Intro to Financial Engineering IEOR W4700

Homework 2

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Problem 1.

We can think of the house as a series of yearly payments x for 20 years whose present value equates 20000. Then,

$$20000 = \frac{x}{0.05}\left(1 - \frac{1}{(1+0.05)^{20}}\right)$$

Solving for x, $x = 1604.852$

```
> f = function(x){(x/0.05) * (1-1/((1+0.05)^20)) - 20000}
> uniroot(f, c(-99999999, 99999999))
$root
[1] 1604.852

$f.root
[1] 4.366302e-08

$iter
[1] 2

$init.it
[1] NA

$estim.prec
[1] 6.103516e-05
```

Now, we know the yearly "coupon" of the roof. Since our current roof only has 5 years left, then the value of the house is the payment for 5 years.

$$PV_R = \sum_{i=1}^{5} \frac{x}{(1+0.05)^i} = \frac{x}{0.05}\left(1 - \frac{1}{(1+0.05)^5}\right) = 6948.168$$

The value of the existing roof if 6948.168

Problem 2.

(a).

The stream of cashflow is

$$PV = -3x + \frac{5}{1+r} + \frac{x}{(1+r)^2}$$

For $PV > 0$,

$$\begin{aligned}
-3x + \frac{5}{1+r} + \frac{x}{(1+r)^2} &> 0 \\
-3x(1+r)^2 + 5(1+r) + x &> 0 \\
-3x(1+r)^2 + x &> -5(1+r) \\
x(-3(1+r)^2 + 1) &> -5(1+r) \\
x &> \frac{-5(1+r)}{-3(1+r)^2 + 1} \\
x &> \frac{5(1+r)}{3(1+r)^2 - 1}
\end{aligned}$$

(b).

IRR is the r such that $PV = 0$ or

$$-3x + \frac{5}{1+r} + \frac{x}{(1+r)^2} = 0$$

Then, let $c = \frac{1}{1+r}$.

$$-3x + 5c + xc^2$$

We need

$$\begin{aligned} r &> 0 \\ 1 + r &> 1 \\ \frac{1}{1+r} &> 1 \\ c &> 1 \end{aligned}$$

Solving quadratically,

$$c = \frac{-5 \pm \sqrt{5^2 - 4(-3x)(x)}}{2x} = \frac{-5 \pm \sqrt{25 + 12x^2}}{2x}$$

Discard the strictly negative root since it won't fulfill our condition of $c > 1$. Using the possibly positive root, we need $c > 1$ or

$$\frac{-5 + \sqrt{25 + 12x^2}}{2x} > 1$$

$x = 0$ or $x = 2.5$

```
> install.packages("rootSolve")
> library(rootSolve)
> g = function(x) {(-5+sqrt(25 + 12*x^2)) / (2*x) - 1}
> uniroot.all(g, c(-9, 9))
[1] 2.5
```

$x > 2.5$ guarantees a strictly positive IRR.

Problem 3.

This can be treated as monthly payments with interest rate 0.01 since rent is paid monthly. Assume monthly compounding.

Compare PV:

- **Stay** $PV = \frac{1000}{0.01}\left(1 - \frac{1}{(1+0.01)^6}\right) = 5795.476$

- **Switch** $PV = 1000 + \frac{900}{0.01}\left(1 - \frac{1}{(1+0.01)^6}\right) = 6215.929$

Within 6 months, the couple should stay since staying costs them less in PV terms.

For 1 year, compare PV again:

- **Stay** $PV = \frac{1000}{0.01}\left(1 - \frac{1}{(1+0.01)^12}\right) = 11255.08$
- **Switch** $PV = 1000 + \frac{900}{0.01}\left(1 - \frac{1}{(1+0.01)^6}\right) = 11129.57$

For 1 year, the couple should switch since switching costs them less in PV terms.

Problem 4.

- At expiration, $P(t, T, K)$ pays $\max\left[0, K - S(T)\right]$
- At expiration, $P(t, T, L)$ pays $\max\left[0, L - S(T)\right]$

Problem 5.

(a).

$$PV = \frac{1}{(1+r)} + \frac{2}{(1+r)^2} + ... + \frac{N}{(1+r)^N}$$

Denote present value with n periods as $P(n)$. Then, $P(0) = 0$.

$$
\begin{aligned}
P(N) &= \frac{1}{(1+r)} + \frac{2}{(1+r)^2} + ... + \frac{N}{(1+r)^N} \\
&= \frac{1}{(1+r)} + \frac{1}{(1+r)^2} + ... + \frac{1}{(1+r)^N} + \frac{N-1}{(1+r)^N} \\
&= \frac{1}{r}\left(1 - \frac{1}{(1+r)^N}\right) + \frac{P(N-1)}{(1+r)} \\
&= \frac{1}{r}\left(1 - \frac{1}{(1+r)^N}\right) + \frac{P(N)}{(1+r)} - \frac{N}{(1+r)^{N+1}} \\
P(N)\left(1 - \frac{1}{1+r}\right) &= \frac{1}{r}\left(1 - \frac{1}{(1+r)^N}\right) - \frac{N}{(1+r)^{N+1}} \\
P(N) &= \frac{\frac{1}{r}\left(1 - \frac{1}{(1+r)^N}\right) - \frac{N}{(1+r)^{N+1}}}{\left(1 - \frac{1}{1+r}\right)}
\end{aligned}
$$

Please don't make me simplify this? It's midnight, and also there's no need to because there's `r`?

(b).

Now,

$$
\begin{aligned}
\lim_{N\to\infty} \frac{N}{(1+r)^{N+1}} &= \frac{\lim_{N\to\infty} N}{\lim_{N\to\infty}(1+r)^{N+1}} \\
&= \frac{\lim_{N\to\infty} 1}{\lim_{N\to\infty}(1+r)^{N+1}\log(1+r)} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\lim_{N\to\infty} P(N) &= \lim_{N\to\infty} \frac{\frac{1}{r}\left(1 - \frac{1}{(1+r)^N}\right) - \frac{N}{(1+r)^{N+1}}}{\left(1 - \frac{1}{1+r}\right)} \\
&= \frac{\frac{1}{r}}{1 - \frac{1}{1+r}} \\
&= \frac{1+r}{r^2}
\end{aligned}
$$

Problem 6.

- To calculate 1 year zero rate, $95 = \frac{100}{(1+r_1)}$, then $r_1 = 0.05262625$
- To calculate 2 year zero rate, $90 = \frac{100}{(1+r_2)^2}$, then $r_2 = 0.05409221$

Cashflow of the two-year bond can be stripped:

$$PV = \frac{10}{1+r_1} + \frac{100+10}{1+r_2} = 113.8552$$

The fair value is 113.8552

Problem 7.

Project 1:

$$0 = -A_1 + \frac{B_1}{1+r_1} + \frac{B_1}{(1+r_1)^2} + ... + \frac{B_1}{(1+r_1)^n} = -A_1 + \frac{B_1}{r_1}\left(1 - \frac{1}{(1+r_1)^n}\right)$$

implies that

$$\begin{aligned}
0 &= -A_1 + \frac{B_1}{r_1}\left(1 - \frac{1}{(1+r_1)^n}\right) \\
\frac{B_1}{A_1} &= \frac{r_1}{\left(1 - \frac{1}{(1+r_1)^n}\right)} \\
&= \frac{r_1(1+r_1)}{(1+r_1)^n} \\
&= \frac{r_1}{(1+r_1)^{n-1}}
\end{aligned}$$

Project 2:

$$0 = -A_2 + \frac{B_2}{1+r_2} + \frac{B_2}{(1+r_2)^2} + ... + \frac{B_2}{(1+r_2)^n} = -A_2 + \frac{B_2}{r_2}\left(1 - \frac{1}{(1+r_2)^n}\right)$$

Since $\frac{B_1}{A_1} > \frac{B_2}{A_2}$, then

$$\frac{r_1}{(1+r_1)^{n-1}} > \frac{r_2}{(1+r_2)^{n-1}}$$
$$r_1 > r_2$$

Hence project 1 will have a higher IRR than project 2.