

ON THE SZEGŐ METRIC

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ABSTRACT. We introduce a new biholomorphically invariant metric based on Fefferman's invariant Szegő kernel and investigate the relation of the new metric to the Bergman and Carathéodory metrics. A key tool is a new absolutely invariant function assembled from the Szegő and Bergman kernels.

1. INTRODUCTION

In this paper we introduce the Szegő metric, which is defined similarly to the Bergman metric using the Szegő kernel instead of the Bergman kernel. The well-known Szegő kernel $S(z, \zeta)$ is a reproducing kernel for $H^2(\partial\Omega)$ (the closure in $L^2(\partial\Omega)$ of the set of holomorphic functions that are continuous up to the boundary); thus

$$f(z) = \int_{\partial\Omega} S(z, \zeta) f(\zeta) d\sigma_E(\zeta), \quad \forall f \in H^2(\partial\Omega)$$

where σ_E stands for the Euclidean surface measure on $\partial\Omega$. The problem with this definition though is that, unlike the volume measure on Ω , the Euclidean surface measure is not transformed nicely under a biholomorphic mapping. To resolve this issue, Fefferman introduced the Fefferman surface area measure, σ_F (p. 259 of [11]).

We define the Szegő metric using the Szegő kernel with respect to the Fefferman surface area measure. Hence it is invariant under biholomorphic mappings. (When Ω is not simply-connected the invariance is subject to a topological condition on the biholomorphic mapping – see Proposition 2 for details.)

In section 2, we provide background information on the Fefferman surface measure and define the Szegő metric. In section 3, we introduce a biholomorphically invariant function $SK_\Omega(z, w)$ which serves to compare the Bergman and Szegő kernels and then proceed to use this function to derive a number of asymptotic results relating the Szegő and Bergman metrics. In section 4, we show that the Szegő metric is always greater than or equal to the Carathéodory metric. In section 5 we show that there is no universal upper bound or positive lower bound for the ratio of the Szegő and Bergman metrics.

Standing assumption. We assume throughout this paper that $\Omega = \{\rho < 0\} \subset \subset \mathbb{C}^n$ is a strongly pseudoconvex domain with C^∞ boundary. (We note however that the Szegő kernel and metric discussed in this paper will be naturally interpretable on many other domains; transformation laws such as Propositions 1, 2, 3 and Theorem 1 below will hold with additional hypotheses on Φ as needed.)

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2. BACKGROUND

Let $H^2(\Omega)$ be the closure in $L^2(\partial\Omega)$ of $A(\Omega) = \mathcal{O}(\Omega) \cap C(\overline{\Omega})$. Then there exists a sesqui-holomorphic Szegő kernel $S(z, \cdot)$ such that

$$(2.1) \quad f(z) = \int_{\partial\Omega} S(z, \zeta) f(\zeta) d\sigma_F(\zeta), \quad \forall f \in H^2(\Omega)$$

where σ_F is the Fefferman measure defined as follows:

$$d\sigma_F \wedge d\rho = c_n \sqrt{-\det \begin{pmatrix} 0 & \rho_{\bar{k}} \\ \rho_j & \rho_{j\bar{k}} \end{pmatrix}_{1 \leq j, k \leq n}} dV$$

or equivalently

$$d\sigma_F = c_n \sqrt{-\det \begin{pmatrix} 0 & \rho_{\bar{k}} \\ \rho_j & \rho_{j\bar{k}} \end{pmatrix}_{1 \leq j, k \leq n}} \frac{d\sigma_E}{\|d\rho\|},$$

where σ_E is the usual Euclidean surface measure and $\rho_j = \frac{\partial \rho}{\partial z_j}$, $\rho_{j\bar{k}} = \frac{\partial \rho}{\partial z_j \partial \bar{z}_k}$.

Note that the surface measure σ_F does not depend on the choice of the defining function ρ ; one can check this letting $\tilde{\rho} = h\rho$, where $h > 0$ is a smooth function, and calculating $d\sigma_F$ with $\tilde{\rho}$.

Remark 1. The constant c_n used above is a dimensional constant which was left unspecified in [11] but has been assigned different values later for convenience in different contexts: for example, $c_n = 2^{2n/(n+1)}$ in [1] and $c_n = 1$ in [14].

Proposition 1. *Let $\Phi : \Omega_1 \longrightarrow \Omega_2$ be a biholomorphic mapping. Then we have*

$$\int_{\partial\Omega_2} |f|^2 d\sigma_F^{\partial\Omega_2} = \int_{\partial\Omega_1} |f \circ \Phi|^2 |\det J_{\mathbb{C}}\Phi|^{\frac{2n}{n+1}} d\sigma_F^{\partial\Omega_1},$$

where $\sigma_F^{\partial\Omega_j}$ denotes the Fefferman measure on Ω_j for $j = 1, 2$ and $J_{\mathbb{C}}\Phi$ is the complex Jacobian matrix of Φ .

Proof. Recall that Φ extends to a diffeomorphism between $\overline{\Omega}_1$ and $\overline{\Omega}_2$ [9].

Let $\Phi : \Omega_1 \longrightarrow \Omega_2$ be a biholomorphic mapping and $\Omega_2 = \{\rho < 0\}$. Then we have

$$d\sigma_F^{\partial\Omega_1} \wedge d(\rho \circ \Phi) = c_n \sqrt{-\det \begin{pmatrix} 0 & (\rho \circ \Phi)_{\bar{k}} \\ (\rho \circ \Phi)_j & (\rho \circ \Phi)_{j\bar{k}} \end{pmatrix}} dV_{\Omega_1}.$$

Since

$$\begin{pmatrix} 0 & (\rho \circ \Phi)_{\bar{k}} \\ (\rho \circ \Phi)_j & (\rho \circ \Phi)_{j\bar{k}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & J_{\mathbb{C}}\Phi \end{pmatrix} \begin{pmatrix} 0 & \rho_{\bar{k}} \\ \rho_j & \rho_{j\bar{k}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \overline{J_{\mathbb{C}}\Phi} \end{pmatrix},$$

we get

$$\det \begin{pmatrix} 0 & (\rho \circ \Phi)_{\bar{k}} \\ (\rho \circ \Phi)_j & (\rho \circ \Phi)_{j\bar{k}} \end{pmatrix} = \det \begin{pmatrix} 0 & \rho_{\bar{k}} \\ \rho_j & \rho_{j\bar{k}} \end{pmatrix} |\det J_{\mathbb{C}}\Phi|^2.$$

Therefore we have

$$\begin{aligned}
d\sigma_F^{\partial\Omega_1} \wedge d(\rho \circ \Phi) &= c_n |\det J_{\mathbb{C}}\Phi|^{2/(n+1)} \cdot \sqrt[n+1]{-\det \begin{pmatrix} 0 & \rho_{\bar{k}} \\ \rho_j & \rho_{j\bar{k}} \end{pmatrix}} |\det J_{\mathbb{R}}\Phi^{-1}|^2 dV_{\Omega_2} \\
&= c_n |\det J_{\mathbb{C}}\Phi|^{2/(n+1)} \cdot |\det J_{\mathbb{C}}\Phi|^{-2} \cdot \sqrt[n+1]{-\det \begin{pmatrix} 0 & \rho_{\bar{k}} \\ \rho_j & \rho_{j\bar{k}} \end{pmatrix}} dV_{\Omega_2} \\
&= |\det J_{\mathbb{C}}\Phi|^{-2n/(n+1)} d\sigma_F^{\partial\Omega_2} \wedge d\rho
\end{aligned}$$

and it follows that $d\sigma_F^{\partial\Omega_2}$ pulls back to $|\det J_{\mathbb{C}}\Phi|^{\frac{2n}{n+1}} d\sigma_F^{\partial\Omega_1}$. \square

Proposition 2. *Let $\Phi : \Omega_1 \longrightarrow \Omega_2$, $\Omega_1, \Omega_2 \subset \mathbb{C}^n$ be a biholomorphic mapping. Assume there exists a well-defined holomorphic branch of $(\det J_{\mathbb{C}}\Phi(z))^{n/(n+1)}$ on Ω_1 . Then we have*

$$(2.2) \quad S_{\Omega_1}(z, w) = S_{\Omega_2}(\Phi(z), \Phi(w)) (\det J_{\mathbb{C}}\Phi(z))^{n/(n+1)} \left(\overline{\det J_{\mathbb{C}}\Phi(w)} \right)^{n/(n+1)},$$

where $S_{\Omega_j}(z, w)$ is the Szegő kernel on Ω_j for $j = 1, 2$.

Proof. It is obvious that the right hand side of (2.2) is anti-holomorphic with respect to w , so it will suffice to show that it also satisfies the reproducing property.

Let $f \in H^2(\Omega_1)$. Then we get

$$\begin{aligned}
&\int_{\partial\Omega_1} S_{\Omega_2}(\Phi(z), \Phi(w)) (\det J_{\mathbb{C}}\Phi(z))^{n/(n+1)} \left(\overline{\det J_{\mathbb{C}}\Phi(w)} \right)^{n/(n+1)} f(w) d\sigma_F^{\partial\Omega_1}(w) \\
&= (\det J_{\mathbb{C}}\Phi(z))^{n/(n+1)} \int_{\partial\Omega_2} S_{\Omega_2}(\Phi(z), \tilde{w}) \left(\overline{\det J_{\mathbb{C}}\Phi(\Phi^{-1}(\tilde{w}))} \right)^{n/(n+1)} \\
&\quad f(\Phi^{-1}(\tilde{w})) |\det J_{\mathbb{C}}\Phi^{-1}(\tilde{w})|^{2n/(n+1)} d\sigma_F^{\partial\Omega_2}(\tilde{w}).
\end{aligned}$$

Note that $\left(\overline{\det J_{\mathbb{C}}\Phi(\Phi^{-1}(\tilde{w}))} \right)^{n/(n+1)} |\det J_{\mathbb{C}}\Phi^{-1}(\tilde{w})|^{2n/(n+1)}$ is holomorphic with respect to \tilde{w} since we have

$$\begin{aligned}
&\left(\overline{\det J_{\mathbb{C}}\Phi(\Phi^{-1}(\tilde{w}))} \right)^{n/(n+1)} |\det J_{\mathbb{C}}\Phi^{-1}(\tilde{w})|^{2n/(n+1)} \\
&= \left(\overline{\det J_{\mathbb{C}}\Phi(\Phi^{-1}(\tilde{w}))} \right)^{n/(n+1)} |\det J_{\mathbb{C}}\Phi(\Phi^{-1}(\tilde{w}))|^{-2n/(n+1)} \\
&= (\det J_{\mathbb{C}}\Phi(\Phi^{-1}(\tilde{w})))^{-n/(n+1)}.
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
&\int_{\partial\Omega_1} S_{\Omega_2}(\Phi(z), \Phi(w)) (\det J_{\mathbb{C}}\Phi(z))^{n/(n+1)} \left(\overline{\det J_{\mathbb{C}}\Phi(w)} \right)^{n/(n+1)} f(w) d\sigma_F^{\partial\Omega_1}(w) \\
&= (\det J_{\mathbb{C}}\Phi(z))^{n/(n+1)} (\det J_{\mathbb{C}}\Phi(\Phi^{-1}(\Phi(z))))^{-n/(n+1)} f(\Phi^{-1}(\Phi(z))) \\
&= f(z)
\end{aligned}$$

as required. \square

Definition 1. We define the Szegő metric on Ω at z in the direction ξ , $F_S^\Omega(z, \xi)$, as follows:

$$F_S^\Omega(z, \xi) = \left(\sum_{j,k=1}^n \frac{\partial^2 \log S_\Omega(z, z)}{\partial z_j \partial \bar{z}_k} \xi_j \bar{\xi}_k \right)^{1/2}.$$

Remark 2. Note that one can write $S_\Omega(z, w) = \sum_\alpha \phi_\alpha(z) \overline{\phi_\alpha(w)}$ where the ϕ_α 's form an orthonormal basis of $H^2(\partial\Omega)$. Hence $S_\Omega(z, z)$ is a positive strongly plurisubharmonic function, ensuring that $F_S^\Omega(z, \xi)$ is a genuine Kähler metric. The orthonormal expansion may also be used to show that

$$(2.3) \quad F_S^\Omega(z, \xi) \geq \gamma_\Omega |\xi| \text{ for some positive constant } \gamma_\Omega.$$

Remark 3. Note that $F_S^\Omega(z, \xi)$ does not depend on the choice of the dimensional constant c_n discussed in Remark 1.

Proposition 3. *The Szegő metric is invariant under biholomorphic mappings satisfying the hypotheses of Proposition 2, i.e., if $\Phi : \Omega_1 \rightarrow \Omega_2$ is such a mapping and $z \in \Omega_1$, $\xi \in T_z \Omega_1$, then*

$$F_S^{\Omega_1}(z, \xi) = F_S^{\Omega_2}(\Phi(z), J_{\mathbb{C}}\Phi(z)\xi).$$

Proof. From (2.2), we have

$$S_{\Omega_1}(z, z) = S_{\Omega_2}(\Phi(z), \Phi(z)) |\det J_{\mathbb{C}}\Phi(z)|^{2n/(n+1)}.$$

Hence we have

$$\log S_{\Omega_1}(z, z) = \log S_{\Omega_2}(\Phi(z), \Phi(z)) + \frac{n}{n+1} \left[\log(\det J_{\mathbb{C}}\Phi(z)) + \log(\overline{\det J_{\mathbb{C}}\Phi(z)}) \right].$$

Let $\Phi(z) = w$. Then

$$\begin{aligned} \sum_{j,k} \frac{\partial^2 \log S_{\Omega_1}(z, z)}{\partial z_j \partial \bar{z}_k} \xi_j \bar{\xi}_k &= \sum_{j,k} \sum_{l,m} \frac{\partial^2 \log S_{\Omega_2}(\Phi(z), \Phi(z))}{\partial w_l \partial \bar{w}_m} \frac{\partial w_l}{\partial z_j} \frac{\partial \bar{w}_m}{\partial \bar{z}_k} \xi_j \bar{\xi}_k \\ &= \sum_{l,m} \frac{\partial^2 \log S_{\Omega_2}(w, w)}{\partial w_l \partial \bar{w}_m} (J_{\mathbb{C}}\Phi(z)\xi)_l (\overline{J_{\mathbb{C}}\Phi(z)\xi})_m. \end{aligned}$$

□

2.1. The Szegő metric on the unit ball. Let $\mathbb{B}^n = \{\rho = |z|^2 - 1 < 0\} \subset \mathbb{C}^n$. Then

$$\det \begin{pmatrix} 0 & \rho_{\bar{k}} \\ \rho_j & \rho_{j\bar{k}} \end{pmatrix} = -1 \quad \text{on } \partial\mathbb{B}^n.$$

Hence $d\sigma_F^{\partial\mathbb{B}^n} = \frac{c_n}{2} d\sigma_E^{\partial\mathbb{B}^n}$ for $S = \{|z|^2 = 1\} \subset \mathbb{C}^n$ and the Szegő kernel for the unit ball in \mathbb{C}^n is given by

$$(2.4) \quad S(z, \zeta) = \frac{1}{c_n} \frac{(n-1)!}{\pi^n} \frac{1}{(1 - z \cdot \bar{\zeta})^n}.$$

One can rewrite (2.4) as follows:

$$\int_{\partial\mathbb{B}^n} S(z, \zeta) f(\zeta) d\sigma_F(\zeta) = \int_{\partial\mathbb{B}^n} \frac{(n-1)!}{\pi^n} \frac{1}{(1 - z \cdot \bar{\zeta})^n} f(\zeta) d\sigma_E(\zeta) = f(z), \quad \forall f \in H^2(\partial\mathbb{B}^n).$$

If we calculate the Szegő metric for \mathbb{B}^n at the origin, we get

$$\log S(z, z) = \log \left(\frac{(n-1)!}{c_n \cdot 2\pi^n} \right) - n \log(1 - |z|^2),$$

and

$$\left. \frac{\partial^2 \log S(z, z)}{\partial z_j \partial \bar{z}_k} \right|_{z=0} = \begin{cases} n \frac{\bar{z}_j z_k}{(1-|z|^2)^2} \Big|_{z=0} = 0, & j \neq k \\ n \frac{|z_j|^2}{(1-|z|^2)^2} + n \frac{1}{(1-|z|^2)} \Big|_{z=0} = n, & j = k \end{cases}.$$

Hence we have

$$(2.5) \quad F_S^{\mathbb{B}^n}(0, \xi) = \sqrt{n} |\xi|.$$

Remark 4. Note that the Bergman metric on the unit ball in \mathbb{C}^n evaluated at the origin is given as

$$(2.6) \quad F_B^{\mathbb{B}^n}(0, \xi) = \sqrt{n+1} |\xi|$$

and the Kobayashi or Carathéodory metric on the unit ball in \mathbb{C}^n at the origin is given as

$$(2.7) \quad F_K^{\mathbb{B}^n}(0, \xi) = F_C^{\mathbb{B}^n}(0, \xi) = |\xi|.$$

Since all four metrics are invariant under the automorphism group of \mathbb{B}^n which acts transitively on \mathbb{B}^n , relations between the metrics at the origin will propagate throughout \mathbb{B}^n . In particular, from (2.5), (2.6) and (2.7) we obtain

$$(2.8) \quad F_S^{\mathbb{B}^n}(z, \xi) = \sqrt{n} F_C^{\mathbb{B}^n}(z, \xi) = \sqrt{n} F_K^{\mathbb{B}^n}(z, \xi) = \sqrt{\frac{n}{n+1}} F_B^{\mathbb{B}^n}(z, \xi), \quad \forall z \in \mathbb{B}^n.$$

3. AN INVARIANT FUNCTION AND SOME BOUNDARY ASYMPTOTICS

Theorem 1. *Let*

$$(3.1) \quad SK_\Omega(z, w) = \frac{S_\Omega(z, w)^{n+1}}{K_\Omega(z, w)^n},$$

where S_Ω and K_Ω are the Szegő and Bergman kernels on Ω . Then $SK_\Omega(z, w)$ is invariant under biholomorphic mappings satisfying the hypotheses of Proposition 2, i.e., if $\Phi : \Omega_1 \rightarrow \Omega_2$ is such a mapping then we have

$$SK_{\Omega_1}(z, w) = SK_{\Omega_2}(\Phi(z), \Phi(w)).$$

Proof. It is a well-known fact (see for example section 6.1 in [7]) that

$$(3.2) \quad K_{\Omega_1}(z, w) = (\det J_{\mathbb{C}} \Phi(z)) K_{\Omega_2}(\Phi(z), \Phi(w)) \left(\det \overline{J_{\mathbb{C}} \Phi(w)} \right).$$

Hence from (2.2) and (3.2), we get

$$\begin{aligned} \frac{S_{\Omega_1}(z, w)^{n+1}}{K_{\Omega_1}(z, w)^n} &= \frac{S_{\Omega_2}(\Phi(z), \Phi(w))^{n+1} (\det J_{\mathbb{C}} \Phi(z))^n \left(\det \overline{J_{\mathbb{C}} \Phi(w)} \right)^n}{K_{\Omega_2}(\Phi(z), \Phi(w))^n (\det J_{\mathbb{C}} \Phi(z))^n \left(\det \overline{J_{\mathbb{C}} \Phi(w)} \right)^n} \\ &= \frac{S_{\Omega_2}(\Phi(z), \Phi(w))^{n+1}}{K_{\Omega_2}(\Phi(z), \Phi(w))^n}. \end{aligned}$$

□

Remark 5. One can easily calculate $SK_{\mathbb{B}^n}(z, z)$, where \mathbb{B}^n is the unit ball in \mathbb{C}^n , using (2.4) and the well known formula

$$K_{\mathbb{B}^n}(z, w) = \frac{n!}{\pi^n} \frac{1}{(1 - z \cdot \bar{w})^{n+1}}$$

for the Bergman kernel on the unit ball to obtain

$$SK_{\mathbb{B}^n}(z, z) = \frac{1}{c_n^{n+1}} \frac{(n-1)!}{(n\pi)^n}, \quad \forall z \in \mathbb{B}^n.$$

For the remainder of this section we assume that the defining function ρ for Ω has been chosen to satisfy Fefferman's approximate Monge-Ampère equation

$$-\det \begin{pmatrix} \rho & \rho_{\bar{k}} \\ \rho_j & \rho_{j\bar{k}} \end{pmatrix}_{1 \leq j, k \leq n} = 1 + O(|\rho|^{n+1})$$

(see [10] – we could also use the not-completely-smooth exact solution to this equation [6, 17]).

We set $r = -\rho$; thus $r > 0$ in Ω .

We have the following asymptotic expansions of the Bergman and Szegő kernels (see [9, 12, 14] and additional references cited in these papers, but the material we are quoting is set forth especially clearly in section 1.1 and Lemma 1.2 from [15]):

$$K_{\Omega}(z, z) = \begin{cases} \frac{n!}{\pi^n r^{n+1}} + \frac{(n-2)! q_{\Omega}}{r^{n-1}} + O\left(\frac{1}{r^{n-2}}\right), & n \geq 3 \\ \frac{2}{\pi^2 r^3} + 3\tilde{q}_{\Omega} \cdot \log r + O(1), & n = 2 \end{cases}$$

$$S_{\Omega}(z, z) = \begin{cases} \frac{(n-1)!}{c_n \pi^n r^n} + \frac{(n-3)! q_{\Omega}}{c_n r^{n-2}} + O\left(\frac{1}{r^{n-3}}\right), & n \geq 4 \\ \frac{2}{c_3 \pi^3 r^3} + \frac{q_{\Omega}}{c_3 r} + O(|\log r|), & n = 3, \\ \frac{1}{c_2 \pi^2 r^2} + \mu_1 + \frac{\tilde{q}_{\Omega}}{c_2} \cdot r \log r + O(r), & n = 2 \end{cases}$$

where $\mu_1 \in C^{\infty}(\overline{\Omega})$ and q_{Ω} and \tilde{q}_{Ω} are certain local geometric boundary invariants – in terms of Moser's normal form [8] we have $q_{\Omega} = \frac{2}{3\pi^n} \|A_{2\bar{2}}^0\|^2$ for $n \geq 3$ and $\tilde{q}_{\Omega} = -\frac{8}{\pi^2} A_{4\bar{4}}^0$ for $n = 2$. Moreover, $r^{n+1} K_{\Omega}(z, z) \in C^{n+1-\epsilon}(\overline{\Omega})$ and $r^n S_{\Omega}(z, z) \in C^{\max\{n, 3\}-\epsilon}(\overline{\Omega})$ for each $\epsilon > 0$. (The remainder terms are equal to a power of r times a first-degree polynomial in $\log r$ with coefficients in $C^{\infty}(\overline{\Omega})$; later in this section the remainder terms have a similar structure but with higher degree in $\log r$.)

Combining these results we obtain the following.

Theorem 2. *The function $SK_{\Omega}(z, z)$ satisfies*

$$SK_{\Omega}(z, z) \in \begin{cases} C^{n-\epsilon}(\overline{\Omega}), & n \geq 3 \\ C^{4-\epsilon}(\overline{\Omega}), & n = 2 \end{cases}$$

with asymptotics

$$SK_{\Omega}(z, z) = \begin{cases} \frac{(n-1)!}{c_n^{n+1}(n\pi)^n} + \frac{(n-3)! \cdot 3q_{\Omega}}{c_n^{n+1}n^n} r^2 + O(r^3), & n \geq 4 \\ \frac{2}{c_3^4(3\pi)^3} + \frac{q_{\Omega}}{c_3^4 \cdot 9} r^2 + O(r^3 |\log r|), & n = 3 \\ \frac{1}{c_2^2 4\pi^2} + \mu_2 r^2 + \mu_3 r^4 \log r + \frac{3\pi^2 q_{\Omega}^2}{c_2^3 16} r^6 \log^2 r + O(r^6 \log r), & n = 2 \end{cases}$$

for z close to the boundary, where $\mu_2, \mu_3 \in C^{\infty}(\overline{\Omega})$.

We will use this result to examine the relation between the Bergman and Szegő metrics. It will be helpful to introduce the quantity

$$E(z, \xi) = (n+1) (F_S^{\Omega}(z, \xi))^2 - n (F_B^{\Omega}(z, \xi))^2.$$

Theorem 3. *For $n \geq 3$ the following hold.*

- (a) $E \in C^{n-2-\epsilon}(T\overline{\Omega})$.
- (b) *There are constants $0 < m_{\Omega} < M_{\Omega} < \infty$ so that*

$$m_{\Omega} F_S^{\Omega}(z, \xi) \leq F_B^{\Omega}(z, \xi) \leq M_{\Omega} F_S^{\Omega}(z, \xi)$$

on $T\Omega$.

- (c) $E(z, \xi) = 0$ when $z \in \partial\Omega$ and ξ lies in the maximal complex subspace of $T_z\partial\Omega$.
- (d) $E(z, \xi) \equiv 0$ on all of $T(\partial\Omega)$ if and only if the boundary is locally spherical.
- (e) *If Ω is simply connected then $E(z, \xi) \equiv 0$ on $T\overline{\Omega}$ if and only if Ω is biholomorphic to the ball.*

Proof. We start by noting that

$$(3.3) \quad E(z, \xi) = \sum_{j,k=1}^n \frac{\partial^2 (\log SK_{\Omega}(z, z))}{\partial z_j \partial \bar{z}_k} \xi_j \bar{\xi}_k.$$

Then (a) follows from the smoothness result in Theorem 2. Statement (b) then follows from (a) and (2.3) along with the Bergman version of (2.3).

For $z \in \partial\Omega$ we use (3.3) and Theorem 2 to conclude that

$$E(z, \xi) = \frac{6\pi^n q_{\Omega}}{(n-1)(n-2)} \sum_{j,k=1}^n r_j r_{\bar{k}} \xi_j \bar{\xi}_k$$

and thus $E(z, \xi) = 0$ when $\sum_{j=1}^n r_j \xi_j = 0$, verifying (c). From the same computation we see that E will vanish on all of $T(\partial\Omega)$ if and only if the invariant q_{Ω} vanishes identically, so from Corollary 2.5 in [5] it follows that (d) holds.

The “if” half of (e) follows from (2.8) and the invariance properties. The “only if” half follows from (d) along with Theorem C in [7] (see also [18] and section 8 of [4]). \square

For $n = 2$ we have instead the following result.

Theorem 4. *For $n = 2$ the following hold.*

- (a) $E \in C^{2-\epsilon}(T\overline{\Omega})$.

(b) *There are constants $0 < m_\Omega < M_\Omega < \infty$ so that*

$$m_\Omega F_S^\Omega(z, \xi) \leq F_B^\Omega(z, \xi) \leq M_\Omega F_S^\Omega(z, \xi)$$

on $T\Omega$.

(c) *If $E \in C^4(T\bar{\Omega})$ then the boundary is locally spherical.*

(d) *If Ω is simply connected then $E \in C^4(T\bar{\Omega})$ if and only if Ω is biholomorphic to the ball, in which case we in fact have $E(z, \xi) \equiv 0$ on $T\bar{\Omega}$.*

Proof. We need to explain part (c), everything else falling into place as before.

If $E \in C^4(T\bar{\Omega})$ then the $r^6 \log^2 r$ term from the expansion in Theorem 2 must disappear, forcing $\tilde{q}_\Omega \equiv 0$. Using an argument of Burns appearing as Theorem 3.2 in Graham's paper [12] along with the previously cited material from [15] we obtain revised expansions

$$\begin{aligned} K_\Omega(z, z) &= \frac{2}{\pi^2 r^3} + \mu_4 + 9q_\Omega^* r \log r + O(r) \\ S_\Omega(z, z) &= \frac{1}{c_2 \pi^2 r^2} + \mu_5 + \frac{1}{c_2} q_\Omega^* r^2 \log r + O(r^2) \\ SK_\Omega(z, z) &= \frac{1}{c_2^3 4\pi^2} + \mu_6 r^2 - \frac{3q_\Omega^*}{c_2^3 2} r^4 \log r + O(r^4) \\ E(z, \xi) &= \mu_7 - 72q_\Omega^* r^2 \log r \sum_{j,k=1}^n r_j r_{\bar{k}} \xi_j \bar{\xi}_k + O(r^2), \end{aligned}$$

where $q_\Omega^* = \frac{8}{15\pi^2} |A_{24}^0|^2$ and $\mu_4, \mu_5, \mu_6 \in C^\infty(\bar{\Omega})$, $\mu_7 \in C^\infty(T\bar{\Omega})$. Our smoothness assumption on E now forces $q_\Omega^* \equiv 0$ and this in turn implies that the boundary is spherical. (We note that by Proposition 1.9 in [12], the condition $\tilde{q}_\Omega \equiv 0$ alone does not guarantee that the boundary is spherical.) \square

For the sake of completeness we also record the corresponding results in one dimension.

Theorem 5. *For $n = 1$ the following hold.*

(a) *$E \in C^\infty(T\bar{\Omega})$.*

(b) *There are constants $0 < m_\Omega < M_\Omega < \infty$ so that*

$$m_\Omega F_S^\Omega(z, \xi) \leq F_B^\Omega(z, \xi) \leq M_\Omega F_S^\Omega(z, \xi)$$

on $T\Omega$.

(c) *If Ω is simply connected then $E(z, \xi) \equiv 0$.*

Proof. (a) follows from Theorem 23.2 in [2] and the well-known fact that $rS_\Omega(z, z)$ and $r^2K_\Omega(z, z)$ are in $C^\infty(\bar{\Omega})$ and are nowhere vanishing on $\bar{\Omega}$. Statement (b) follows from (a) as in the proof of Theorem 3 above.

(c) follows from (2.8), invariance properties and the Riemann mapping theorem. \square

4. COMPARISON WITH THE CARATHÉODORY METRIC

In this section we discuss the comparison between the Carathéodory and Szegő metrics and show that the Szegő metric is always greater than or equal to the Carathéodory metric. The proof follows the same method that was used to show that the Bergman metric is greater than or equal to the Carathéodory metric in [13].

We define the Carathéodory metric on a domain $\Omega \subset \mathbb{C}^n$ at $p \in \Omega$ in the direction $\xi \in \mathbb{C}^n$, $F_C^\Omega(p, \xi)$, as

$$F_C^\Omega(p, \xi) = \sup \left\{ \left(\sum_{j=1}^n \left| \frac{\partial \phi(p)}{\partial z_j} \xi_j \right|^2 \right)^{1/2} : \phi \in \mathcal{O}(\Omega, \Delta), \phi(p) = 0 \right\},$$

where $\mathcal{O}(\Omega, \Delta)$ denotes the set of holomorphic mappings from Ω to Δ , the unit disc in \mathbb{C} .

Theorem 6. *The Szegő metric is greater than or equal to the Carathéodory metric.*

Proof. One can show that

$$(4.1) \quad (F_S^\Omega(p, \xi))^2 = \frac{\sup \left\{ |\xi g(p)|^2 : g \in H^2(\partial\Omega), g(p) = 0, \|g\|_{L^2(\partial\Omega)} = 1 \right\}}{S(p, p)}$$

using the Hilbert space method. Refer to Theorem 6.2.5 in [16] for further details.

Let $p \in \Omega$. We have

$$\|S(\cdot, p)\|_{L^2(\partial\Omega)}^2 = \|S(p, \cdot)\|_{L^2(\partial\Omega)}^2 = \int_{\partial\Omega} \overline{S(p, \zeta)} S(p, \zeta) d\sigma_F(\zeta) = \overline{S(p, p)} = S(p, p).$$

Let $\phi : \Omega \rightarrow \Delta$ be a holomorphic function with $\phi(p) = 0$. Define a holomorphic function $g : \Omega \rightarrow \Delta$ as follows:

$$g(z) = \frac{S(z, p)}{\sqrt{S(p, p)}} \phi(z).$$

Then $\|g\|_{L^2(\partial\Omega)} \leq 1$ and $g(p) = 0$. Hence from (4.1) we get

$$(4.2) \quad (F_S^\Omega(p, \xi))^2 \geq \frac{|\xi g(p)|^2}{S(p, p)} = \frac{S(p, p)^2 |\xi \phi(p)|^2}{S(p, p)^2} = |\xi \phi(p)|^2.$$

Therefore we get $F_S^\Omega(p, \xi) \geq F_C^\Omega(p, \xi)$. □

Remark 6. This argument works on any smoothly bounded pseudoconvex domain where the Szegő metric is defined.

Remark 7. The equation (2.8) shows that the inequality $F_S^\Omega(p, \xi) \geq F_C^\Omega(p, \xi)$ is sharp even in some cases where $F_C^\Omega(p, \xi) > 0$, whereas we have $F_B^\Omega(p, \xi) \gtrsim F_C^\Omega(p, \xi)$ if $F_C^\Omega(p, \xi) > 0$ [16].

5. COMPARISON WITH THE BERGMAN METRIC

In this section we carry out some computations on annuli to show that the constants m_Ω and M_Ω in Theorem 5 must depend on Ω .

Theorem 7. *There are no constants $0 < m < M < \infty$ independent of Ω with the property that*

$$m F_S^\Omega(z, \xi) \leq F_B^\Omega(z, \xi) \leq M F_S^\Omega(z, \xi)$$

on $T\Omega$.

Proof. The results of Proposition 4 below show that

$$F_S^{\Omega_r}(\sqrt{r}, 1)/F_B^{\Omega_r}(\sqrt{r}, 1) \rightarrow \infty$$

and

$$F_S^{\Omega_r}(\sqrt[5]{r}, 1)/F_B^{\Omega_r}(\sqrt[5]{r}, 1) \rightarrow 0$$

as $r \rightarrow 0$, where $\Omega_r = \{z \in \mathbb{C} : r < |z| < 1\}$. □

Proposition 4. *Let $\Omega_r = \{r < |z| < 1\} \subset \mathbb{C}$ and $r \in (0, 1)$. We have*

$$\lim_{r \rightarrow 0} \frac{F_B^{\Omega_r}(\sqrt{r}, 1)}{\sqrt{\log(1/r)}} = 2 \quad \text{and} \quad \lim_{r \rightarrow 0} \sqrt{r} \cdot F_S^{\Omega_r}(\sqrt{r}, 1) = \frac{1}{2}.$$

Also,

$$\lim_{r \rightarrow 0} \frac{F_B^{\Omega_r}(\sqrt[5]{r}, 1)}{\sqrt{\log(1/r)}} = \sqrt{2} \quad \text{and} \quad \lim_{r \rightarrow 0} F_S^{\Omega_r}(\sqrt[5]{r}, 1) = 1.$$

Proof. On the boundary of a planar domain, the Fefferman measure is $\frac{c_1}{2} ds$, where ds denotes the element of arclength. In view of Remark 3, we may set $c_1 = 2$ so that $d\sigma_F = ds$.

The Szegő and Bergman spaces of Ω_r admit orthonormal bases $\{a_n(r)z^n\}_{n \in \mathbb{Z}}$ and $\{b_n(r)z^n\}_{n \in \mathbb{Z}}$ with $a_n(r)$ and $b_n(r) \geq 0$; thus $B_r(z, \zeta) = \sum_{n \in \mathbb{Z}} (b_n(r))^2 z^n \bar{\zeta}^n$ and $S_r(z, \zeta) = \sum_{n \in \mathbb{Z}} (a_n(r))^2 z^n \bar{\zeta}^n$.

One can calculate $a_n(r)$ and $b_n(r)$ as follows:

$$\begin{aligned} \int_{\partial\Omega_r} |a_n(r)z^n|^2 ds &= \int_{|z|=r} r^{2n} |a_n(r)|^2 ds + \int_{|z|=1} |a_n(r)|^2 ds \\ &= |a_n(r)|^2 2\pi(r^{2n+1} + 1) = 1, \end{aligned}$$

hence

$$|a_n(r)|^2 = \frac{1}{2\pi(1 + r^{2n+1})}, \quad n \in \mathbb{Z}.$$

Also we have

$$\begin{aligned} \int_{\Omega_r} |b_n(r)z^n|^2 dA &= \int_0^{2\pi} \int_r^1 |b_n(r)|^2 t^{2n} t dt d\theta \\ &= |b_n(r)|^2 2\pi \frac{1}{2n+2} (1 - r^{2n+2}) = 1, \quad n \neq -1, \\ \int_{\Omega_r} |b_{-1}(r)z^{-1}|^2 dA &= \int_0^{2\pi} \int_r^1 |b_{-1}(r)|^2 \frac{1}{t} dt d\theta = |b_{-1}(r)|^2 2\pi \ln(1/r) = 1, \end{aligned}$$

and so

$$|b_n(r)|^2 = \begin{cases} \frac{n+1}{\pi} \cdot \frac{1}{1-r^{2n+2}}, & n \neq -1, \\ \frac{1}{2\pi \ln(1/r)}, & n = -1. \end{cases}$$

Let $B_r(z, \zeta)$ and $S_r(z, \zeta)$ be the Bergman and Szegő kernel on Ω_r respectively and $z \in \Omega_r$. We have

$$\begin{aligned} (F_B^{\Omega_r}(z, 1))^2 &= \frac{\partial \bar{\partial} \log B_r(z, z)}{\partial z \partial \bar{z}} \\ &= \frac{B_r(z, z) \cdot (B_r(z, z))_{z\bar{z}} - |(B_r(z, z))_z|^2}{(B_r(z, z))^2} \\ &= \frac{\beta_0(z, r) \cdot \beta_2(z, r) - |\beta_1(z, r)|^2}{(\beta_0(z, r))^2}, \end{aligned}$$

where

$$\beta_0(z, r) = B_r(z, z), \quad \beta_1(z, r) = (B_r(z, z))_z, \quad \text{and} \quad \beta_2(z, r) = (B_r(z, z))_{z\bar{z}}.$$

We also get

$$(F_S^{\Omega_r}(z, 1))^2 = \frac{\alpha_0(z, r) \cdot \alpha_2(z, r) - |\alpha_1(z, r)|^2}{(\alpha_0(z, r))^2},$$

where

$$\alpha_0(z, r) = S_r(z, z), \quad \alpha_1(z, r) = (S_r(z, z))_z, \quad \text{and} \quad \alpha_2(z, r) = (S_r(z, z))_{z\bar{z}}.$$

Let us calculate $\alpha_j(r^q, r)$ for $j = 0, 1, 2$, $q > 0$ and estimate $F_S^{\Omega_r}(r^q, 1)$:

$$\begin{aligned} 2\pi \alpha_0(r^q, r) &= \sum_{n \in \mathbb{Z}} \frac{1}{(1 + r^{2n+1})} r^{2nq}, \\ 2\pi \alpha_1(r^q, r) &= \sum_{n \in \mathbb{Z}} \frac{1}{(1 + r^{2n+1})} \cdot n \cdot r^{(2n-1)q}, \\ 2\pi \alpha_2(r^q, r) &= \sum_{n \in \mathbb{Z}} \frac{1}{(1 + r^{2n+1})} \cdot n^2 \cdot r^{2(n-1)q}. \end{aligned}$$

Note that

$$\begin{aligned} 2\pi \alpha_0(\sqrt{r}, r) &= \frac{2}{1 + \sqrt{r}} + \frac{2r}{1 + r^3} + O(r^2), \\ 2\pi \alpha_1(\sqrt{r}, r) &= \frac{-1}{\sqrt{r}(1 + r)} - \frac{\sqrt{r}}{1 + r^3} + O(r^{3/2}), \\ 2\pi \alpha_2(\sqrt{r}, r) &= \frac{1}{r(1 + r)} + \frac{5}{1 + r^3} + O(r), \end{aligned}$$

and that

$$\begin{aligned} 2\pi \alpha_0(\sqrt[5]{r}, r) &= \frac{1}{1 + r} + \frac{r^{2/5}}{1 + r^3} + O(r^{3/5}), \\ 2\pi \alpha_1(\sqrt[5]{r}, r) &= \frac{r^{1/5}}{1 + r^3} - \frac{r^{2/5}}{1 + r} + O(r^{3/5}), \\ 2\pi \alpha_2(\sqrt[5]{r}, r) &= \frac{1}{1 + r^3} + \frac{r^{1/5}}{1 + r} + O(r^{2/5}), \end{aligned}$$

which one can verify easily using the comparison test with the geometric series. Therefore we get

$$\lim_{r \rightarrow 0} r \cdot (F_S^{\Omega_r}(\sqrt{r}, 1))^2 = \frac{1}{4}, \quad \text{and} \quad \lim_{r \rightarrow 0} (F_S^{\Omega_r}(\sqrt[5]{r}, 1))^2 = 1.$$

One can calculate $\beta_j(r^q, r)$'s for $j = 1, 2, 3$ and estimate $F_B^{\Omega_r}(r^q, 1)$ in a similar way:

$$\begin{aligned} \pi \beta_0(r^q, r) &= \left(\sum_{n \in \mathbb{Z} \setminus \{-1\}} \frac{(n+1)}{1 - r^{2n+2}} r^{2nq} \right) + \frac{1}{2r^{2q} \log(1/r)}, \\ \pi \beta_1(r^q, r) &= \left(\sum_{n \in \mathbb{Z} \setminus \{-1\}} \frac{(n+1)}{1 - r^{2n+2}} \cdot n \cdot r^{(2n-1)q} \right) - \frac{1}{2r^{3q} \log(1/r)}, \\ \pi \beta_2(r^q, r) &= \left(\sum_{n \in \mathbb{Z} \setminus \{-1\}} \frac{(n+1)}{1 - r^{2n+2}} n^2 r^{(2n-2)q} \right) + \frac{1}{2r^{4q} \log(1/r)}. \end{aligned}$$

We have

$$\begin{aligned} \pi \beta_0(\sqrt{r}, r) &= \frac{1}{2r \log(1/r)} + \frac{2}{1 - r^2} + O(r), \\ \pi \beta_1(\sqrt{r}, r) &= -\frac{1}{2r^{3/2} \log(1/r)} - \frac{2}{\sqrt{r}(1 - r^2)} + O(\sqrt{r}), \\ \pi \beta_2(\sqrt{r}, r) &= \frac{1}{2r^2 \log(1/r)} + \frac{4}{r(1 - r^2)} + O(1), \end{aligned}$$

and

$$\begin{aligned}\pi \beta_0(\sqrt[5]{r}, r) &= \frac{1}{2r^{2/5} \log(1/r)} + \frac{1}{1-r^2} + O(r^{2/5}), \\ \pi \beta_1(\sqrt[5]{r}, r) &= -\frac{1}{2r^{3/5} \log(1/r)} + \frac{2r^{1/5}}{1-r^4} + O(r^{3/5}), \\ \pi \beta_2(\sqrt[5]{r}, r) &= \frac{1}{2r^{4/5} \log(1/r)} + \frac{2}{1-r^4} + O(r^{2/5}).\end{aligned}$$

Therefore we get

$$\lim_{r \rightarrow 0} \frac{(F_B^{\Omega_r}(\sqrt{r}, 1))^2}{\log(1/r)} = 4 \quad \text{and} \quad \lim_{r \rightarrow 0} \frac{(F_B^{\Omega_r}(\sqrt[5]{r}, 1))^2}{\log(1/r)} = 2.$$

□

Remark 8. We note that the Szegő and Bergman kernels of Ω_r can be written in closed form in terms of elliptic functions (see for example [3]) though that is not particularly helpful for the computations above.

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