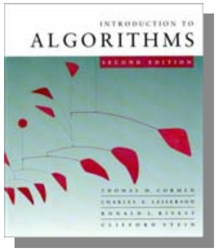
Design and Analysis of Algorithms

CSE 5311

Lecture 2 Asymptotic Notation and Solving Recurrences

Song Jiang, Ph.D.

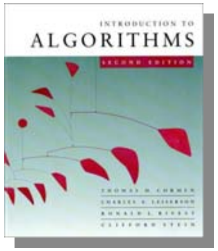
Department of Computer Science and Engineering



Asymptotic notation

O-notation (upper bounds):

We write $f(n) = O(g(n))$ if there exist constants $c > 0$, $n_0 > 0$ such that $0 \leq f(n) \leq cg(n)$ for all $n \geq n_0$.



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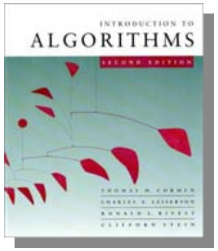
Example: $2n^2 = O(n^3)$ $\quad (c = 1, n_0 = 2)$



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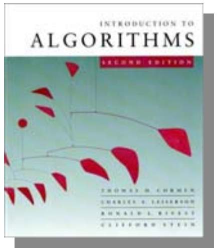
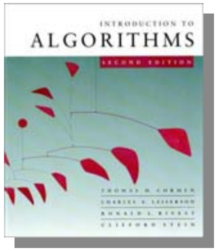
funny, "one-way" equality



Set definition of O-notation

$O(g(n)) = \{ f(n)$: there exist constants $c > 0$, $n_0 > 0$ such that $0 \le f(n) \le cg(n)$ for all $n \ge n_0$ $\}$



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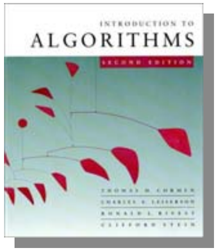
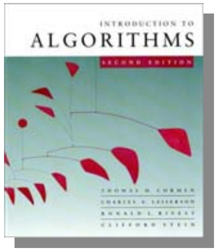
EXAMPLE: $2n^2 \in O(n^3)$



Macro substitution

Convention: A set in a formula represents an anonymous function in the set.



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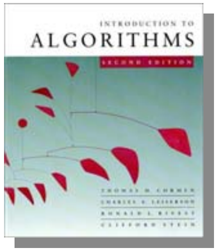
EXAMPLE:

$f(n) = n^3 + O(n^2)$

means

$f(n) = n^3 + h(n)$

for some $h(n) \in O(n^2)$.



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EXAMPLE: $n^2 + O(n) = O(n^2)$

means

for any $f(n) \in O(n)$:

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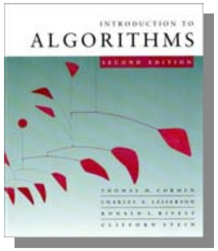
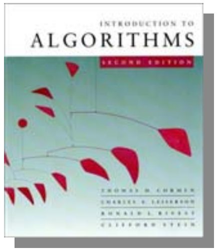
for some $h(n) \in O(n^2)$.



Ω-notation (lower bounds)

O-notation is an *upper-bound* notation. It makes no sense to say $f(n)$ is at least $O(n^2)$.



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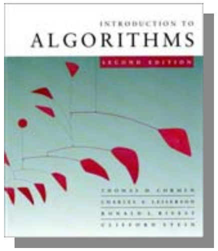

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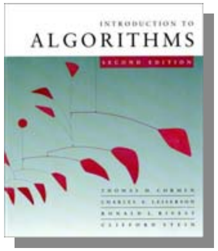
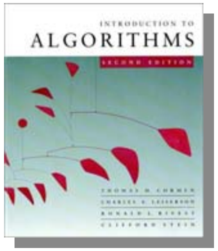
Example: $\sqrt{n} = \Omega(\lg n)$ $\quad (c = 1, n_0 = 16)$



Θ-notation (tight bounds)

$$\Theta(g(n)) = O(g(n)) \cap \Omega(g(n))$$



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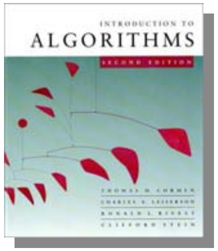
Example: $\frac{1}{2}n^2 - 2n = \Theta(n^2)$



o-notation and ω-notation

O-notation and Ω-notation are like $\leq$ and $\geq$.
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$o(g(n)) = \{ f(n)$: for any constant $c > 0$,
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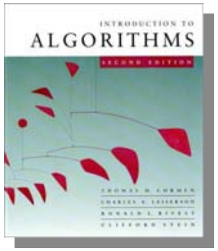
EXAMPLE: $2n^2 = o(n^3)$ $\quad (n_0 = 2/c)$



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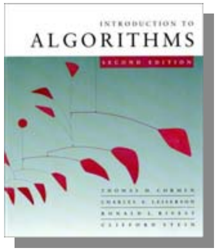
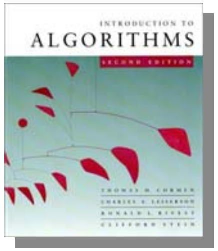


Solving recurrences

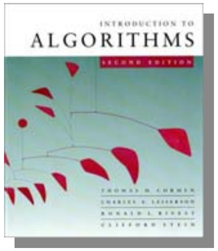
- The analysis of merge sort from ***Lecture 1*** required us to solve a recurrence.
- Recurrences are like solving integrals, differential equations, etc.
  - Learn a few tricks.
- ***Lecture 3***: Applications of recurrences to divide-and-conquer algorithms.



Substitution method

The most general method:

1. ***Guess*** the form of the solution.
2. ***Verify*** by induction.
3. ***Solve*** for constants.



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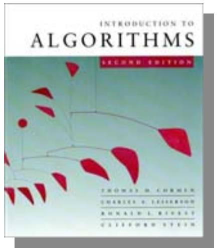
EXAMPLE: $T(n) = 4T(n/2) + n$

- [Assume that $T(1) = \Theta(1)$.]
- Guess $O(n^3)$. (Prove O and Ω separately.)
- Assume that $T(k) \le ck^3$ for $k < n$.
- Prove $T(n) \le cn^3$ by induction.

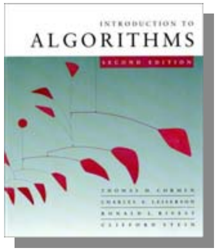


Example of substitution

$$
\begin{aligned}
T(n) &= 4T(n/2) + n \\
&\leq 4c(n/2)^3 + n \\
&= (c/2)n^3 + n \\
&= cn^3 - ((c/2)n^3 - n) \longleftarrow \textit{desired} - \textit{residual} \\
&\leq cn^3 \longleftarrow \textit{desired}
\end{aligned}
$$

whenever $(c/2)n^3 - n \geq 0$, for example, if $c \geq 2$ and $n \geq 1$.

residual



Example (continued)

- We must also handle the initial conditions, that is, ground the induction with base cases.
- ***Base:*** $T(n) = \Theta(1)$ for all $n < n_0$, where n_0 is a suitable constant.
- For $1 \leq n < n_0$, we have "$\Theta(1)$" $\leq cn^3$, if we pick c big enough.

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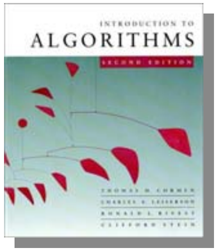

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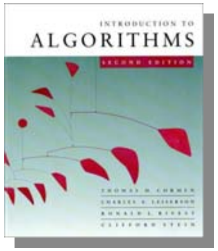
Wrong! We must prove the I.H.



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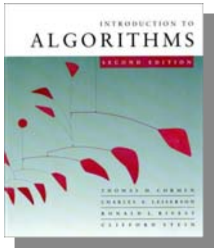
$$= cn^2 - (-n)$$

[desired – residual]

$$\le cn^2$$

for ***no*** choice of $c > 0$. Lose!



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IDEA: Strengthen the inductive hypothesis.

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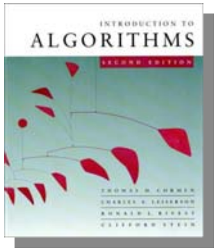

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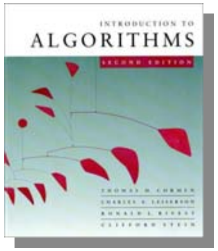

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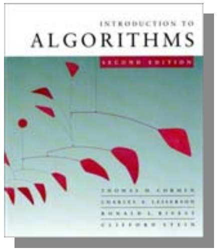
Pick c_1 big enough to handle the initial conditions.



Recursion-tree method

- A recursion tree models the costs (time) of a recursive execution of an algorithm.
- The recursion-tree method can be unreliable, just like any method that uses ellipses (…).
- The recursion-tree method promotes intuition, however.
- The recursion tree method is good for generating guesses for the substitution method.

Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:

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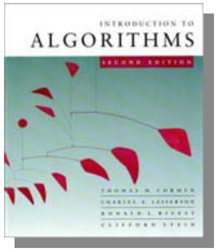
$$T(n)$$



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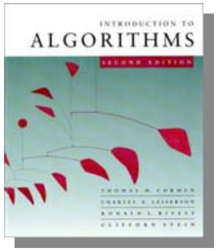
$T(n/4)$ $\quad$ $T(n/2)$



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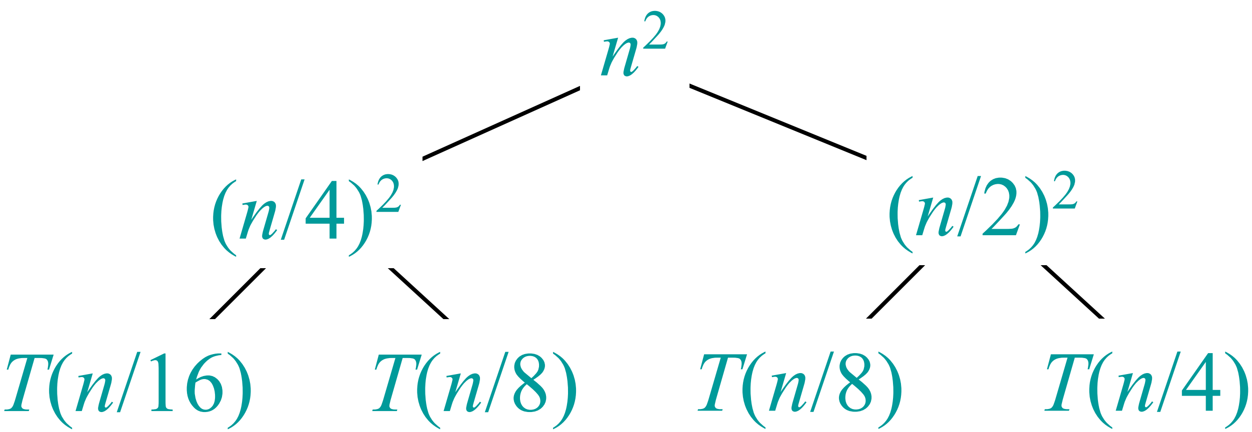

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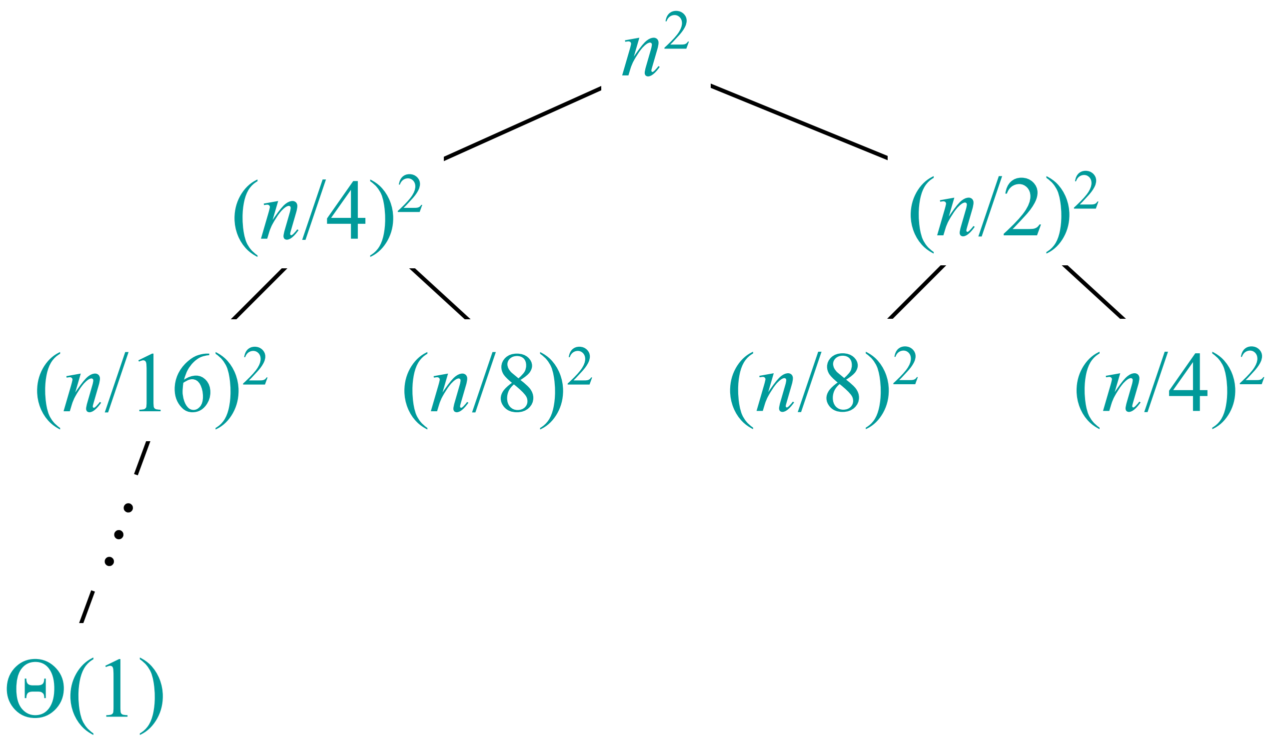

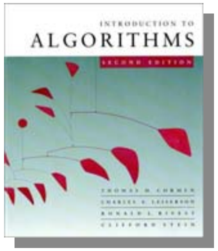

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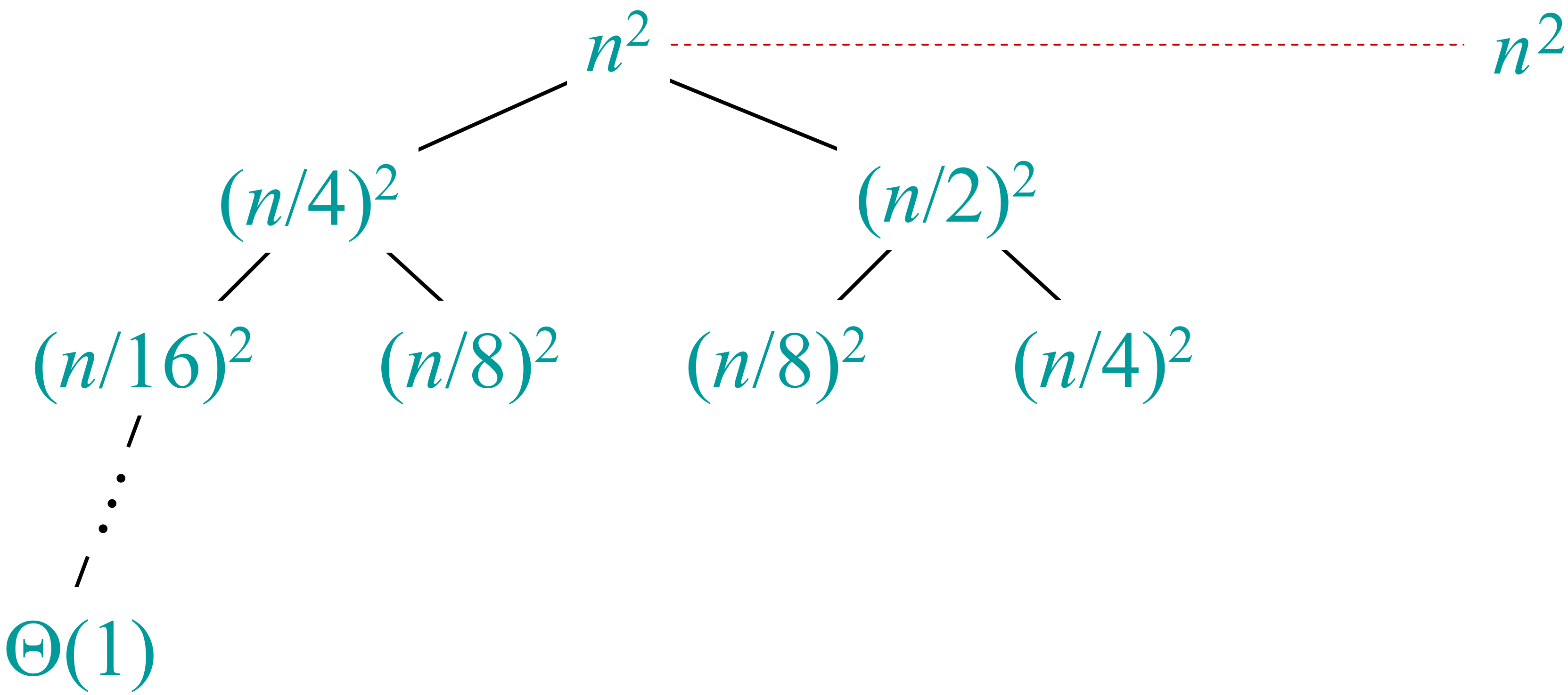

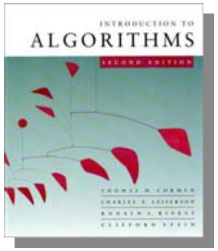

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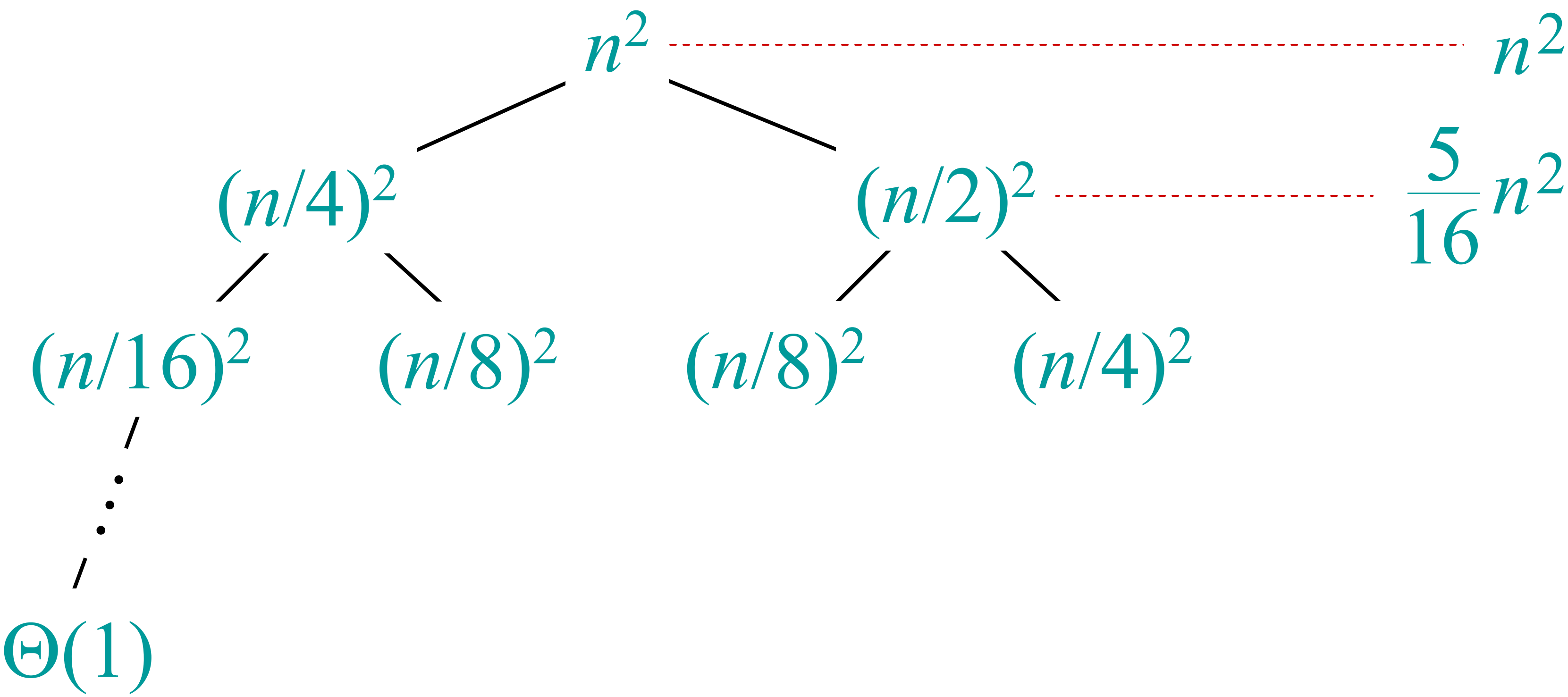

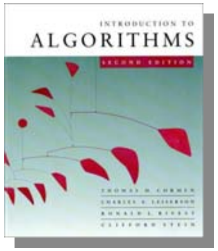

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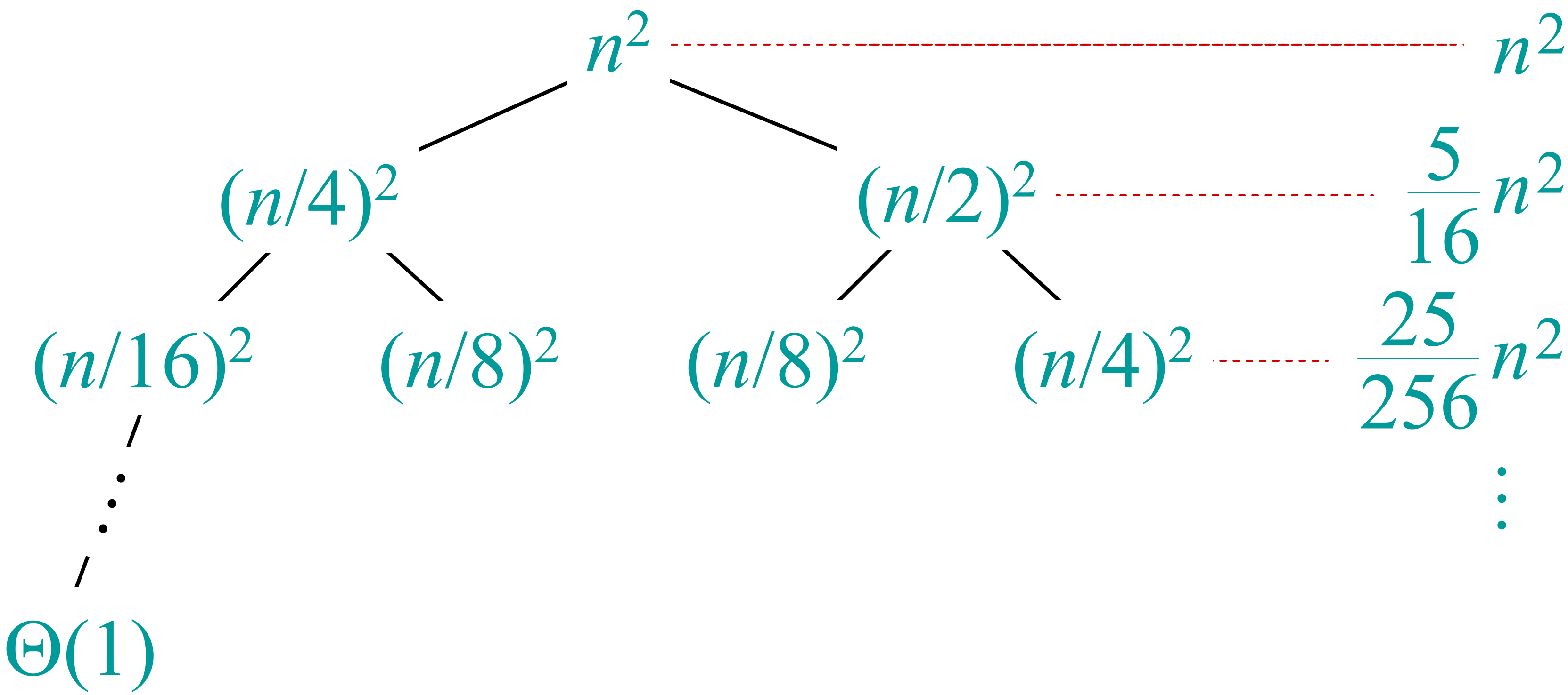

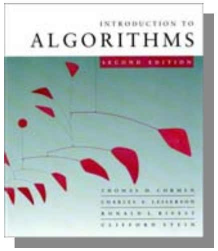

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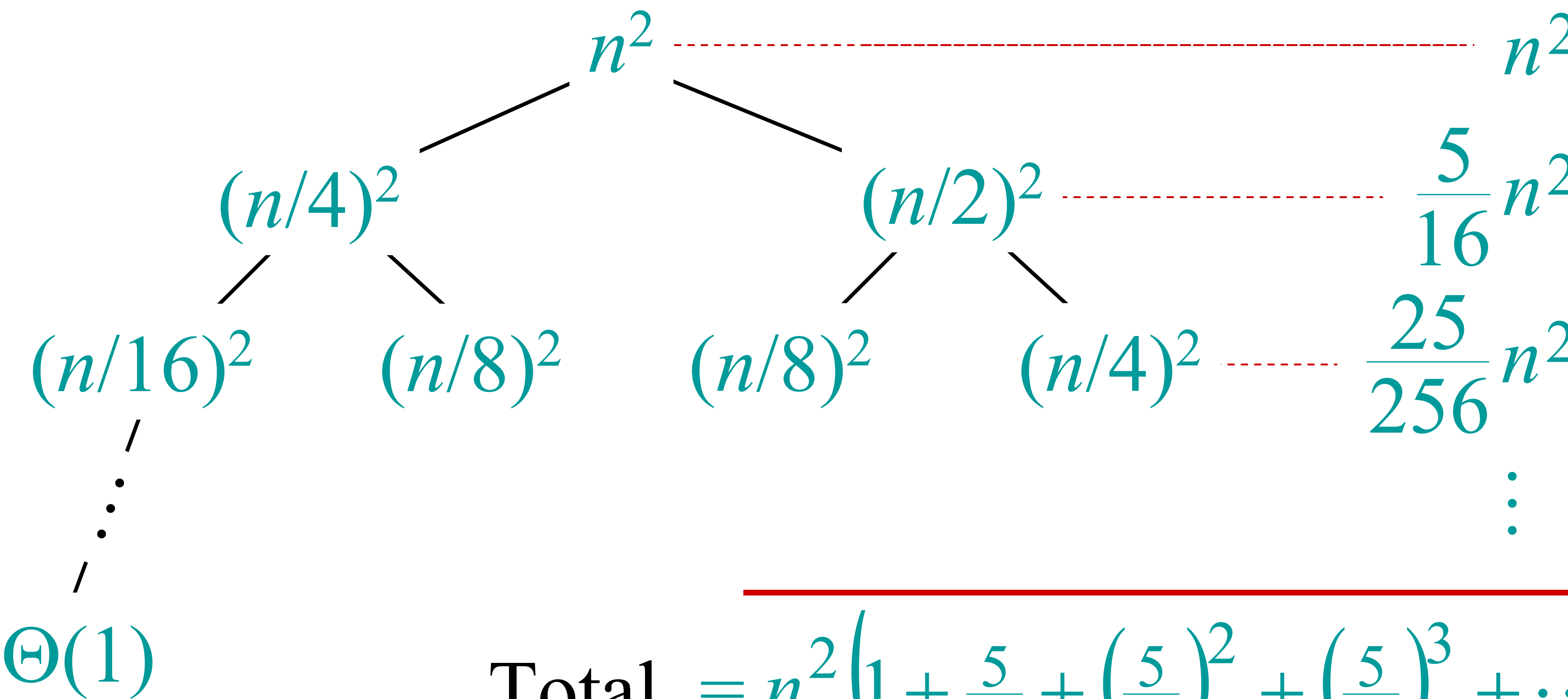


Total $= n^2\left(1 + \frac{5}{16} + \left(\frac{5}{16}\right)^2 + \left(\frac{5}{16}\right)^3 + \cdots\right)$

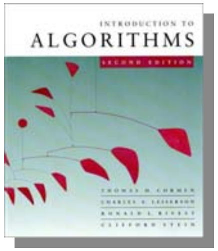
$= \Theta(n^2)$ *geometric series*

The master method

The master method applies to recurrences of the form

$$T(n) = a\,T(n/b) + f(n)\ ,$$

where $a \geq 1$, $b > 1$, and f is asymptotically positive.

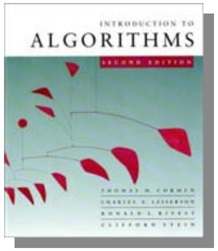


Three common cases

Compare $f(n)$ with $n^{\log_b a}$:

1. $f(n) = O(n^{\log_b a - \varepsilon})$ for some constant $\varepsilon > 0$.
   - $f(n)$ grows polynomially slower than $n^{\log_b a}$ (by an n^{ε} factor).

Solution: $T(n) = \Theta(n^{\log_b a})$.



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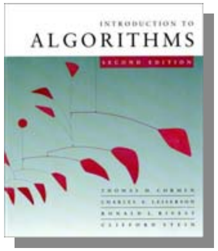
2. $f(n) = \Theta(n^{\log_b a} \lg^k n)$ for some constant $k \geq 0$.
   - $f(n)$ and $n^{\log_b a}$ grow at similar rates.

   Solution: $T(n) = \Theta(n^{\log_b a} \lg^{k+1} n)$.



Three common cases (cont.)

Compare $f(n)$ with $n^{\log_b a}$:

3. $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some constant $\varepsilon > 0$.
   - $f(n)$ grows polynomially faster than $n^{\log_b a}$ (by an n^{ε} factor),

   and $f(n)$ satisfies the ***regularity condition*** that $af(n/b) \le cf(n)$ for some constant $c < 1$.

   Solution: $T(n) = \Theta(f(n))$.

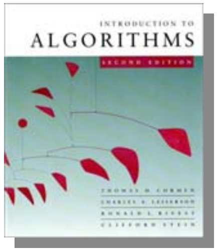
Examples

Ex. $T(n) = 4T(n/2) + n$

$a = 4,\ b = 2 \Rightarrow n^{\log_b a} = n^2;\ f(n) = n.$

Case 1: $f(n) = O(n^{2-\varepsilon})$ for $\varepsilon = 1$.

$\therefore\ T(n) = \Theta(n^2)$.



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Ex. $T(n) = 4T(n/2) + n^2$
$a = 4,\ b = 2 \Rightarrow n^{\log_b a} = n^2;\ f(n) = n^2.$
CASE 2: $f(n) = \Theta(n^2 \lg^0 n)$, that is, $k = 0$.
$\therefore\ T(n) = \Theta(n^2 \lg n).$

Examples

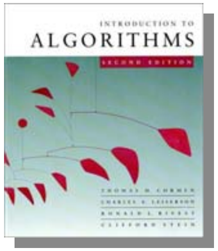
Ex. $T(n) = 4T(n/2) + n^3$

$a = 4,\ b = 2 \Rightarrow n^{\log_b a} = n^2;\ f(n) = n^3.$

CASE 3: $f(n) = \Omega(n^{2+\varepsilon})$ for $\varepsilon = 1$

and $4(n/2)^3 \le cn^3$ (reg. cond.) for $c = 1/2$.

$\therefore\ T(n) = \Theta(n^3)$.



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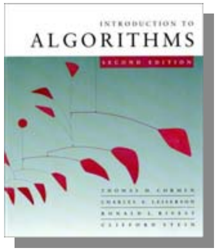
Ex. $T(n) = 4T(n/2) + n^2/\lg n$

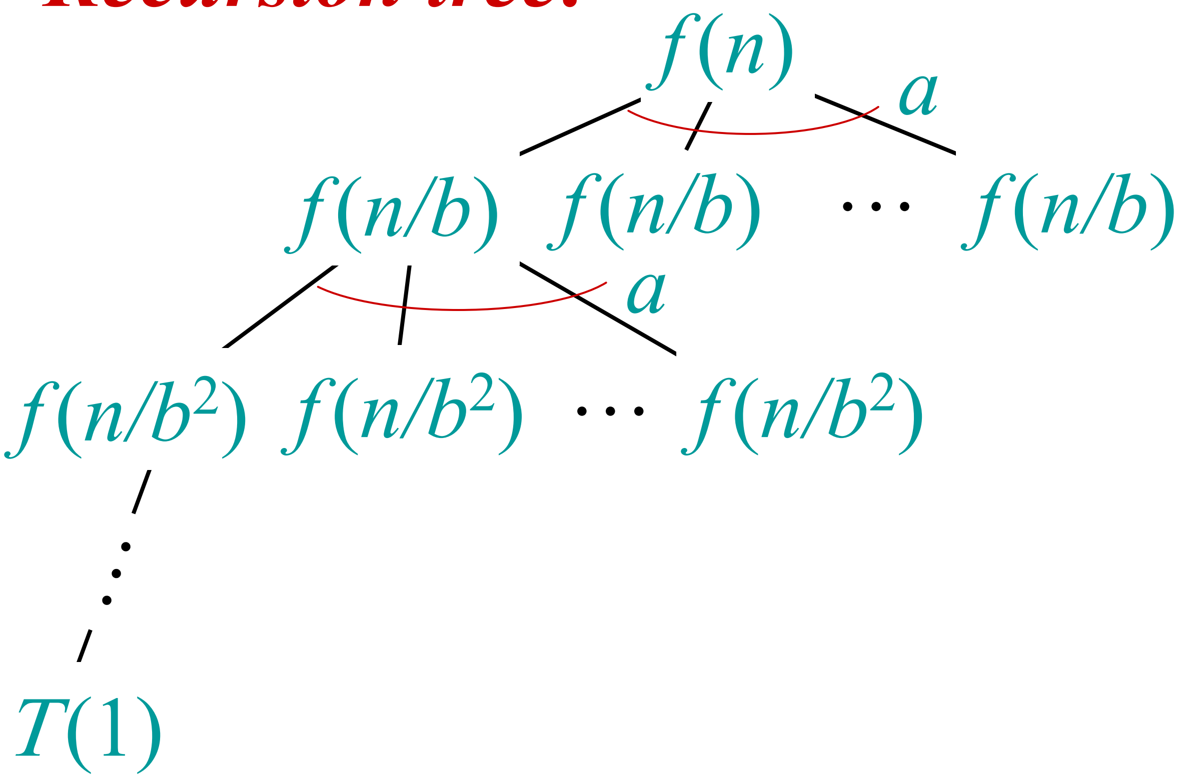
$a = 4,\ b = 2 \Rightarrow n^{\log_b a} = n^2;\ f(n) = n^2/\lg n.$

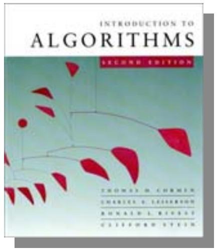
Master method does not apply. In particular, for every constant $\varepsilon > 0$, we have $n^{\varepsilon} = \omega(\lg n)$.



Idea of master theorem

Recursion tree:





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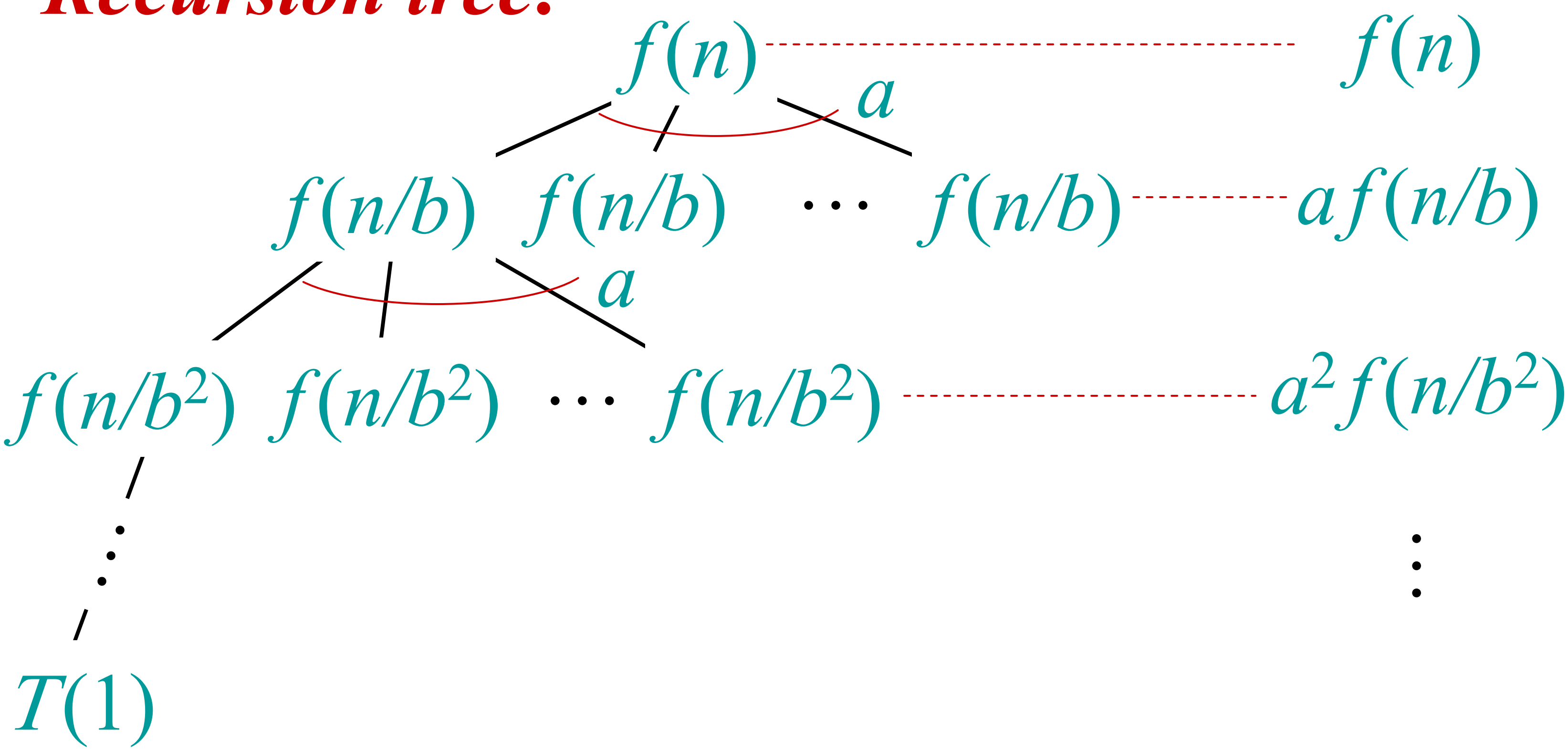

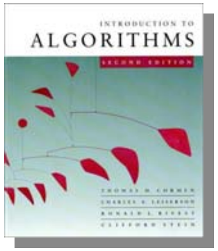

Idea of master theorem

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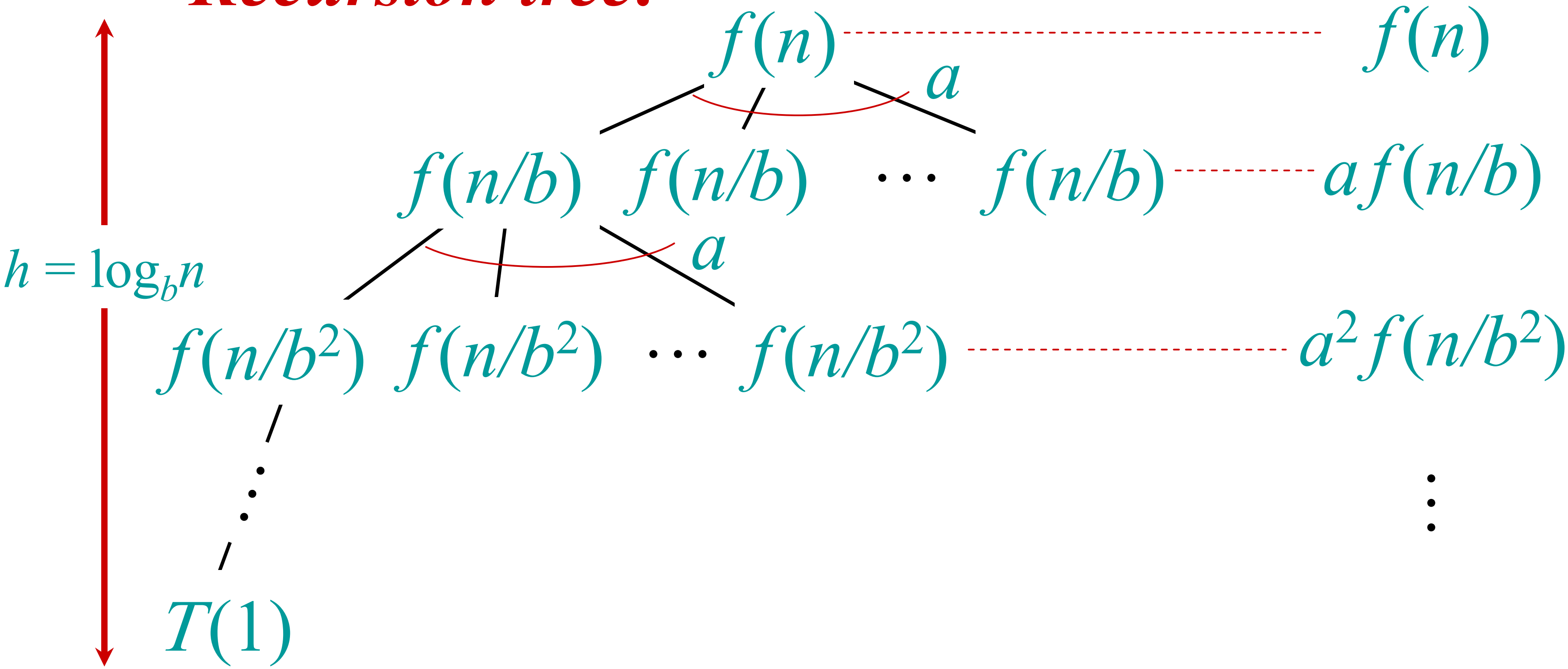
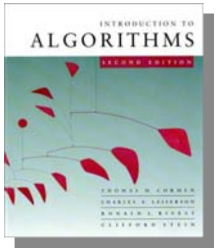




Idea of master theorem

Recursion tree:

$$f(n) \quad\cdots\cdots\quad f(n)$$

a

$$f(n/b) \quad f(n/b) \quad \cdots \quad f(n/b) \quad\cdots\cdots\quad a\,f(n/b)$$

a

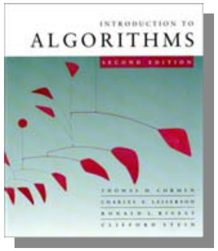
$h = \log_b n$

$$f(n/b^2) \quad f(n/b^2) \quad \cdots \quad f(n/b^2) \quad\cdots\cdots\quad a^2 f(n/b^2)$$

$\vdots$ $\qquad\qquad$ $\vdots$

$$T(1) \quad\cdots\cdots\quad n^{\log_b a}\,T(1)$$

$$\begin{aligned}\#\text{leaves} &= a^h \\ &= a^{\log_b n} \\ &= n^{\log_b a}\end{aligned}$$



Idea of master theorem

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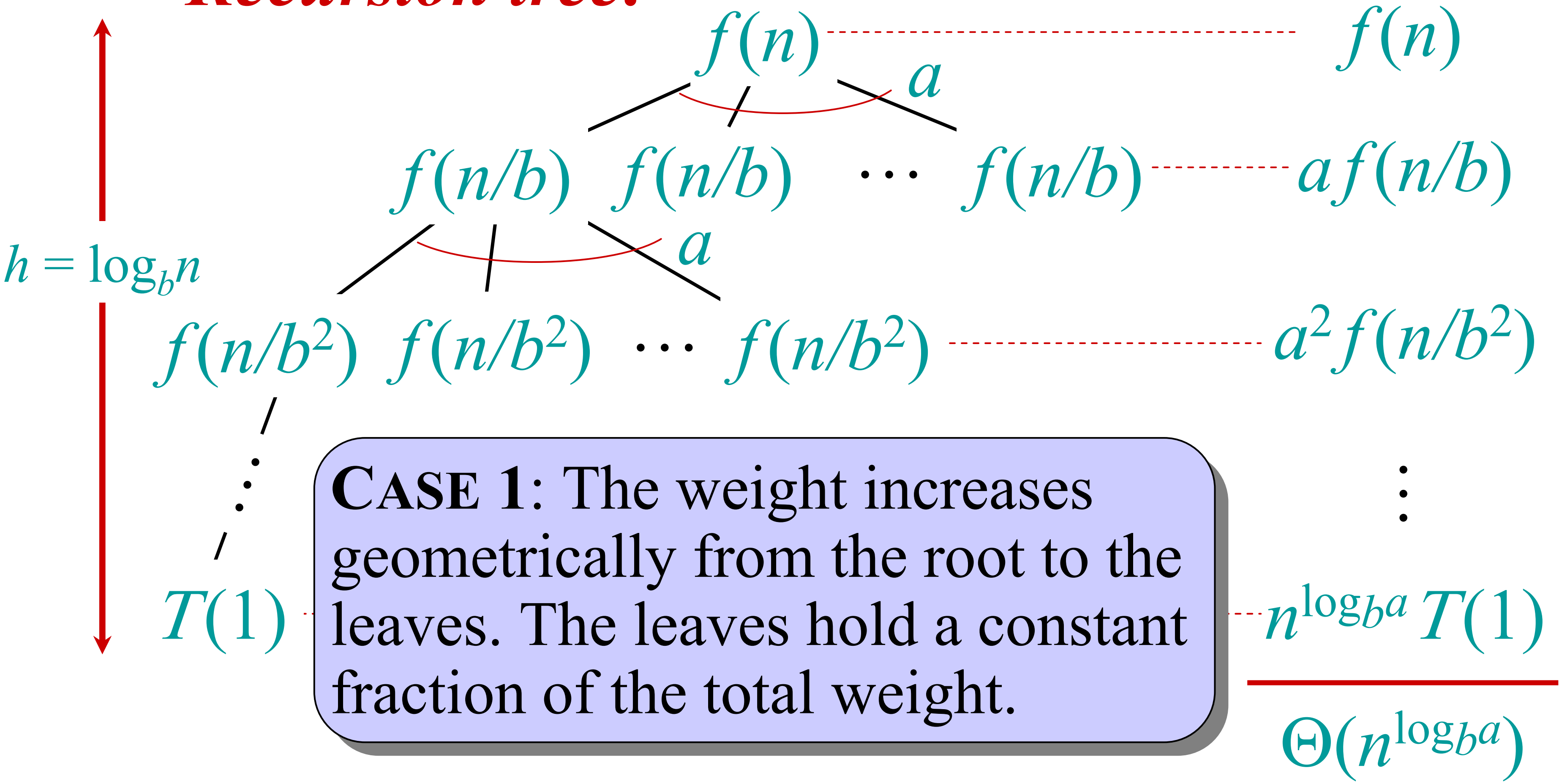
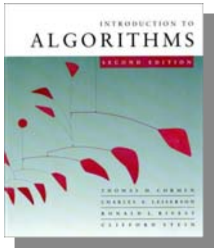


CASE 1: The weight increases geometrically from the root to the leaves. The leaves hold a constant fraction of the total weight.



Idea of master theorem

Recursion tree:

$f(n)$ .... $f(n)$

a

$f(n/b)$ $f(n/b)$ $\cdots$ $f(n/b)$ .... $a\,f(n/b)$

a

$h = \log_b n$

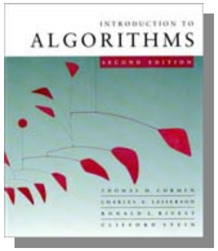
$f(n/b^2)$ $f(n/b^2)$ $\cdots$ $f(n/b^2)$ .... $a^2 f(n/b^2)$

$\vdots$

$T(1)$ .... $n^{\log_b a}\,T(1)$

CASE 2: ($k = 0$) The weight is approximately the same on each of the $\log_b n$ levels.

$\Theta(n^{\log_b a} \lg n)$



Idea of master theorem

Recursion tree:

$f(n)$ .... $f(n)$

a

$f(n/b) \quad f(n/b) \quad \cdots \quad f(n/b)$ .... $a f(n/b)$

a

$h = \log_b n$

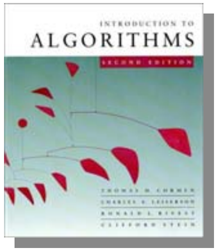
$f(n/b^2) \quad f(n/b^2) \quad \cdots \quad f(n/b^2)$ .... $a^2 f(n/b^2)$

$\vdots$

$T(1)$ .... $n^{\log_b a}\, T(1)$

Case 3: The weight decreases geometrically from the root to the leaves. The root holds a constant fraction of the total weight.

$\Theta(f(n))$



Appendix: geometric series

$$1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x} \quad \text{for } x \neq 1$$

$$1 + x + x^2 + \cdots = \frac{1}{1 - x} \quad \text{for } |x| < 1$$

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