Homework 1

Information about homework:

Task 2-8 are mandatory and can give bonus points if they are handed in before the bonus deadline. If solutions to questions 2-8 are handed in before the date are not correct, they have to be redone, but the second time without yielding bonus points for the exam. Solutions must be clearly written, and easy to follow. If not, they will not generate bonus points, and must be redone. It is highly recommended that the answers are done on a computer (for instance LaTeX or word). Task 2-8 (but not 5) can be done in groups of two (single person groups are allowed). Please submit:

- written solutions (in the form of a PDF-file)
- your matlab (or julia) programs for the task

The files should be uploaded in

CANVAS → SF2526 → Assignments → Homework 1

You need to create a homework group in order to submit the assignment.

The wiki part (Task 1) should be handed in here:

CANVAS → SF2526 → Assignments → Wiki part homework 1

Note: The wiki deadline is different from the homework deadline.

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1. Create problems and solutions on the course training wiki: `http://gragg.math.kth.se/sf2524/merge_group_pages2.php?name=97359`
   This task is optional but can increase your bonus. This task is individual (do work in groups). See how the work influences your bonus points under

   `http://kth.instructure.com/courses/17791/pages/homework-slash-bonus-points-rules`.

   You should solve x problems and create $y \geq x$ problems in the current part of the course (block 1-2). If $x > 1$ the problems need to be from different blocks. Do not solve your own problem.

   The course training area is formative examination, meaning it is mainly intended for you to practice and get feedback for learning, continuously throughout the course.

In order to get good problem ideas, you might want to do the wiki-problems in parallell or after solving the problems below. Save your wiki work often such that nobody else starts working on your problem.

2. We shall investigate Gram-Schmidt by column elimination (Algorithm 1 in block1.pdf).

   (a) Consider the matrix

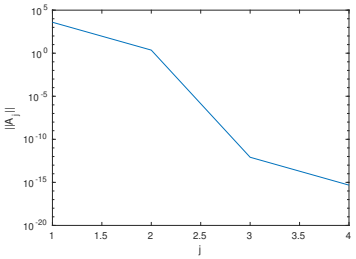
   $$A = \begin{pmatrix} 1 & 2 & 2003 & 2005 \\ 2 & 2 & 2002 & 2004 \\ 3 & 2 & 2001 & 2003 \\ 4 & 7 & 7005 & 7012 \end{pmatrix}$$

   Show by hand that the matrix has rank three. (Hint: What is the sum of the two middle columns?)

   (b) Implement and apply Algorithm 1 (standard) and apply it to the matrix in (a). Verify that after $i = 3$, the "error" $\|A_j\|$ is small. Plot $\|A_j\|$ against iteration j. Compare with figure in margin.

   (c) Carry out the same simulation as in (b) but with the larger matrix: `load_mat_hw1(1000,100)` where `load_mat_hw1.m` can be downloaded from course web page. Provide the error plot.

   (d) Carry out the same simulation as in (c) but with Algorithm 1 (greedy). Provide the error plot. Why is this algorithm better, if we want an approximation of the span of the columns?



Warning: (c) will not be as smooth as convergence as (b)

3. Consider the following matrix

   $$A = \begin{bmatrix} 5 & -1 \\ 5 & 7 \end{bmatrix}$$

   Let $A = USV^T$ be the singular value decomposition of A.

   (a) Given that

   $$S = \begin{bmatrix} \sqrt{80} & 0 \\ 0 & \sqrt{20} \end{bmatrix}, V = \beta \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

   determine β and U with a closed formula (no computer solution). Verify that U is an orthogonal matrix (you may use computer).

   (b) Determine the matrix $X = uv^T \in \mathbb{R}^{2\times 2}$ where u and v are vectors, such that

   $$X = \underset{\text{rank}(X)=1}{\text{argmin}} \|A - X\|$$

   How big is the error $\|A - X\|$ according to theory? Compare with your computed solution.

4. Use the matrix `load_mat_hw1.m` and compute and SVD-of that matrix by using the technique "Getting SVD from the QR factorization". You may use the matlab command `svd` for the small matrix. What is the size of the small matrix, if we want accuracy 10^{-10}?

5. Do Quiz 1 on CANVAS. This is mandatory and should be done individually (not in a group).

6. Follow these steps to analyze the testbild video. Only written answers to d-f need to be handed it.

   (a) Download `testbild_snapshots.zip`

   (b) Familiarize yourself with plotting through julia/matlab by plotting the first snapshot, and compare in performance with the `testbild.mkv` video.

   (c) Write a program which puts all the snapshots into one matrix $A \in \mathbb{R}^{n\times m}$, where m=number of columns=number of snapshots

   (d) Form a rank-one matrix as follows. Let $u = a_1 \in \mathbb{R}^n$ be the first column of A, and let $v^T = [1, \dots, 1] \in \mathbb{R}^m$. Analyze $A - uv^T$. Is it as expected?

   (e) Compute an approximate SVD the A matrix using your program in the previous task. Provide an error plot until $p = 4$. What is the numerical rank (tolerance 10^{-14}.)? Relate (d)-(e) to the video.

   (f) How does your implementation in (e) compare with MATLAB `svd(A,0)`? (This will heavily depend on your computer. If you get an error message explain which one, not much explanation is necessary.)

7. Adapt your program for task 6a-6e to the video corresponding to `roundabout.mkv` and `roundabout_snapshots.zip`.

   (a) In your QR factorization, provide an error plot until $p = 20$.

   (b) Use it to compute the vectors u and v such that

   $$uv^T = B = \operatorname*{argmin}_{\text{rank}(B)=1} \|A - B\|$$

   Let b_1 be the first column of B. (Compute it without computing the entire matrix B.) Treat b_1 as an image and visualize it for $p = 5$, $p = 10$ and $= 20$.

   (c) Explain your image. What happened to the cars?

8. For the matrix in task 7, use the "basic randomized SVD computation" to compute the SVD corresponding to the 5 largest eigenvalues. Provide an image of b_1 of the rank one approximation, as well as CPU-time comparison to task 7.

In task 6 you need to use image loading in MATLAB. A short introduction to the main tools you need can be found here: `https://kth.instructure.com/courses/17791/pages/video-editing-in-matlab-for-sf2526`

The video tasks require computer memory. If you get an "out of memory" error, please contact a teacher such that we can make a lower resolution video available or other solution.

Extra/fun: An additional more difficult video is also available: market.mkv