Lorentz

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1 Lorentz Attractor

The Lorentz System can be described by the following system of nonlinear ODE's.

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} a(y-x) \\ rx - y - xz \\ xy - bz \end{bmatrix}$$

Note the jacobian is defined:

$$J(x, y, z) = \begin{bmatrix} -a & a & 0 \\ r & -1 & -x \\ y & x & -b \end{bmatrix}$$

The third order arbitrary characteristic polynomial are generally quite complex, however the appearance of imaginary lypunov exponents regardless of position indicate rotational attraction in the manifolds of the phase space. Similarly, the negative magnitude of these exponents also reveals a slight dampening effect. This is all observation, however, as the lypunov exponents are dependent on state position within the phase portrait. Figure 1 shows the 3D plot of the lorentz attractor with initial conditions [-13, -12, 52]. This plot is generated using an adaptive time step. The primary algorithm is as follows:

- Compute the next state using time step h
- Compute the next state using two steps of size $\frac{h}{2}$.
- Compute error ($p_{h/2}$ is the h/2 resulting state p_h is the h resulting state):

  $$Error = ||p_{h/2} - p_h||$$

- If $error <$ tolerance, while $error <$ tolerance, repeate the above steps with $h/2$ as the new h.
- If $error >$ tolerance, while $error >$ tolerance, repeat the above steps with h as the new h.

- Update state with the most accurate state (smallest time step used in the above sequence)

Figure 2 compares the adaptive and stationary fixed point time series method for a small subset of the lorentz system. Noticeably, the adaptive time step has much larger jumps, this is because it is possible to make these large jumps, the adaptive integration technique updates the time step according the the mathematical curvature of the curve.

Some interesting topological elements of the attractor exist for different values of r.

For values of r in the set $(0, 1]$ ends in a single fixed point. As r surpasses this set, it begins to oscilate towards a damped orbit, converging on another fixed point. This damped orbit continues at $r = 15$ and begins to rotate, but does not change in topological shape (not a bifurcation that is). In between $r = 26, 26$, another bifurcation occurs, creating two orbital attractors. Of course, paths of individual trajectories as the initial condition changes, deviate from partners starting at slightly different initial conditions (attributed to sensitivity to initial conditions), the topological shape generally remains the same. Even drastic initial conditions ([100, 100, 100], [0.1, 0, 0]) tend towards similar topological shapes. However, plotting two trajectories at the same time, the two trajectories tend to deviate fairly rapidly from one another (again attributed to SIC).

2 Rossler System

The Rossler can be described by the following system of nonlinear ODE's.

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -(y+z) \\ x + ay \\ b + z(x-c) \end{bmatrix}$$

Two different figures are shown below. Unlike the lorentz attractor, I found initial conditions themselves cause bifurcations, ie changes in the topology of the system. Figure 3 shows one set of initial conditions and figure 4 shows a second set of initial conditions. The topology of these two curves are quite different, but they maintain the same system parameters.

The adaptive time step of the ODE solver can, however, be broken, similar to the time step of the stationary system. By slowly incrementing the error bound, the structure of the curve would get more loose. However, unlike changing the stationary time step size, it takes more to break the dynamics of the system. I hypothesise this is due to the high dimensional of the state system. An error of 1 is not actually that significant. Of course, this is relative to the curvature of the system, but the state system error bound is proportional to $(\delta t)^2$. Therefore, the increment of tolerance and dynamical changes in the system is a square root as slow as an increment in δt. That is, if δt doubles, *error* quadruples. If error, however, doubles, error of course only doubles. Therefore, a change in time step does much more to affect the state system than changing the tolerance of

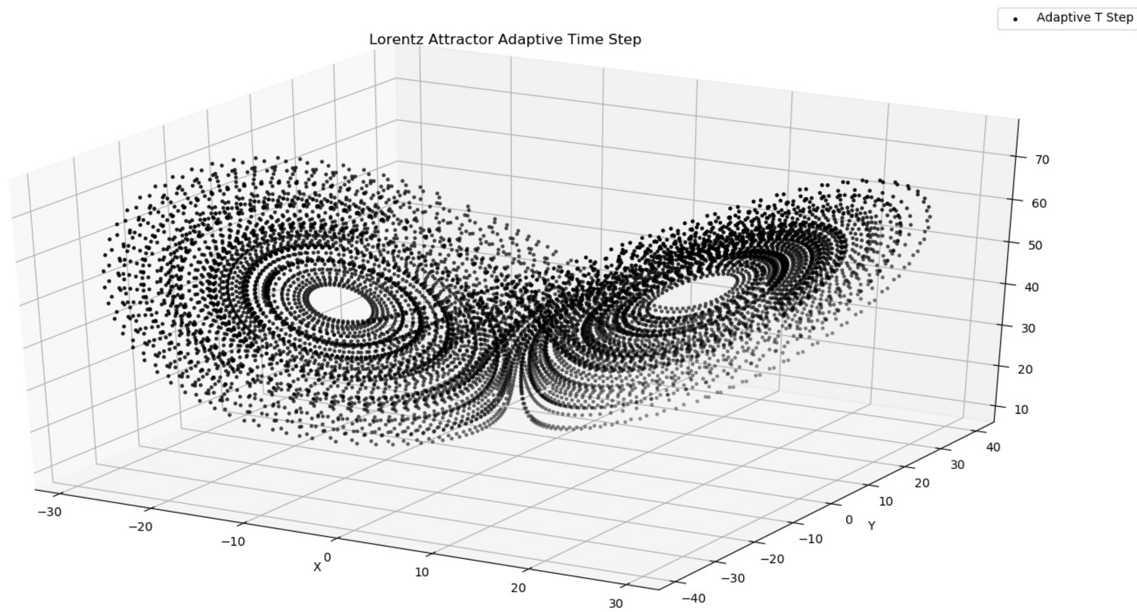

Figure 1: Lorentz Attractor with Adaptive Time step with error bound 0.001.

an adaptive solver. Figure 5 shows the dynamics breaking down of the Rossler system with tolerance equal to 2.

The above explanation comes directly from the taylor expansion of a state system. Our nonlinear ODE can be represented as a series taylor expansion:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + O((x - x_0)^2)$$
$$error \propto (x - x_0)^2 = h^2$$

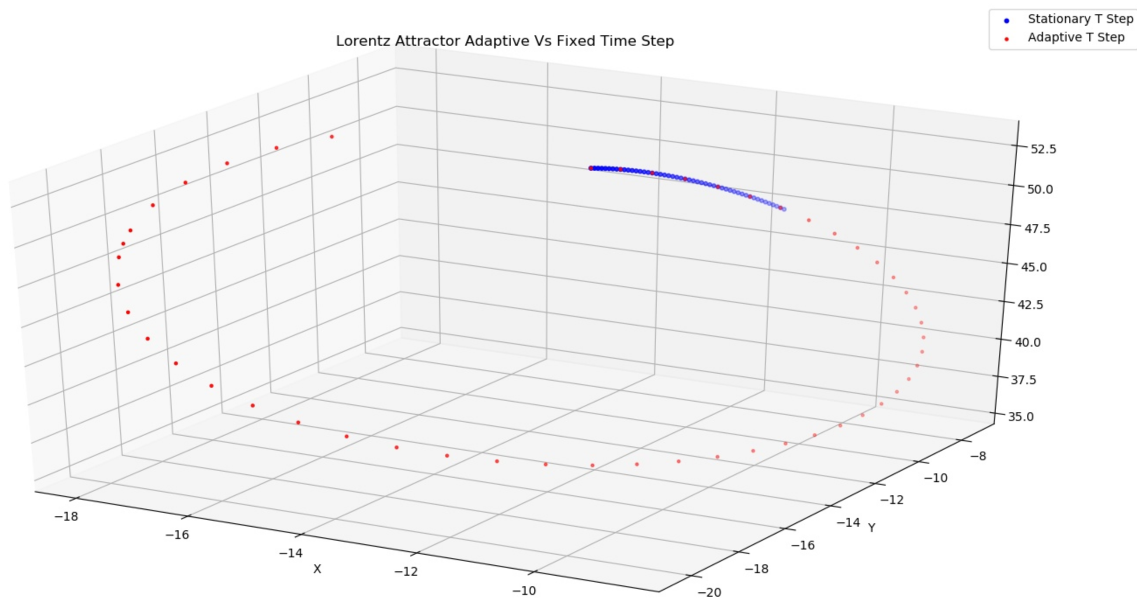

Figure 2: Lorentz Attractor Comparing Adaptive and Fixed Time Steps

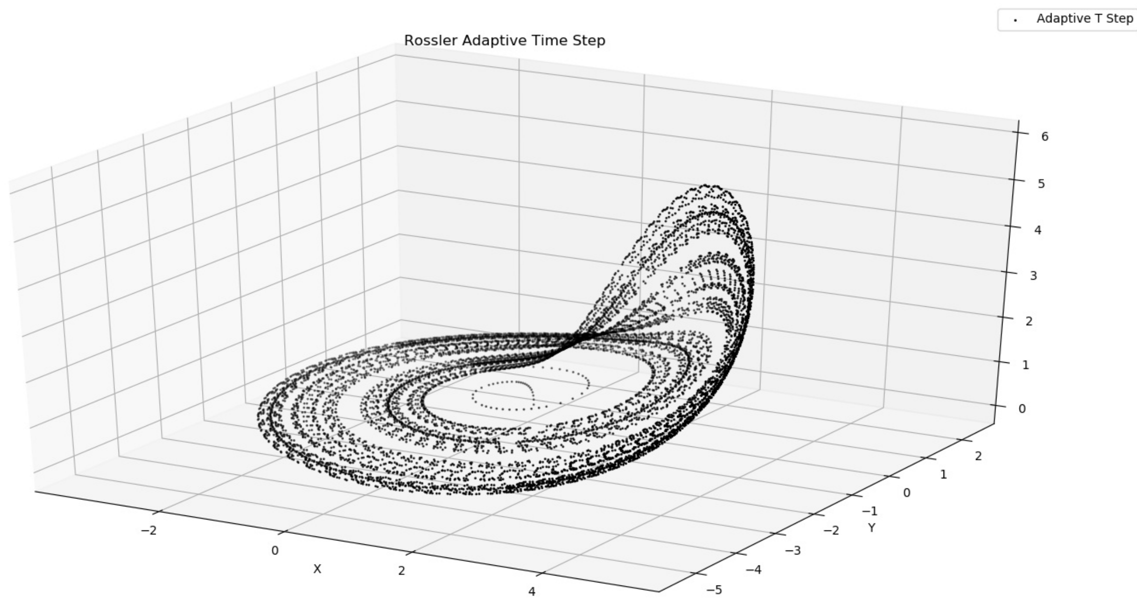

Figure 3: Rossler Curve with initial condition [0.1, 0, 0]

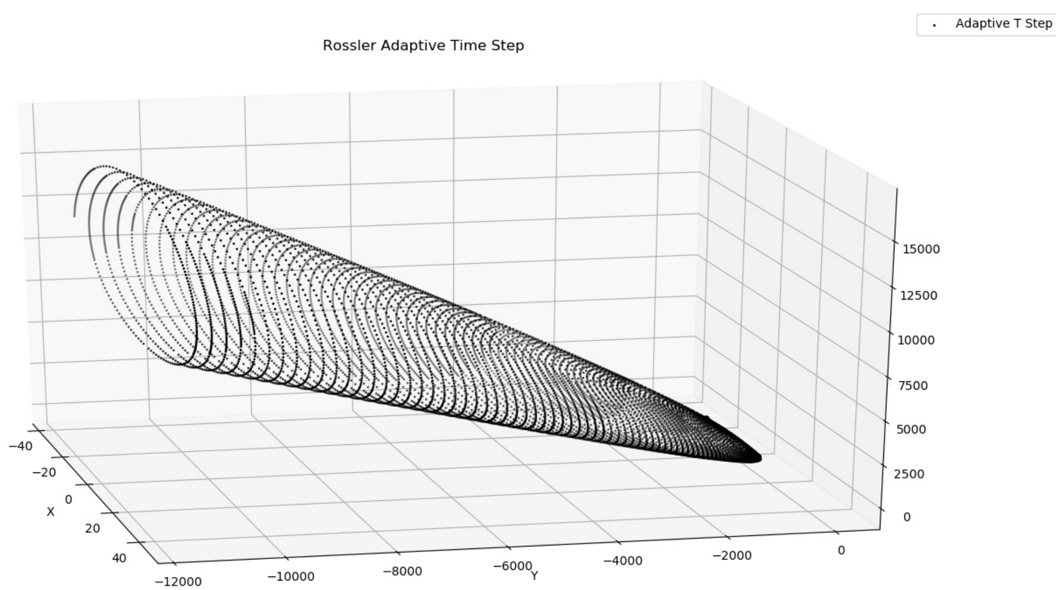

Figure 4: Rossler Curve with initial condition [-13, -12, 52]

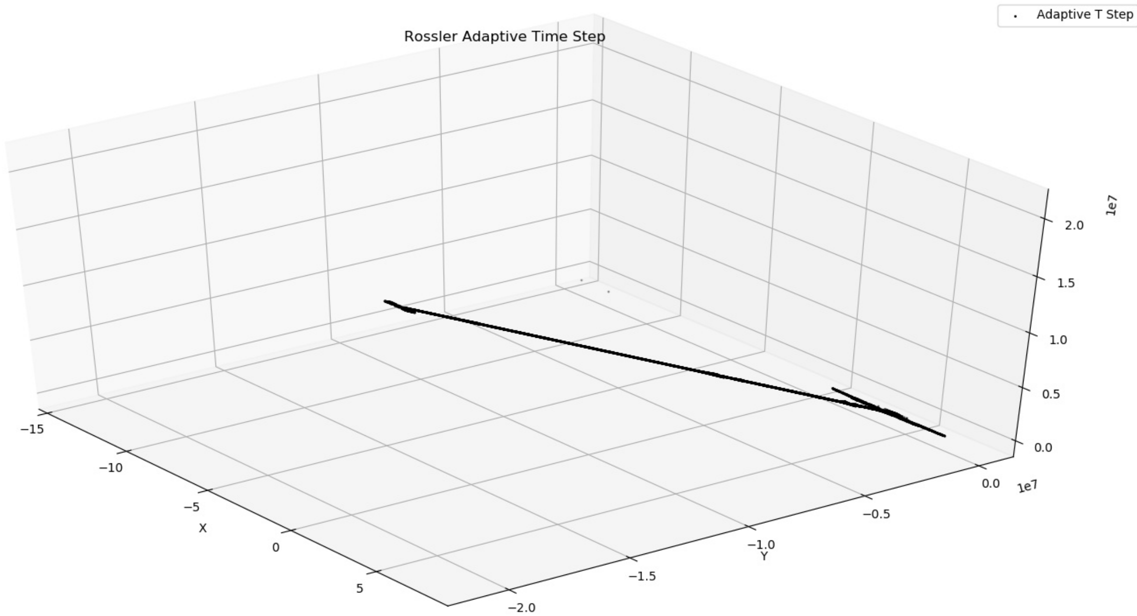

Figure 5: Numerical Error Rossler Curve, t error = 2