Pendulum

Theo Lincke

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1 Ordinary Differential Equation for the Nonlinear Pendulum

The general equation for a single pendulum is shown in equation 1.

$$ml\frac{d^2}{dt^2}\theta(t) + \beta l\frac{d}{dt}\theta(t) + mgsin\theta(t) = Acos(\alpha t) \tag{1}$$

The same system in a nonlinear representation $\dot{x} = f(x)$ is:

$$\dot{\theta} = \omega$$
$$\dot{\omega} = \frac{1}{ml}(-\beta l\omega - mgsin(\theta) + Acos(\alpha t))$$

Note that the above equation is non autonomous, and therefore for the driven case, a third variable may be added:

$$\dot{t} = 1$$

Figure 1 shows the state space trajectory for an undamped undriven pendulum ($\beta = 0$ and $A = 0$). From the plot it is clear that the starting point $[\theta, \omega] = [3, 0.1]$ is a hyperbolic fixed point, and is unstable due to the locally linear eigenvectors at that point. Intuitively, it is unstable because solutions tend away from the point. The jacobian of the state space expressed in its non autonomous form is:

$$J_{\dot{\theta},\dot{\omega}} = \begin{pmatrix} \frac{\partial\dot{\theta}}{\partial\theta} & \frac{\partial\dot{\theta}}{\partial\omega} \\ \frac{\partial\dot{\omega}}{\partial\theta} & \frac{\partial\dot{\omega}}{\partial\omega} \end{pmatrix}$$

$$J_{\dot{\theta},\dot{\omega}} = \begin{pmatrix} 0 & 1 \\ \frac{1}{ml}(-mgcos(\theta) - \alpha Asin(\alpha\theta)) & -\frac{1}{ml}\beta lm \end{pmatrix}$$

Evaluating the eigenvalues of the jacobian for the undriven undamped pendulum with conditions $[\theta, \omega] = [3, 0.1]$ yeilds an eigenvalue greater than 0 ($\lambda > 0$) and thus is unstable about this initial condition. The process of expanding the nonlinear equations into a system of linear ones based on the system jacobian

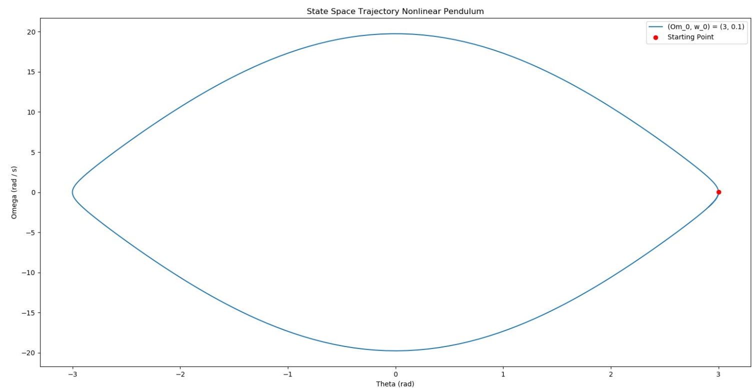

Figure 1: State Space Trajectory with initial conditions 3, 0.1. $\delta t = 0.005$

allows one to examine the stable / unstable manifolds of the state vector x. This particular point is a Heteroclinic orbit, as one unstable manifold initial condition (3 0.1) is another stable manifolds initial condition (-3 0.1).

Figure 2 shows the same state space trajectory with initial conditions $[\theta, \omega] = [0.01, 0]$. From the circular nature of the plot, it is clear that this initial condition maintains a periodic orbit attractor. Without dampening, this is intuitive as the pendulum will continue to oscilate at infinitum.

Figure 3 shows a state space portrait of the system with representative curves plotted to aid in visualization. The periodic occurrence of periodic orbit attractors at multiples of 2π indicate a full revolution of the pendulum plus some extra. Figures similar to that of Figure 2 exist at these initial conditions. Figures similar to that of Figure 1 exist on periods with a shift of pi, exhibiting the same unstable manifolds as before. As ω approaches infinity, intuitively, the systems maintains harmonic motion, and as ω reaches very high numbers, the plot may look entirely linear, representing the case in which the pendulum is rotating at a constant velocity about the origin.

Figure 4 shows the same state space portrait for a damped pendulum with $\beta = 0.25$. The entire plot shifts so that all initial conditions tend toward one fixed point (at multiples of $2pi$). This is intuitively explained as the pendulum approaches rest about the stable fixed point at the base. Of course, it is possible to approach and end at an unstable fixed point given perfect initial conditions, however, this is rare if not problematically impossible as the trajectory must match perfectly with the real valued trajectory and one may argue that it is impossible to match a real state element perfectly in the real domain. All initial conditions, then approach the same set of fixed points, regardless of the properties of the initial condition.

If β was greater, this convergence would occur more rapidly, and if β was smaller, this convergence would occur less rapidly.

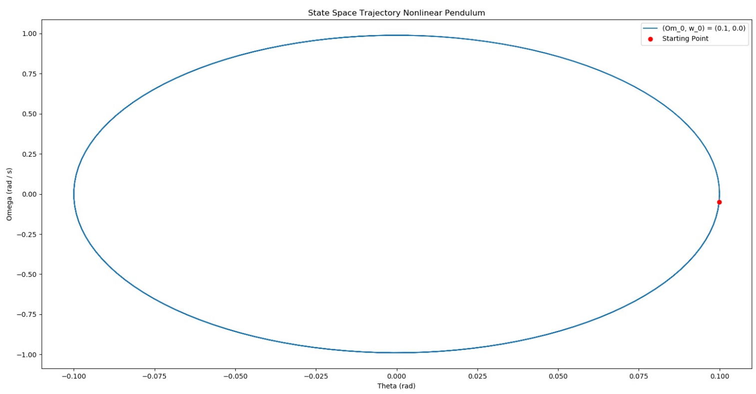

Figure 2: State Space Trajectory with initial conditions 0.1, 0. $\delta t = 0.005$

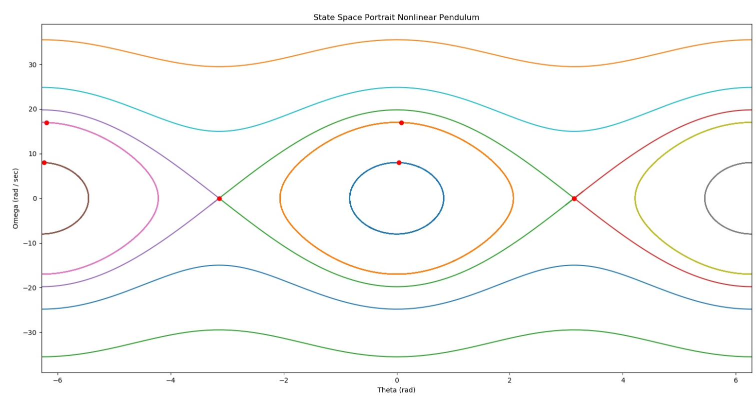

Figure 3: State Space Portrait. Undamped

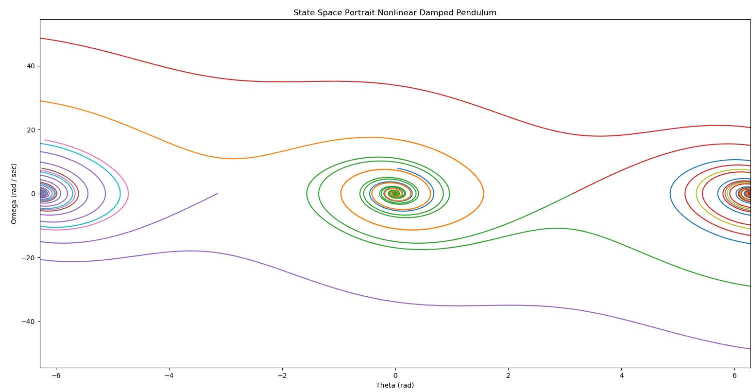

Figure 4: State Space Portrait. Damped

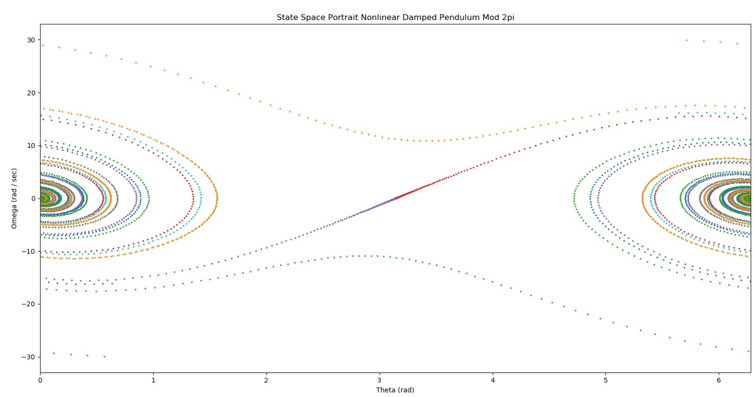

Figure 5: State Space Portrait Mod 2π. Damped

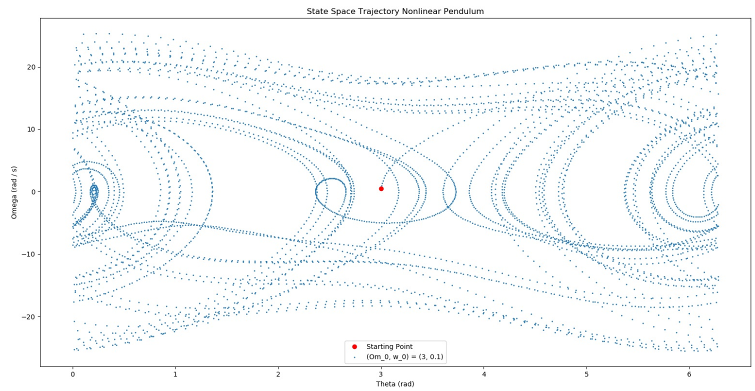

Figure 6: Chaotic Nature of a Driven Pendulum

2 Driven Pendulum

The case for the driven pendulum shows a more interesting story. A driven pendulum, given the correct push may result in seemingly chaotic orbits. After experimenting with the driven pendulum, if the drive frequency is within 0.25 of the natural frequency of the pendulum, the system will exhibit a quasi periodic orbit. I.e the pendulum will orbit, but a phase shift will occur with every revolution. At $\alpha = 1.1 * f$ where f denotes the natural frequency, a bifurcation occurs chaos ensures as shown visually in figure 6. Similarly, while keeping the frequency at $0.75f$ and increasing amplitude, amplitudes above 1.4 tend towards chaotic trajectories. Before the bifurcation at $A = 1.4$, quasi periodic orbits occur and the pendulum has appears almost steady. Above $A = 1.4$, orbits appear non steady.

I noticed at drastically large amplitudes ($A = 500$), of course, the plot is simply just an array of randomized dots, however, clusters often appeared in locations scattered throughout the plot. I would explain this as more of a numerical inaccuracy as the time step needed to adjust for a very large curvature is necessary. However, I am interested in investigating this property more analytically.

3 Numerical Analysis

As a final note, when increasing the time step δt, noticeable dynamics begin to occur. For the case of RK4, figure 7 shows what happens when $\delta t = 0.1$. After altering the time step even more, RK4 breaks around 0.7, causing an interesting fixed point to occur, then entirely unreadable plots begin to form. Figure 8 shows the exact same initial condition and undamped nature as that of

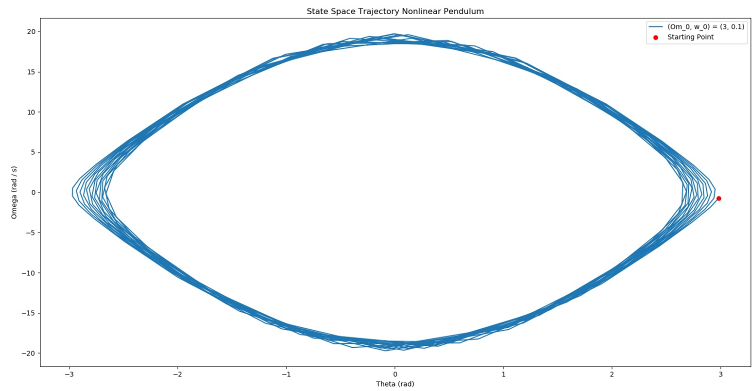

Figure 7: Change in Time Step Runga Kutta

2, however, it uses euler's method, showing a strange inversely damped system that builds in speed as the pendulum oscillates (leading to a numerically induced driven pendulum). I attempted to mathematically formulate the exact drive needed to correct this shift, however, the orbit often corrected itself for the initial portion that drastically fell from its desired orbit.

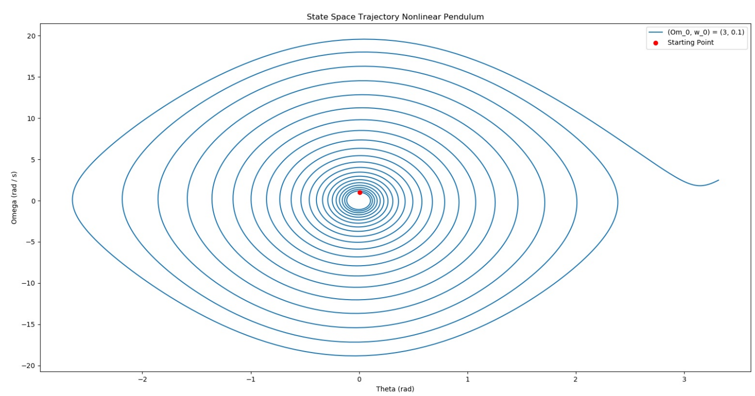

Figure 8: Numerical Dynamics (Undamped Pendulum with Euler)