Laplace Transform Lab: Solving ODEs using Laplace Transform in MATLAB

This lab will teach you to solve ODEs using a built in MATLAB Laplace transform function `laplace`.

There are five (5) exercises in this lab that are to be handed in. Write your solutions in a separate file, including appropriate descriptions in each step.

Include your name and student number in the submitted file.

Student Information

```
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```

Using symbolic variables to define functions

In this exercise we will use symbolic variables and functions.

```
syms t s x y

f = cos(t)
```

f = $\cos(t)$

```
h = exp(2*x)
```

h = e^{2x}

Laplace transform and its inverse

```
% The routine |laplace| computes the Laplace transform of a function

F=laplace(f)
```

F =

$$\frac{s}{s^2+1}$$

By default it uses the variable `s` for the Laplace transform But we can specify which variable we want:

```
H=laplace(h)
```

H =

$$\frac{1}{s-2}$$

```
laplace(h,y)
```

ans =

$$\frac{1}{y-2}$$

```
% Observe that the results are identical: one in the variable |s| and the
% other in the variable |y|
```

We can also specify which variable to use to compute the Laplace transform:

```
j = exp(x*t)
```

j = $e^{t\,x}$

```
laplace(j)
```

ans =

$$\frac{1}{s-x}$$

```
laplace(j,x,s)
```

ans =

$$\frac{1}{s-t}$$

```
% By default, MATLAB assumes that the Laplace transform is to be computed
% using the variable |t|, unless we specify that we should use the variable
% |x|
```

We can also use inline functions with `laplace`. When using inline functions, we always have to specify the variable of the function.

```
l = @(t) t^2+t+1
```

l = *function_handle with value:*
    @(t)t^2+t+1

```
laplace(l(t))
```

ans =

$$\frac{s+1}{s^2}+\frac{2}{s^3}$$

MATLAB also has the routine `ilaplace` to compute the inverse Laplace transform

```
ilaplace(F)
```

ans = $\cos(t)$

```
ilaplace(H)
```

ans = e^{2t}

```
ilaplace(laplace(f))
```

ans = $\cos(t)$

If `laplace` cannot compute the Laplace transform, it returns an unevaluated call.

```
g = 1/sqrt(t^2+1)
```

g =

$$\frac{1}{\sqrt{t^2+1}}$$

```
G = laplace(g)
```

G =

$$\mathrm{laplace}\left(\frac{1}{\sqrt{t^2+1}}, t, s\right)$$

But MATLAB "knows" that it is supposed to be a Laplace transform of a function. So if we compute the inverse Laplace transform, we obtain the original function

```
ilaplace(G)
```

ans =

$$\frac{1}{\sqrt{t^2+1}}$$

The Laplace transform of a function is related to the Laplace transform of its derivative:

```
syms g(t)
laplace(diff(g,t),t,s)
```

ans = $s\,\mathrm{laplace}(g(t), t, s) - g(0)$

Exercise 1

Objective: Compute the Laplace transform and use it to show that MATLAB 'knows' some of its properties.

Details:

(a) Define the function `f(t)=exp(2t)*t^3`, and compute its Laplace transform `F(s)`. (b) Find a function `f(t)` such that its Laplace transform is `(s - 1)*(s - 2))/(s*(s + 2)*(s - 3)` (c) Show that MATLAB 'knows' that if `F(s)` is the Laplace transform of `f(t)`, then the Laplace transform of `exp(at)f(t)` is `F(s-a)` (in your answer, explain part (c) using comments).

Observe that MATLAB splits the rational function automatically when solving the inverse Laplace transform.

```
% ============================================================================
% Exercise 1 Submission
% ============================================================================

% Define symbolic variables
syms t s

% (a)
% Define inline function
f(t) = exp(2*t)*(t^3);
% Compute (and display) laplace transform
F = laplace(f)
```

F =

$$\frac{6}{(s-2)^4}$$

```
% (b)
% Define Laplace transformed function
G(s) = ((s-1)*(s-2))/(s*(s+2)*(s-3));
% Compute (and display) inverse
g = ilaplace(G)
```

g =

$$\frac{6\,e^{-2t}}{5} + \frac{2\,e^{3t}}{15} - \frac{1}{3}$$

```
% (c)
syms q(t) p(t) a t

% Define function
q(t) = exp(a*t)*p(t);
% Compute laplace transforms
P = laplace(p)
```

P = $\mathrm{laplace}(p(t), t, s)$

```
Q = laplace(q)
```

Q = $\mathrm{laplace}(p(t), t, s-a)$

```
% When displaying the laplace transforms of the arbitrary functions p(t) and
% q(t) = exp(a*t)*p(t) (represented by P and Q, respectively),
% MATLAB simply changes the last argument of the |laplace| routine from
```

```
% |s| to |s-a|. In other words, MATLAB simply replaces the |s| parameter in
% P(s) with |s-a| to evaluate Q, meaning that MATLAB "knows" we can write
% Q = P(s-a).

% ============================================================================
```

Heaviside and Dirac functions

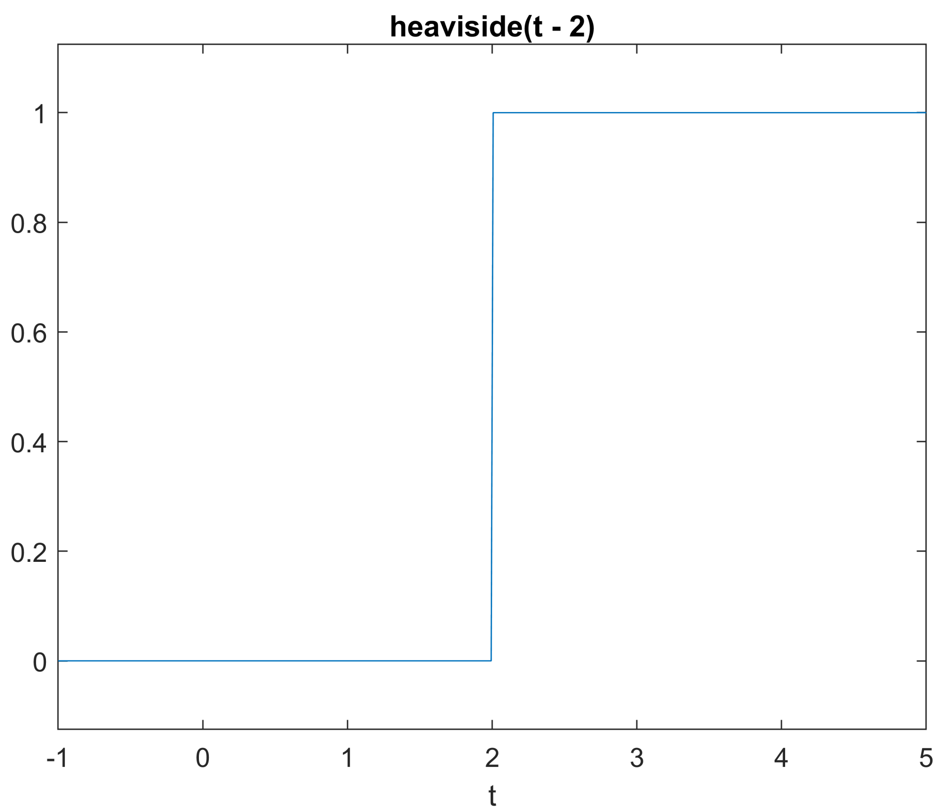
These two functions are builtin to MATLAB: `heaviside` is the Heaviside function `u_0(t)` at `0`

To define `u_2(t)`, we need to write

```
f=heaviside(t-2)
```

f = $\mathrm{heaviside}(t-2)$

```
ezplot(f,[-1,5])
```




```
% The Dirac delta function (at |0|) is also defined with the routine |dirac|

g = dirac(t-3)
```

g = $\delta(t-3)$

```
% MATLAB "knows" how to compute the Laplace transform of these functions

laplace(f)
```

ans =

$$\frac{e^{-2s}}{s}$$

```
laplace(g)
```

ans = e^{-3s}

Exercise 2

Objective: Find a formula comparing the Laplace transform of a translation of `f(t)` by `t-a` with the Laplace transform of `f(t)`

Details:

- Give a value to a
- Let `G(s)` be the Laplace transform of `g(t)=u_a(t)f(t-a)` and `F(s)` is the Laplace transform of `f(t)`, then find a formula relating `G(s)` and `F(s)`

In your answer, explain the 'proof' using comments.

```
% ============================================================================
% Exercise 2 Submission
% ============================================================================

syms u_a(t) f(t) g(t) t

for i = 1:5
    a = i
    % Define Heaviside function u_a(t)
    u_a(t) = heaviside(t-a);

    % Define function g(t)
    g(t) = u_a(t)*f(t-a);

    % Compute and display Laplace transforms for several values of a
    F = laplace(f);
    G = laplace(g)
end
```

a = 1

G = $e^{-s}\,\mathrm{laplace}(f(t), t, s)$

a = 2

G = $e^{-2s}\,\mathrm{laplace}(f(t), t, s)$

a = 3

G = $e^{-3s}\,\mathrm{laplace}(f(t), t, s)$

```
a = 4
```

G = $e^{-4s}\,\mathrm{laplace}(f(t), t, s)$

```
a = 5
```

G = $e^{-5s}\,\mathrm{laplace}(f(t), t, s)$

```
% As shown by the MATLAB output display, the Laplace transform G of g(t) =
% u_a(t)*f(t-a) is simply exp(-a*s)*F, where F is the Laplace transform of
% f(t) and a E [1,5] in this example.
% This outcome is corroborated by Theorem 5.5.1 in the textbook,
% which states that L{u_a(t)*f(t-a)} = exp(-a*s)*L{f(t)}. The proof of this
% theorem lies in a change of variables used to transform the integral
% (from c to infinity) of exp(-st)*f(t-c)dt to the integral of
% exp(-(z+c)*s)*f(x)dz, where z = t - c. Then, the exp(-c*s) term can be
% taken out of the integral since the integral is with respect to z, which
% results in L{u_a(t)*f(t-a)} = exp(-c*s) * integral from 0 to infinity of
% exp(-sz)*f(z)dz = exp(-c*s)*F(s).
% ========================================================================
```

Solving IVPs using Laplace transforms

Consider the following IVP, `y''-3y = 5t` with the initial conditions `y(0)=1` and `y'(0)=2`. We can use MATLAB to solve this problem using Laplace transforms:

```
% First we define the unknown function and its variable and the Laplace
% tranform of the unknown

syms y(t) t Y s

% Then we define the ODE

ODE=diff(y(t),t,2)-3*y(t)-5*t == 0
```

```
ODE =
```

$$\frac{\partial^2}{\partial t^2}\, y(t) - 5\,t - 3\,y(t) = 0$$

```
% Now we compute the Laplace transform of the ODE.

L_ODE = laplace(ODE)
```

```
L_ODE =
```

$$s^2\,\mathrm{laplace}(y(t), t, s) - s\,y(0) - \left(\left(\frac{\partial}{\partial t}\, y(t)\right)\Big|_{t=0}\right) - \frac{5}{s^2} - 3\,\mathrm{laplace}(y(t), t, s) = 0$$

```
% Use the initial conditions

L_ODE=subs(L_ODE,y(0),1)
```

L_ODE =

$$s^2\,\mathrm{laplace}(y(t),t,s) - s - \left(\left(\frac{\partial}{\partial t}\;y(t)\right)\Big|_{t=0}\right) - \frac{5}{s^2} - 3\,\mathrm{laplace}(y(t),t,s) = 0$$

```
L_ODE=subs(L_ODE,subs(diff(y(t), t), t, 0),2)
```

L_ODE =

$$s^2\,\mathrm{laplace}(y(t),t,s) - s - \frac{5}{s^2} - 3\,\mathrm{laplace}(y(t),t,s) - 2 = 0$$

```
% We then need to factor out the Laplace transform of |y(t)|

L_ODE = subs(L_ODE,laplace(y(t), t, s), Y)
```

L_ODE =

$$Y\,s^2 - s - 3\,Y - \frac{5}{s^2} - 2 = 0$$

```
Y=solve(L_ODE,Y)
```

Y =

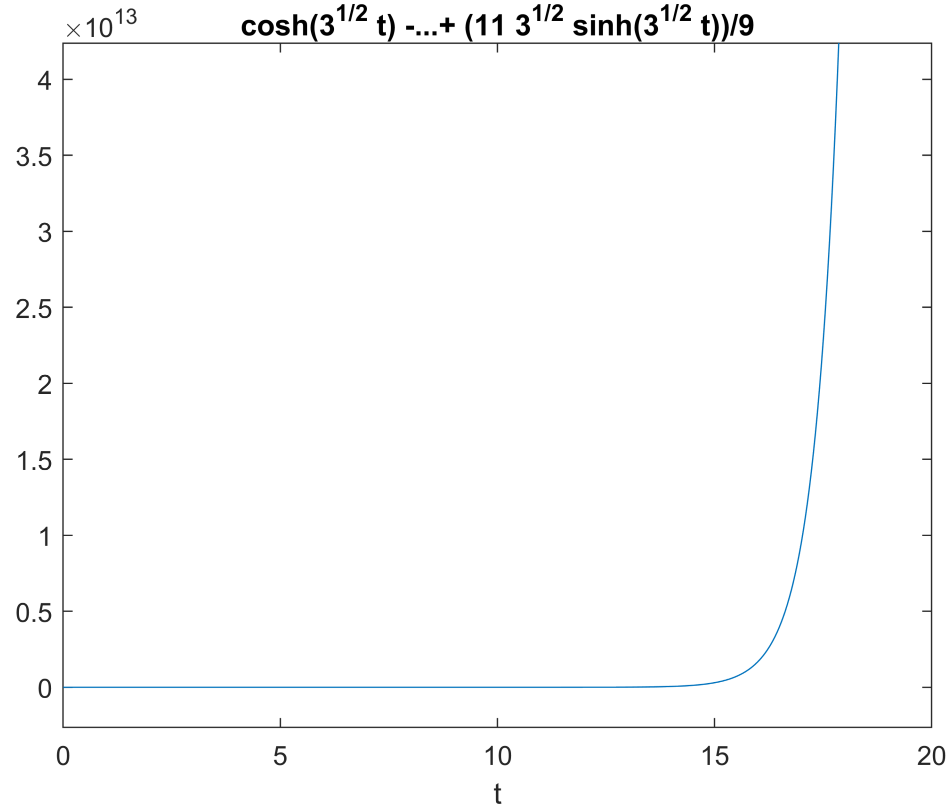
$$\frac{s + \frac{5}{s^2} + 2}{s^2 - 3}$$

```
% We now need to use the inverse Laplace transform to obtain the solution
% to the original IVP

y = ilaplace(Y)
```

y =

$$\cosh\left(\sqrt{3}\;t\right) - \frac{5\,t}{3} + \frac{11\;\sqrt{3}\;\sinh\left(\sqrt{3}\;t\right)}{9}$$

```
% We can plot the solution

ezplot(y,[0,20])
```




```
% We can check that this is indeed the solution

diff(y,t,2)-3*y
```

ans = $5\,t$

Exercise 3

Objective: Solve an IVP using the Laplace transform

Details: Explain your steps using comments

- Solve the IVP
- `y'''+2y''+y'+2*y=-cos(t)`
- `y(0)=0`, `y'(0)=0`, and `y''(0)=0`
- for `t` in `[0,10*pi]`
- Is there an initial condition for which y remains bounded as `t` goes to infinity? If so, find it.

```
% ============================================================================
% Exercise 3 Submission
% ============================================================================

% Define unknown function, independent variable and Laplace transform of
% the unknown function
```

```
syms y(t) t Y s

% Define ODE
ODE = diff(y(t),t,3) + 2*diff(y(t),t,2) + diff(y(t),t,1) + 2*y(t) + cos(t) == 0;

% Compute Laplace transform of ODE
L_ODE = laplace(ODE);

% Use initial conditions
L_ODE = subs(L_ODE,y(0),0);
L_ODE = subs(L_ODE,subs(diff(y(t),t),t,0),0);
L_ODE = subs(L_ODE,subs(diff(y(t),t,2),t,0),0);

% Factor out Laplace transform of y(t) and solve for it
L_ODE = subs(L_ODE,laplace(y(t), t, s), Y)
```

L_ODE =

$$2\,Y + Y\,s + \frac{s}{s^2+1} + 2\,Y\,s^2 + Y\,s^3 = 0$$

```
Y=solve(L_ODE,Y)
```

Y =

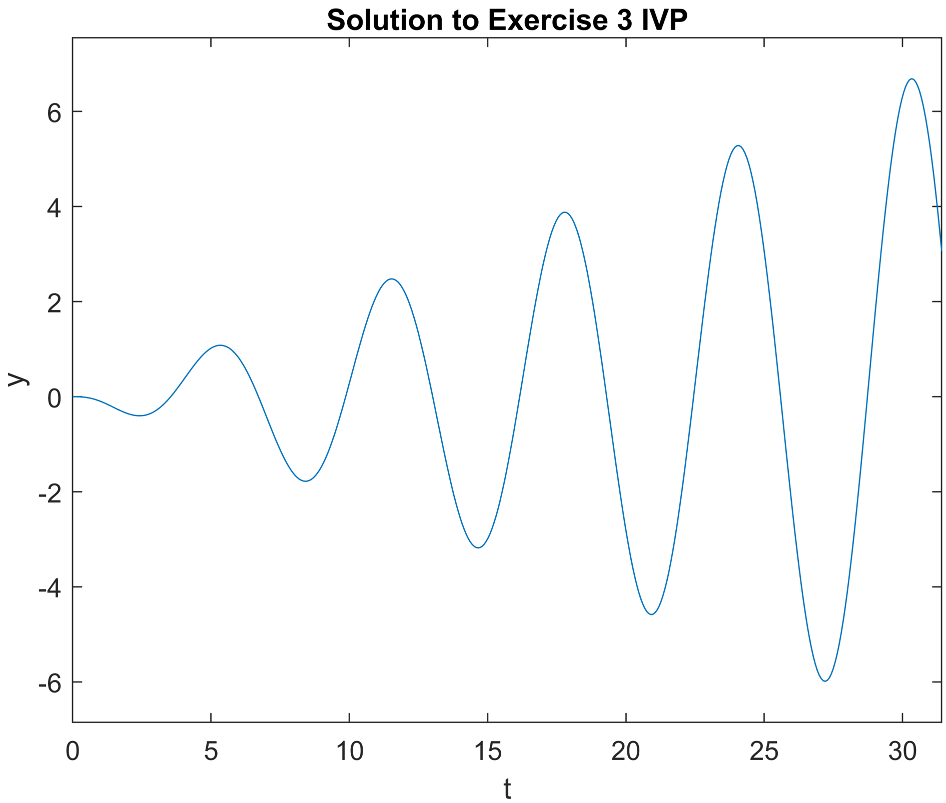
$$-\frac{s}{(s^2+1)\,(s^3+2\,s^2+s+2)}$$

```
% Get (and display) the solution to the original IVP using the inverse Laplace transform
y = ilaplace(Y)
```

y =

$$\frac{2\,e^{-2\,t}}{25} - \frac{2\cos(t)}{25} + \frac{3\sin(t)}{50} + \frac{t\cos(t)}{10} - \frac{t\sin(t)}{5}$$

```
% Plot solution
ezplot(y,[0,10*pi]);
title('Solution to Exercise 3 IVP')
xlabel('t');
ylabel('y');
```




```
% Based on the solution to the problem, there is no intial condition for
% which y remains bounded as t goes to infinity. This is because the
% solution contains terms containing t multiplied by a sinusoidal function,
% which will always oscillate while increasing to infinity as t goes to
% infinity. Changing the initial conditions will not change this fundamental
% form of the solution.
% ===========================================================================
```

Exercise 4

Objective: Solve an IVP using the Laplace transform

Details:

- Define
- `g(t) = 3 if 0 < t < 2`
- `g(t) = t+1 if 2 < t < 5`
- `g(t) = 5 if t > 5`
- Solve the IVP
- `y''+2y'+5y=g(t)`
- `y(0)=2 and y'(0)=1`
- Plot the solution for `t` in `[0,12]` and y in `[0,2.25]`.

In your answer, explain your steps using comments.

```
% ============================================================================
% Exercise 4 Submission
% ============================================================================

% Define unknown function, independent variable and Laplace transform of
% the unknown function
syms y(t) t Y s

% Define heaviside functions for g(t)
u_0(t) = heaviside(t);
u_2(t) = heaviside(t-2);
u_5(t) = heaviside(t-5);

% Define g(t) using heaviside functions, whose "coefficients" can be found
% by graphing each piecewise component.
g(t) = 3*u_0(t) + (t-2)*u_2(t) + (4-t)*u_5(t);

% Define ODE
ODE = diff(y(t),t,2) + 2*diff(y(t),t,1) + 5*y(t) - g(t) == 0;

% Compute Laplace transform of ODE
L_ODE = laplace(ODE);

% Use initial conditions
L_ODE = subs(L_ODE,y(0),2);
L_ODE = subs(L_ODE,subs(diff(y(t),t),t,0),1);

% Factor out Laplace transform of y(t) and solve for it
L_ODE = subs(L_ODE,laplace(y(t), t, s), Y)
```

L_ODE =

$$5\,Y - 2\,s + 2\,Y\,s - \frac{e^{-2\,s}}{s^2} + Y\,s^2 - \frac{3}{s} + \frac{e^{-5\,s}\,(s+1)}{s^2} - 5 = 0$$

```
Y=solve(L_ODE,Y)
```

Y =

$$\frac{2\,s + \frac{e^{-2\,s}}{s^2} + \frac{3}{s} - \frac{e^{-5\,s}\,(s+1)}{s^2} + 5}{s^2 + 2\,s + 5}$$

```
% Get (and display) the solution to the original IVP using the inverse Laplace transform
y = ilaplace(Y)
```

y =

$$\mathrm{heaviside}(t-2)\left(\frac{t}{5}+\frac{2\,e^{2-t}\left(\cos(2\,t-4)-\frac{3\sin(2\,t-4)}{4}\right)}{25}-\frac{12}{25}\right)-\mathrm{heaviside}(t-5)\left(\frac{t}{5}+\frac{2\,e^{5-t}\left(\sigma_3-\frac{3}{\,}\right.}{25}\right.$$

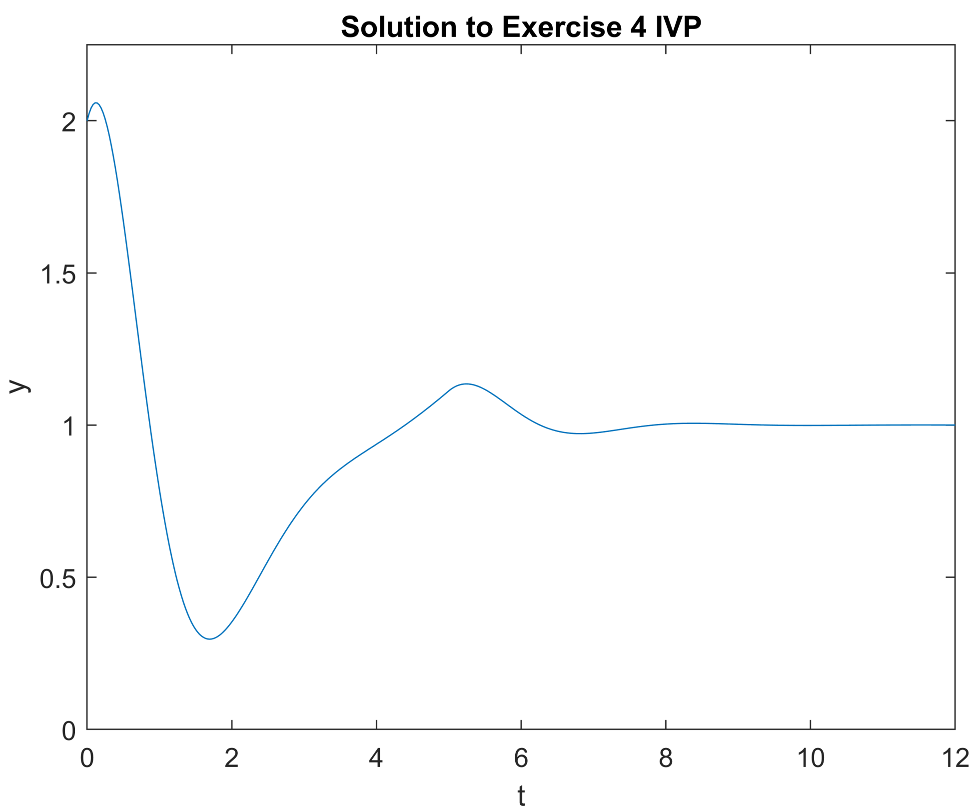
where

$$\sigma_1=\frac{\sin(2\,t)}{2}$$

$$\sigma_2=\sin(2\,t-10)$$

$$\sigma_3=\cos(2\,t-10)$$

```
% Plot solution
ezplot(y,[0,12,0,2.25]);
title('Solution to Exercise 4 IVP')
xlabel('t');
ylabel('y');
```




```
% ============================================================================
```

Exercise 5

Objective: Use the Laplace transform to solve an integral equation

Verify that MATLAB knowns about the convolution theorem by explaining why the following transform is computed correctly.

```
% ==============================================================================
% Exercise 5 Submission
% ==============================================================================

syms t tau y(tau) s
I=int(exp(-2*(t-tau))*y(tau),tau,0,t)
```

I =

$$\int_0^t e^{2\tau-2t}\, y(\tau)\mathrm{d}\tau$$

```
laplace(I,t,s)
```

ans =

$$\frac{\mathrm{laplace}(y(t),t,s)}{s+2}$$

```

% The Convolution Theorem (5.8.3) states that for two functions f and g,
% the Laplace transform of f convolved with g is equal to the product of
% the individual Laplace transforms of f and g (i.e. L{h(t)} = F(s)*G(s)
% where h(t) = integral from 0 to t of f(t-tau)*g(tau)d(tau)).

% In the case of the above transform, we can take f(t) = exp(-2*t) as convolved
% with some arbitrary function g(t) = y(t). The laplace transform of this
% convolution is taken in the following line and the result is displayed.

% We can prove that MATLAB computes the transform correctly by writing out
% explicitly the product of the two individual Laplace transforms:

syms f(t) g(t)

f(t) = exp(-2*t);

F = laplace(f)
```

F =

$$\frac{1}{s+2}$$

```
G = laplace(y)
```

G = $\mathrm{laplace}(y(\tau),\tau,s)$

```

H = F * G
```

H =

$$\frac{\mathrm{laplace}(y(\tau), \tau, s)}{s+2}$$

```
% Since H(s) = F(s)*G(s) is the same as the above output (given by laplace(I,t,s),
% by the convolution theorem, we know that MATLAB computed the transform correctly.

% ============================================================================
```