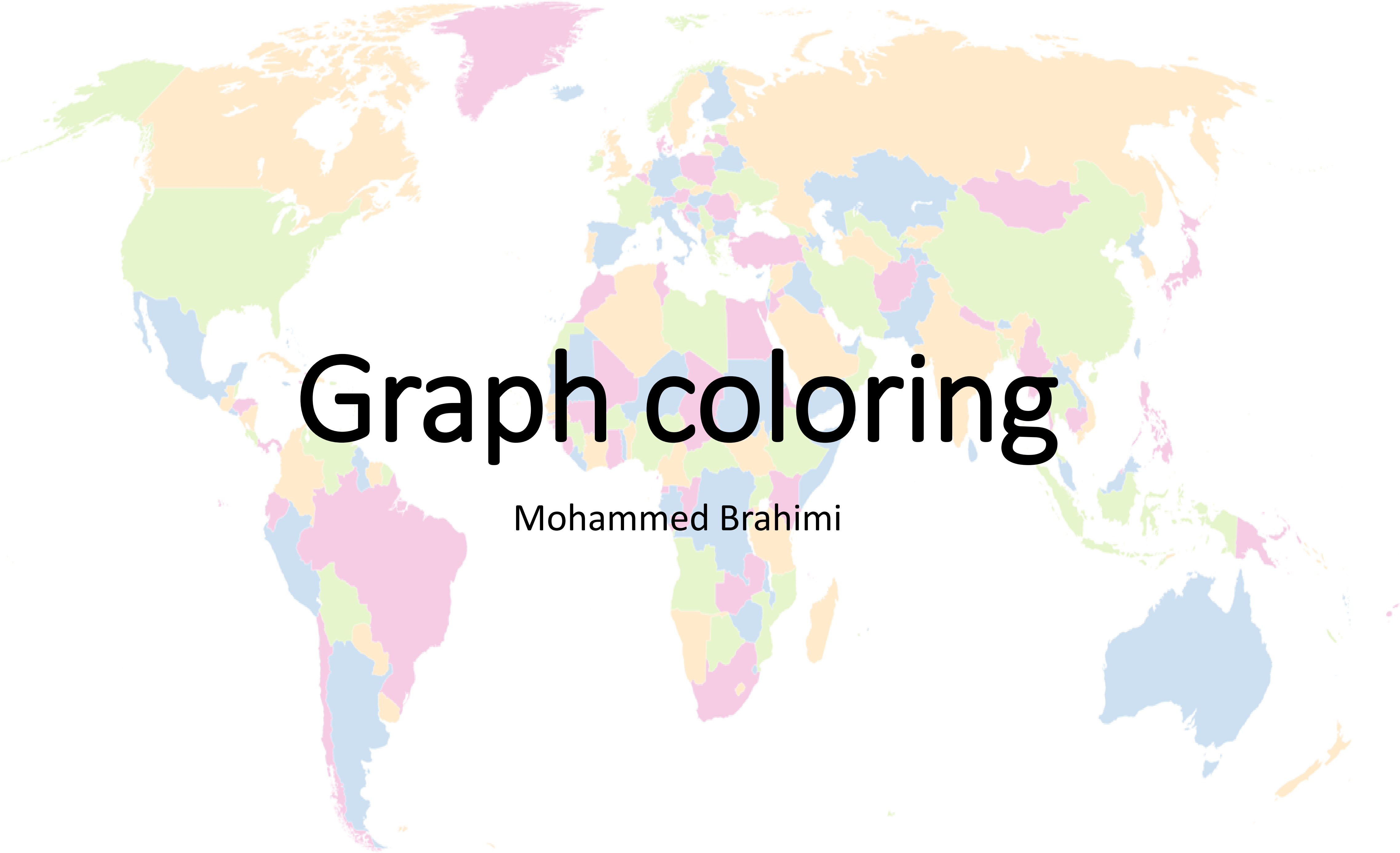

Graph coloring

Mohammed Brahimi

Module distribution concerns raised by students

- Issues
  - Multiple difficult modules scheduled in the same term
  - Unbalanced distribution of workload between terms
- Goal
  - Achieving a balanced load distribution between terms
- Approach:
  - Represent modules as vertices
  - High-load modules are linked by edges
  - Assign colors to terms
  - Ensure that difficult and high-load modules are not scheduled in the same term.

Find optimal module-term assignment to minimize load imbalance and avoid clustering high-load modules.

How to avoid challenging modules in the same term?

		1	2	3	4	5	6	7	8
Analysis 3	1			X		X		X	
Data structure	2				X	X			
Mathematical Logic	3	X					X		
Architecture 1	4		X						X
Complexity	5	X	X				X		
Linear Algebra 2	6			X		X			
Probability and Statistics	7	X							X
Databases	8				X			X	

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		1	2	3	4	5	6	7	8
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Architecture 1	4		X						X
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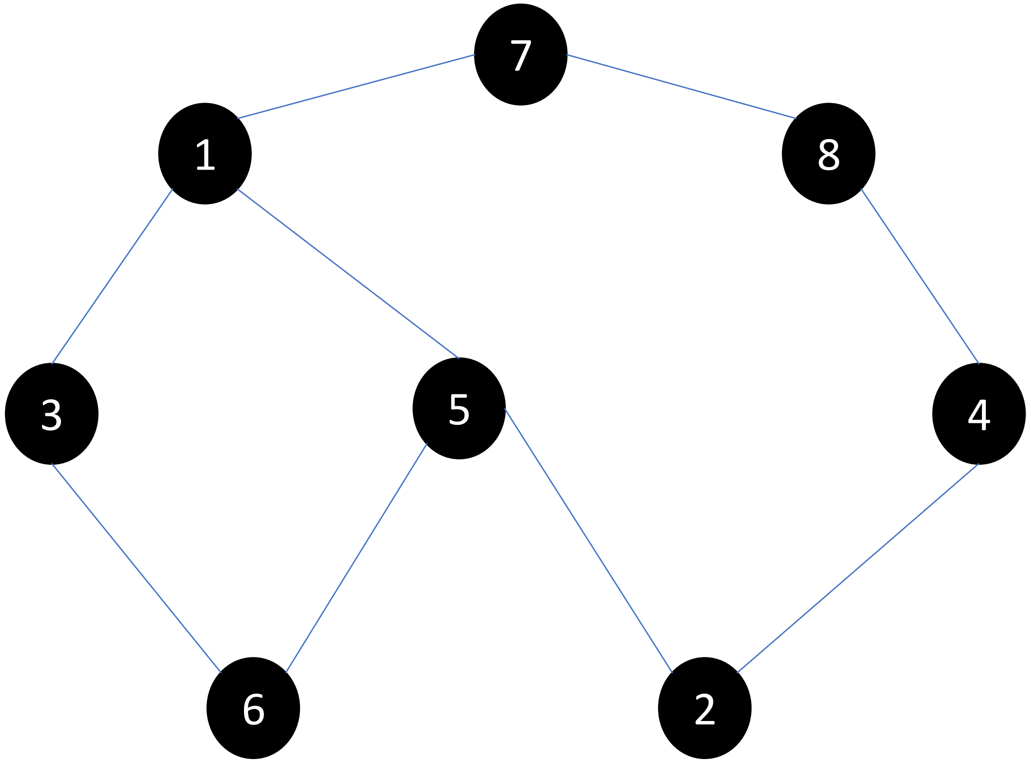
Linear Algebra 2	6			X		X			
Probability and Statistics	7	X							X
Databases	8				X			X	



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Data structure	2				X	X			
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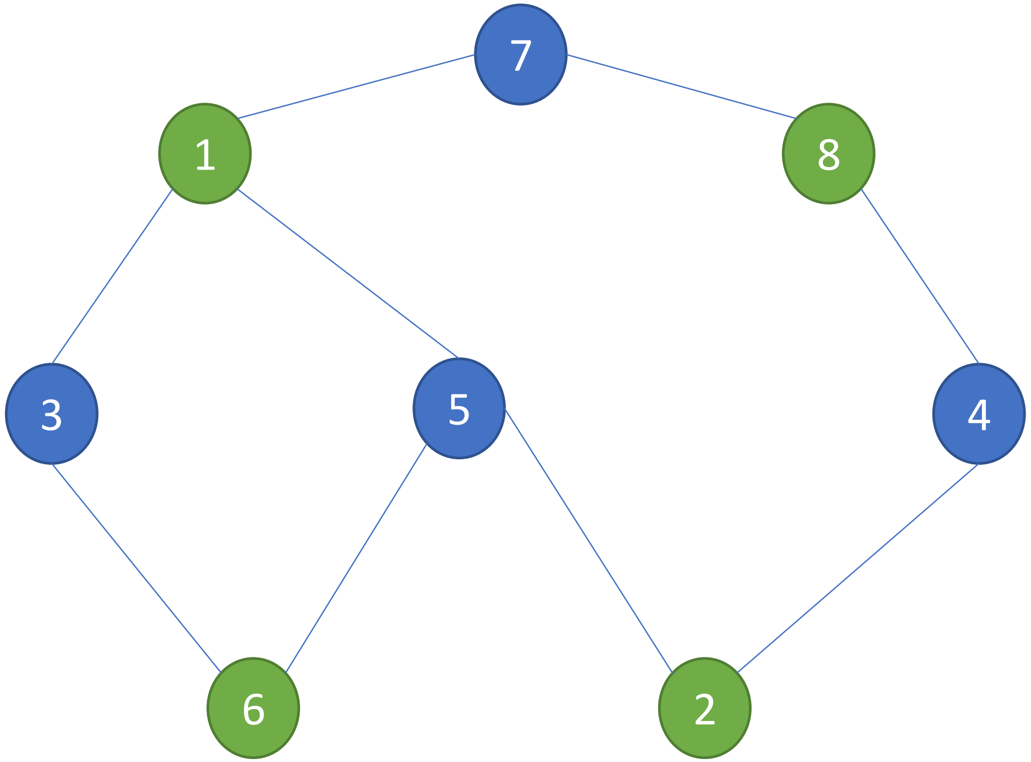
Architecture 1	4		X						X
Complexity	5	X	X				X		
Linear Algebra 2	6			X		X			
Probability and Statistics	7	X							X
Databases	8				X			X	



How to avoid challenging modules in the same term?

		1	2	3	4	5	6	7	8
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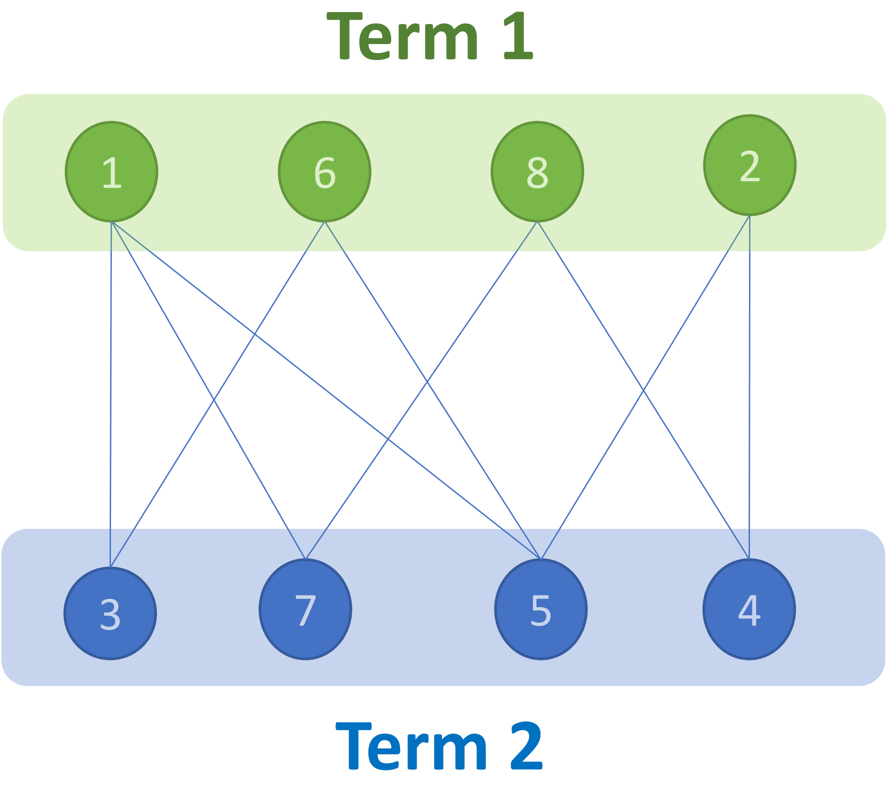
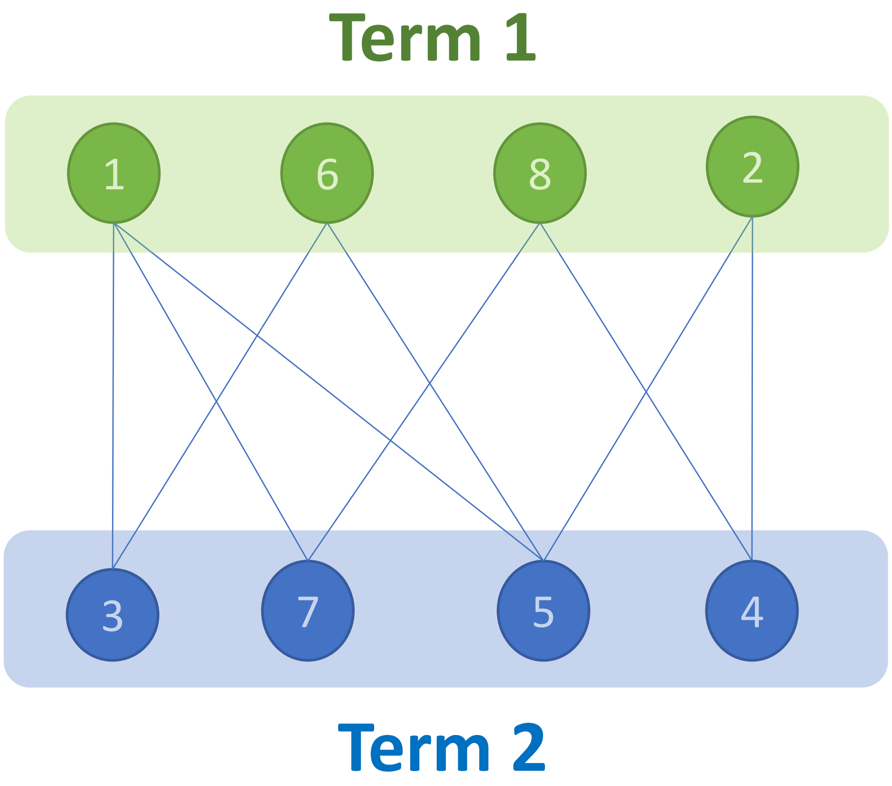
Mathematical Logic	3	X					X		
Architecture 1	4		X						X
Complexity	5	X	X				X		
Linear Algebra 2	6			X		X			
Probability and Statistics	7	X							X
Databases	8				X			X	



How to avoid challenging modules in the same term?

- **Term 1**
  - Data structure
  - Linear Algebra 2
  - Databases
  - Analysis 3
- **Term 2**
  - Architecture 1
  - Mathematical Logic
  - Complexity
  - Probability and Statistics



Problem extension

- What if a teacher teach two modules together:
  - Assign a unique color to each teacher.
  - Modules taught by the same teacher are adjacent vertices in the graph.
  - Find a suitable coloring for the graph representing the modules and teachers.

- What about other time table constraints:
  - Sessions represented as vertices.
  - Make two vertices adjacent based on any constraints.
  - Ensure that conflicting sessions, are not scheduled in parallel.

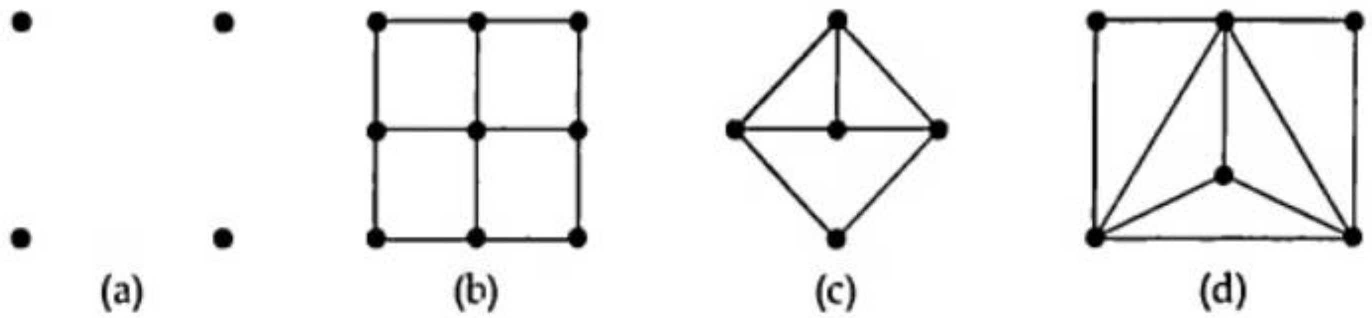
Create a well-structured time table that considers teacher availability, room constraints, and avoids conflicts between modules.

What is graph coloring ?

- In the context of a simple graph G, a **k-coloring** refers to the assignment of at most k colors to its vertices.
- **k-coloring** should color the adjacent vertices in G by different colors.
- If G has a valid **k-coloring**, we say G is **k-colorable**.
- The **chromatic number** of G, denoted by $\chi(G)$, is the smallest number k for which G is k-colorable.

- Reason for restricting coloring to simple graphs:
  - Vertex with loop cannot be assigned a different color from itself.
  - The presence of one or multiple edges between two vertices has a similar effect, requiring them to be colored differently.

What is the **chromatic number $\chi(G)$** ?



a) $\chi(G) = 1$
- One color is sufficient
- Absence of edges

b) $\chi(G) = 2$
- +1 color is required (presence of edges)
- It exists 2-coloring.

a) $\chi(G) = 3$
- + 3 colors are required (presence of triangle)
- It exists 3-coloring.

b) $\chi(G) = 4$
- +4 colors are required (presence of k_4)
- It exists 4-coloring.

Subgraphs and the Chromatic Number

- Subgraphs play a crucial role in understanding the chromatic number of a graph.
- If a subgraph H of G has a chromatic number χ(H), then χ(G) must also be at least that value.
- The presence of complete subgraphs (cliques) can increase the chromatic number of a graph.

The subgraphs with the highest chromatic numbers can provide insights into the chromatic number of the entire graph.

THEORME 1 (coloring bipartite graph)

A graph is bipartite if and only if it is $\chi(G)$=2

- Direction 1: Bipartite graph ⟹ Chromatic number is 2.
  - In a bipartite graph, vertices can be split into two sets, A and B, where all edges connect vertices from different sets.
    - Assign **color 1 to set A**
    - Assign color 2 **to set B**.
  - This yields a valid 2-coloring.
  - Thus, the chromatic number of a bipartite graph is 2.

- Direction 2: Chromatic number is 2 ⟹ Graph is bipartite.
  - If a graph can be colored with only two colors:
    - Put all the vertices of color 1 to the set A
    - Put all the vertices of color 2 to the set B
  - Two vertices from A (or B) are not adjacent because they have the same color.
  - This results in a partition of the graph into sets A and B where the vertices of each set are not adjacent.

What is the chromatic number ?

- Complete graph K_n
  - All the vertices are connected
  - $\boldsymbol{\chi(G)}$=n
- Path graph P_n(n>1)
  - Alternate the colors
  - $\boldsymbol{\chi(G)}$=2
- Cycle graph C_n (n>2)
  - $\boldsymbol{\chi(G)}$=2, if n =2k
  - $\boldsymbol{\chi(G)}$=3, if n =2k+1
- Wheel graph W_n(n>1)
  - $\boldsymbol{\chi(G)}$ = 4, if n =2k
  - $\boldsymbol{\chi(G)}$ = 3, if n =2k+1

THEOREME 2

let G be a simple graph whose maximum vertex degree is d. Then

$$\chi(G) \leq \boldsymbol{d} + \mathbf{1}$$

- Intuition
  - If the maximum vertex degree of a graph is low, we can determine a tight upper bound for the chromatic number.
  - However, for graphs with a high maximum degree, this theorem becomes less effective or loses its usefulness.

Proof

- Proof by induction on the number of vertices n

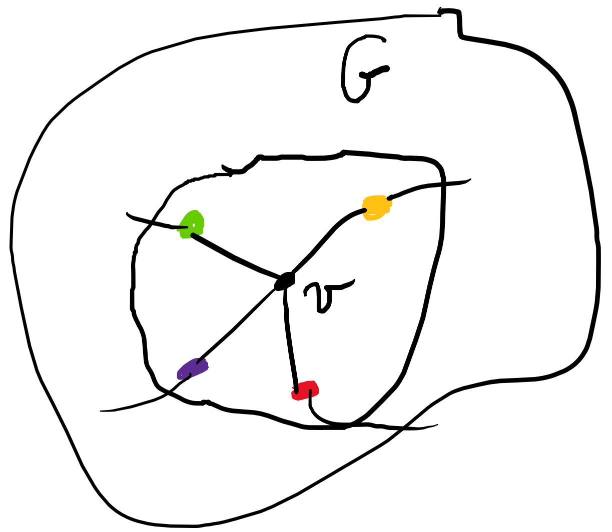
- The statement is true for K_1 the simple graph with one vertex
  - $\chi(K_1) = 1$ and $d = O$.

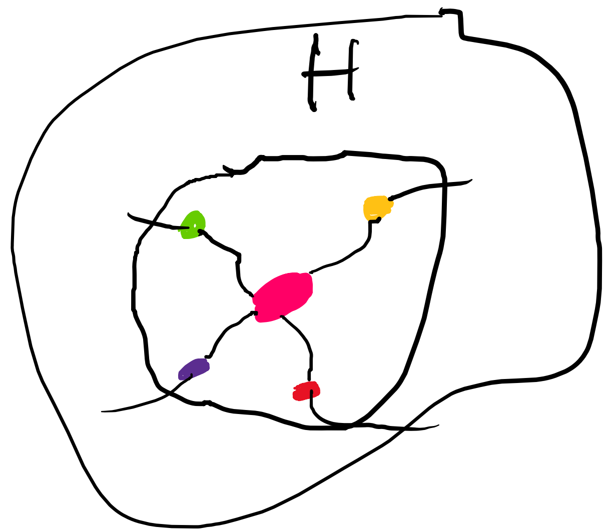
- We assume that $\chi(G) \leq d + 1$ for all simple graphs H with fewer than or equal n vertices.

- We should show that $\chi(G) \leq d + 1$ for all simple graphs G with $n + 1$ vertices.

Proof

- Let G be a simple graph with $n+1$ vertices and maximum vertex degree d.
- Let H be a graph obtained from G by removing a vertex v and its incident edges.
- H has at most n vertices and a maximum vertex degree of d or less. By our assumption, we have $\chi(H) \leq d+1$.
- **By assigning an unassigned color to v, we can create a (d+1)-coloring of G. Therefore, $\boldsymbol{\chi(G) \leq d+1}$.**
- In the worst-case scenario, if $\deg(\boldsymbol{v}) = \boldsymbol{d}$ and all vertices have unique colors, exactly one unassigned color remains.





THEOREM 3 (Brooks 1941)

*Let G be a connected simple graph whose **maximum vertex degree is d**. If G is **neither a cycle graph with an odd number of vertices, nor a complete graph**, then :*

$$\chi(G) \leq d.$$

- Intuition
  - The theorem does not works for K_n and C_{2k+1}
  - This theorem provides a stricter upper limit for the chromatic number.

Chromatic Number $\chi(G)$ and Bounds

- To determine the chromatic number $\chi(G)$ of a graph G, find an equal upper and lower bound, which becomes the chromatic number $\chi(G)$.

- Possible upper bounds for $\chi(G)$
  - Total number of vertices in G.
  - Number of colors in an explicit vertex coloring of G.
  - Maximum degree (d) in G plus one (Theorem 2).
  - Maximum degree (d) in G, if G is not an odd cycle or complete graph K_n (Brooks' theorem).

- Possible lower bound for $\chi(G)$
  - Number of vertices in the largest complete subgraph in G.

Coloring Planar Graphs

- **THEOREM 4 (**Six Colors Theorem for Planar Graphs)

  *The vertices of any simple connected planar graph G can be coloured with **six (or fewer) colours**.*

- **Intuition:**
  - For planar graphs, it is possible to color them using 6 colors or fewer.
  - Even highly complex planar graphs can be colored using a maximum of 6 colors.

Proof

- Proof by induction on the number of vertices n

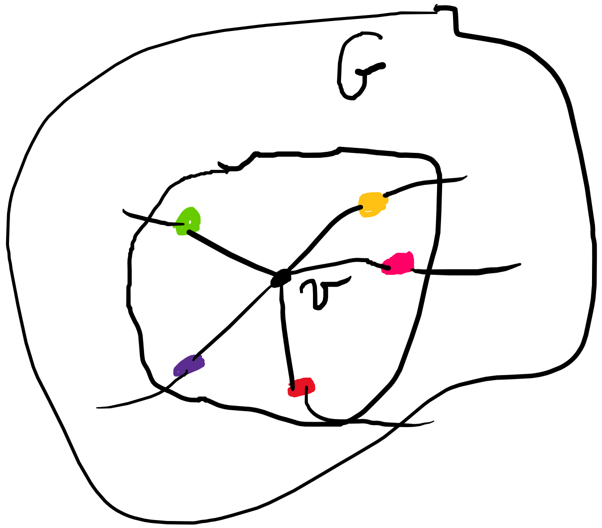
- The statement is trivially true when $n = 1$

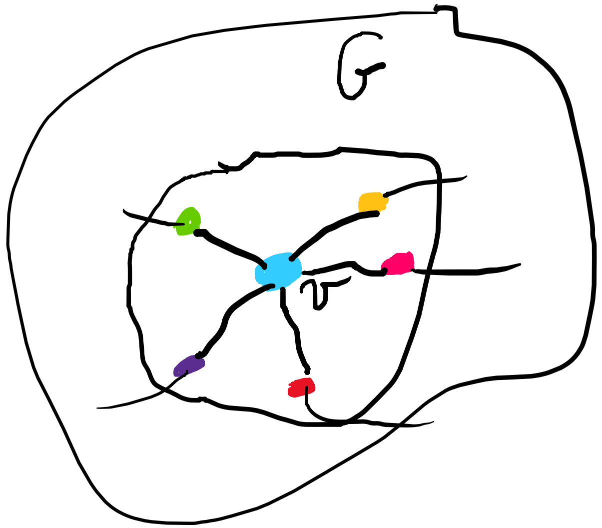
- Assuming simple connected planar graphs with fewer or equal than n vertices can be colored with 6 or fewer colors

- We aim to prove the same for simple connected planar graphs with $n + 1$ vertices.

Proof

- Let G be a simple graph with n+1 vertices
- G contains a vertex v of degree 5 or less (Previous lecture).
- Create graph H by removing vertex v and its incident edges from G.
- By our assumption, the vertices of H can be colored with 6 colors.
- Reintroduce vertex v.
  - An unassigned color is available because v has a degree ≤ 5 and there are 6 available colors.
  - We color v with this unsaigned color.
- This results in a 6-coloring of the vertices in G.





THEORME 5 (Five Colors Theorem for Planar Graphs)

The vertices of any simple connected planar graph G can be colored with five (or fewer) colors.

- **Intuition:**
  - For planar graphs, it is possible to color them using only 5 colors or fewer.
  - Even highly complex planar graphs can be colored using a maximum of 5 colors.

Proof

- Proof by induction on the number of vertices n

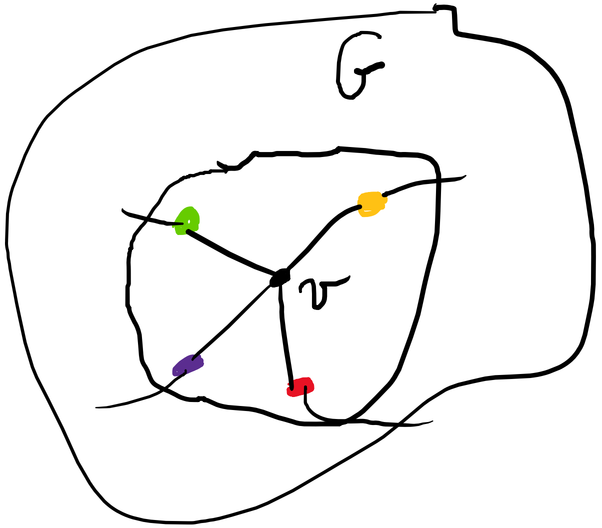
- The statement is trivially true when $n = 1$

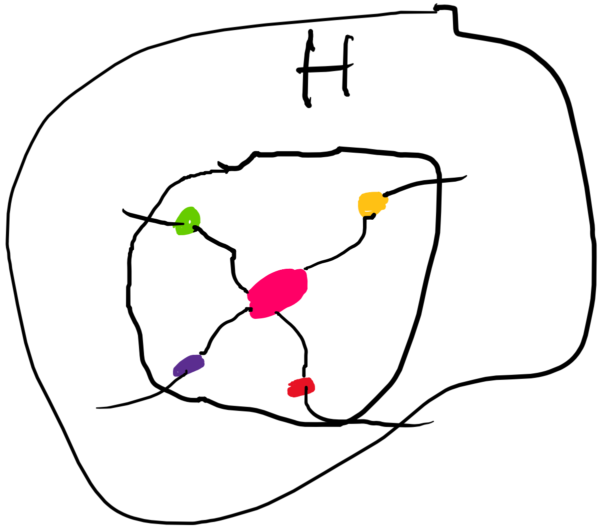
- Assuming simple connected planar graphs with fewer or equal than n vertices can be colored with 5 or fewer colors

- We aim to prove the same for simple connected planar graphs with $n + 1$ vertices.

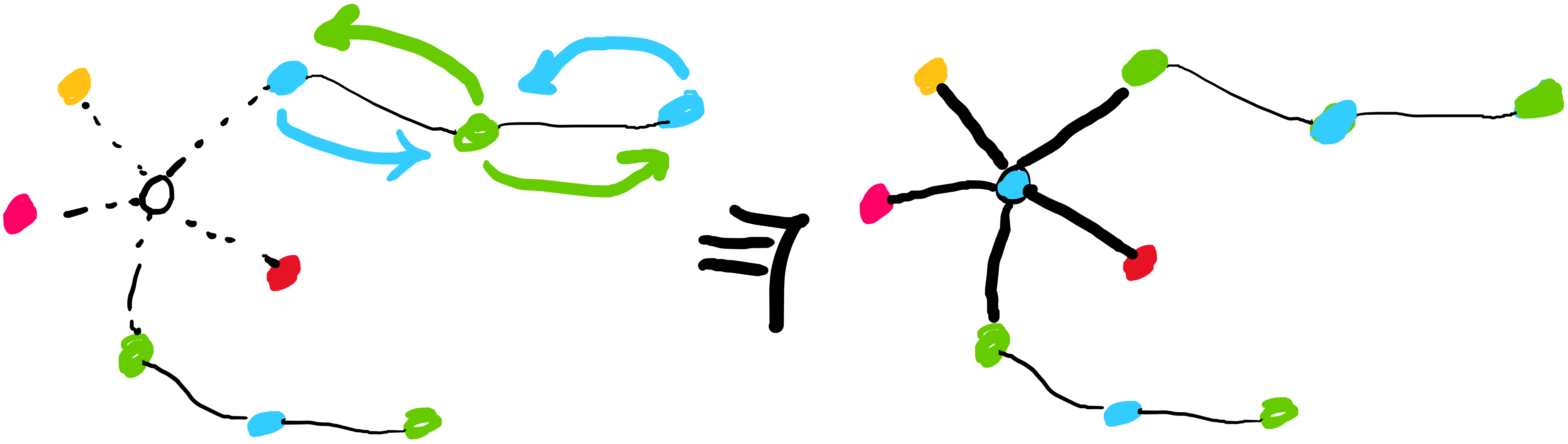
Proof

- Let G be a simple graph with n+1 vertices
- G contains a vertex v of degree 5 or less (Previous lecture).
- Create graph H by removing vertex v and its incident edges from G.
- By our assumption, the vertices of H can be colored with 5 colors.
- If we reintroduce vertex v, three cases appear
  - Case 1: If there is an unassigned color, we are done
  - Case 2: two colors are not linked path
  - Case 3: two colors are connected with a path



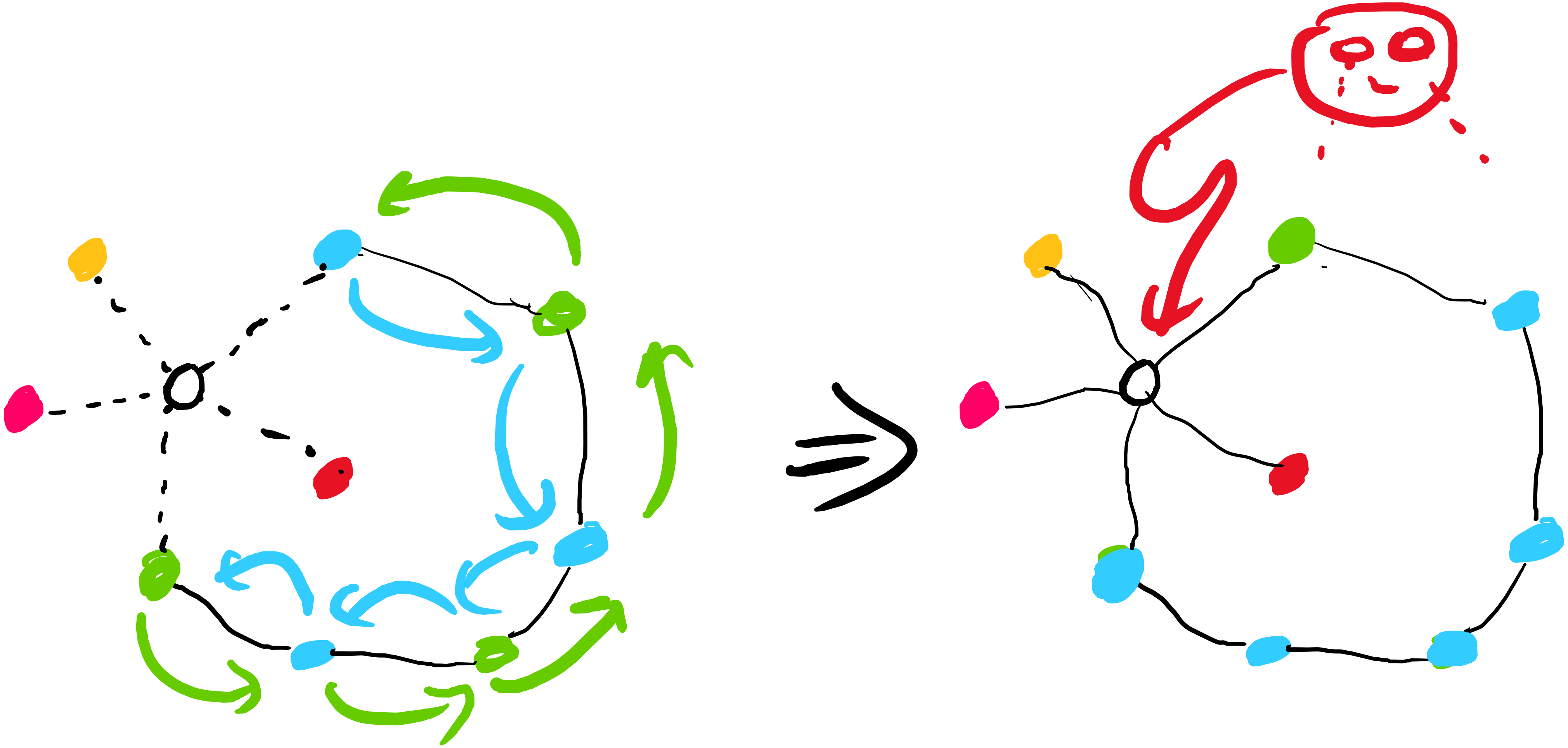


Case 2: two colors are not linked path



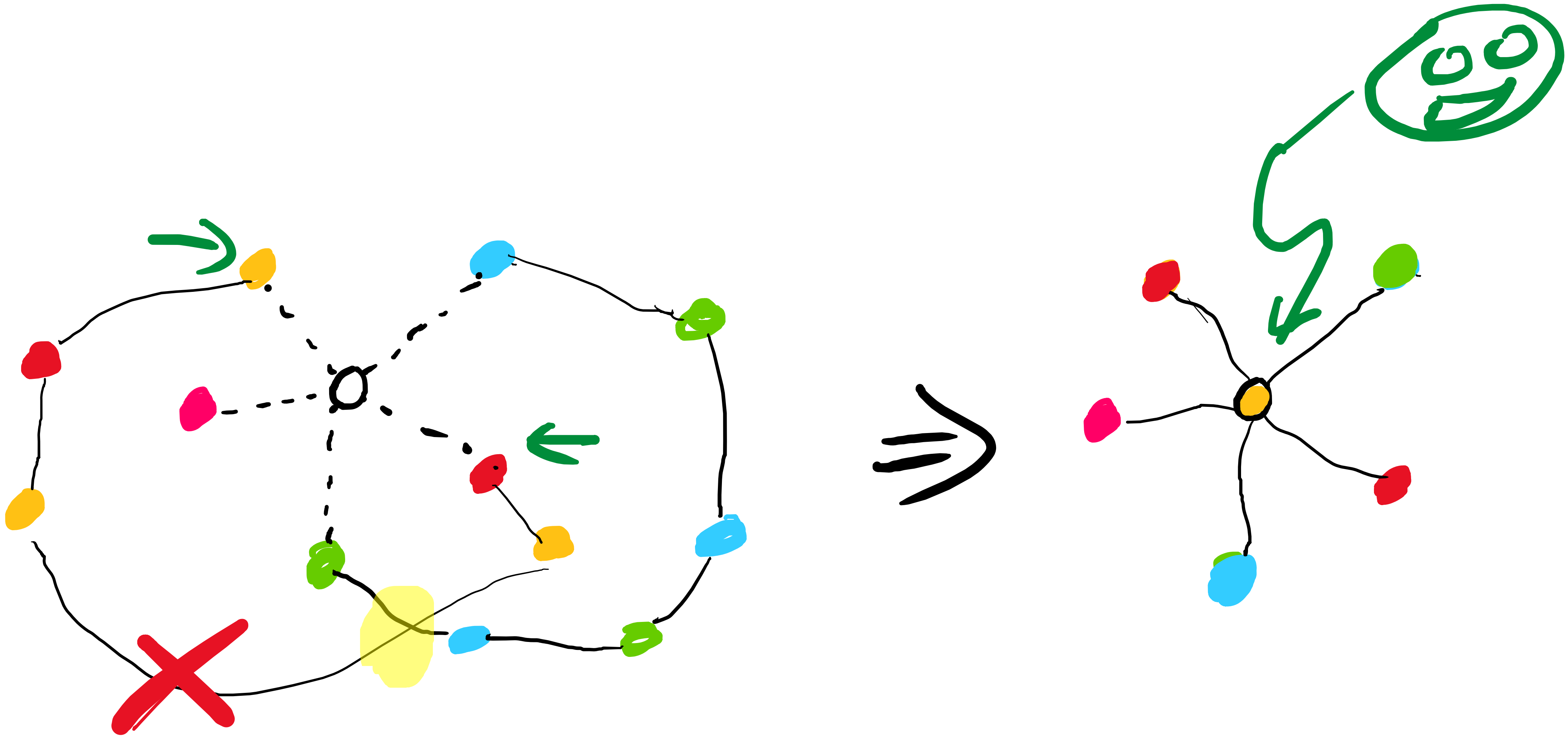
By switching one color to be the same as another color, we created an unassigned color.

Case 3: two colors are connected with a path



We simply switch two colors without creating an unassigned color.

Case 3: two colors are connected with a path



The planarity prevents the colors from forming a path due to existing path

Another proof of case 3

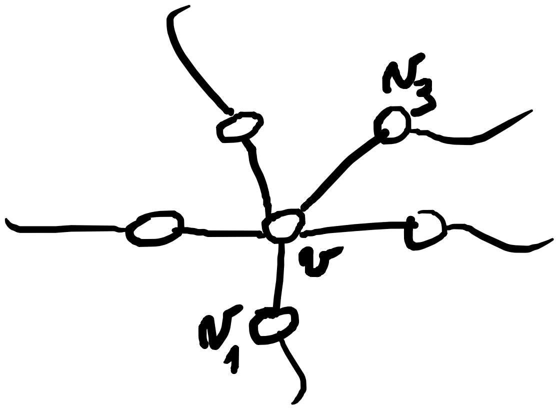
- The graph is contracted by removing two edges, resulting in a planar graph with fewer than n vertices.

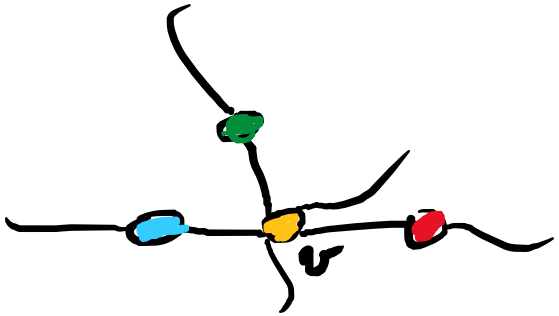
- The contracted graph is 5-colorable.

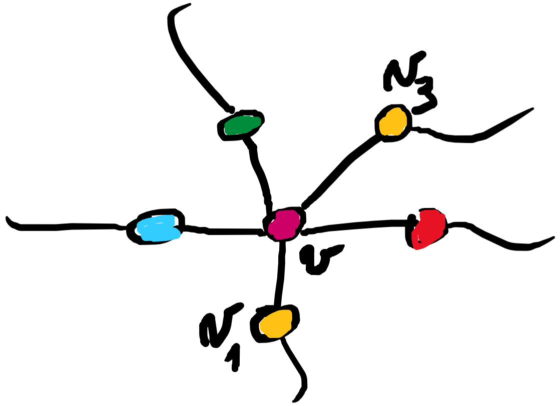
- The two edges are reinstated, assigning the original color of v to both v_1 and v_3.

- Assign to v a an unassigned color in the contracted grpah

- A 5-coloring of the graph G is achieved by coloring.







THEORME 6 (Four Colour Theorem for Planar Graphs)

The vertices of any simple connected planar graph can be colored with four (or fewer) colours.

Intuition

- It states that only 4 colors are needed to color the regions oof any map.
- Conjectured in the 19th century.
- Proven in 1976 with computer assistance.
- It demonstrates the inherent simplicity within complex spatial configurations.

Greedy Algorithm for Vertex Colouring

- Start with graph G and a list of colors 1, 2, 3, ...

- **Step 1:**
  - Label the vertices as a, b, c, ... in any manner.

- **Step 2:**
  - Identify the uncolored vertex labeled with the earliest letter in the alphabet.
  - Color this vertex with the first color from the list that is not used by any adjacent colored vertex.
- Repeat **Step 2** until all vertices are colored.

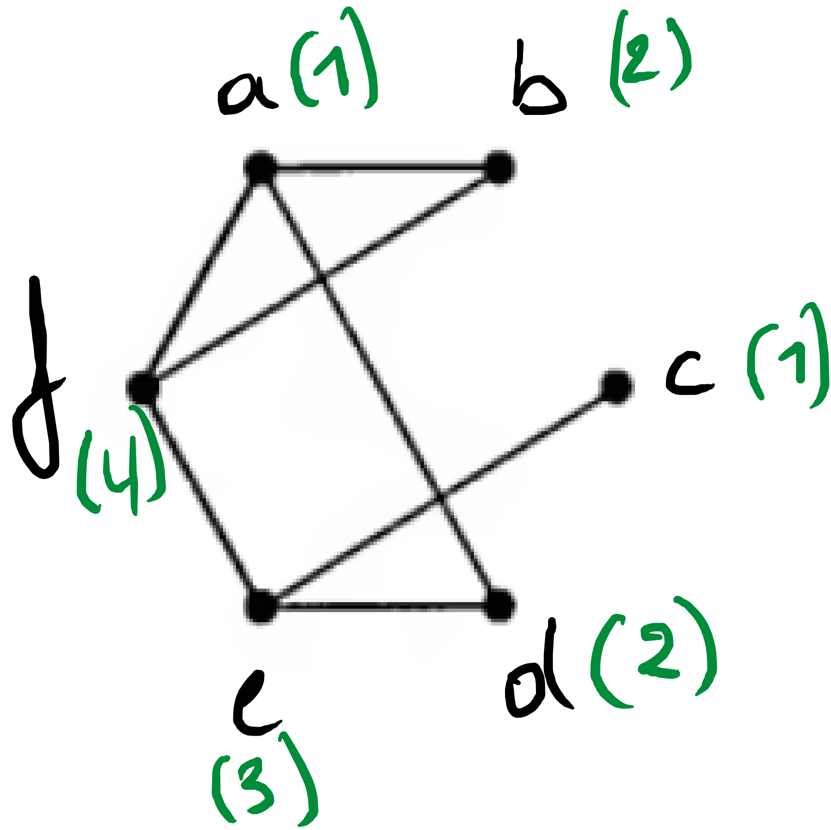
- Stop. A vertex coloring of G has been obtained.

- The number of colors used depends on the labeling chosen for the vertices in **Step 1**.

Example

Find a vertex colouring of the following graph G.

- **Step 1:**
  - Label the vertices as a, b, c, ... in any manner.
- **Step 2**:
  - Identify the uncolored vertex labeled with the earliest letter in the alphabet.
  - Color this vertex with the first color from the list that is not used by any adjacent colored vertex.



Example

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- **Step 1:**
  - Label the vertices as a, b, c, ... in any manner.
- **Step 2**:
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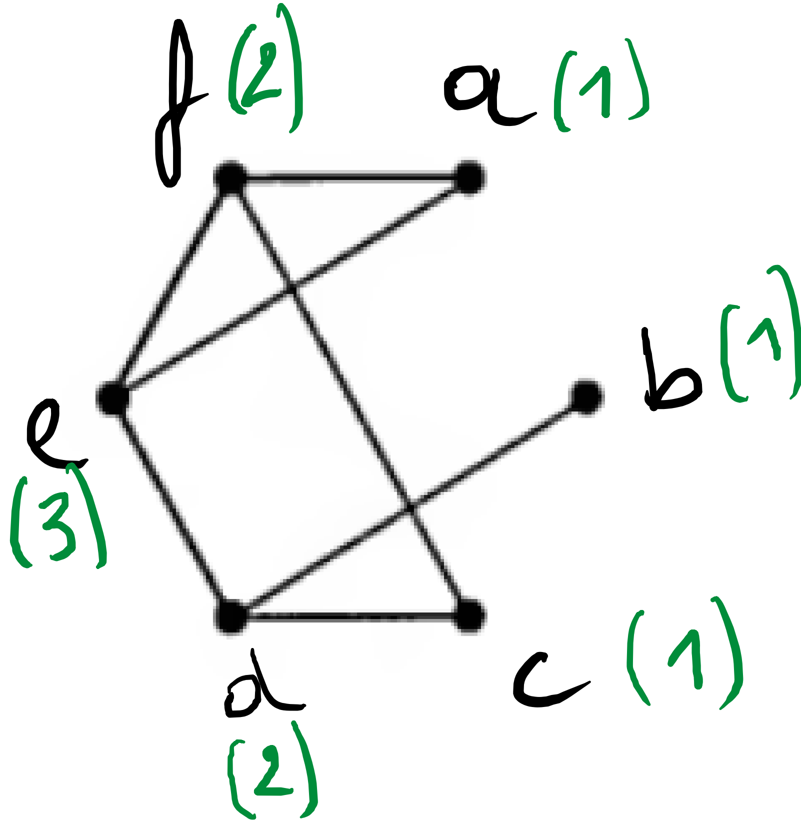
  - Color this vertex with the first color from the list that is not used by any adjacent colored vertex.



THEORME 6

For any graph G, there is a labelling of the vertices for which the greedy algorithm yields a vertex colouring with $\chi(G)$ colours.

- Proof sketch
  - Take any vertex coloring of G with $\chi(G)$ colors, denoted by 1, 2, 3, ...
  - Sequentially label the vertices colored 1 as a, b, c, ...
  - Label the vertices colored 2 starting from the next available label after the last label used for color 1.
  - Continue this labeling pattern for the vertices colored 3, 4, and so on.
  - The greedy algorithm assigns colors 1, 2, 3, ... in order.
  - As a result, only $\chi(G)$ colors are needed for this labeling.

Coloring Problems: Storing Chemicals

- Certain chemicals react violently when they are in contact.
- The manufacturer plans to divide the warehouse into regions to separate dangerous chemical pairs.
- The dangerous pairs of chemicals are marked with an asterisk in a table.

	a	*b*	*c*	*d*	*e*	*f*	*g*
a	–	*	*	*	–	–	*
b	*	–	*	*	*	–	*
c	*	*	–	*	–	*	–
d	*	*	*	–	–	*	–
e	–	*	–	–	–	–	–
f	–	–	*	*	–	–	*
g	*	*	–	–	–	*	–

The objective is to find the minimum number of areas required to safely store the chemicals.

Coloring Problems

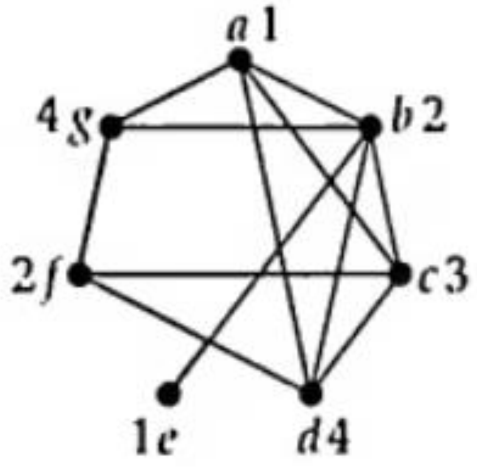
- To determine the minimum number of areas required for safe chemical storage, a graph was created.
    - Each vertex represent chemical.
    - Vertices are adjacent when the corresponding chemicals need to be separated.

Assigning chemicals to areas is a vertex coloring problem, with colors corresponding to the areas.

- A vertex coloring leads to a vertex decomposition of the graph, where **no adjacent vertices** are in the same subset.

- The subsets **{a, e}, {b, j}, {c}, {d, g},** representing the chemicals in the four areas.

- The minimum number of subsets required for this problem is the **chromatic number** $\chi(G)$ of this graph.



Coloring Problems: Map Colouring

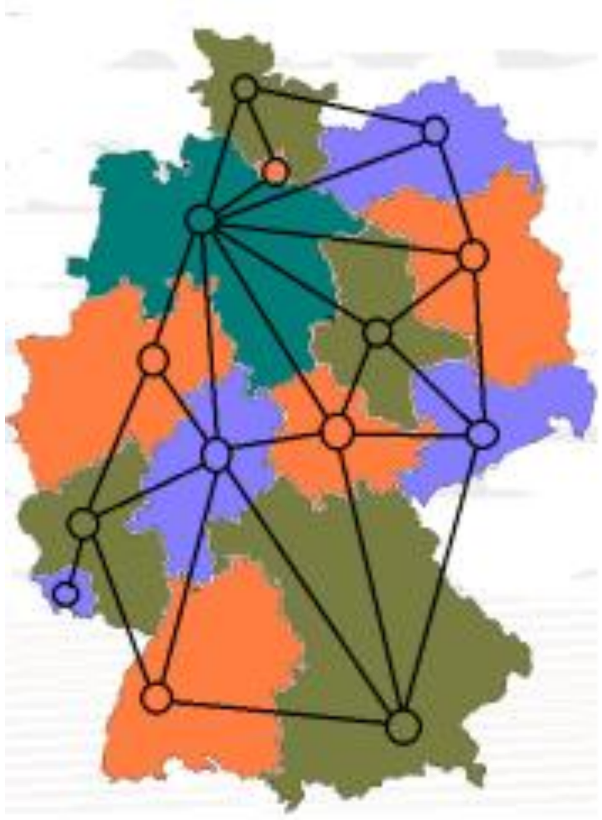
- In 1852, Francis Guthrie proposed the four-color problem:
  - Can all maps be colored with four colors so that neighboring countries have different colors?
- Mathematicians, including De Morgan, Cayley, and Kempe, studied the problem.
- In 1976, Appel and Haken provided a proof using nearly 2000 country configurations and extensive computer analysis.

Even today, no "simple" proof has been discovered for the four-color problem.

Coloring Problems: Map Colouring

- Objective:
  - Assign colors to countries on a map, ensuring adjacent countries have different colors.
- Representation:
  - Use the geometrical dual to represent countries as vertices in a graph, with edges connecting adjacent countries.

Determine the minimum number of colors needed to avoid adjacent countries sharing the same color.



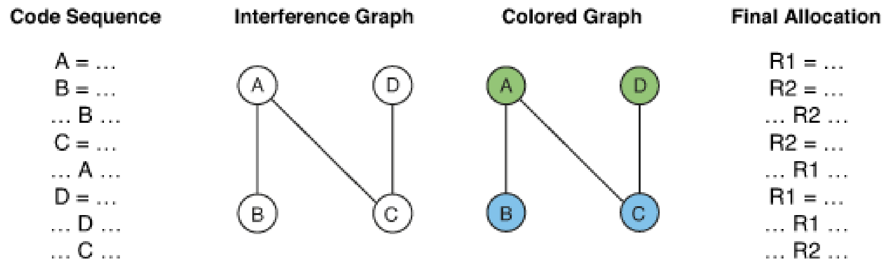
Coloring Problems: Map Colouring

A map is planar graph

- **Theorem 4**: Any map can be colored with 6 colors .
- **Theorem 5**: Any map can be colored with 5 colors .
- **Theorem 6**: Any map can be colored with 4 colors .

Coloring Problems: Register allocation

- Register allocation can be seen as a graph coloring problem.
- Nodes in the graph represent the live ranges of variables.
- Edges indicate connections between two live ranges.
- The objective is to assign colors to nodes such that adjacent nodes have different colors.



The chromatic number represents the minimum number of registers required.

Let's play Sudoku

- Objective:
  - Complete a 9x9 matrix.

- Criteria:
  - In each row, column, and marked 3x3 square, the numbers 1 to 9 should occur exactly once.

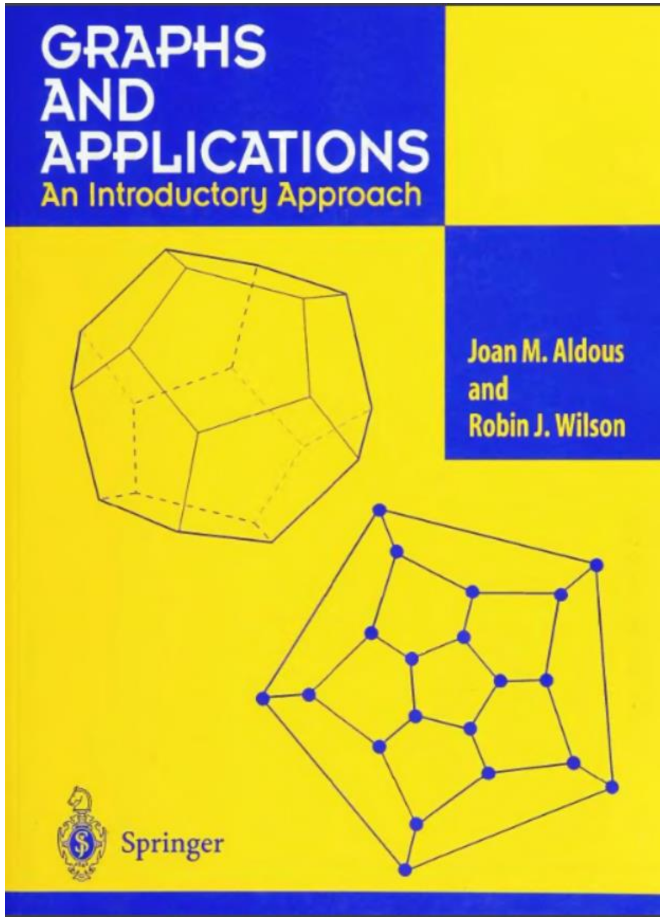
Can we model this as coloring problem ?!

			6		3			
	3			1			5	
		9				2		
7			1		6			9
	2						8	
1			4		9			3
		8				1		
	5			9			7	
			7		4			

Conclusion

- Graph coloring is a fundamental concept in graph theory with diverse applications.
- It enables the assignment of colors to vertices by representing entities as vertices and their relationships as edges.
- The goal is to ensure adjacent vertices have different colors, minimizing conflicts.
- Efficient graph coloring algorithms are crucial for optimizing resource allocation and improving system performance.
- Understanding graph coloring techniques helps solve complex optimization problems in various domains.
- Graph coloring has applications beyond computer science, contributing to advancements in diverse fields.

Reference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