數值分析

Chapter 8

Ordinary Differential Equations: Boundary Value Problems

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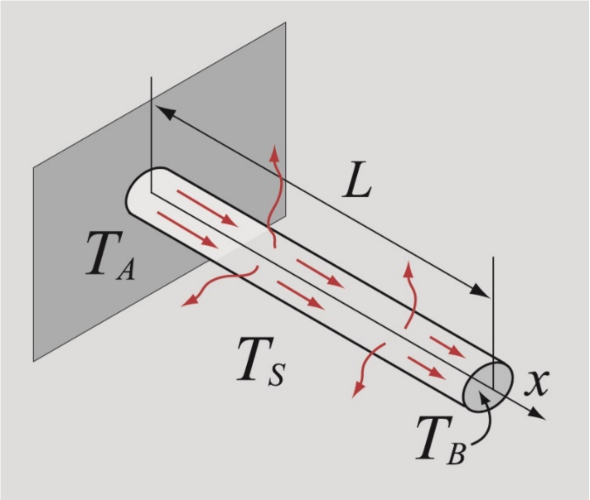
Outline

- Background (8.1)
- The shooting method (8.2)
- The finite difference method (8.3)
- MATLAB built-in functions (8.4)

8.1 Background

- Differential equations of second and higher order that have constrains specified at different values of the independent variables
- Boundary value problems (BVP): the constrains are often specified at the **endpoints** or boundaries of the domain of the solution (**boundary conditions**)



$$\frac{d^2T}{dx^2} - \alpha_1(T - T_s) - \alpha_2(T^4 - T_s^4) = 0$$

Problem statement

- A second order boundary value problem:

$$\frac{d^2y}{dx^2} = f(x, y, \frac{dy}{dx})$$

- Two values of y are given at x=a and x=b (**Dirichlet boundary condition**)

$$y(a) = Y_a \qquad y(b) = Y_b$$

- Two values of dy/dx are given (**Neumann boundary condition**):

$$\left.\frac{dy}{dx}\right|_{x=a} = D_a \qquad \left.\frac{dy}{dx}\right|_{x=b} = D_b$$

- Mixed boundary conditions
- Higher order ODEs require additional boundary conditions

Solving Boundary Value Problem

- Shooting methods
  - Reduce the second order (or higher order) ODE to an initial value problem
  - Transforming the equation into a system of first order ODEs

- Finite differences methods
  - The derivatives in the differential equation are approximated with finite difference formulas (Ch 5)
  - The difference between various finite difference methods is in the finite difference formulas

8.2 Shooting Method

- Transform a boundary value problem into a system of initial value problems

$$\frac{d^2y}{dx^2} = f(x, y, \frac{dy}{dx}) \qquad y(a) = Y_a \quad y(b) = Y_b$$

$$\Rightarrow \frac{dy}{dx} = w \qquad y(a) = Y_a$$

$$\Rightarrow \frac{dw}{dx} = f(x, y, w)$$ Initial condition?

- Two estimates are made for the initial value of the second equation:

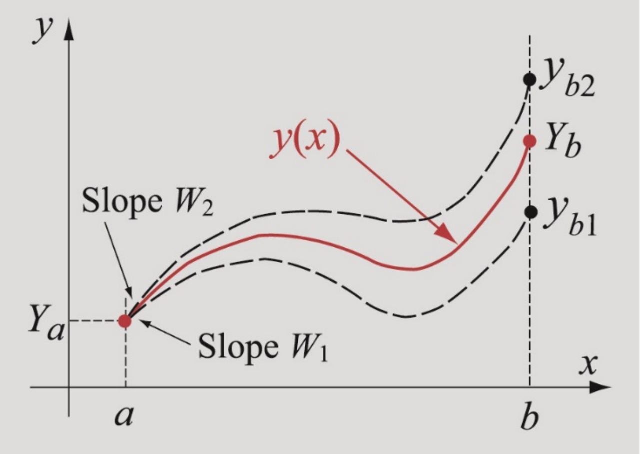
$$w(a) = \left.\frac{dy}{dx}\right|_{x=a} = W_1 \qquad w(a) = \left.\frac{dy}{dx}\right|_{x=a} = W_2$$

8.2 Shooting Method

- A new estimate for the initial value is determined by using the results of the previous two solutions:
  - Interpolation, Bisection method, Secant method…

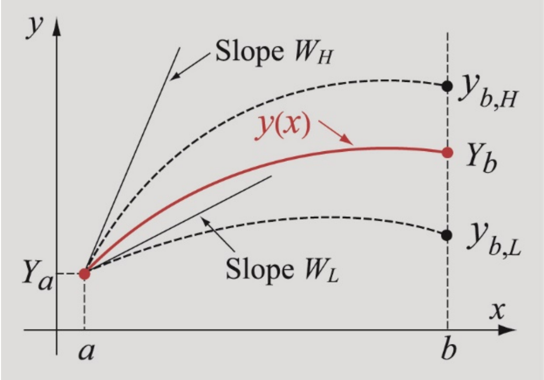
$$w(a) = \left.\frac{dy}{dx}\right|_{x=a} = W_3$$

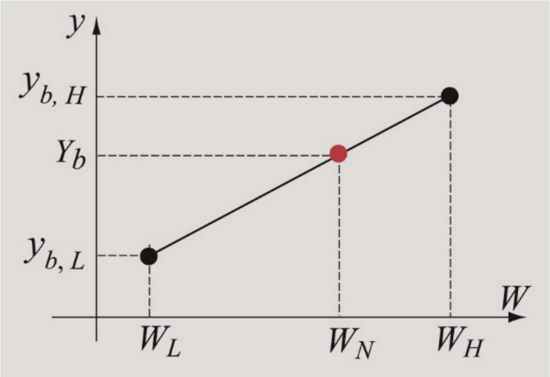
- Repeat the following steps until the numerical accuracy is reached



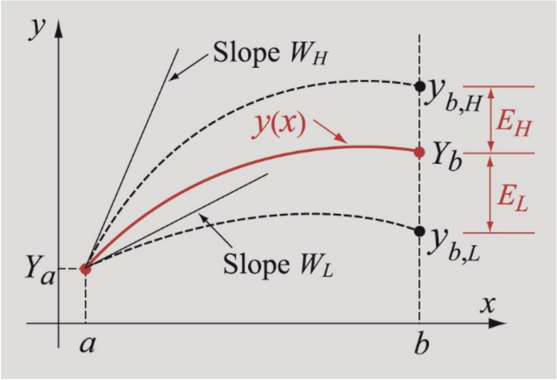
Estimating the slope

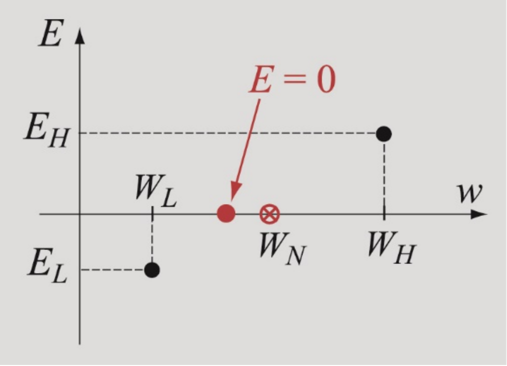
- Linear Interpolation: $W_N = W_L + (Y_b - y_{b,L})\dfrac{W_H - W_L}{y_{b,H} - y_{b,L}}$





- Bisection method:





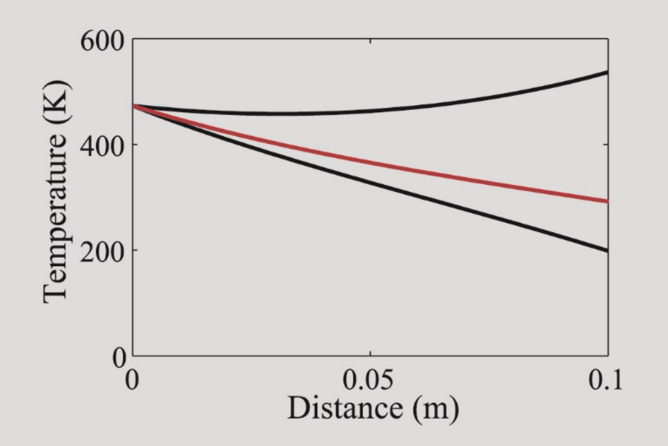
Example 8-1:2nd order ODE

- Solve the following 2nd order ODEs:

$$\frac{d^2T}{dx^2} - \alpha_1(T - T_s) - \alpha_2(T^4 - T_s^4) = 0$$

$$\alpha_1 = 500/3$$
$$\alpha_2 = 9.467 \times 10^{-8}$$
$$T_s = 293$$

- With the boundary condition:
- T(0)=473 T(0.1)=293



- Transform the equation:

$$\frac{dT}{dx} = w \qquad T(0) = 473$$

$$\frac{dw}{dx} = \alpha_1(T - T_s) + \alpha_2(T^4 - T_s^4) \qquad w(0) = ???$$

Example 8-2: 2nd Order ODEs with Bisection Method

- Solve the following 2nd order ODEs:

$$\frac{d^2T}{dx^2} - \alpha_1(T - T_s) - \alpha_2(T^4 - T_s^4) = 0$$

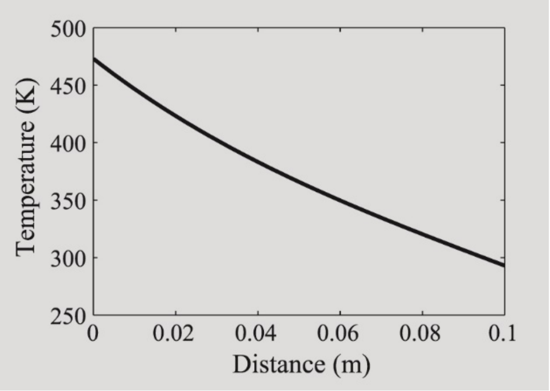
$$\alpha_1 = 500/3$$
$$\alpha_2 = 9.467\times 10^{-8}$$
$$T_s = 293$$

- With the boundary condition:
- T(0)=473 T(0.1)=293

$$\frac{dT}{dx} = w \qquad T(0) = 473$$

$$\frac{dw}{dx} = \alpha_1(T - T_s) + \alpha_2(T^4 - T_s^4) \qquad w(0) = ???$$

- Start with w(0) = -1000 and -3500



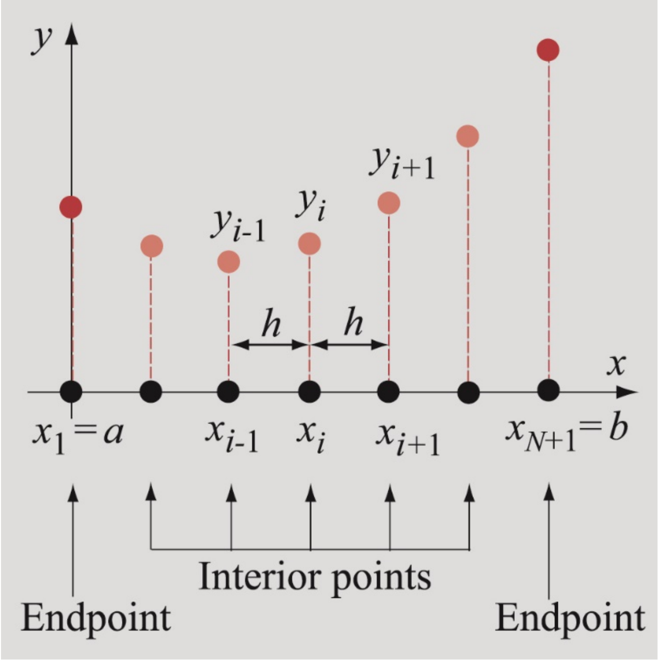
8.3 Finite Difference Method

- The derivatives in the differential equation are replaced with finite difference approximations
- The domain [a,b] is divided into N subintervals of equal length h, that are defined by (N+1) **grid points**
- The differential equation is then written at **each** of the interior points which gives a **system** of algebraic equations

- The central difference formula:

$$\frac{dy}{dx} = \frac{y_{i+1} - y_{i-1}}{2h}$$

$$\frac{d^2y}{dx^2} = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}$$



Example 8-3: Finite Difference Method

- Solve the following ODEs:

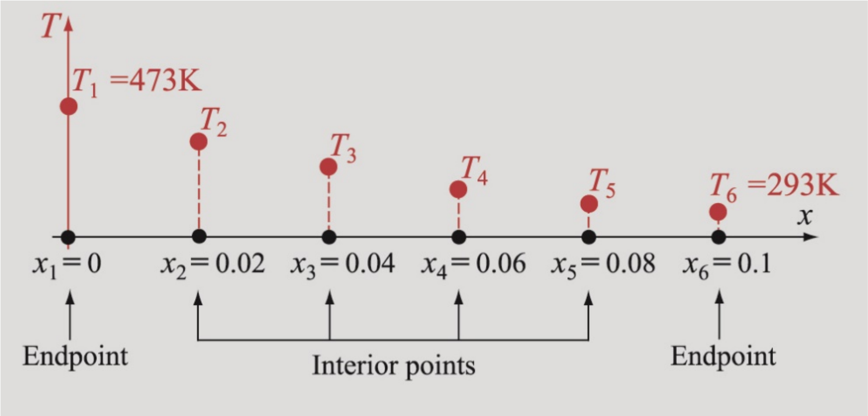
$$\frac{d^2T}{dx^2} - \alpha_1 (T - T_s) = 0$$

- With the boundary condition: T(0)=473 T(0.1)=293
- Divide the domain of the solution into given subintervals
- Using the central difference formula:

$$\frac{T_{i-1} - 2T_i + T_{i+1}}{h^2} - \alpha_1 (T_i - T_s) = 0$$

- Therefore:

$$T_{i-1} - (2 + \alpha_1 h^2) T_i + T_{i+1} = -\alpha_1 h^2 T_s$$



Example 8-3: Finite Difference Method

- The equation is written for each of the interior points:
- For i=2, 3, 4, 5

$$T_1 - (2+\alpha_1 h^2)T_2 + T_3 = -\alpha_1 h^2 T_s \qquad T_2 - (2+\alpha_1 h^2)T_3 + T_4 = -\alpha_1 h^2 T_s$$

$$T_3 - (2+\alpha_1 h^2)T_4 + T_5 = -\alpha_1 h^2 T_s \qquad T_4 - (2+\alpha_1 h^2)T_5 + T_6 = -\alpha_1 h^2 T_s$$

- The system of linear algebraic equations can be solved:

$$\begin{bmatrix} -(2+\alpha_1 h^2) & 1 & 0 & 0 \\ 1 & -(2+\alpha_1 h^2) & 1 & 0 \\ 0 & 1 & -(2+\alpha_1 h^2) & 1 \\ 0 & 0 & 0 & -(2+\alpha_1 h^2) \end{bmatrix} \begin{bmatrix} T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix} = \begin{bmatrix} -(\alpha_1 h^2 T_s + T_1) \\ -\alpha_1 h^2 T_s \\ -\alpha_1 h^2 T_s \\ -(\alpha_1 h^2 T_s + T_6) \end{bmatrix}$$

Mixed Boundary Conditions

- The system of algebraic equations that is obtained cannot be solved since the solution at the endpoints is not given

- The boundary conditions are discretized using finite difference method, and incorporating the resulting equations for the interior points

Ex. 8-5: BVP with mixed boundary conditions

- Consider the following ODE:

$$-2\frac{d^2y}{dx^2} + y = e^{-0.2x} \qquad 0 \le x \le 1$$

- With the boundary conditions

$$y(0) = 1 \quad \left.\frac{dy}{dx}\right|_{x=1} = -y$$

- Divided the solution domain into eight subintervals and use the central difference approximation.
- Compare the results with the exact solution:

$$y = -0.2108e^{x/\sqrt{2}} + 0.1238e^{-x/\sqrt{2}} + \frac{e^{-0.2x}}{0.92}$$

Ex. 8-5: BVP with mixed boundary conditions

- Use the central difference scheme:

$$-2\left(\frac{y_{i-1}-2y_i+y_{i+1}}{h^2}\right)+y_i=e^{-0.2x_i}$$

- For the interior points: i=2, 3, 4, …, 8

$$-2y_2+(4+h^2)y_3-2y_4=h^2e^{-0.2x_3}$$
$$-2y_3+(4+h^2)y_4-2y_5=h^2e^{-0.2x_4}$$
$$\vdots$$
$$-2y_6+(4+h^2)y_7-2y_8=h^2e^{-0.2x_7}$$
$$-2y_7+(4+h^2)y_8-2y_9=h^2e^{-0.2x_8}$$

- The solution at the right point (y_9) is unknown, therefore using three point backward difference:

$$\frac{dy}{dx}=\frac{y_{i-2}-4y_{i-1}+3y_i}{2h} \quad and \quad \left.\frac{dy}{dx}\right|_{x=1}=-y \quad \Rightarrow \quad \frac{y_7-4y_8+3y_9}{2h}=-y_9$$

Ex. 8-5: BVP with mixed boundary conditions

- The linear equations can be expressed by:

$$\begin{bmatrix} (4+h^2) & -2 & 0 & 0 & 0 & 0 & 0 \\ -2 & (4+h^2) & -2 & 0 & 0 & 0 & 0 \\ 0 & -2 & (4+h^2) & -2 & 0 & 0 & 0 \\ 0 & 0 & -2 & (4+h^2) & -2 & 0 & 0 \\ & & & -2 & (4+h^2) & -2 & 0 \\ & & & & -2 & (4+h^2) & -2 \\ & & & & & \left(\dfrac{2}{3+2h}-2\right) & \left(4+h^2-\dfrac{8}{3+2h}\right) \end{bmatrix} \begin{bmatrix} y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \\ y_8 \end{bmatrix} = \begin{bmatrix} 2+h^2 e^{-0.2x_2} \\ h^2 e^{-0.2x_3} \\ h^2 e^{-0.2x_4} \\ h^2 e^{-0.2x_5} \\ h^2 e^{-0.2x_6} \\ h^2 e^{-0.2x_7} \\ h^2 e^{-0.2x_8} \end{bmatrix}$$

8.4 MATLAB built-in Functions

- For solving a first order ODE: bvp4c

- sol = bvp4c(odefun, bcfun, solinit)

- odefun: *dydx = odefun(x, yw)*
- bcfun: *res = bcfun(ya, yb)*
- solinit: *solinit = bvpinit(x, yinit)*

The Residuals

- For the mixed boundary condition

Boundary condition:

$$y(a) = Y_a \quad \text{and} \quad \left.\frac{dy}{dx}\right|_{x=b} = D_b$$

vector `res` is: $\begin{bmatrix} ya(1) - Y_a \\ yb(2) - D_b \end{bmatrix}$

Boundary condition:

$$\left.\frac{dy}{dx}\right|_{x=a} = D_a \quad \text{and} \quad y(b) = Y_b$$

vector `res` is: $\begin{bmatrix} ya(2) - D_a \\ yb(1) - Y_b \end{bmatrix}$

Boundary condition (general case):

$$c_1 \left.\frac{dy}{dx}\right|_{x=a} + c_2 y(a) = C_a \quad \text{and}$$

$$c_3 \left.\frac{dy}{dx}\right|_{x=b} + c_4 y(b) = C_b$$

vector `res` is (for $c_1, c_3 \neq 0$):

$$\begin{bmatrix} ya(2) - \dfrac{C_a}{c_1} + \dfrac{c_2}{c_1} ya(1) \\ yb(2) - \dfrac{C_b}{c_3} + \dfrac{c_4}{c_3} yb(1) \end{bmatrix}$$

Example 8-6: Solving a BVP problem

- Solve the following two-point BVP:

$$\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + 5y - \cos(3x) = 0 \qquad 0 \le x \le \pi$$

- With the boundary conditions: y(0) = 1.5 and y(π)=0

- Solution:
- The equation is rewritten as:

$$\frac{dy}{dx} = w$$

$$\frac{dw}{dx} = -2xw - 5y + \cos(3x)$$