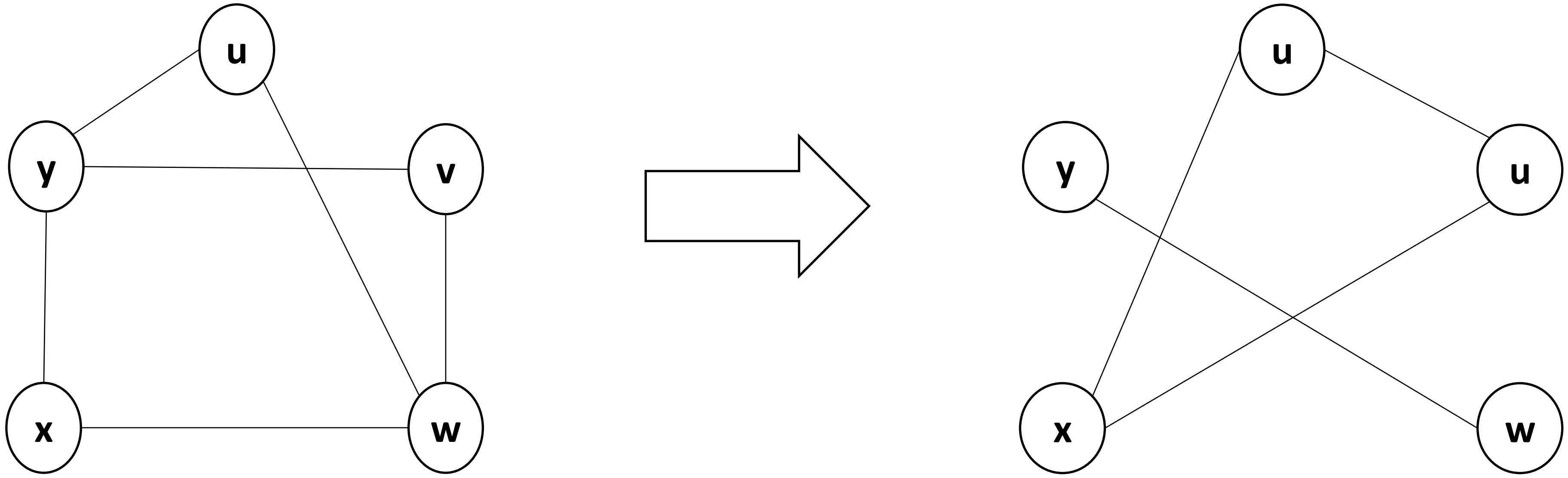
The complement of a simple graph

The ***complement of graph*** $\boldsymbol{G}$ *is the simple graph* ***with vertex-set*** $\boldsymbol{V(G)}$ *in which* ***two vertices are adjacent*** *if and only if* ***they are not adjacent*** *in* G**.**



The complement of a simple graph

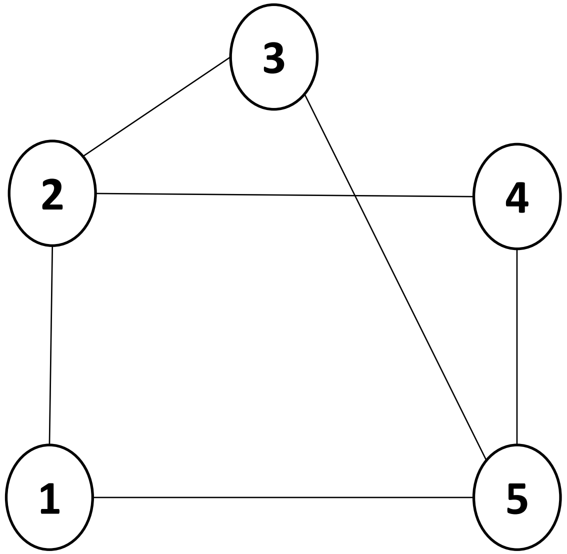
*The **complement of graph** $\boldsymbol{G}$ is the simple graph **with vertex-set** $\boldsymbol{V(G)}$ in which **two vertices are adjacent** if and only if **they are not adjacent** in G.*



Matrix representations

- Matrix representations of graphs provide
  - Useful tool for studying and analysing graphs.
  - To be stored/processed by computer
- Adjacency matrix
  - Vertices labelled $\{1, 2, \dots, n\}$
  - $n \times n$ matrix A.
  - A_{ij} entry is the number of edges joining vertex i and j.
- Incidence matrix
  - Vertices labelled $\{1, 2, \dots, n\}$, Edges labelled $\{1, 2, \dots, m\}$
  - $n \times m$ matrix M.
  - $M_{ij} = 1$ if vertex i is incident to edge j, and is 0 otherwise

Matrix representations



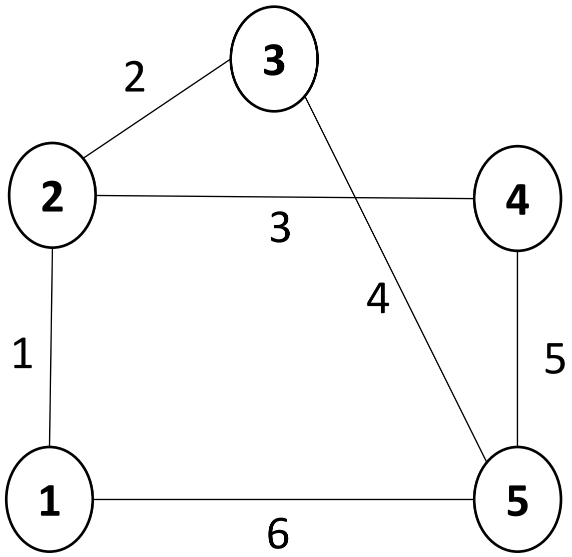
Vertices

Vertices

0	1	0	0	1
1	0	1	1	0
0	1	0	0	1
0	1	0	0	1
1	0	1	1	0

Adjacency matrix

Matrix representations

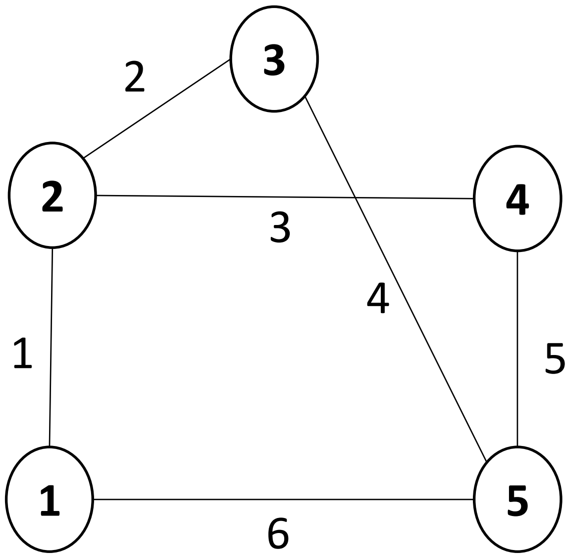


Edges

Vertices						
	1	0	0	0	0	1
	1	1	1	0	0	0
	0	1	0	1	0	0
	0	0	1	0	1	0
	0	0	0	1	1	1

Incidence matrix

Matrix representations



Vertices

Vertices					
	0	1	0	0	1
	1	0	1	1	0
	0	1	0	0	1
	0	1	0	0	1
	1	0	1	1	0

Adjacency matrix

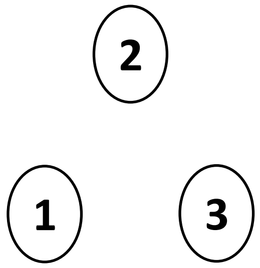
Edges

Vertices						
	1	0	0	0	0	1
	1	1	1	0	0	0
	0	1	0	1	0	0
	0	0	1	0	1	0
	0	0	0	1	1	1

Incidence matrix

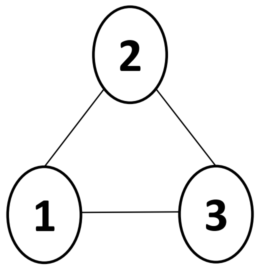
Graph types

- Null graphs
  - Edge-set is empty.
  - N_n: null graph of n vertices

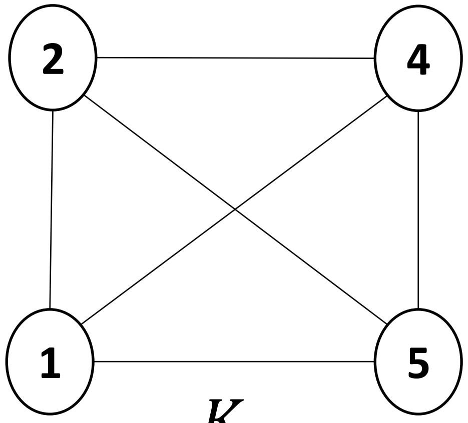


N_3

- Complete graphs
  - Each two distinct vertices are adjacent
  - All vertices are connected
  - K_n: complete graph of n vertices
  - Number of edges is $\frac{n(n-1)}{2}$



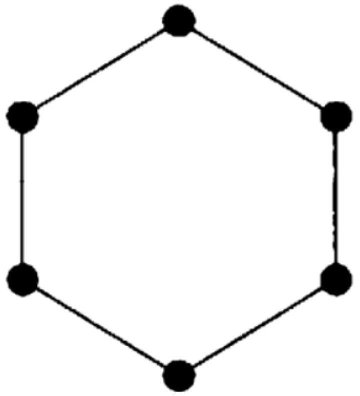
K_3



K_4

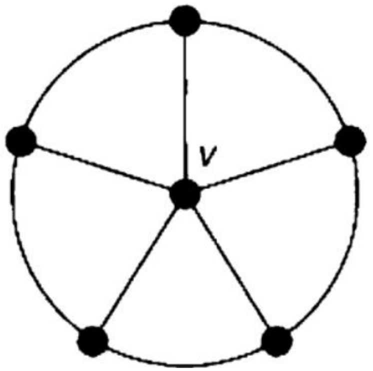
Graph types

- Cycle graphs
  - All vertices has the degree 2
  - C_n: cycle graph of n vertices



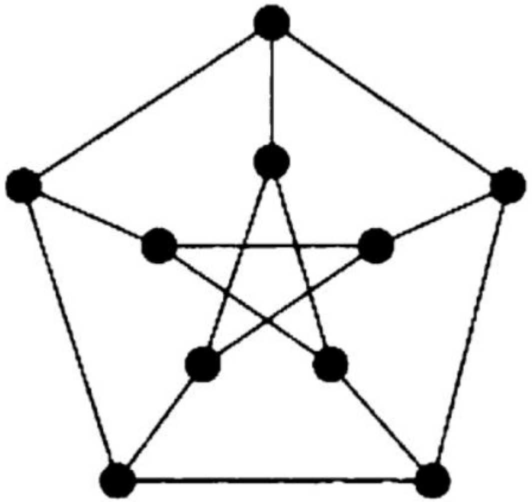
- Path graphs
  - Removing one edge from C_n
  - P_n: path graph of n vertices

- Wheel graphs
  - Joining all vertices of C_{n-1} by a new vertex V
  - W_n: wheel graph of n vertices



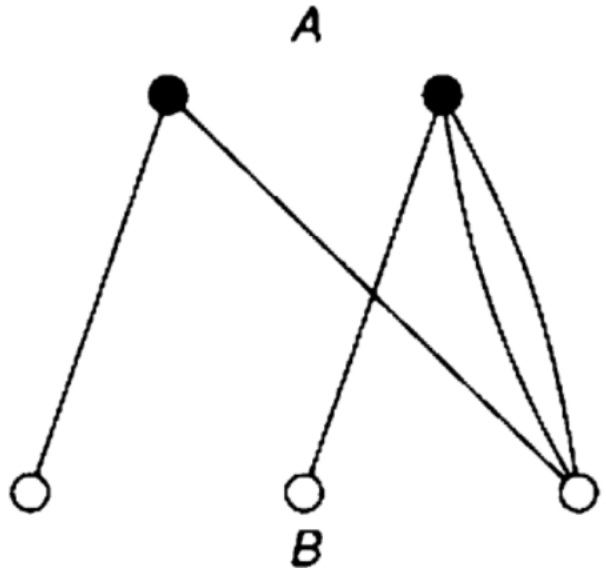
Graph types

- Regular graphs
  - A graph in which each vertex has the same degree.
- If each vertex has degree r, the graph is **r-regular**.
  - Null graph is **0-regular**
  - Cycle graph is **2-regular**
  - Complete graph K_n is regular of degree $\mathbf{n-1}$.
- Cubic graphs, which are regular of degree 3
  - Cubic graph called also the Petersen graph.



Graph types

- Bipartite graphs
  - Vertex-set can be divided into **two disjoint sets A and B**, and **each edge connects a vertex of A and a vertex of B**.
  - Bipartite graph if its vertices **can be coloured black and white** such that **each edge connects a black vertex in A** and **a white vertex in B**.
  - Notation: G = G(A, B).



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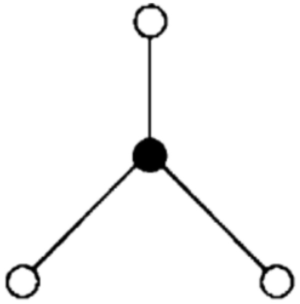
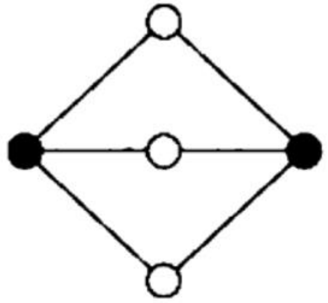
Graph types

- Complete bipartite graph:
  - Each vertex in set A is connected to every vertex in set B with a single edge.

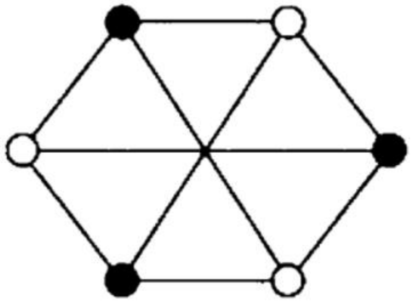
- $\boldsymbol{K_{rj}}$: is complete bipartite graph with $\boldsymbol{r}$ black vertices and $\boldsymbol{s}$ white vertices



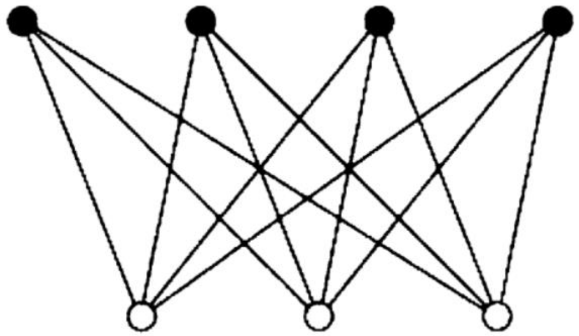
$K_{1,3}$



$K_{2,3}$



$K_{3,3}$



$K_{4,3}$

Directed graph (digraph)

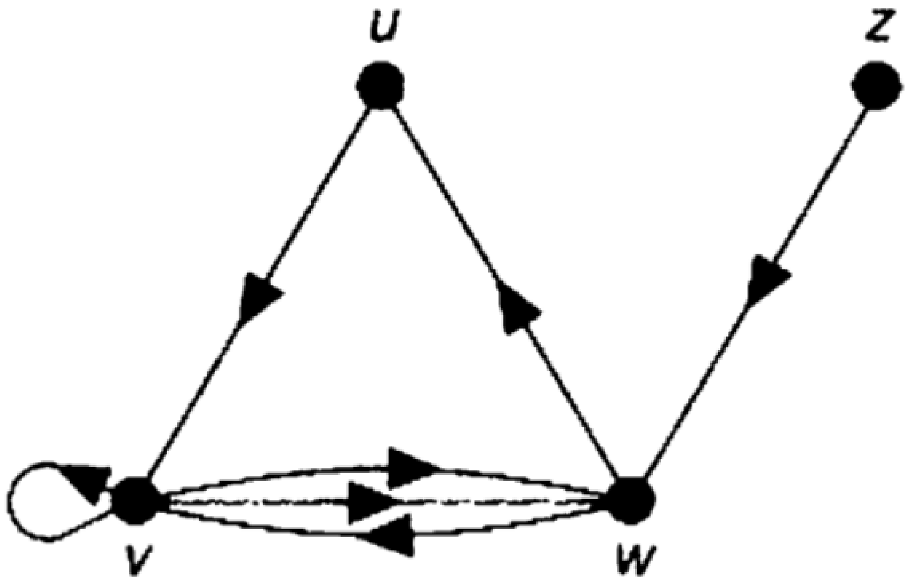
- Directed graph (digraph) D consists of finite **vertex-set $\boldsymbol{V(D)}$** and a finite family arc-family $A(D)$ of **ordered pairs of vertices called arcs.**

- An arc is abbreviated to $\boldsymbol{vw}$, where $\boldsymbol{v}$ and $\boldsymbol{w}$ are vertices.

- The ordering of the vertices in an arc is indicated by an arrow.

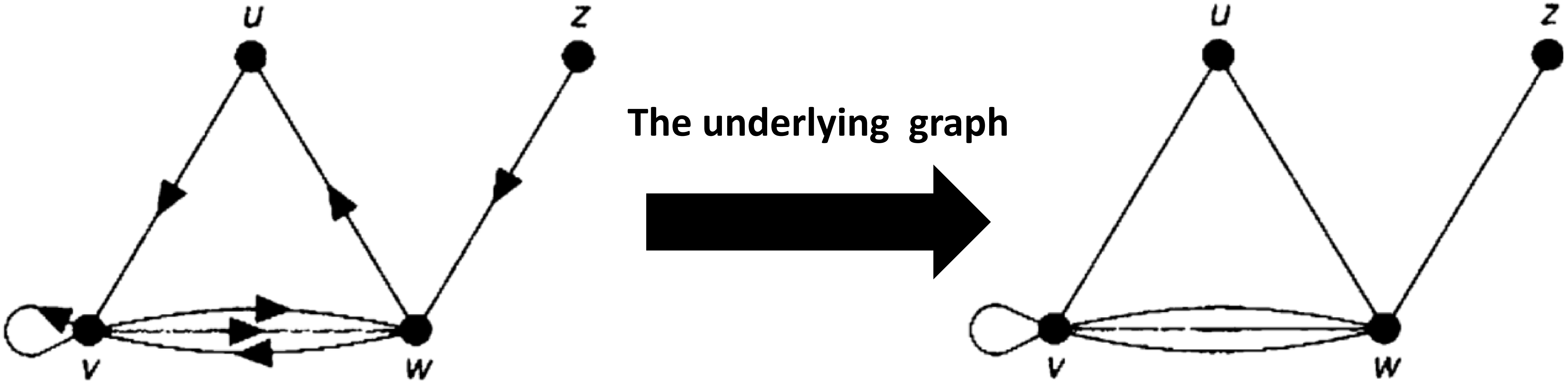
- **The underlying graph** of D is the graph obtained from D by **'removing the arrows'**

Directed graph (digraph)



- $V(D) = \{u, v, w, z\}$
- $A(D) = \{uv, w, vw\ (twice), wv, wu, zw\}$

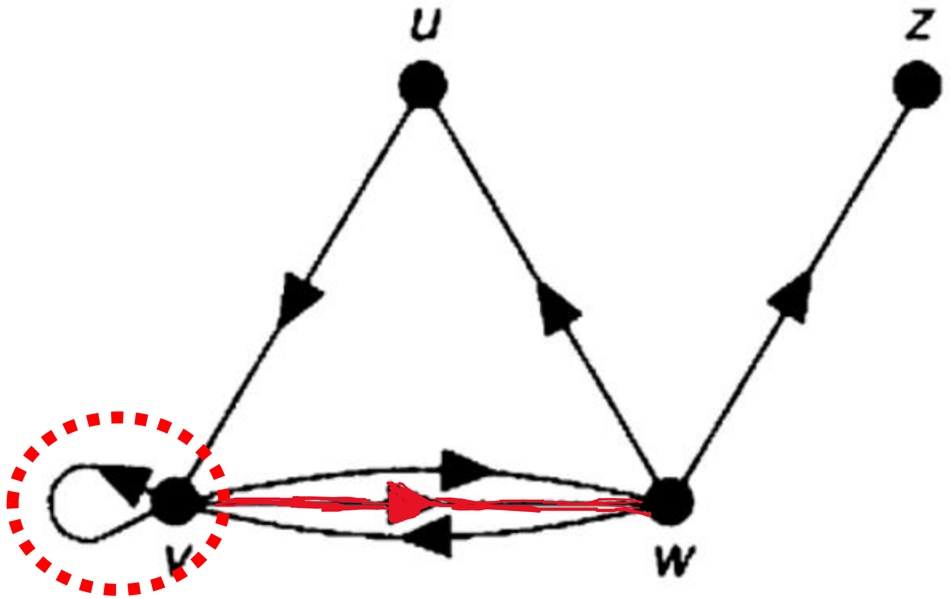
Directed graph (digraph)

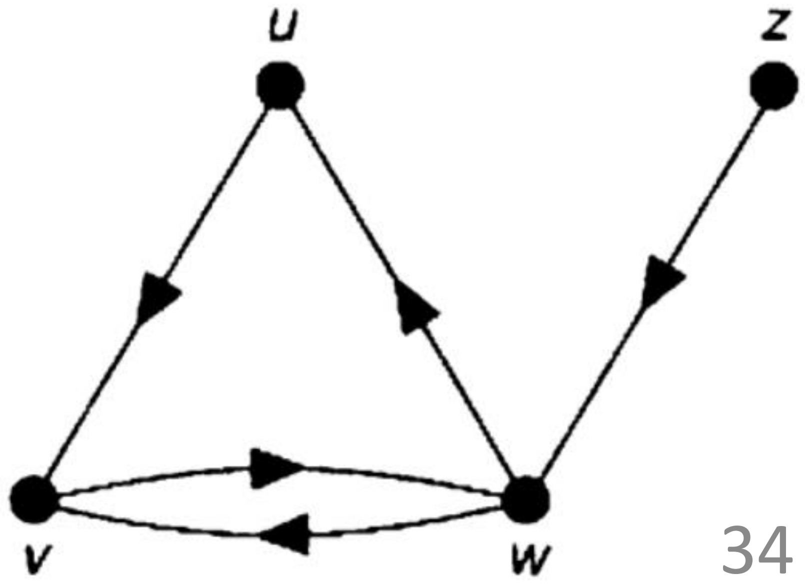


- $V(D) = \{u, v, w, z\}$
- $A(D) = \{uv, w, vw\ (twice), wv, wu, zw\}$

Simple digraph

- D is a simple digraph if :
  - The arcs of D are all distinct
  - There are no "**loops**" arcs of the form vv).
- The underlying graph of a simple digraph need not be a simple graph





Connectedness and Degrees in Directed Graphs

- A digraph D is (weakly) connected
  - if it cannot be expressed as the union of two digraphs
  - The underlying graph of D is a connected graph.

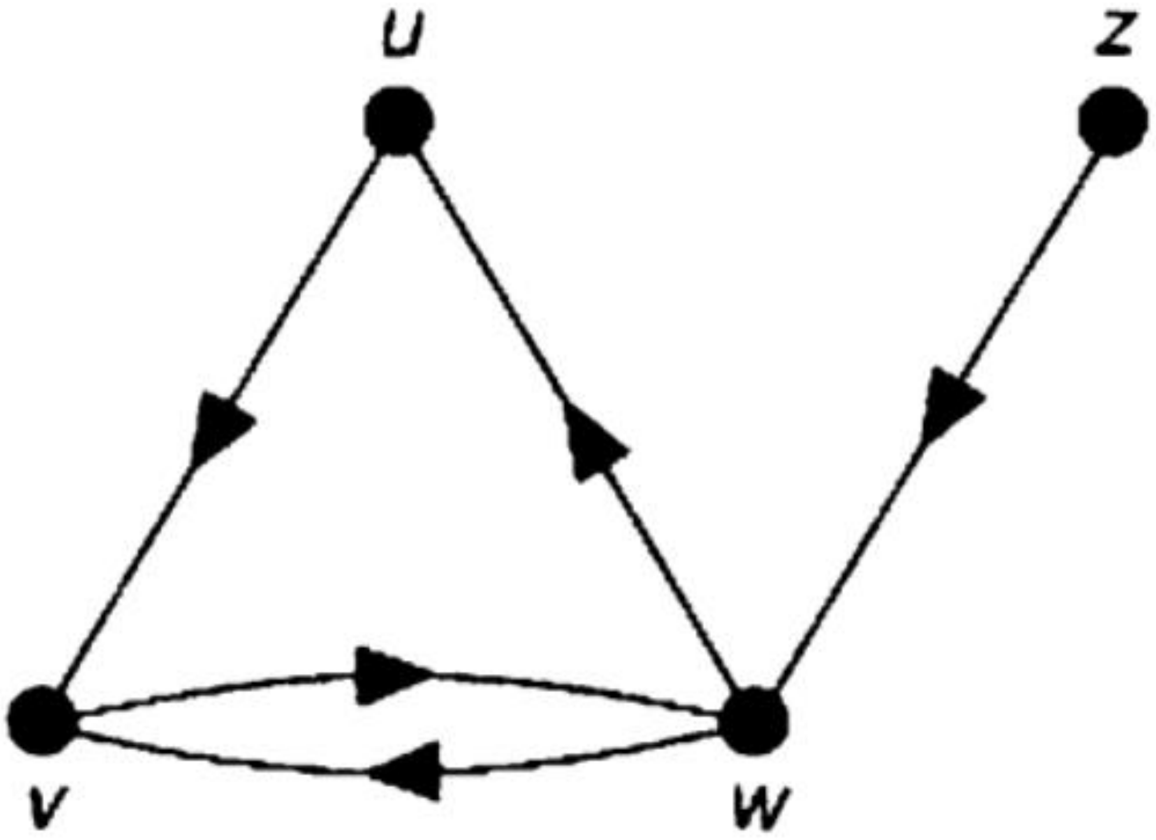
- Two vertices u and v of a digraph D are adjacent if there is an arc in $A(D)$ of the form $u \rightarrow v$ or $v \rightarrow u$

- The vertices u and v are incident with the arc $u \rightarrow v$ or $v \rightarrow u$.

- **Out-degree** of a vertex v
  - Number of arcs of the form $v \rightarrow w$, denoted by $\boldsymbol{outdeg(v)}$.

- **In-degree** of a vertex v
  - Number of arcs of the form $u \rightarrow v$, denoted by $\boldsymbol{indeg(v)}$.

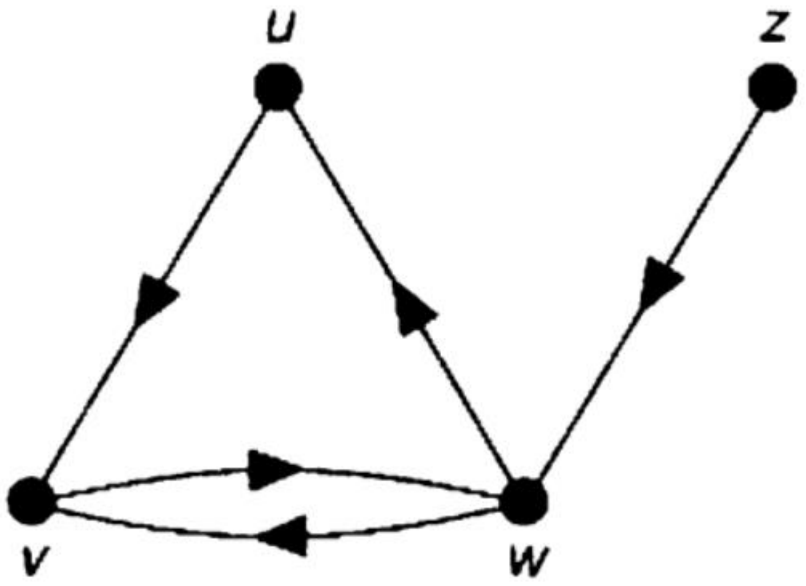
Connectedness and Degrees in Directed Graphs



Connectedness and Degrees in Directed Graphs

- u and v are adjacent
- u and v are incident of the arc uv

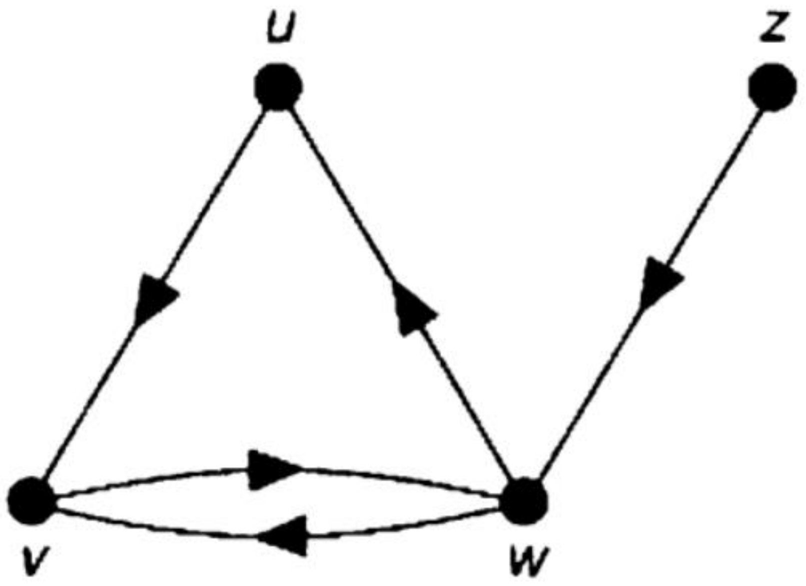
Vertices	$indeg(v)$	$outdeg(v)$
u		
v		
w		
z		
Sum		



Connectedness and Degrees in Directed Graphs

- u and v are adjacent
- u and v are incident of the arc uv

Vertices	$indeg(v)$	$outdeg(v)$
u	1	1
v	2	1
w	2	2
z	0	1
Sum	**5**	**5**



Handshaking lemma in Directed graphs

For any directed graph, the sum of all the in-degrees is equal to the sum of all the out-degrees.

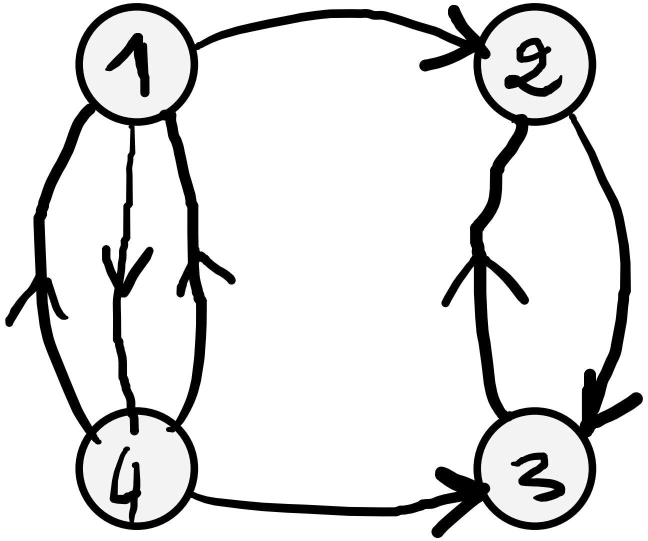
$$\sum_{v \in V(G)} \text{outdeg}(v) = \sum_{v \in V(G)} \text{indeg}(v)$$

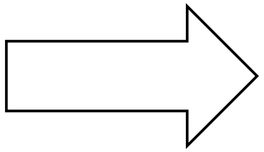
Proof

- Each arc contributes exactly 1 to the two sums.
- The two sums equal to the number of arcs.

Adjacency Matrix of a Digraph

- A digraph's adjacency matrix is:
    - $n \; x \; n$ matrix, where the A_{ij} entry represents the number of arcs from vertex i to vertex j.

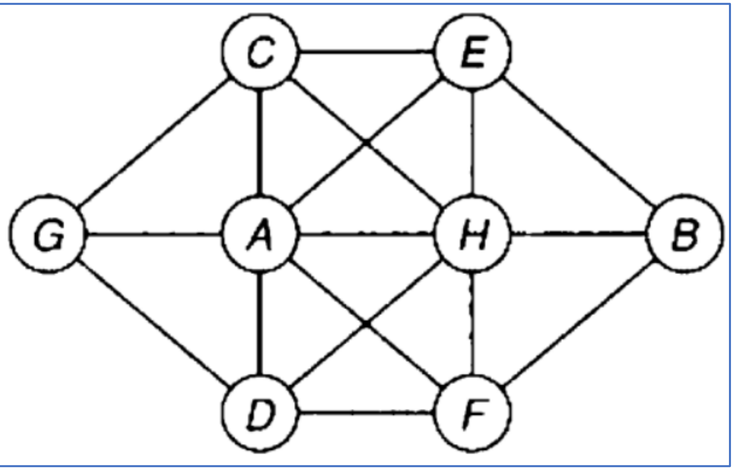


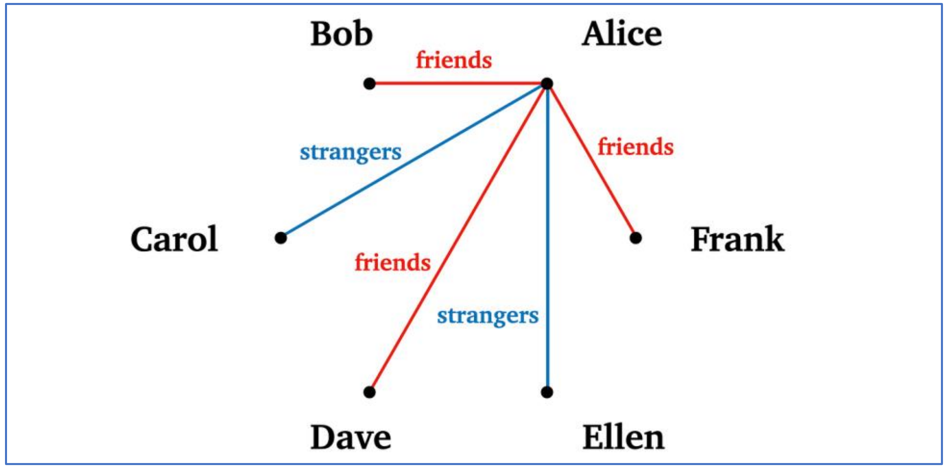


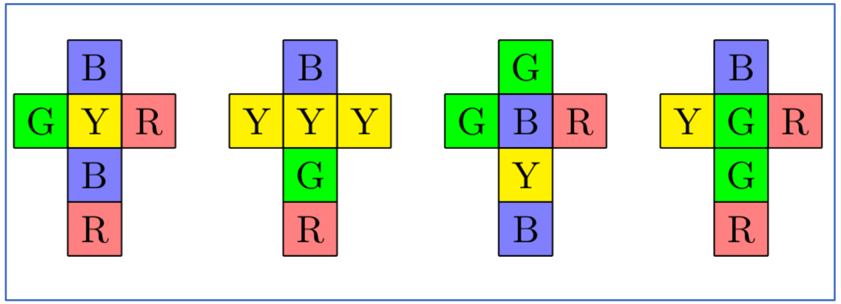
	1	2	3	4
1	0	1	0	1
2	0	0	1	0
3	0	1	0	0
4	2	0	1	0

Three puzzles

- Graph theory make some problems easier to understand and solve

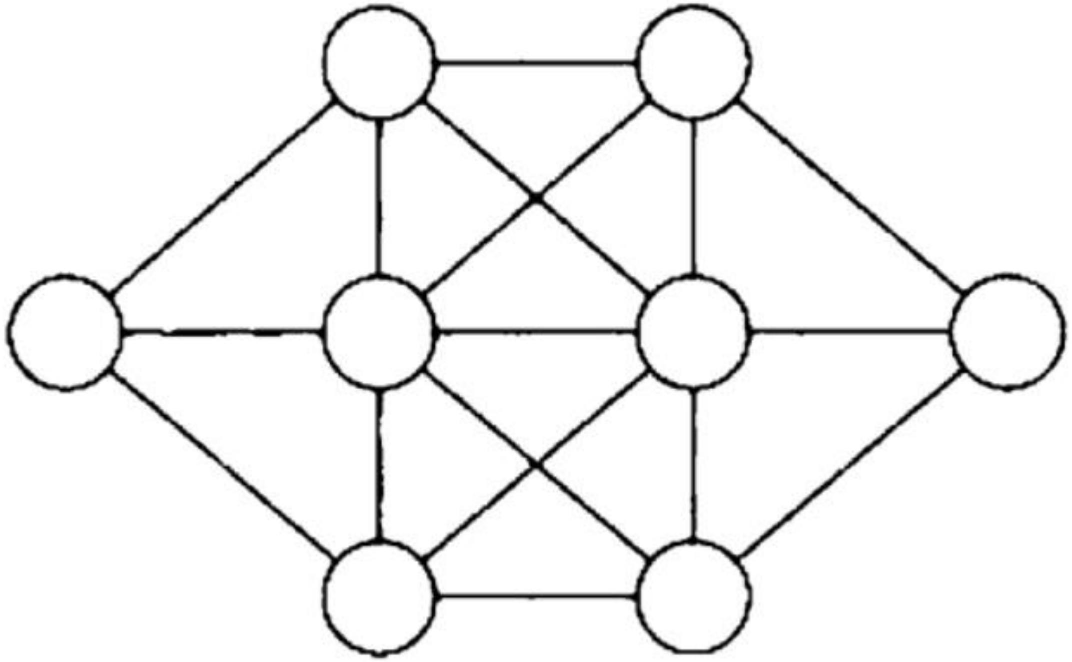






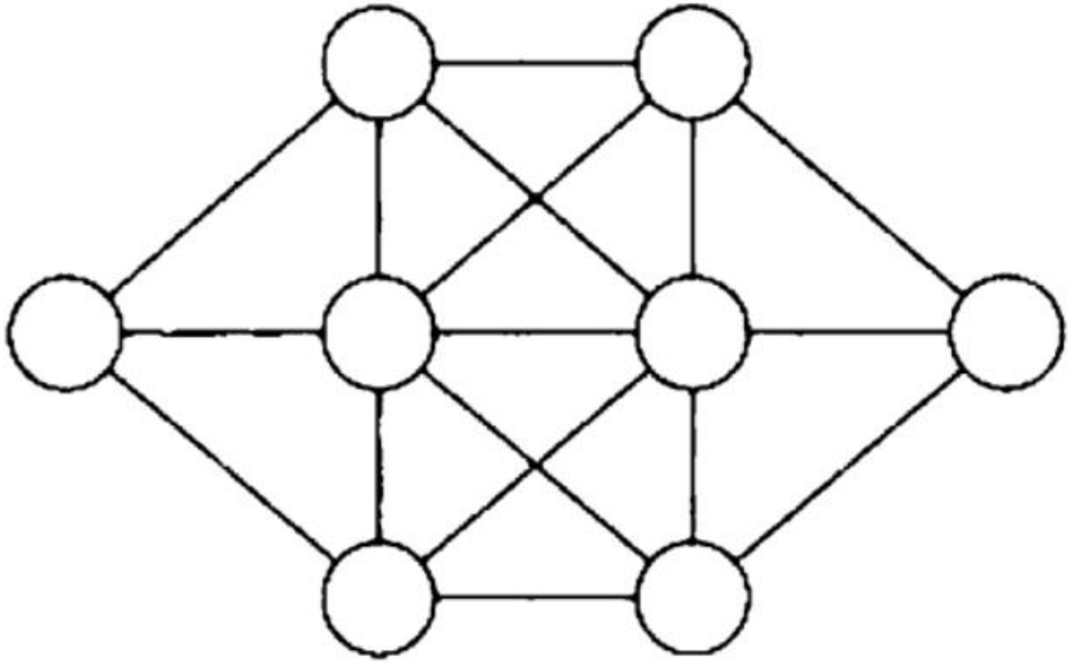
The eight-circles problem

- Place the letters $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}, \boldsymbol{E}, \boldsymbol{F}, \boldsymbol{G}, \boldsymbol{H}$ into the eight vertices of G
- No letter is adjacent to a letter that is next to it in the alphabet.



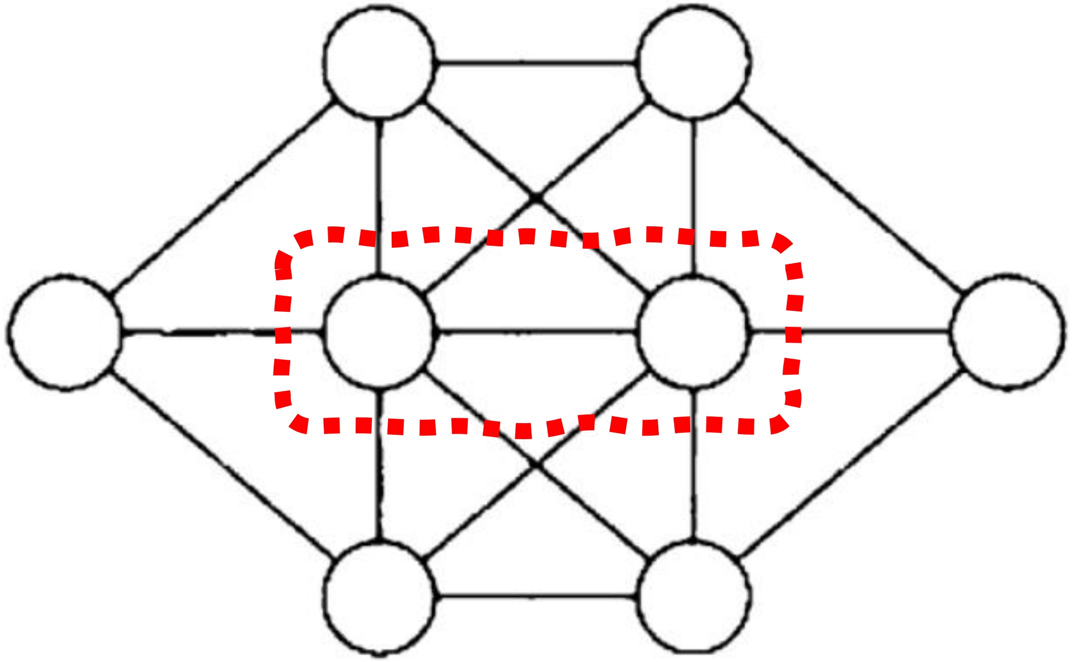
The eight-circles problem

- Trying all possibilities is not feasible
  - $8! = 40320$
- More systematic approach is required



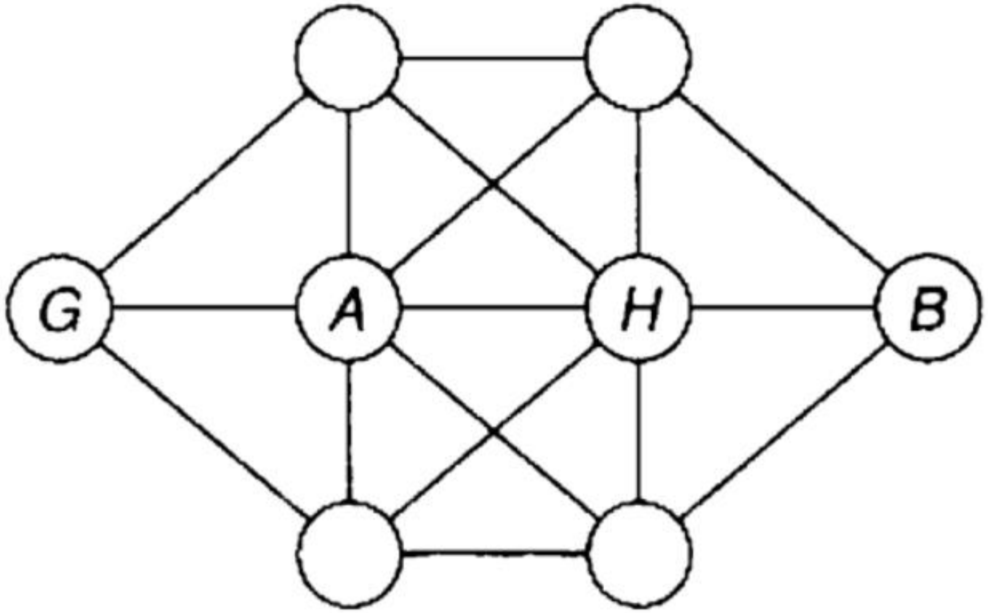
The eight-circles problem

- The letters $\boldsymbol{A}$ and $\boldsymbol{H}$ are the easiest to place
  - They have only one letter each to which they cannot be adjacent, $\boldsymbol{B}$ and $\boldsymbol{G}$ respectively.
- The hardest circles to fill are those in the middle, as each is adjacent to six others.



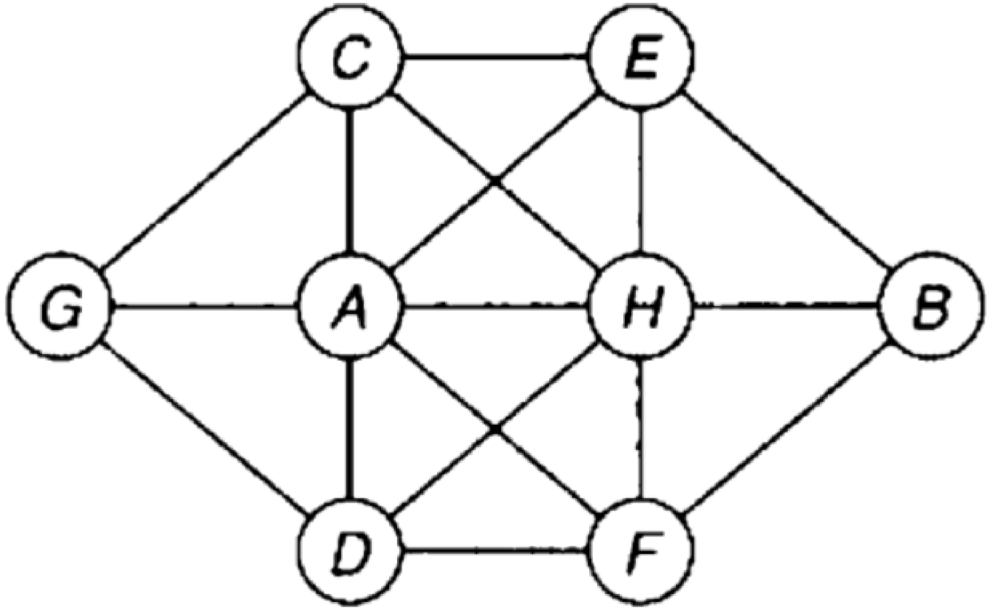
The eight-circles problem

- This suggests that we place $\boldsymbol{A}$ and $\boldsymbol{H}$ in the middle circles.
- **Only one possiblity for $\boldsymbol{B}$ and $\boldsymbol{G}$.**
  - **A**, **B**, C, D, E, F, **G**, **H.**



The eight-circles problem

- C must now be placed Above A
- F must be placed bellow H.
- It is easy to place the remaining letters



Six people at a party

In any group of six people, there will always be either :

- *A subgroup of three people who all know each other*

OR

- *A subgroup of three people where none of them knows the other two.*

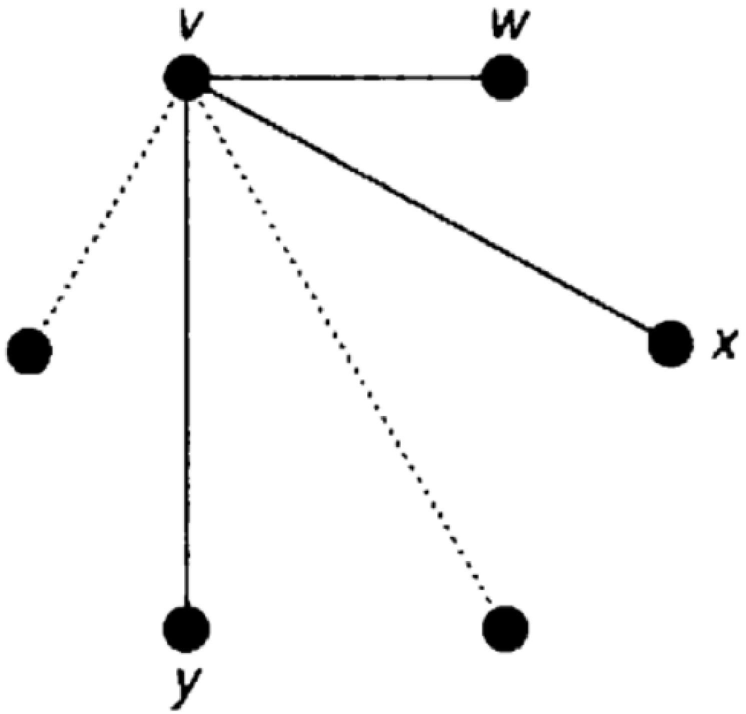
Six people at a party

- To solve the problem:
  - A graph is created with people represented as vertices
  - Solid/Dotted edges indicating whether they know each other or no.

- The goal is to prove that there will always be
  - A solid or dotted triangle present in the graph.

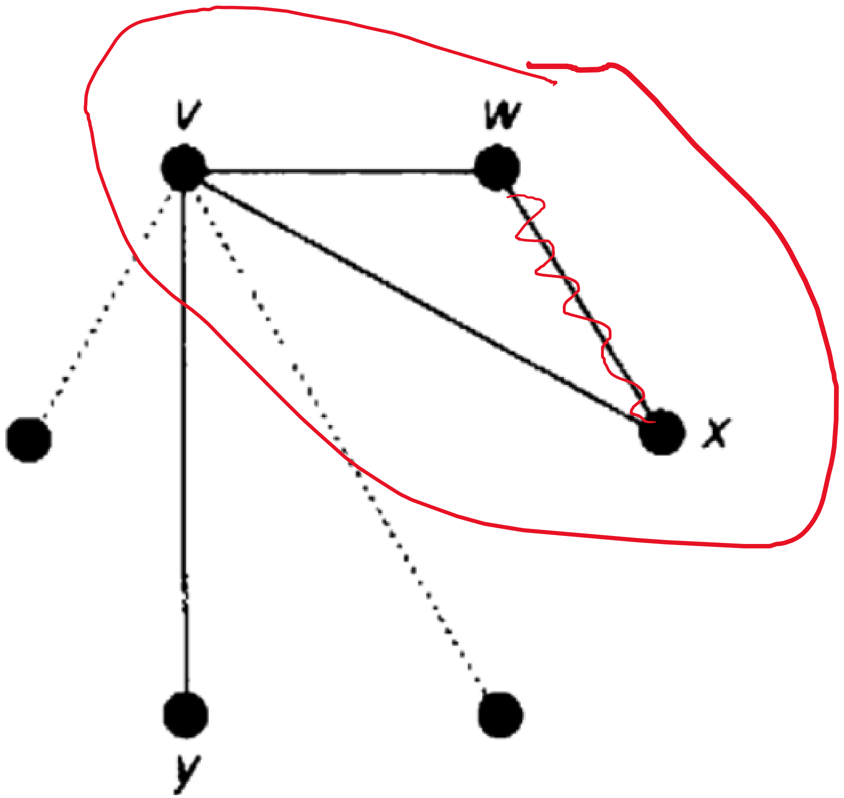
Six people at a party

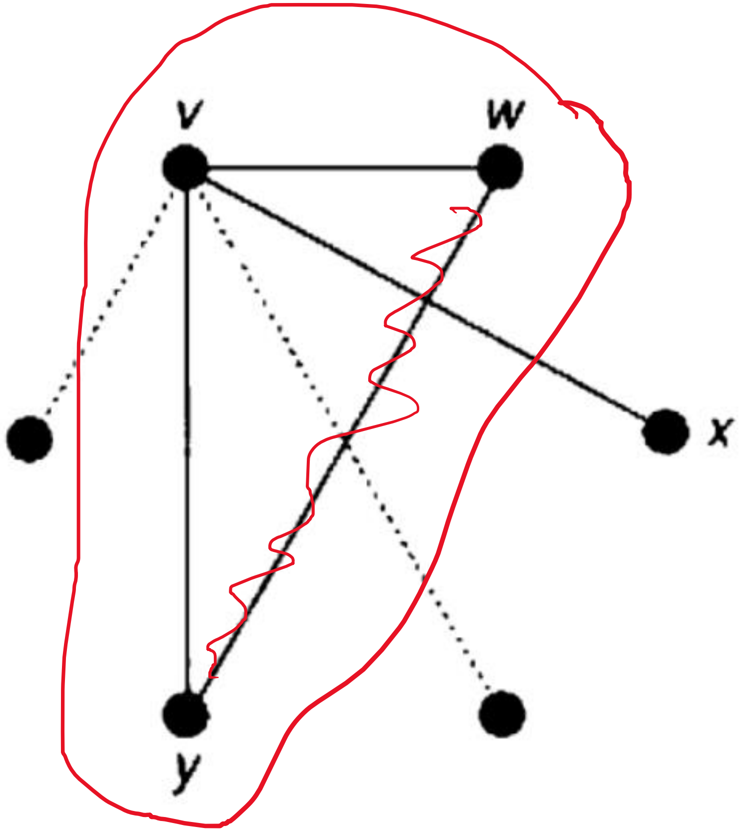
- If v is a vertex in a graph, it will have five incident edges (solid or dotted).
- At least three of these edges will be of the same type.
- Assuming that there are three solid edges (the same applies if there are at least three dotted edges).



Six people at a party

- There are 4 possibilities





Six people at a party

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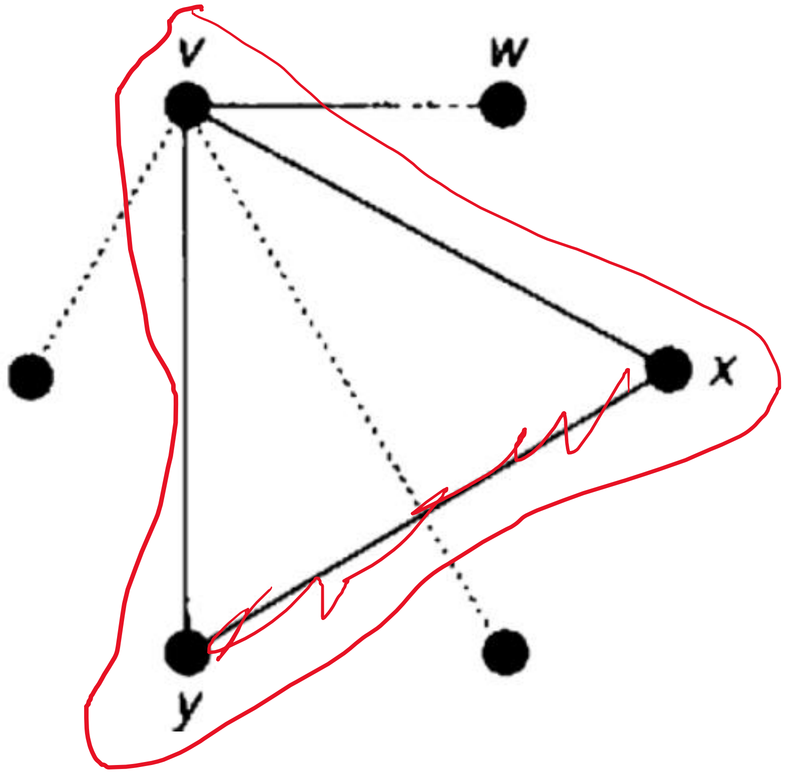
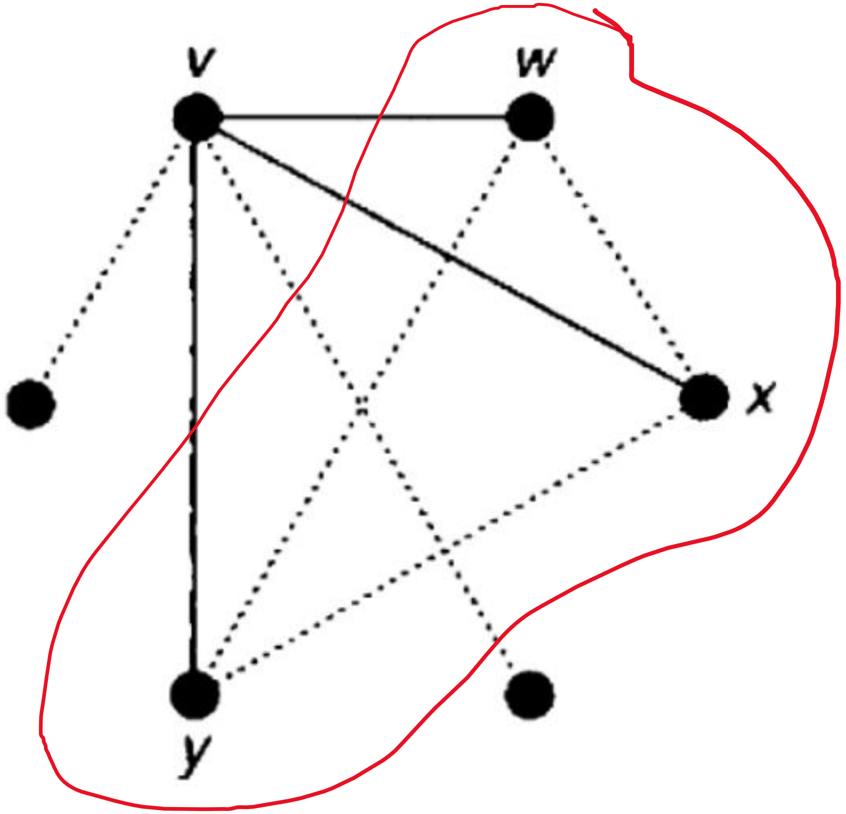
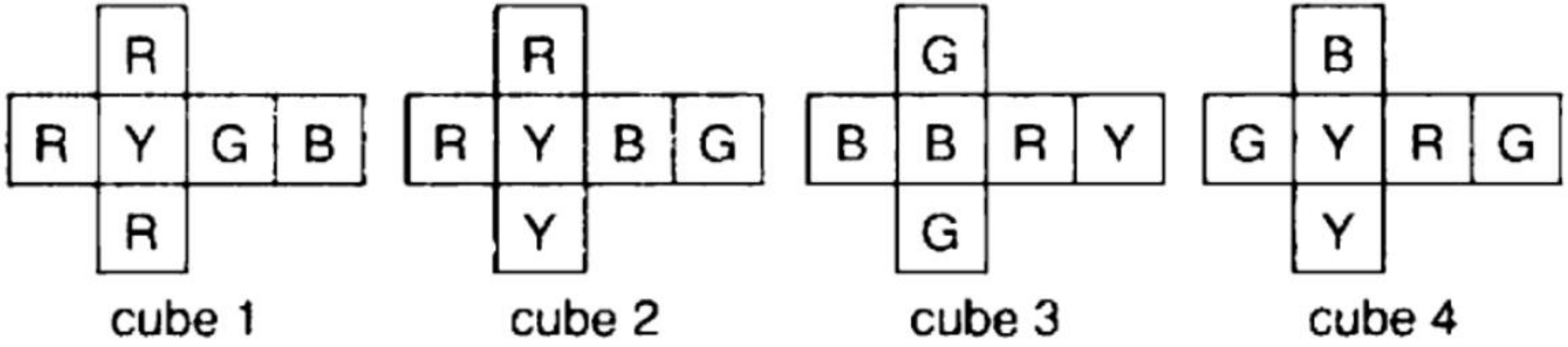
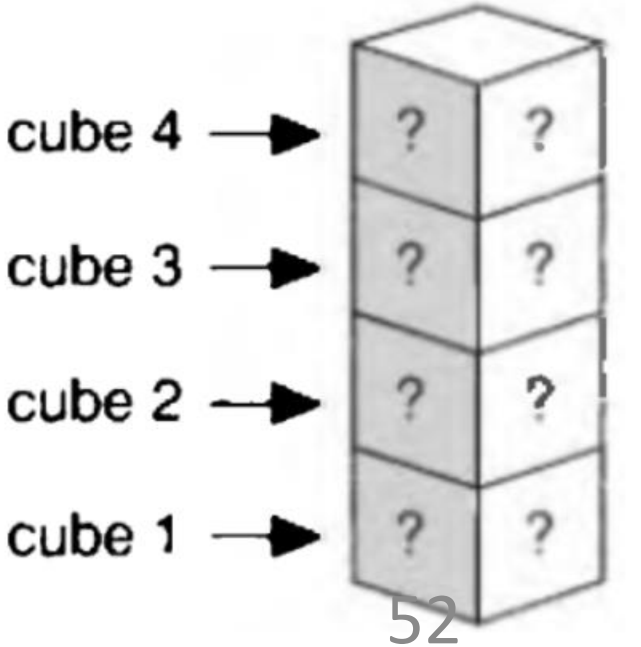




The four-cubes problem

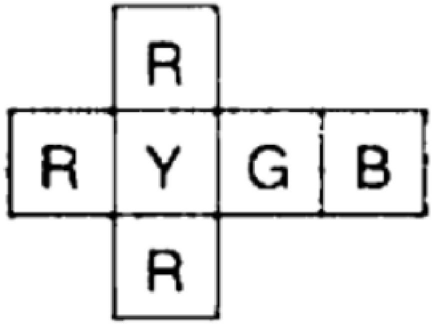
- Given four cubes whose faces are coloured red blue (B), green (G) and yellow (Y), can we pile them up so that all four colours appear on each side of the resulting 4x1 stack?



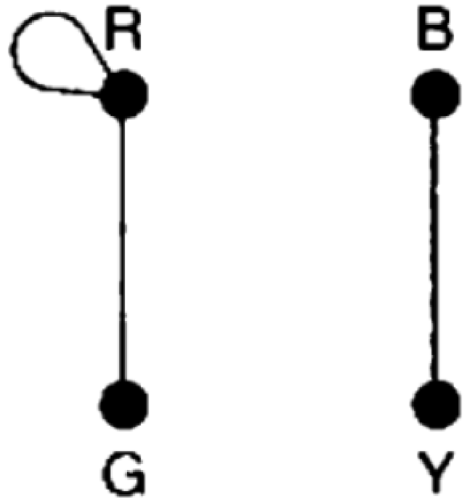


The four-cubes problem

- To solve this problem, we represent each cube by a graph using R, B, G, and Y as vertices.
- Two vertices are connected if the corresponding colours are on opposite faces.



cube 1



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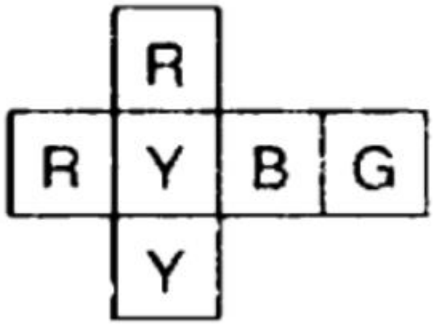
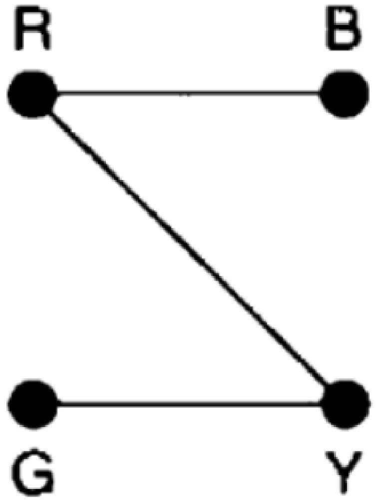


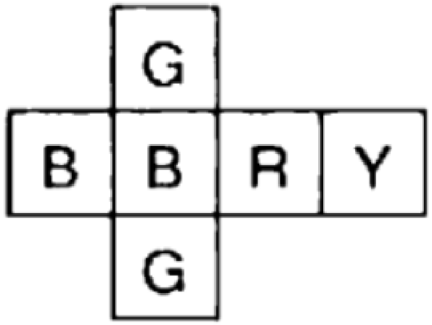
cube 2



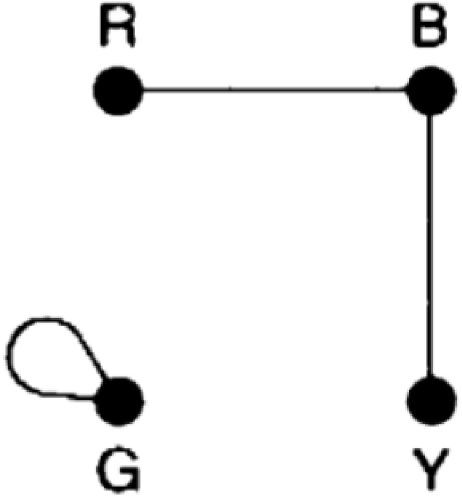
cube 2

The four-cubes problem

- To solve this problem, we represent each cube by a graph using R, B, G, and Y as vertices.
- Two vertices are connected if the corresponding colours are on opposite faces.



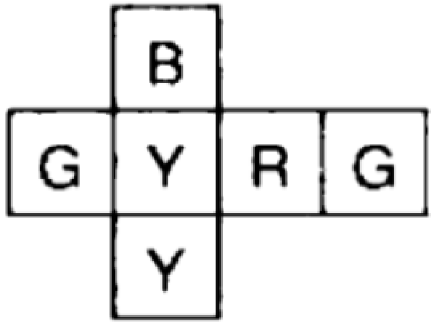
cube 3



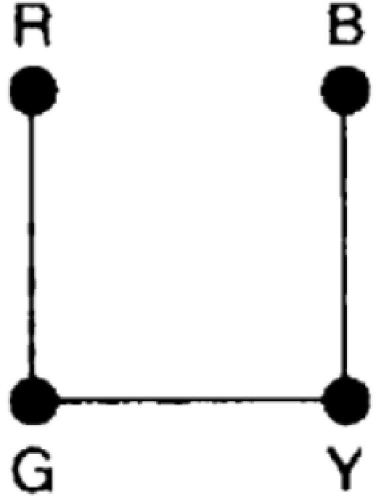
cube 3

The four-cubes problem

- To solve this problem, we represent each cube by a graph using R, B, G, and Y as vertices.
- Two vertices are connected if the corresponding colours are on opposite faces.

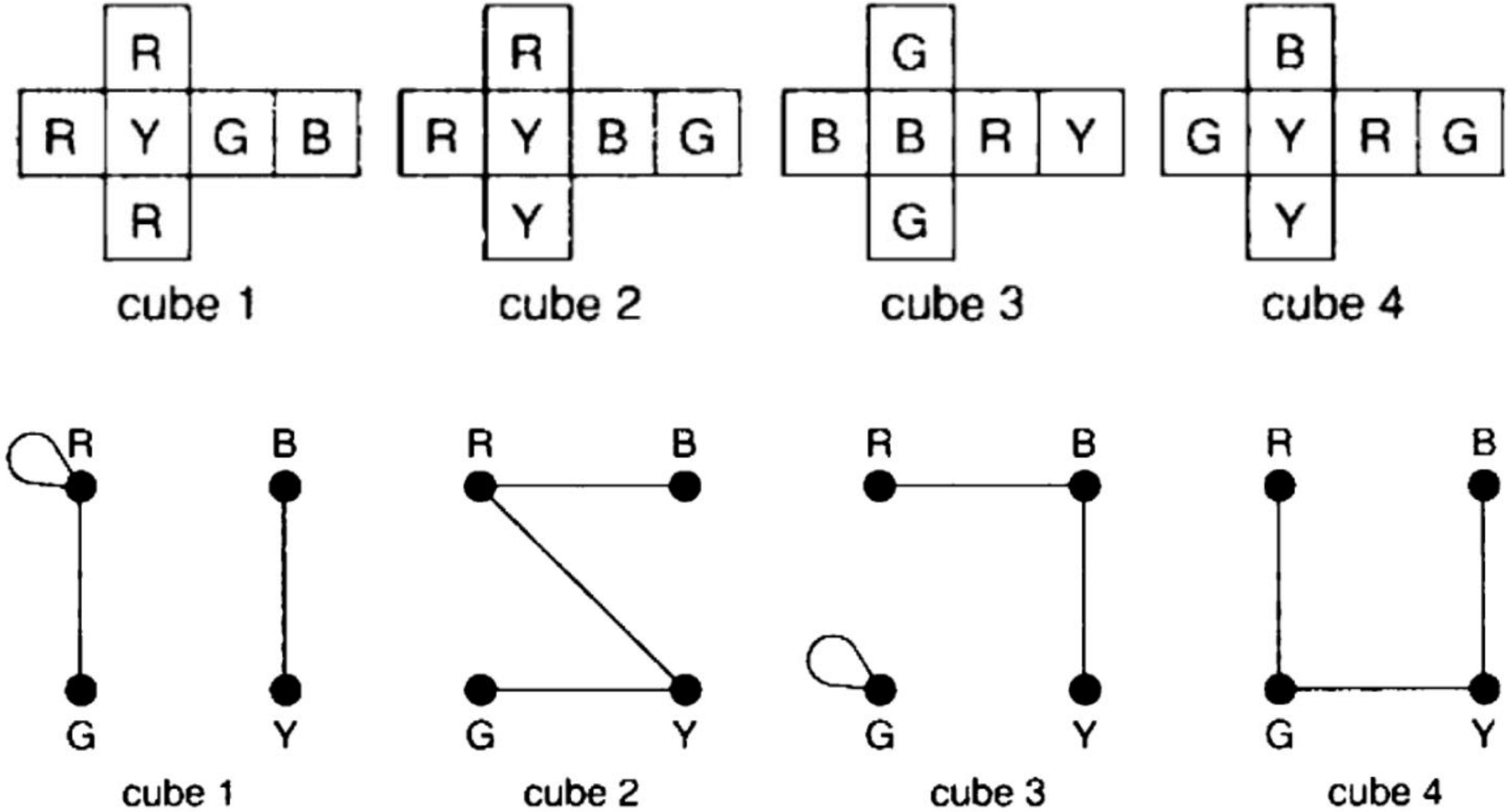


cube 4



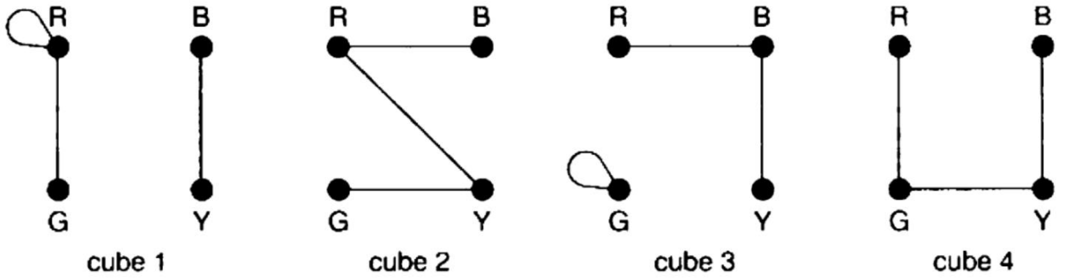
cube 4

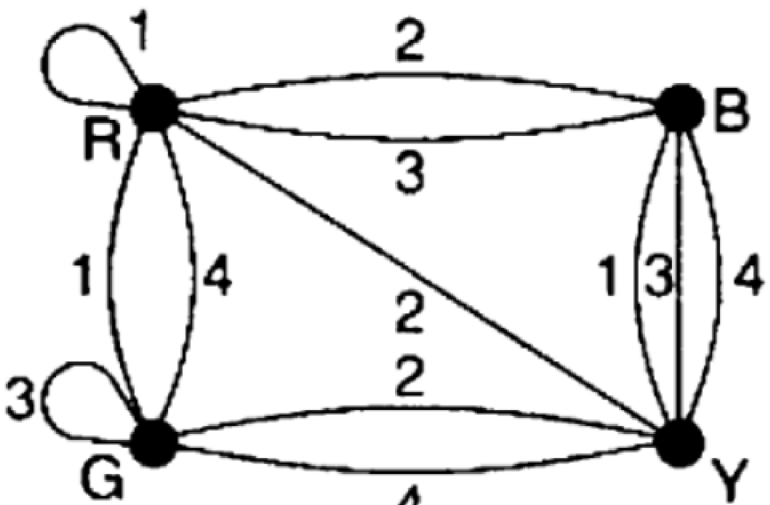
The four-cubes problem



The four-cubes problem

- We next superimpose these graphs to form a new graph G

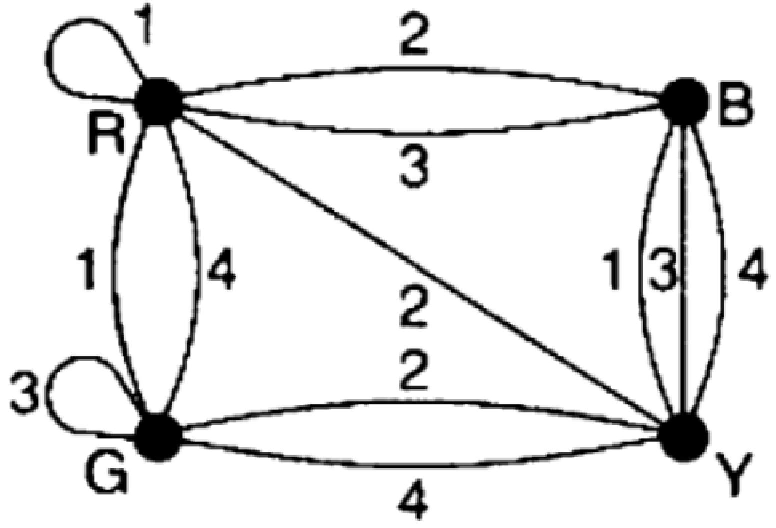


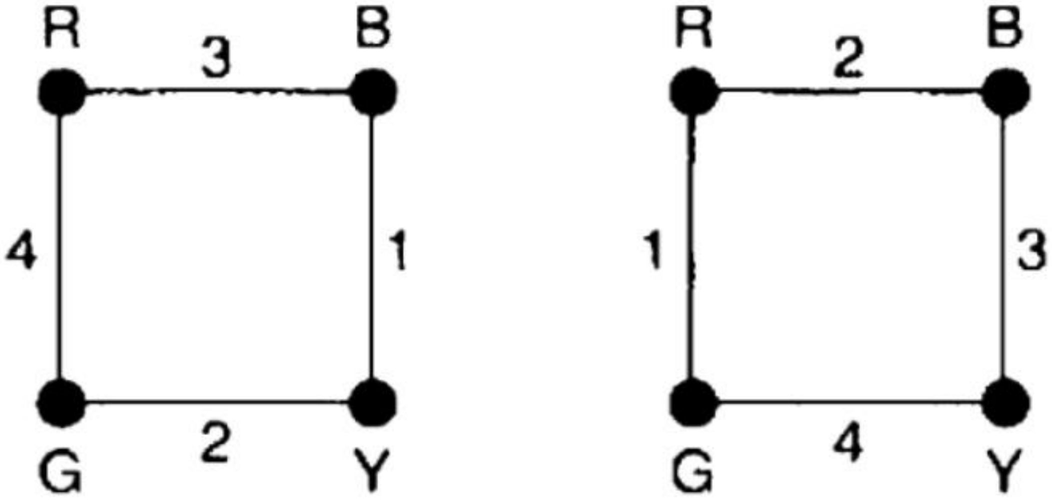


The four-cubes problem

- Objective:
  - Find two subgraphs H_1 and H_2 of G that give information about the front, back, left and right faces of each cube in the stack.
- Properties of subgraphs H1 and H2:
  - Each subgraph contains exactly one edge from each cube, indicating the pairs of colours on front, back, left and right faces.
  - The subgraphs have no edges in common, indicating that front/back faces are different from the side faces.
  - Each subgraph is regular of degree 2, indicating that each colour appears exactly twice on the sides of the stack and exactly twice on the front and back faces.

The four-cubes problem

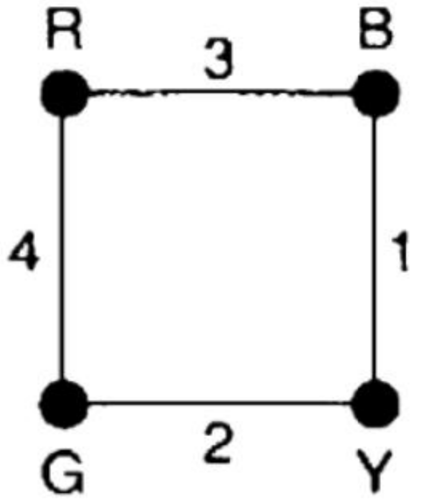




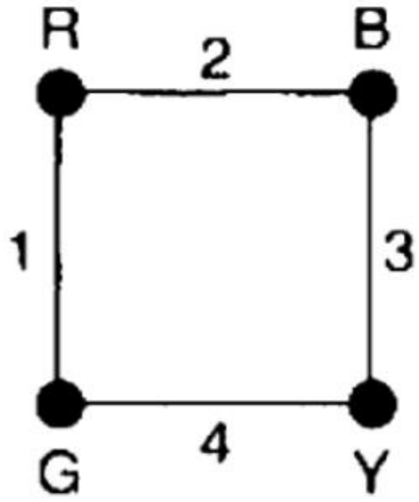
front and back
H_1

left and right
H_2

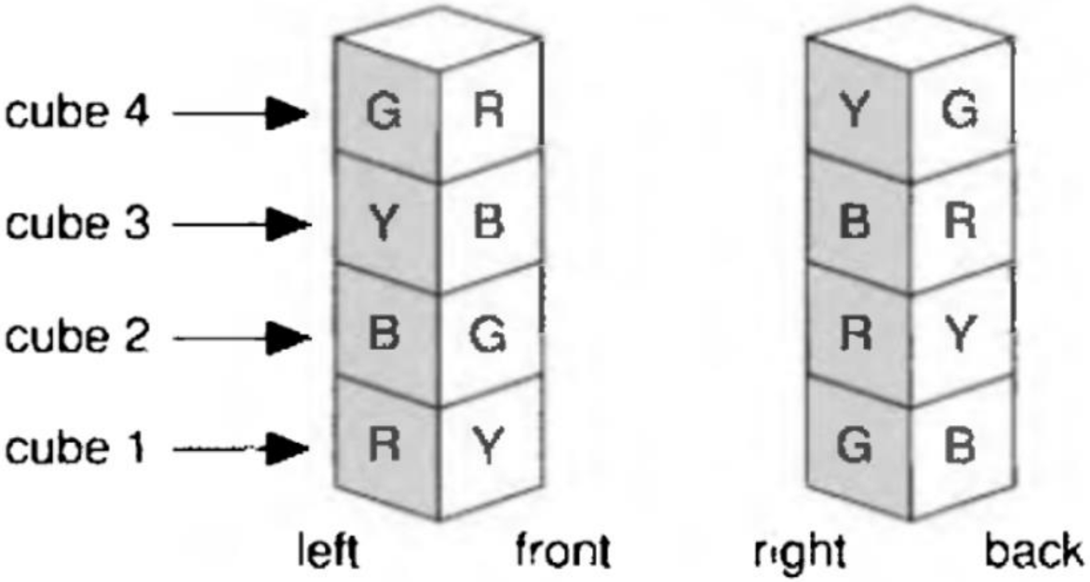
The four-cubes problem



front and back
H_1



left and right
H_2



Conclusion

- Graph theory is an abstraction of some problems that :
  - Provides a systematic approach to problem-solving.
  - Enables solving a wide range of problems, e.g., Königsberg bridges, traveling salesman, and four-color theorem.
  - Essential for developing problem-solving skills in mathematics and other fields.
  - Has significant applications in diverse areas of study.

Reference text book