# Similarity / Distance Measures

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(Slides courtesy of Pang-Ning Tan, Michael Steinbach and Vipin Kumar)

# Similarity and Dissimilarity

## Similarity

- Numerical measure of how alike two data objects are
- Is higher when objects are more alike
- Often falls in the range [0,1]

## Dissimilarity

- Numerical measure of how different are two data objects
- Lower when objects are more alike
- Minimum dissimilarity is often 0
- Upper limit varies

## Proximity refers to a similarity or dissimilarity

# Similarity and Dissimilarity

## Similarity and Dissimilarity of Simple Attributes

## Dissimilarity between Objects

- Distance
- Set Difference
- …

## Similarity between Objects

- Binary Vectors
- Vectors
- …

# Similarity/Dissimilarity for Simple Attributes

p and q are the attribute values for two data objects.

| Attribute Type | Dissimilarity | Similarity |
|---|---|---|
| Nominal | $d = \begin{cases} 0 & \text{if } p = q \\ 1 & \text{if } p \neq q \end{cases}$ | $s = \begin{cases} 1 & \text{if } p = q \\ 0 & \text{if } p \neq q \end{cases}$ |
| Ordinal | $d = \frac{\lvert p-q \rvert}{n-1}$ (values mapped to integers 0 to $n-1$, where $n$ is the number of values) | $s = 1 - \frac{\lvert p-q \rvert}{n-1}$ |
| Interval or Ratio | $d = \lvert p - q \rvert$ | $s = -d$, $s = \frac{1}{1+d}$ or $s = 1 - \frac{d - min\_d}{max\_d - min\_d}$ |

**Table 5.1.** Similarity and dissimilarity for simple attributes

# Euclidean Distance

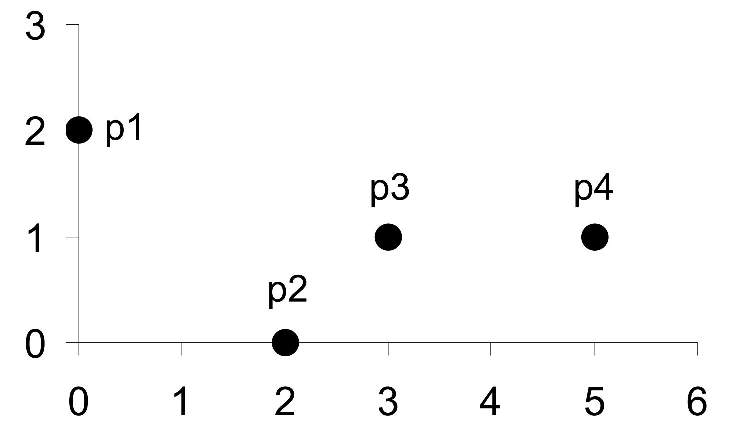
## Euclidean Distance

$$dist = \sqrt{\sum_{k=1}^{n} (p_k - q_k)^2}$$

Where $n$ is the number of dimensions (attributes) and $p_k$ and $q_k$ are, respectively, the $k^{th}$ attributes (components) or data objects $p$ and $q$.

Standardization is necessary, if scales differ.

# Euclidean Distance




| point | x | y |
|---|---|---|
| p1 | 0 | 2 |
| p2 | 2 | 0 |
| p3 | 3 | 1 |
| p4 | 5 | 1 |

| | p1 | p2 | p3 | p4 |
|---|---|---|---|---|
| p1 | 0 | 2.828 | 3.162 | 5.099 |
| p2 | 2.828 | 0 | 1.414 | 3.162 |
| p3 | 3.162 | 1.414 | 0 | 2 |
| p4 | 5.099 | 3.162 | 2 | 0 |

**Distance Matrix**

# Minkowski Distance

## Minkowski Distance is a generalization of Euclidean Distance

$$dist = \left( \sum_{k=1}^{n} | p_k - q_k |^r \right)^{\frac{1}{r}}$$

Where $r$ is a parameter, $n$ is the number of dimensions (attributes) and $p_k$ and $q_k$ are, respectively, the k-th attributes (components) or data objects $p$ and $q$.

# Minkowski Distance: Examples

- $r = 1$. City block (Manhattan, taxicab, $L_1$ norm) distance.

  A common example of this is the Hamming distance, which is just the number of bits that are different between two binary vectors

- $r = 2$. Euclidean distance

- $r \rightarrow \infty$. "supremum" ($L_{max}$ norm, $L_\infty$ norm) distance.

  This is the maximum difference between any component of the vectors

- Do not confuse $r$ with $n$, i.e., all these distances are defined for all numbers of dimensions.

# Minkowski Distance

## Data

| point | x | y |
|---|---|---|
| p1 | 0 | 2 |
| p2 | 2 | 0 |
| p3 | 3 | 1 |
| p4 | 5 | 1 |

## Distance Matrix

| L1 | p1 | p2 | p3 | p4 |
|---|---|---|---|---|
| p1 | 0 | 4 | 4 | 6 |
| p2 | 4 | 0 | 2 | 4 |
| p3 | 4 | 2 | 0 | 2 |
| p4 | 6 | 4 | 2 | 0 |

| L2 | p1 | p2 | p3 | p4 |
|---|---|---|---|---|
| p1 | 0 | 2.828 | 3.162 | 5.099 |
| p2 | 2.828 | 0 | 1.414 | 3.162 |
| p3 | 3.162 | 1.414 | 0 | 2 |
| p4 | 5.099 | 3.162 | 2 | 0 |

| $L_\infty$ | p1 | p2 | p3 | p4 |
|---|---|---|---|---|
| p1 | 0 | 2 | 3 | 5 |
| p2 | 2 | 0 | 1 | 3 |
| p3 | 3 | 1 | 0 | 2 |
| p4 | 5 | 3 | 2 | 0 |

# Common Properties of a Distance

- Distances, such as the Euclidean distance, have some well known properties.

  1. $d(p, q) \geq 0$ for all $p$ and $q$ and $d(p, q) = 0$ only if $p = q$. (Positive definiteness)
  2. $d(p, q) = d(q, p)$ for all $p$ and $q$. (Symmetry)
  3. $d(p, r) \leq d(p, q) + d(q, r)$ for all points $p$, $q$, and $r$.
     (Triangle Inequality)

  where $d(p, q)$ is the distance (dissimilarity) between points (data objects), $p$ and $q$.

- A distance that satisfies these properties is a metric

# Common Properties of a Similarity

- Similarities, also have some well known properties.

  1. $s(p, q) = 1$ (or maximum similarity) only if $p = q$.

  2. $s(p, q) = s(q, p)$ for all $p$ and $q$. (Symmetry)

  where $s(p, q)$ is the similarity between points (data objects), $p$ and $q$.

# Similarity Between Binary Vectors

- Common situation is that objects, *p* and *q*, have only binary attributes

- Compute similarities using the following quantities
  $M_{01}$ = the number of attributes where p was 0 and q was 1
  $M_{10}$ = the number of attributes where p was 1 and q was 0
  $M_{00}$ = the number of attributes where p was 0 and q was 0
  $M_{11}$ = the number of attributes where p was 1 and q was 1

- Simple Matching and Jaccard Coefficients
  SMC = number of matches / number of attributes
  $= (M_{11} + M_{00}) / (M_{01} + M_{10} + M_{11} + M_{00})$

  J = number of 11 matches / number of not-both-zero attributes values
  $= (M_{11}) / (M_{01} + M_{10} + M_{11})$

# SMC versus Jaccard: Example

$p =$ 1 0 0 0 0 0 0 0 0 0
$q =$ 0 0 0 0 0 0 1 0 0 1

$M_{01} = 2$ (the number of attributes where p was 0 and q was 1)
$M_{10} = 1$ (the number of attributes where p was 1 and q was 0)
$M_{00} = 7$ (the number of attributes where p was 0 and q was 0)
$M_{11} = 0$ (the number of attributes where p was 1 and q was 1)

$$SMC = (M_{11} + M_{00})/(M_{01} + M_{10} + M_{11} + M_{00}) = (0+7) / (2+1+0+7) = 0.7$$

$$J = (M_{11}) / (M_{01} + M_{10} + M_{11}) = 0 / (2 + 1 + 0) = 0$$

# Cosine Similarity

If $d_1$ and $d_2$ are two document vectors, then

$$\cos(d_1, d_2) = (d_1 \bullet d_2) / ||d_1||\ ||d_2||,$$

where $\bullet$ indicates vector dot product and $||\ d\ ||$ is the length of vector $d$.

Example:

$d_1 =$ **3 2 0 5 0 0 0 2 0 0**

$d_2 =$ **1 0 0 0 0 0 0 1 0 2**

$d_1 \bullet d_2 = 3*1 + 2*0 + 0*0 + 5*0 + 0*0 + 0*0 + 0*0 + 2*1 + 0*0 + 0*2 = 5$

$||d_1|| = (3*3+2*2+0*0+5*5+0*0+0*0+0*0+2*2+0*0+0*0)^{\mathbf{0.5}} = (42)^{\mathbf{0.5}} = 6.481$

$||d_2|| = (1*1+0*0+0*0+0*0+0*0+0*0+0*0+1*1+0*0+2*2)^{\mathbf{0.5}} = (6)^{\mathbf{0.5}} = 2.245$

$\cos(d_1, d_2) = .3150$

# Pearson Correlation Coefficient

❖ Correlation measures the linear relationship between objects

standard deviation

$$\mathrm{E}[X] = \mu.$$

$$\begin{aligned}\sigma &= \sqrt{\mathrm{E}[(X-\mu)^2]} \\ &= \sqrt{\mathrm{E}[X^2] + \mathrm{E}[-2\mu X] + \mathrm{E}[\mu^2]} = \sqrt{\mathrm{E}[X^2] - 2\mu\,\mathrm{E}[X] + \mu^2} \\ &= \sqrt{\mathrm{E}[X^2] - 2\mu^2 + \mu^2} = \sqrt{\mathrm{E}[X^2] - \mu^2} \\ &= \sqrt{\mathrm{E}[X^2] - (\mathrm{E}[X])^2}\end{aligned}$$

covariance

$$\begin{aligned}\operatorname{cov}(X,Y) &= \mathrm{E}[(X-\mathrm{E}[X])\,(Y-\mathrm{E}[Y])] \\ &= \mathrm{E}[XY - X\,\mathrm{E}[Y] - \mathrm{E}[X]Y + \mathrm{E}[X]\,\mathrm{E}[Y]] \\ &= \mathrm{E}[XY] - \mathrm{E}[X]\,\mathrm{E}[Y] - \mathrm{E}[X]\,\mathrm{E}[Y] + \mathrm{E}[X]\,\mathrm{E}[Y] \\ &= \mathrm{E}[XY] - \mathrm{E}[X]\,\mathrm{E}[Y].\end{aligned}$$

# Pearson Correlation Coefficient

❖ Correlation measures the linear relationship between objects

population correlation

$$\rho_{X,Y} = \mathrm{corr}(X,Y) = \frac{\mathrm{cov}(X,Y)}{\sigma_X \sigma_Y} = \frac{E[(X-\mu_X)(Y-\mu_Y)]}{\sigma_X \sigma_Y},$$

sample correlation

$$r = \frac{\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i-\bar{x})^2}\sqrt{\sum_{i=1}^{n}(y_i-\bar{y})^2}}$$

$$r = r_{xy} = \frac{n\sum x_i y_i - \sum x_i \sum y_i}{\sqrt{n\sum x_i^2 - (\sum x_i)^2}\ \sqrt{n\sum y_i^2 - (\sum y_i)^2}}.$$

$$r = r_{xy} = \frac{\sum x_i y_i - n\bar{x}\bar{y}}{\sqrt{(\sum x_i^2 - n\bar{x}^2)}\ \sqrt{(\sum y_i^2 - n\bar{y}^2)}}.$$

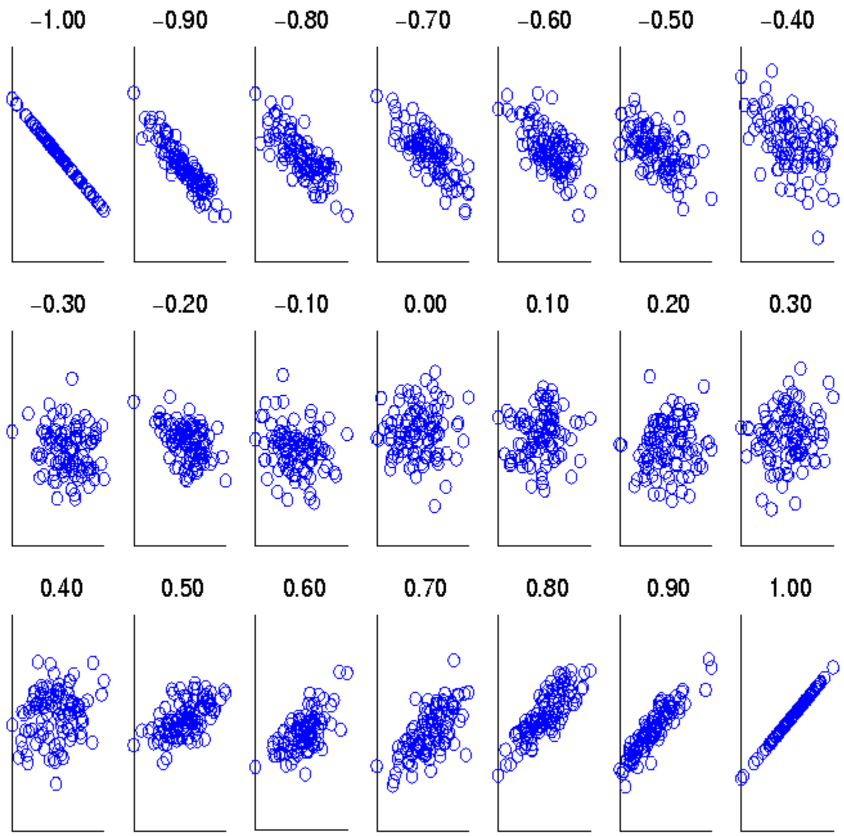
$$r = r_{xy} = \frac{1}{n-1}\boxed{\sum_{i=1}^{n}\left(\frac{x_i-\bar{x}}{s_x}\right)\left(\frac{y_i-\bar{y}}{s_y}\right)}$$

← dot product

$$r = r_{xy} = \frac{\sum x_i y_i - n\bar{x}\bar{y}}{(n-1)s_x s_y}$$

$$s_x = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2}$$

# Visually Evaluating Correlation




Scatter plots showing the similarity from –1 to 1.

# General Approach for Combining Similarities

- Sometimes attributes are of many different types, but an overall similarity is needed.

1. For the $k^{th}$ attribute, compute a similarity, $s_k$, in the range $[0, 1]$.
2. Define an indicator variable, $\delta_k$, for the $k_{th}$ attribute as follows:

$$\delta_k = \begin{cases} 0 & \text{if the } k^{th} \text{ attribute is a binary asymmetric attribute and both objects have} \\ & \text{a value of } 0\text{, or if one of the objects has a missing values for the } k^{th} \text{ attribute} \\ 1 & \text{otherwise} \end{cases}$$

3. Compute the overall similarity between the two objects using the following formula:

$$similarity(p, q) = \frac{\sum_{k=1}^{n} \delta_k s_k}{\sum_{k=1}^{n} \delta_k}$$

# Using Weights to Combine Similarities

## ❖ May not want to treat all attributes the same.

- Use weights $w_k$ which are between 0 and 1 and sum to 1.

$$similarity(p,q) = \frac{\sum_{k=1}^{n} w_k \delta_k s_k}{\sum_{k=1}^{n} \delta_k}$$

$$distance(p,q) = \left( \sum_{k=1}^{n} w_k |p_k - q_k|^r \right)^{1/r}$$