Intro to Financial Engineering IEOR W4700

Homework 1

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Problem 1.

If S settles on $T+2$, then $P_{t+2} = 50$.

Then, $P_{t+1} = \frac{P_{t+2}}{d}$ where d is the discount factor for 1 month.

$d = \left(1 + \frac{r}{12}\right) = \left(1 + \frac{0.1}{12}\right)$ due to annual compounding.

Then, $P_{t+1} = \frac{50}{\left(1+\frac{0.1}{12}\right)} = 49.58678$

I'd pay $49.58678 for the stock.

Problem 2.

Recall that the Taylor series for $f(x)$ at $x = a$ is

$$f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ...$$

Applying this for $\log 1 + x$ function at $1 + x = 1$, $x = 0$ is

$$0 + x - \frac{x^2}{2} + \frac{x^3}{3} + ...$$

$$
\begin{aligned}
(1+r)^n &= 2 \\
n \log(1+r) &= \log 2 \\
n\left(r - \frac{r^2}{2}\right) &= 0.69
\end{aligned}
$$

Solving for n,

$$
\begin{aligned}
n &= \frac{0.69}{\left(r - \frac{r^2}{2}\right)} \\
&= \frac{0.69}{r\left(1 - \frac{r}{2}\right)} \\
&= \frac{0.69\left(1 + \frac{r}{2}\right)}{r\left(1 - \frac{r}{2}\right)\left(1 + \frac{r}{2}\right)} \\
&\approx \frac{0.69\left(1 + \frac{r}{2}\right)}{r} \\
&= \frac{69(1 + \frac{i}{200}}{i}
\end{aligned}
$$

The approximation that $\left(1 - \frac{r}{2}\right)\left(1 + \frac{r}{2}\right) = 1 - \frac{r^2}{4} \approx 1$ works because r is assumed to be small (close to 0), hence $\frac{r^2}{4} \approx 0$

Problem 3.

Express both equations in terms of annual 365-day interest rate r:

$$
\begin{aligned}
r &= (1 + \frac{y_{SB}}{2})^2 \\
r &= (1 + y_{AM})\frac{360}{365}
\end{aligned}
$$

Then equating both,

$$
\begin{aligned}
(1 + y_{AM}) &= \left(1 + \frac{y_{SB}}{2}\right)^2 \frac{365}{360} \\
y_{AM} &= \left(1 + \frac{y_{SB}}{2}\right)^2 \frac{365}{360} - 1
\end{aligned}
$$

Problem 4.

(a).

Let R be the compound interest rate.

(i).

$$
\begin{aligned}
100(1+R)^N &= 100(1+rN) \\
N\log(1+R) &= \log(1+rN) \\
\log(1+R) &= \frac{\log(1+0.05*10)}{10} \\
R &= \exp\left(\frac{\log(1+0.05*10)}{10}\right) - 1 = 0.04137974
\end{aligned}
$$

(ii).

$$
\begin{aligned}
100\left(1+\frac{R}{4}\right)^{4N} &= 100(1+rN) \\
4N\log\left(1+\frac{R}{4}\right) &= \log(1+rN) \\
\log\left(1+\frac{R}{4}\right) &= \frac{\log(1+0.05*10)}{40} \\
R &= 4\left(\exp\left(\frac{\log(1+0.05*10)}{40}\right) - 1\right) = 0.04075271
\end{aligned}
$$

(iii).

$$
\begin{aligned}
100 \exp(RN) &= 100(1+rN) \\
RN &= \log(1+rN) \\
R &= \frac{\log(1+rN)}{N} = \frac{\log(1+0.05*10)}{10} = 0.04054651
\end{aligned}
$$

(b).

$$
\begin{aligned}
N\log(1+R) &= \log(1+rN) \\
R &= \exp\left(\frac{\log(1+rN)}{N}\right) - 1 \\
&= (1+rN)^{\frac{1}{N}} - 1
\end{aligned}
$$

To find $\lim_{N\to\infty}(1+rN)^{\frac{1}{N}}$, we let $x = (1+rN)^{\frac{1}{N}}$. Then, $\log x = \frac{\log(1+rN)}{N}$. Now we apply L'hospital's rule:

$$
\begin{aligned}
\frac{d\log(1+rN)}{dN} &= \frac{r}{1+rN} \\
\frac{dN}{dN} &= 1
\end{aligned}
$$

Then,

$$
\begin{aligned}
\lim_{N\to\infty} \log\left((1+rN)^{\frac{1}{N}}\right) &= \lim_{N\to\infty} \frac{r}{1+rN} = 0 \\
\lim_{N\to\infty}(1+rN)^{\frac{1}{N}} &= \exp 0 = 1 \\
\lim_{N\to\infty}(1+rN)^{\frac{1}{N}} - 1 &= 0
\end{aligned}
$$

Problem 5.

Solve by replication.

The forward contract at time t with delivery price K at time T is$F(t,T,K)$. At expiration, payoff is $S(T)-K$.

To replicate this, buy $\frac{S(t)}{1+b}$ of stock S, and borrow $\frac{K}{1+r}$ dollars. At expiration, I will have $S(T)$ of stocks and K dollars to be returned, making my eventual position $S(T)-K$.

Then, $F = \frac{S(t)}{1+b} - \frac{K}{1+r}$. To make the forward value 0,

$$\begin{aligned}
\frac{S(t)}{1+b} - \frac{K}{1+r} &= 0 \\
K &= \frac{S(t)}{(1+r)}(1+b)
\end{aligned}$$

Problem 6.

The forward contract is still worth $S(T) - K$ at expiration.

Now, each $S(t)$ entitles the holder to D amount of dividend at time T. Then, buying $\frac{S(t)}{1+b}$ of stock in time t will result in $\frac{D}{1+b}$ worth of dividend in time T. To offset that, borrow $\frac{D}{(1+b)(1+r)}$ worth of dollars in time t.

Then, to replicate, buy $\frac{S(t)}{1+b}$ worth of stocks, borrow $\frac{K}{1+r} + \frac{D}{(1+r)(1+d)}$ dollars at time t. Then, at time T, this would be worth $S(T) - K$.

Then the forward contract is worth $F = \frac{S(t)}{1+b} - \frac{K}{1+r} - \frac{D}{(1+b)(1+r)}$. Setting the forward contract to be value,

$$\begin{aligned}
\frac{S(t)}{1+b} &= \frac{K}{1+r} + \frac{D}{(1+b)(1+r)} \\
K &= \frac{1+r}{1+b}S(t) - \frac{D}{1+b}
\end{aligned}$$

Problem 7.

The forward contract pays $X(T) - K$ at time T, where $X(t)$ is the dollar-yen rate.

(a).

At expiration, the forward contract pays $X(T) - K$.

To replicate, buy $\frac{X(t)}{1+r_X}$ yen today for $X(t)$ and invest in Japan at rate r_X. At the same time, borrow $\frac{K}{1+r_{US}}$. At expiration time T, this position will be worth $X(T) - K$.

Then, the forward contract at time t is worth $F = \frac{X(t)}{1+r_X} - \frac{K}{1+r_{US}}$. To make the forward contract value 0,

$$\frac{X(t)}{1+r_X} = \frac{K}{1+r_{US}}$$
$$K = \frac{1+r_{US}}{1+r_X}X(t)$$

(b).

From the derivation process above, one can basically replicate the forward contract by

At time t,

- Buy $\frac{X(t)}{1+r_X}$ yen and earn yen money rate.
- Borrow $\frac{K}{1+r_{US}}$ dollars and pay dollar money rate.

making the whole position worth $\frac{X(t)}{1+r_X} - \frac{K}{1+r_{US}}$.
At time T, the position will be worth $X(T) - K$. This combination replicates a forward contract for dollar-yen one year forward.

Problem 8.

Assuming borrowing cost $r = 0$ and 0 carry cost,

(a).

Total profit π is

$$\pi = 1000 * (70.50 - 68.30) = 2200$$

(b).

$$\pi_{2015} = 1000 * (69.10 - 68.30) = 800$$

$$\pi_{2016} = 1000 * (70.50 - 69.10) = 1400$$

Problem 9.

Since the company is Japanese, we calculate profits in ¥.

For contract 1, the forward contract buys \$1m with ¥at rate F_1 ¥/\$. Then, for \$1m, the company will pay F_1 ¥and receive S ¥per \$. Hence, in ¥, the company profits $(S - F_1) * 1,000,000$ ¥.

For contract 2, the forward contract sells \$1m with ¥at rate F_2 ¥/\$. Then, for \$1m, the company will receive F_2 ¥and pay S ¥per \$. Hence, in ¥, the company profits $(F_2 - S) * 1,000,000$ ¥.

The company's overall profits $\pi_{¥} = 1,000,000 * (S - F_1 + F_2 - S) = 1,000,000 * (F_2 - F_1)$¥.

The company is essentially making a curve (divergence between F_1 and F_2 bet instead of a directional bet, and the PnL is independent of the final spot rate S.