Solving Linear Programming Problems

The Simplex Method

Mohammed Brahimi

ENSIA/Intelligent Systems Enginnering

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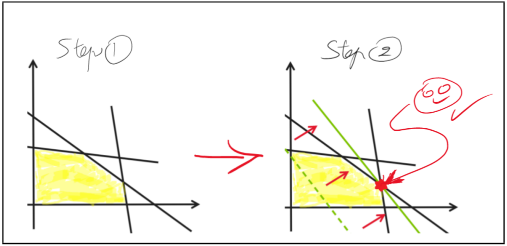
Outline

Key Takeaways from Last Lecture

- If an optimal solution exists for the LP, then at least **one of the corner points is optimal**.

Key Takeaways from Last Lecture

- If an optimal solution exists for the LP, then at least **one of the corner points is optimal**.
- The feasible region for any LP is a **convex set**.



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LP Standard Form

$$
\begin{aligned}
\textbf{Maximize} \quad & z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\
\textbf{Subject to} \quad & a_{11}x_1 + a_{21}x_2 + \cdots + a_{n1}x_n = b_1, \\
& a_{12}x_1 + a_{22}x_2 + \cdots + a_{n2}x_n = b_2, \\
& \quad \vdots \\
& a_{1m}x_1 + a_{2m}x_2 + \cdots + a_{nm}x_n = b_m, \\
& x_i \geq 0 \quad (1 \leq i \leq n)
\end{aligned}
$$

But ...

Why this LP Standard Form ?!

LP Standard Form

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\end{array}
$$

Because ...

To solve a system of equations.

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How to convert to the LP Standard Form

- Introduce non-negative slack and excess variable variables, $S_i \geq 0$, to transform the inequality constraints into equality constraints.
  - $\sum_{i=1}^{n} a_{ij}x_i \leq b_j$, add a slack variable $S_j \geq 0$ such that $\left(\sum_{i=1}^{n} a_{ij}x_i\right) + S_j = b_j$.
  - $\sum_{i=1}^{n} a_{ij}x_i \geq b_j$, add a surplus variable $S_j \geq 0$ such that $\left(\sum_{i=1}^{n} a_{ij}x_i\right) - S_j = b_j$.

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- When the Slack/Surplus/Original variable is null, it means that a constraint is satisfied with equality (Binding constraint).

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- When the Slack/Surplus/Original variable is null, it means that a constraint is satisfied with equality (Binding constraint).
- Introduce two non-negative artificial variables, $S_i^+ \geq 0$ and $S_i^- \geq 0$, to transform unrestricted variable x_i (no sign restriction).
  - $x_i = S_i^+ - S_i^-$

How to convert to the LP Standard Form

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- When the Slack/Surplus/Original variable is null, it means that a constraint is satisfied with equality (Binding constraint).
- Introduce two non-negative artificial variables, $S_i^+ \geq 0$ and $S_i^- \geq 0$, to transform unrestricted variable x_i (no sign restriction).
  - $x_i = S_i^+ - S_i^-$
- All the original variables x_i and the added variables S_i are positive.

LP Standard Form

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\end{aligned}
$$

But ...

More variables than equations ($m \ll n$).

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& x_i \geq 0 \quad (1 \leq i \leq n)
\end{aligned}
$$

But ...

Take *m* variables and solve the system.

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Solving LP problem: A Naive Algorithm

- Generate all feasible corner points
  - Determine all the intersection points between constraints.
  - Test whether it is feasible.

- Use the objective function to determine which corner point is the optimal solution.

But ...

What is the **number of intersection points** for a LP problem with *m* constraints and *n* variables?

Solving LP problem: A Naive Algorithm

- Generate all feasible corner points
  - Determine all the intersection points between constraints.
  - Test whether it is feasible.

- Use the objective function to determine which corner point is the optimal solution.

But ...

$$\binom{n}{m} = \frac{n!}{m!(n-m)!}$$

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Solving LP problem: A Naive Algorithm

- Generate all feasible corner points
  - Determine all the intersection points between constraints.
  - Test whether it is feasible.
- Use the objective function to determine which corner point is the optimal solution.

But ...

$$\binom{30}{10} = 30045015 \text{ !!!}$$

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Basic and Nonbasic Variables

- A **Basic Solution** is obtained by setting $(n - m)$ variables to 0 and solving the remaining system of m variables.
- The remaining variables are the **Basic variables**, and the removed ones are the **Non-basic variables**.
- The non basic variables set to zero represent **the fully satisfied constraints.**
- A **Basic Feasible Solution (BFS)** is a basic solution that satisfies all of the constraints.

Key Idea

A corner point in the feasible region of an LP is a Basic Feasible Solution (BFS).

Basic Variables **Nonbasic Variables**

$$x_1 = b_1 + \sum_{i=m+1}^{n} a_{1i}x_i$$

$$\vdots$$

$$x_m = b_m + \sum_{i=m+1}^{n} a_{mi}x_i$$

Basic Variables **Nonbasic Variables**

$$x_1 = b_1 + \sum_{i=m+1}^{n} a_{1i}x_i$$

$$\vdots$$

$$x_m = b_m + \sum_{i=m+1}^{n} a_{mi}x_i$$

Assign to 0

Basic Variables **Nonbasic Variables**

Assign to b's

$$x_1 = b_1 + \sum_{i=m+1}^{n} a_{1i} x_i$$

$$\vdots$$

$$x_m = b_m + \sum_{i=m+1}^{n} a_{mi} x_i$$

Assign to 0

Example: Basic Feasible solutions (BFS's)

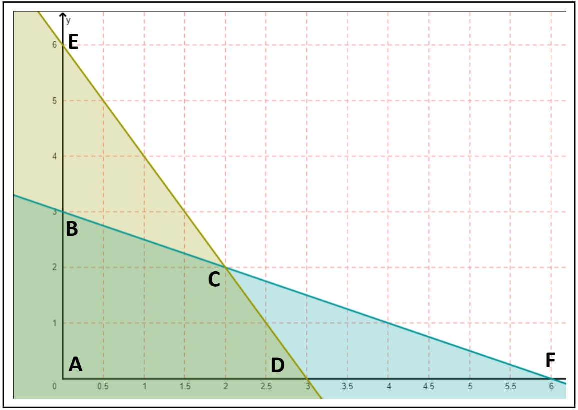
$$\begin{aligned} \textbf{Maximize} \quad & x + y \\ \textbf{subject to} \quad & x + 2y \leq 6 \\ & 2x + y \leq 6 \end{aligned}$$

Example: Basic Feasible Solutions (BFS's)

Maximize $z = x + y$

subject to $x + 2y \leq 6$

$2x + y \leq 6$



Example: Basic Feasible Solutions (BFS's)

Maximize $z = x + y$

subject to

$$x + 2y + S_1 = 6$$
$$2x + y + S_2 = 6$$
$$x, y, S_1, S_2 \geq 0$$

- Variables $\{x, y, S_1, S_2\}$

Example: Basic Feasible Solutions (BFS's)

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\textbf{Maximize} \quad & z = x + y \\
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& x, y, S_1, S_2 \geq 0
\end{aligned}
$$

- Variables $\{x, y, S_1, S_2\}$
- Number of variables $n = 4$
- Number of constraints $m = 2$

Example: Basic Feasible Solutions (BFS's)

Maximize $z = x + y$

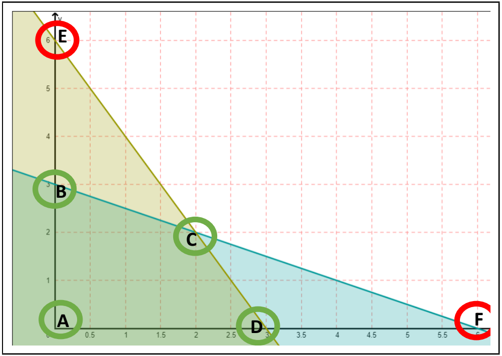
subject to $x + 2y + S_1 = 6$

$2x + y + S_2 = 6$

$x, y, S_1, S_2 \geq 0$

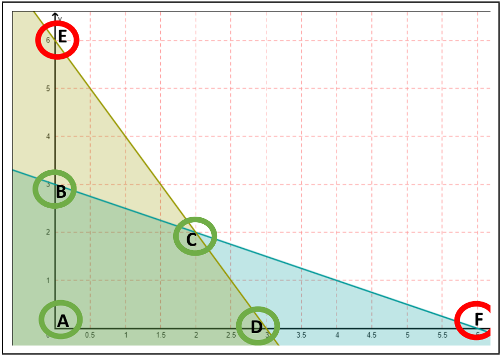
- Variables $\{x, y, S_1, S_2\}$
- Number of variables $n = 4$
- Number of constraints $m = 2$
- Number of Basic solutions $\binom{n}{m}$
- $\binom{4}{2} = 6$ Basic Solutions

Example: Basic Feasible Solutions (BFS's)



- Basic solutions
  - A, B, C, D, E, F
- Feasible Basic Solutions
  - A, B, C, D

Example: Basic Feasible Solutions (BFS's)



- Basic solutions
  - A, B, C, D, E, F
- Feasible Basic Solutions
  - A, B, C, D
- **How to find these FBS's ?!**

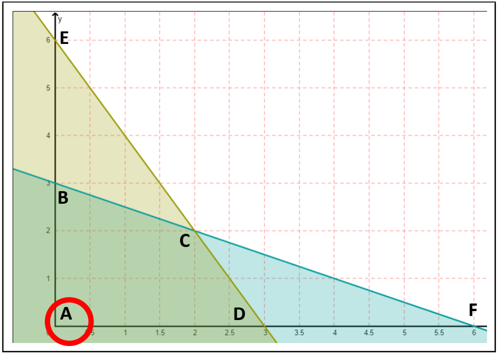
Example: BFS (S_1, S_2) - A -

Maximize $z = x + y$

subject to $S_1 = 6 - x - 2y$

$S_2 = 6 - 2x - y$

$x, y, S_1, S_2 \geq 0$



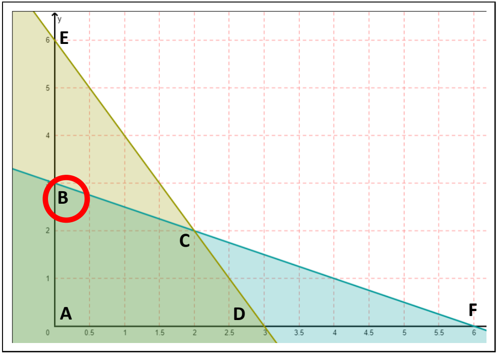
Example: BFS (y, S_2)- B -

Maximize $$z = 3 + \frac{1}{2}x - \frac{1}{2}S_1$$

subject to $$y = 3 - \frac{1}{2}x - \frac{1}{2}S_1$$

$$S_2 = 3 - \frac{3}{2}x + \frac{1}{2}S_1$$

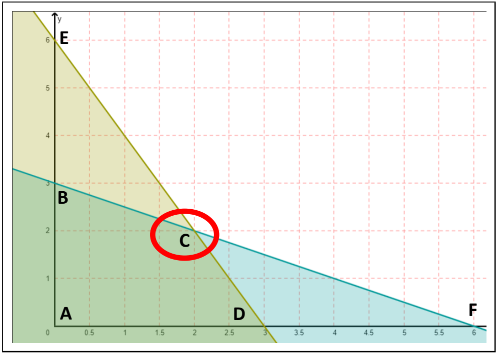
$$x, y, S_1, S_2 \geq 0$$



Example: BFS (x, y)- C -

Maximize $$z = 4 - \frac{1}{3}S_1 - \frac{1}{3}S_2$$

subject to
$$x = 2 + \frac{1}{3}S_1 - \frac{2}{3}S_2$$
$$y = 2 - \frac{2}{3}S_1 + \frac{1}{3}S_2$$
$$x, y, S_1, S_2 \geq 0$$



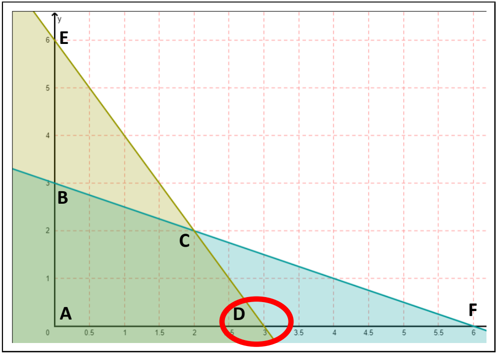
Example: BFS (x, S_1)- D -

Maximize $z = 3 + \frac{1}{2}y - \frac{1}{2}S_2$

subject to $x = 3 - \frac{1}{2}y - \frac{1}{2}S_2$

$S_1 = 3 - \frac{3}{2}y + \frac{1}{2}S_2$

$x, y, S_1, S_2 \geq 0$

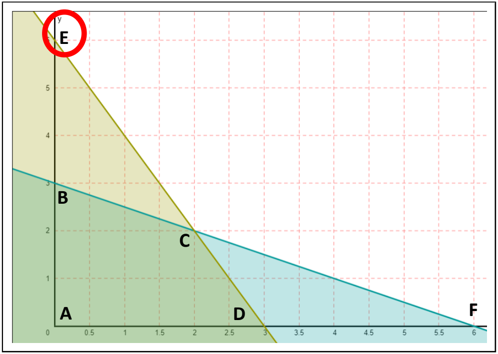


Example: Infeasible solution (y, S_1)- E -

Maximize $z = 6 - x - s_2$

subject to

$$y = 6 - 2x - S_2$$
$$S_1 = -6 + 3x + 2S_2$$
$$x, y, S_1, S_2 \geq 0$$



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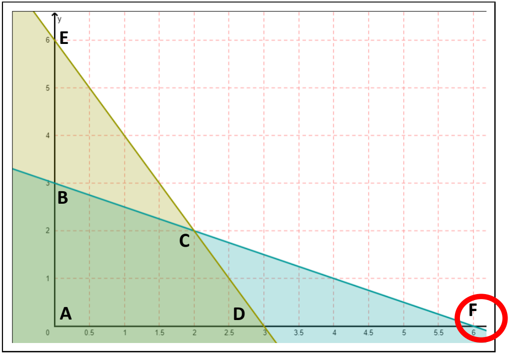
Example: Infeasible solution (x, S_2)- F -

Maximize $z = 6 - y - s_1$

subject to $x = 6 - 2y - S_1$

$S_2 = -6 - 3y + 2S_1$

$x, y, S_1, S_2 \geq 0$



Example: Summary

Nonbasic (zero) variables	Basic variables	Basic solution	Corner	Feasible ?	Objective
x,y	S_1, S_2	(0,0)	A	Yes	0
x, S_1	y, S_2	(0,3)	B	Yes	3
S_1, S_2	x, y	(2,2)	C	Yes	**4 (0ptimum)**
y, S_2	x, S_1	(3,0)	D	Yes	3
x, S_2	y, S_1	(0,6)	E	No	$\cancel{6}$
y, S_1	x, S_2	(6,0)	F	No	$\cancel{6}$

State 1: Basic Feasible Solution

Maximize $$z = 3 + \frac{1}{2}x - \frac{1}{2}s_1$$

subject to $$y = 3 - \frac{1}{2}x - \frac{1}{2}S_1$$

$$S_2 = 3 - \frac{3}{2}x + \frac{1}{2}S_1$$

$$x, y, S_1, S_2 \geq 0$$

State 1: Basic Feasible Solution

$$\textbf{Maximize} \quad z = 3 + \frac{1}{2}x - \frac{1}{2}s_1$$

$$\textbf{subject to} \quad y = \mathbf{3} - \frac{1}{2}x - \frac{1}{2}S_1$$

$$S_2 = \mathbf{3} - \frac{3}{2}x + \frac{1}{2}S_1$$

$$x, y, S_1, S_2 \geq 0$$

State 2:Infeasible Solution

$$\begin{aligned} \textbf{Maximize} \quad & z = 6 - x - s_2 \\ \textbf{subject to} \quad & y = 6 - 2x - S_2 \\ & S_1 = -6 + 3x + 2S_2 \\ & x, y, S_1, S_2 \geq 0 \end{aligned}$$

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State 2:Infeasible Solution

$$\begin{aligned} \textbf{Maximize} \quad & z = 6 - x - s_2 \\ \textbf{subject to} \quad & y = 6 - 2x - S_2 \\ & S_1 = \textbf{-6} + 3x + 2S_2 \\ & x, y, S_1, S_2 \geq 0 \end{aligned}$$

State 3:Optimal Basic Feasible Solution

Maximize $z = 4 - \frac{1}{3}S_1 - \frac{1}{3}S_2$

subject to

$$x = 2 + \frac{1}{3}S_1 - \frac{2}{3}S_2$$

$$y = 2 - \frac{2}{3}S_1 + \frac{1}{3}S_2$$

$$x, y, S_1, S_2 \geq 0$$

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State 3:Optimal Basic Feasible Solution

Maximize $z = 4 - \frac{1}{3}S_1 - \frac{1}{3}S_2$

subject to

$$x = 2 + \frac{1}{3}S_1 - \frac{2}{3}S_2$$

$$y = 2 - \frac{2}{3}S_1 + \frac{1}{3}S_2$$

$$x, y, S_1, S_2 \geq 0$$

Simplex Algorithm

Maximize $z = b_0 + \sum_{i=m+1}^{n} c_i x_i$

subject to $x_1 = b_1 + \sum_{i=m+1}^{n} a_{1i} x_i$

$$\vdots$$

$$x_m = b_m + \sum_{i=m+1}^{n} a_{mi} x_i$$

- Convert to the standard form
- While (The BFS is not optimal)
  - Move to a new Basic Feasible Solution
- Output the optimal solution.

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Simplex Algorithm

Maximize $z = b_0 + \sum_{i=m+1}^{n} c_i x_i$

subject to $x_1 = b_1 + \sum_{i=m+1}^{n} a_{1i} x_i$

$\vdots$

$x_m = b_m + \sum_{i=m+1}^{n} a_{mi} x_i$

- While $(\exists C_i > 0)$
  - Select a non-basic variable with a positive coefficient: **Entering variable**
  - Introduce this variable in the basis by removing a basic variable: **Leaving variable**
  - Perform Gaussian elimination
- Output the optimal solution.

Simplex Algorithm

```
Initialize a feasible basis and corresponding basic feasible solution.
while the objective function has positive coefficients do
    Choose a non-basic variable with a positive coefficient as the entering variable.
    Choose a basic variable to leave the basis using the minimum ratio test.
    Update the basis by replacing the leaving variable with the entering variable.
    Recalculate the basic feasible solution.
end while
Output the optimal solution.
```

The minimum ratio test

$$
\begin{aligned}
\textbf{Maximize} \quad & z = x + y \\
\textbf{subject to} \quad & S_1 = 6 - x - 2y \\
& S_2 = 6 - 2x - y \\
& x, y, S_1, S_2 \geq 0
\end{aligned}
$$

How to choose the leaving variable?

- We must maintain feasibility.

The minimum ratio test

$$\begin{aligned} \textbf{Maximize} \quad & z = x + y \\ \textbf{subject to} \quad & S_1 = 6 - x - 2y \\ & S_2 = 6 - 2x - y \\ & x, y, S_1, S_2 \geq 0 \end{aligned}$$

Chose the leaving variable based on the ration $\frac{b_i}{-a_i} \quad (a_i \leq 0)$

- x is the Entering variable (y also can be selected).
- S_2 is the Leaving variable.
  - Ratio(S_1) = $\frac{6}{1} = 6$.
  - Ratio(S_2)=$\frac{6}{2} = 3$.
- We only consider constraints with Negative coefficients ($a_i \leq 0$) for the entering variable, so the ratio is necessarily positive.

The minimum ratio test

$$\begin{aligned} \textbf{Maximize} \quad & z = x + y \\ \textbf{subject to} \quad & S_1 = 6 - x - 2y \\ & S_2 = 6 - 2x - y \\ & x, y, S_1, S_2 \geq 0 \end{aligned}$$

Old BFS is

- Basic Variables = $\{S_1, S_2\}$
- Non Basic variables = $\{x, y\}$

New BFS is

- Basic Variables = $\{x, S_1\}$
- Non Basic variables = $\{y, s_2\}$

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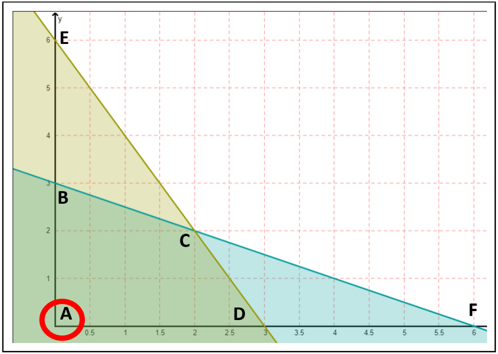
Example: BFS (S_1, S_2) - A -

Maximize $z = x + y$

subject to
$$S_1 = 6 - x - 2y$$
$$S_2 = 6 - 2x - y$$
$$x, y, S_1, S_2 \geq 0$$



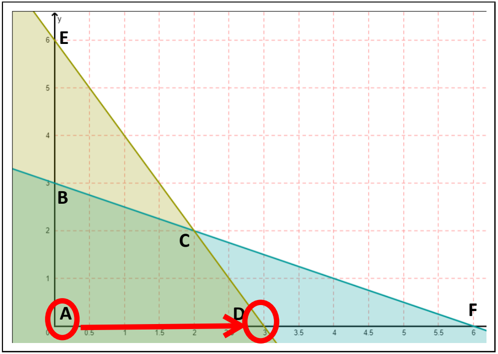
Example: BFS (x, S_1)- A $\rightarrow$ D -

Maximize $$z = 3 + \frac{1}{2}y - \frac{1}{2}S_2$$

subject to $$x = 3 - \frac{1}{2}y - \frac{1}{2}S_2$$

$$S_1 = 3 - \frac{3}{2}y + \frac{1}{2}S_2$$

$$x, y, S_1, S_2 \geq 0$$



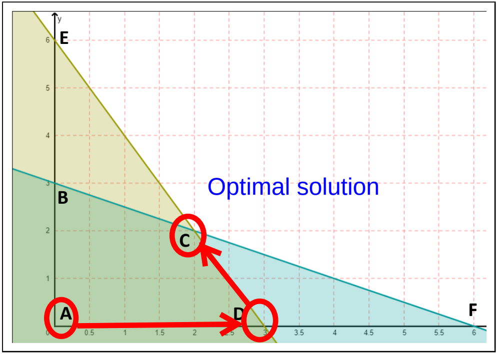
Example: BFS (x, y)- A $\rightarrow$ D $\rightarrow$ C -

Maximize $$z = 4 - \frac{1}{3}S_1 - \frac{1}{3}S_2$$

subject to $$x = 2 + \frac{1}{3}S_1 - \frac{2}{3}S_2$$

$$y = 2 - \frac{2}{3}S_1 + \frac{1}{3}S_2$$

$$x, y, S_1, S_2 \geq 0$$



Simplex Tableau Method

Maximize	$z = -(-x - y)$
subject to	$S_1 + x + 2y = 6$
	$S_2 + 2x + y = 6$
	$x, y, S_1, S_2 \geq 0$

Base	z	x	y	S_1	S_2	b
	1	-1	-1	0	0	0
S_1	0	1	2	**1**	0	6
S_2	0	2	1	0	**1**	6

- The simplex should be updated to consider the sign modifications (optimality test, how to choose leaving/entering variables, minimum ratio test).

Simplex Tableau Method

Base	z	x	y	S1	S2	b	**Ratio**
	1	-1	-1	0	0	0	
S1	0	1	2	**1**	0	6	$\frac{6}{1} = 6$
S2	0	2	1	0	**1**	6	$\frac{6}{3} = 2$

Pivot

- **Entering variable:** x
- **Leaving variable:** S_2

Simplex Tableau Method

Base	z	x	y	S1	S2	b	**Ratio**
	1	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	3	
S_1	0	0	$\frac{3}{2}$	**1**	$-\frac{1}{2}$	3	$3 \times \frac{2}{3} = 2$
x	0	**1**	$\frac{1}{2}$	0	$\frac{1}{2}$	3	$3 \times \frac{2}{1} = 6$

Pivot

- **Entering variable:** y
- **Leaving variable:** S_1

Simplex Tableau Method

Base	z	x	y	S1	S2	b	**Ratio**
	1	0	0	$\frac{1}{3}$	$\frac{1}{3}$	4	
y	0	0	**1**	$\frac{2}{3}$	$-\frac{1}{3}$	2	
x	0	**1**	0	$-\frac{1}{3}$	$\frac{2}{3}$	2	

Optimality test satisfied

- All the objective coefficients are positive.
- **Optimal Basic Feasible Solution detected**

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Simplex: Special cases

- **Unboundedness:** occurs when the objective function can be increased indefinitely without violating any of the constraints.
- **Infeasibility:** occurs when there is no feasible solution that satisfies all of the constraints.
- **Degeneracy:** occurs when one or more basic variables become zero during the iteration process.

Conclusion

- Simplex method is an efficient algorithm for finding optimal solutions to LP problems by navigating through the corner points of the feasible region.
- It iteratively moves from one Basic Feasible Solution (BFS) to a better neighborhood BFS until the optimal BFS is reached.
- By detecting the optimal BFS, the simplex method provides the optimal values of the decision variables and the objective function.