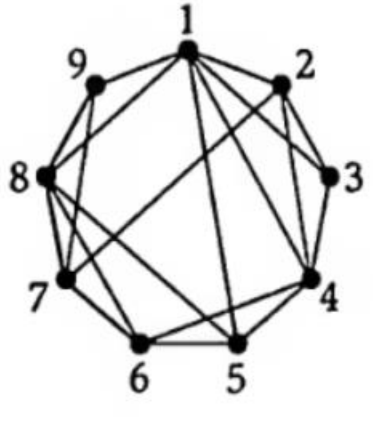
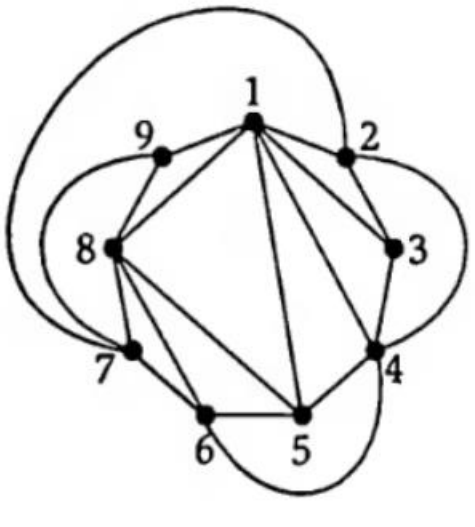
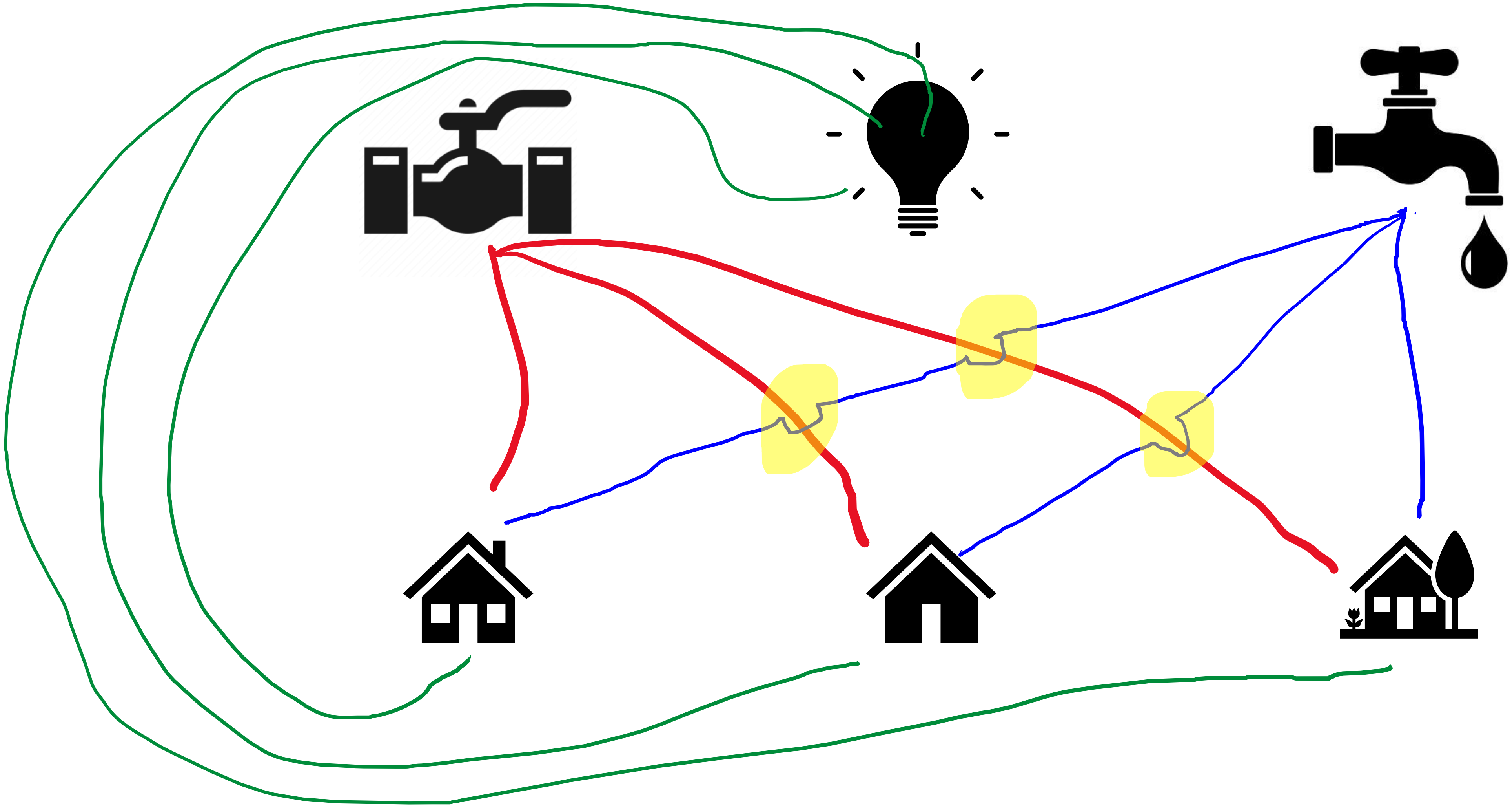
Planar graphs

Mohammed Brahimi





Utilities problem



Applications of Planar Graphs

- Urban Planning
  - Planar graphs model utility line layouts, ensuring non-crossing connections between houses.
- Computer-Aided Design
  - They are used in electronic system design, such as integrated circuits, for optimal layout and minimized total area.
- Network Design
  - Network topologies are represented as planar graphs to aim for non-interfering connections.
- Subway Route Planning
  - Planar graphs aid in efficient route determination for subway lines, reducing intersections to prevent congestion and bottlenecks.

Applications of Planar Graphs

- Graph Drawing
  - Planar graphs are crucial in graph drawing for software engineering, databases, data visualization, and more.
- Wireless Communication
  - In wireless ad hoc networks, planar graphs represent possible communication paths, minimizing interference and ensuring efficient communication.
- Scheduling
  - Certain scheduling problems can be modelled as planar graphs, helping to organize overlapping tasks.
- Maze Solving Algorithms
  - Planar graphs are useful for maze generation and solving algorithms, representing corridors and junctions in the maze.

Any situation involving connected "points" without overlap can potentially use planar graphs.

What is planar graph ?

- Definition:

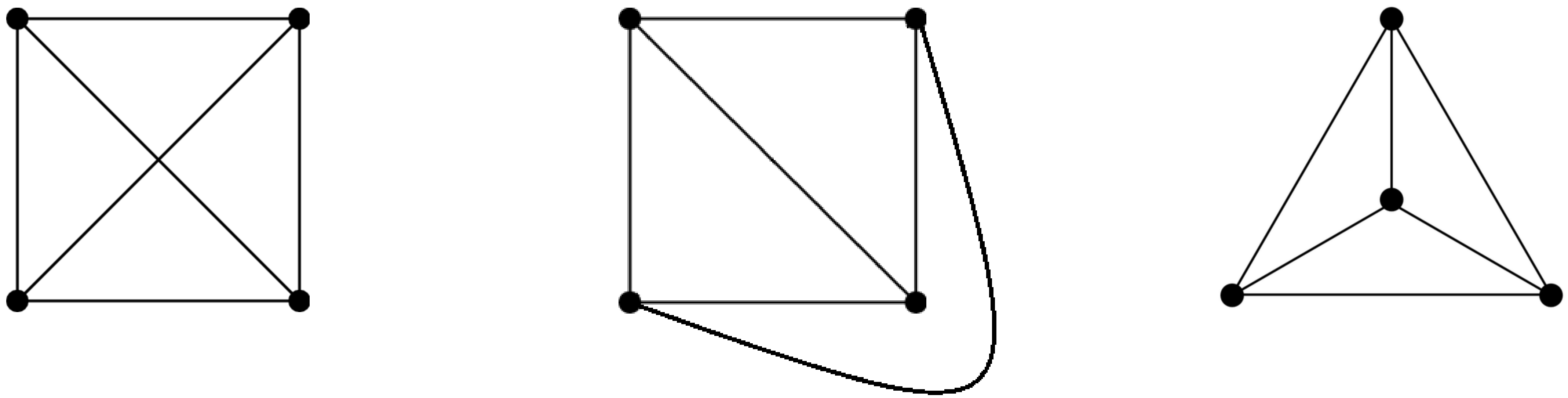
A planar graph can be drawn in a plane without edge crossings, with edges only intersecting at vertices they're incident to.

- **Plane Drawing (embedding)**
  - Any drawing of a planar graph without crossings.
- **Plane Graph**
  - An abbreviation used for a plane drawing of a planar graph.

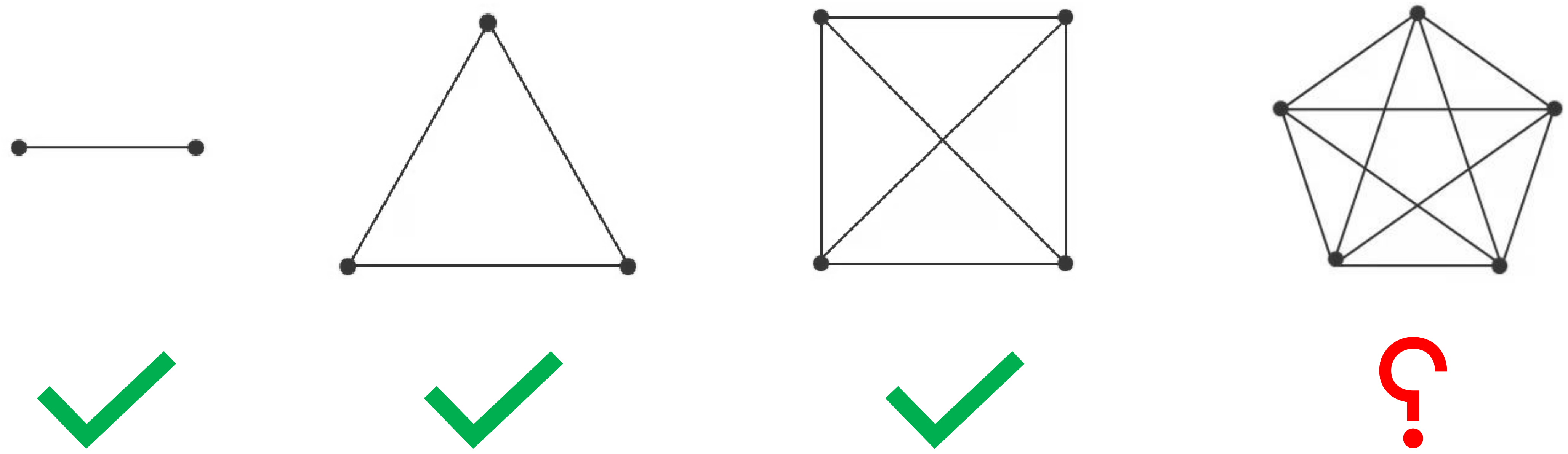
Not all drawings of a planar graph are plane graphs. Only those without edge crossings qualify as plane graphs.

Example

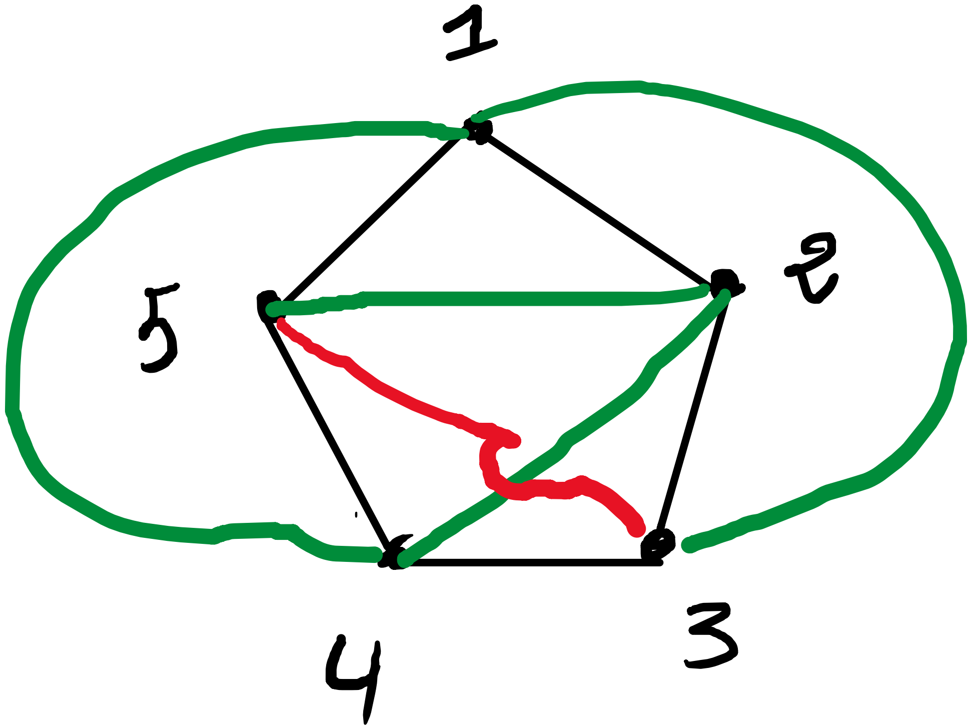
Is K_4 graph planar ?



What's the largest complete planar graph !?

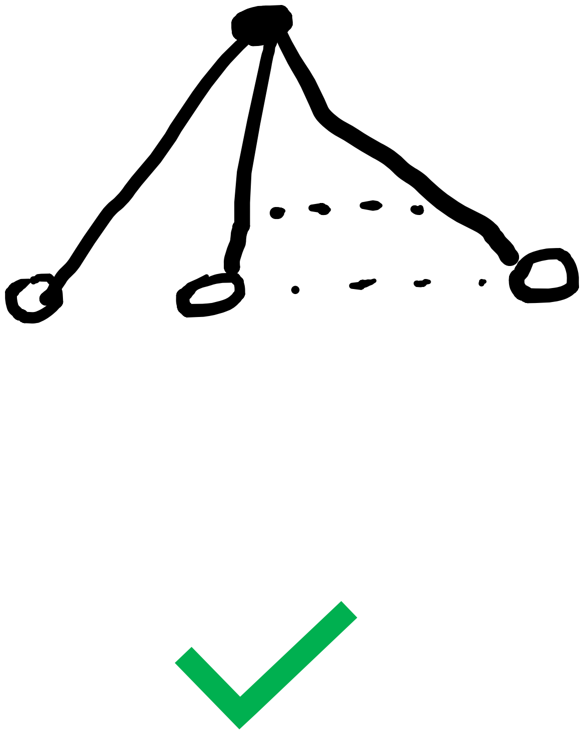


Is K_5 a planar graph ?

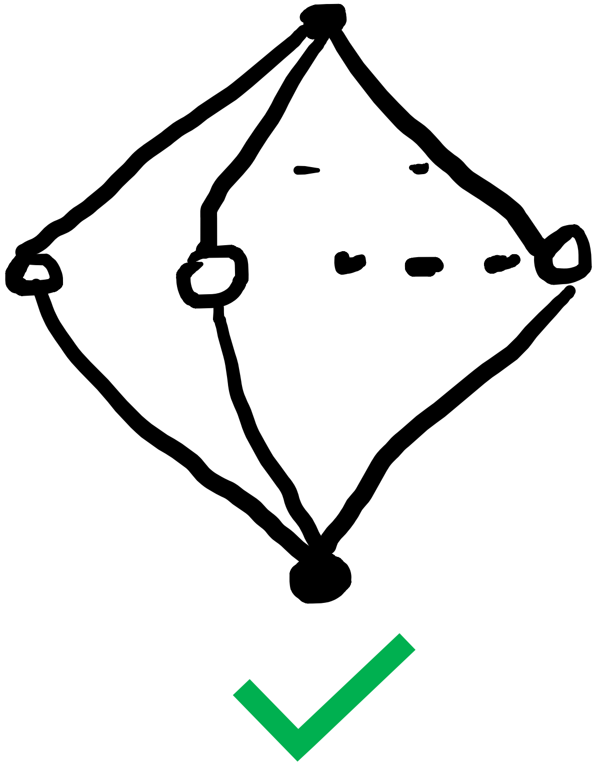


What's the largest bipartite complete planar graph !?

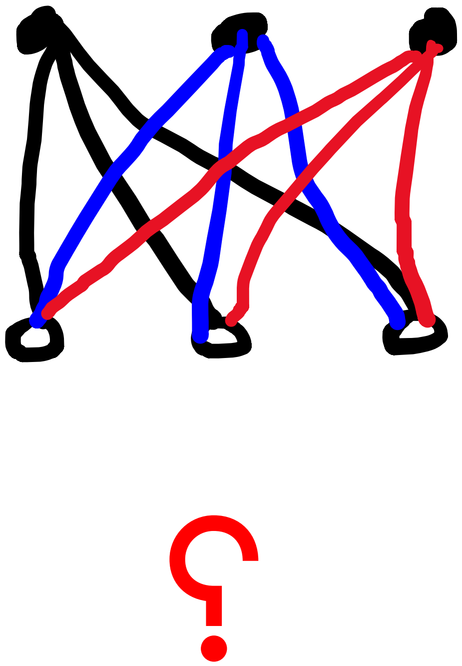
$K_{1,n}$



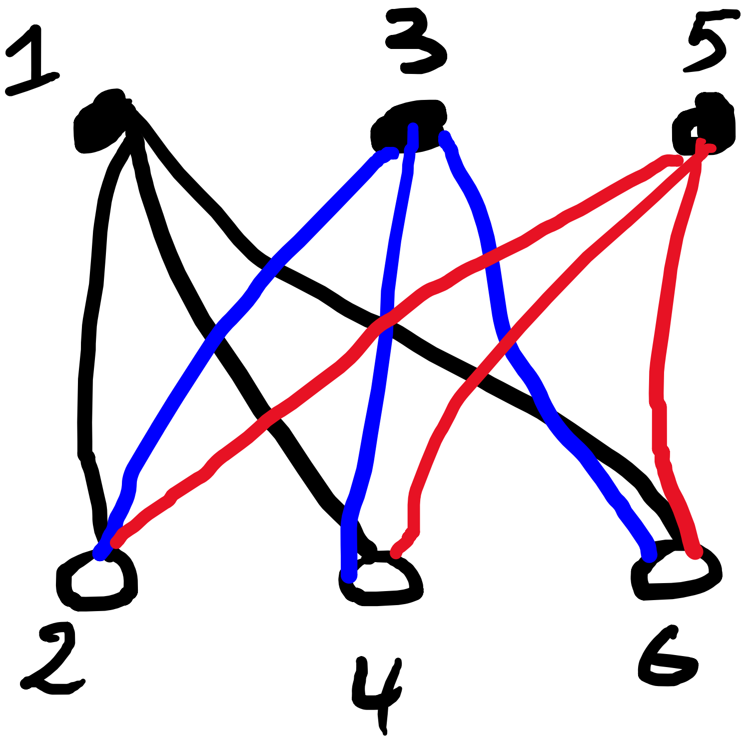
$K_{2,N}$

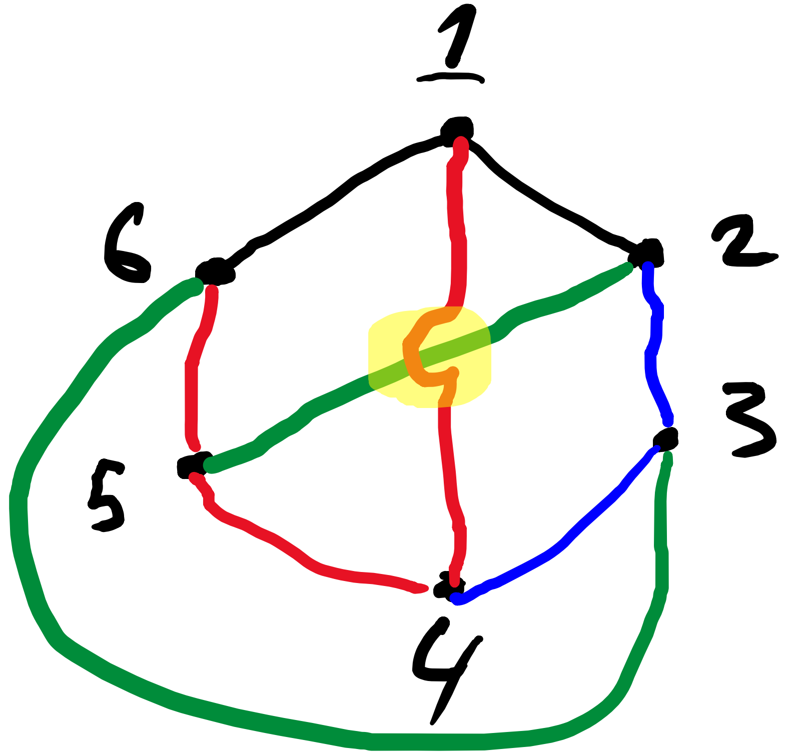


$K_{3,3}$

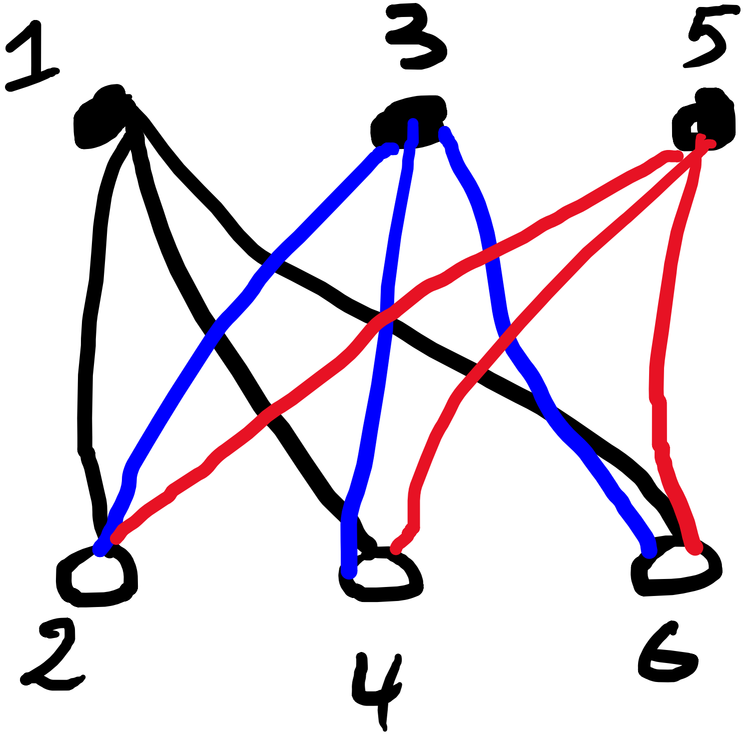


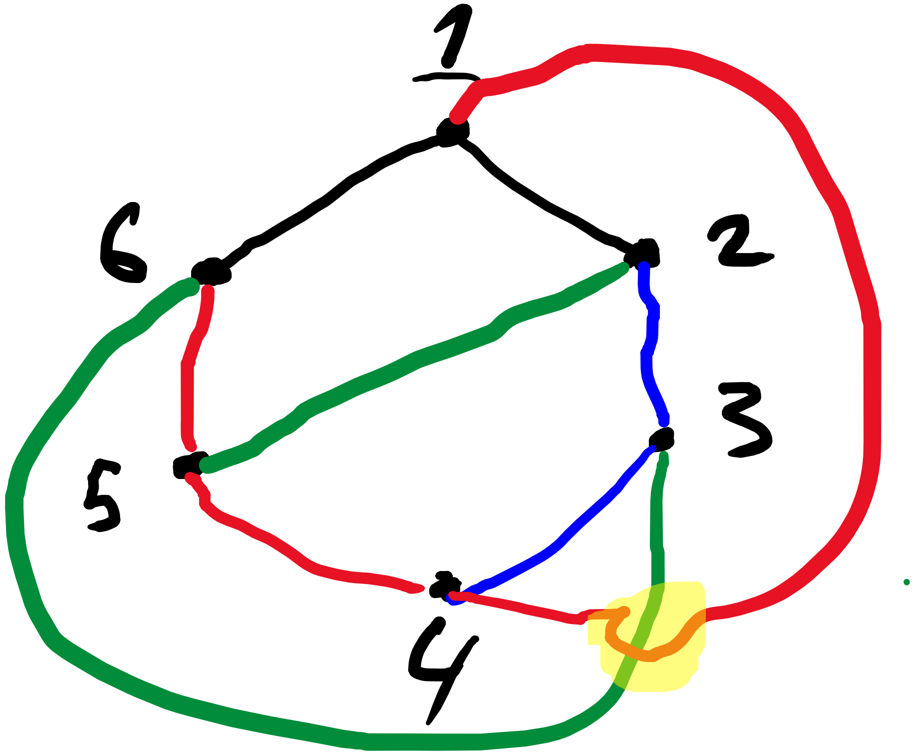
Is $K_{3,3}$ planar graph ?





Is $K_{3,3}$ planar graph ?





THEORME 1

The graphs $\boldsymbol{K_5}$ *and the graph* $\boldsymbol{K_{3,3}}$ *are non-planar*

Proof

- Later in the lecture.

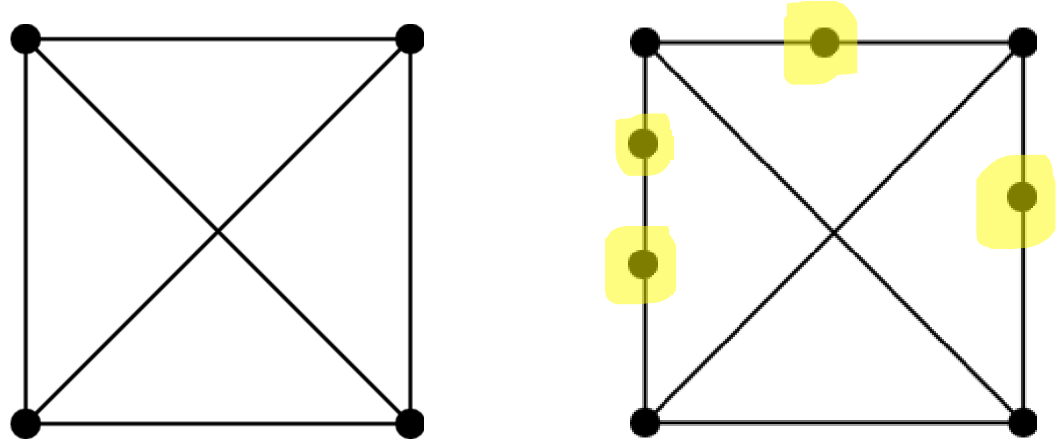
Questions ???

- Can a graph be planar even if it contains a non-planar subgraph?
- Is a subgraph of a planar graph also planar?
- Can we identify basic non-planar subgraphs within every non-planar graph?
- How can we tell if a graph is planar ?

Subdivision of a graph

- A **subdivision** of a graph is the resultant graph obtained by **inserting vertices of degree 2** in its edges.

- Two graphs are **homeomorphic** if they can both be obtained by subdividing the same graph.



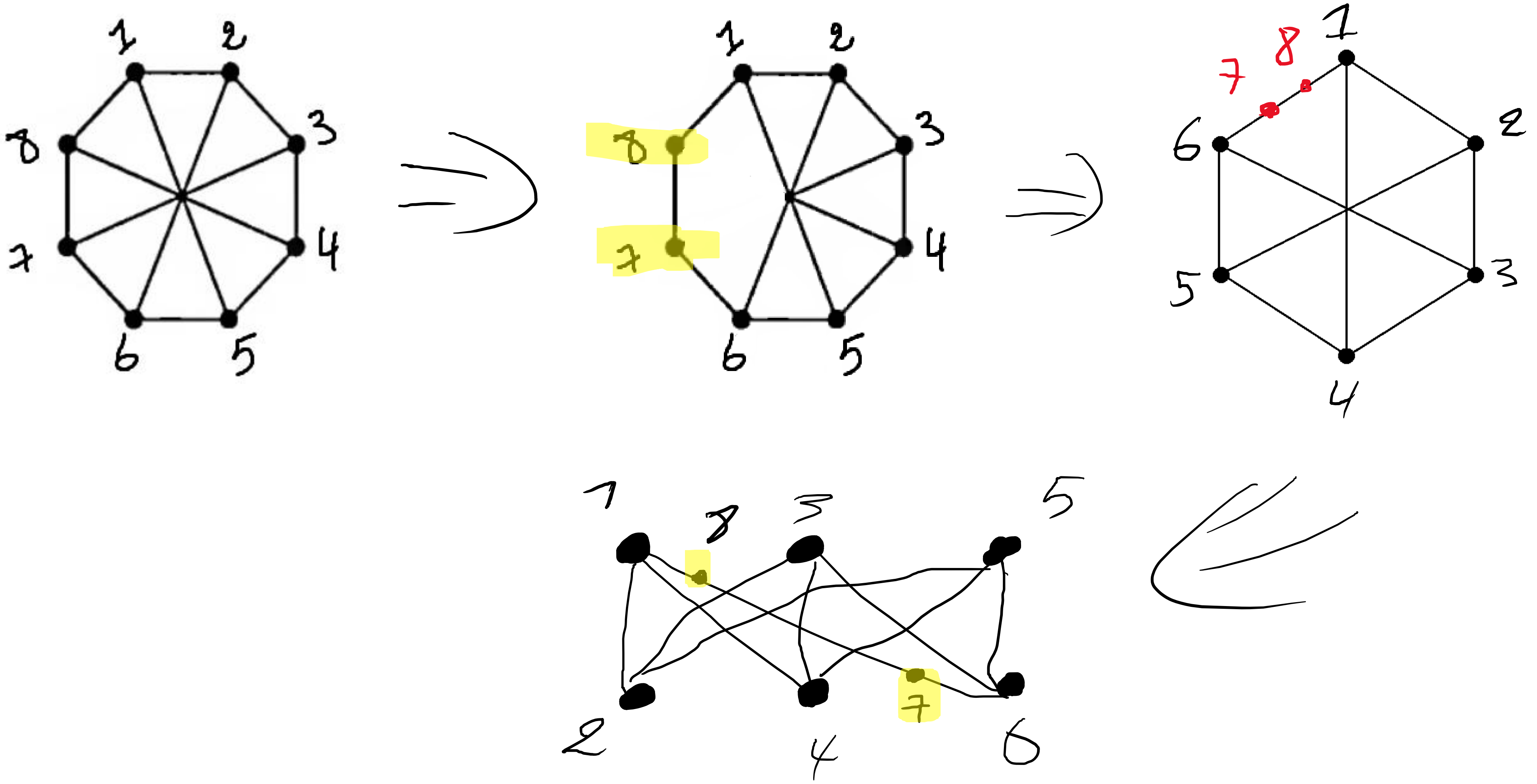
THEOREM 2 (Kuratowski. 1930)

A graph is planar if and only if it contains no subgraph homoeomorphic to K_5 or $K_{3,3}$.

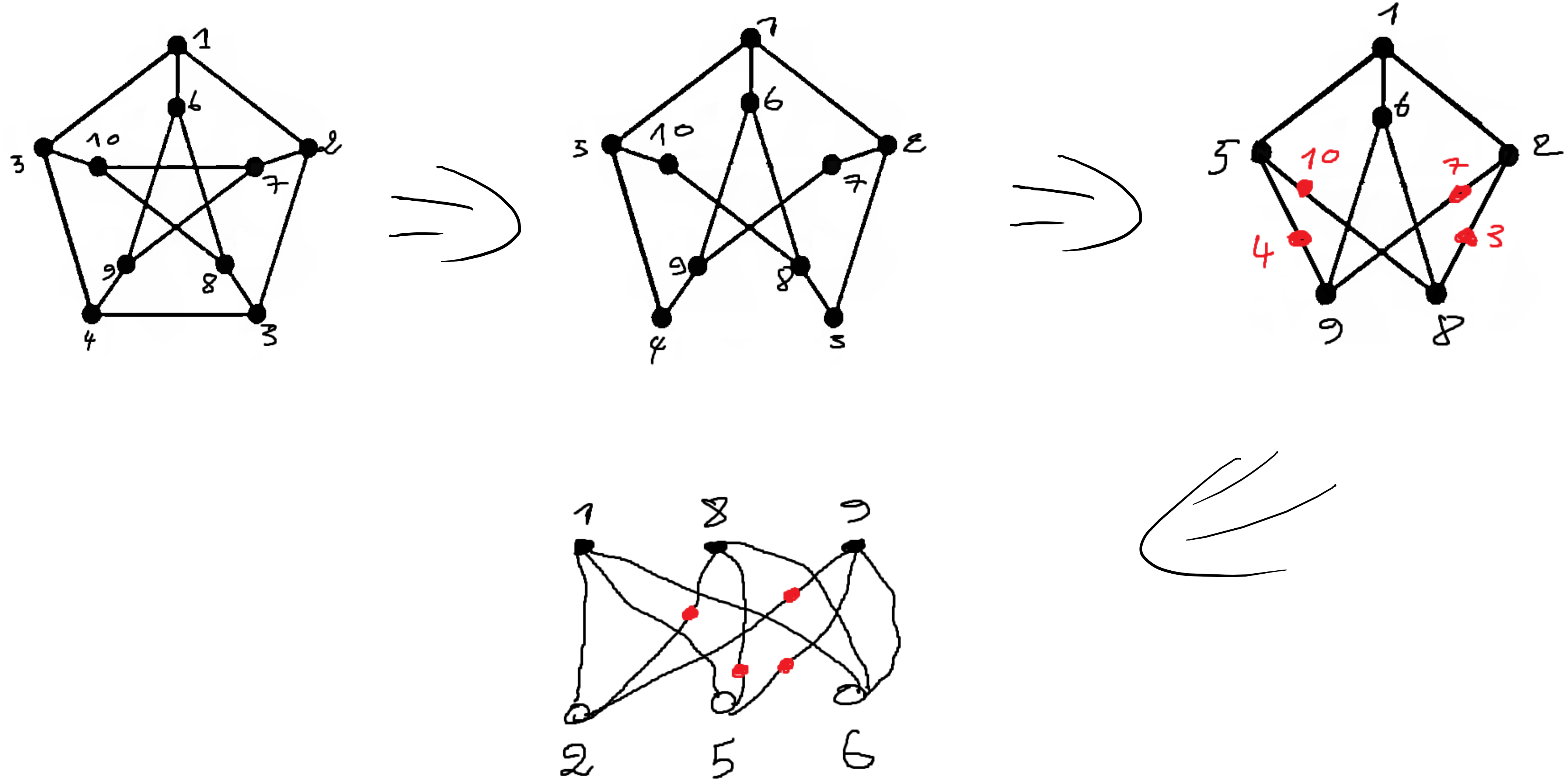
- Proof (not covered)
- **Intuition**

The theorem often confirms a graph's non-planarity by identifying a subgraph homeomorphic to K5 or K3,3.

Example 1

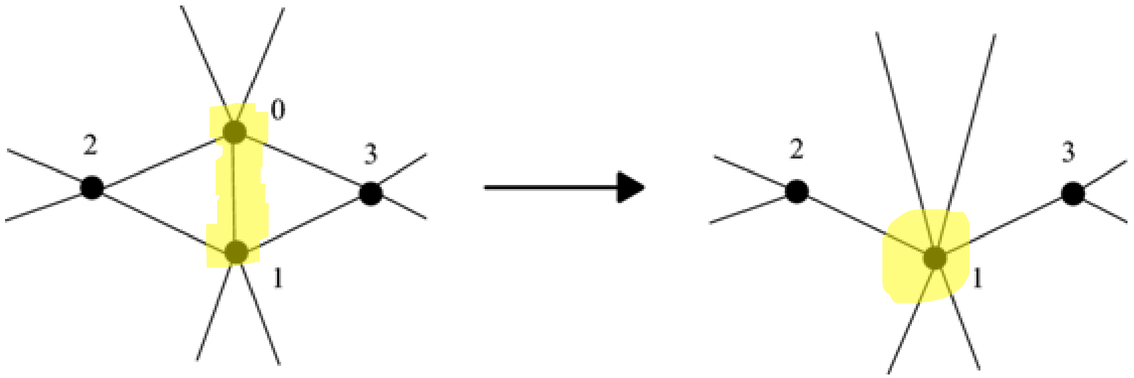


Example 2 (Petersen graph)

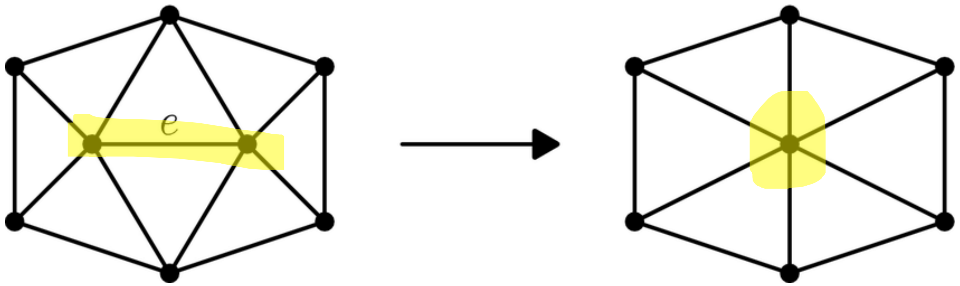


Edge contraction

- **Edge contraction** is the process of removing an edge and merging the **incident vertices** into a single vertex, while maintaining the remaining adjacency relationships.



- A graph H is **contractible** to a graph G, if we can obtain H by successively contracting the edges of G.



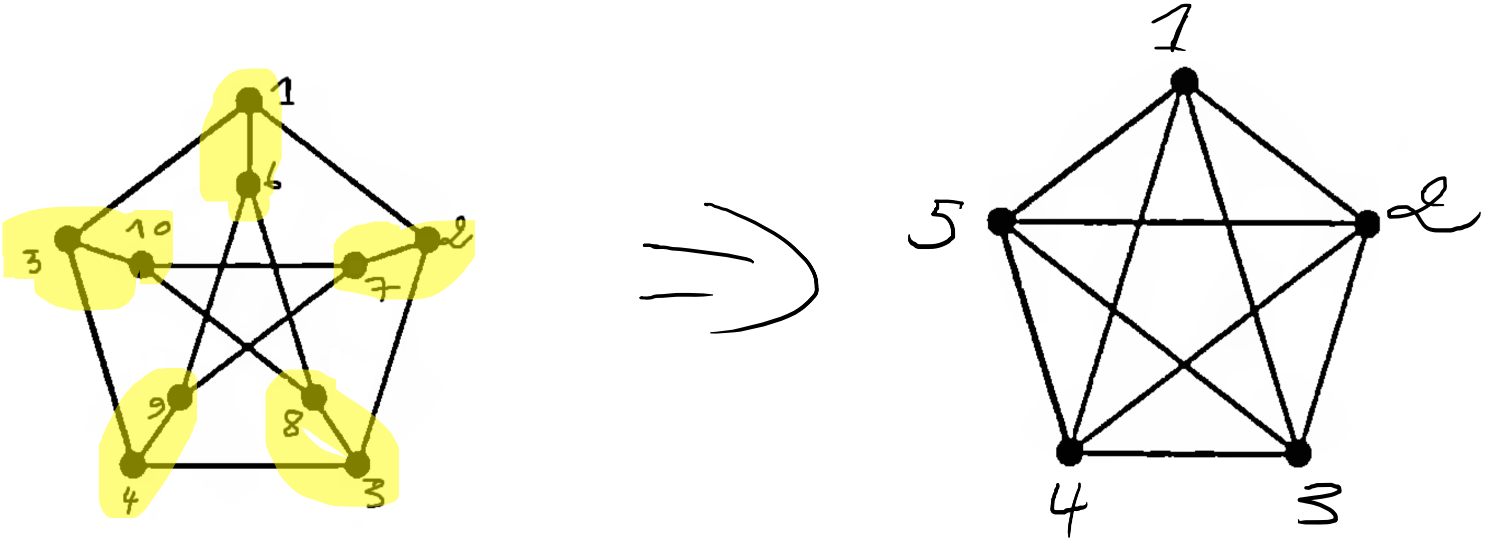
THEOREM 3

A graph is planar if and only if it contains no subgraph ***contractible*** *to K_5 or $K_{3,3}$.*

- Proof (not covered)
- **Intuition**

The theorem often confirms a graph's non-planarity by identifying a subgraph contractible to K_5 or $K_{3,3}$.

Example



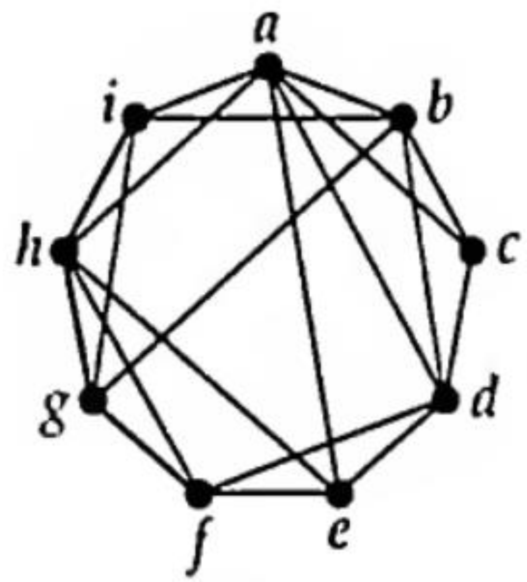
Cycle Method for Planarity Testing

- The Cycle Method is a heuristic algorithm used to test whether a given graph is planar.
- It is applicable to small graphs with a Hamiltonian cycle, providing a quick and intuitive planarity test.

- Steps of the Cycle Method
    1. Find a Hamiltonian cycle C in the graph G.
    2. Draw C as a regular polygon and list the remaining edges.
    3. Divide the remaining edges into two sets, A and B:
        1. A: Edges that can be drawn inside C without crossing.
        2. B: Edges that can be drawn outside C without crossing.
    4. If it is possible to allocate all remaining edges to A and B without crossings, G is planar.
        1. Use sets A and B to obtain a plane drawing of G.
    5. If it is not possible to allocate the remaining edges without crossings, G is non-planar.

- Incompatibility of edges
    - Incompatible edges cannot both be drawn inside C or both be drawn outside C without crossings.
    - Compatible edges can be drawn inside or outside C without crossings.

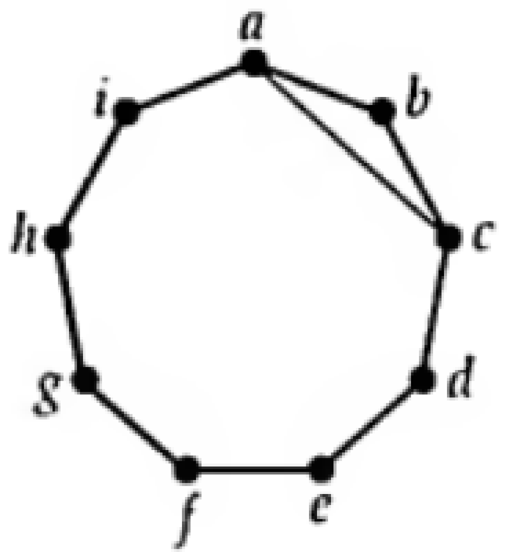
Example

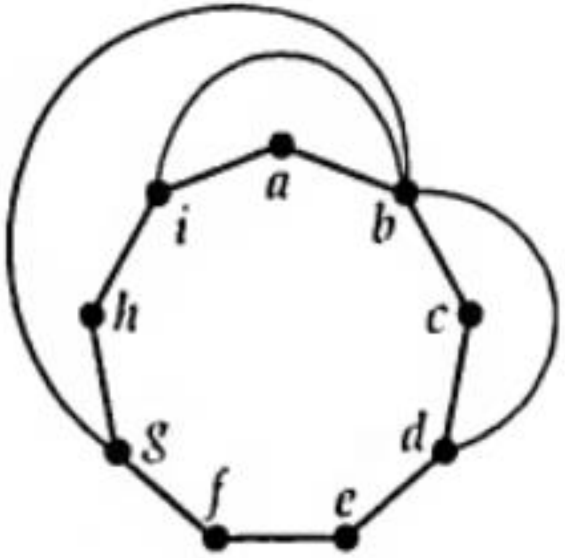


Example

- ac bd df
- ad bg eh
- ae bi fh
- ah gi

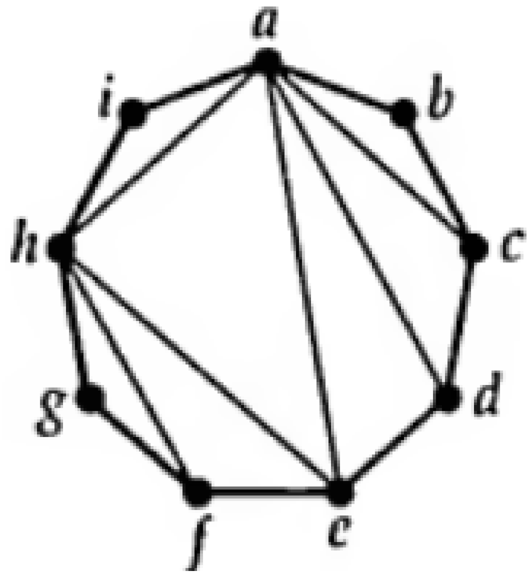
- **A = {ac}**
- We put incompatible with A in B
- **B = { bd, bg, bi}**

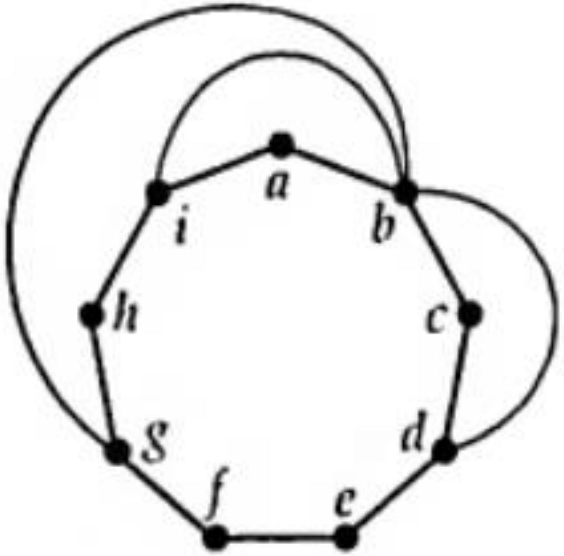




Example

	bd	df
• ad	bg	eh
• ae	bi	fh
• ah	gi	





- **B = { bd, bg, bi}**
- We add incompatible edges with B in A
- **A = {ac, ad, ae, eh, fh, ah}**

Example

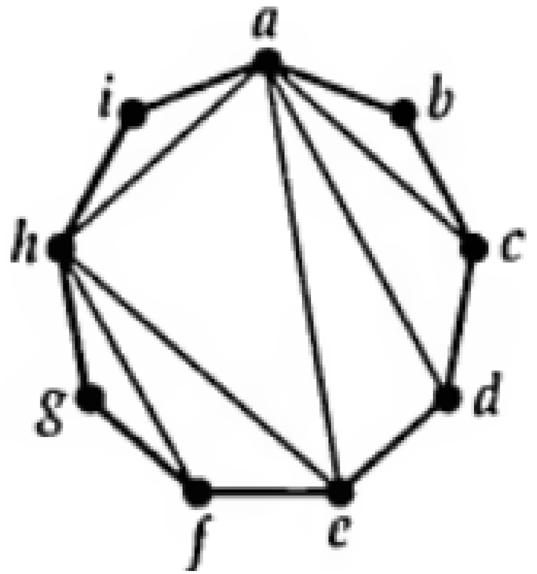
- ad
- ae
- ah

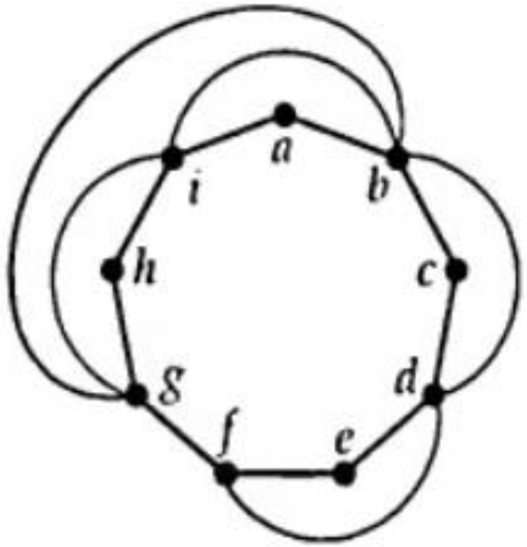
gi

df

eh

fh





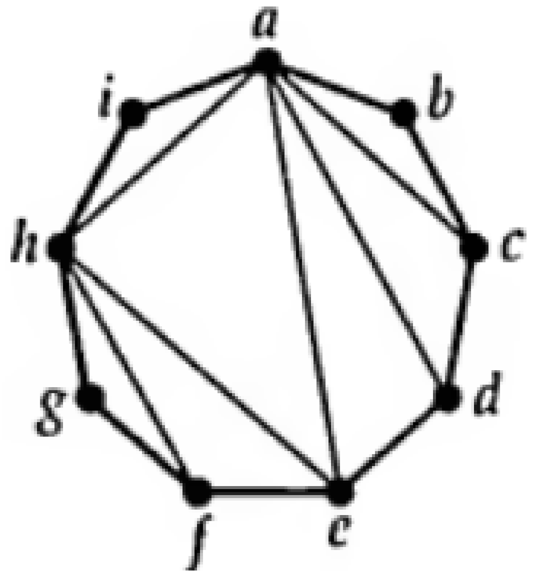
- **A = {ac, ad, ae, eh, fh, ah}**
- We add incompatible edges with A in B
- **B = { bd, bg, bi,df,gi}**

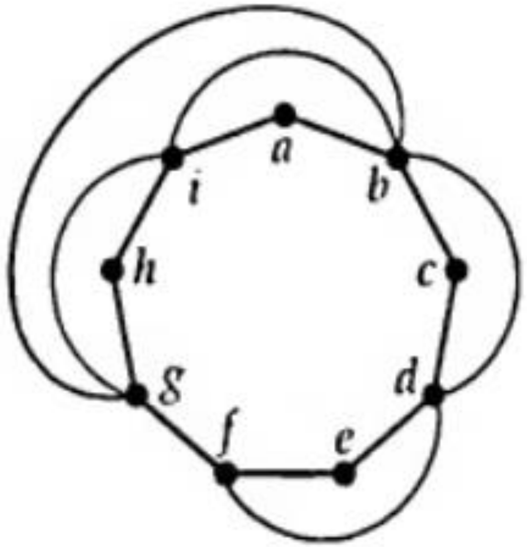
Example

df

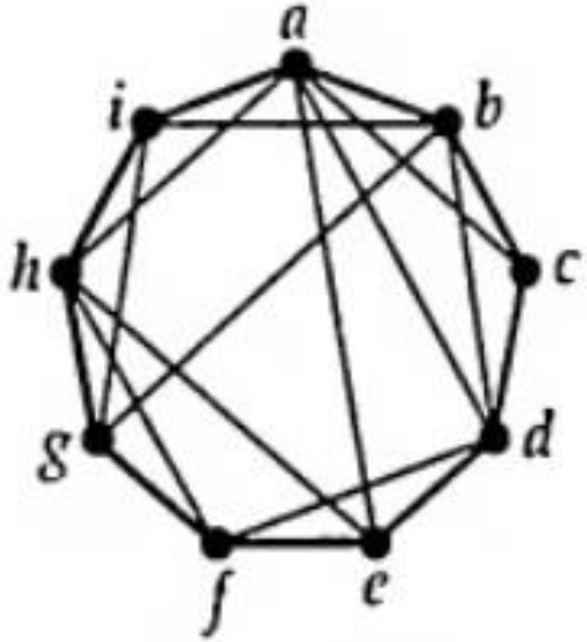
- ad eh
- ae fh
- ah gi

- **A = {ac, ad, ae, eh, fh, ah}**
- We add incompatible edges with A in B
- **B = { bd, bg, bi,df,gi}**

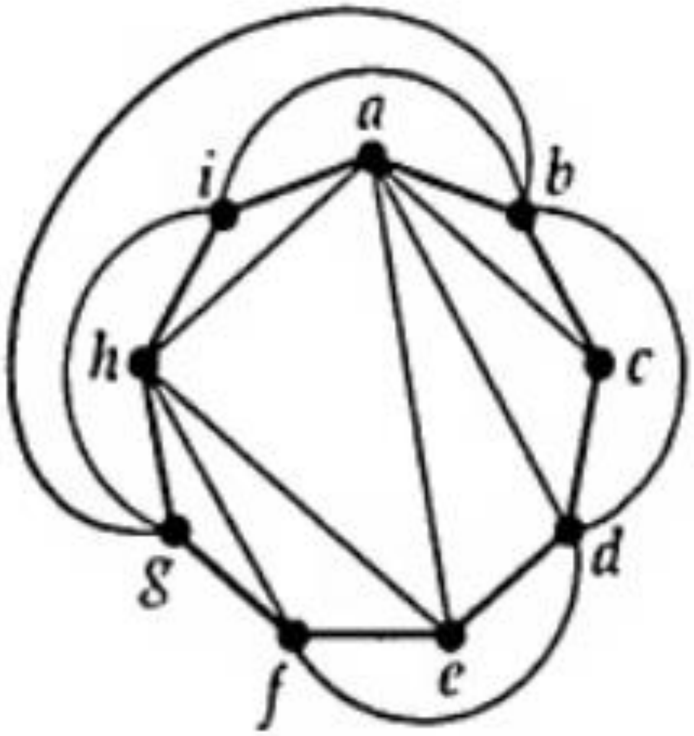




Example



graph G



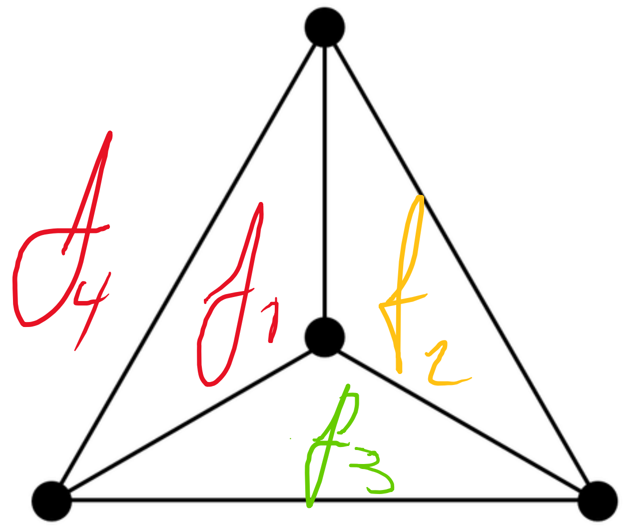
plane drawing of G

Faces in planar graph

- Any plane drawing of planar graph G divides the set of points of the plane into regions, **called faces**.

- One face is unbounded and called **infinite face**.

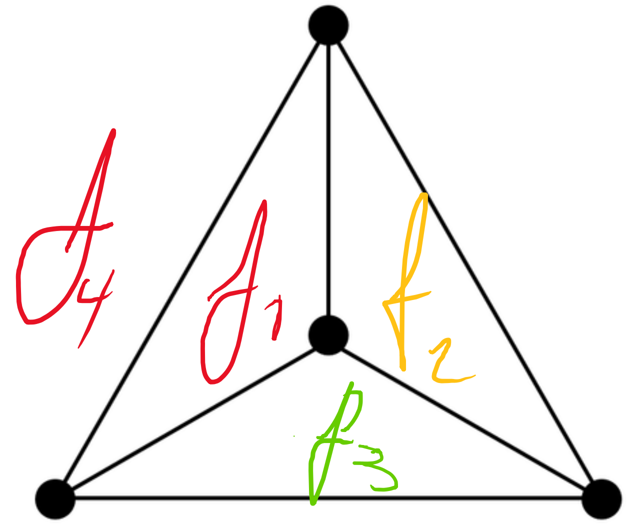
- Example:
  - Faces of the K_4 : f_1, f_2, f_3, f_4.
  - f_4 is infinite face.



Degree of face

- The degree of f, denoted by $\boldsymbol{deg(f)}$, is the number of edges encountered in a walk around the boundary of the face f.

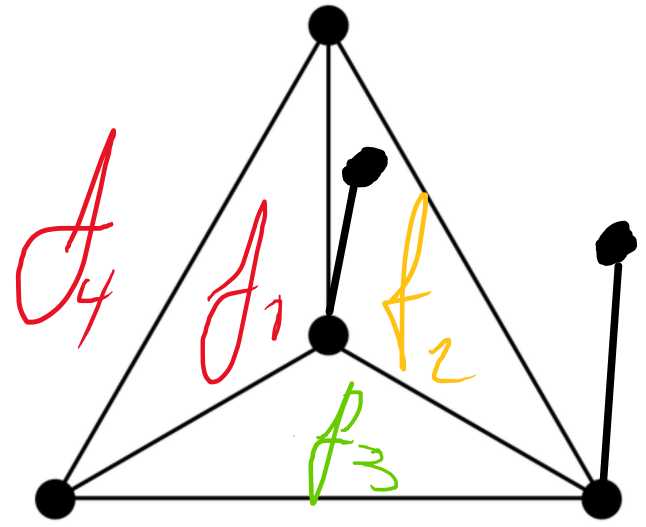
- $deg(f_1) = 3$
- $deg(f_2) = 3$
- $deg(f_3) = 3$
- $deg(f_4) = 3$



Degree of face

- The degree of f, denoted by $\boldsymbol{deg(f)}$, is the number of edges encountered in a walk around the boundary of the face f.

- $deg(f_1) = 3$
- $deg(f_2) = 5$
- $deg(f_3) = 3$
- $deg(f_4) = 5$



Handshaking Lemma for Faces

- In any plane drawing of a planar graph, the sum of all the face degrees is equal to twice the number of edges.

$$\sum \deg(f_i) = 2 \times |E(G)|$$

Proof

- In any plane drawing of a planar graph, each edge has two sides:
  - The edge may lie on the boundary of a single face.
  - The edge can be in the boundaries of two different faces.

- Each edge contributes exactly 2 to the sum of the face degrees.

THEOREM 4 (Euler. 1750)

Let G *be a plane drawing of a connected planar graph and let* n, m *and* f *denote respectively the number of vertices, edges and faces of* G. *Then*

$$\boldsymbol{n - m + F = 2}$$

- $\boldsymbol{n - m + F = 2}$ is called Euler's Formula.

Proof of Euler's Formula

- Proof by induction on the number of edges (m) in graph G.

- Base Case
  - If $m = 0$, then $n = 1$ (as G is connected) and $F = 1$ (the infinite face). Hence, the theorem holds true.

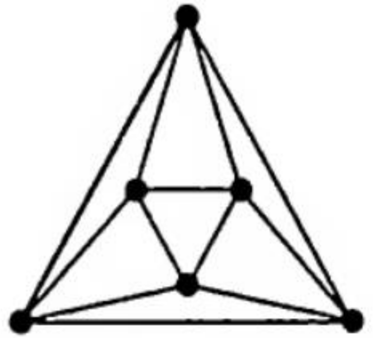
- Inductive Step
  - Assume the theorem holds for all plane graphs with at most $m - 1$ edges.

- Consider G, a plane graph with m edges.
  - If G is a tree, then $m = n - 1$ and $F = 1$, satisfying $n - m + F = 2$.
  - If G is not a tree, choose an edge (e) in some cycle of G.
  - The graph $\boldsymbol{G - e}$ is connected, with $\boldsymbol{n}$ vertices, $\boldsymbol{m - 1}$ edges, and $\boldsymbol{F - 1}$ faces.

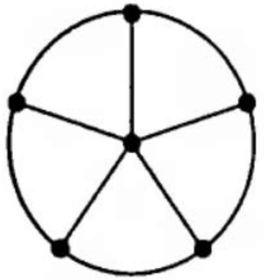
- By the induction hypothesis, we can write: $\boldsymbol{n - (m - 1) + (F - 1) = 2}$.
- Hence, simplifying, we obtain: $\boldsymbol{n - m + F = 2}$, which completes the proof.

Examples

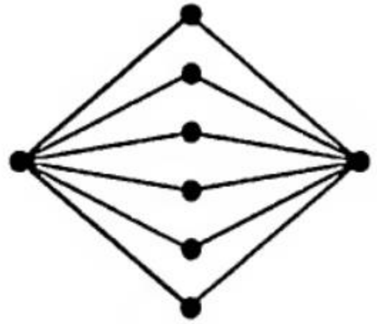
Graph	n	m	F	N-m+F
Octahedron				
W_6				
$K_{2,6}$				
4×4 square lattice				



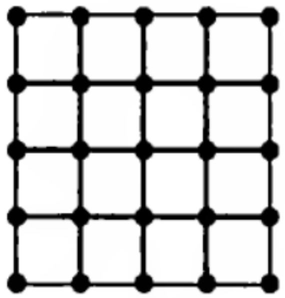
octahedron



wheel with
5 spokes



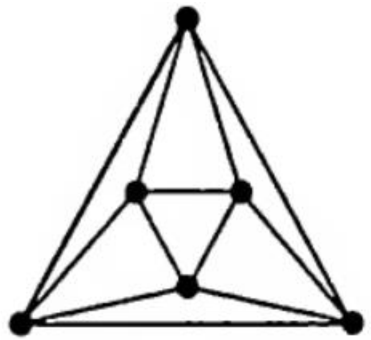
$K_{2,6}$



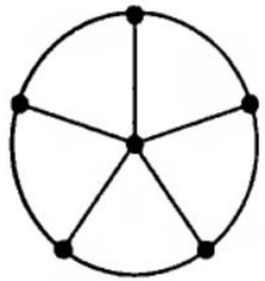
4 x 4
square lattice

Examples

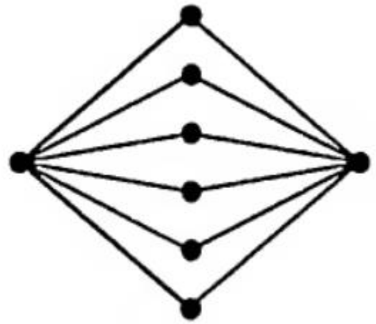
Graph	n	m	F	N-m+F
Octahedron	**6**	**12**	**8**	**2**
W_6	**6**	**10**	**6**	**2**
$K_{2,6}$	**8**	**12**	**6**	**2**
4×4 square lattice	**25**	**40**	**17**	**2**



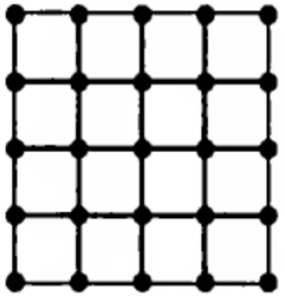
octahedron



wheel with
5 spokes



$K_{2,6}$



4 x 4
square lattice

COROLLARY 1

- If G is a simple connected planar graph With n (≥ 3) vertices and m edges. then

$$m \leq 3n - 6.$$

- If, in addition, G has no triangles then

$$m \leq \ 2n - 4.$$

Significance

- This corollary can be utilized ***to identify non-planar graphs***, **eliminating the need for the number of faces**.

Proof ($m \leq 3n - 6$)

- If a planar graph has m edges and F faces.
- $\sum \deg(f_i) = 2m$ (Handshaking Lemma for Faces)
- $\deg(f_i) \geq 3$ (Each face is bounded at least by 3 edges)

$$\sum \deg(f_i) \geq 3F$$

- $2m \geq 3F$
- $n - m + F = 2$
- $3F = 6 - 3n + 3m$
- $6 - 3n + 3m \leq 2m$

$$m \leq 3n - 6$$

Proof ($m \leq \ 2n - 4$)

- If the a planar graph has m edges and F faces.
- $\sum \deg(f_i) = 2m$ (Handshaking Lemma for Faces)
- $\deg(f_i) \geq 4$ (no triangle in the graph)

$$\sum \deg(f_i) \geq 4F$$

- $2m \geq 4F$
- $n - m + F = 2$
- $4F = 8 - 4n + 4m$
- $8 - 4n + 4m \leq 2m$

$$m \leq 2n - 4$$

Proof of Theorem 1

The graphs $\boldsymbol{K_5}$ and the graph $\boldsymbol{K_{3,3}}$ are non-planar

- **Proof**
  - $\boldsymbol{K_5}$
    - Number of vertices $\boldsymbol{n} = \mathbf{5}$
    - Number of edges $\boldsymbol{m} = \mathbf{10}$
    - $\mathbf{3}\boldsymbol{n} - \mathbf{6} = \mathbf{9} < \boldsymbol{m}$ ➔ **The graph is non-planar.**
  - $\boldsymbol{K_{3,3}}$ (**absence of triangle**)
    - Number of vertices $\boldsymbol{n} = \mathbf{6}$
    - Number of edges $\boldsymbol{m} = \mathbf{9}$
    - $\mathbf{2}\boldsymbol{n} - \mathbf{4} = \mathbf{8} < \boldsymbol{m}$ ➔ **The graph is non-planar.**

COROLLARY 2

Let G be a simple connected planar graph. Then G contains a vertex of degree 5 or less.

- Proof

  $d_i \geq 6 \Rightarrow \sum d_i \geq 6n$

  $\Rightarrow 2m \geq 6n$

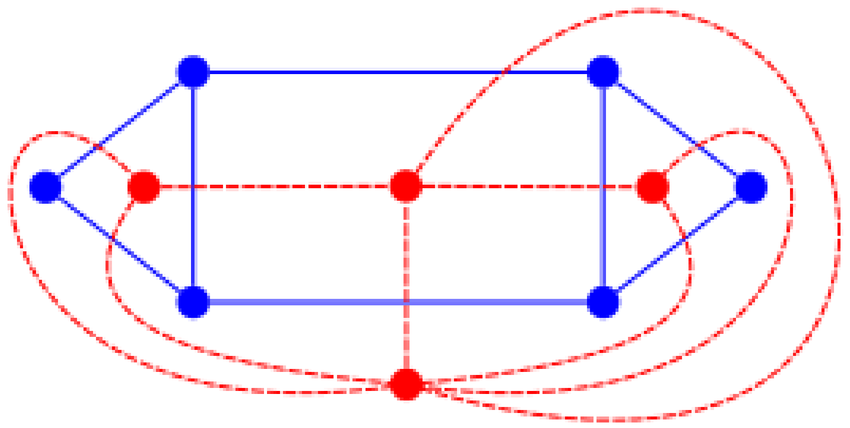
  $\Rightarrow m \geq 3n$

  $\Rightarrow m > 3n - 6$

  $\Rightarrow$ **Contradiction with $\boldsymbol{m \leq 3n - 6}$**

Dual graph

- Dual graphs **capture the relationships between the faces of planar graph**.
- Associate each vertex with a face of the original graph.
- Connect the vertices if the corresponding faces share an edge.
- The geometrical dual graph of planar graph is planar



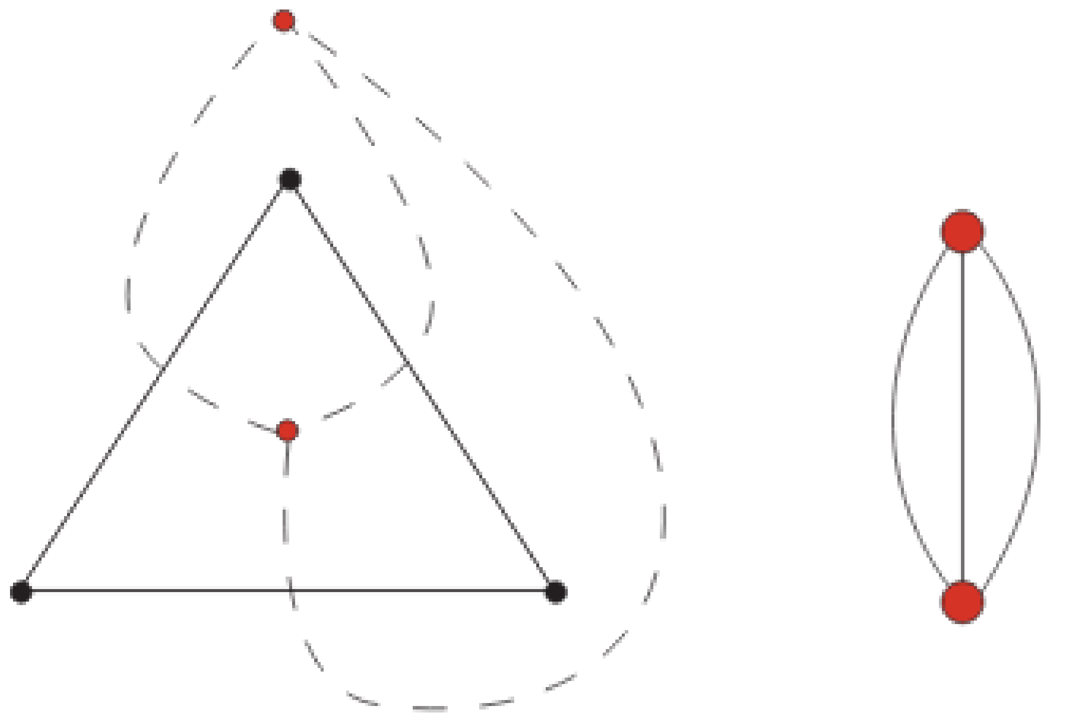
Dual Graph Construction

- Given a plane drawing of a planar graph G, its geometric dual graph G* is constructed using the following two stages:

1. Vertices of dual graph
   - Inside each face of G, we choose a point $\boldsymbol{v}^*$.
   - These points will serve as the vertices of G^*.

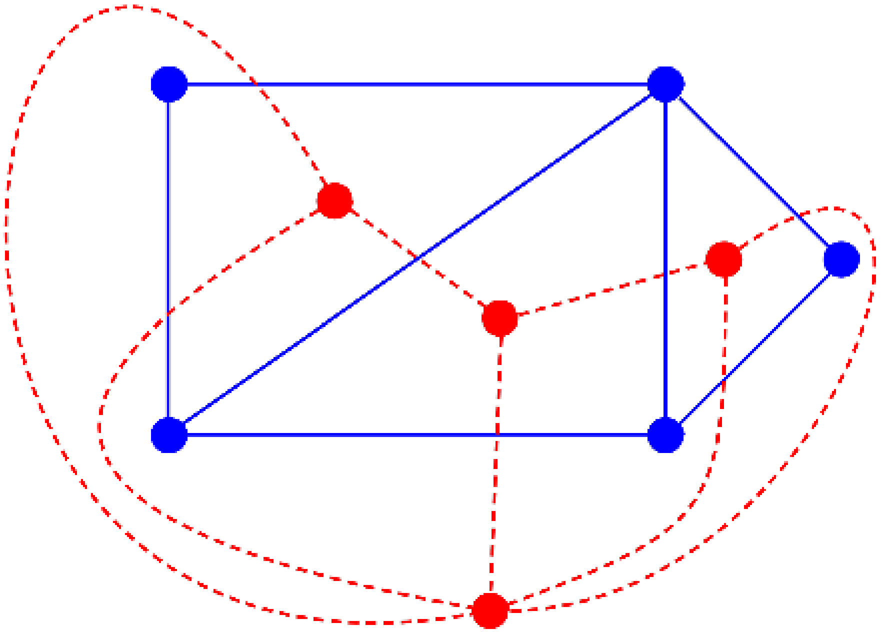
2. Edges of dual graph
   - We connect two vertices $\boldsymbol{v}^*$ related to the faces sharing an edge e in G by drawing an edge $\boldsymbol{e}^*$ that only crosses e.
   - The edges $\boldsymbol{e}^*$ represent the edges of the dual graph G*.

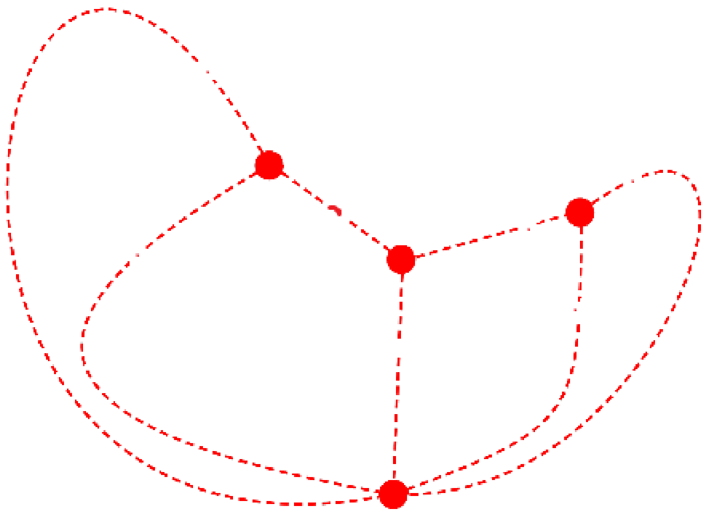
Example



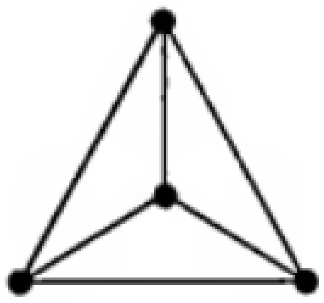
G G^*

Example

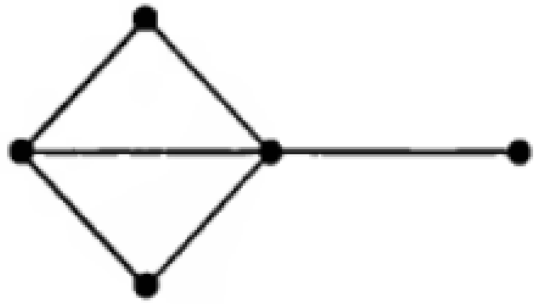




Examples



(a)



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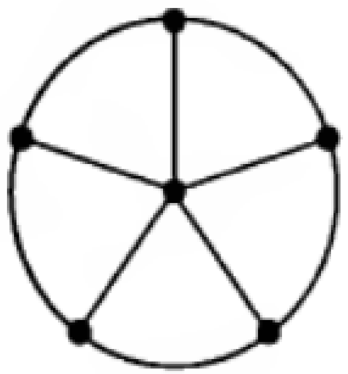


(c)

LEMMA 1

- Let G be a connected planer graph with n vertices, m edges and F faces, and let its geometric dual G^* have n^* vertices, m^* edges and F^* faces. Then

$$\boldsymbol{n}^* = \boldsymbol{F}, \boldsymbol{m}^* = m \text{ and } \boldsymbol{F}^* = \boldsymbol{n}\ .$$

Proof

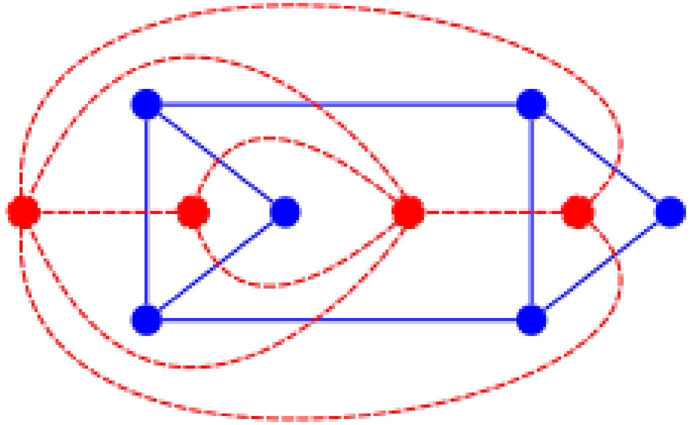
- $\boldsymbol{n}^* \ = \boldsymbol{F}$
  - The first relation mentioned in the statement is a consequence of the definition of G*.

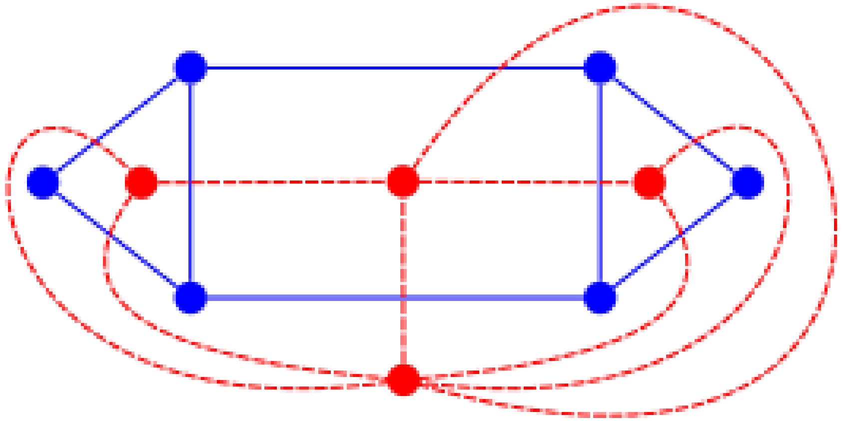
- $\boldsymbol{m}^* = m$
  - A consequence of the definition of G*.
  - According to the definition, vertices in G* are adjacent if the related faces in G share an edge.
  - The number of edges in G* is determined by the number of shared edges between faces in G.

- $\boldsymbol{F}^* = \boldsymbol{n}$
  - By substituting the first two relations into Euler's formula for $\boldsymbol{G}$ and $\boldsymbol{G}^*$.

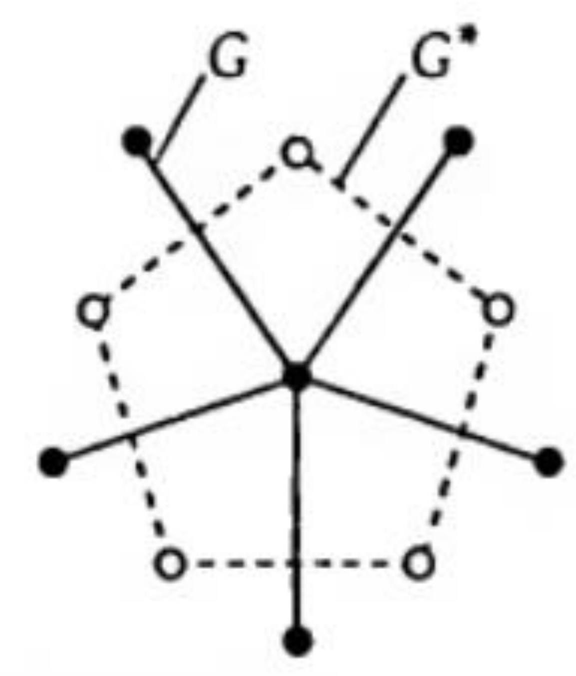
Is the geometrical dual graph unique ?

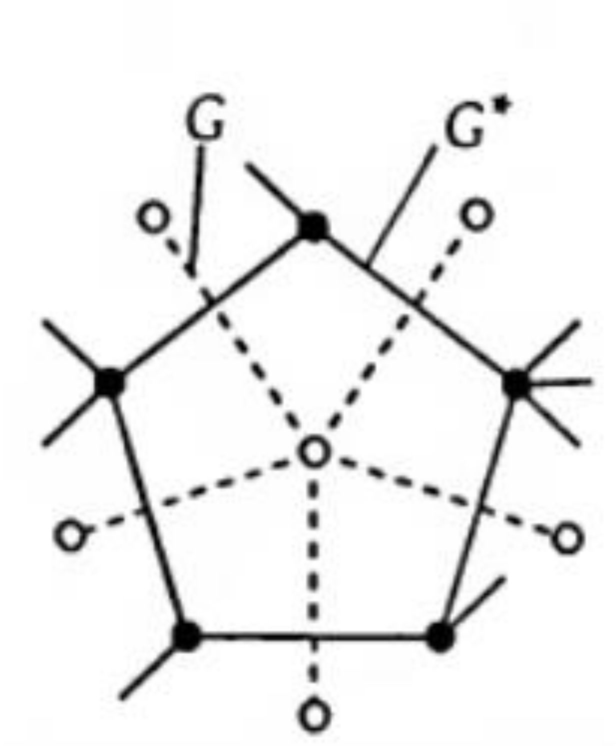
- Different plane drawings of a planar graph G may give rise to non-isomorphic dual graphs G*.





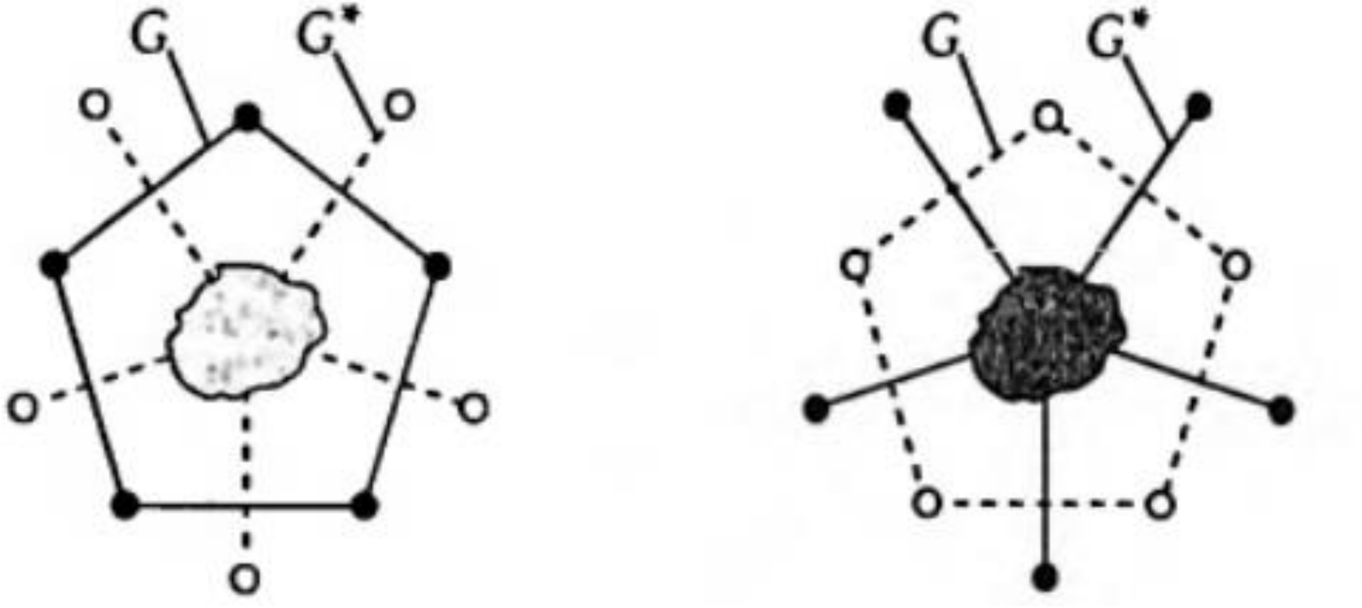
Degrees in dual graph





The degree of a face in graph G corresponds to the degree of the associated vertex in the dual graph G*.

Cuteset and cycle in dual graph



A cycle in graph G is equivalent to a cutset in the dual graph G*.

Plane graph vs dual graph

Plane drawing G	Dual graph G*
Edge of G	Edge of G*
Vertex of degree **k** in G	Face of degree **k** in G*
Face of degree **k** in G	a vertex of degree **k** in G *
Cycle of length **k** in G	Cutset of G* with **k** edges
Cutset of G with **k** edges	Cycle of length **k** in G*

Theorems and corollaries of dual graph

- The theorems and corollaries of planar graphs remain valid in the dual graph:

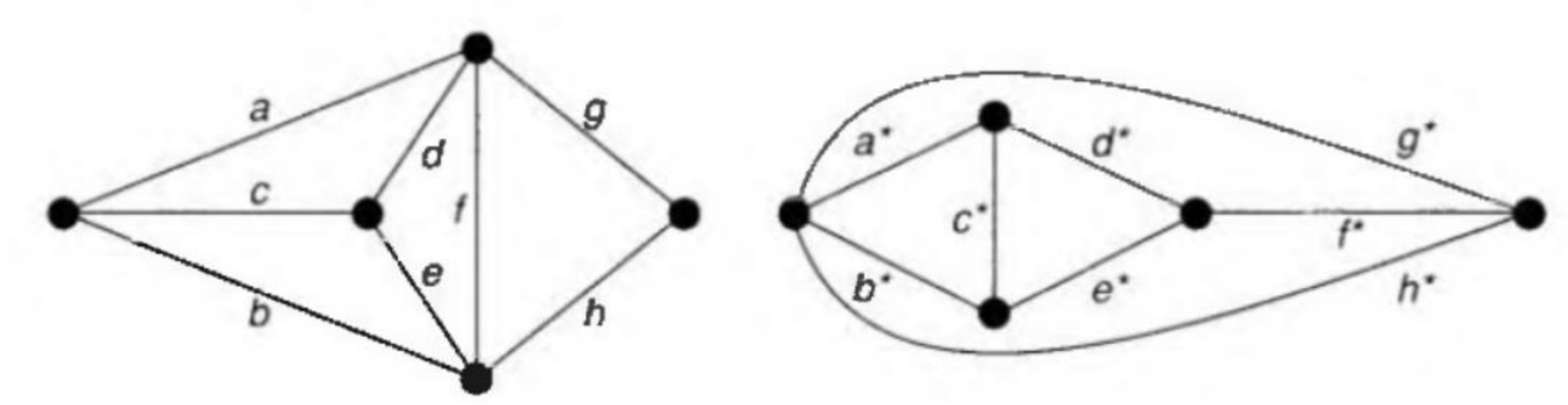
- Let G* be a connected planar graph with **f** faces and **m** edges, and with no cutest with 1 or 2 edges.

  Then $m \leq 3f - 6$

- Let G* be a connected planar graph with no cutset with 1 or 2 edges. Then G* **has a face of degree 5 or less**.

Why dual graph is important ?

- An **abstract dual** of a graph G is a graph G* that:
  - Has a one-to-one correspondence with the edges of G
  - If a set of edges of G forms a cycle in G if and only if the corresponding set of edges of G* forms a cutset in G*.



THEORME

A graph is planar if and only if it has an abstract dual.

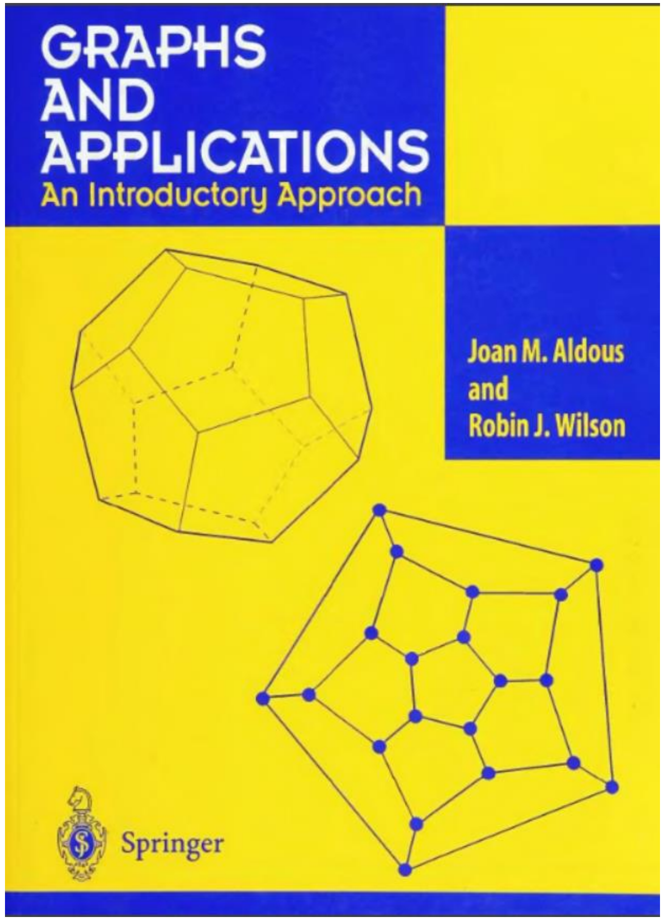
Intuition

- Finding an abstract dual graph indicates planarity.
- A test on the previous slide checks abstract duality without constructing the geometric dual.

Conclusion

- Planarity in graphs is a fundamental concept in computer science, mathematics, and network design.
- Planar graphs can be visually represented without any edges crossing each other.
- Kuratowski's theorem provides a useful criterion for identifying non-planar graphs by detecting the presence of specific subgraphs.
- Determining planarity involves examining planar embeddings and applying Euler's formula: **V - E + F = 2**.
- Duality plays a significant role in understanding the relationships between a graph and its dual representation.
- Abstract dual graphs can be used to test for planarity without constructing the geometrical dual.

References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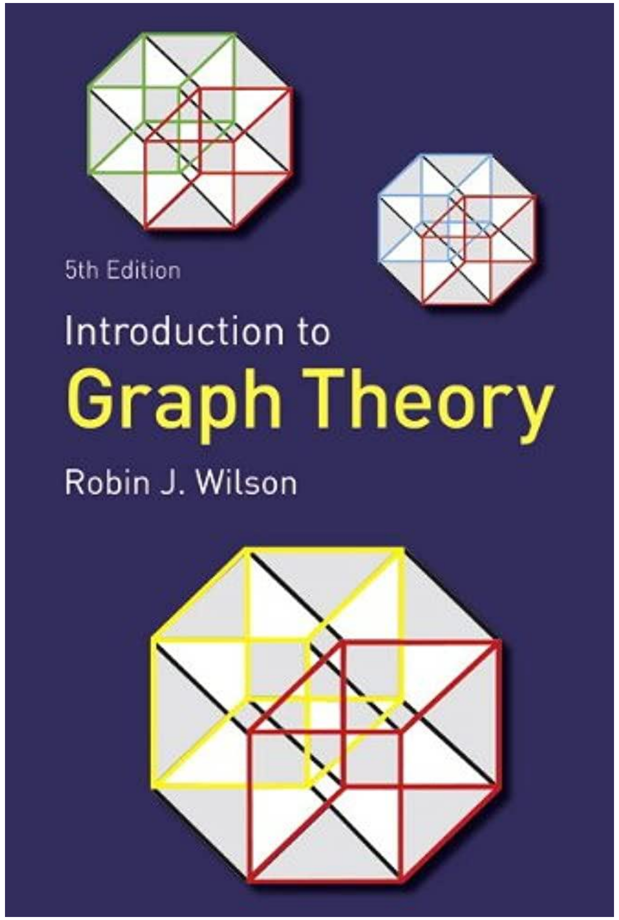
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