Applications of the Riemann Hilbert Problem and Inverse Scattering to Nonlinear Waves

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Abstract

In this paper, the Riemann Hilbert problem is examined in depth, with analysis of its application to the solution of solitons and shock waves. Specifically, the Korteweg-de-Vries initial value problem for solitary and interactive solitary waves is examined with solutions being presented using sectionally analytic solutions. The focus of this paper is to employ direct scattering to the KdV equation and to pose the associated Riemann Hilbert Problem. At the end, the focus is shifted towards Riemann Hilbert Problem Generalizations such as the n dimensional problem. This project offers a higher level overview of the aforementioned topics. The field of nonlinear waves and partial differential equations is vast and complex, and thus many aspects of nonlinear waves are neglected. At the end is a brief discussion on computational techniques to solving the KdV.

1 Introduction

A remarkable result of complex analysis is the idea that a function can be uniquely described by the properties of its singular points. For rational functions, it is fairly easy to deduce a function by fractional decomposition and the coefficient residues at its poles. However, Riemann Hilbert problems generalize the notion of constructing a **sectionally analytic** function dictated by its behavior at the boundary of regions of analyticity. There are many astounding results of the Riemann Hilbert problem, of which we will cover primarily its application to solving important nonlinear PDEs like the Time Independent Schrödinger equation and the Korteweg-de Vries equation through inverse scattering.

2 The Riemann Hilbert Problem

The standard Riemann Hilbert Problem can be expressed using two jump functions:

$$G : \Gamma \to \mathbb{C}^{m\times m} \qquad F : \Gamma \to \mathbb{C}^{n\times m}$$

Find a function $\Phi(z)$ analytic everywhere except Γ that is bounded at ∞ that satisfies the jump condition:

$$\phi^+(t) = G(t)\phi^-(t) + F(t) \qquad t \in \Gamma$$

For the scalar Riemann Hilbert problem, we will be referring to G and F as g and f and calling the RH problem **homogeneous** when F = 0.

Throughout this paper, we will be using the notation $\phi^+(z)$ and $\phi^-(z)$ denoting the limiting function of some arbitrary function ϕ as z approaches some arbitrary contour Γ. We make this formulation precise by defining Γ to be a **complete** contour, separating regions Ω_+ and Ω_-.

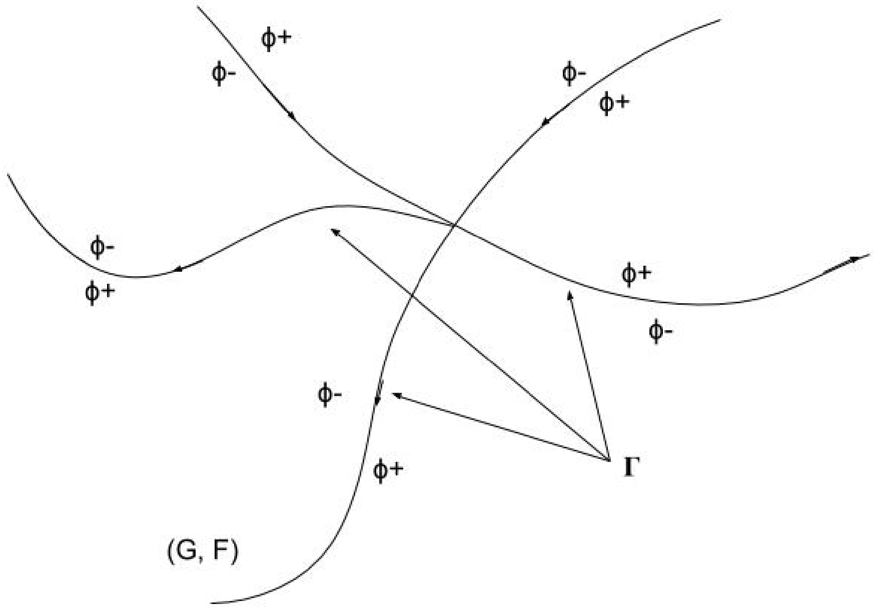

Figure 1: A generic contour Γ with a pair of jump functions (G, F). This figure was based on a figure found in [6].

$$\phi_+(t) \equiv \lim_{\substack{z\to t\\ z\in\Omega_+}} \phi(z) \qquad \phi_-(t) \equiv \lim_{\substack{z\to t\\ z\in\Omega_-}} \phi(z)$$

2.1 Cauchy Type Integrals

To introduce the machinery for Riemann Hilbert Problems, we must first consider the Cauchy type integral in equation (1).

$$\Phi(z) = \mathcal{C}_\Gamma(\varphi(z)) = \frac{1}{2\pi i}\int_\Gamma \frac{\varphi(\tau)}{\tau - z} d\tau \tag{1}$$

equation (1) maps functions on a contour to analytic functions off the contour. For the purposes of RH problems, we will consider functions φ with the following properties:

1. Given Ω, a function $f : \Omega \to \mathbb{C}$ is λ-Hölder continuous on Ω if for each $\tau_1 \in \Omega$, there exists $\Lambda(\tau_1)$, $C(\tau_1) > 0$ such that:

   $$|\varphi(\tau) - \varphi(\tau_1)| \leq \Lambda|\tau - \tau_1|^\lambda \quad for \quad |\tau - \tau_1| < C(\tau_1) \qquad 0 < \lambda \leq 1 \quad \Lambda > 0$$

   Note that if $\lambda = 1$, f is lipschitz, and if $\lambda > 1$, f must be constant. Also, if C can be chosen independent of τ_1, f is considered *uniformly λ-Hölder continuous.*

2. For each λ-Hölder continuous function on a bounded curve Γ with constants Λ, C satisfies

   $$\sup_{\tau_1 \neq \tau_2, \tau_1, \tau_2 \in \Gamma} \left(\frac{|f(\tau_1) - f(\tau_2)|}{|\tau_1 - \tau_2|^\lambda} < \delta C < \infty\right)$$

   Where δ depends on Λ and C

That is, φ is Holder continuous.

Note that as $|z| \to \infty$, equation (1) can be evaluated as:

$$\Phi(z) = (\frac{1}{z})(\frac{-1}{2\pi i} \oint_C \frac{\varphi(\tau)}{1 - \frac{\tau}{z}} d\tau) \to \frac{c}{z}$$

where $c = \frac{-1}{2\pi i} \oint_C \varphi(\tau)$.

This fact comes in handy when finding the solutions to the Scalar Riemann Hilbert Problem for verifying the fundamental solution to the problem $(X(z))$.

For the Riemann Hilbert Problem, we are specifically interested in the sectionally analytic function $\Phi(z)$ composed of $\Phi^+(z)$ and $\Phi^-(z)$ on the (+) and (-) sides of the contour respectively. To solve for $\Phi^+(z)$ and $\Phi^-(z)$ when Φ is a cauchy type integral expressed in (1) and $\varphi(z)$ satisfies a Hölder Condition on some closed contour C, we will most often use the results of the **Plemelj Formula**:

$$\Phi^{\pm}(t) = \pm\frac{1}{2}\Phi(t) + \frac{1}{2\pi i}\int \frac{\phi(t)}{t-z}dt \tag{2}$$

$$\Phi^{+}(z) - \Phi^{-}(z) = \varphi(z) \tag{3}$$

Note that the derivation of equation (3) is left out, but may be found in Ablowitz, Fokas [2] in section (7.3.1) for the case that $\varphi(z)$ is analytic on the contour C and $\varphi(t)$ with t on the contour.

2.2 Solving the Scalar Riemann Hilbert Problem

In this section we will solve the Riemann Hilbert Problem for Closed contours, then open contours.

1. The simple closed contour Riemann Hilbert Problem:

   find the sectionally analytic function $\phi(z)$ composed of $\phi^{+}(z)$ and $\phi^{-}(z)$ analytic inside and outside (respectively) a closed contour c satisfying the form:

   $$\phi^{+}(t) = g(t)\phi^{-}(t) + f(t) \tag{4}$$

   for t on the contour c and with g(t) satisfying a hölder condition.

   First, consider the case where $f(t) = 0$ (the homogeneous case).

   To solve the problem, we will reduce equation (5) to a similar form as equation (3) to find the cauchey type integral function $\Gamma(z)$.

   If g is λ-Hölder continuous and $index(g)_C = 0$, then $log(g(t))$ is also λ-Hölder continuous. If $index(g)_C = \kappa \neq 0$, and without a loss of generality, $z = 0$ is in C_+, (ie C encloses the origin), then $index(t^{-\kappa}g(t)) = 0$. Therefore, we wish to solve the newly posed problem:

$$log(\Phi^+(t)) = log(t^{-\kappa}g(t)) + log(t^\kappa \Phi^-(t))$$

Note that $\Phi^\pm(t)$ is analytic everywhere inside / outside the contour respectively, so $log(t^\kappa \Phi^\pm(t))$ is analytic everywhere assuming the contour C does not pass through the origin.

So we let $\varphi(t) = log(t^{-\kappa}g(t))$ and find $\Gamma(z) = \frac{1}{2\pi i}\oint_C \frac{log(\tau^{-\kappa}g(t))}{\tau - z}d\tau$ with $\Gamma^-(z) = log(z^\kappa \Phi^-(t))$ and $\Gamma^+(z) = log(\Phi^+(t))$. This leaves us with the final solution to equation (5):

$$X(z) = \begin{cases} e^{\Gamma(z)} & z \in D^+ \\ z^{-\kappa}e^{\Gamma(z)} & z \in D^- \end{cases}$$

and $P_{m+\kappa}(z)$ is an arbitrary polynomial of degree $m + \kappa$.

$$\Phi(z) = X(z)P_{m+\kappa}(z)$$

When f(t) does not equal 0, we substitute $g(t) = \frac{X^+(t)}{X^-(t)}$, reducing equation (5) to:

$$\frac{\Phi^+(t)}{X^+(t)} - \frac{\Phi^-(t)}{X^-(t)} = \frac{f(t)}{X^+(t)}$$

We solve for $\frac{\Phi}{X}(t)$ using the plemelj formulae, and find:

$$\Phi(z) = X(z)(P_{m+\kappa}(z) + \Psi(z)) \qquad \Psi(z) = \frac{1}{2\pi i}\int_C \frac{f(\tau)}{X^+(\tau)(\tau - z)}d\tau$$

2. Smooth, bounded and open curve Riemann Hilbert Problem

Find the sectionally analytic function $\phi(z)$ composed of $\phi^+(z)$ and $\phi^-(z)$ analytic on either sides of a smooth, bounded, and open curve C extending from z = a to z = b and g(t) $\neq$ 0

$$\phi^+(t) = g(t)\phi^-(t) + f(t) \tag{5}$$

for t on the contour c and with g(t) satisfying a hölder condition.

Just like before, we look for solutions of the form $\Phi(z) = \frac{1}{2\pi i}\int_a^b \frac{\varphi(t)}{t-z}dt$ where $\varphi(z)$ satisfies a Hölder condition on any subset of the contour from a to b. However, one change includes the existence of a pole of order γ on either ends, ie:

$$\varphi(t) = \frac{\tilde{\varphi(z)}}{(t-a)^{\gamma}} \quad \text{or} \quad \varphi(t) = \frac{\tilde{\varphi(z)}}{(t-b)^{\gamma}} \qquad \gamma = \alpha + i\beta, \qquad 0 \leq \alpha < 1$$

The following gives the definition of Φ for various cases and the proof can be found in Muskhelishvili [5].

When $\gamma = 0$,

As $z \to a, b$

$$\Phi(z) = \frac{\varphi(a)}{2\pi i} log(\frac{1}{z-a}) + \Phi_0(z) \qquad or \qquad \Phi(z) = -\frac{\varphi(b)}{2\pi i} log(\frac{1}{z-b}) + \Phi_0(z)$$

As $t \to c$ on the contour,

$$\Phi(t) = \frac{\varphi(a)}{2\pi i} log(\frac{1}{t-a}) + \Psi_0(t) \qquad or \qquad \Phi(t) = -\frac{\varphi(b)}{2\pi i} log(\frac{1}{t-b}) + \Psi_0(t)$$

And Φ_0 is bounded and tends to any limit as $z \to a, b$ along any path. Ψ_0 satisfies a Hölder condition near c.

When $\gamma \neq 0$

As $z \to a, b$

$$\Phi(z) = \frac{e^{\gamma\pi i}}{2\pi i sin(\gamma)} \frac{\tilde{\varphi(a)}}{(z-a)^{\gamma}} + \Phi_0(z) \qquad or \qquad \Phi(z) = -\frac{e^{\gamma\pi i}}{2\pi i sin(\gamma)} \frac{\tilde{\varphi(b)}}{(z-b)^{\gamma}} + \Phi_0(z)$$

As $t \to c$ on the contour,

$$\Phi(t) = \frac{cot(\gamma\pi)}{2i} \frac{\tilde{\varphi(a)}}{(t-a)^{\gamma}} + \Phi_0(t) \qquad or \qquad \Phi(t) = -\frac{cot(\gamma\pi)}{2i} \frac{\tilde{\varphi(b)}}{(t-b)^{\gamma}} + \Phi_0(t)$$

where

$$|\Phi_0(z)| < \frac{A_0}{|z-c|^{\alpha_0}} \qquad |\Phi_0(z)| < \frac{\tilde{\Psi_0(t)}}{|z-c|^{\alpha_0}} \qquad \alpha_0 < \alpha$$

and $\tilde{\Psi_0(t)}$ satisfies a Hölder condition near a or b.

The Riemann Hilbert problem, as stated, is not limited only to scalars. We can form the vectorized RH problem with the sectionally analytic vector $\Phi(z) \in \mathbb{C}^n$, comprised of analytic functions on either side of a contour:

$$\Phi^{\pm} = \begin{bmatrix} \Phi_1^{\pm} \\ \Phi_2^{\pm} \\ \dots \\ \Phi_n^{\pm} \end{bmatrix}$$

with a corresponding jump condition $G(t) \in \mathbb{C}^{n \times n}$ with elements $G_{ij}(t)$.

Note that the result of the vector RH problem is of the form:

$$\begin{gathered} \Phi_1^+(t) = G_{11}(t)\Phi_1^-(t) + G_{12}(t)\Phi_2^-(t) + \cdots + G_{1n}(t)\Phi_n^-(t) \\ \Phi_2^+(t) = G_{21}(t)\Phi_1^-(t) + G_{22}(t)\Phi_2^-(t) + \cdots + G_{nn}(t)\Phi_n^-(t) \\ \dots \\ \Phi_n^+(t) = G_{n1}(t)\Phi_1^-(t) + G_{n2}(t)\Phi_2^-(t) + \cdots + G_{nn}(t)\Phi_n^-(t) \end{gathered}$$

Which is a clearly challenging and often unsurprisingly insolvable equation. However, an important uniqueness theorem exists for the solution of a vector RH problem. First, note that a matrix $G = G_{ij}$ is positive / negative definite if for every non zero vector $\Phi = \Phi_j$, the scalar

$$\bar{\Phi}^T G \Phi = \bar{\Phi}_i G_{ij} \Phi_j$$

is a real, strictly positive / negative number respectively.

Theorem 2.1 *Let $G(t)$ denote a square matrix taking values on $t \in \mathbb{R}$ and suppose that either $G + \bar{G}^T or G - \bar{G}^T$ is definite for all t. Then the Riemann Hilbert problem for $\Phi(z)$ satisfying:*

$$\Phi^+(t) = G(t)\Phi^-(t) \qquad -\infty < t < \infty$$

with the boundary condition:

$$\Phi(z) = \Psi_0 + O(\frac{1}{z}) \qquad at|z| = \infty$$

where Ψ_0 is a specified constant vector, has a unique solution.

3 Inverse Scattering

3.1 Motivation for the Inverse Scattering Reducing to an RH Problem

After laying the groundwork for the Riemann Hilbert Problem, we take a brief aside to discuss the inverse scattering transformation. However, before this, in order to not lose track of the Riemann Hilbert Problem we mention the result we obtain for a specific inverse scattering transformation on the KdV equation. After undergoing inverse scattering, the KdV equation can be reduced to a quite familiar form:

$$\begin{bmatrix} \frac{M(x,k)}{a(k)} \\ N(x,k)e^{-2ikx} \end{bmatrix} = \begin{bmatrix} 1 + r(k)\bar{r}(k) & r(k)e^{2ikx} \\ \bar{r}(k)e^{-2ikx} & 1 \end{bmatrix} \begin{bmatrix} \bar{N}(x,k) \\ -\frac{\bar{M}(x,k)}{\bar{a}(k)}e^{-2ikx} \end{bmatrix} \tag{6}$$

Where we discuss the significance / derivation of M, a, N, r, k in the following section. This is in fact a Riemann Hilbert Problem of the form:

$$\phi^+(x,k) = \begin{bmatrix} \frac{M(x,k)}{a(k)} \\ N(x,k)e^{-2ikx} \end{bmatrix} \qquad \phi^-(x,k) = \begin{bmatrix} \bar{N}(x,k) \\ -\frac{\bar{M}(x,k)}{\bar{a}(k)}e^{-2ikx} \end{bmatrix}$$

$$G(x,k) = \begin{bmatrix} 1+|r(k)|^2 & r(k)e^{2ikx} \\ \bar{r}(k)e^{-2ikx} & 1 \end{bmatrix}$$

where the $\Phi^{\pm}$ are analytic on the upper and lower half planes. We wish to find a final vector Φ that satisfies the given jump conditions.

Now we discuss the general inverse scattering method, followed by the RH formulation on the KdV equation. The motivation behind the scattering method is to find the solution of nonlinear waves. It can be thought of as a direct nonlinear analogy to the Fourier analysis method of solving linear wave equations. It is also an interesting application to discuss the singularities of certain disperse shock waves and classify their behavior using Riemann surfaces.

3.2 The General Scattering Transformation

Waves are a fairly frequently visited topic in nonlinear applied mathematics, beginning with the linear wave equation:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

we may take a Fourier transform and find that the general solution is of the form:

$$u(x,t) = f(x-ct) + g(x+ct)$$

This simple linear approach to the wave equation offers solutions to a traversing wave: a wave that maintains its shape in space through time. In this paper, we are more interested in wave dispersion, dissipation, or nonlinearity. If we consider the most simple form of dispersive wave equation:

$$u_t + u_x + u_{xxx} = 0$$

We find solutions of the form:

$$u(x,t) = e^{i(kx-\omega t)}$$

Where $\omega(k)$ is the dispersion relation for our dispersive wave equation:

$$\omega(k) = k - k^3$$

And k is the wave number, with ω being the frequency. We also find that the speed of propagation of a wave with number k is equal to $\frac{\omega}{k} = 1 - k^2$. For the general form of the dispersive wave equation, we simply sum over all k:

$$u(x,t) = \int_{-\infty}^{\infty} A(k)e^{i(k-w(k)t)}dk$$

Notice that the speed of propagation, depending quadratically on k, means that different wave numbers travel faster than others. One of the major results of Kruskal and Zabousky is the fact that certain solutions to the (non linear) KdV overlap, causing waves (described as singular solitons) to interact with one another and pass by unchanged. We show this visually in our computational section.

Note that A(k) is a function of k that is determined from the initial conditions presented. More generally, we can describe the dispersion relation as a function of partial derivatives with respect to x. For example, the general equation:

$$u_t - i\omega(i\partial_x)u = 0$$

Where $\omega(i\partial_x)$ is a polynomial of partial derivatives with respect to x acting on u.

We find that $u(k,t) = u_0(k)e^{-i\omega(k)t}$

Thus, the solution is of the form:

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u_0(k)e^{i(kx-\omega(k)t)}dk$$

This shows that the solution to the linearized KdV equation:

$$u_t + u_{xxx} = 0$$

is

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u_0(k)e^{i(kx+k^3t)}dk$$

Of course, the Fourier method for solving for a dispersive wave equation can only be applied to linear wave equations. For the non linear case, we use a similar idea:

1. Move into "scattering space" using a spatial evolution opperator (L) by associating our desired non linear equation with the Sturm Liouiville problem (more specifically, the Time Independent Schrodinger Equation, often abbreviated as TISE). This is referred to as Direct Scattering.
2. Solve the time evolution to obtain scattering data for arbitrary t using the time evolution operator (M). This is the time evolution step to scattering.
3. Obtain the inverse scattering transformation.

Consider the time independent Schrodinger Equation:

$$Lv := (\frac{\partial^2}{\partial x^2} - u)\Psi = -\lambda\Psi \tag{7}$$

(The negative signs on λ and u are conventions we adopt in this paper, but care should be taken when discussing the sign of λ, as the solutions rely heavily on this.) We now consider only $u(x; t = t_0)$ as a function only on x. We later find the time evolution of our solution to relate time into $u(x,t)$. The conditions on $u(x)$ imply that $u(x)$ must be bounded on the real line:

$$\int_{-\infty}^{\infty} |u(x)|dx < \infty$$

as well as satisfy the Faddeev condition:

$$\int_{-\infty}^{\infty} (1 + |x|)|u(x)|dx < \infty$$

Asymptotically, as $|x| \to \infty$, we note that $|\Psi_{xx}| \to -\lambda\Psi$. Therefore, asymptotically, the eigenfunction $\Psi(x)$ has the form for $\lambda > 0$:

$$\Psi(x) \to Ae^{-i\sqrt{\lambda}x} + Be^{i\sqrt{\lambda}x} \tag{8}$$

For all λ, this solution is bounded, so we refer to $\lambda > 0$ as the **continuous spectrum of eigenvalues**.

If $\lambda < 0$, then Ψ will exhibit exponential growth or decay in this solution. This is because $|e^{\pm i\sqrt{\lambda}x}| = 1$ for $\lambda > 0$ and $|e^{\pm i\sqrt{-|\lambda|}x}| = e^{\pm-\sqrt{|\lambda|}x}| \neq 1$. Consider the **real** solution as $x \to +\infty$:

$$\Psi(x) \to Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x} \tag{9}$$

Notice that the above solution is not bounded when $x \to +\infty$ unless $A = 0$. We solve for the values of λ that make $A = 0$ and refer to them as the **discrete spectrum of eigenvalues**.

For $\lambda < 0$, the solution to the eigenfunction must decay, and for $\lambda > 0$, the eigenfunction oscillates infinitely. It is also very possible that the discrete spectrum doesn't exist at all, as is the case for $u(x) \geq 0$, and for $u(x) \leq 0$, there are a finite number of discrete eigenvalues λ.

Now we chose to solve for the discrete spectrum. We define $\kappa_n = \sqrt{-\lambda_n}$ for all finite discrete eigenvalues $\lambda_n, n = 1, 2...N$. The bounded solution will be characterised at $x \to \infty$ as

$$\Psi_n(x) \to c_n e^{-\kappa_n x}$$

.

We fix c_n so that $\int_{-\infty}^{\infty} \Psi_n^2 dx = 1$.

For the continuous spectrum, we let $k = \sqrt{\lambda}$. The oscillatory behaviors at infinity are the same as equation (8), with a constants $a(k), b(k)$ satisfing the same requirements as above:

$$\hat{\Psi}(x;k) \to \begin{cases} e^{-ikx} + be^{ikx} & x \to +\infty \\ ae^{-ikx} & x \to -\infty \end{cases} \tag{10}$$

Consider two different discrete eigenfunctions Ψ_n, Ψ_m for $m \neq n$,

$$\begin{gathered} \Psi_n'' - (\kappa_n^2 + u)\Psi_n = 0 \\ \Psi_m'' - (\kappa_m^2 + u)\Psi_m = 0 \\ (\kappa_n^2 - \kappa_m^2)\Psi_n \Psi_m = \Psi_m \Psi_n'' - \Psi_n \Psi_m'' = \frac{d}{dx} W(\Psi_m, \Psi_n) \end{gathered} \tag{11}$$

Where $W(a, b) = ab' - ba'$ is the Wronskian operator. Now, we find that:

$$W(\Psi_m, \Psi_n)|_{-\infty}^{\infty} = (\kappa_n^2 - \kappa_m^2) \int_{-\infty}^{\infty} \Psi_n \Psi_m dx$$

Which, because we know that Ψ_m, Ψ_n decay as x goes to ∞, equals 0. This implies the functions Ψ_m and Ψ_n are orthogonal. We also note that the continuous eigenfunction $\hat{\Phi}$ is also orthogonal to all discrete eigenfunctions Ψ_m. Therefore, from equation (11), if Ψ_1 and Ψ_2 are solutions to the TISE, then $\frac{d}{dx} W(\Psi_1, \Psi_2) = 0$. Therefore, $W(\Psi_1, \Psi_2) = constant$.

Now, consider $\hat{\Psi}$ and $\bar{\hat{\Psi}}$ to be the continuous and complex conjugate of the continuous eigenfunction respectively. Using equation (10), we find for $x \to \pm\infty$:

$$W(\hat{\Psi}, \bar{\hat{\Psi}}) = 2ik(1 - b\hat{b}) = 2ika\bar{a}$$

solving so that:

$$|a|^2 + |b|^2 = 1$$

Define the function

$$F(X) = \sum_{n=1}^{N} c_n^2 e^{-\kappa_n X} + \frac{1}{2\pi} \int_{-\infty}^{\infty} b(k) e^{ikX} dk$$

and we find that

$$u(x) = -2\frac{d}{dx} K(x, x)$$

where $K(x, x)$ is the solution to the Marchenko equation:

$$K(x, y) + F(x + z) + \int_{x}^{\infty} K(x, y) F(y + z) dy = 0$$

If b(k) (often refered to as the reflection potential) is 0, then, this reduces to a simple infinite sum, and we find solution of N discrete solitons of the form:

$$u_n(x, t) \to 2\kappa_n^2 sech^2(\kappa_n(x - 4\kappa_n^2 + log(c_n^2(0)))), \qquad n = 1, 2...N$$

as $t \to \pm\infty$

In summary, we conduct three steps in inverse scattering:

1. Direct Scattering: At time $t = 0$, we solve the schrodinger equation for $u(x, 0)$ and find a set of finite discrete eigenvalues $\lambda = \kappa_n^2$ for $\lambda > 0$. We find a continuum, $\lambda = k^2$ for $\lambda < 0$. Asymptotically, for the discrete spectrum:

$$\Psi(x,t) \to c_n(t)e^{-\kappa_n x}$$

   and for the continuous spectrum:

$$\begin{aligned} \Psi(x,t) &\to e^{-ikx} + b(k,t)e^{ikx}, \qquad x \to +\infty \\ \Psi(x,t) &\to a(k,t)e^{-ikx}, \qquad x \to -\infty \end{aligned}$$

2. Time Evolution: It can be shown that

$$\begin{aligned} \kappa_n &= const \\ c_n(t) &= c_n(0)e^{4\kappa_n^3 t} \\ a(k,t) &= a(k,0) \\ b(k,t) &= b(k,0)e^{8ik^3 t} \end{aligned}$$

3. Inverse Problem: Given the scattering data for time evolution, we reconstruct u(x, t) by $F(x;t)$ and $K(x,x,t)$. This step, as we shall explore, is reduced to a Riemann Hilbert Problem.

3.3 Riemann Hilbert Problem Approach to Inverse Scattering

In this section, we discuss how to reduce the inverse scattering transform to a Riemann Hilbert problem. The motivation behind introducing Inverse Scattering in context to the Riemann Hilbert problem is the fact that the last step: the "inverse scattering" step can be reduced (arguably more fundamentally) to a Riemann Hilbert problem. The KdV equation will be our primary motivating example:

$$u_t - 6uu_x + \epsilon^2 u_{xxx} = 0, \qquad \epsilon > 0 \tag{12}$$

With the initial condition $u(x,0)$ that decays to 0 as $|x| \to \infty$. For this paper, we will set ϵ equal to 1.

The outlined method above is used to solve the KdV equation by reducing the origional KdV equation to an eigenvalue, eigenfunction equation like the TISE.

That reduction is shown here:

First, consider the Miura Transformation: $u = v^2 + v_x$. Substitute this into equation (12) to obtain:

$$2vv_t + v_{xt} - 6(v^2 + v_x)(2vv_x + v_{xx}) + 6v_x v_{xx} + 2vv_{xxx} + v_{xxxx} = 0$$

Simplified to:

$$(2v + \frac{\partial}{\partial x})(v_t - 6v^2 v_x + v_{xxx}) = 0$$

Therefore, if v is a solution of the mKDV:

$$v_t - 6v^2 v_x + v_{xxx} = 0$$

then $v^2 + v_x$ is a solution of the Kdv, equation (12). Notice that the mirua transformation is simply a Riccati equation of v. Therefore, we can linearize the problem by substituting $v = \frac{\Psi_x}{\Psi}$ for some $\Psi(x,t) \neq 0$. We then get the form:

$$\Psi_{xx} - u\Psi = 0$$

and observing that the KdV equation is Galilean invariant, we find that if we transform $u(x,t)$ to $\lambda + u(x + 6\lambda t, t)$, it leaves equation (12) the same for any real λ. Therefore, we can replace any u with $u - \lambda$, leaving

$$\Psi_{xx} + (\lambda - u)\Psi = 0$$

which is the original TISE analysed in the previous section. Therefore, if we solve for Ψ, we can make the transformation $v = \Psi_{xx}/\Psi$ and $u = v^2 + v_x$. To undergo scattering, we must consider the potential function u defined as $u(x;t)$, or some potential u with respect to time t. We also find that in the TISE, $\lambda = \lambda(t)$.

For the KdV equation, we find a time evolution operator (not so trivially as discussed in the next section):

$$\Psi_t = (\gamma - u_x)\Psi + (4\lambda + 2u)\Psi_x$$

(note that the expression above often uses $-(4\lambda - 2u)$, but with our choice of $-\lambda$, we take a slightly different form.)

We now have a lax pair for the KdV equation:

$$L\Psi = -\lambda\Psi = (\partial_x^2 - u)\Psi + \lambda\Psi \tag{13}$$

$$M\Psi = \Psi_t = ((\gamma - u_x) + (4\lambda + 2u)\frac{\partial}{\partial_x})\Psi \tag{14}$$

We show the way to derive M in the next section. Note that for a more generalized KdV equation:

$$u_t - 6uu_x + \epsilon^2 u_{xxx} = 0 \quad \epsilon > 0$$

,

$L = (\epsilon^2\partial_x^2 - u)$. But for our purposes, we chose to elect $\epsilon = 1$. It is common to examine the limit for which $\epsilon \to 0$ to find the dispersionless limit of the KdV equation, but that is not within the scope of this paper.

3.3.1 The Jost Solutions

Let $\lambda = k^2$ as before in direct scattering, and $u(x) = u(x,0)$, $\Psi(x) = \Psi(x,0)$, which yields:

$$\begin{aligned} \Psi_{xx} - u\Psi &= -k^2\Psi \\ \Psi(x,k) &\to Ae^{ikx} + Be^{-ikx} \end{aligned} \tag{15}$$

Let $\Psi(x,k) = \Phi(x,k)$ be a solution to equation (15). We break the solution up into four distinct parts, each describing the long term asymptotic behavior of the solution Φ as $|x| \to \pm\infty$. Note that these four eigenfunctions are merely asymptotic descriptors, and not the actual solution.

$$\phi(x,k) \to e^{-ikx}, \quad \bar{\phi}(x,k) \to e^{ikx} \qquad x \to -\infty \tag{16}$$

$$\psi(x,k) \to e^{ikx}, \quad \bar{\psi}(x,k) \to e^{-ikx} \qquad x \to \infty \tag{17}$$

Analysing the long term assymptotics, it is clear that ψ and $\bar{\psi}$ are linearly independent. Because equation (15) is an ode of the second order, we can use them to construct a more general solution for $\phi(x,k)$ and $\bar{\phi}(x,k)$:

$$\phi(x,k) = a(k)\bar{\psi}(x,k) + b(k)\psi(x,k) \tag{18}$$

$$\phi(\bar{x},k) = -\bar{a}(k)\psi(x,k) + \bar{b}(k)\bar{\psi}(x,k) \tag{19}$$

where $a(k)$ and $b(k)$ are functions describing the scattering data. Note that the negative sign on $\bar{a}$ is used for convenience.

For convenience, we define the four auxiliary functions $M, N, \bar{M}, \bar{N}$ by multiplying equation (17) and equation (16) by e^{ikx}.

$$M(x,k) = \phi e^{ikx}, \quad \bar{M}(x,k) = \bar{\phi} e^{ikx}$$
$$N(x,k) = \psi e^{ikx}, \quad \bar{M}(x,k) = \bar{\psi} e^{ikx}$$

Multiplying equation (18) and equation (19) by e^{ikx} and dividing by a and $\bar{a}$ we obtain:

$$\frac{M(x,k)}{a(k)} = \bar{N}(x,k) + r(k)N(x,k)$$
$$\frac{\bar{M}(x,k)}{\bar{a}(k)} = -N(x,k) + \bar{r}(k)\bar{N}(x,k)$$

$$r(k) = \frac{b(k)}{a(k)} \qquad \bar{r}(k) = \frac{\bar{b}(k)}{\bar{a}(k)}$$

We manipulate this expression to the familiar matrix form:

$$\begin{bmatrix} \frac{M(x,k)}{a(k)} \\ N(x,k)e^{-2ikx} \end{bmatrix} = \begin{bmatrix} 1 + r(k)\bar{r}(k) & r(k)e^{2ikx} \\ \bar{r}(k)e^{-2ikx} & 1 \end{bmatrix} \begin{bmatrix} \bar{N}(x,k) \\ -\frac{\bar{M}(x,k)}{\bar{a}(k)}e^{-2ikx} \end{bmatrix} \tag{20}$$

Note that $\bar{\alpha}$ now explicitly indicates the complex conjugate of α. This is now a vector Riemann Hilbert problem comprised of Φ^+ and Φ^- in the upper and lower half planes respectively. The functions in G satisfy a holder condition for $k \in \mathbb{R}$. Our contour Γ is the real axis.

We briefly note that it can be shown [7] that Ne^{-2ikx} and $\bar{M}e^{-2ikx}$ are analytic in the upper half and lower half planes, respectively. And that M and $\bar{N}$ are analytically extendable off the real axis into the upper and lower half planes. Also, $a(k)$ has only a finite number of simple zeros in the upper half plane and they must all lie on the imaginary axis. The proof is found in [7].

After undergoing time evolution using equation (14), the time evolution generalization of equation (21) is:

$$\Phi^+(x,k,t) = G(x,k,t)\Phi^-(x,k,t) \tag{21}$$

Suppose we find a vector $\Phi \in \mathbb{C}^2$ that satisfies the stated RH problem for the KdV equation. Observe that:

$$\frac{1}{2}[G + \bar{G}^T] = \begin{bmatrix} 1 - |r(k)|^2 & 0 \\ 0 & 1 \end{bmatrix}$$

Because $|a|^2 - |b|^2 = 1$, $|r|^2 = 1 - \frac{1}{|a|^2}$, so $|r|^2 < 1$ so $G + \bar{G}^T$ is positive definite and therefore, by theorem 2.1, the solution is unique.

3.4 Finding the Time Propigation Operator and Lax Pair

Consider the Lax Pair required for inverse scattering:

$$Lv = \lambda v \tag{22a}$$
$$v_t = Mv \tag{22b}$$

In taking the time derivative of Lv, we equate equation (22) with the following relation:

$$L_t + [L, M] = 0 \tag{23}$$

$$[L, M] := (LM - ML)$$

We note that equation (23) is only valid for an invariant temporal eigenvalue, i.e. $\lambda_t = 0$.

The "Lax Pair" problem involves finding M, time time propigator in the ist. Generally, reducing a nonlinear equation to its Lax pairs is no easy or straight forward task. However, we briefly review the derivation of two cases of nonlinear functions, facilitated by the work of Ablowitz, Kaup, Newell and Segur [3].

We consider the time evolving pair of equations:

$$v_x = Xv$$
$$v_t = Tv$$

and expand the time spacial derivatives to equate $v_{xt} = v_{tx}$ to end up with a relationship similar to equation (23):

$$X_t - T_x + [X, T] = 0 \tag{24}$$

The results of equation (24) are slightly more generalized than the requirements of equation (23). An algebraic approach to finding the pair L, M involves solving the undetermined function time evolving matrix for A, B, C, and D in the below system:

$$v_x = \begin{bmatrix} -ik & q(x,t) \\ r(x,t) & ik \end{bmatrix} v$$
$$v_t = \begin{bmatrix} A(x,t) & B(x,t) \\ C(x,t) & D(x,t) \end{bmatrix} v$$

Where k is an eigenvalue of the associated eigenfunction in v_x. When we expand expression equation (24), we end up with a system of equation of A, B, and C (noting that A = -D), which we can solve by allowing

$$A = \sum^{n} A_i k^i \quad B = \sum^{n} B_i k^i \quad C = \sum^{n} C_i k^i$$

(allowing n to be negative), we solve the associated system for A, B, and C. We specifically neglect the precise form of A, B, and C. Note that some common examples include setting r = -1, and n = 2, which reduces to the Non linear Schrodinger eigenfunction problem. Setting n = 3 and solving for A, B and C reduces to the KdV equation and setting n = -1 and solving for A, B and C reduces to the Sine Gordon problem.

This process has been generalized further, with the following two cases:

r(x, t) and q(x, t) decay to 0 as $|x| \to \infty$.

$$\begin{pmatrix} r_t \\ -q_t \end{pmatrix} = -2A_0(L) \begin{pmatrix} r \\ q \end{pmatrix}$$

Where

$$A_0 = \frac{1}{2} i w_r(2k) = -\frac{1}{2} i w_q(-2k)$$

and $w_i(k)$ is the dispersion relation of the associated linearized problem.

and

$$L = \frac{1}{2i} \frac{\partial}{\partial x} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

r = -1 (Schrodinger Scattering Problem) If r and q do not vanish, and we wish to solve a form of the Schrodinger scattering problem:

$$v_{xx} + (k^2 + q)v = 0$$

The time evolution can be solved using the following relation:

$$\begin{pmatrix} r_t \\ -q_t \end{pmatrix} = -\gamma(L) \begin{pmatrix} r_x \\ q_x \end{pmatrix}$$

Where

$$\gamma(k^2) = \frac{w(2k)}{2k}$$

and

$$L = -\frac{1}{4}\partial_x^2 - q + \frac{1}{2}q_x I_+$$

$$(I_+ f)(x) = \int_x^\infty f(y)dy$$

Of course, the dependence on r_x and q_x is fine because we already have a spatial evolution.

4 Computational Korteweg-de Vries

Briefly, we discuss the numerical appproach to examine the long term behaviour of the KdV equation. First, employing a simple finite differences method, as discussed in [4], we find very large numerical error on figure (2) over a short period of time T = 3 seconds. We instead deploy a fourth order runge kutta algorithm described in [1] to generate the cleaner solution in figure (3).

The first numerical solution used a simple finite difference scheme illustrated by Morgan in [4]. The curve used a Gaussian curve as the initial data, and at $T = 3$ seconds, there are clearly oscillatory fringes near the left of the soliton. The results of figure (2) illustrate that it is not practical to use conventional single step methods to capture the solution to the KdV equation for longer periods. The solutions only grow very rapidly for larger time periods to extraordinarily complex curves that have no physical meaning. It should be noted that there are better raw numerical solutions than a simple differences approach, but the main point is that classical techniques are not sufficient enough to generate solutions over large time periods.

We use a fourth order time difference scheme that operates on a PDE of the form:

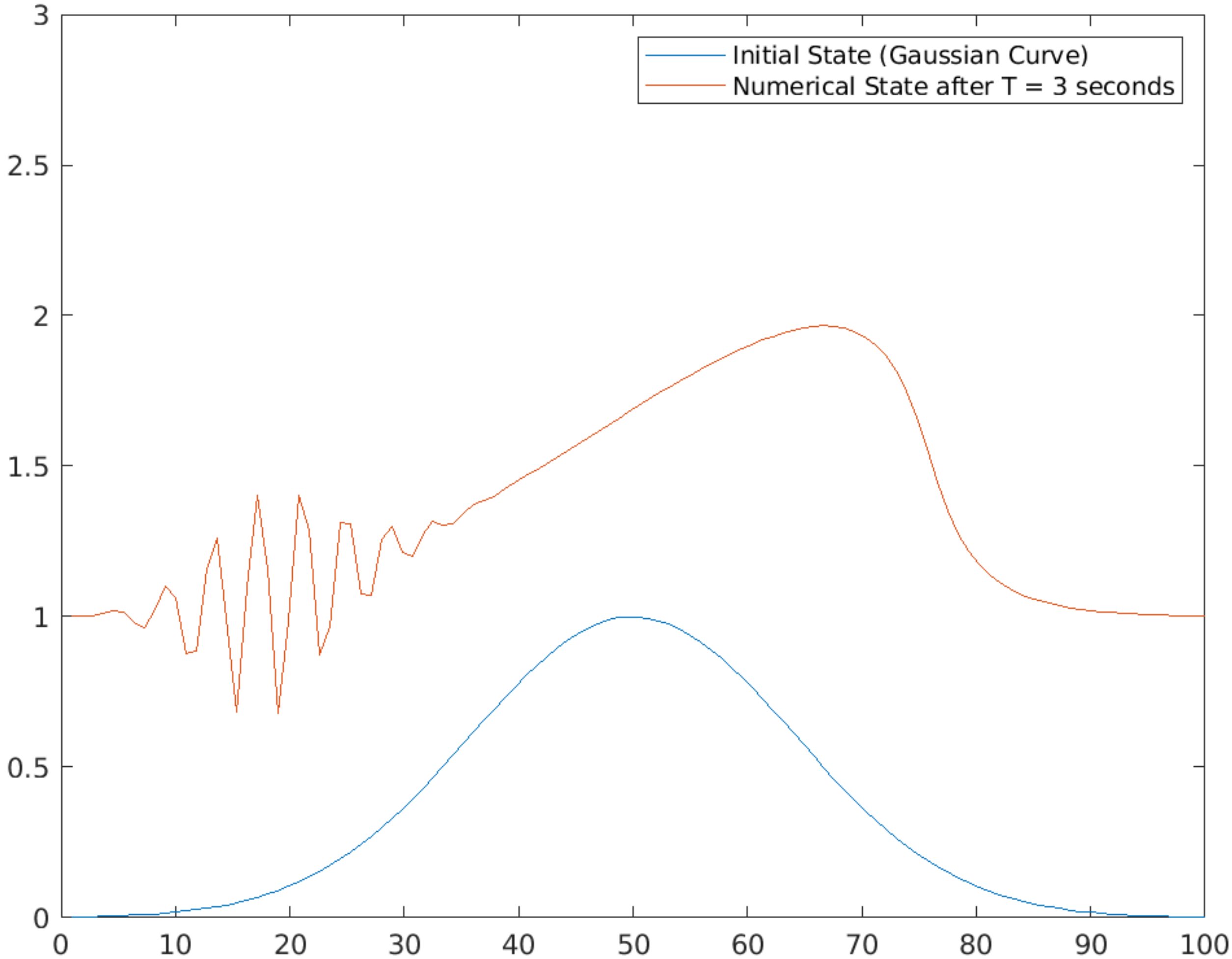

Figure 2: Numerical Solution for the KdV equation using Gaussian initial state. The graph above is at T = 3 seconds.

.

$$u_t = Lu + N(u,t)$$

where L and N are linear and nonlinear operators. If we discretize the spatial part of the PDE, we get a system of ODEs. We used python to write a simple time stepping fourth order Runge Kutta scheme that accepts an input initial state and propagates the data forward in time. We used an initial set of wave numbers, so this solution is not guaranteed on the continuous spectrum, however, it accurately modeled the behavior of the wave over long periods of time.

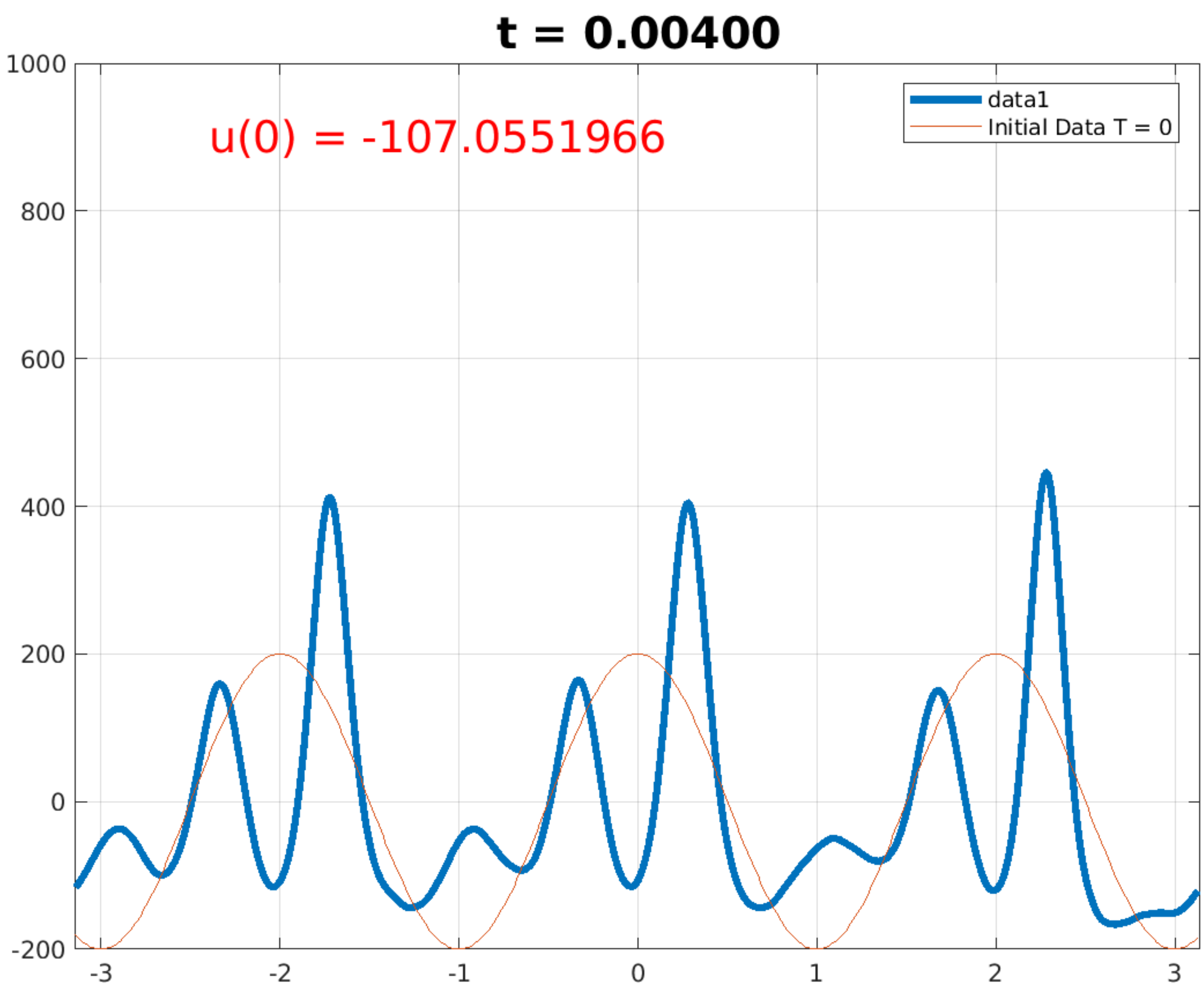

Figure 3: KdV Numerical Solution using Exponential Time Decay RK4

5 Conclusions

In this project, we discussed one of the applications of the Riemann Hilbert problem in complex analysis with regards to inverse scattering. The scattering Transformation is a very powerful method for solving various nonlinear PDEs and providing fundamental results in the field of nonlinear waves. The Riemann Hilbert problem allows for a convenient alternative to the final step of the inverse scattering transform, as well as being a fundamental tool in the field of complex analysis. We primarily focused on the solution to the KdV equation, a nonlinear wave equation with a third order dispersive term that describes the movement of shallow water waves in long canals. There are many very interesting properties of the KdV equation and the study of dispersive waves can be considered a topic in itself, this paper was merely an introductory discussion of some of the final results found by inverse scattering and Riemann Hilbert Factorization.

Although we posed the inverse scattering transformation as a Riemann Hilbert problem, the act of reconstructing the potential $u(x,t)$ was not undergone. With more time, we would have been interested in examining this reconstruction and further understand some of the more analytic properties of the sectionally analytic vector function Φ. This would be done by finding a green's function that decays to zero at $\pm\infty$. We could then reconstruct the function Φ_1 and Φ_2 as components of the vector Φ.

As mentioned, there are many interesting properties of non linear water waves (not only limited to the Kdv). We did not discuss the zero dispersion limit ($\epsilon \to 0$ in the generalized KdV), nor did we discuss higher order dispersion, which have non analytic solutions in scattering space. The KdV, after an analysis of the phase portrait, does not contain any hetero clinic orbits, and thus some analysis of the discontinuous dispersion limits don't work. Specifically, we were interested in incorporating the method of Whitham averaging, but we did not have time, nor the scope to discuss this.

References

[1] "Lloyd N. Trefethen" "Aly-Khan Kassam". Fourth oder time-stepping for stiff pdes. 2005.

[2] Mark J. Ablowitz Athanassios S. Fokas. *Complex Variables, Introduction and Applications*. Cambridge Texts in Applied Mathematics. Cambridge University Press, 2nd edition, 2003.

[3] "P.A. Clarkson" "M.J. Ablowitz". *Solitons, Nonlinear Evolution Equations and Inverse Scattering*. Cambridge University Press, 1991.

[4] Hannah Morgan. Numerical solutions to the kdv equation.

[5] "Muskhelishvili". *Singular Integral Equations*. Noordhoff International Publishing, Leyden The Netherlands, 1977.

[6] Sheehan Olver Thomas Trogdon. *Riemann Hilbert Problems, Their Numerical Solutions and the Computation of Nonlinear Special Functions*. Society for Industrial and Applied Mathematics, 2016.

[7] "Michael Wheeler". An introduction to riemann-hilbert problems and their applications, 2005.

Appendices

A Divided Differences (First figure result)

```
dx=1;
dt=.01;
nts=1000;
a=[1];
b=[1 -1];
c=[1];
d=[1 -3 3 -1];

r=1:dx:100;

% Gaussian initial data:
u0=1*exp(-(.05*(r-50)).^2);

u=u0;

for k=1:nts
       u=u-(dt/dx)*filter(b, a, 3*(u .* u))-(dt/(dx*dx*dx))*filter(d, c, u);
end

h = figure();
plot(r,u0,r,u+1);

print(h, "plot.pdf", "-dpdflatexstandalone");
```

B RK4 exponential time decay (Second figure result)

```
clear
clc

% Initial data
```

```
N = 512;
x = (2*pi/N)*(-N/2:N/2-1)';
A = 25; B = 16;
u = 200 * cos(pi * x);

% Plot the initial data
p = plot(x,u,'linewidth',3);
axis([-pi pi -200 1000]), grid on



% Fourth order time stepper for stiff pdes
   http://people.maths.ox.ac.uk/trefethen/publication/PDF/2005_111.pdf

% Time step
h = 5e-6;

% Wave numbers
k = [0:N/2-1 0 -N/2+1:-1]';

% Fourier Multipliers
L = 1i*k.^3;
E = exp(h*L); E2 = exp(h*L/2);
M = 64;
r = exp(2i*pi*((1:M)-0.5)/M);
LR = h*L(:,ones(M,1))+r(ones(N,1),:);
Q  = h*mean((exp(LR/2)-1)./LR ,2);
f1 = h*mean((-4-LR+exp(LR).*(4-3*LR+LR.^2))./LR.^3,2);
f2 = h*mean((4+2*LR+exp(LR).*(-4+2*LR))./LR.^3,2);
f3 = h*mean((-4-3*LR-LR.^2+exp(LR).*(4-LR))./LR.^3,2);
g = -.5i*k;

set(gcf,'doublebuffer','on')
disp('press <return> to begin'), pause % wait for user input
t = 0; step = 0; v = fft(u);


while t+h/2 < 0.004
      step = step+1;
      t = t+h;
```

```
    Nv = g.*fft(real(ifft(v)).^2);
    a = E2.*v+Q.*Nv;      Na = g.*fft(real(ifft(a)).^2);
    b = E2.*v+Q.*Na;      Nb = g.*fft(real(ifft(b)).^2);
    c = E2.*a+Q.*(2*Nb-Nv); Nc = g.*fft(real(ifft(c)).^2);
    v = E.*v+(Nv.*f1+(Na+Nb).*f2+Nc.*f3);

    if mod(step,25)==0
      u = real(ifft(v));
      set(p,'ydata',u)
      title(sprintf('t = %7.5f',t),'fontsize',18), drawnow
    end

end

hold on;
plot(x, 200 * cos(pi * x), 'DisplayName', "Initial Data T = 0");

text(-2.4,900,sprintf('u(0) = %11.7f',u(N/2+1)),...
                'fontsize',18,'color','r')

legend;
hold off;
```