Variable Claim Model

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1 Introduction

This document is the design document for a simple policy model that was created with the purposes of a singular policy for a client having multiple scenarios covered by the policy that are probabilistically independent.

2 Overview

This model is a simple model that assumes that the policy covers n different scenarios. Each scenario has an average number of events $\{\lambda_i\}$ for each scenario $i \leq n$ that happen to the relevant policy holders for the duration of the policy. The simplest model assumes that the events would be problematically independent of one another and can occur at any point of time during the duration. The distribution that theoretically describes these scenarios is the Poisson distribution. For each event that occurs to a policy holder, the policy holder has an average cost of damages $\{\theta_i\}$ for scenario $i \leq n$. Assuming that the highest entropy state of probability distribution describes the underlying distribution of this curve, we would model the cost of damages to be an exponential distribution for each claim.

3 Deriving cost distribution per scenario

The distributions that will be used for this derivation will be the Poisson distribution and the Gamma distribution. The Gamma distribution is the convolution of multiple identically independent exponential distributions. If for a given period had y_i number of events happen with its average of λ_i, all of the claims would have independent values for the cost. This makes the convolution of these independent claims be the total cost. The Poisson distribution for an average event number during policy duration λ_i is

$$P(y) = \frac{\lambda_i^y}{y!} e^{-\lambda_i} \tag{1}$$

The total cost distribution (Gamma) with an average cost θ_i per an event would be

$$\rho(x|y) = \frac{x^{y-1}}{(y-1)!\theta_i^y} e^{-\frac{x}{\theta_i}} \tag{2}$$

From here we can use the Poisson distribution of calculating that the total number of possible events occuring is zero would be

$$P(0) = e^{-\lambda_i} \tag{3}$$

making the total cost 0 for that percentage of time. Now the probability density given that $y > 0$ would be

$$\begin{aligned} \rho(x, y > 0) &= \sum_{y=1}^{\infty} \frac{\lambda_i^y}{y!} e^{-\lambda_i} \frac{x^{y-1}}{(y-1)!\theta_i^y} e^{-\frac{x}{\theta_i}} \\ &= \frac{\lambda_i}{\theta_i} e^{-\lambda_i - \frac{x}{\theta_i}} \sum_{y=0}^{\infty} \frac{(\frac{\lambda x}{\theta_i})^{y_i}}{y!(y+1)!} \\ &= \sqrt{\frac{\lambda_i}{\theta_i x}} e^{-\lambda_i - \frac{x}{\theta_i}} I_1(2\sqrt{\frac{\lambda_i x}{\theta_i}}) \end{aligned} \tag{4}$$

Which the last result is computed from wolfram alpha with the modified Bessel function I.

4 Adding deductible, policy rate and policy limit to distribution

Let us define the deductible as the amount of money per a claim that must be made in order for the insurance to start paying out. When it pays out, it pays $x - d$ where x is the total cost and d is the deductible. If there is a policy rate where the policy only covers a certain percentage of the cost after the deductible then the policy pays out $\alpha(x - d)$ where $\alpha \in [0, 1]$ is the rate of payout. Once the policy pays out pass a limit L then the policy pays out only L when $\alpha(x - d) > L$.

Because the cost distribution for the damages would be modeled to be exponential with a mean cost of θ_i for one claim, then we can model the probability of a singular claim being within certain limits.

The probability of the claim costing less than the deductible d is

$$P(x < d) = 1 - e^{-\frac{d}{\theta_i}} \tag{5}$$

The probability of the claim cost being greater than the policy limit is

$$P(\alpha(x - d) > L) = e^{-\frac{\alpha^{-1}L + d}{\theta_i}} \tag{6}$$

This makes the variable claim cost to have a probability of

$$P(\alpha(x-d) \in [0,L]) = e^{-\frac{d}{\theta_i}} - e^{-\frac{\alpha^{-1}L+d}{\theta_i}} \tag{7}$$

To model the cost of an event with respect to the insurance agency, we must define a variable r where r is the cost to the insurance company. This means for $x \in [d, d+\alpha^{-1}L]$ that $r = \alpha(x-d)$. The cumulative distribution for this function will be

$$F(r) = \left\{ \begin{array}{lr} 0, & \text{if } r < 0 \\ 1 - e^{-\frac{\alpha^{-1}r+d}{\theta_i}}, & \text{if } r \in [0, L) \\ 1, & \text{if } r \geq L \end{array} \right\} \tag{8}$$

And the survival function would be

$$S(r) = \left\{ \begin{array}{lr} 1, & \text{if } r < 0 \\ e^{-\frac{\alpha^{-1}r+d}{\theta_i}}, & \text{if } r \in [0, L) \\ 0, & \text{if } r \geq L \end{array} \right\} \tag{9}$$

Writing the probability density function with Dirac Deltas would be

$$\rho(r) = (1 - e^{-\frac{d}{\theta_i}})\delta(r) + \phi(r) + e^{-\frac{\alpha^{-1}L+d}{\theta_i}}\delta(r-L) \tag{10}$$

with $\phi(r)$ defined as a piecewise function

$$\phi(r) = \left\{ \begin{array}{lr} 0, & \text{if } r \notin (0, L) \\ \frac{1}{\alpha\theta_i} e^{-\frac{\alpha^{-1}r+d}{\theta_i}}, & \text{if } r \in (0, L) \end{array} \right\} \tag{11}$$

From here, we can convolute the probability density by doing $\rho(r_1, ..., r_y) = \prod_{j=1}^{y} \rho(r_j)$. Since we want to find a total $r = \sum_{j=1}^{y} r_j$ distribution. It would be easier to calculate the convolution using the Fourier transform of this probability density. This would end up being

$$\tilde{\rho}(k) = (1 - e^{-\frac{d}{\theta_i}}) + \tilde{\phi}(k) + e^{-\frac{\alpha^{-1}L+d}{\theta_i}} e^{-ikL} \tag{12}$$

Where

$$\tilde{\phi}(k) = \frac{e^{-\frac{d}{\theta_i}}}{ik\theta_i\alpha + 1}(1 - e^{-\frac{L}{\theta_i\alpha}(ik\theta_i\alpha+1)}) \tag{13}$$

Note: this is using the definition of the Fourier Transform being

$$\mathcal{F}[\rho](k) = \int_{-\infty}^{\infty} \rho(r) e^{-ikr} dr \tag{14}$$

$$\mathcal{F}^{-1}[\tilde{\rho}](r) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{\rho}(k) e^{ikr} dk \tag{15}$$

From here the density of the total cost function (convolution of densities) in the frequency domain will is just a product of itself y times due to convolution theorem.

$$
\begin{aligned}
\tilde{\rho}_y(k) &= (\tilde{\rho}(k))^y \\
&= [(1-e^{-\frac{d}{\theta_i}}) + \tilde{\phi}(k) + e^{-\frac{\alpha^{-1}L+d}{\theta_i}}e^{-ikL}]^y \\
&= \sum_{a+b+c=y} \frac{y!}{a!b!c!}(1-e^{-\frac{d}{\theta_i}})^a[\tilde{\phi}(k)]^b[e^{-\frac{\alpha^{-1}L+d}{\theta_i}}e^{-ikL}]^c
\end{aligned} \tag{16}
$$

Using the trinomial expansion in the last step of equation 16, we can use the Fourier inverse transform 14 to calculate the probability distribution function formed from the convolution of the independent identically distributed (i.i.d) density. because the $\tilde{\phi}(k)$ function from equation 13 has a pole at $k = \frac{i}{\theta\alpha}$ We can utilize contour integrals to evaluate these Fourier transforms. We first foil the equation $[\tilde{\phi}(k)]^b$ and evaluate each component of addition separately with the appropriate contour that makes the none real line component converge to zero of the integral. Due to Residue theorem, the only components that will contribute to the total integral are where $b = 0, 1$. All other components where $b > 1$ will evaluate to zero since the pole at $k = \frac{i}{\theta\alpha}$ is not simple. That being said our integral would be evaluated as such

$$
\begin{aligned}
\rho_y(r) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{\rho}_y(k)e^{ikr}dk \\
&= \frac{1}{2\pi}\sum_{a+b+c=y}\frac{y!}{a!b!c!}\int_{-\infty}^{\infty}dke^{ikr}(1-e^{-\frac{d}{\theta_i}})^a[\tilde{\phi}(k)]^b[e^{-\frac{\alpha^{-1}L+d}{\theta_i}}e^{-ikL}]^c \\
&= \frac{1}{2\pi}\sum_{a+c=y}\frac{y!}{a!c!}\int_{-\infty}^{\infty}dke^{ikr}(1-e^{-\frac{d}{\theta_i}})^a[e^{-\frac{\alpha^{-1}L+d}{\theta_i}}e^{-ikL}]^c \\
&+ \frac{1}{2\pi}\sum_{a+c=y-1}\frac{y!}{a!c!}\int_{-\infty}^{\infty}dke^{ikr}(1-e^{-\frac{d}{\theta_i}})^a[e^{-\frac{\alpha^{-1}L+d}{\theta_i}}e^{-ikL}]^c\tilde{\phi}(k) \\
&= \sum_{a+c=y}\frac{y!}{a!c!}(1-e^{-\frac{d}{\theta_i}})^a[e^{-\frac{\alpha^{-1}L+d}{\theta_i}}]^c\delta(r-cL) \\
&+ \sum_{b=1}^{y}\sum_{a+c=y-b}\frac{y!}{a!b!c!}(1-e^{-\frac{d}{\theta_i}})^a[e^{-\frac{\alpha^{-1}L+d}{\theta_i}}]^c e^{-\frac{d}{\theta_i}}\frac{1}{2\pi}\oint dke^{ikr}[\tilde{\phi}(k)]^b
\end{aligned} \tag{17}
$$

To evaluate the contour integral, we can replace $r - cL$ or $r - (c+1)L$ with the variable β. If $\beta < 0$, then the contour that is convergent to the real integral has no simple pole, in the interior of the Jordan curve. Hence to calculate a non-zero value, it only evaluates for $\beta > 0$ over the Jordan curve. for now and we will get

$$\begin{aligned}
\frac{1}{2\pi}\oint dk e^{ikr}[\tilde{\phi}(k)]^b &= \frac{1}{2\pi}\oint dk e^{ikr}[\frac{e^{-\frac{d}{\theta_i}}}{ik\theta_i\alpha+1}(1-e^{-\frac{L}{\theta_i\alpha}(ik\theta_i\alpha+1)})]^b \\
&= \frac{e^{-\frac{r}{\theta_i\alpha}-\frac{bd}{\theta_i}}}{2\pi(i\alpha\theta_i)^b}\oint dk[\frac{1}{k-\frac{i}{\theta_i\alpha}}]^b\sum_{j=0}^{b}(-1)^j\binom{b}{j}e^{i(r-jL)(k-\frac{i}{\theta_i\alpha})} \\
&= \frac{e^{-\frac{r}{\theta_i\alpha}-\frac{bd}{\theta_i}}}{2\pi(i\alpha\theta_i)^b}\oint dk\sum_{j=0}^{b}(-1)^j\binom{b}{j}\sum_{q=0}^{\infty}(k-\frac{i}{\theta_i\alpha})^{q-b}\frac{[i(r-jL)]^q}{q!} \\
&= \frac{e^{-\frac{r}{\theta_i\alpha}-\frac{bd}{\theta_i}}}{(\alpha\theta_i)^b}\sum_{j=0}^{b}\Phi(r-jL,b,j)
\end{aligned} \tag{18}$$

where

$$\Phi(\beta,b,j) = \left\{\begin{array}{ll} 0, & \text{if } \beta\notin(0,\infty) \\ (-1)^j\frac{b[\beta]^{b-1}}{j!(b-j)!}, & \text{if } \beta\in(0,\infty)\end{array}\right\} \tag{19}$$

On the condition of $\beta = 0$ there isn't necessarily a formal argument but it can be approximated as $\frac{\pi}{\theta_i\alpha}$. However we can ignore this correction since there is only a finite number of these discontinuities in this density contribution which a standard integral would ignore.

Looking at our equation 18 we can see that the evaluation has an exponential term similar to the constant just before the two sets of contour integrals in equation 17. Interestingly enough this will make the terms where $c > 0$ cancel when the evaluated r is passed a certain threshold in our last two sums of equation 17. If we combine the last two sums, we would have an equation that looks like this

$$\begin{aligned}
\rho_y(r) = &\sum_{a+c=y}\frac{y!}{a!c!}(1-e^{-\frac{d}{\theta_i}})^a[e^{-\frac{\alpha^{-1}L+d}{\theta_i}}]^c\delta(r-cL) \\
&+\sum_{b=1}^{y}\{\frac{e^{-\frac{r}{\theta_i\alpha}-\frac{bd}{\theta_i}}}{b!(\alpha\theta_i)^b}[\sum_{j=0}^{b}\Phi(r-jL,b,j)][\sum_{a+c=y-b}\frac{y!}{a!c!}(1-e^{-\frac{d}{\theta_i}})^a[e^{-\frac{\alpha^{-1}L+d}{\theta_i}}]^c]\}
\end{aligned} \tag{20}$$

This $\rho_y(r)$ is the conditional probability density function of the cost to the insurance company if there were y events that occurred during the duration of the policy. We should be able to combine this with a Poisson distribution to get the total cost of the policy.

4.1 Combining PDF with Poisson

The probability that there are no events that occur is $e^{-\lambda_i}$, for this condition there is a hundred percent chance that $r = 0$ since there is nothing to claim.

Hence we can say that our total cost joint probability density function for this case $\eta_i(r, y = 0)$ would be this equation

$$\eta_i(r, y = 0) = \delta(r)e^{-\lambda_i} \tag{21}$$

The joint distribution for when $y \neq 0$ is of this form

$$\eta_i(r, y) = \rho_y(r)\frac{\lambda_i^y}{y!}e^{-\lambda_i} \tag{22}$$

where $y \in \mathbb{N}/\{0\}$. From this point we can use these two facts to calculate the total cost probability density function over all $y \in \mathbb{N}$ which we will call $\eta_i(r)$.

$$\begin{aligned}
\eta_i(r) &= \sum_{y\in\mathbb{N}} \eta_i(r, y) \\
&= \delta(r)e^{-\lambda_i} + \sum_{y=1}^{\infty} \rho_y(r)\frac{\lambda_i^y}{y!}e^{-\lambda_i} \\
&= \delta(r)e^{-\lambda_i} \\
&+ e^{-\lambda_i}\sum_{y=1}^{\infty}\sum_{a+c=y}\frac{\lambda_i^y}{a!c!}(1 - e^{-\frac{d}{\theta_i}})^a[e^{-\frac{\alpha^{-1}L+d}{\theta_i}}]^c\delta(r - cL) \\
&+ e^{-\lambda_i}\sum_{y=1}^{\infty}\sum_{a+c=y-1}\frac{\lambda_i^y}{a!c!}(1 - e^{-\frac{d}{\theta_i}})^a[e^{-\frac{\alpha^{-1}L+d}{\theta_i}}]^c\phi(r - cL)
\end{aligned} \tag{23}$$

5 Adding a maximal buyout for cost distribution and Maximal claims

For most items in this world, there is a maximal value for that item that it could be replaced with. That maximal value is going to be denoted as M_i. So far in this paper, we have been assuming the maximal value of an item is $M_i = \infty$. This will adjust our distributions a bit where θ_i is no longer the true mean of the distribution. The normalization constant will also adjust due to the space of cost for the client being in $[0, M]$. Even though this might adjust our computation a little bit, it can be easily substituted in with defining M_i and the true mean μ_i. The highest entropy solution for a function on an interval $[0, M_i]$ that has a definite mean is still an exponential distribution. In the case that $M_i = \infty$ the distribution for the cost of the client is

$$\rho(x) = \frac{1}{\theta_i}e^{-\frac{x}{\theta_i}} \tag{24}$$

However when this is adjusted for $x \in [0, M_i]$, the probability distribution has a different normalization constant

$$\rho(x) = \frac{e^{-\frac{x}{\theta_i}}}{\theta_i(1 - e^{-\frac{M_i}{\theta_i}})} \tag{25}$$

From this distribution we can calculate what the true mean would be relative to M_i and θ_i

$$
\begin{aligned}
\mu_i &= \int_0^M x\rho(x)dx \\
&= \theta_i[1 - \frac{\frac{M_i}{\theta_i}}{e^{\frac{M_i}{\theta_i}} - 1}] \qquad (26) \\
&= \theta_i - \frac{M_i}{e^{\frac{M_i}{\theta_i}} - 1}
\end{aligned}
$$

In a real case scenario would know what the maximal buyout M_i is and we would have acquired empirical data for μ_i. Hence to solve for the value of θ_i we must evaluate it by inverting this equation. To invert the equation, we would need to solve for θ_i as the root of the equation derived from 26,

$$
\mu_i - \theta_i + \frac{M_i}{e^{\frac{M_i}{\theta_i}} - 1} = 0 \qquad (27)
$$

this can be done by using Newton's method where $\Theta_{i,n}$ is a particular iteration. such that $\Theta_{i,0} = \mu_i$ and $\lim_{n\to\infty} \Theta_{i,n} = \theta_i$. The recursive formula for $\Theta_{i,n}$ is this

$$
\Theta_{i,n+1} = \Theta_{i,n} - \Theta_{i,n}^2[(\mu_i - \Theta_{i,n})(e^{\frac{M_i}{\Theta_{i,n}}} - 1) + M_i] \frac{e^{\frac{M_i}{\Theta_{i,n}}} - 1}{M_i^2 e^{\frac{M_i}{\Theta_{i,n}}} + \Theta_{i,n}^2(2e^{\frac{M_i}{\Theta_{i,n}}} - e^{\frac{2M_i}{\Theta_{i,n}}} - 1)} \qquad (28)
$$

Using this will give us an approximation of θ_i to be used in equation 25. However from the insurance company perspective we care about our $\rho(r)$ where r is the cost for the insurance company with policy limits, deductibles and rates. So we must find our adjusted versions of equations 10 and 11.

For any given insurance policy, the insurance would generally be priced where $L \leq \alpha(M_i - d)$. Thus the cost function for the policy would not have any weird added implications if it were to payout more than the buyout. If we set a general limit L for the policy, we will add functional logic where if the generalized $L > \alpha(M - d)$, it will be replaced with $L = \alpha(M_i - d)$ for the distribution. The cumulative distribution should look very similar equation 8 with the only difference being in the region of $r \in [0, L)$ the function should be multiplied by $\frac{1}{1-e^{-\frac{M_i}{\theta_i}}}$

$$
F(r) = \left\{ \begin{array}{ll} 0, & \text{if } r < 0 \\ \frac{1-e^{-\frac{\alpha^{-1}r+d}{\theta_i}}}{1-e^{-\frac{M_i}{\theta_i}}}, & \text{if } r \in [0, L) \\ 1, & \text{if } r \geq L \end{array} \right\} \qquad (29)
$$

This creates a slight adjustment to our PDF

$$
\rho(r) = \frac{1 - e^{-\frac{d}{\theta_i}}}{1 - e^{-\frac{M_i}{\theta_i}}}\delta(r) + \frac{\phi(r)}{1 - e^{-\frac{M_i}{\theta_i}}} + \frac{e^{-\frac{\alpha^{-1}L+d}{\theta_i}} - e^{-\frac{M_i}{\theta_i}}}{1 - e^{-\frac{M_i}{\theta_i}}}\delta(r - L) \qquad (30)
$$

with $\phi(r)$ defined is the same piecewise function we had before.

$$\phi(r) = \left\{ \begin{array}{ll} 0, & \text{if } r \notin (0, L) \\ \frac{1}{\alpha\theta_i} e^{-\frac{\alpha^{-1}r+d}{\theta_i}}, & \text{if } r \in (0, L) \end{array} \right\} \tag{31}$$

This only will slightly modify our distribution of multiple claims

$$\begin{aligned} \rho_y(r) = & \sum_{a+c=y} \frac{y!}{a!c!} \left(\frac{1-e^{-\frac{d}{\theta_i}}}{1-e^{-\frac{M_i}{\theta_i}}}\right)^a \left[\frac{e^{-\frac{\alpha^{-1}L+d}{\theta_i}} - e^{-\frac{M_i}{\theta_i}}}{1-e^{-\frac{M_i}{\theta_i}}}\right]^c \delta(r - cL) \\ & + \sum_{b=1}^{y} \left\{ \frac{e^{-\frac{r}{\theta_i \alpha} - \frac{bd}{\theta_i}}}{(1-e^{-\frac{M_i}{\theta_i}}) b! (\alpha\theta_i)^b} \left[\sum_{j=0}^{b} \Phi(r - jL, b, j)\right] \left[\sum_{a+c=y-b} \frac{y!}{a!c!} \left(\frac{1-e^{-\frac{d}{\theta_i}}}{1-e^{-\frac{M_i}{\theta_i}}}\right)^a \left(\frac{e^{-\frac{\alpha^{-1}L+d}{\theta_i}} - e^{-\frac{M_i}{\theta_i}}}{1-e^{-\frac{M_i}{\theta_i}}}\right)^c\right]\right\} \end{aligned} \tag{32}$$

where

$$\Phi(\beta, b, j) = \left\{ \begin{array}{ll} 0, & \text{if } \beta \notin (0, \infty) \\ (-1)^j \frac{b[\beta]^{b-1}}{j!(b-j)!}, & \text{if } \beta \in (0, \infty) \end{array} \right\} \tag{33}$$

As a final note, we have discussed that for when $y = 0$ we just use the dirac delta function evaluated at $r = 0$ for the function. This means that $\rho_0(r) = \delta(r)$. The convolution of $\rho_0(r)$ with any other $\rho_y(r)$ is $\rho_y(r)$. In general, we can assert an important theorem about $\rho_y(r)$.

$$\rho_{\sum_i y_i}(r) = [\rho_{y_1} * \rho_{y_2} * ... * \rho_{y_n}](r) \tag{34}$$

This is due to the fact that $\rho_y(r) = (\rho * \rho ... * \rho)(r)$ for y convolutions by its definition. This theorem works for all of the various i.i.d distributions.

5.1 Total cost with maximal claims

In the combining PDF with Poisson chapter, we discussed that the joint distribution follows equation 22. This would be a totally valid distribution if the total number of claims that can be made is infinite during the period. We can adjust equation 23 by substituting our new version of $\rho_y(r)$.

$$
\begin{aligned}
\eta_i^{\infty}(r) &= \sum_{y\in\mathbb{N}} \eta_i(r,y) \\
&= \sum_{y=0}^{\infty} \rho_y(r) \frac{\lambda_i^y}{y!} e^{-\lambda_i} \\
&= \delta(r) e^{-\lambda_i} \\
&+ e^{-\lambda_i} \sum_{y=1}^{\infty} \sum_{a+c=y} \frac{\lambda_i^y}{a!c!} \Big(\frac{1-e^{-\frac{d}{\theta_i}}}{1-e^{-\frac{M_i}{\theta_i}}}\Big)^a \Big[\frac{e^{-\frac{\alpha^{-1}L+d}{\theta_i}} - e^{-\frac{M_i}{\theta_i}}}{1-e^{-\frac{M_i}{\theta_i}}}\Big]^c \delta(r-cL) \\
&+ e^{-\lambda_i} \sum_{y=1}^{\infty} \sum_{b=1}^{y} \Big\{ \frac{e^{-\frac{r}{\theta_i \alpha} - \frac{bd}{\theta_i}}}{(1-e^{-\frac{M_i}{\theta_i}}) b! (\alpha\theta_i)^b} \Big[\sum_{j=0}^{b} \Phi(r-jL,b,j)\Big] \Big[\sum_{a+c=y-b} \frac{\lambda_i^y}{a!c!} \Big(\frac{1-e^{-\frac{d}{\theta_i}}}{1-e^{-\frac{M_i}{\theta_i}}}\Big)^a \Big[\frac{e^{-\frac{\alpha^{-1}L+d}{\theta_i}} - e^{-\frac{M_i}{\theta_i}}}{1-e^{-\frac{M_i}{\theta_i}}}\Big]^c \Big]
\end{aligned} \tag{35}
$$

This equation would work with any number of clients given that λ_i is the average number of claims over the whole population of clients per duration. This distribution is perfectly fine to define a theoretical policy for it. However it is too computationally complex to be done with computers without relying on approximation theory. One correction we can do to our policy generation is that we can induce a maximal claim limit G_i for this policy. The maximal claim limit will define the max number of claims a single client can use for this i^{th} condition in the policy. If we had N clients for our policy, then we could calculate the total distribution for this particular event covered in the policy. Let us define our $\eta_i^{G_i,N}(r)$ the distribution for this particular event category for N clients. If λ_i is the average number of events over an entire population, then $\Lambda_i = \frac{\lambda_i}{N}$ is the average number of events over a single client. This would be used for our Poisson distribution as it applies to $\eta_i^{G_i,1}(r)$ which we can convolute with itself N times to find $\eta_i^{G_i,N}(r)$. We start with our probabilty mass function $P_1(y)$ of there being y events per a client. This mass function would be

$$
P_1(y) = \frac{\Lambda_i^y}{y!} e^{-\Lambda_i} \tag{36}
$$

And let us define the symbol $P_1(y \geq G_i)$ as the probability that there were greater than or equal to G_i events per a client

$$
\begin{aligned}
P_1(y \geq G_i) &= \sum_{y=G_i}^{\infty} \frac{\Lambda_i^y}{y!} e^{-\Lambda_i} \\
&= 1 - e^{-\Lambda_i} \sum_{y=0}^{G_i-1} \frac{\Lambda_i^y}{y!}
\end{aligned} \tag{37}
$$

The computational cost per a client would have a distribution of

$$
\begin{aligned}
\eta_i^{G_i,1}(r) &= P_1(y \geq G_i)\rho_{G_i}(r) + \sum_{y=0}^{G_i-1} P_1(y)\rho_y(r) \\
&= (1 - e^{-\Lambda_i} \sum_{y=0}^{G_i-1} \frac{\Lambda_i^y}{y!})\rho_{G_i}(r) + e^{-\Lambda_i} \sum_{y=0}^{G_i-1} \frac{\Lambda_i^y}{y!}\rho_y(r)
\end{aligned}
\tag{38}
$$

From this point we want to calculate the total cost function of N clients subscribed to the policy. For computational purposes we can assume that all of the P_1 is precalculated before moving on to calculate the total cost function. The total cost function for N clients would be the convolution of $\eta_i^{G_i,1}(r)$ for N times, to do this it is easier to do this as a product of the Fourier transform of $\eta_i^{G_i,1}(r)$ as $\tilde{\eta}_i^{G_i,1}(k)$ for N times.

$$
\begin{aligned}
\tilde{\eta}_i^{G_i,1}(k) &= P_1(y \geq G_i)\tilde{\rho}_{G_i}(k) + \sum_{y=0}^{G_i-1} P_1(y)\tilde{\rho}_y(k) \\
&= (1 - e^{-\Lambda_i} \sum_{y=0}^{G_i-1} \frac{\Lambda_i^y}{y!})\tilde{\rho}_{G_i}(k) + e^{-\Lambda_i} \sum_{y=0}^{G_i-1} \frac{\Lambda_i^y}{y!}\tilde{\rho}_y(k)
\end{aligned}
\tag{39}
$$

Hence let us calculate the N client distribution

$$
\begin{aligned}
\tilde{\eta}_i^{G_i,N}(k) &= (\tilde{\eta}_i^{G_i,1}(k))^N \\
&= [P_1(y \geq G_i)\tilde{\rho}_{G_i}(k) + \sum_{y=0}^{G_i-1} P_1(y)\tilde{\rho}_y(k)]^N \\
&= \sum_{\sum_{y=0}^{G_i} q_y = N} \frac{N!}{\prod_{y=0}^{N} q_y!}(P_1(y \geq G_i)\tilde{\rho}_{G_i}(k))^{q_{G_i}} \prod_{y=0}^{G_i-1} (P_1(y)\tilde{\rho}_y(k))^{q_y} \\
&= \sum_{\sum_{y=0}^{G_i} q_y = N} [\frac{N!}{\prod_{y=0}^{N} q_y!}(P_1(y \geq G_i))^{q_{G_i}} \prod_{y=0}^{G_i-1} (P_1(y))^{q_y}]\tilde{\rho}_{\sum_{y=0}^{G_i} q_y y}(k)
\end{aligned}
\tag{40}
$$

So when you Fourier transform it back to the cost domain, we will find the total distribution to be

$$
\eta_i^{G_i,N}(r) = \sum_{\sum_{y=0}^{G_i} q_y = N} [\frac{N!}{\prod_{y=0}^{N} q_y!}(P_1(y \geq G_i))^{q_{G_i}} \prod_{y=0}^{G_i-1} (P_1(y))^{q_y}]\rho_{\sum_{y=0}^{G_i} q_y y}(r)
\tag{41}
$$

This equation will work with either equations 32 or 20 as $\rho_y(r)$ depending if a buyout is needed.

6 Moment generating functions, expectation values and variance

6.1 Raw Moment Generating functions

For this section, I will focus on the main attributes of these probability distributions. We will start with the probabilty distribution of the cost to the insurer of a claim with a bound of M_i enforced by equation 30.

$$\rho(r) = \frac{1 - e^{-\frac{d}{\theta_i}}}{1 - e^{-\frac{M_i}{\theta_i}}}\delta(r) + \frac{\phi(r)}{1 - e^{-\frac{M_i}{\theta_i}}} + \frac{e^{-\frac{\alpha^{-1}L+d}{\theta_i}} - e^{-\frac{M_i}{\theta_i}}}{1 - e^{-\frac{M_i}{\theta_i}}}\delta(r - L) \tag{42}$$

With its $\phi(r)$ function being from equation 31

$$\phi(r) = \left\{ \begin{array}{ll} 0, & \text{if } r \notin (0, L) \\ \frac{1}{\alpha\theta_i}e^{-\frac{\alpha^{-1}r+d}{\theta_i}}, & \text{if } r \in (0, L) \end{array} \right\} \tag{43}$$

The moment generating function is calculated as

$$\begin{aligned} E[e^{tr}] &= \int_{\mathbb{R}} dr\rho(r)e^{tr} \\ &= \frac{1 - e^{-\frac{d}{\theta_i}}}{1 - e^{-\frac{M_i}{\theta_i}}} + \frac{e^{-\frac{\alpha^{-1}L+d}{\theta_i}} - e^{-\frac{M_i}{\theta_i}}}{1 - e^{-\frac{M_i}{\theta_i}}}e^{tL} \\ &+ \int_{\mathbb{R}} dr\frac{\phi(r)e^{tr}}{1 - e^{-\frac{M_i}{\theta_i}}} \\ &= \frac{1 - e^{-\frac{d}{\theta_i}}}{1 - e^{-\frac{M_i}{\theta_i}}} + \frac{e^{-\frac{\alpha^{-1}L+d}{\theta_i}} - e^{-\frac{M_i}{\theta_i}}}{1 - e^{-\frac{M_i}{\theta_i}}}e^{tL} \\ &+ \frac{e^{-\frac{d}{\theta_i}}}{\alpha\theta t - 1}(e^{(t-\frac{1}{\alpha\theta_i})L} - 1) \end{aligned} \tag{44}$$

We can use this moment generating function to define how the derivative moment generating functions act. The first moment generating function derived should be $\rho_y(r)$ which is a convolution of $\rho(r)$ y times. the generating moment function of an i.i.d has an analogues theorem to the convolution theorem from Fourier transforms where the total moment generating function for $\rho_y(r)$ is the moment generating function for $\rho(r)$ to the power of y. Hence the moment generating function is

$$E_y[e^{tr}] = [\frac{1 - e^{-\frac{d}{\theta_i}}}{1 - e^{-\frac{M_i}{\theta_i}}} + \frac{e^{-\frac{\alpha^{-1}L+d}{\theta_i}} - e^{-\frac{M_i}{\theta_i}}}{1 - e^{-\frac{M_i}{\theta_i}}}e^{tL} + \frac{e^{-\frac{d}{\theta_i}}}{\alpha\theta t - 1}(e^{(t-\frac{1}{\alpha\theta_i})L} - 1)]^y \tag{45}$$

From this moment generating function, we can utilize the linearity of expectation values to determine what the moment generating function of $\eta_i^{G_i,1}(r)$ is from equation 38.

$$\eta_i^{G_i,1}(r) = P_1(y \geq G_i)\rho_{G_i}(r) + \sum_{y=0}^{G_i-1} P_1(y)\rho_y(r) \tag{46}$$

Will have the raw moment generating function be

$$E_{\eta,G_i,i,1}[e^{tr}] = P_1(y \geq G_i)E_{G_i}[e^{tr}] + \sum_{y=0}^{G_i-1} P_1(y)E_y[e^{tr}] \tag{47}$$

Where $E_{G_i}[e^{tr}], E_y[e^{tr}]$ are calculated for their y value described in 45. The final moment generating function from all of this would be for $\eta_i^{G_i,N}(r)$ where you have N policy holders for this particular claim type making the final raw moment function be

$$E_{\eta,G_i,i,N}[e^{tr}] = (P_1(y \geq G_i)E_{G_i}[e^{tr}] + \sum_{y=0}^{G_i-1} P_1(y)E_y[e^{tr}])^N \tag{48}$$

6.2 Means and Variance

for the cost of a single claim distribution, the mean is the first derivative of $E[e^{tr}]$ evaluated at $t = 0$,

$$\begin{aligned} E[r] = &\frac{e^{-\frac{d}{\theta_i}}}{1 - e^{-\frac{M}{\theta_i}}}(\alpha\theta_i - (L + \alpha\theta_i)e^{-\frac{L}{\alpha\theta_i}}) \\ &+ \frac{e^{-\frac{\alpha^{-1}L+d}{\theta_i}} - e^{-\frac{M_i}{\theta_i}}}{1 - e^{-\frac{M_i}{\theta_i}}}L \end{aligned} \tag{49}$$

And we can take the second derivative to find $E[r^2]$ which is

$$\begin{aligned} E[r^2] = &e^{-\frac{d}{\theta_i}}[2(\alpha\theta_i)^2 - (2 + \frac{1}{\alpha^2\theta_i^2} + 2(\alpha\theta_i)^2)e^{-\frac{L}{\alpha\theta_i}}] \\ &+ \frac{e^{-\frac{\alpha^{-1}L+d}{\theta_i}} - e^{-\frac{M_i}{\theta_i}}}{1 - e^{-\frac{M_i}{\theta_i}}}L^2 \end{aligned} \tag{50}$$

to find the variance of the cost being $\sigma_r^2 = E[r^2] - E[r]^2$ which I will leave as an exercise for the computer. As for the other distributions, we can derive them by from these calculated expectations and variance. For the iid distribution following equation 32 the mean is

$$E_y[r] = yE[r] \tag{51}$$

and the variance is

$$\sigma_y^2 = y\sigma_r^2 \tag{52}$$

From this point we use the law of total variance $Var[x] = E[Var[x|y]] + Var[E[x|y]]$ and the double expectation $E[x] = E[E[x|y]]$ to find $\eta_i^{G_i,1}$ which is

Then use the fact that $\eta_i^{G_i,N}$ is an i.i.d of $\eta_i^{G_i,1}$ to find