Intro to Financial Engineering IEOR W4700

Homework 8

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November 23, 2015

Problem 1.

$$\Gamma = \frac{\delta^2 C}{\delta S^2} = \frac{e^{\left(-\frac{d_1^2}{2}\right)}}{S\sigma\sqrt{2\pi t}}$$

Now,

$$\frac{\delta d_1}{\delta S} = \frac{1}{S\sigma\sqrt{t}}$$

Then, at maximum Γ,

$$\begin{aligned}
\frac{\delta \Gamma}{\delta S} &= \frac{-e^{\left(-\frac{d_1^2}{2}\right)}}{S^2\sigma\sqrt{2\pi t}} - \frac{e^{\left(-\frac{d_1^2}{2}\right)}}{S\sigma\sqrt{2\pi t}} d_1 \frac{\delta d_1}{\delta S} \\
&= \frac{-e^{\left(-\frac{d_1^2}{2}\right)}}{S^2\sigma\sqrt{2\pi t}} \left(1 + \frac{d_1}{\sigma\sqrt{t}}\right) = 0 \\
-1 &= \frac{d_1}{\sigma\sqrt{t}} = \frac{\log\left(\frac{S}{Ke^{-rt}}\right) + 0.5\sigma^2 t}{\sigma^2 t}
\end{aligned}$$

Solving for S,

$$
\begin{aligned}
-\sigma^2 t - 0.5\sigma^2 t &= \log\left(\frac{S}{Ke^{-rt}}\right) \\
&= \log S - \log K + rt \\
-1.5\sigma^2 t - rt + \log K &= \log S \\
S &= Ke^{\left(-(1.5\sigma^2 + r)t\right)}
\end{aligned}
$$

Problem 2.

The value of a put option can be calculated via:

```
d1 = function(S, K, r, sigma, T) {
  return((log(S/K) + (r+(sigma^2)/2)*T)/(sigma*sqrt(T)))
}

d2 = function(S, K, r, sigma, T) {
  return(d1(S, K, r, sigma, T) - sigma*sqrt(T))
}

bsm_put = function(sigma, K, S, r, T) {
  d1_val = d1(S, K, r, sigma, T)
  d2_val = d2(S, K, r, sigma, T)
  return(price=K*exp(-r*T)*pnorm(-d2_val) - S*pnorm(-d1_val)
    )
}
```

Given a quoted price, we want to find the value of `sigma` that makes `bsm_put` equal to the quoted price.

```
sigma_root_put = function(sigma, S, K, r, T, price) {
  return(bsm_put(sigma=sigma, K=K, S=S, r=r, T=T) - price)
}
```

This can be done via `uniroot` in `R`

```
implied_volatility_put = function(S, K, r, T, price) {
  return(uniroot(sigma_root_put, c(-10^99, 10^99), S=S, K=K,
      r=r, T=T, price=price)$root)
}
```

Solving the problem and plotting is then trivial:

```
strike = c(91:110)
putprice = c(9.57,9.78,10.01,10.24,10.49,10.75,
    11.02,11.31,11.61,11.92,12.25,12.60,12.96,
    13.344,13.74,14.16,14.60,15.05,15.53,16.02)
volatility = mapply(implied_volatility_put, K=strike, price=
    putprice, S=100, r=0, T=1)

library(ggplot2)

data = as.data.frame(list(strike=strike, volatility=
    volatility))
plot = ggplot(data=data) + geom_line(aes(x=strike, y=
    volatility)) + geom_point(aes(x=strike, y=volatility)) +
    labs(title="Implied Volatility Smile", x="Strike Price",
    y="Implied Volatility")
```

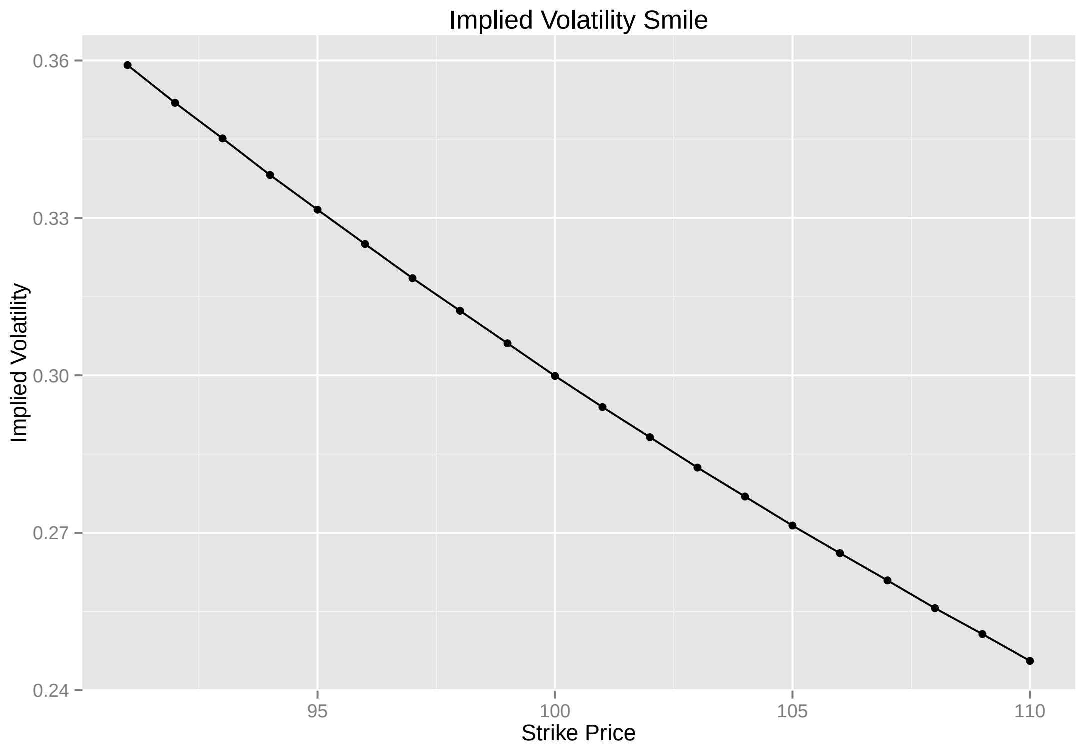

Figure 1: Plot of implied volatility against strike price.

Problem 3.

Let π be the value of the portfolio. Then $\Delta\Pi$ is the price change of the portfolio. Assume that the volatility of the underlying asset S is constant. Then, Taylor series expansion gives

$$\Delta\Pi = \frac{\delta\Pi}{\delta S}\Delta S + \frac{\delta\Pi}{\delta t}\Delta t + \frac{1}{2}\frac{\delta^2\Pi}{\delta S^2}\Delta S^2 + \frac{1}{2}\frac{\delta^2\Pi}{\delta t^2}\Delta t^2 + \frac{\delta^2\Pi}{\delta S\delta t}\Delta S\Delta t + ...$$

If the portfolio Π is delta-hedged, the first term $\frac{\delta\Pi}{\delta S}\Delta S$ is eliminated. The second term, $\Theta = \frac{\delta\Pi}{\delta t}$ is non stochastic. Hence for a delta neutral portfolio,

$$\Delta\Pi = \Theta\Delta t + \frac{1}{2}\Gamma\Delta S^2$$

when terms of order higher than Δt are ignored.

For a portfolio that is only long call / put, the delta of the portfolio is always increasing as S increases, hence $\Gamma \geq 0$. Then, when ΔS is greater, $\Delta\Pi$ is greater and hence profits are greater. Hence, wild movements in exchange rates will be more profitable than virtually constant exchange rates if the portfolio is perfectly delta hedged.

Problem 4.

Data entry intern task.

```
positions = c(-5000, -500, -2000, -500)
deltas = c(0.5, 0.7, -0.4, 0.7)
gammas = c(2.2, 0.6, 1.3, 1.9)
vegas = c(1.8, 0.2, 0.8, 1.4)
to_delta = 0.7
to_gamma = 1.4
to_vega = 0.9
sterling_delta = 1
sterling_gamma = 0
sterling_vega = 0
```

Problem 4(a).

Let w_t be position in traded option and w_s be position in sterling.

To make the portfolio delta neutral,

$$w_t 0.7 + w_s = -\sum_{i \in \text{OTCs}} w_i * \Delta_i$$

To make the portfolio gamma neutral we note that the Γ of sterling is 0, since sterling's Δ is always 1,

$$w_t 1.4 = -\sum_{i \in \text{OTCs}} w_i * \Gamma_i$$

To solve this,

```
> # delta gamma
> # solve a %*% x = b
> b = c(-sum(positions*deltas), -sum(positions*gammas))
> a = matrix(c(to_delta, to_gamma, sterling_delta, sterling_
    gamma), nrow=2, ncol=2)
> colnames(a) = c('traded_option', 'sterling')
> rownames(a) = c('delta', 'gamma')
> a
      traded_option sterling
delta           0.7        1
gamma           1.4        0
> solve(a, b)
traded_option      sterling
     10607.14      -5025.00
```

Problem 4(b).

To make portfolio delta neutral,

$$w_t 0.7 + w_s = -\sum_{i \in \text{OTCs}} w_i * \Delta_i$$

To make portfolio vega[1] neutral we note that sterling's κ is always 0 since the value of the asset does not change with volatility,

$$w_t 0.9 = -\sum_{i \in \text{OTCs}} w_i * \kappa_i$$

To solve this,

```
> # delta vega
> # solve a %*% x = b
> b = c(-sum(positions*deltas), -sum(positions*gammas))
> a = matrix(c(to_delta, to_vega, sterling_delta, sterling_vega), nrow=2, ncol=2)
> colnames(a) = c('traded_option', 'sterling')
> rownames(a) = c('delta', 'vega')
> a
      traded_option sterling
delta           0.7        1
vega            0.9        0
> solve(a, b)
traded_option      sterling
        16500         -9150
```

Problem 5.

t	Stock Price	Delta	Shares Purchased	Cost of Shares	Cash Position
Setup	-	-	-	-	-20.510
0.000	100.000	0.750	-0.750	-75.000	54.490
0.250	116.180	0.890	-0.140	-16.265	72.415
0.500	100.000	0.710	0.180	18.000	56.620
0.750	86.070	0.190	0.520	44.756	13.588
1.000	100.000	1.000	-0.810	-81.00	**95.002**

In the final period, we exercise the option at receive \$5. This leaves us with \$100.002 in cash. Then, we close out the long position of $\Delta_1 = 1$ shares (technically we could have just not longed 0.81 shares in the last period and closed out the short position, but I did this to stick with the example in the slides and also for clarity, assuming 0 transaction

[1] I forgot for a moment that finance is weird and Vega is totally not a Greek letter. LaTeX gave me an error when I tried to type `\Vega`

costs). The closing out reduces cash by $\$-100.000$, resulting in remaining cash of \$0.002, reasonably close to zero.

Since we could not make money by delta hedging, the \$20.51 price for the option was fair given the parameters and trees provided.