ASYMPTOTIC BEHAVIOR OF THE KOBAYASHI METRIC IN THE NORMAL DIRECTION

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Abstract. In this paper we construct a pseudoconvex domain in $\mathbb{C}^3$ where the Kobayashi metric does not blow up at a rate of one over distance to the boundary in the normal direction.

1. Introduction

The asymptotic behaviour of invariant metrics has been studied by several authors, Graham [3], Catlin [1], Krantz [4], Diederich-Herbort [2], Lee [7], and others.

In this paper, we study the behavior of the Kobayashi metric, which is defined as follows:

Definition 1. Let $\Omega \subset \mathbb{C}^n$ be a domain, $Q \in \Omega$ and $X \in T_Q(\Omega)$. The Kobayashi metric $F_K : T\Omega \longrightarrow \mathbb{R}^+ \cup \{0\}$ is defined as

$$F_K(Q,X) = \inf \left\{ \alpha > 0 : \exists \phi \in \Omega(\mathbb{D}),\, \phi(0) = Q,\, \phi'(0) = X/\alpha \right\},$$

where $A(B)$ denotes the family of holomorphic mappings from B to A and $\mathbb{D}$ is the unit disc in $\mathbb{C}$.

Let $\Omega = \{r < 0\} \subset\subset \mathbb{C}^n$ be a smoothly bounded domain and $P \in \partial\Omega$. Let $P_\delta = P - \delta\nu$, where ν is the unit outward normal vector to $\partial\Omega$ at P, i.e., $\nu = \nabla r(P)/\left\|\nabla r(P)\right\|$.

We want to estimate $F_K(P_\delta, \nu)$ as $\delta > 0$ goes to 0, i.e., as the point $P_\delta \in \Omega$ approaches the boundary point P along the normal line to the boundary.

We see that the mapping $\phi(\zeta) = P_\delta + \zeta\delta\nu$ lies in Ω for all $\zeta \in \mathbb{D}$ for $\delta > 0$ small enough. Hence we see that

$$F_K(P_\delta, \nu) \leq \frac{1}{\delta},$$

for $\delta > 0$ small enough.

The question is whether the other direction is true, i.e., whether there exists some constant $C > 0$ such that

$$F_K(P_\delta, \nu) \geq C\frac{1}{\delta} \tag{1}$$

for $\delta > 0$ small enough.

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The answer is "No". It is not true in general. Krantz [4] showed that if P is a strongly pseudoconcave point, then one has the following estimate:

$$F_K(P_\delta, \nu) \approx \frac{1}{\delta^{3/4}}.$$

Ian Graham [3] proved that if Ω is a strongly pseudoconvex domain in $\mathbb{C}^n$, $n \geq 2$, then (1) holds. Catlin [1] showed (1) is true when Ω is a finite type pseudoconvex domain in $\mathbb{C}^2$ and Lee [7] proved it for convex domains in $\mathbb{C}^n$.

So it has been conjectured that the estimate (1) must hold for a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$, $n \geq 2$.

In this paper, we give a counter example to the conjecture. We construct a smoothly bounded infinite type pseudoconvex domain in $\mathbb{C}^3$ where we can find sequences $\delta_n \searrow 0$ and $a_n \nearrow \infty$ such that

$$F_K(P_{\delta_n}, \nu) \leq \frac{1}{a_n \delta_n}, \quad \text{for all } n \in \mathbb{N}. \tag{2}$$

To construct such a domain, we modify the domain in Krantz [5], which is a smoothly bounded pseudoconvex domain in $\mathbb{C}^2$, where there exist sequences $b_n \nearrow \infty$, $c_n \nearrow \infty$ and $\delta_n \searrow 0$ such that

$$F_K(P_{\delta_n}, \nu + b_n T) \leq \frac{1}{c_n \delta_n}, \quad T \in T_P^{\mathbb{C}}(\partial\Omega), \forall n \in \mathbb{N}. \tag{3}$$

We see that the vector $\nu + b_n T$ has a very large tangential component for a large number n. Therefore, as P_{δ_n} approaches the boundary point P, the estimate above gives the estimate for an almost tangential vector, not the normal vector.

In section 2, we give a detailed proof of (3) since the proof is very sketchy in [5]. In section 3, we modify the example in section 2 and construct a smoothly bounded pseudoconvex domain in $\mathbb{C}^3$, where (2) holds.

2. Construction of a domain in $\mathbb{C}^2$

Proposition 2. *For a given increasing sequence $a_n \nearrow \infty$, we can construct a smoothly bounded pseudoconvex domain $\Omega = \{r < 0\} \subset \mathbb{C}^2$ and find a boundary point $P \in \partial\Omega$ and a sequence $\delta_n \searrow 0$ such that*

$$F_K(P_{\delta_n}, X_n) \leq \frac{1}{a_n \delta_n}, \quad P_{\delta_n} = P - \delta_n \nu,$$

where X_n's are vectors such that $(X_n, \nu) = 1$ and ν is a unit outward normal vector to $\partial\Omega$ at P.

Proof. Let $\Omega \subset \mathbb{C}^2$ be a pseudoconvex domain defined as follows:

$$\Omega = \big\{(z, w) \in \mathbb{C}^2 : r(z, w) = \operatorname{Re} w + \rho(z) < 0\big\} \cap B(0, 2),$$

where $\rho(z)$ vanishes to infinite order at 0 and satisfies the following:

$$\rho(z) < \delta_n - a_n \frac{\delta_n}{r_n} \operatorname{Re} z, \quad \forall |z| \leq r_n, \ r_n \searrow 0. \tag{4}$$

Then the analytic disc

$$\phi(\zeta) = (r_n\zeta, -\delta_n + a_n\delta_n\zeta), \quad \zeta \in \mathbb{D}$$

lies inside Ω since, by (4), we have

$$\begin{aligned} r(\phi(\zeta)) &= -\delta_n + a_n\delta_n \mathrm{Re}\,\zeta + \rho(r_n\zeta) \\ &< -\delta_n + a_n\delta_n \mathrm{Re}\,\zeta + \delta_n - a_n\delta_n \mathrm{Re}\,\zeta = 0, \quad \forall \zeta \in \mathbb{D}. \end{aligned} \tag{5}$$

Hence, letting $P = 0$, $P_{\delta_n} = P - \delta_n\nu = (0, -\delta_n)$ and

$$X_n = \frac{r_n}{a_n\delta_n}\frac{\partial}{\partial z} + \frac{\partial}{\partial w},$$

we have

$$\phi'(0) = r_n\frac{\partial}{\partial z} + a_n\delta_n\frac{\partial}{\partial w} = a_n\delta_n X_n.$$

Therefore we get

$$F_K(P_{\delta_n}, X_n) \le \frac{1}{a_n\delta_n}.$$

Construction of ρ.

Choose r_n's such that

$$r_n = a_n^{-1} r_{n-1}^2, \quad r_n \le \frac{1}{4} \tag{6}$$

and let

$$u_n(z) = \frac{1}{8} - \mathrm{Re}\,z + \frac{\log|z|}{4\log a_n}.$$

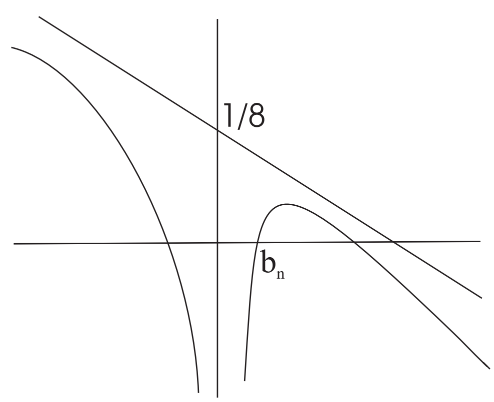
Define a function $R_n(z)$ as follows:

$$R_n(z) = \begin{cases} \max\{u_n(z), 0\}, & \mathrm{Re}\,z \le b_n \\ u_n(z), & \mathrm{Re}\,z > b_n, \end{cases} \tag{7}$$

where $0 < b_n \le 1$ is the smallest positive number such that

$$\frac{1}{8} - b_n + \frac{\log b_n}{4\log a_n} = 0.$$

Such b_n exists for large enough a_n and we may assume a_1 is large enough.



Note that $b_n \searrow 0$ and that $b_n \leq 1/8$.

We show that $R_n(z)$ is subharmonic on $\mathbb{C}$. From (7), we see that the function $R_n(z)$ is subharmonic on $\mathbb{C} \setminus \{\mathrm{Re}\, z = b_n\}$. We can check that that R_n is continuous near $\{\mathrm{Re}\, z = b_n\}$. On the line $\mathrm{Re}\, z = b_n$, we have that $|z| \geq b_n$ and that

$$\frac{1}{8} - \mathrm{Re}\, z + \frac{\log|z|}{4\log a_n} \geq \frac{1}{8} - b_n + \frac{\log b_n}{4\log a_n} = 0, \quad \mathrm{Re}\, z = b_n.$$

Hence we conclude that R_n is continuous and hence $R_n(z)$ is subharmonic on $\mathbb{C}$.

Calculating $u_n(z)$ for $|z| < 2r_n$, we can prove that $R_n(z) \equiv 0$ for all $|z| < 2r_n$.

If $|z| < 2r_n$, $(2r_n \leq b_n)$ then we get

$$\begin{aligned} u_n(z) = \frac{1}{8} - \mathrm{Re}\, z + \frac{\log|z|}{4\log a_n} &\leq \frac{1}{8} + 2a_n^{-1}r_{n-1}^2 + \frac{\log 2a_n^{-1}r_{n-1}^2}{4\log a_n} \\ &\leq \frac{1}{8} + 2a_n^{-1} + \frac{-\log a_n + \log 2r_{n-1}^2}{4\log a_n} \leq -\frac{1}{8} + 2a_n^{-1} < 0. \end{aligned} \tag{8}$$

Therefore we check that

$$R_n(z) \equiv 0, \quad \forall |z| < 2r_n. \tag{9}$$

Also we can easily check that $R_n(z) \leq 3/8 - \mathrm{Re}\, z$ for all $|z| < a_n$. Calculate $u_n(z)$ for $|z| < a_n$:

$$u_n(z) = \frac{1}{8} - \mathrm{Re}\, z + \frac{\log|z|}{4\log a_n} \leq \frac{3}{8} - \mathrm{Re}\, z, \quad \forall |z| < a_n$$

Since b_n is a small number, we have that

$$0 \leq \frac{3}{8} - \mathrm{Re}\, z, \quad \mathrm{Re}\, z \leq b_n.$$

Hence, from (7), we have

$$R_n(z) \leq \frac{3}{8} - \mathrm{Re}\, z, \quad \forall |z| < a_n. \tag{10}$$

Now we make R_n smooth using convolution with a smooth function. Choose a nonconstant C^∞ function $\chi : \mathbb{C} \to \mathbb{R}^+ \cup \{0\}$ such that

$$0 \leq \chi \leq 1, \quad \chi(z) = \chi(|z|), \quad \chi(z) \equiv 0, \text{ if } |z| \geq 1$$

and let

$$m = \int_{\mathbb{C}} \chi(z)\,dxdy.$$

Let $d\mu = dxdy/m$. We define the function $\tilde{R}_n(z)$ as follows

$$\tilde{R}_n(z) = \int_{\mathbb{C}} R_n(z - \epsilon_n w)\chi(w)\,d\mu(w), \quad \epsilon_n < \frac{r_n}{2}. \tag{11}$$

From (9), we have that

$$R_n(z - \epsilon_n w) = 0, \quad \forall |z| < r_n,\ |w| < 1.$$

Hence

$$\tilde{R}_n(z) \equiv 0, \quad |z| < r_n. \tag{12}$$

Note that

$$|u_n(z - \epsilon_n w) - u_n(z)| \le \epsilon_n + \frac{\log\left(1 + \frac{\epsilon_n}{r_n}\right)}{4 \log a_n}, \quad |z| \ge r_n. \tag{13}$$

Therefore, for $|z| \ge r_n$, from (13) we have

$$|R_n(z - \epsilon_n w) - R_n(z)| \le \frac{1}{8} \tag{14}$$

uniformly in $|w| \le 1$ and $|z| \ge r_n$ for small enough ϵ_n. Together with

$$R_n(z) = \tilde{R}_n(z) = 0, \quad \forall |z| < r_n,$$

we get

$$0 \le \tilde{R}_n(z) - R_n(z) \le \frac{1}{8}, \quad \forall z \in \mathbb{C}. \tag{15}$$

The lower estimate $\tilde{R}_n(z) \ge R_n(z)$ holds by subharmonicity of R_n.

Therefore, from (10) and (15), we have

$$\tilde{R}_n(z) \le \frac{1}{2} - \operatorname{Re} z, \quad \forall |z| < a_n. \tag{16}$$

Let

$$\rho_n(z) = \tilde{R}_n\left(\frac{a_n z}{r_n}\right).$$

Then, from (16), we have

$$\rho_n(z) \le \frac{1}{2} - \frac{a_n}{r_n} \operatorname{Re} z, \quad \forall |z| < r_n \tag{17}$$

and, from (15),

$$\left|\tilde{R}_n\left(\frac{a_n z}{r_n}\right) - R_n\left(\frac{a_n z}{r_n}\right)\right| \le \frac{1}{8}, \quad \forall z \in \mathbb{C}.$$

By (12), we have

$$\rho_n(z) = \tilde{R}_n\left(\frac{a_n z}{r_n}\right) \equiv 0, \quad \forall |z| < r_n^2 a_n^{-1}. \tag{18}$$

Also, from (6), we get

$$r_{n+1} = a_{n+1}^{-1} r_n^2 < r_n^2 a_n^{-1}. \tag{19}$$

Therefore, from (18) and (19), we have

$$\rho_n(z) \equiv 0, \quad \forall |z| < r_{n+1}. \tag{20}$$

We define $\rho(z)$ as follows:

$$\rho(z) = \sum_{k=1}^{\infty} \delta_k \rho_k(z), \tag{21}$$

where δ_k's are positive numbers that will be chosen later. Since $r_n \searrow 0$, from (20), we see that

$$\rho(z) = \sum_{k=n}^{\infty} \delta_k \rho_k(z), \quad \forall |z| < r_n.$$

Therefore, from (17), we have

$$\rho(z) = \sum_{k=n}^{\infty} \delta_k \rho_k(z) \leq \frac{\delta_n}{2} - a_n \frac{\delta_n}{r_n} \operatorname{Re} z + \sum_{k=n+1}^{\infty} \delta_k \rho_k(z), \quad |z| < r_n. \tag{22}$$

Now we evaluate $\rho_k(z)$ for $|z| < r_n$, $n < k$.

$$\begin{aligned}\rho_k(z) = \tilde{R}_k\left(\frac{a_k z}{r_k}\right) &\leq R_k\left(\frac{a_k z}{r_k}\right) + \frac{1}{8} \leq \left| u_k\left(\frac{a_k z}{r_k}\right)\right| + \frac{1}{8} \\ &\leq \frac{1}{4} + \frac{a_k}{r_k} r_n + \frac{\log |a_k r_n / r_k|}{4 \log a_k} \leq \frac{1}{4} + \frac{a_k}{r_k} + \frac{\log a_k + \log(r_n/r_k)}{4 \log a_k} \\ &\leq \frac{1}{2} + \frac{a_k}{r_k} + \frac{\log(1/r_k)}{4 \log a_k}, \quad |z| < r_n\end{aligned} \tag{23}$$

Let

$$A_k = \frac{1}{2} + \frac{a_k}{r_k} + \frac{\log(1/r_k)}{4 \log a_k}$$

and choose δ_k's as follows:

$$\delta_k \leq \frac{\delta_{k-1}}{A_k} \frac{1}{2^k}, \quad \delta_k < 1.$$

Then we see that $\delta_k \searrow 0$ and

$$\delta_k \rho_k(z) \leq \delta_k A_k \leq \delta_{k-1} \frac{1}{2^k} \leq \delta_n \frac{1}{2^k}, \quad |z| < r_n, \ k > n.$$

Hence we have

$$\sum_{k=n+1}^{\infty} \delta_k \rho_k(z) \leq \delta_n \sum_{k=n+1}^{\infty} \frac{1}{2^k} \leq \frac{\delta_n}{2}, \quad |z| < r_n. \tag{24}$$

Therefore, from (22) and (24), we get

$$\rho(z) < \delta_n - a_n \frac{\delta_n}{r_n} \operatorname{Re} z, \quad \forall |z| < r_n. \tag{25}$$

Note that we can choose δ_k's small enough that ρ is smooth. □

3. Construction of a domain in $\mathbb{C}^3$

Theorem 3. *For a given increasing sequence $a_n \nearrow \infty$, we can construct a smoothly bounded infinite type pseudoconvex domain $\Omega \subset\subset \mathbb{C}^3$ and find a sequence $\delta_n \searrow 0$ such that with a suitable point $P \in \partial\Omega$, one has*

$$F_K(P_{\delta_n}, \nu) \leq \frac{1}{a_n \delta_n}, \quad P_{\delta_n} = P - \delta_n \nu,$$

where ν is the unit outward normal vector to $\partial\Omega$ at $P \in \partial\Omega$.

Proof. Let $\Omega \subset \mathbb{C}^3$ be a pseudoconvex domain defined as follows:

$$\Omega = \left\{(s,t,w) \in \mathbb{C}^3 : r(s,t,w) = \mathrm{Re}\, w + \tilde{\rho}(s,t) < 0\right\} \cap B(0,2).$$

We construct $\tilde{\rho}(s,t)$ such that for a given sequence $a_n \nearrow \infty$, $a_n \geq 4$, we can find a sequence $\delta_n \searrow 0$ and $r_n \searrow 0$ such that $\tilde{\rho}(s,t)$ satisfies

$$\tilde{\rho}(\zeta^3,\zeta^2) < \delta_n - a_n \frac{\delta_n}{r_n} \mathrm{Re}\, \zeta, \quad \forall |\zeta| \leq r_n.$$

Then the analytic disc

$$\phi(\zeta) = (r_n^3\zeta^3, r_n^2\zeta^2, -\delta_n + a_n\delta_n\zeta), \quad \zeta \in \mathbb{D}$$

lies inside Ω since

$$\begin{aligned} (26) \quad r(\phi(\zeta)) &= -\delta_n + a_n\delta_n \mathrm{Re}\, \zeta + \tilde{\rho}(r_n^3\zeta^3, r_n^2\zeta^2) \\ &< -\delta_n + a_n\delta_n \mathrm{Re}\, \zeta + \delta_n - a_n\delta_n \mathrm{Re}\, \zeta = 0, \quad \forall \zeta \in \mathbb{D}. \end{aligned}$$

Hence, letting $P = 0$ and $P_{\delta_n} = P - \delta_n\nu = (0, -\delta_n)$, we have

$$\phi(0) = P_{\delta_n}, \quad \phi'(0) = a_n\delta_n \frac{\partial}{\partial w}.$$

Therefore we get

$$F_K(P_{\delta_n}, \nu) \leq \frac{1}{a_n\delta_n}.$$

Construction of $\tilde{\rho}$

We modify the function ρ we defined in section 2: Refer (21), (20), and (25).

Here we restate the definition and the properties of ρ:

$$\rho(\zeta) = \sum_{j=1}^{\infty} \delta_j \rho_j(\zeta),$$

where $\rho_j(\zeta)$ satisfies

$$\rho_j(\zeta) \equiv 0, \quad |\zeta| < r_{j+1} \tag{27}$$

$$\rho_j(\zeta) \leq \frac{1}{2} - \frac{a_j}{r_j} \mathrm{Re}\, \zeta, \quad \forall |\zeta| < r_j. \tag{28}$$

Let $z = (s,t) \in \mathbb{C}^2$ and $V = \left\{s^2 - t^3 = 0\right\} \subset \mathbb{C}^2$. We define

$$\tilde{\rho}_n(s,t) = \rho_n(s/t) = \rho_n(\zeta), \quad \text{if } (s,t) = (\zeta^3, \zeta^2) \in V.$$

Now we extend $\tilde{\rho}_n$ to $\mathbb{C}^2$.

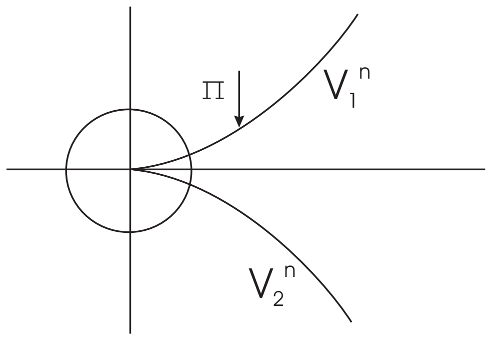
Let $\tilde{r}_n = r_{n+1}^3$ and $B_n = B(0, \tilde{r}_n) \subset \mathbb{C}^2$. If $(s,t) \in B_n \cap V$, then $(s,t) = (\zeta^3, \zeta^2)$ and $|\zeta^3| \leq r_{n+1}^3$ and $|\zeta^2| \leq r_{n+1}^3$. Hence $|\zeta| \leq r_{n+1}$ $(r_{n+1} < 1)$. So we know that

$$\tilde{\rho}_n(s,t) = 0, \quad \forall (s,t) \in V \cap B_n.$$

We let

$$\tilde{\rho}_n(s,t) \equiv 0, \quad \forall (s,t) \in B_n.$$

Let $B'_n = B(0, 3\tilde{r}_n/4) \subset B_n$ and choose a small neighborhood U_n of V such that the projection $\pi : U_n \longrightarrow V$ is well defined on $U_n \setminus B'_n$ and that $U_n \setminus B'_n = \left\{p \in \mathbb{C}^2 : |p - \pi(p)| < d_n\right\}$ for suitable small positive numbers d_n.



We define the projection π more precisely. Let V_1 and V_2 be the two sheets of V: $V_1 = \left\{(r^{3/2}e^{i(3\theta/2+\pi)}, re^{i\theta}) : r, \theta \in \mathbb{R}\right\}$, $V_2 = \left\{(r^{3/2}e^{i(3\theta/2)}, re^{i\theta}) : r, \theta \in \mathbb{R}\right\}$ and let $V_1^n = V_1 \setminus B'_n$ and $V_2^n = V_2 \setminus B'_n$. We consider a biholomorphic mapping

$$\Phi : (s, t) \longrightarrow (s, t - s^{2/3})$$

in a small neighborhood U_n of $V_1^n \cup V_2^n$. Then $\Phi(z) = (s, 0)$ if $z \in V_1^n \cup V_2^n$. We define the projection π as follows:

$$\pi(z) = \Phi^{-1}(\pi_1(\Phi(z))),$$

where $\pi_1(s, t) = (t, 0)$.

We define $\tilde{\rho}_n$ on $U_n \setminus B_n$ as follows:

$$\tilde{\rho}_n(z) = \rho_n(\pi(z)), \quad z \in U_n \setminus B_n.$$

Then the function $\tilde{\rho}_n$ is well defined on $B_n \cup U_n$. Now we extend $\tilde{\rho}_n$ to $\mathbb{C}^2$ Choose a smooth function $h : \mathbb{R} \longrightarrow [0, 1]$ such that

$$h(x) = \begin{cases} 0, & x \in [0, (3/4)^2] \\ 1, & x \geq 1 \end{cases}$$

Let $\chi : \mathbb{R} \longrightarrow [0, 1]$ be a smooth function defined as follows:

$$\chi(x) = \begin{cases} 1, & x \in [0, 1/2] \\ 0, & x \geq 1 \end{cases}$$

and let

$$\chi_n(z) = \chi\left(h\left(\frac{|z|^2}{\tilde{r}_n^2}\right)\frac{|z - \pi(z)|^2}{d_n^2}\right).$$

Then χ_n satisfies

$$\chi_n(z) = \begin{cases} \chi\left(\frac{|z-\pi(z)|^2}{d_n^2}\right), & z \in \mathbb{C}^2 \setminus B_n \\ \chi\left(h\left(\frac{|z|^2}{\tilde{r}_n^2}\right)\frac{|z-\pi(z)|^2}{d_n^2}\right), & z \in B_n \setminus B'_n \\ 1, & z \in B'_n \end{cases}$$

Consider the function $\tilde{\rho}_n\chi_n$. Then $\tilde{\rho}_n\chi_n$ is a smooth extension of $\tilde{\rho}_n$ on $\mathbb{C}^2$ and it satisfies the following:

$$\tilde{\rho}_n\chi_n(z) = \begin{cases} 0, & z \in B_n \\ \tilde{\rho}_n, & z \in U \setminus B_n,\ |z-\pi(z)|^2 \leq \frac{d_n^2}{2} \\ \tilde{\rho}_n\chi\left(\frac{|z-\pi(z)|^2}{d_n^2}\right), & z \in U \setminus B_n,\ \frac{d_n^2}{2} \leq |z-\pi(z)|^2 \leq d_n^2 \\ 0, & z \notin U_n \cup B_n \end{cases}$$

Let

$$p_n(z) = \tilde{\rho}_n\chi_n(z).$$

We can find $C_n \geq 0$ such that

$$\partial\overline{\partial}p_n(z)(L,\overline{L}) \geq -C_n \|L\|^2, \quad \forall z \in B(0,2).$$

Now we add a strictly plurisubharmonic function to make it plurisubharmonic. Note that p_n is plurisubharmonic everywhere except on

$$A_n = \left\{ z \in U_n \setminus B_n : \frac{d_n^2}{2} \leq |z-\pi(z)|^2 \leq d_n^2 \right\}.$$

Let

$$q(z) = e^{|z|^2}|s^2 - t^3|^2, \quad z = (s,t).$$

Then $q(z)$ is strictly plurisubharmonic outside a small neighborhood of V since $\log q$ is strictly plurisubharmonic. Hence we can find a number $c_n > 0$ such that

$$\partial\overline{\partial}q(z)(L,\overline{L}) \geq c_n \|L\|^2, \quad \forall z \in A_n \cap B(0,2)$$

Choose $K_n > 0$ large enough that

$$-C_n + K_n c_n \geq 0$$

Then $p_n + K_n q$ is plurisubharmonic on $A_n \cup D_n$ and hence in $\mathbb{C}^2$. Let

$$r_n = p_n + K_n q.$$

Then

$$r_n(\zeta^3,\zeta^2) = p_n(\zeta^3,\zeta^2) = \tilde{\rho}_n(\zeta) = \rho_n(\zeta).$$

Hence

$$r_n(\zeta^3,\zeta^2) = \begin{cases} 0, & |\zeta| < r_{n+1} \\ \leq \frac{1}{2} - \frac{a_n}{r_n}\operatorname{Re}\zeta, & |\zeta| < r_n. \end{cases}$$

Let

$$\tilde{\rho}(z) = \sum_j \delta_j r_j,$$

and choose δ_j's small enough that $\tilde{\rho}$ is smooth in a small neighborhood of 0. $\square$

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