# CSE 5311 Homework Assignment 2 (Fall 2019)

Due date: 9/11 (Wednesday) (type, print, and hand-in in class)

**(1) [10 points] Exercise 2.2-1 on Page 29.**
**Express the function $n^3/1000 - 100n^2 - 100n + 3$ in terms of $\Theta$-notation.**

**Answer:** $\Theta(n^3)$

**(2) [10 points] Exercise 3.1-1 on Page 52.**
**Let $f(n)$ and $g(n)$ be asymptotically nonnegative functions. Using the basic definition of $\Theta$-notation, prove that $\max(f(n), g(n)) = \Theta(f(n) + g(n))$.**

**Answer:**
First, let's clarify what the function $\max(f(n), g(n))$ is. Let's define the function $h(n) = \max(f(n), g(n))$. Then

$$h(n) = \begin{cases} f(n), & f(n) \geq g(n) \\ g(n), & f(n) < g(n) \end{cases}$$

Since $f(n)$ and $g(n)$ are asymptotically nonnegative, there exists $n_0$ such that $f(n) \geq 0$ and $g(n) \geq 0$ for all $n \geq n_0$. Thus for $n \geq n_0$, $f(n) + g(n) \geq f(n) \geq 0$ and $f(n) + g(n) \geq g(n) \geq 0$. Since for any particular n, $h(n)$ is either $f(n)$ or $g(n)$, we have $f(n) + g(n) \geq f(n) \geq 0$, which shows that $h(n) = \max(f(n), g(n))$ $\leq c_2(f(n) + g(n))$ for all $n \geq n_0$ (with $c_2 = 1$ in the definition of $\Theta$).
Similarly, since for any particular n, $h(n)$ is the larger of $f(n)$ and $g(n)$, we have for all $n \geq n_0$, $0 \leq f(n) \leq h(n)$ and $0 \leq g(n) \leq h(n)$. Adding these two inequalities yields $0 \leq f(n) + g(n) \leq 2h(n)$, or equivalently $0 \leq (f(n) + g(n))/2 \leq h(n)$, which shows that $h(n) = \max(f(n), g(n)) \geq c_1(f(n) + g(n))$ for all $n \geq n_0$ (with $c_2 = 1/2$ in the definition of $\Theta$).

**(3) [10 points] Exercise 3.1-2 on Page 52.**
**Show that for any real constants a and b, where b > 0, $(n + a)^b = \Theta(n^b)$.**

**Answer:**
To show that $(n + a)^b = \Theta(n^b)$, we want to find constants $c_1, c_2, n_0 > 0$ such that $0 \leq c_1 n^b \leq (n + a)^b \leq c_2 n^b$ for all $n \geq n_0$.
Note that
$n + a \leq n + |a| \leq 2\mathrm{n}$ when $|a| \leq n$,
and

$n + a \geq n - |a| \geq \frac{1}{2}\mathrm{n}$ when $|a| \leq \frac{1}{2}n$.

Thus, when $n \geq 2|a|$, $0 \leq \frac{1}{2}n \leq n + a \leq 2\mathrm{n}$.

Since b > 0, the inequality still holds when all parts are raised to the power b:

$0 \le (\frac{1}{2}n)^b \le (n+a)^b \le (2n)^b$ ,

Thus, $c_1 = (1/2)^b, c_2 = 2^b$, and $n_0 = 2|a|$ satisfy the definition.

**(4) [10 points] Problem 3.4 (b) (h) on Page 62.**
**Let $f(n)$ and $g(n)$ be asymptotically positive functions. Prove or disprove each of the following conjectures.**
***b.*** $\boldsymbol{f(n) + g(n) = \Theta(\min(f(n), g(n)))}$.
***h.*** $\boldsymbol{f(n) + o(f(n)) = \Theta(f(n))}$.

**Answer:**
**b.** False. Counterexample: $n + n^2 \ne \Theta(\min(n, n^2)) = \Theta(n)$.

**h.** True. Let $g(n) = o(f(n))$. Then $\exists c, n_0 : \forall n \ge n_0, 0 \le g(n) \le cf(n)$.

We need to prove that

$\exists c_1, c_2, n_0 : \forall n \ge n_0, 0 \le c_1 f(n) \le f(n) + g(n) \le c_2 f(n)$.

Thus, if we pick $c_1 = 1$ and $c_2 = c + 1$, it holds.

**(5) [10 points] Exercise 4.4-2 on Page 92.**
**Use a recursion tree to determine a good asymptotic upper bound on the recurrence $\boldsymbol{T(n) = T(n/2) + n^2}$. Use the substitution method to verify your answer.**

**Answer:**

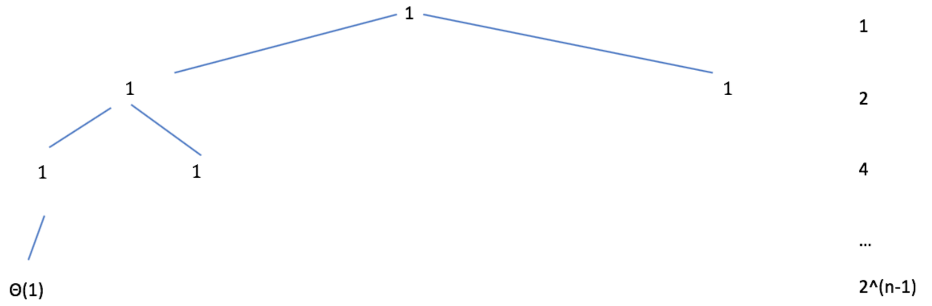
| | |
|---|---|
| $n^2$ | $n^2$ |
| $(n/2)^2$ | $\frac{1}{4}n^2$ |
| $(n/4)^2$ | $\frac{1}{16}n^2$ |
| … | … |
| $\Theta(1)$ | $(\frac{1}{4})^{lgn-1}n^2$ |

The height of the tree is lg$n$, and there are $1^{lgn} = 1$ leaf.
T(n) = $\sum_{i=0}^{lgn-1} (\frac{1}{4})^i n^2 = \Theta(n^2)$.
Guess T(n) = $\Theta(n^2)$.
Assume that $T(k) \le ck^2$.

Then $T(n) \le c(n/2)^2 + n^2 = (c/4 + 1)n^2 \le cn^2$, when $c \ge \frac{4}{3}$.

**(6) [10 points] Exercise 4.4-4 on Page 93.**
**Use a recursion tree to determine a good asymptotic upper bound on the recurrence $T(n) = 2T(n-1) + 1$. Use the substitution method to verify your answer.**

**Answer:**




Depth is n and leaf number is $2^{n-1}$
$T(n) = 1 + 2 + \cdots + 2^{n-1} = 2^n - 1 = \Theta(2^n)$
Assume that $T(k) \leq c2^k$ for k < n
$Then\ T(n) = 2T(n-1) + 1 \ \leq 2c2^{n-1} + 1 = c2^n + 1 = O(2^n)$

**(7) [10 points] Exercise 4.5-4 on Page 97.**

Can the master method be applied to the recurrence $T(n) = 4T(n/2) + n^2 \lg n$? Why or why not? Give an asymptotic upper bound for this recurrence.

Answer:
In the given recurrence, a=4 and b=2. Hence, $n^{\log_b a} = n^{\log_2 4} = n^2$ and $f(n) = \Theta(n^2 \lg n)$.
Based on Case 2 of master method, $f(n) = \Theta(n^2 \lg n)$ for some constant $k=1 \geq 0$. f(n) and $n^2$ grow at similar rates.
So we **can** use master method to calculate time complexity of T(n):
Solution: $T(n) = \Theta(n^{\log_b a} \lg^{k+1} n) = \Theta(n^2 \lg^2 n)$

Assume that $T(n) \leq c(n^2(lgn)^2)$, for k < n
$T(n) = 4T\left(\frac{n}{2}\right) + n^2 lgn \ \leq 4\left(c\left(\frac{n}{2}\right)^2\left(\lg\left(\frac{n}{2}\right)\right)^2\right) + n^2 lgn$
$\leq \ cn^2(lgn - 1)^2 + n^2 lgn$
$\leq \ cn^2(lgn)^2 + n^2(c + (1 - 2c)lgn)$

Just let $c = 1$, whenever n is large enough, $T(n) \leq \ cn^2(lgn)^2$

**(8) [30 points] Problem 4.1 on Page 107.**

***4-1 Recurrence examples***

Give asymptotic upper and lower bounds for $T(n)$ in each of the following recurrences. Assume that $T(n)$ is constant for $n \le 2$. Make your bounds as tight as possible, and justify your answers.

*a.* $T(n) = 2T(n/2) + n^4$.

*b.* $T(n) = T(7n/10) + n$.

*c.* $T(n) = 16T(n/4) + n^2$.

*d.* $T(n) = 7T(n/3) + n^2$.

*e.* $T(n) = 7T(n/2) + n^2$.

*f.* $T(n) = 2T(n/4) + \sqrt{n}$.

*g.* $T(n) = T(n-2) + n^2$.

**Answer:**

a. $T(n) = 2T(n/2) + n^4$: $T(n) = \Theta(n^4)$: in terms of the Master Theorem, $a = 2$, $b = 2$, so $\log_b a = \log_2 2 = 1$. $f(n) = n^4 = n^{\log_b a + 3}$; this is looking like case 3. We need to check $af\left(\frac{n}{b}\right) \le cf(n)$ for some $c < 1$ and $n$ large enough. In fact, $af\left(\frac{n}{b}\right) = 2\left(\frac{n}{2}\right)^4 = \frac{1}{8}n^4 = \frac{1}{8}f(n)$, and case 3 applies.

Conclude $T(n) = \Theta(n^4)$.

b. $T(n) = T(7n/10) + n$: $T(n) = \Theta(n)$ : in terms of the Master Theorem, $a = 1$, $b = \frac{10}{7}$, so $\log_b a = \log_{\frac{10}{7}} 1 = 0$. $f(n) = n = n^{\log_b a + 1}$; this is looking like case 3. We need to check $af\left(\frac{n}{b}\right) \le cf(n)$ for some $c < 1$ and $n$ large enough. In fact, $af\left(\frac{n}{b}\right) = 1\left(\frac{7n}{10}\right)^1 = \frac{7}{10}n = \frac{7}{10}f(n)$, and case 3 applies.

Conclude $T(n) = \Theta(n)$.

c. $T(n) = 16T(n/4) + n^2$: $T(n) = \Theta(n^2 \lg n)$: in terms of the Master Theorem, $a = 16$, $b = 4$, so $\log_b a = \log_4 16 = 2$. $f(n) = n^2 = n^{\log_b a}$; this is case 2.

Conclude $T(n) = \Theta(n^2 \lg n)$.

d. as $\log_3 7 < 2$, based on master method, as case 3 applies, $T(n) = \Theta(n^2)$

e. $T(n) = 7T(n/2) + n^2$: $T(n) = \Theta(n^{\log_2 7})$: in terms of the Master Theorem, $a = 7$, $b = 2$, so $2 < \log_b a = \log_2 7 < 3$. $f(n) = n^2 = n^{\log_b a - \varepsilon}$; this is case 1.

Conclude $T(n) = \Theta(n^{\log_2 7})$.

f. $T(n) = 2T(n/4) + \sqrt{n}$: $T(n) = \Theta(\sqrt{n} \lg n)$: in terms of the Master Theorem, $a = 2$, $b = 4$, so $\log_b a = \log_4 2 = \frac{1}{2}$. $f(n) = n^{\frac{1}{2}} = n^{\log_b a}$; this is case 2.

Conclude $T(n) = \Theta(\sqrt{n} \lg n)$.

g. Recursion-tree method. $T(n) = T(n-2) + n^2 = \sum_{i=0}^{n/2}(n-2i)^2 = \sum_{i=0}^{n/2}(n^2 - 4ni + 4i^2) = n^2\sum_{i=0}^{n/2} 1 - 4n\sum_{i=0}^{n/2} i + 4\sum_{i=0}^{n/2} i^2 = \frac{n^3}{2} - 4n\left(\frac{n}{4} * \frac{n+2}{2}\right) + 4 * \frac{\frac{n}{2} * \left(\frac{n}{2}+1\right) * (n+1)}{6} = \Theta(n^3)$.