Homework 0

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Let

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \ B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \ x = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

1. $B_{2,1}$ is 3

2. $A + B = \begin{bmatrix} 2 & 4 \\ 5 & 8 \end{bmatrix}$

3. $AB = \begin{bmatrix} 1*1+2*3 & 1*2+2*4 \\ 2*1+4*3 & 2*2+4*4 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 14 & 20 \end{bmatrix}$

4. $rank(A) = 1$

$$Av = \lambda v$$
$$(A - \lambda I)v = 0$$
$$det(A - \lambda I) = 0$$
$$= (1 - \lambda)(4 - \lambda) - 4$$
$$= \lambda^2 - 5\lambda$$
$$= \lambda(\lambda - 5)$$

5. Largest eigenvalue of A is 5

$$(A - \lambda I)v = 0$$
$$(A - 5I)v = 0$$
$$\begin{bmatrix} 1-5 & 2 \\ 2 & 4-5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

Then,

$$-4v_1 + 2v_2 = 0$$
$$2v_1 - 1v_2 = 0$$

Then, let $v_2 = t$, $v_1 = -2t$. Then, the eigenspace corresponding to $\lambda = 5$ is given by the span of $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

6. Eigenvector associated is the span of $v = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

7. $|B| = 1 * 4 - 2 * 3 = -2$

8. $x^T A x = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 16$

9. $x^T x = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 5$

10. $x x^T = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$

11. $||x||_2 = \sqrt{2^2 + 1^2} = \sqrt{5}$

$$\nabla_x x^T A x = x^T A^T + x^T A = x^T (A^T + A)$$

For the special case that A is symmetric (as it is given),

$$\nabla_x x^T A x = 2x^T A$$

12. $\nabla_x x^T A x = 2x^T A$

$$\nabla_x^2 x^T A x = \nabla_x \left(x^T A^T + x^T A \right) = A + A^T$$

For the special case that A is symmetric (as it is given),

$$\nabla_x^2 x^T A x = 2A$$

13. $\nabla_x^2 x^T A x = 2A$

2

14. $n = 2$

$$Var(y) = E(y)^2 - E(y^2)$$
$$E(y^2) = E(y)^2 - Var(y)$$
$$= \mu^2 - \sigma^2$$

15. $E(y^2) = \mu^2 - \sigma^2$

16. $y + w \sim N(2.7 + 3.1, \sqrt{8^2 + 15^2}) = N(5.8, 17)$

17. The normalizing constant is $(2\pi)^{\frac{p}{2}} |\Sigma|^{\frac{1}{2}}$ where $p = 2$ and $|\Sigma|$ is the determinant of the Σ given.

18. The support of a Bernoulli random variable is $\{0, 1\}$

19. Define the Bernoulli process as $N(k, n)$. Then $N(k, n) = \binom{n}{k} = \frac{n!}{k!(n-k)!}$

$$h(x_1) = \frac{1}{3}x_1^3 - \frac{1}{2}x_1^2 - 6x_1 + \frac{27}{2}$$
$$h'(x_1) = x_1^2 - x_1 - 6$$
$$h''(x_1) = 2x_1 - 1$$

Minima/maxima is at

$$h'(x_1) = x_1^2 - x_1 - 6 = 0$$
$$(x_1 - 3)(x_1 + 2) = 0$$

$x_1 = 3$ is a minima since $h''(3) > 0$. $x_1 = -2$ is a maxima since $h''(-2) < 0$. To compare with boundary values,

$$h(3) = 0$$
$$h(-2) = 20.83$$
$$h(-4) = 8.17$$
$$h(4) = 2.83$$

20. $x_1 = -2$

21. $x_1 = 3$

The indefinite integral

$$\int \frac{1}{Z} \left(\frac{1}{3}x_1^3 - \frac{1}{2}x_1^2 - 6x_1 + \frac{27}{2} \right) = \frac{1}{Z} \left(\frac{1}{12}x^4 - \frac{1}{6}x^3 - 3x^2 + \frac{27}{2}x + C \right)$$

3

Set C and Z such that

$$\int_0^1 \frac{1}{Z}\left(\frac{1}{3}x_1^3 - \frac{1}{2}x_1^2 - 6x_1 + \frac{27}{2}\right) = 1$$

$$\frac{1}{Z}\left(\frac{1}{12} - \frac{1}{6} - 3 + \frac{27}{2}\right) = 1$$

$$Z = \frac{1}{12} - \frac{1}{6} - 3 + \frac{27}{2}$$

22. $Z = \frac{1}{12} - \frac{1}{6} - 3 + \frac{27}{2}$

$$\int_A b(x) = \int_0^3 \int_0^2 x_1 x_2^3 \, dx_2 dx_1$$

$$= \int_0^3 \left[x_1 \frac{1}{4}x_2^4\right]_0^2 dx_1$$

$$= \int_0^3 4x_1 \, dx_1$$

$$= \left[2x_1^2\right]_0^3$$

$$= 18$$

23. 18

$$L(x_1, x_2; \lambda) = x_1 + \sqrt{3}x_2 + \lambda(x_1^2 + x_2^2 - 1)$$

$$\frac{\delta L}{\delta x_1} = 1 + \lambda 2x_1$$

$$\frac{\delta L}{\delta x_2} = \sqrt{3} + \lambda 2x_2$$

$$\frac{\delta L}{\delta \lambda} = x_1^2 + x_2^2 - 1$$

Using first 2 equations, $x_1 = \frac{1}{\sqrt{3}}x_2$. Solving trivially, we have

$$x_1 = \pm\frac{1}{2}$$

$$x_2 = \pm\frac{\sqrt{3}}{2}$$

24. The x that maximizes $c(x)$ subject to the constraints specified is $x = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}$

$$L(x_1, x_2; \lambda) = -x_1 \log x_1 - x_2 \log x_2 + \lambda(x_1 + x_2 - 1)$$

$$\frac{\delta L}{\delta x_1} = -1 - \log x_1 + \lambda$$

$$\frac{\delta L}{\delta x_2} = -1 - \log x_2 + \lambda$$

$$\frac{\delta L}{\delta \lambda} = x_1 + x_2 - 1$$

By visual inspection, if $\log x_1 = \log x_2$, then $x_1 = x_2 = 0.5$

25. $x = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$