Intro to Financial Engineering IEOR W4700

Homework 5

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Problem 1.

Problem 1(i).

$$
\begin{aligned}
B(y) &= \int_0^\infty e^{-yt}\,dt \\
&= -\frac{1}{y}\int_0^\infty -ye^{-yt}\,dt \\
&= -\frac{1}{y}\left(0 - e^0\right) \\
&= \frac{1}{y}
\end{aligned}
$$

Problem 1(ii).

Since $B(y) = \frac{1}{y}$,

$$
\begin{aligned}
\frac{\delta B}{\delta t} &= 0 \\
\frac{\delta B}{\delta y} &= -\frac{1}{y^2} \\
\frac{\delta^2 B}{\delta y^2} &= \frac{2}{y^3}
\end{aligned}
$$

Then by Ito's Lemma,

$$\begin{aligned}
dB &= \left(\frac{\delta B}{\delta y} a(m-y) + \frac{\delta B}{\delta t} + \frac{1}{2}\frac{\delta^2 B}{\delta y^2} b^2 y^2 \right) dt + \frac{\delta B}{\delta y} by dZ \\
&= \left(-\frac{1}{y^2} a(m-y) + \frac{1}{2}\frac{2}{y^3} b^2 y^2 \right) dt - \frac{1}{y^2} by dZ \\
&= \left(-\frac{1}{y^2} a(m-y) + \frac{b^2}{y} \right) dt - \frac{b}{y} dZ
\end{aligned}$$

We find that the expected increase in value is:

$$E(dB) = \left(-\frac{1}{y^2} a(m-y) + \frac{b^2}{y} \right) dt$$

Interest per unit time (which is non stochastic) is:

$$1dt$$

Total return is:

$$\frac{E(dB) + 1dt}{B(y)dt} = \frac{\left(-\frac{1}{y^2} a(m-y) + \frac{b^2}{y} + 1 \right)}{\frac{1}{y}} = -\frac{1}{y} a(m-y) + b^2 + y$$

Problem 2.

Given that $G(t) = X(t)Y(t)$, and omitting the argument t, $G = XY$

$$\begin{aligned}
&\frac{\delta G}{\delta t} = 0 & & \\
&\frac{\delta G}{\delta x} = Y & &\frac{\delta^2 G}{\delta x^2} = 0 \\
&\frac{\delta G}{\delta y} = X & &\frac{\delta^2 G}{\delta y^2} = 0 \\
&\frac{\delta^2 G}{\delta x \delta y} = 1 & &
\end{aligned}$$

Then by Ito's Lemma, and omitting higher powers of dt,

$$
\begin{aligned}
dG &= YdX + XdY + dXdY \\
&= YX(\mu_X dt + \sigma_X dW) + XY(\mu_Y dt + \sigma_Y dW) + XY(\mu_X dt + \sigma_X dW)(\mu_Y dt + \sigma_Y dW) \\
&= XY(\mu_X + \mu_Y)dt + XY(\sigma_X + \sigma_Y)dW + XY(\sigma_X\sigma_Y)(dW)^2 \\
&= XY(\mu_X + \mu_Y)dt + XY(\sigma_X + \sigma_Y)dW + XY(\sigma_X\sigma_Y)dt \\
&= G(\mu_X + \mu_Y + \sigma_X\sigma_Y)dt + G(\sigma_X + \sigma_Y)dW
\end{aligned}
$$

This is a GBM with drift $= \mu_x + \mu_Y$ and variance $(\sigma_X + \sigma_Y + \sigma_X\sigma_Y)^2$

Problem 3.

Let stock A be A and stock B be B

$$
\begin{aligned}
dA &= \mu_A A dt + \sigma_A A dZ \\
dB &= \mu_B B dt + \sigma_B B dZ
\end{aligned}
$$

Then let $G = A + B$ is the value of hte portfolio.

$$
\begin{aligned}
dG &= dA + dB \\
&= \mu_A A dt + \sigma_A A dZ + \mu_B B dt + \sigma_B B dZ \\
&= (\mu_A A + \mu_B B)dt + (\sigma_A A + \sigma_B B)dZ
\end{aligned}
$$

$(A + B) = G$ cannot be factorized from the coefficient of dt or dZ. In other words,

$$
\begin{aligned}
(\mu_A A + \mu_B B) &\neq \mu(A + B) \\
(\sigma_A A + \sigma_B B) &\neq \sigma(A + B)
\end{aligned}
$$

For for all μ_A μ_B σ_A and σ_B. Hence, not GBM.

Problem 4.

Again, $G = XY$

$$\frac{\delta G}{\delta t} = 0$$
$$\frac{\delta G}{\delta x} = Y \qquad \frac{\delta^2 G}{\delta x^2} = 0$$
$$\frac{\delta G}{\delta y} = X \qquad \frac{\delta^2 G}{\delta y^2} = 0$$
$$\frac{\delta^2 G}{\delta x \delta y} = 1$$

Then, by Ito's Lemma,

$$\begin{aligned}
dG &= \frac{\delta G}{\delta t}dt + \frac{\delta G}{\delta x}dX + \frac{\delta G}{\delta y}dY + \frac{1}{2}\frac{\delta^2 G}{\delta x^2}dX^2 + \frac{1}{2}\frac{\delta^2 G}{\delta y^2}dY^2 + \frac{\delta^2 G}{\delta x \delta y}dXdY \\
&= YdX + XdY + dXdY \\
&= YX(\mu_X dt + \sigma_X dZ) + XY(\mu_Y dt + \sigma_Y dW) + XY(\mu_X dt + \sigma_X dZ)(\mu_Y dt + \sigma_Y dW) \\
&= XY(\mu_X + \mu_Y)dt + XY(\sigma_X dZ + \sigma_Y dW) + XY\sigma_X\sigma_Y dZdW \\
&= G(\mu_X + \mu_Y)dt + G(\sigma_X dZ + \sigma_Y dW) + G\sigma_X\sigma_Y\rho dt \\
&= G(\mu_X + \mu_Y + \sigma_X\sigma_Y\rho)dt + G(\sigma_X dZ + \sigma_Y dW)
\end{aligned}$$

$$dG = G(\mu_X + \mu_Y + \sigma_X\sigma_Y\rho)dt + G(\sigma_X dZ + \sigma_Y dW) = G(\mu_X + \mu_Y + \sigma_X\sigma_Y\rho)dt + GdA$$

Let $dA = \sigma_X dZ + \sigma_Y dW$. Now $E(dA) = 0$, $Var(dA) = (\sigma_X^2 + \sigma_Y^2 + 2\rho\sigma_X\sigma_Y)dt$, and $SD(dA) = \sqrt{\sigma_X^2 + \sigma_Y^2 + 2\rho\sigma_X\sigma_Y}\sqrt{dt}$ hence dA is also a Wiener process.

Then, dG is a GBM with drift $\mu_X + \mu_Y + \sigma_X\sigma_Y\rho$ and variance rate $(\sigma_X^2 + \sigma_Y^2 + 2\rho\sigma_X\sigma_Y)$.

Problem 5.

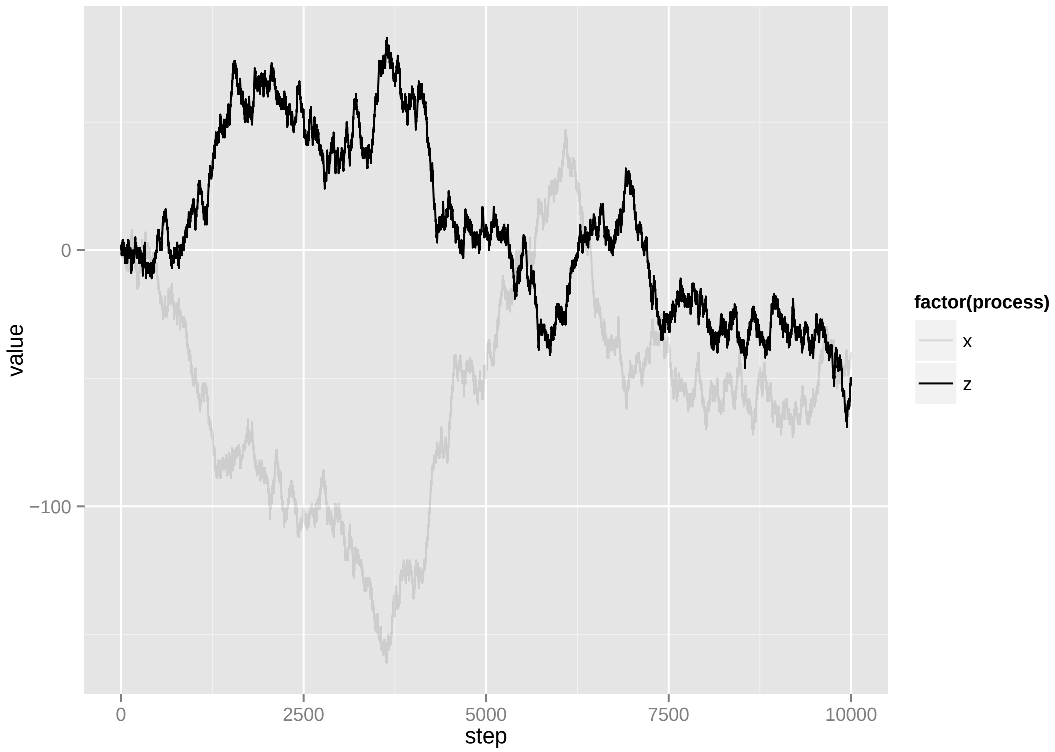

Figure 1: Plot of cumulative value of X and cumulative value of Z

```r
rho = -0.5
steps = 10000
x = vector(mode = 'numeric', length = steps)
z = vector(mode = 'numeric', length = steps)
dx = vector(mode = 'numeric', length = steps)
dz = vector(mode = 'numeric', length = steps)
random = runif(steps, 0, 1)

for(i in 2:steps) {
  if(random[i] < (1-rho) / 4) {
    x[i] = x[i-1] - 1;
    z[i] = z[i-1] + 1;
    dx[i] = -1;
    dz[i] = 1;
  } else if (random[i] < 0.5) {
    x[i] = x[i-1] + 1;
```

```
    z[i] = z[i-1] + 1;
    dx[i] = 1;
    dz[i] = 1;
  } else if (random[i] < 0.5 + (1-rho) / 4) {
    x[i] = x[i-1] + 1;
    z[i] = z[i-1] - 1;
    dx[i] = 1;
    dz[i] = -1;
  } else {
    x[i] = x[i-1] - 1;
    z[i] = z[i-1] - 1;
    dx[i] = -1;
    dz[i] = -1;
  }
}

library(ggplot2)
library(reshape)
data = cbind(x, z)
colnames(data) = c('x', 'z')
melted = melt(data, id=c('x', 'z'))
colnames(melted) = c('step', 'process', 'value')
plot = ggplot(data = melted) + geom_line(aes(x=step, y=value, colour=process))
plot

cor(dx, dz)
```

$cor(dx, dz) = -0.5003817$, fits chosen ρ value pretty well.

Problem 6.

Let V be the value of the portfolio. Since B is borrowed and used to buy S,

$$V = S + B - B = S$$

Then since B amount of stocks are added,

$$
\begin{aligned}
dV &= \left(\frac{S+B}{S} dS - rBdt\right) \\
&= (S\mu + B\mu - rB)\, dt + (S+B)\sigma dZ
\end{aligned}
$$

Since we're interested in excess returns $= S\mu + B\mu - rB - Vr = S\mu + B\mu - rB - rS$,

$$
\lambda_V = \frac{S\mu + B\mu - rB - rS}{(S+B)\sigma} = \frac{\mu - r}{\sigma} = \lambda
$$

Problem 7.

Let i be index. Let p be portfolio.

$$
\begin{aligned}
\mu_p &= (1-w)\mu_f + w\mu_i \\
\sigma_p &= w\sigma_i
\end{aligned}
$$

Problem 7(i).

For $\sigma_p = 0.15$, $w = 0.5$. Then,

$$
\mu_p = (0.5)(0.04) + 0.5(0.24) = 0.14
$$

Problem 7(ii).

You'd borrow -0.5 the value of the whole portfolio. ie. you'd lend as much as you invest in the index.