數值分析

Chapter 3
System of Linear Equations

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Outline

- Background
- Gauss elimination(3.2)
- Gauss elimination with pivoting (3.3)
- Gauss Jordan(3.4)
- LU decomposition(3.5)
- Inverse of a matrix
- Iterative methods (Jacobi,Gauss-Seidel) (3.7)
- MATLAB built-in functions (3.8)

Background

- Solving a system of linear equations:
- $$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases}$$

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{bmatrix}$$

- Solved by
- 1.Direct method: Gauss elimination(3.2,3.3)
- Gauss Jordan(3.4)
- LU decompositim(3.5)
- 2.(Indirect) Iterative: Jacobi,Gauss-Seidel(3.7)

- Direct method: solved the equations by **upper triangular**, **lower triangular**, and **diagonal** forms

Upper Triangular Form

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

Lower Triangular Form

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

Diagonal Form

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & a_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_2 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_2 \\ b_4 \end{bmatrix}$$

3.2 Gauss elimination method

- Rewrite the equation in **upper-triangular** form
- Solve the equation by back substitution
- Procedure: eliminate the terms(x_1) in the other equations except 1st equation
- Pivot equation, pivot coefficient (a_{11})

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a'_{22} & a'_{23} & a'_{24} \\ 0 & 0 & a'_{33} & a'_{34} \\ 0 & 0 & 0 & a'_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b'_3 \\ b'_4 \end{bmatrix}$$

Gauss elimination

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

Pivot equation

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a'_{22} & a'_{23} & a'_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b_3 \\ b_4 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a'_{22} & a'_{23} & a'_{24} \\ 0 & a'_{32} & a'_{33} & a'_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b'_3 \\ b_4 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a'_{22} & a'_{23} & a'_{24} \\ 0 & a'_{32} & a'_{33} & a'_{34} \\ 0 & a'_{42} & a'_{43} & a'_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b'_3 \\ b'_4 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a'_{22} & a'_{23} & a'_{24} \\ 0 & 0 & a''_{33} & a''_{34} \\ 0 & a'_{42} & a'_{43} & a'_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b''_3 \\ b'_4 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a'_{22} & a'_{23} & a'_{24} \\ 0 & 0 & a''_{33} & a''_{34} \\ 0 & 0 & a''_{43} & a''_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b''_3 \\ b''_4 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a'_{22} & a'_{23} & a'_{24} \\ 0 & 0 & a''_{33} & a''_{34} \\ 0 & 0 & 0 & a'''_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b''_3 \\ b'''_4 \end{bmatrix}$$

Gauss elimination

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

Initial set of equations.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ \cancel{a_{21}} & a'_{22} & a'_{23} & a'_{24} \\ \cancel{a_{31}} & a'_{32} & a'_{33} & a'_{34} \\ \cancel{a_{41}} & a'_{42} & a'_{43} & a'_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x'_2 \\ x'_3 \\ x'_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b'_3 \\ b'_4 \end{bmatrix}$$

Step 1.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a'_{22} & a'_{23} & a'_{24} \\ 0 & \cancel{a'_{32}} & a''_{33} & a''_{34} \\ 0 & \cancel{a'_{42}} & a''_{43} & a''_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x'_2 \\ x''_3 \\ x''_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b''_3 \\ b''_4 \end{bmatrix}$$

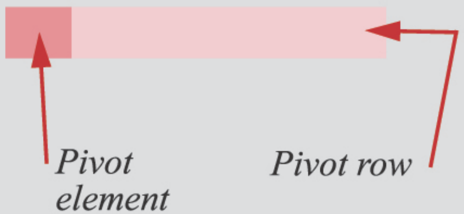
Step 2.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a'_{22} & a'_{23} & a'_{24} \\ 0 & 0 & a''_{33} & a''_{34} \\ 0 & 0 & \cancel{a''_{43}} & a'''_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x'_2 \\ x''_3 \\ x'''_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b''_3 \\ b'''_4 \end{bmatrix}$$

Step 3.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a'_{22} & a'_{23} & a'_{24} \\ 0 & 0 & a''_{33} & a''_{34} \\ 0 & 0 & 0 & a'''_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x'_2 \\ x''_3 \\ x'''_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b''_3 \\ b'''_4 \end{bmatrix}$$

Equations in upper triangular form.



Example 3-2

- Write a user-defined MATLAB function for solving a system of linear equations [a][x] = [b]
- For function name and arguments, use x = Gauss(a,b)
- a is the matrix of coefficients and b is the right-hand-side column vector of constants
- x is a column vector of solution

$$
\begin{aligned}
4x_1 - 2x_2 - 3x_3 + 6x_4 &= 12 \\
-6x_1 + 7x_2 + 6.5x_3 - 6x_4 &= -6.5 \\
x_1 + 7.5x_2 + 6.25x_3 + 5.5x_4 &= 16 \\
-12x_1 + 22x_2 + 15.5x_3 - x_4 &= 17
\end{aligned}
$$

Potential difficulties of Gauss elimination

- The pivot element is zero
  - Corrected by changing the order of rows
- The pivot element is small relative to the other terms in the pivot row
  - May cause rounding errors
  - It can be corrected by exchanging the order of the equations
- The pivot equation should have the largest possible pivot element

3.3 Gauss elimination with pivoting

- If pivot element=0 ➔ pivoting
- If the pivot element is zero, the equation is exchanged with one of the equations that are below

After the first step, the second equation has a pivot element that is equal to zero.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & 0 & a'_{23} & a'_{24} \\ 0 & a'_{32} & a'_{33} & a'_{34} \\ 0 & a'_{42} & a'_{43} & a'_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b'_3 \\ b'_4 \end{bmatrix}$$

Using pivoting, the second equation is exchanged with the fourth equation.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a'_{32} & a'_{33} & a'_{34} \\ 0 & 0 & a'_{23} & a'_{24} \\ 0 & a'_{42} & a'_{43} & a'_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_3 \\ b'_2 \\ b'_4 \end{bmatrix}$$

3.4 Gauss-Jordan elimination

- Change the system to diagonal form
- $$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x1 \\ x2 \\ x3 \\ x4 \end{bmatrix} = \begin{bmatrix} b1 \\ b2 \\ b3 \\ b4 \end{bmatrix}$$
- Procedure:
- (1) Normalized the pivot equation, a_{11}=1, a_{22}=1, a_{33}=1
- (2) Similar to Gauss elimination procedure, but eliminate the equations above and below the pivot equation

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & b_1 \\ a_{21} & a_{22} & a_{23} & a_{24} & b_2 \\ a_{31} & a_{32} & a_{33} & a_{34} & b_3 \\ a_{41} & a_{42} & a_{43} & a_{44} & b_4 \end{bmatrix} \xrightarrow{\text{Gauss–Jordan procedure}} \begin{bmatrix} 1 & 0 & 0 & 0 & b'_1 \\ 0 & 1 & 0 & 0 & b'_2 \\ 0 & 0 & 1 & 0 & b'_3 \\ 0 & 0 & 0 & 1 & b'_4 \end{bmatrix}$$

(a) (b)

3.7 Iterative method

- Solving the linear equations by an iterative approach

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 &= b_3 \\ a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 &= b_4 \end{aligned}$$

(*a*)

Writing the equations in an explicit form. $\longrightarrow$

$$\begin{aligned} x_1 &= [b_1 - (a_{12}x_2 + a_{13}x_3 + a_{14}x_4)]/a_{11} \\ x_2 &= [b_2 - (a_{21}x_1 + a_{23}x_3 + a_{24}x_4)]/a_{22} \\ x_3 &= [b_3 - (a_{31}x_1 + a_{32}x_2 + a_{34}x_4)]/a_{33} \\ x_4 &= [b_4 - (a_{21}x_1 + a_{42}x_2 + a_{43}x_3)]/a_{44} \end{aligned}$$

(*b*)

- $x_i = \frac{1}{a_{ii}}[b_i - (\sum_{\substack{j \neq i \\ j=1}}^{j=n} a_{ij}x_j)]$ i=1,2,……,n <3.51>
- Assume initial value ➔ first estimate ➔ second estimate…
- Condition for convergence:
- $|a_{ii}| > \sum_{j \neq i}^{j=n} |a_{ij}|$
- Sufficient but not necessary for convergence when the iteration method is used

3.7.1 Jacobi iterative method

- 1st estimate $x_1^{(1)}, x_2^{(1)}, \ldots, x_n^{(1)}$

- 2nd estimate $x_1^{(2)}, x_2^{(2)}, \ldots, x_n^{(2)}$

- $x_i^{k+1} = \frac{1}{a_{ii}}[b_i - (\sum_{\substack{j \neq i \\ j=1}}^{j=n} a_{ij}\, x_j^{(k)})]$ i=1,2,….,n

- Stopped when $\left|\frac{x_i^{(k+1)} - x_i^{(k)}}{x_i^{(k)}}\right| < \in$ i=1,2,…,n

3.7.2 Gauss-Seidel Iterative Method

- Use the new estimate in the same iteration

$$
\begin{cases}
x_1^{(k+1)} = \dfrac{1}{a_{11}}[b_1 - \displaystyle\sum_{j=2}^{j=n} a_{1j}x_j^{(k)}] \\
x_i^{(k+1)} = \dfrac{1}{a_{ii}}[b_i - (\displaystyle\sum_{j=1}^{j=i-1} a_{ij}x_j^{(k+1)} + \sum_{j=i+1}^{j=n} a_{ij}x_j^{(k)})] \\
x_n^{(k+1)} = \dfrac{1}{a_{nn}}[b_n - \displaystyle\sum_{j=1}^{j=n-1} a_n x_j^{(k+1)}]
\end{cases}
$$

- For i=1 ➔ x_1 ➔ calculate x_2 ➔ calculate x_3 ➔ calculate x_4………..
- The new value (estimate) is updated every iteration

Example 3-8

- Solve the following linear equations using Gauss-Seidel iteration method

$$
\begin{aligned}
9x_1 - 2x_2 + 3x_3 + 3x_4 &= 54.5 \\
2x_1 + 8x_2 - 2x_3 + 3x_4 &= -14 \\
-3x_1 + 2x_2 + 11x_3 - 4x_4 &= 12.5 \\
-2x_1 + 3x_2 + 2x_3 + 10x_4 &= -21
\end{aligned}
$$

3.8 MATLAB Built-in function

- Left division \ : $[a][x] = [b]$ ⇨ $x = a\backslash b$

$\gg a = [4\ -2\ -3\ 6;\ -6\ 7\ 6.5\ -6;\ 1\ 7.5\ 6.25\ 5.5; -12\ 22\ 15.5\ -1]$

$\gg\ b = [12; -6.5; 16; 17]$ $\gg x = a\backslash b$

- Right division / : $[x][a] = [b]$ ⇨ $x = b/a$

$\gg a = [4\ -6\ 1\ -12; -2\ 7\ 7.5\ 22; -3\ 6.5\ 6.25\ 15.5; 6\ -6\ 5.5\ -1]$

$\gg b = [12\ -6.5\ 16\ 17]$ $\gg x = b/a$

- Inverse operation $[a][x] = [b]$ ⇨ $[x] = [a]^{-1}[b]$

$[a]^{-1}$ ⇨ a^-1 or inv(a)

≫ x=a^-1 * b

3.9 Tridiagonal systems of equations

- A matrix of coefficients with zero as their entries except along the **diagonal**, **above-diagonal**, and **below-diagonal** elements

$$\begin{bmatrix} d_1 & a_1 & 0 & \cdots & \cdots & 0 \\ b_2 & d_2 & a_2 & 0 & \cdots & 0 \\ 0 & b_3 & d_3 & a_3 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & a_{n-2} & 0 \\ 0 & \cdots & 0 & b_{n-1} & d_{n-1} & a_{n-1} \\ 0 & \cdots & \cdots & 0 & b_n & d_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{bmatrix}$$

- Solved by Thomas algorithm
- Transform the matrix to upper-triangular with 1 along diagonal

$$\begin{bmatrix} 1 & a'_1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & a'_2 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & a'_3 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & a'_{n-1} \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} B"_1 \\ B"_2 \\ \vdots \\ B"_n \end{bmatrix}$$

Procedure

- (1) Define $b = [0, b_2, b_3, \dots, b_n]$ $d = [d_1, d_2, \dots, d_n]$
  $a = [a_1, a_2, \dots, a_n]$ $B = [B_1, B_2, \dots, B_n]$
- (2) $a_1 = \frac{a_1}{d_1}$ $B_1 = \frac{B_1}{d_1}$
- (3) for $i = 2,3, \dots, n-1$
- $$a_i = \frac{a_i}{d_i - b_i a_{i-1}} \qquad B_i = \frac{B_i - b_i B_{i-1}}{d_i - b_i a_{i-1}}$$
- (4) $B_n = \frac{B_n - b_n B_{n-1}}{d_n - b_n a_{n-1}}$
- (5) Calculate the solution by back substitution.

3.12 Eigenvalues and Eigenvectors

$[a][u] = \lambda[u]$ λ: eigenvalue of the matrix
$[u]$: eigenvector

$d = eig(A)$

$[V, D] = eig(A)$

Eigenvector Eigenvalue

Ex: $\gg A = [6,7,2; 4,-5,2; 1,-1,1]$

$\gg lambd = eig(A)$

$\gg [eVec\ eVal] = eig(A)$