Intro to Financial Engineering IEOR W4700

Homework 2

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Problem 1.

We can think of the house as a series of yearly payments x for 20 years whose present value equates 20000. Then,

$$20000 = \frac{x}{0.05}\left(1 - \frac{1}{(1+0.05)^{20}}\right)$$

Solving for x, $x = 1604.852$

```
> f = function(x){(x/0.05) * (1-1/((1+0.05)^20)) - 20000}
> uniroot(f, c(-99999999, 99999999))
$root
[1] 1604.852

$f.root
[1] 4.366302e-08

$iter
[1] 2

$init.it
[1] NA

$estim.prec
[1] 6.103516e-05
```

Now, we know the yearly "coupon" of the roof. Since our current roof only has 5 years left, then the value of the house is the payment for 5 years.

$$PV_R = \sum_{i=1}^{5} \frac{x}{(1+0.05)^i} = \frac{x}{0.05}\left(1 - \frac{1}{(1+0.05)^5}\right) = 6948.168$$

The value of the existing roof if 6948.168

Problem 2.

(a).

The stream of cashflow is

$$PV = -3x + \frac{5}{1+r} + \frac{x}{(1+r)^2}$$

For $PV > 0$,

$$\begin{aligned}
-3x + \frac{5}{1+r} + \frac{x}{(1+r)^2} &> 0 \\
-3x(1+r)^2 + 5(1+r) + x &> 0 \\
-3x(1+r)^2 + x &> -5(1+r) \\
x(-3(1+r)^2 + 1) &> -5(1+r) \\
x &> \frac{-5(1+r)}{-3(1+r)^2 + 1} \\
x &> \frac{5(1+r)}{3(1+r)^2 - 1}
\end{aligned}$$

(b).

IRR is the r such that $PV = 0$ or

$$-3x + \frac{5}{1+r} + \frac{x}{(1+r)^2} = 0$$

Then, let $c = \frac{1}{1+r}$.

$$-3x + 5c + xc^2$$

We need

$$\begin{aligned} r &> 0 \\ 1 + r &> 1 \\ \frac{1}{1+r} &> 1 \\ c &> 1 \end{aligned}$$

Solving quadratically,

$$c = \frac{-5 \pm \sqrt{5^2 - 4(-3x)(x)}}{2x} = \frac{-5 \pm \sqrt{25 + 12x^2}}{2x}$$

Discard the strictly negative root since it won't fulfill our condition of $c > 1$. Using the possibly positive root, we need $c > 1$ or

$$\frac{-5 + \sqrt{25 + 12x^2}}{2x} > 1$$

$x = 0$ or $x = 2.5$

```
> install.packages("rootSolve")
> library(rootSolve)
> g = function(x) {(-5+sqrt(25 + 12*x^2)) / (2*x) - 1}
> uniroot.all(g, c(-9, 9))
[1] 2.5
```

$x > 2.5$ guarantees a strictly positive IRR.

Problem 3.

This can be treated as monthly payments with interest rate 0.01 since rent is paid monthly. Assume monthly compounding.

Compare PV:

- **Stay** $PV = \frac{1000}{0.01}\left(1 - \frac{1}{(1+0.01)^6}\right) = 5795.476$

- **Switch** $PV = 1000 + \frac{900}{0.01}\left(1 - \frac{1}{(1+0.01)^6}\right) = 6215.929$

Within 6 months, the couple should stay since staying costs them less in PV terms.

For 1 year, compare PV again:

- **Stay** $PV = \frac{1000}{0.01}\left(1 - \frac{1}{(1+0.01)^12}\right) = 11255.08$
- **Switch** $PV = 1000 + \frac{900}{0.01}\left(1 - \frac{1}{(1+0.01)^6}\right) = 11129.57$

For 1 year, the couple should switch since switching costs them less in PV terms.

Problem 4.

x_K is the number of put options purchased whose strike price is K. Let x_L is the number of put options purchased whose strike price is L. To arb this, find x_K and x_L such that

$$\begin{aligned} -P(t,T,K)x_K - P(t,T,L)x_L &\geq 0 \\ \max\left[0,(K-S(T))\right]x_K + \max\left[0,(L-S(T))\right]x_L &\geq 0 \end{aligned}$$

Both inequalities need to hold true (since one represents time t and the other time of expiration T). Furthermore, one of the inequalities need to be strict (for us to be able to profit not just break even).

The max notation is really annoying, so let's get rid of it by specifying ranges of $S(T)$.

- When $L < S(T)$

  x_K and x_L are 0, holding both inequalities true.
- When $K < S(T) < L$

  The two inequalities become:

  $$\begin{aligned} -P(t,T,K)x_K - P(t,T,L)x_L &\geq 0 \\ (L-S(T))x_L &\geq 0 \end{aligned}$$

  Then, $x_L \geq 0$ from the second inequality.

- When $S(T) < K$

  The two inequalities become:

$$
\begin{aligned}
-P(t,T,K)x_K - P(t,T,L)x_L \geq 0 \\
(K - S(T))x_K + (L - S(T))x_L \geq 0
\end{aligned}
$$

  From the first inequality, $x_K \leq \frac{-P(t,T,L)x_L}{P(t,T,K)}$

  From the second inequality, $x_K \geq \frac{-(L-S(T))x_L}{(K-S(T))}$. Then, we require that

$$\frac{-(L - S(T))x_L}{(K - S(T))} \leq x_K \leq \frac{-P(t,T,L)x_L}{P(t,T,K)}$$

  Now if $\frac{P(t,T,L)}{P(t,T,K)} > \frac{L}{K}$, then $\frac{-P(t,T,L)}{P(t,T,K)} < \frac{-L}{K}$. Then, we can find a small $S(T)$ such that $\frac{-P(t,T,L)}{P(t,T,K)} < \frac{-(L-S(T))}{(K-S(T))}$. Then there would be no solution for x_K from the above two inequalities.

Thus there is no arbitrage strategy.

Problem 5.

(a).

$$PV = \frac{1}{(1+r)} + \frac{2}{(1+r)^2} + ... + \frac{N}{(1+r)^N}$$

Denote present value with n periods as $P(n)$. Then, $P(0) = 0$.

$$
\begin{aligned}
P(N) &= \frac{1}{(1+r)} + \frac{2}{(1+r)^2} + ... + \frac{N}{(1+r)^N} \\
&= \frac{1}{(1+r)} + \frac{1}{(1+r)^2} + ... + \frac{1}{(1+r)^N} + \frac{N-1}{(1+r)^N} \\
&= \frac{1}{r}\left(1 - \frac{1}{(1+r)^N}\right) + \frac{P(N-1)}{(1+r)} \\
&= \frac{1}{r}\left(1 - \frac{1}{(1+r)^N}\right) + \frac{P(N)}{(1+r)} - \frac{N}{(1+r)^{N+1}} \\
P(N)\left(1 - \frac{1}{1+r}\right) &= \frac{1}{r}\left(1 - \frac{1}{(1+r)^N}\right) - \frac{N}{(1+r)^{N+1}} \\
P(N) &= \frac{\frac{1}{r}\left(1 - \frac{1}{(1+r)^N}\right) - \frac{N}{(1+r)^{N+1}}}{\left(1 - \frac{1}{1+r}\right)}
\end{aligned}
$$

Please don't make me simplify this? It's midnight, and also there's no need to because there's `r`?

(b).

Now,

$$
\begin{aligned}
\lim_{N\to\infty} \frac{N}{(1+r)^{N+1}} &= \frac{\lim_{N\to\infty} N}{\lim_{N\to\infty}(1+r)^{N+1}} \\
&= \frac{\lim_{N\to\infty} 1}{\lim_{N\to\infty}(1+r)^{N+1}\log(1+r)} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\lim_{N\to\infty} P(N) = \lim_{N\to\infty} & \frac{\frac{1}{r}\left(1 - \frac{1}{(1+r)^N}\right) - \frac{N}{(1+r)^{N+1}}}{\left(1 - \frac{1}{1+r}\right)} \\
&= \frac{\frac{1}{r}}{1 - \frac{1}{1+r}} \\
&= \frac{1+r}{r^2}
\end{aligned}
$$

Problem 6.

- To calculate 1 year zero rate, $95 = \frac{100}{(1+r_1)}$, then $r_1 = 0.05262625$
- To calculate 2 year zero rate, $90 = \frac{100}{(1+r_2)^2}$, then $r_2 = 0.05409221$

We now have a yield curve for 2 years.

Cashflow of the two-year bond can be stripped:

$$PV = \frac{10}{1+r_1} + \frac{100+10}{(1+r_2)^2} = 108.5$$

The fair value is 108.5

Problem 7.

Project 1:

$$0 = -A_1 + \frac{B_1}{1+r_1} + \frac{B_1}{(1+r_1)^2} + ... + \frac{B_1}{(1+r_1)^n} = -A_1 + \frac{B_1}{r_1}\left(1 - \frac{1}{(1+r_1)^n}\right)$$

implies that

$$\begin{aligned}
0 &= -A_1 + \frac{B_1}{r_1}\left(1 - \frac{1}{(1+r_1)^n}\right) \\
\frac{B_1}{A_1} &= \frac{r_1}{\left(1 - \frac{1}{(1+r_1)^n}\right)} \\
&= \frac{r_1(1+r_1)}{(1+r_1)^n} \\
&= \frac{r_1}{(1+r_1)^{n-1}}
\end{aligned}$$

Project 2:

$$0 = -A_2 + \frac{B_2}{1+r_2} + \frac{B_2}{(1+r_2)^2} + ... + \frac{B_2}{(1+r_2)^n} = -A_2 + \frac{B_2}{r_2}\left(1 - \frac{1}{(1+r_2)^n}\right)$$

Since $\frac{B_1}{A_1} > \frac{B_2}{A_2}$, then

$$\frac{r_1}{(1+r_1)^{n-1}} > \frac{r_2}{(1+r_2)^{n-1}}$$
$$r_1 > r_2$$

Hence project 1 will have a higher IRR than project 2.

Problem 8.

Monthly payment A is

$$203150 = \frac{A}{\left(\frac{0.08083}{12}\right)} \left(1 - \frac{1}{\left(1+\frac{0.08083}{12}\right)^{30*12}}\right)$$

Solving for A, $A = 1504.76$.

```
> f = function(a) {(a/(0.08083/12)) * (1-1/(1+0.08083/12)^360) - 203150}
> uniroot(f, c(-99999999999, 99999999999))
$root
[1] 1502.414

$f.root
[1] -7.888876e-05

$iter
[1] 2

$init.it
[1] NA

$estim.prec
[1] 6.103516e-05
```

The principal that corresponds with A and rate of 7.875% is

$$P = \frac{A}{\left(\frac{0.07875}{12}\right)} \left(1 - \frac{1}{\left(1+\frac{0.07875}{12}\right)^{30*12}}\right)$$

Solving for P using $A = 1502.414$, $P = 207209.7$.

Then, total fee is $207209.7 - 203150 = 4059.7$.

Problem 9.

Coupon $C = 5\%$. Assume, without loss of generality, that the bond face value is 1. Let yield $r = 2x$.

After 5 years, the price of the bond should be

$$P = \frac{0.05}{x}\left(1 - \frac{1}{(1+x)^{30}}\right) + \frac{1}{(1+x)^{30}}$$

If the company exercises the call option, the company would have to pay 1.05 per \$1 of bonds. That means x would have to be such that $1.05 < P$ or

$$1.05 < \frac{0.05}{x}\left(1 - \frac{1}{(1+x)^{30}}\right) + \frac{1}{(1+x)^{30}}$$

Solving for x, we get $x < 0.04686262$ or $r < 2x = 0.09372524$.

```
> f = function(x) {(0.05/x)*(1-1/(1+x)^30) + 1/(1+x)^30 -
   1.05}
> uniroot(f, c(0.000000001,9999999999))
$root
[1] 0.04686262

$f.root
[1] 3.102862e-06

$iter
[1] 41

$init.it
[1] NA

$estim.prec
[1] 0.0001155707

> x = uniroot(f, c(0.000000001,9999999999))
> x = x$root
> 2*x
[1] 0.09372524
```

Yield rates would have to be less than 9.372524%

Problem 10.

Ten year bonds have semi-annual coupon payments. Let price of bond be P.

$$P = \frac{4}{\frac{y}{2}}\left(1 - \frac{1}{(1+\frac{y}{2})^{20}}\right) + \frac{100}{(1+\frac{y}{2})^{20}} = 87.53779$$

Substituting in $y = 0.1$,

$$P = \frac{4}{0.05}\left(1 - \frac{1}{1.05^{20}}\right) + \frac{100}{1.05^{20}} = 87.53779$$

Calculating numerical derivative at $y = 0.1$, $\frac{dP}{dy} = -570.2769$. Then, modified duration

$$D = -\frac{1}{P}\frac{dP}{dy} = -\frac{1}{87.53779}(-570.2769) = 6.514637$$

```
> install.packages("numDeriv")
> library(numDeriv)
> f = function(y) {(4/(y/2)) * (1-1/(1+y/2)^20) + 100/(1+y/
  2)^20}
> z = grad(f, y, method="Richardson")
> z
[1] -570.2769
```