# Riemann Hilbert Problems and the Inverse Scattering Transform

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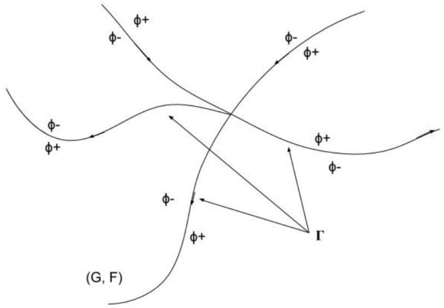
# Riemann Hilbert (RH) Problems:

Many problems in mathematics and the physical sciences involve the solution of the RH problem..

Some background:

We look into Cauchy type integrals:




$$\Phi(z) = \frac{1}{2\pi i}\int_L \frac{\phi(\tau)}{\tau - z} d\tau,$$

Where out function satisfies the Hölder condition on L:

$$|\phi(\tau_1) - \phi(\tau_2)| \leq \kappa |\tau_1 - \tau_2|^{\lambda}.$$

What about when z $\in$ L? We consider:

$$\Phi^+(t) \equiv \lim_{z\to t^+} \frac{1}{2\pi i}\int_L \frac{\phi(\tau)}{\tau - z} d\tau,$$

$$\Phi^-(t) \equiv \lim_{z\to t^-} \frac{1}{2\pi i}\int_L \frac{\phi(\tau)}{\tau - z} d\tau,$$

We use the Plemelj Formula extensively to solve the Scalar RH prob

$$\Phi^{\pm}(t) = \pm\frac{1}{2}\phi(t) + \frac{1}{2\pi i}\int_L \frac{\phi(\tau)}{\tau - t} d\tau,$$

which yields

$$\Phi^+(t) - \Phi^-(t) = \phi(t),$$

# The scalar RH problem for a closed curve:

A sectionally analytic function is the main object of importance when studying RH problems.

The RH problem: $\Phi^{+}(t) = G(t)\Phi^{-}(t) + g(t)$. We consider a closed arc L and the homogenous case where g(t)=0.

Seek a sectionally analytic function that satisfies:

$$X^{+}(t) = G(t)X^{-}(t)$$

Want to use Plemelj so we take log:

$$\log X^{+}(t) - \log X^{-}(t) = \log G(t)$$

However, need Ind(G(t)) = 0 for Hölder c state:

$$\log X(z) = \frac{1}{2\pi i}\int_{L}\frac{\log G(\tau)}{\tau - z}d\tau$$

If Ind(G(t)) = k, we introduce: e which yields:

$$\Phi^{+}(t) = (t^{-\kappa}g(t))t^{\kappa}\Phi^{-}(t)$$

$$\log \Phi^{+}(t) - \log(t^{\kappa}\Phi^{-}(t)) = \log(t^{-\kappa}g(t))$$

Then using Plemelj we get our solution:

Where:

$$X(z) \equiv \begin{cases} e^{\Gamma(z)}, & z \text{ in } D^{+} \\ z^{-\kappa}e^{\Gamma(z)}, & z \text{ in } D^{-} \end{cases}$$

$$\Phi(z) = X(z)P_{m+\kappa}(z)$$

$$\Gamma(z) \equiv \frac{1}{2\pi i}\int_{C}\frac{d\tau\,\log(\tau^{-\kappa}g(\tau))}{\tau - z}$$

Non-homogenous case: $\Phi^+(t) = G(t)\Phi^-(t) + g(t).$

For this case, we make the change of variables: $G(t) = \frac{X^+(t)}{X^-(t)}$: $\frac{\Phi^+(t)}{X^+(t)} - \frac{\Phi^-(t)}{X^-(t)} = \frac{g(t)}{X^+(t)}$

Applying Plemelj again, we obtain our solution: $\Phi(z) \quad = \quad X(z)\left[\frac{1}{2\pi i}\int_L \frac{g(\tau)}{X^+(\tau)(\tau - z)}d\tau + P(z)\right]$

These are our solutions for the scalar riemann problem on a closed arc L. We can further generalize and simplify our solutions by considering the cases where X > 0, X < 0, X = 0.

We can also look at points in our cauchy type integrals that are on the endpoints of the arc, or we can look at open arcs, etc.

Riemann Hilbert Motivation - Results First:

$$\begin{bmatrix} \frac{M(x,k)}{a(k)} \\ N(x,k)e^{-2ikx} \end{bmatrix} = \begin{bmatrix} 1+r(k)\bar{r}(k) & r(k)e^{2ikx} \\ \bar{r}(k)e^{-2ikx} & 1 \end{bmatrix} \begin{bmatrix} \bar{N}(x,k) \\ -\frac{\bar{M}(x,k)}{\bar{a}(k)}e^{-2ikx} \end{bmatrix}$$

Which is just a Homogeneous RH Problem separated by the Upper and Lower k planes (separated by the real axis)

$$\Phi^{+}(x,k) = \begin{bmatrix} \frac{M(x,k)}{a(k)} \\ N(x,k)e^{-2ikx} \end{bmatrix}$$

$$\Phi^{-}(x,k) = \begin{bmatrix} \bar{N}(x,k) \\ -\frac{\bar{M}(x,k)}{\bar{a}(k)}e^{-2ikx} \end{bmatrix}$$

$$G(x,k) = \begin{bmatrix} 1+r(k)\bar{r}(k) & r(k)e^{2ikx} \\ \bar{r}(k)e^{-2ikx} & 1 \end{bmatrix}$$

# Inverse Scattering

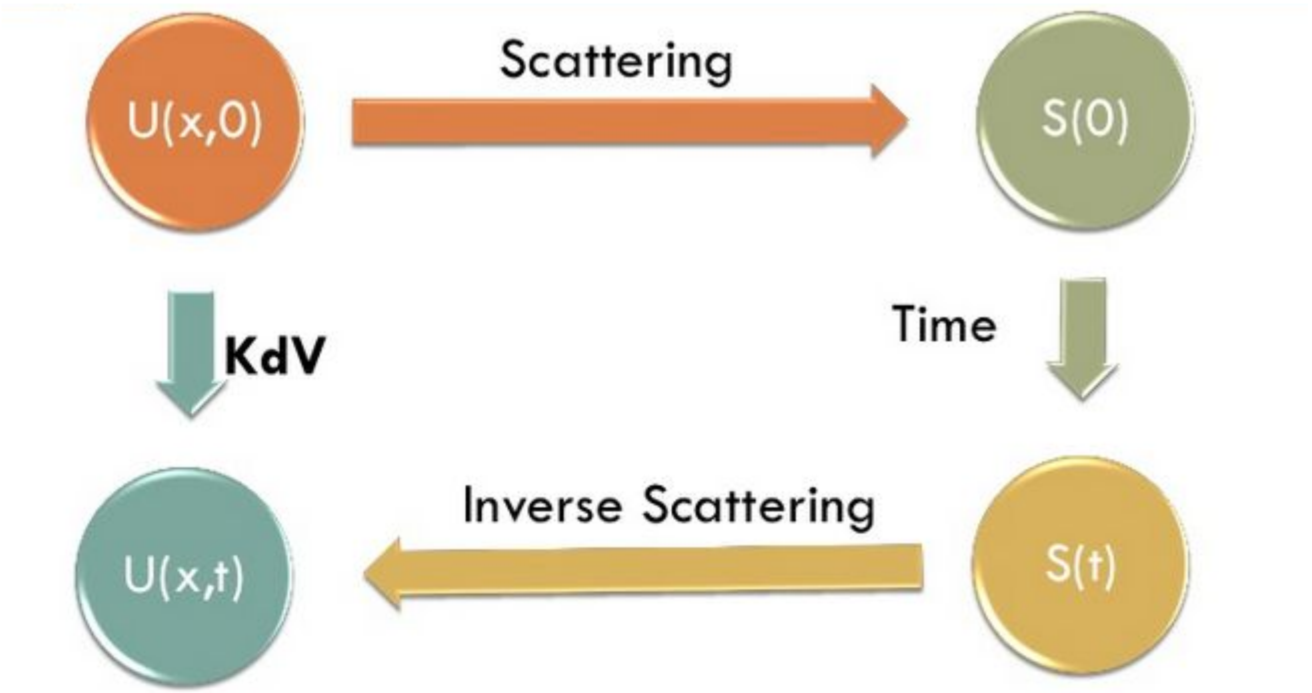



Image courtesy of https://www.slideshare.net/rupantor/an-excellent-boring-presentation-presentation/46-Inverse_Scattering_ProblemAlmost_Done_Scattering

Fit the desired equation into the TISE (A Sturm Liouville Problem).

$$\psi_{xx} + (\lambda - u)\psi = 0$$

$$\int_{-\infty}^{\infty} u(x;t)dx < \infty$$

Find the long term behavior of the eigenfunctions as x goes to infinity.

$$|u| \to 0 \implies \psi_{xx} \to -\lambda\psi$$

$$\psi \to \alpha e^{ikx} + \beta e^{-ikx}$$

$$k = \sqrt{\lambda}$$

Direct Scattering

$$0 > \lambda = -\kappa_n^2, \quad n = 1, 2 ... N \qquad 0 < \lambda = k^2$$

Two cases:

1. Discrete Spectrum
2. Continuous Spectrum

$$\psi(x,0) \to c_n(0)e^{-\kappa_n x}$$

$$\psi(x,0) \to e^{-ikx} + b(k)e^{ikx} \quad x \to \infty \quad (2)$$

$$\psi(x,0) \to a(k)e^{-ikx} \quad x \to -\infty$$

Find the discrete and Continuous spectrum of eigenvalues.

$$\psi \to \alpha e^{ikx} + \beta e^{-ikx}$$

$$k = \sqrt{\lambda} \quad (1)$$

# Time Evolution

Previously we found a time evolution operator (M).

Use this result to find the time propagated results.

$$\kappa_n = const$$

$$c_n(t) = c_n(0)e^{4\kappa_n^3 t}$$

$$a(k,t) = a(k,0)$$

$$b(k,t) = b(k,0)e^{8ik^3 t}$$

$$L\Psi = -\lambda\Psi = (\partial_x^2 - u)\Psi + \lambda\Psi$$

$$M\Psi = \Psi_t = ((\gamma - u_x) + (4\lambda + 2u)\frac{\partial}{\partial_x})\Psi$$

Define the four eigenfunctions asymptotically:

$$\phi(x;k) \to e^{-ikx}, \quad \bar{\phi}(x;k) \to e^{ikx} \qquad x \to \infty$$
$$\psi(x;k) \to e^{ikx}, \quad \bar{\psi}(x;k) \to e^{-ikx} \qquad x \to -\infty$$

From long term asymptotics, a solution, $\phi, \bar{\phi}$ can be constructed by a linear combination of the linearly independent functions $\psi, \bar{\psi}$ and hence, by the invariance of k = -k:

$$\phi(x,k) = \bar{\phi}(x,-k) = a(k)\bar{\psi}(x,k) + b(k)\psi(x,k)$$
$$\bar{\phi}(x,k) = \phi(x,-k) = \bar{a}(k)\psi(x,k) + \bar{b}(k)\bar{\psi}(x,k)$$

If we take the Wronskian of the solutions:

$$W(\phi(x,k),\phi(\bar{x},k)) = (a(k)\bar{a}(k) + b(k)\bar{b}(k))W(\psi(x,k),\bar{\psi}(x,k))$$

From the asymptotic behavior:

$$W(\phi(x,k),\phi(\bar{x},k)) = 2ik = -W(\psi(x,k),\bar{\psi}(x,k))$$

So,

$$|a|^2 - |b|^2 = 1$$

Define the auxiliary functions

$$M(x,k) = \phi e^{ikx}, \qquad \bar{M}(x,k) = \bar{\phi} e^{ikx}$$
$$N(x,k) = \psi e^{ikx}, \qquad \bar{N}(x,k) = \bar{\psi} e^{ikx}$$

Multiply Original equation by exp(ikx) and divide by a:

$$\frac{M(x,k)}{a(k)} = \bar{N}(x,k) + r(k)N(x,k)$$

$$\frac{\bar{M}(x,k)}{\bar{a}(k)} = -N(x,k) + \bar{r}(k)\bar{N}(x,k)$$

$$r(k) = \frac{b}{a}(k)$$

$$N(x,k) = \bar{N}(x;-k)e^{2ikx}$$

$$\frac{M(x,k)}{a(k)} = \bar{N}(x,k) + r(k)e^{2ikx}\bar{N}(x,-k)$$

It can be shown that:

$$\begin{bmatrix} \frac{M(x,k)}{a(k)} \\ N(x,k)e^{-2ikx} \end{bmatrix} = \begin{bmatrix} 1+r(k)\bar{r}(k) & r(k)e^{2ikx} \\ \bar{r}(k)e^{-2ikx} & 1 \end{bmatrix} \begin{bmatrix} \bar{N}(x,k) \\ -\frac{\bar{M}(x,k)}{\bar{a}(k)}e^{-2ikx} \end{bmatrix}$$

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$$G(x,k) = \begin{bmatrix} 1+r(k)\bar{r}(k) & r(k)e^{2ikx} \\ \bar{r}(k)e^{-2ikx} & 1 \end{bmatrix}$$

**Theorem 2.1** *Let $G(t)$ denote a square matrix taking values on $t \in \mathbb{R}$ and suppose that either $G + \bar{G}^T or G - \bar{G}^T$ is definite for all $t$. Then the Riemann Hilbert problem for $\Phi(z)$ satisfying:*

$$\Phi^+(t) = G(t)\Phi^-(t) \qquad -\infty < t < \infty$$

*with the boundary condition:*

$$\Phi(z) = \Psi_0 + O(\frac{1}{z}) \qquad at |z| = \infty$$

*where $\Psi_0$ is a specified constant vector, has a unique solution.*

$$\frac{1}{2}[G + \bar{G}^T] = \begin{bmatrix} 1 - |r(k)|^2 & 0 \\ 0 & 1 \end{bmatrix}$$

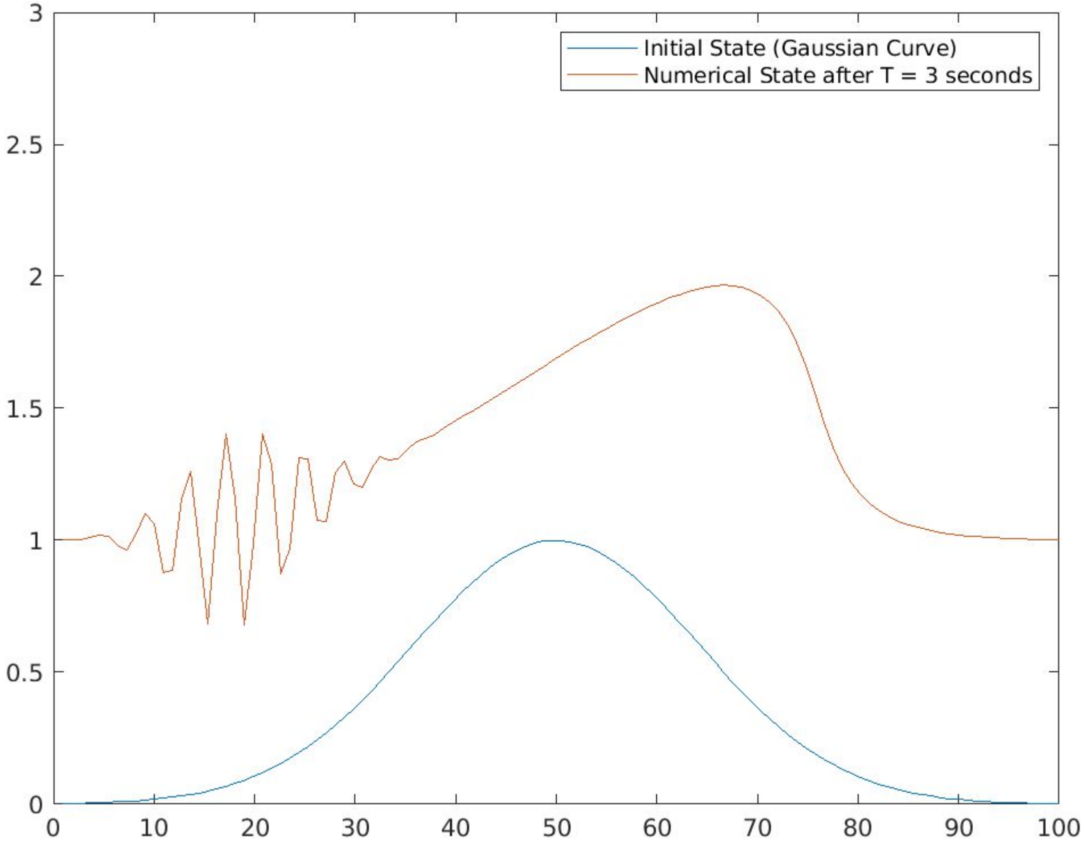
$$|a|^2 - |b|^2 = 1$$

$$|r|^2 = 1 - \frac{1}{|a|^2}$$

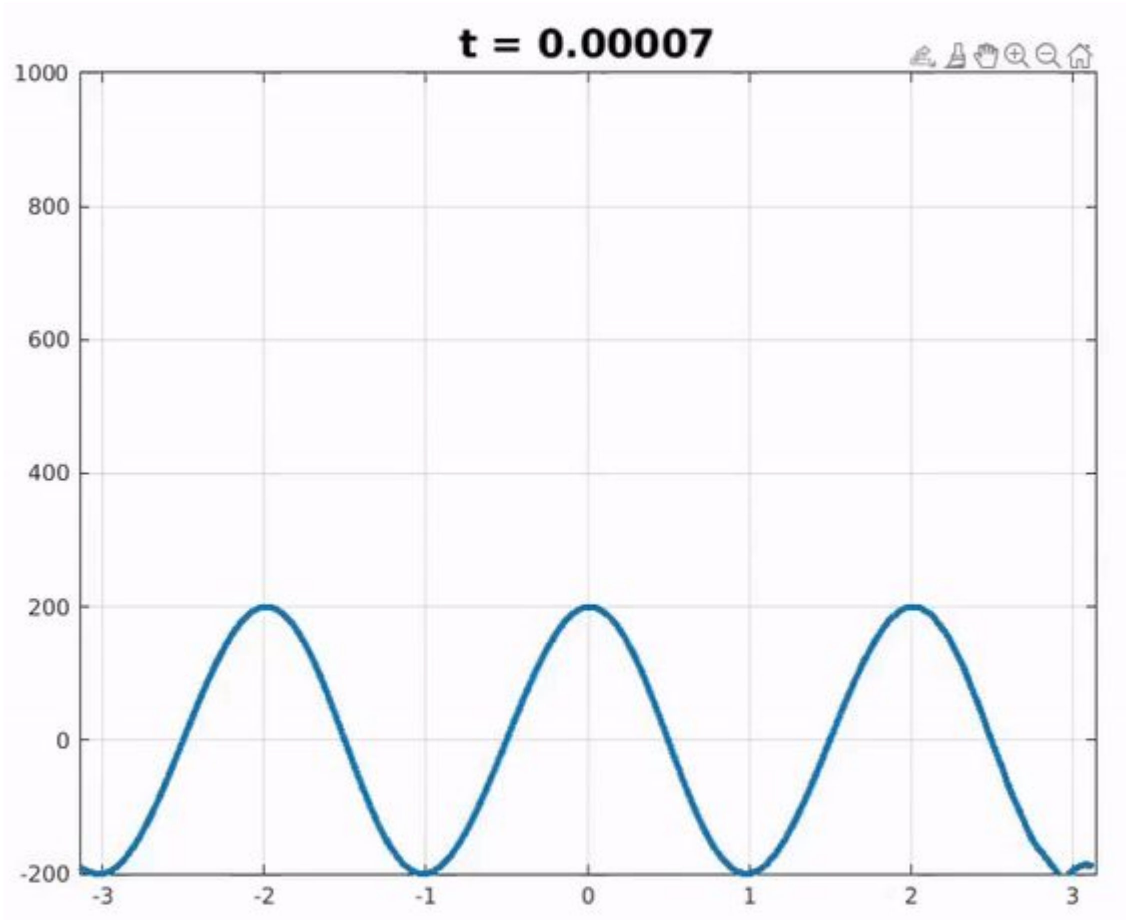
$$|r|^2 < 1$$

This Problem has a Unique Solution!

# KdV In Depth - Numerics



# Discovery of Solitons - Kruskal and Zabusky

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# Long Term Behavior

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