2/5/14

18.06 Linear Algebra

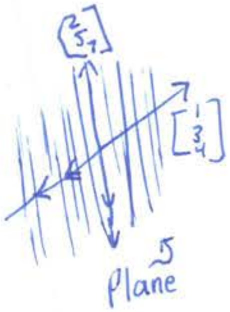
Linear Combinations

$Ax = b$

Linear Combination

$$\underbrace{c\overset{v}{\begin{bmatrix}1\\3\\4\end{bmatrix}} + d\overset{w}{\begin{bmatrix}2\\5\\7\end{bmatrix}}}_{\text{Linear Combination}} = \begin{bmatrix}1&2\\3&5\\4&7\end{bmatrix}\begin{bmatrix}c\\d\end{bmatrix} = \begin{bmatrix}c+2d\\3c+5d\\4c+7d\end{bmatrix}$$

dot product w/ rows



"v and w span the column space"

space of all linear combinations of vectors

$Ax = b$

looking for c and d

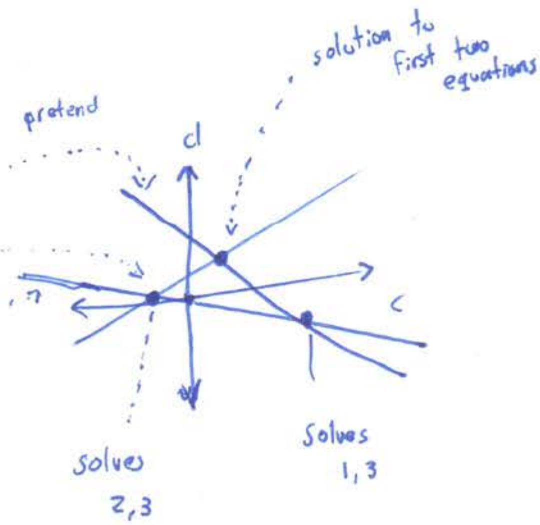
$$\begin{bmatrix}1&2\\3&5\\4&7\end{bmatrix}\begin{bmatrix}c\\d\end{bmatrix} = \begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix} \longrightarrow \begin{aligned} c + 2d &= b_1\\ 3c + 5d &= b_2\\ 4c + 7d &= b_3\end{aligned}$$

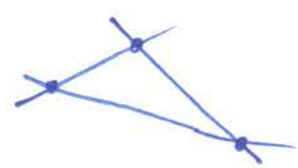
Solve.

Is there a solution?

$Ax = b$

Find c, d to get b





where is the solution?

18.06 Quizes

Mar 7
Apr 7
May 7

Vectors in $\mathbb{R}^n$

↖ higher dimentions

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 4 \\ 5 \\ 7 \\ 11 \end{bmatrix} + x_2 \begin{bmatrix} \\ \end{bmatrix} + x_3 \begin{bmatrix} \\ \end{bmatrix} + x_4 \begin{bmatrix} \\ \end{bmatrix} = A_x = \begin{bmatrix} 6 \times 4 \end{bmatrix} \begin{bmatrix} 4 \times 1 \end{bmatrix} = \begin{bmatrix} 6 \times 1 \end{bmatrix}$$

$1x_1 + 3x_2 + 11x_3 - 4x_4 = b_1$

↳ 1 eq, 4 unknowns = 3D hyperplane in $\mathbb{R}^4$

- Now include eq 2

$2x_1 + \ldots + x_4 = b_2$

- intersection of 2 3D hyperplanes intersecsecs in a 2D plane in $\mathbb{R}^3$

- add anther equation

- can now intersect in a beautiful line

add another

- ???

- Point

2/2

2/9/14

18.06 Lecture 1 Video

Linear Equations

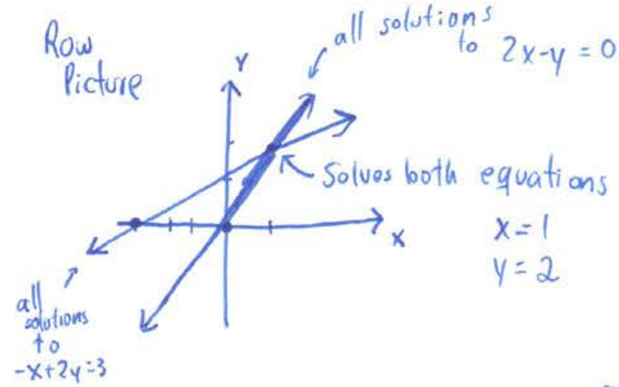
$2x - y = 0$

$-x + 2y = 3$

$$\begin{bmatrix}2 & -1\\-1 & 2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0\\3\end{bmatrix}$$

$$A \quad x = b$$

Row Picture



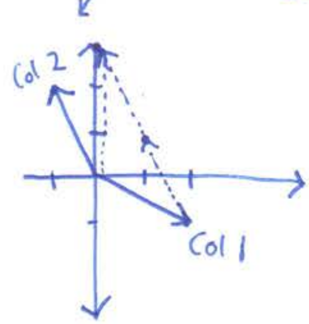
Column Picture

$$x\underbrace{\begin{bmatrix}2\\-1\end{bmatrix}}_{\text{col 1}} + y\underbrace{\begin{bmatrix}-1\\2\end{bmatrix}}_{\text{col 2}} = \begin{bmatrix}0\\3\end{bmatrix}$$

- Combine in the right amounts to get $\begin{bmatrix}0\\3\end{bmatrix}$

* Linear Combination of columns

$\begin{bmatrix}2\\-1\end{bmatrix}+2\begin{bmatrix}-1\\2\end{bmatrix}=\begin{bmatrix}0\\3\end{bmatrix}$



Example 2

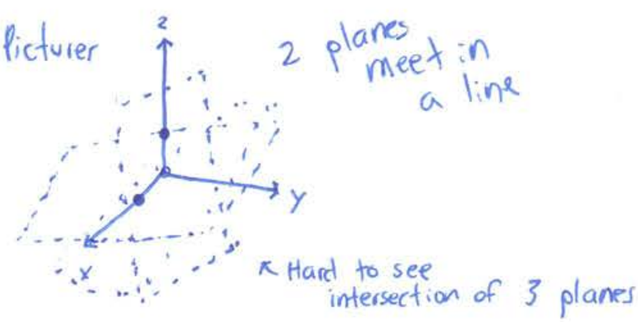
$2x - y = 0$ ← plot of all points = plane

$-x + 2y - z = -1$

$-3y + 4z = 4$

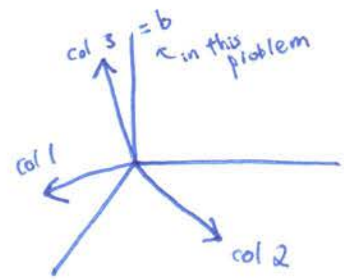
$$A = \begin{bmatrix}2 & -1 & 0\\-1 & 2 & -1\\0 & -3 & 4\end{bmatrix} \qquad b = \begin{bmatrix}0\\-1\\4\end{bmatrix}$$

Row Picturer



Column Picture

$$x\begin{bmatrix}2\\-1\\0\end{bmatrix} + y\begin{bmatrix}-1\\2\\-3\end{bmatrix} + z\begin{bmatrix}0\\-1\\4\end{bmatrix} = \begin{bmatrix}0\\-1\\4\end{bmatrix}$$



$x = 0$

$y = 0$

$z = 1$

Matrix * Vector = combination of columns

Can I solve $\underset{\text{matrix}}{A}\,\underset{\text{vector}}{x} = b$ for every b

aka Do the Linear combinations of the columns fill 3D Space

for this A = YES
- non singular / invertible

- Can do matrix multiplication by rows or columns.

1

$Ax = b$ ← matlab

$x = A \backslash b$

-bet matlab

Elimination $0, 1, \infty$ solutions?

Elimination matrices

Matrix Multiplication

$$AX = I = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$ ← identity

$X = A \backslash I = A^{-1}$ ← matlab

$AA^{-1} = I$

Elimination

$$c\begin{bmatrix}1\\3\\4\end{bmatrix} + d\begin{bmatrix}2\\5\\7\end{bmatrix} = \begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix} \quad \begin{bmatrix}1 & 2 & b_1\\3 & 5 & b_2\\4 & 7 & b_3\end{bmatrix} \rightarrow \begin{bmatrix}1 & 2 & b_1\\0 & -1 & b_2-3b_1\\0 & -1 & b_3-4b_1\end{bmatrix}$$

want to make it 0

pivot

$$\downarrow$$

$$\begin{bmatrix}1 & 2 & b_1\\0 & -1 & b_2-3b_1\\0 & 0 & \underbrace{b_3-4b_1-(b_2-3b_1)}_{b_3-b_2-b_1}\end{bmatrix}$$

$c + 2d = b_1$

$-d = b_2 - 3b_1$

$0 = b_3 - b_2 - b_1$

$z - y - x = 0$ → plane that satisfies the equation.

$[1, 1, -1]$ ← vector perpendicular to plane

Example

$$c\begin{bmatrix}1\\3\\8\end{bmatrix} + d\begin{bmatrix}2\\5\\14\end{bmatrix} = \begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix} \xrightarrow{\text{elim}} \begin{bmatrix}1 & 2 & b_1\\0 & -1 & b_2-3b_1\\0 & 0 & b_3-2b_2-2b_1\end{bmatrix}$$

$[2, 2, -1]$ ← perp

Elimination Matrices

$$\begin{bmatrix}1&0&0\\0&1&0\\0&-2&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\-3&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\-2&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&1&-1\\2&5&1\\3&9&4\end{bmatrix}\xrightarrow{E_{21}}\begin{bmatrix}1&1&-1\\0&3&3\\3&9&4\end{bmatrix}\xrightarrow{E_{31}}\begin{bmatrix}1&1&-1\\0&3&3\\0&6&7\end{bmatrix}$$

E_{32} E_{31} E_{21} A

Almost Like identity matrix

$$\downarrow E_{32}$$

$$\begin{bmatrix}1&1&-1\\0&3&3\\0&0&1\end{bmatrix}=U$$

Start w/ $A \to E_{21} \to E_{31} \to E_{32} \to U$

$$E_{32}\underbrace{(E_{31}E_{21})}_{\text{combine}}A=U$$

$$\underset{E_{32}}{\begin{bmatrix}1&0&0\\0&1&0\\0&-2&1\end{bmatrix}}\underset{E_{31}}{\begin{bmatrix}1&0&0\\0&1&0\\-3&0&1\end{bmatrix}}=\begin{bmatrix}1&0&0\\-2&1&0\\-3&0&1\end{bmatrix}$$

Matrix Multiplication

$$(AB)C = A(BC)$$

$$E_{32}E_{21} \neq E_{21}E_{32}$$

matrix mult not commutative

2/9/14 18.06 Lecture 2 Video

Elimination — Success / Failure

Back-substitution

Elimination Matrices

Matrix Multiplication

$$x + 2y + z = 2$$
$$3x + 8y + z = 12$$
$$4y + z = 2$$

$$Ax = b$$

pivot

$$\underset{A}{\begin{bmatrix} \boxed{1} & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix}} \xrightarrow[-(2,1)]{\times 3} \begin{bmatrix} \boxed{1} & 2 & 1 \\ 0 & \boxed{2} & -2 \\ 0 & 4 & 1 \end{bmatrix} \xrightarrow{3,2} \underset{U}{\begin{bmatrix} \boxed{1} & 2 & 1 \\ 0 & \boxed{2} & -2 \\ 0 & 0 & \boxed{5} \end{bmatrix}}$$

21

Elimination: Get rid of x in equation 2 by mutiplying eq 1 and subtracting

$A \to U$

Pivots can't be 0!

FUN FACT det(A) = Mutiply Pivots!

How can elimination Fail? bad!

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 1 \\ 0 & 4 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 \\ 0 & \boxed{0} & -2 \\ 0 & 4 & 1 \end{bmatrix}$$

can do row exchange

Back-substitution

augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 3 & 8 & 1 & 12 \\ 0 & 4 & 1 & 2 \end{array}\right] \to \left[\begin{array}{cccc} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 4 & 1 & 2 \end{array}\right] \to \left[\begin{array}{cccc} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 0 & 5 & -10 \end{array}\right]$$

A b U c

$\therefore Ux = c$

$$x + 2y + z = 2 \qquad x = 2$$
$$2y - 2z = 6 \qquad y = 1$$
$$5z = -10 \qquad z = -2$$

Elimination Matrices

$$\begin{bmatrix} & ? & \end{bmatrix}\begin{bmatrix}1&2&1\\3&8&1\\0&4&1\end{bmatrix} = \begin{bmatrix}1&2&1\\0&2&-2\\0&4&1\end{bmatrix}$$

What matrix takes care of step 1?
- subtract 3×row 1 from row 2

$$\begin{bmatrix}1&0&0\\-3&1&0\\0&0&1\end{bmatrix} \leftarrow E_{21}$$

Fixes 21 position - makes it 0

takes one of the third and none of the other

Identity Matrix would do nothing $\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$

Step 2: Subtract 2 × row 2 from row 3

$$E_{32}\ \begin{bmatrix}1&0&0\\0&1&0\\0&-2&1\end{bmatrix}\begin{bmatrix}1&2&1\\0&2&-2\\0&4&1\end{bmatrix} = \begin{bmatrix}1&2&1\\0&2&-2\\0&0&5\end{bmatrix}$$

$E_{32}(E_{21}A) = U$ ... Associative Law

★ - Can't change order but

$$\underbrace{(E_{32}E_{21})}_{\text{magic matrix that would take care of both}}A = U$$

Reminders

$$\begin{bmatrix}-&-&-\\-&-&-\\-&-&-\end{bmatrix}\begin{bmatrix}3\\4\\5\end{bmatrix} = \begin{matrix}3\times \text{col } 1 \cancel{+4\times\text{col}2+5}\times\text{col}3\\ +\,4\times\text{col}2\\ +\,5\times\text{col}3\end{matrix}$$

Matrix Vector Combination of column of Matrix

matrix·column = column

Also works with rows!

$$\begin{bmatrix}1&2&7\end{bmatrix}\begin{bmatrix}-&-&-\\-&-&-\\-&-&-\end{bmatrix}$$

1×3 Row matrix 3×3

How to check specific entry in matrix multiplication

$$\begin{bmatrix}1&0&0\\ \boxed{-3\ \ 1\ \ 0}\\0&0&1\end{bmatrix}\begin{bmatrix}1&2&\boxed{1}\\3&8&\boxed{1}\\0&4&\boxed{1}\end{bmatrix} = \begin{bmatrix}1&2&1\\0&2&?\\0&4&1\end{bmatrix}$$

dot product of row 2 and col 3

$-3(1)+1(1)+0(1) = -2$

Permutation Matrix

★ Exchange rows 1 and 2

$$P \rightarrow \begin{bmatrix} ? \end{bmatrix}\begin{bmatrix}a&b\\c&d\end{bmatrix} = \begin{bmatrix}c&d\\a&b\end{bmatrix}$$

$$P = \begin{bmatrix}0&1\\1&0\end{bmatrix}$$

Left = row operations

exchange rows of identity matrix

- Exchange col 1 and 2

$$\begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix} ? \end{bmatrix} = \begin{bmatrix}b&a\\d&c\end{bmatrix}$$

$$\begin{bmatrix}0&1\\1&0\end{bmatrix}$$

right = column operations

Inverses

$$\begin{bmatrix}1&0&0\\3&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\-3&1&0\\0&0&1\end{bmatrix} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \leftarrow I$$

matrix that will undo that step.

$$A^{-1}A = I$$

2/10/14 18.06 Lecture Fernando Trujano

1. Multiply Matrices
2. Solve $Ax = b$
3. Invert A

Multiply Matrices

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} P & Q \\ R & S \end{bmatrix} = \begin{bmatrix} & \cdot \\ & \end{bmatrix} \quad AQ+BS$$

$$\begin{bmatrix} \cdot & \cdot \\ & \end{bmatrix} \quad \underset{AP+BR}{A[P \ Q]} + \underset{AQ+BS}{B[R \ S]}$$

By columns

$$Q\begin{bmatrix} A \\ C \end{bmatrix} + S\begin{bmatrix} B \\ D \end{bmatrix} = \underset{\text{column}}{\begin{bmatrix} & \cdot \\ & \cdot \end{bmatrix}}$$

Column times row

$$\begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix} = \underset{\text{col 1}}{\begin{bmatrix} A \\ C \end{bmatrix}}[P \ Q] + \underset{\text{col 2}}{\begin{bmatrix} B \\ D \end{bmatrix}}\underset{\text{row 2}}{[R \ S]}$$

$2\times1 \cdot 1\times2 = 2\times2$

Solving $Ax = b$

pivot

$$\begin{bmatrix} 2 & 3 & 4 \\ 4 & 8 & 12 \\ 6 & 6 & 6 \end{bmatrix}\begin{matrix} b_1 \\ b_2 \\ b_3 \end{matrix} \rightarrow \begin{bmatrix} 2 & 3 & 4 \\ 0 & 2 & 4 \\ 0 & -3 & -6 \end{bmatrix}\begin{matrix} b_1 \\ b_2 - 2b_1 \\ \ \end{matrix}$$

* 0 can't be pivot

$$\begin{bmatrix} 2 & 3 & 4 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix} \text{ a singular matrix}$$

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

subtracts 2 — Reverse — Adds 2 back

$$\underset{\det = 12}{\begin{bmatrix} 2 & 3 & 4 \\ 4 & 6 & 10 \\ 6 & 6 & 7 \end{bmatrix}} \rightarrow \underset{\det = 12}{\begin{bmatrix} 2 & 3 & 4 \\ 0 & 0 & 2 \\ 0 & -3 & -5 \end{bmatrix}} \rightarrow \begin{bmatrix} 2 & 3 & 4 \\ 0 & -3 & 5 \\ 0 & 0 & 2 \end{bmatrix}$$

OH NO! There is a 0 on the pivot!

but you can switch the order of equations

det = multiply pivots
= -12

row exchange changes ~~order~~ sign of det.

1

Row exchange
P_{23}

$$\begin{bmatrix} & ? & \end{bmatrix}\begin{bmatrix} 2 & 3 & 4 \\ 0 & 0 & 2 \\ 0 & -3 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 4 \\ 0 & -3 & -5 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

- Do permutation to the identity matrix.

Inverses

$AA^{-1} = I$

$A^{-1}A = I$ n

$B = A^{-1}$ ① pivot test: n pivot $\neq 0$

② det $\neq 0$

③ If $Ax = 0$ for some $x \neq 0$ (nonzero vector)

Then A is singular

$$\underset{A}{\begin{bmatrix} 2 & 3 & -5 \\ 4 & -4 & 0 \\ 1 & 1 & -2 \end{bmatrix}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

2/11/13 — 18.06 Recitation — Fernando Trujano

Gonsalo Tabuada

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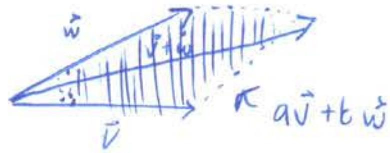


Linear combination of both vectors.

Now in 3D

(1,1,2) (1,2,3) (3,5,8)

write as linear combination of

$$a(1,1,2)+b(1,2,3)=(3,5,8) \qquad a=? \quad b=?$$

Brute Force

$$\begin{cases} a+b=3 \\ a+2b=5 \\ 2a+3b=8 \end{cases}$$

$a=1$
$b=2$

$$(1,1,2)+2(1,2,3)-(3,5,8)=(0,0,0)$$

$$\downarrow$$

$$a(1,1,2)+b(1,2,3)+c(3,5,8)=(0,0,0)$$

$a=?$ $b=?$ $c=?$

$$\begin{cases} a+b+3c=0 \\ a+2b+5c=0 \\ 2a+3b+8c=0 \end{cases} \iff \begin{bmatrix} 1 & 1 & 3 \\ 1 & 2 & 5 \\ 2 & 3 & 8 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \iff \begin{bmatrix} 1 & 1 & 3 & 0 \\ 1 & 2 & 5 & 0 \\ 2 & 3 & 8 & 0 \end{bmatrix}$$

Gauss Elimination

infinately many solutions

Triangulate: Make this 0

$$\begin{bmatrix} \boxed{1} & 1 & 3 & 0 \\ 1 & 2 & 5 & 0 \\ 2 & 3 & 8 & 0 \end{bmatrix} \xrightarrow{col2=col2-col1} \begin{bmatrix} 1 & 1 & 3 & 0 \\ 0 & \boxed{1} & 2 & 0 \\ 2 & 3 & 8 & 0 \end{bmatrix} \xrightarrow{col_3=col_3-2col_1} \begin{bmatrix} 1 & 1 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix} \xrightarrow{3: col3-col2} \begin{bmatrix} 1 & 1 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- switch rows
- add mults of rows

$$\begin{bmatrix} 1 & 1 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{cases} a+b+3c=0 \\ b+2c=0 \quad \therefore b=-2c \\ 0a+0b+0c=0 \end{cases}$$

$a-2c+3c=0$

can choose c

$$\{(-c,-2c,c) \mid c\}$$

Matrix Multiplication. -Different ways

$$m\Big[\overset{n}{\big[\ \big]}\big[\ \big] = \big[\ c_{34}\ \big]$$

$C_{34} = (\text{row } 3 \text{ of } A) \overset{\text{dot}}{\bullet} (\text{col } 4 \text{ of } B)$

$$\underset{m\times n}{A}\ \underset{n\times p}{B} = \underset{m\times p}{C}$$

rule

$$\underset{A}{[\ ]}\ \underset{B}{\overset{\text{col } 1}{[\ ]}} = \underset{C}{[\ ]}$$

$= A(\text{col } 1)$

Columns of C are combinations of columns of A

$$\underset{m\times 1}{\overset{\text{column } A}{\begin{bmatrix}2\\3\\4\end{bmatrix}}}\ \underset{1\times p}{\overset{\text{row } p}{\begin{bmatrix}1 & 6\end{bmatrix}}} = \begin{bmatrix}2 & 12\\3 & 18\\4 & 24\end{bmatrix}$$

rows multiples combinations of B

$AB =$ sum of (col A)(row B)

$$\begin{bmatrix}2 & 7\\3 & 8\\4 & 9\end{bmatrix}\begin{bmatrix}1 & 6\\0 & 0\end{bmatrix} = \begin{bmatrix}2\\3\\4\end{bmatrix}\begin{bmatrix}1 & 6\end{bmatrix} + \begin{bmatrix}7\\8\\9\end{bmatrix}\begin{bmatrix}0 & 0\end{bmatrix} = \begin{bmatrix}2 & 12\\3 & 18\\4 & 24\end{bmatrix}$$

<u>Block</u>

$$\underset{A}{\left[\begin{array}{c|c}A_1 & A_2\\ \hline A_3 & A_4\end{array}\right]}\ \underset{B}{\left[\begin{array}{c|c}B_1 & B_2\\ \hline B_3 & B_4\end{array}\right]} = \left[\begin{array}{c|c}A_1B_1 + A_2B_3 & \\ \hline & \end{array}\right]$$

Inverses

$$A^{-1}A = I = AA^{-1}$$

-For square matrix
left inverse = right inverse

if it exists

↳ invertable, non-singular

$$A = \begin{bmatrix}1 & 3\\2 & 6\end{bmatrix}$$

No inverse! because you can find $Ax = 0$ (vector x)

$$\begin{bmatrix}1 & 3\\2 & 6\end{bmatrix}\begin{bmatrix}3\\-1\end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$A \qquad A^{-1}$

$$A \times \text{col}_j \text{ of } A^{-1} = \text{col}_j \text{ of } I$$

Gauss-Jordan (solve 2 eqns at once)

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

augmented matrix

$$\left[\begin{array}{cc:cc} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right] \xrightarrow{\text{Elimination}} \left[\begin{array}{cc:cc} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right] \xrightarrow{\text{elimination}} \left[\begin{array}{cc:cc} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

$A \quad I \qquad\qquad I \quad A^{-1}$

$$E[A\,I] = [I\ \ ?]$$

$EA = I$ tells us $E = A^{-1}$

$$E[A\,I] = [I\,A^{-1}]$$

Factorization into $A = LU$

$AA^{-1} = I = A^{-1}A$

$(AB)(B^{-1}A^{-1}) = I$

$B^{-1}A^{-1}AB = I$

$AA^{-1} = I$

$(A^{-1})^T A^T = I$

↑ This is $(A^T)^{-1}$

$$\underset{E_{21}}{\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}} \underset{A}{\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}} = \underset{U}{\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}}$$

$E_{21}A = U$

$L = E_{21}^{-1} \leftarrow A = \frac{1}{E_{21}} U$

lower triangular, upper triangular

$$A = L\,U$$

$$\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$$

inverse of E_{21}

$E_{32}E_{31}E_{21}A = U$ (no row exchanges)

$A = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}U$

$= L \quad U$

$EA = U$

$$\underset{E_{32}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix}} \underset{E_{21}}{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 10 & -5 & 1 \end{bmatrix} = E \text{ (left of } A)$$

not nice

reverse order!

$$\underset{E_{21}^{-1}}{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{E_{32}^{-1}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix} = L$$

add 2 instead of subtracting

inverse!

nice!

$A = LU$

$A = LU$

If no row exchanges, multipliers go directly into L

How many operations (multiply + subtract) on an $n \times n$ matrix A?

$\boxed{n = 100}$

about 100^2 → about 99^2 → $\Rightarrow n^2 + (n-1)^2 + (n-2)^2 \dots 1$

$\approx \boxed{\frac{1}{3}n^3}$ ← on A $\qquad n^2$ ← on b

What if there were row exchanges?
↳ zero in the pivot position

Transposes and Permutations

3×3 Permutation Matrixes 6 P's

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}, \begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}, \begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}, \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}, \begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}, \begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}$$

- Multiply any two together or inverse and get another in the list. $P^{-1} = P^T$

4×4
Permutation Matrixes 24 P's

2/14/14 18.06 Lecture Fernando Trujano

Transpose $(Ax)^T y$
Symmetric
Permutation
Rotation
Orthogonal

Transpose

$$(Ax)^T y = x^T (A^T y)$$

Permutations

- Row exchanges

$P_{12} \rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \rightarrow$ odd only switches 2 rows

$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow$ even # of row exchanges

Elimination For an invertable matrix

1. Hope for $A = LU$ - no zero pivots - no exchanges
2. With exchange $PA = LU$ - with exchanges
3. Singular ... $\det(A) = 0$

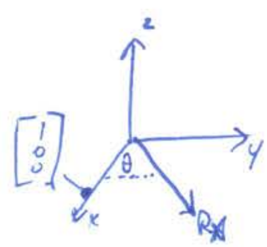
Symmetry switch rows and columns $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}^T = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$ $(AB)^T = B^T A^T$

$$A^T = A$$

Show that $A^T A$ is always symmetrical.

$$(A^T A)^T = A^T A^{TT} = A^T A$$

Rotation



$R = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ rotation in xy plane $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$R^T R = I$$
$$R^T = R^{-1}$$

Orthogonal

- the transpose is inverse

$$A^T = A^{-1}$$

$$Q^T Q = I$$

$$\begin{bmatrix} — u^T \\ — v^T \\ — w^T \end{bmatrix} \begin{bmatrix} ① & ② & ③ \\ | & | & | \\ u & v & w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$u^T u = 1$
$v^T v = 1$
$w^T w = 1$
unit vector B

$u^T v = u^T w = v^T w = 0$
perpendicular

2/2

18.06 Lecture 5 ocw

Transposes, Permutations, Spaces R^n

Permutations P: execute row exchanges

$$A = LU = \begin{bmatrix} 1 & & 0 \\ \cdot & 1 & \\ \cdot & \cdot & 1 \end{bmatrix} \begin{bmatrix} \diagdown & \cdot & \cdot \\ 0 & \diagdown & \cdot \\ & & \diagdown \end{bmatrix}$$

$\downarrow$

$PA = LU$ // any invertable A

Permutations

P = identity matrix w/ reordered rows

$n!$ possible reorderings

↳ all $n \times n$ permutation

$$P^{-1} = P^T \rightarrow P^T P = I$$

Transpose

$$\begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 3 & 1 \end{bmatrix}$$

- Rows become columns

$$\Rightarrow (A^T)_{ij} = A_{ji}$$

Symmetric Matrix

- $A^T = A$
- Transposing does not change the matrix

Ex $\begin{bmatrix} 3 & 1 & 7 \\ 1 & 2 & 9 \\ 7 & 9 & 4 \end{bmatrix}$

$$R = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 3 & 1 \end{bmatrix} \quad R^T = \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}$$

$R^T R$ = Always symmetric!!

$$R^T R = \begin{bmatrix} 10 & 11 & 7 \\ 11 & \cdot & \cdot \\ 7 & \cdot & \cdot \end{bmatrix}$$

Why? → Take Transpose

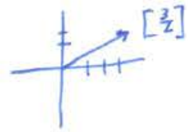
$$\underbrace{(R^T R)^T}_{\text{reverse}} = R^T \underbrace{R^{TT}}_{\text{R cancels out}}$$

$$= R^T R$$

Vector Spaces ← a bunch of vectors

Examples: R^2 = all 2D real vectors $\begin{bmatrix}3\\2\end{bmatrix}\begin{bmatrix}0\\0\end{bmatrix}\begin{bmatrix}\pi\\e\end{bmatrix}$

"x-y plane"



- can operate and stay within same space

R^3 = all 3D vectors

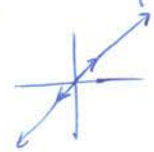
∴ vectors w/ 3 real components

$\begin{bmatrix}3\\2\\0\end{bmatrix}$

R^n = all column vectors with n components

 = not a vector space
- can't mutiply by anything and stay in space.

vectorspace inside R^2 => subspace of R^2



- line in R^2 through zero vector

- Every subspace must contain the zero vector
  - Have to be able to multiply by 0 and stay in space

Subspaces of R^2

① All of R^2

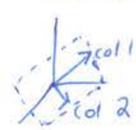
② Any line through $\begin{bmatrix}0\\0\end{bmatrix}$ ~ L

③ zero vector only ~ Z

☆ $A = \begin{bmatrix}1 & 3\\2 & 3\\4 & 1\end{bmatrix}$ columns in R^3

↳ all their linear combinations form a subspace

~ Column Space C(A)



= Plane through origin!

2/2

18.06 Lecture 6 OCW

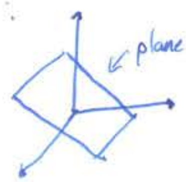
Column Space and Nullspace

- Vector spaces and subspaces
- Columnspace of A
- Nullspace of A

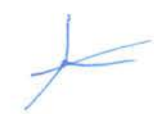
Vector Space

- $v+w$ and cv are in the space
- all combinations $cv+dw$ are in the space.

Examples:



plane through origin. $\begin{bmatrix}0\\0\\0\end{bmatrix}$ - Subspace of R^3



ℓ - Subspace of R^3

2 subspaces: P and L

$P \cup L$ = all vectors in P or L or both
union

This is not a subspace!

$P \cap L$ = all vectors in both P and L
intersection — Is a subspace

Subspaces S and T
Intersection $S \cap T$ is a subspace!

Columnspace of A - 4×3

↳ a subspace of R^4 $C(A)$

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}$$ = all linear combinations of columns

Does $\underbrace{Ax=b}_{\text{4 equations 3 unknowns}}$ have a solution for every b? NO

↓

1/2

$$A x = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

Which b's allow this system to be solved ??

$b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ etc

$x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ in

Can solve $Ax = b$ exactly when b is in $C(A)$

in R^4

col 3 = col 1 + col 2

↳ not useful

vector space

Nullspace of A

in R^3

- All solutions $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ to $Ax = 0$

$$A x = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$N(A)$ contains $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix}$

$c \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

Check that solutions to $Ax = 0$ always give a subspace

- If $Av = 0$ and $Aw = 0$

then $A(v+w) = 0$

$Av + Aw$ ✓

then $A(12v) = 0$

$12Av = 0$

line in R^3 through origin

$$A x = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

zero vector does not solve!

Solutions do NOT form a subspace.

$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

18.06 Lecture 7 ocw

Computing the Nullspace ($Ax=0$)
Pivot variables - free variables
Special Solutions $\quad rref(A) = R$

$m \times n$

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \xrightarrow{\text{Elimination}} \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

Echelon

Free colums

Pivot Colums

row 3 is a combination of other rows.

rank of A = # of pivots
= 2

$n - r = 4-2$ Free Variables

$Ux = 0$

- Can assign anything to free columns x_2, x_4

$x_1 + 2x_2 + 2x_3 + 2x_4 = 0$

$2x_3 + 4x_4 = 0$

$x \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

set to 0

$x_3 = 0$

$$x = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$x = c\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$ Line in the null space

OR

another solution

$$x = d\begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$ - set different free variables and solve

Special solutions

Solution:

$$x = c\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d\begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

- Nullspace contains all combinations of the special solutions

R = reduced row echelon Form
- zeroes above and below pivots
- pivots = 1

$$U = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{1} & 2 & 0 & -2 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R = rref(A)$$

matlab

notice $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ in pivot rows/colums

$$I \quad \left[\begin{array}{cc|cc} 1 & 0 & 2 & -2 \\ 0 & 1 & 0 & 2 \end{array}\right] \; F$$

Pivot cols — Free cols

0 0 0 0

$Rx = 0$

rref Form

$$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} \leftarrow r \text{ pivot rows}$$

r pivot cols — n - r free cols

nullspace matrix $N = \begin{bmatrix} -F \\ I \end{bmatrix}$

(columns = special solution)

$Rx = 0$

$$\begin{bmatrix} I & F \end{bmatrix} \begin{bmatrix} x_{pivot} \\ x_{free} \end{bmatrix} = 0$$

$$x_{pivot} I + F x_{free} = 0$$

$$\boxed{x_{pivot} = -F x_{free}}$$

Example

Transpose of first A

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 6 & 8 \\ 2 & 8 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{1} & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 4 & 4 \end{bmatrix} \xrightarrow{\text{row exchange}} \begin{bmatrix} \boxed{1} & 2 & 3 \\ 0 & \boxed{2} & 2 \\ 0 & 0 & 0 \\ 0 & 4 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{1} & 2 & 3 \\ 0 & \boxed{2} & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

2 pivot cols

U — 1 Free column

r = 2 again

3 - 2 = 1 Free col

$$x_1 + 2x_2 + 3x_3 = 0$$
$$2x_2 + 2x_3 = 0$$

$$x = c\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

subtract row1 = row1 - row2

divide row2 by 2

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = R$$

I — F

$$N = \begin{bmatrix} -F \\ I \end{bmatrix} = c\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

18.06 Lecture 8 ocw

Solving $Ax = b$: Row Reduced Form R

Complete solution of $Ax=b$

Rank r

$r = m$: Solution exists $r = n$: Solution is unique

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$x_1 + 2x_2 + 2x_3 + 2x_4 = b_1$$
$$2x_1 + 4x_2 + 6x_3 + 8x_4 = b_2$$
$$3x_1 + 6x_2 + 8x_3 + 10x_4 = b_3$$

Intuition
$b_3 = b_1 + b_2$

$$\left[\begin{array}{cccc|c} \boxed{1} & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{array}\right] \rightarrow \begin{bmatrix} \boxed{1} & 2 & 2 & 2 & b_1 \\ 0 & 0 & \boxed{2} & 4 & b_2 - 2b_1 \\ 0 & 0 & 2 & 4 & b_3 - 3b_1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - 3b_1 - b_2 - 2b_1 \end{bmatrix}$$

Augmented Matrix $= [A \; b]$

$$\downarrow$$

$$\begin{bmatrix} \boxed{1} & 2 & 2 & 2 & b_1 \\ 0 & 0 & \boxed{2} & 4 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_1 - b_2 \end{bmatrix} \rightarrow 0 = b_3 - b_1 - b_2$$

Soluability: Condition on right hand side

$Ax = b$ solvable when b is in $C(A)$

- b has to be a combination of cols

IF a combination of rows of A gives zero row then same combination of entries of b must give 0.

Pick an OK $b = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}$

Pivot cols

$$U = \begin{bmatrix} \boxed{1} & 2 & 2 & 2 & 1 \\ 0 & 0 & \boxed{2} & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

x_2, x_4 = Free variables

To Find complete solution to $Ax = b$

① $X_{particular}$: Set all free variables to 0 → $x_2 = 0$, $x_4 = 0$
Solve $ax = b$ for pivot variables

$$x_1 + 2x_3 = 1$$
$$2x_3 = 3$$

$x_3 = 3/2$
$x_1 = -2$

$$x_p = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix}$$

② $x_{nullspace}$

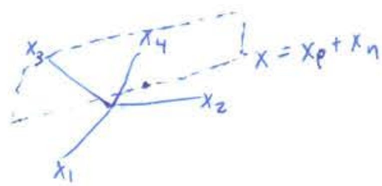
complete solution: $x_p + x_n$

$$A x_p = b$$
$$A x_n = 0$$
$$A(x_p + x_n) = b$$

$$x_{complete} = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix} + \underbrace{c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}}_{\text{Nullspace}}$$

Nullspace
↳ 2D subspace

Plot all solutions x in R^4



set of solutions to $Ax = b$ does not form a subspace
- Shifted away from the origin
- Does not go through 0

$m \times n$ matrix A of rank r (# of pivots) (know $r \leq m$, $r \leq n$)

Full column rank → $\boxed{r = n}$
- n pivot variables → no free variables

$N(A) = \begin{bmatrix} \text{zero} \\ \text{vector} \end{bmatrix} = [0]$

Solution to $Ax = b$: $x = x_p$
- unique solution if it exists

$\boxed{\text{(0 or 1 solutions)}}$

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 6 & 1 \\ 5 & 1 \end{bmatrix} \rightarrow R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$r = 2$

Full row rank → $\boxed{r = m}$
- m pivots → $n-r$ Free variables
- can solve $Ax = b$ for every b

$\boxed{\text{Can Always Solve}}$

$$A = \begin{bmatrix} 1 & 2 & 6 & 5 \\ 3 & 1 & 1 & 1 \end{bmatrix}$$

$r = 2$

$$R = \begin{bmatrix} \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} & F \end{bmatrix}$$

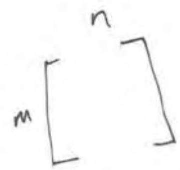
2/13

Full Rank
$r = m = n$
- invertible

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$$

$$R = I$$

$$N = [0]$$



- No conditions on b
- unique solution

$r = m = n$	$r = n < m$	$r = m < n$ $\quad \frac{I}{}F$	$r < m,\ r < n$
$R = I$	$R = \begin{bmatrix} I \\ 0 \end{bmatrix}$	$R = \begin{bmatrix} I & F \end{bmatrix}$	$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$
1 solution	0 or 1 solution	∞ solutions	0 or ∞ solutions

- Rank tells you everything about the # of solutions

18.06 Lecture 9 acw

Independence, Basis and Dimension

- Linear Independence
- Spanning a space
- BASIS and dimension

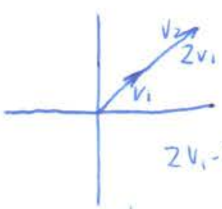
- Suppose A is m by n with $m < n$
  - Then there are non-zero solutions to $Ax = 0$ There is something in the nullspace
    (more unknown than equations)

  Reason: There will be free variables!

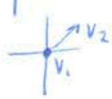
Independence

- Vectors $x_1, x_2 \dots x_n$ are linearly independent if no combination gives the zero vector. (all $c_n \neq 0$)

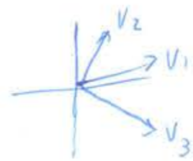
$$c_1 x_1 + c_2 x_2 + \dots c_n x_n \neq 0$$



$2v_1 - v_2 = 0 \rightarrow$ dependent



$0v_2 + 9001 v_1 = 0$



$\Rightarrow$ dependent

$$A = \begin{array}{c} \begin{matrix} v_1 & v_2 & v_3 \end{matrix} \\ \begin{bmatrix} 2 & 1 & 2.5 \\ 1 & 2 & -1 \end{bmatrix} \end{array} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

2×3

Repeat when $v_1, \dots v_n$ are cols of A
- Independent if nullspace of A is only zero vector   no free vars, rank $= n$
- Dependent if $Ac = 0$ for some nonzero c rank $< n$

Spanning a space

- Vector $v_1 \dots v_\ell$ span a space
  - $\rightarrow$ Space consists of all combs of those vectors

Basis for a space is a sequence of vectors $v_1, v_2, \dots v_d$

① They are independent

② They span the space.

Example:

Space is $\mathbb{R}^3$ standard basis → x, y, z vector

One basis is $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$

- Independent

Another basis $\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}2\\2\\5\end{bmatrix}, \begin{bmatrix}3\\3\\8\end{bmatrix}$? nope / 2 rows are equal → not invertible

R^n n vectors give basis if $n \times n$ matrix with cols is invertable → not singular

Given a space:

Every basis for the space has the same # of vectors

→ Dimension of the space

Definitions

Independence - combinations not 0

Spanning - all the combinations

Dimension of space - # of vectors in any basis

Space is $C(A)$

$$\begin{bmatrix}1&2&3&1\\1&1&2&1\\1&2&3&1\end{bmatrix}$$

Pivot cols

Form a basis $\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}2\\1\\2\end{bmatrix}$

solution to $Ax = 0$

$N(A) = \begin{bmatrix}-1\\-1\\1\\0\end{bmatrix}$

Another basis $\begin{bmatrix}2\\2\\2\end{bmatrix}, \begin{bmatrix}7\\5\\7\end{bmatrix}$

$2 = \text{rank}(A) = \#$ of pivot cols = dimension of column space $C(A)$

not $C(A)$ / not (A)

$\boxed{\dim(C(A)) = r}$

Nullspace

$\begin{bmatrix}-1\\-1\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\0\\1\end{bmatrix}$

2 special solutions

- Free vars col 3 and 4 x_3 x_4

$\dim N(A) = \#$ free variables $= n - r$

$\boxed{\dim N(A) = n - r}$

18.06 Lecture 10 ocw

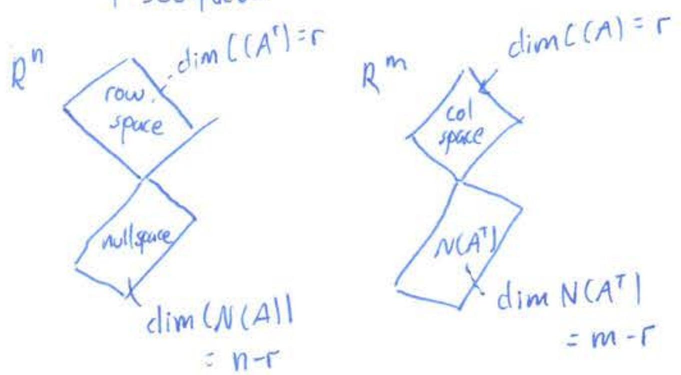
Four Fundamental subspaces (for matrix A)

A is $m \times n$

① Column Space $C(A)$ .. in $\mathbb{R}^m$
② Nullspace $N(A)$ .. in $\mathbb{R}^n$
③ rowspace $C(A^T)$ .. in $\mathbb{R}^n$
- all combinations of rows of A = all combinations of colums of A^T $C(A^T)$
- rows span rowspace
- rows are basis for rowspace if independent

④ Nullspace of $A^T = N(A^T)$ = left nullspace of A .. in $\mathbb{R}^m$

4 subspaces



	$C(A)$	$C(A^T)$	$N(A)$	$N(A^T)$
basis ?	pivot cols		special solutions	
dimension ?	r	r	$n-r$ (Free vars)	$m-r$

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{array}{c} \quad I \qquad F \\ \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} = R$$

$C(R) \neq C(A)$
$C(R^T) = C(A^T)$

Combinations of vectors in R can get you back to A

Basis for row space is first r rows of R

combination of rows still in the row space.
$\Rightarrow$ Elimination does not change row space

④ Nullspace of A^T

$(A^T y = 0)^T$

$$\left(\begin{bmatrix} \quad \end{bmatrix} \begin{bmatrix} \; \end{bmatrix} = 0 \right)^T \qquad y^T A^{TT} = 0^T$$

$$y^T A$$

$$\begin{bmatrix} y^T \end{bmatrix} \begin{bmatrix} \quad \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

left

Gauss Jordan

$$E \; \cancel{\text{rref}} \begin{bmatrix} A_{n \times m} & I_{m \times m} \end{bmatrix} \rightarrow \begin{bmatrix} R_{m \times n} & E_{m \times n} \end{bmatrix}$$

$$EA = R$$

$A \to R$ to identity

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} -1 & 2 & 0 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = E$$

$$EA = R$$

$$\begin{bmatrix} -1 & 2 & 0 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

combinations of rows to get zero row

New vector space! M $\quad A + B,\ cA$ (not AB for now)

All 3×3 matrices

Subspaces of M:

- All upper triangular matrices
- All symmetric matrices $\quad$ Dim?
- Diagonal Matrices

D

$\dim(D) = 3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

3/5/14

18.06 Lecture Exam Review

Midterm Friday 11-12 in 10-250

* Emphasize Chapter 3

$C(A)$, $N(A)$, 4 subspaces – bases – dimension – rank

$R = \text{rref}(A)$

- Networks and Graphs → not on quiz

Example

not in column space

$Ax = \begin{bmatrix} | \\ | \\ | \end{bmatrix}$ no solution

$Ax = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ only 1 solution

in column space

nothing in the nullspace

$N(A) = [0]$

columns are independent

$r = n < 3$

① m, n, r of A:

$m = 3$

$r < 3$

$A = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

② Find all solutions to $Ax = 0$

$A_{5\times3} = \begin{bmatrix} & \\ & \end{bmatrix}$

Is $b = \begin{bmatrix} | \\ | \\ | \end{bmatrix}$ a combination of columns of a?

How to decide? = Does $Ax = b$ have solution

- Augment A with b
- Elimination → Reduce

$[A\ b] \to [R\ d] = \begin{bmatrix} = & = & d_1 \\ = & = & d_2 \\ & & \vdots \\ 0 & 0 & d_5 \end{bmatrix}$

$d_5 = 0$

For rows?

$\begin{bmatrix} = \\ = \\ ||||| \end{bmatrix} A^T \xrightarrow{\text{Elimination}}$ if last row $= [0\ 0\ 0\ 0\ 0] \to$ YES

else → no

$$A = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 3 & c & 2 & 8 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

For each c
Find
Basis For $C(A)$

$$\downarrow$$

$$\begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & c-3 & -4 & -4 \\ 0 & 0 & 2 & 2 \end{bmatrix}_{/2} \rightarrow \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & c-3 & -4 & -4 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

If $c \neq 3$
First 3 cols → basis

If $c = 3$
1^{st}, 3^{rd}

If $A_{3\times5}$ what do you know about $N(A)$

Must have dimension → at least 2

know $r \le 3$

$5 - r \ge 2$

Basis for space of all vectors in R^4 with $x_1 + x_2 + x_3 + x_4 = 0$

hyperplane

H

$$\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

Independent

rowspace perpendicular to H

Find a matrix ~~that~~

$C(A) = H$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$C(B) = H$

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$$

$Bx = 0$

$x_1 + x_2 + x_3 + x_4 = 0$

The only solution to $Ax = 0$ is $x = 0$
aka The columns are independent

Rank = n
Nullspace = [0]

$$A = \begin{bmatrix} \boxed{1} & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 6 & 8 \\ 2 & 8 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 2 & 6 & 8 \\ 2 & 8 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{1} & 2 & 3 \\ 2 & \boxed{6} & 8 \\ 0 & 0 & 0 \\ 2 & 8 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 2 & 6 & 8 \\ 0 & 0 & 0 \\ 0 & 2 & 2 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} \boxed{1} & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 4 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{1} & 2 & 3 \\ 0 & \boxed{2} & 2 \\ 0 & 0 & 0 \\ 0 & 4 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{cc} I & F \end{array}$$
$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \qquad \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Special Solutions

$$N(A) = c\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + d\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} \boxed{1} & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{1} & 2 & 2 & 2 & b_1 \\ 0 & 0 & \boxed{2} & 4 & b_2 - 2b_1 \\ 0 & 0 & 2 & 4 & b_3 - 3b_1 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 2 & 0 & -2 & b_1 - (b_2 - 2b_1) \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - 3b_1 - (b_2 - 2b_1) \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} \boxed{1} & 2 & 0 & -2 & \underline{b_1 - b_2 + 3b_1} \\ 0 & 0 & \boxed{1} & 2 & (b_2 - 2b_1)(.5) \\ 0 & 0 & 0 & 0 & \overline{b_3 - b_1 - b_2} \end{bmatrix}$$

Complete solution $Ax = b$

$x_p + x_n$

$\underline{x_p}$

Say $b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

- Set Free vars $= 0$

$x_2 = x_4 = 0$

$1x_1 + \cancel{2x_2} + -\cancel{2x_4} = 1$

$x_3 + \cancel{2x_4} = 0$

$$\downarrow$$

$$\begin{bmatrix} 1 & 2 & 0 & -2 & 1 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_p = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$N(A) = c\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix}$$

$$\text{Solution} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix}$$

c)

$$Ax = 0$$

$$\begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{1} & 0 & 5/2 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$x = \begin{bmatrix} -5/2 \\ 0 \\ 1 \end{bmatrix}$$

a)

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 2 & 4 & 5 \\ -2 & 0 & -5 \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\left[\begin{array}{cccc} \boxed{2} & 3 & 5 & b_1 \\ 2 & 4 & 5 & b_2 \\ -2 & 0 & -5 & b_3 \end{array}\right] \rightarrow \left[\begin{array}{cccc} 2 & 3 & 5 & b_1 \\ 0 & \boxed{1} & 0 & b_2 - b_1 \\ 0 & 3 & 0 & b_3 + b_1 \end{array}\right] \rightarrow \left[\begin{array}{cccc} 2 & 3 & 5 & b_1 \\ 0 & 1 & 0 & b_2 - b_1 \\ 0 & 0 & 0 & (b_3 + b_1) - 3(b_2 - b_1) \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{cccc} 2 & 3 & 5 & b_1 \\ 0 & 1 & 0 & b_2 - b_1 \\ 0 & 0 & 0 & b_3 + 4b_1 - 3b_2 \end{array}\right]$$

$$U = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad E = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 4 & -3 & 1 \end{bmatrix}$$

$$EA = U$$

$$A = E^{-1}U$$

$$E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ & 3 & 1 \end{bmatrix}$$

$$EE^{-1} = I$$

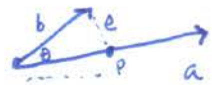
$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 4 & -3 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 3 & 1 \end{bmatrix}}_{L} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

b)

$c(A)$ $\quad b_3 + 4b_1 - 3b_2 = 0$

3/10/14 18.06 Lecture

Orthogonal / Projections



Projection p of b on line through a

$$\|p\| = \|b\|\cos\theta = \|b\| \frac{b^T a}{\|a\|\,\|b\|}$$

$\sqrt{a^T a}$

$|\cos\theta| \le 1 \rightarrow |b^T a| \le \|a\|\,\|b\|$

$|dot| \le \|a\|\,\|b\|$

$$p = \underbrace{\frac{b^T a}{\|a\|}}_{x \text{ length}} \underbrace{\frac{a}{\|a\|}}_{\text{unit vector}} = \underbrace{\frac{b^T a}{a^T a}}_{x} a$$

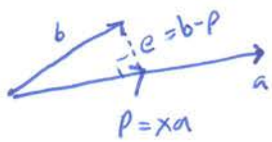
$$\begin{bmatrix} 2 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = 12 \le \sqrt{13}\sqrt{13}$$

$\sqrt{2^2+3^2}$

$$\boxed{p = \frac{b^T a}{a^T a} a}$$

|— Cauchy Schwartz —| Inequality

Different Derivation



- Distance from b to line is $\|e\|$

$e^T a = 0$

e is perpendicular to a

$b^T a - p^T a = 0$

$b^T a - x a^T a = 0$

$$x = \frac{b^T a}{a^T a}$$

$p = xa$

Orthogonal vectors $x^T y = 0$

orthogonal subspace: every x in V (subspace) perp. (orthogonal) to every y in W (sp)

orthogonal complement: start w/ V, construct $V^\perp$ "V perp"

all vectors that are orthogonal to V

- $C(A) = N(A^T)^\perp$
- orthogonal complement of 0 in R^3 = R^3

rowspace - $C(A^T) = N(A)^\perp$

Orthogonal Basis for V

$v_1, v_2, v_3, \dots v_d$ are orthogonal

+ orthonormal – scales all v to length 1

$$v^T v = 1$$

Orthogonal Matrix Q

orthogonal basis: $q_1, q_2, q_3, \dots q_n$

$$\begin{bmatrix} | & | & | & & | \\ q_1 & q_2 & q_3 & \cdots & q_n \\ | & | & | & & | \end{bmatrix}$$

- Identity Matrix is orthogonal
- Rotation

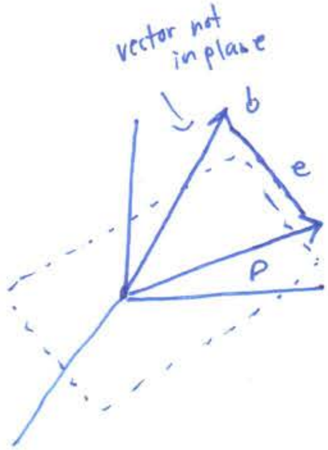
$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Why are orthogonal matrices so good?

$$\begin{bmatrix} - q_1 - \\ - q_2 - \\ - q_3 - \end{bmatrix} \begin{bmatrix} q_1 & q_2 & q_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

☆ $$\boxed{Q^T Q = I}$$

$$Q^T = Q^{-1}$$



e is orthogonal to p and the plane

- Given a basis for a subspace a_1, a_2

combination of basis

$$p = \underbrace{x_1 a_1 + x_2}_{?} a_2$$

3/12/13

18.06 Lecture

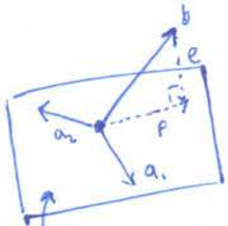
Fernando Trujano

Projections onto $C(A)$ + Projection matrix P

$p = Pb$

Pset 4 Hints

Start least squares



basis of the plane

$$p = \underbrace{x_1 a_1 + x_2 a_2}_{\text{Find}} = A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

p is a combination of $a_1 + a_2$

Key: $e \perp a_1 \quad e \perp a_2$

$p + e = b$

Plane is columnspace of $A = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots \\ | & | & | & \end{bmatrix}$

$a_1^T e = 0$

$a_2^T e = 0$

$$\begin{bmatrix} a_1^T & \text{---} \\ a_2^T & \text{---} \end{bmatrix} \begin{bmatrix} | \\ b - A\hat{x} \\ | \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

e

p

$$A^T(b - A\hat{x}) = 0$$

$$\boxed{A^T A \hat{x} = A^T b}$$

normal equation

$$\underbrace{A^T A}_{\text{invertible}} = \begin{bmatrix} a_1^T \\ a_2^T \end{bmatrix} \begin{bmatrix} a_1 & a_2 \end{bmatrix} = \begin{bmatrix} a_1^T a_1 & a_1^T a_2 \\ a_2^T a_1 & a_2^T a_2 \end{bmatrix}$$

$$\hat{x} = (A^T A)^{-1} A^T b$$

4A Formula $\rightarrow$

$$p = A\hat{x} = \underbrace{A(A^T A)^{-1} A^T}_{P} b$$

P projection matrix

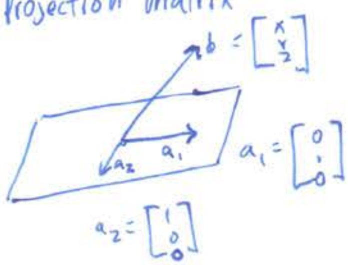
<u>1D</u> imention

$$p = \frac{b^T a}{a^T a} a$$

onto line through

$$p = A\hat{x} = Pb$$

Projection matrix P



$$P\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad P = \underset{3\times 2}{A}\,\underset{2\times 2}{(A^TA)^{-1}}\,\underset{2\times 3}{A^T}$$

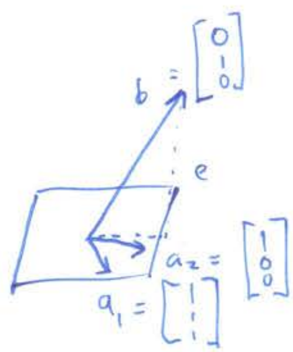
$P = 3 \times 3$

$$P^2 b = P(Pb) = Pp = p$$

- Nothing happens when you project more than once

$\boxed{P^2 = P}$ $\quad P^T = P$

↑ symmetric



Find A^TA, p, P

① $A^TA = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$

② $p = A^TA\hat{x} = A^Tb$

$\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \hat{x} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$3x_1 + x_2 = 1$

$x_1 + x_2 = 0$

$x = 1/2$

$x_2 = -1/2$

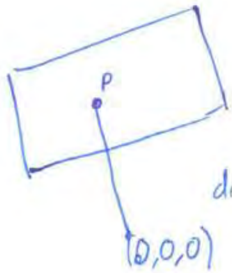
$$p = x_1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/2 \\ -1/2 \end{bmatrix}$$

check $e = b - p = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1/2 \\ -1/2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/2 \\ -1/2 \end{bmatrix}$

det

③ $P = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}}_{A} \underbrace{\frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}}_{(A^TA)^{-1}} \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}}_{A^T} = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

check $P^2 = P$



$$Ax + By + Cz = D$$

on the plane ⟶ solves equation

$$p = \alpha \begin{bmatrix} A \\ B \\ C \end{bmatrix}$$

Perpendicular to plane

$$\begin{bmatrix} A & B & C \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} D \end{bmatrix}$$

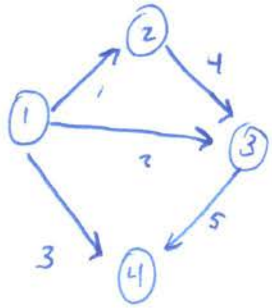
$$\alpha (A^2 + B^2 + C^2) = D$$

$$p = \frac{D}{A^2+B^2+C^2} \begin{bmatrix} A \\ B \\ C \end{bmatrix}$$

$$\|p\| = \frac{|D|}{\sqrt{A^2+B^2+C^2}}$$

3/11/14

18.06 Recitation



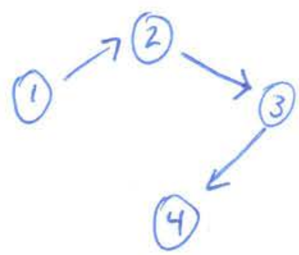
Incidence Matrix

$$\begin{array}{c} \begin{array}{cccc} ① & ② & ③ & ④ \end{array} \\ \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \end{array}$$

Arrow points from 3 to 4

Gauss Elimination $\longrightarrow$

$$\begin{bmatrix} \boxed{-1} & 1 & 0 & 0 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{-1} & 1 & 0 & 0 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & \boxed{-1} & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \rightarrow \begin{array}{c} \begin{array}{cccc} ① & ② & ③ & ④ \end{array} \\ \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

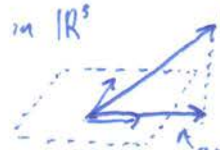


Problem 7 ✓ on HW

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{A_{4\times 3}}$$

a) Project $b = (1,2,3,4)$ onto the column space of A

suppose in $\mathbb{R}^3$



(a,b,c)
$\downarrow$
$(a,b,0)$

Two vectors make a plane in $\mathbb{R}^3$

Linear combinations of cols

$$a\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + b\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + c\begin{pmatrix}0\\0\\1\\0\end{pmatrix} \cancel{two} = \begin{pmatrix}a\\b\\c\\0\end{pmatrix}$$

$$= [1, 2, 3, 0]$$

b) Write the projection matrix P

$$P \Rightarrow \begin{bmatrix} & ? & \\ & & \end{bmatrix}_{4\times 4} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x - 3y - 4z = 0$ Plane P in $\mathbb{R}^3$

$A = ?$

a) P = nullspace of matrix $A = ?$

Lives in

$$Av = 0 \iff v \in P$$

has 3 components

$$A_{1\times 3} \begin{bmatrix} \\ \end{bmatrix}_{3\times 1} = 0_{1\times 1}$$

From equation $\rightarrow$

$$\underset{a}{\begin{bmatrix} 1 & -3 & -4 \end{bmatrix}_{1\times 3}} \begin{bmatrix} \\ \\ \end{bmatrix}_{3\times 1} = 0$$

b) Find 2 special solutions

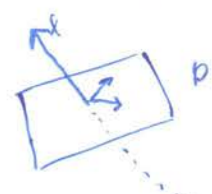
$$x = 3y + 4z$$

$$\{ (3y+4z, y, z) \mid y, z \in \mathbb{R} \} = \langle (3,1,0), (4,0,1) \rangle$$

$y = 1$ and $z = 0 \rightsquigarrow (3,1,0)$

$y = 0$ and $z = 1 \rightsquigarrow (4,0,1)$

c) Find basis for the line $\ell \perp P$



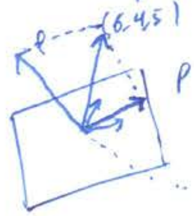
$(?,?,?) \perp (3,1,0)$

$(?,?,?) \perp (4,0,1)$

$\rightarrow (1,-3,4)$ ← From equation

$\langle (1,-3,4) \rangle$

☆ d) Split $v = (6,4,5)$ into $v_p \in P$ and $v_\ell \in \ell$



$(3,1,0)$ $\quad (4,0,1)$ $\quad (1\ -3\ -4)$

Generate a plane $\qquad$ Perpendicular to others

- All three are linearly independent
  - unique a, b, c that makes this work

$$(6, \overset{4}{\cancel{8}}, \overset{5}{\cancel{4}}) = a(3,1,0) + b(4,0,1) + c(1,-3,-4)$$

$$\begin{bmatrix} 3 & 4 & 1 \\ 1 & 0 & -3 \\ 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 5 \end{bmatrix}$$

$a = 1$
$b = 1$
$c = -1$

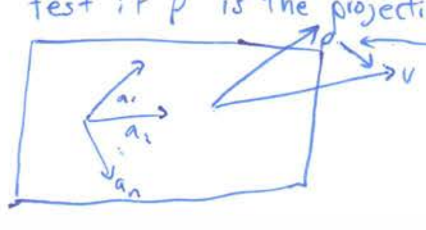
$$(6,4,5) = \underbrace{(3,1,0) + (4,0,1)}_{v_p} + \underbrace{-(1,-3,-4)}_{v_\ell}$$

$v_p = (7,1,1)$
$v_\ell = (-1,3,4)$

Problem 8 ← on HW HINT

$\mathbb{R}^m$ $\quad \vec{a}_1 \dots \vec{a}_n + \vec{p} + \vec{v}$

- How to test if $\vec{p}$ is the projection of $\vec{v}$ onto $\langle \vec{a}_1 \dots a_n \rangle$?



$\vec{v} - \vec{p}$
↳ Is it perpendicular?

3/14/14

18.06 Lecture

- Orthogonal Projections on column space of Q
- Least Squares
  - Fit a line
  - Fit a plane

projection $= A\hat{x} = p$

$= \underbrace{A(A^TA)^{-1}A^T}b$

$= \; P b$

$(A^TA)\hat{x} = A^Tb$

ortho matrix

$Q^TQ = I$

Q not square

$\hat{x} = Q^Tb$

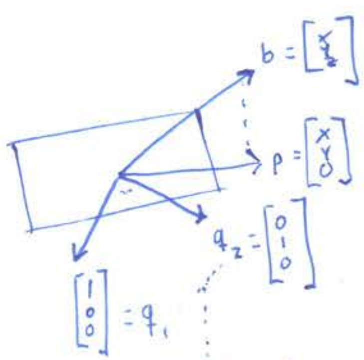
$P = Q(Q^TQ)^{-1}Q^T$

$= QQ^T$

Projection Matrix P

$P^2 = P$

$Q\underbrace{Q^TQ}_{I}Q^T = QQ^T$



ortho normal

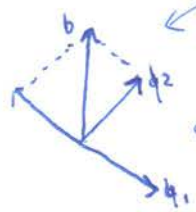
$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

$Q^TQ = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$QQ^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = P$

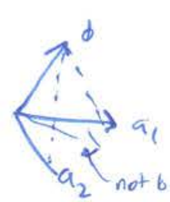
$p = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix}$

2 1D projections onto lines



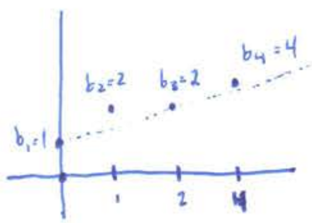
$q_1^Tq_2 = 0$

0000



Least Squares

Find closest line $C + Dt$ to m points
$n=2$



If 4 points on a line

when $t=4$

$$C + 4D = 3$$
$$C + 2D = 2$$
$$C + 1D = 2$$
$$C + 0D = 1$$

$Ax = b$ with no solution

$$Ax = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 3 \end{bmatrix}$$

det = 35

$$A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 7 \\ 7 & 21 \end{bmatrix}$$

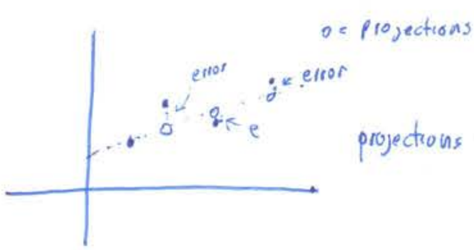
$$(A^TA)^{-1} = \frac{1}{35}\begin{bmatrix} 21 & -7 \\ -7 & 4 \end{bmatrix}$$ ← Use if it is easy to calculate

$$A^Tb = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 18 \end{bmatrix}$$

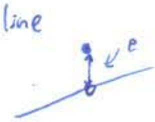
$$(A^TA)^{-1}(A^Tb) = \frac{1}{35}\begin{bmatrix} 21 & -7 \\ -7 & 4 \end{bmatrix} \begin{bmatrix} 8 \\ 18 \end{bmatrix} = \frac{1}{35}\begin{bmatrix} 42 \\ 16 \end{bmatrix} = \begin{matrix} C \\ D \end{matrix}$$

$C = 42/35$

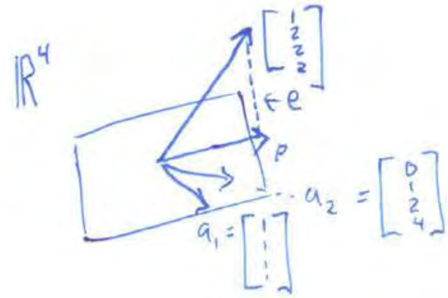
$D = 16/35$



projections are on the line



· 4 points did not lie on the line



↑ Samething

$Ct^N \sim b$

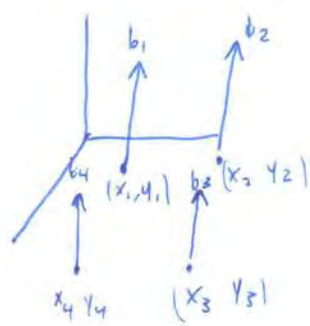
$\ln(C) + N \log t \sim \log_e b$

$c \quad d \sim b$

fit by plane .

$y = \underline{\underline{C}} + \underline{\underline{D}}x + \underline{\underline{E}}y$

straight plane picture



$A_{\times 3}$

① Gram - Schmidt
orthogonalize vectors

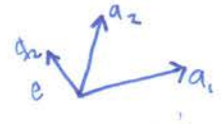
② Start Determinants

Gram - Schmidt

$A \to \to Q$

Start with independent vectors $a_1, a_2 \dots a_n$

End with orthogonal vectors $q_1, q_2, \dots q_n$

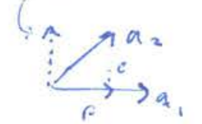


$q_1^T q_2 = 0$

$\|q_1\| = 1 \quad \|q_2\| = 1$

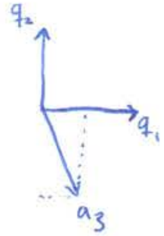
$q_1 = \dfrac{a_1}{\|a_1\|}$

$$q_2 = a_2 - \underbrace{\frac{a_2^T q_1}{q_1^T q_1}}_{p} q_1 = \frac{a_2 - (a_2^T q_1) q_1}{\|a_2 - (a_2^T q_1) q_1\|} = \frac{B}{\|B\|}$$



divide by unit

Projection of a_2 onto q_1



$$q_3 = \frac{a_3 - (a_3^T q_1) q_1 - (a_3^T q_2) q_2}{\| \dots \|}$$

orthogonal to q_1 and q_2

$q_1^T a_3 = 0$

$$q_1^T q_3 = \overset{\text{hope for}}{0} = a_3^T q_1 - (a_3^T q_1)(q_1^T q_1) - (a_3^T q_2)(\cancel{q_2^T q_1}) = 0$$

$$A = \begin{bmatrix} | & | & | \\ a_1 & a_2 & a_3 \\ | & | & | \end{bmatrix} \to Q = \begin{bmatrix} | & | & | \\ q_1 & q_2 & q_3 \\ | & | & | \end{bmatrix}$$

upper triangular

$$A = Q \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} = QR$$

$A = QR$

$$\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} = \begin{bmatrix} q_1 & q_2 & \dots \end{bmatrix} \begin{bmatrix} a_1^T q_1 & a_2^T q_1 & \\ & a_2^T q_2 & \\ & & \end{bmatrix} \leftarrow R$$

- gives all the dot products

$a_2 = (a_2^T q_1) q_1 + (a_2^T q_2)(q_2)$

Determinant

Properties:

$\det(AB) = (\det(A))(\det(B))$

$\det(A^T) = \det(A)$

Defining Properties

① $\det(I) = 1$

② Exchange 2 rows → det flips sign

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $A' = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$

$\det(A) = ad - bc$

$\det(A') = bc - da$

③ Det is Linear is linear in row 1 with other rows stay same.

$$\begin{bmatrix} a+A & b+B \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} A & B \\ c & d \end{bmatrix}$$

$(a+A)d - (b+B)c$ $\qquad ad - bc + Ad - Bc$

$ad + Ad - bc - Bc$ $\qquad ad + Ad - bc - Bc$

- Elimination steps dont change determinant

3×3 determinant → 6 terms $\qquad$ 4×4 det → 24 terms

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Orthonormal basis

a) Check that

$\vec{\ell}_1 = \frac{1}{\sqrt{2}}(1,1)$

$\vec{\ell}_2 = \frac{1}{\sqrt{2}}(-1,1)$

is an orthonormal basis

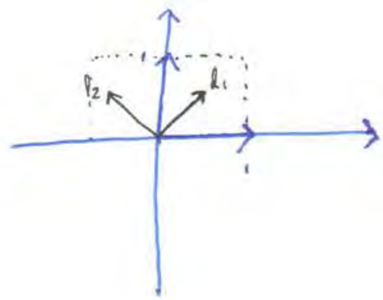
orthogonal $\quad$ normal

$\ell_1 \cdot \ell_2 = 0 \qquad ||\ell_1|| = ||\ell_2|| = 1$

$\ell_1 \cdot \ell_2 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \cdot \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = -\frac{1}{2} + \frac{1}{2} = 0$ ✓

$||\ell_1|| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{1} = 1$ ✓

$||\ell_2|| = \sqrt{\left(-\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{1} = 1$ ✓



b) compute the components $(0,1) = a\vec{\ell}_1 + \vec{b}\,\ell_2$

$a = (0,1) \cdot \vec{\ell}_1$ → only works w/ orthonormal basis

$b = (0,1) \cdot \vec{\ell}_2$

$(0,1) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}$

$(0,+1) \cdot \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}$

$(0,1) = \frac{1}{\sqrt{2}}\vec{\ell}_1 + \frac{1}{\sqrt{2}}\vec{\ell}_2$

(4) $q_1 \ldots q_m$ are orthonormal vectors in $\mathbb{R}^m$

a) what combination of the q's produces a given vector b

b) How do you know that the q's form a basis

a) $b = a_1 q_1 + \overset{a_i q_i}{\cdots\cdots} a_n q_n$

$a_i = b \cdot q_i$

because they are orthonormal

b) - They are all linearly independent

Projection Matrix

$x - 3y = 0$ describes a plane P in $\mathbb{R}^3$

a) Find an orthonormal basis of P

$x = 3y$

$$\{ 3y, y, z \ \vdots \ \underbrace{y \ z}_{\text{only 2 variables}} \} = P =$$

$y = 1 \quad z = 0 \rightarrow (3, 1, 0)$
$y = 0 \quad z = 1 \rightarrow (0, 0, 1)$

$$P = \langle (3, 1, 0), (0\ 0\ 1) \rangle$$

Check orthonormal

$\|(0, 0, 1)\| = 1$ ✓
$\|(3, 1, 0)\| \neq 1$

$(3, 1, 0) \cdot (0, 0, 1) = 0$ ✓

→ Force the norm to be 1

$\|3, 1, 0\| = \sqrt{10}$

$$P = \langle \tfrac{1}{\sqrt{10}}(3, 1, 0), (0, 0, 1) \rangle$$

b) Find the projection matrix P

$$P = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}_{3 \times 3}$$

$$A = \begin{bmatrix} 0 & 3/\sqrt{10} \\ 0 & 1/\sqrt{10} \\ 1 & 0 \end{bmatrix} \qquad P = A(A^TA)^{-1}A^T$$

basis of plane

$(A^TA)^{-1} = I$ if A orthonormal basis

norm dotproduct

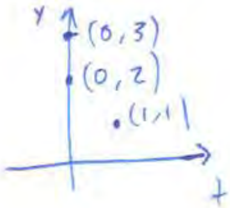
$$P = AA^T \qquad \begin{bmatrix} 0 & 0 & 1 \\ 3/\sqrt{10} & 1/\sqrt{10} & 0 \end{bmatrix} \begin{bmatrix} 0 & 3/\sqrt{10} \\ 0 & 1/\sqrt{10} \\ 1 & 0 \end{bmatrix} = \begin{pmatrix} 1 & 0 \\ & 1 \end{pmatrix} = \underline{I}$$

$$P = \begin{bmatrix} 0 & 3/\sqrt{10} \\ 0 & 1/\sqrt{10} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 3/\sqrt{10} & 1/\sqrt{10} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3^2/10 & 3/10 & 0 \\ 3/10 & 1/10 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Least Squares

t-y plane



Find the closest symetric parabola

$y = C(1-t^2) + D$ to these points

$$\begin{array}{l} \text{Plug in} \\ (1,1) \\ (0,2) \\ (0,3) \end{array} \begin{cases} 1 = D \\ 2 = C + D \\ 3 = C + D \end{cases} \text{- No solution } 2 \neq 3$$

$$\underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} C \\ D \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}_{b}$$

★ Minimize $\overbrace{\underbrace{\|Ax-b\|}_{e}{}^2}^{e^2}$

$$A^TAx = A^Tb$$

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \iff \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$$\left[\begin{array}{cc:c} 2 & 2 & 5 \\ 2 & 3 & 6 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{cc:c} 2 & 2 & 5 \\ 0 & 1 & 1 \end{array}\right]$$

$2C + 2D = 5$

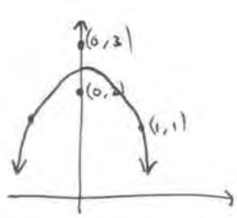
$D = 1$

$C = 3/2$

$$\boxed{y = \frac{3}{2}(1-t^2) + 1}$$

$t = 0 \Rightarrow y = 5/2$

$t = 1 \Rightarrow y = 1$



LEAST SQUARES

3/19/14 18.06 Lecture Fernando Trujano

Determinant

① det $I = 1$

② det switches signs with row exchange

③ det is linear in each row
- other rows stays the same

④ 2 equal rows $\rightarrow \det A = 0$

A: Switch: det changes sign ↖ must be 0
Also: det stays the same ↙

Column of all $0 \rightarrow \det = 0$

⑤ Elimination step does not change the determinant

$$\begin{bmatrix} \text{row1} \\ \text{row2} \\ \text{row3} \end{bmatrix} + \begin{bmatrix} -\ell\,\text{row2} \\ \text{row2} \\ \text{row3} \end{bmatrix} \overset{③}{=} \begin{bmatrix} \text{row1} - \text{row2} \\ \text{row2} \\ \text{row3} \end{bmatrix}$$

$$\underset{A}{\begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix}} \rightarrow \rightarrow \rightarrow \underset{\text{same det}}{\begin{bmatrix} p_1 & & 0 \\ & p_2 & \\ 0 & & p_3 \end{bmatrix}} \quad \det(A) = \pm p_1 p_2 p_3$$

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = 6 \text{ terms} \quad +++\ ---$$

$$= \begin{bmatrix} a_{11} & 0 & 0 \\ & \text{---} & \\ & \text{---} & \end{bmatrix} + \begin{bmatrix} \cancel{0} & a_{12} & 0 \\ a_{21} & a_{22} & \text{---} \\ & \text{---} & \end{bmatrix} + \begin{bmatrix} 0 & 0 & a_{13} \\ & \text{---} & \\ & \text{---} & \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & 0 & 0 \\ & \text{row 3} & \end{bmatrix} + \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ & \text{row 3} & \end{bmatrix} + \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{32} \\ & \text{row 3} & \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & 0 & 0 \\ a_{31} & 0 & 0 \end{bmatrix} + \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{bmatrix} + \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

Column of $0 \rightarrow \det = 0$

BIG FORMULA

$$\det A = \sum_{n!\text{ term}} \left(\pm\, a_{1c_1} a_{2c_2} \dots a_{nc_n} \right)$$

column

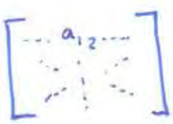
$c_1, c_2 \dots c_n$ = Permutation of $1, 2 \dots n$

3×3
6 terms

$$\begin{aligned} &a_{11}\, a_{22}\, a_{33} \\ -\,&a_{11}\, a_{23}\, a_{32} \end{aligned}$$

$$\underbrace{a_{11}}_{\text{Factor}} \underbrace{\left(a_{22} a_{33} - a_{23} a_{32} \right)}_{\text{cofactor}}$$

* Like 18.02



$$-a_{12}\left(a_{21} a_{33} - a_{23} a_{31} \right)$$

$$\begin{aligned} &-a_{12}\, a_{21}\, a_{33} \\ &+a_{12}\, a_{23}\, a_{31} \end{aligned}$$

Cofactor Formula

$$\det A = a_{11} C_{11} + a_{12} C_{12} + \dots a_{1n} C_{1n}$$

cofactors

$\downarrow$

smaller determinant

2/12

3/21/14

18.06 Lecture

Determinants

det(A) A^{-1} $A^{-1}b$ volume

1. Product of Pivots
2. BIG formula
3. Co-Factors

$$A_4 = \begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix} \quad 16-4-4-4+1=5$$

$\downarrow$

$$\begin{bmatrix} \boxed{2} & -1 & 0 & 0 \\ 0 & \boxed{3/2} & -1 & 0 \\ 0 & 0 & \boxed{4/3} & -1 \\ 0 & 0 & 0 & \boxed{5/4} \end{bmatrix} \quad 2\cdot\frac{3}{2}\cdot\frac{4}{3}\cdot\frac{5}{4}=5$$

Cofactors

$2\cdot C_{11} - 1C_{12}$

$A_2 = \begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix}$

$A_3 = \begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix}$

$\det A_4 = 2\det(A_3) \quad -1\det(-(-\det A_2)$

$8-3=5$

$$\boxed{\begin{matrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{matrix}}$$

Signs for cofactors

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & C_{31} & C_{41} \\ C_{12} & C_{22} & C_{32} & C_{42} \\ C_{13} & C_{23} & C_{33} & C_{43} \\ C_{14} & C_{24} & C_{34} & C_{44} \end{bmatrix} = \begin{bmatrix} \det(A) & 0 & 0 & 0 \\ 0 & \det(A) & 0 & 0 \\ 0 & 0 & \det(A) & 0 \\ 0 & 0 & 0 & \det(A) \end{bmatrix}$$

$A \qquad C^T$

$= (\det(A)) I$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}$$

$$A \qquad C^T \qquad = (ad-bc)I$$

$$A\frac{C^T}{\det(A)} = I$$

$$\boxed{A^{-1} = \frac{C^T}{\det(A)}}$$ Inverse Formula

$Ax = b$

$$x = A^{-1}b = \frac{C^T}{\det A} b$$ Cramer's Rule

$$x_1 = \frac{\left| b \quad \text{cols } 2 \ldots n \text{ of } A \right|}{\det(A)} \qquad x_2 = \frac{\left| \text{col } 1 \quad b \quad \text{cols } 3 \text{ to } n \text{ of } A \right|}{\det A}$$

$x, y \to r, \theta$

$$\iint f \, dx\, dy = \iint f \, ? \, dr\, d\theta$$

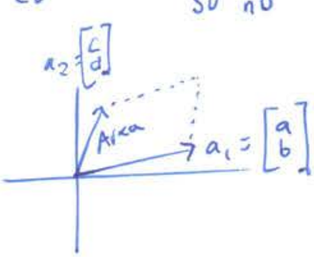
$x = r\cos\theta$
$y = r\sin\theta$

$$\left| \begin{bmatrix} \partial x/\partial r & \partial x/\partial \theta \\ \partial y/\partial r & \partial y/\partial \theta \end{bmatrix} \right| = \left| \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} \right| = r$$

"Jacobian"

$dx\,dy \to r\,dr\,d\theta$

(Area) or (Volume) of a box
2D   3D nD



$$A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix}$$

$$\text{Area} = \text{determinant} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = |ad - bc|$$

Unit cube

$$Q = \left| \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} \right| = \pm 1$$

18.06 Notes

Gram - Schmidt Example

$$a = \begin{bmatrix}1\\-1\\0\end{bmatrix} \quad b = \begin{bmatrix}2\\0\\-2\end{bmatrix} \quad c = \begin{bmatrix}3\\-3\\3\end{bmatrix}$$

a, b, c independent, non orthogonal

Find q_1, q_2, q_3.

$$\boxed{A = a}$$

$$\boxed{B = b - \frac{A^Tb}{A^TA}A}$$

$$(A^TA) = \begin{bmatrix}1 & -1 & 0\end{bmatrix}\begin{bmatrix}1\\-1\\0\end{bmatrix} = 2$$

$$(A^Tb) = \begin{bmatrix}1 & -1 & 0\end{bmatrix}\begin{bmatrix}2\\0\\-2\end{bmatrix} = 2$$

$$B = b - \frac{2}{2}A$$

$$= \begin{bmatrix}2\\0\\-2\end{bmatrix} - \begin{bmatrix}1\\-1\\0\end{bmatrix}$$

$$B = \begin{bmatrix}1\\1\\-2\end{bmatrix}$$

$$\boxed{C = c - \frac{A^Tc}{A^TA}A - \frac{B^Tc}{B^TB}B}$$

$$(A^TA) = 2$$

$$(A^Tc) = \begin{bmatrix}1 & -1 & 0\end{bmatrix}\begin{bmatrix}3\\-3\\3\end{bmatrix} = 6$$

$$(B^TB) = \begin{bmatrix}1 & 1 & -2\end{bmatrix}\begin{bmatrix}1\\1\\-2\end{bmatrix} = 6$$

$$(B^Tc) = \begin{bmatrix}1 & 1 & -2\end{bmatrix}\begin{bmatrix}3\\-3\\3\end{bmatrix} = -6$$

$$C = c - \frac{6}{2}A + \frac{6}{6}B$$

$$= \begin{bmatrix}3\\-3\\3\end{bmatrix} - \begin{bmatrix}6/2\\-6/2\\0\end{bmatrix} + \begin{bmatrix}1\\1\\-2\end{bmatrix}$$

$$C = \begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$q_1 = \frac{A}{\|A\|} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\\0\end{bmatrix}$$

$$q_2 = \frac{B}{\|B\|} = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\1\\-2\end{bmatrix}$$

$$q_3 = \frac{C}{\|C\|} = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

Factorization $A = QR$

$$A = QR$$
$$Q^TA = Q^TQR$$
$$Q^TA = IR$$
$$Q^TA = R$$

① Do Gram-Schmidt

② Get R

$$R = Q^TA$$

③ Pass the test :)

18.06 Professor Edelman Quiz 2 November 6, 2013

Grading

1

2

3

4

Your PRINTED name is: ____________________

Please circle your recitation:

1	T 9	Dan Harris	E17-401G	3-7775	`dmh`
2	T 10	Dan Harris	E17-401G	3-7775	`dmh`
3	T 10	Tanya Khovanova	E18-420	4-1459	`tanya`
4	T 11	Tanya Khovanova	E18-420	4-1459	`tanya`
5	T 12	Saul Glasman	E18-301H	3-4091	`sglasman`
6	T 1	Alex Dubbs	32-G580	3-6770	`dubbs`
7	T 2	Alex Dubbs	32-G580	3-6770	`dubbs`

1 (25 pts.)

Compute the determinant of

a) (10 pts.) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1806 & 1806 & 0 \\ 2013 & 2014 & 2015 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -1806 \\ 2013 & 2014 & 2015 \end{bmatrix}$

$(-1)(-1806)(2014-2013)$

$\boxed{= 1806}$

b) (15 pts.)

The $n \times n$ matrix A_n has ones in every element off the diagonal, and also $a_{11} = 1$ as well. The rest of the diagonal elements are 0: $a_{22} = a_{33} = \ldots = a_{nn} = 0$. For example

$$A_5 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Write the determinant of A_n in terms of n in simplest form. Argue briefly but convincingly your answer is right.

$$\rightarrow \begin{bmatrix} 1 & 1 & \cdots & & 1 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & & 0 \\ \vdots & & & \ddots & \\ \vdots & & 0 & & \ddots \end{bmatrix}$$

det = multiply pivots

$= 1 \cdot (-1) \cdot (-1)$

$\boxed{= (-1)^{n-1}}$

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2 (30 pts.)

Let $Q = [q_1\ q_2\ q_3]$ be an $m \times 3$ real matrix with $m > 3$ and $Q^TQ = I_3$, the 3×3 identity.

Let $P = QQ^T$.

a) (7 pts.) What are all possible values of $\det(P)$?

b) (7 pts.) What are all the eigenvalues of the $m \times m$ matrix P including multiplicities?

c) (8 pts.) Find one eigenvalue, eigenvector pair of the non-symmetric $m \times m$ matrix $q_1 q_2^T$.

d) (8 pts.) What are the four fundamental subspaces of $M = I - P$ in terms of the column space of P?

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3 (20 pts.)

Let A be a 4×4 general matrix and x a scalar variable. Circle your answers and provide a very brief explanation.

a) (5 pts.) What kind of polynomial in x best describes $\det(A - xI)$?

constant linear quadratic cubic (degree 3) quartic (degree 4)

b) (5 pts.) What kind of polynomial in A_{11} best describes $\det(A - xI)$?

constant linear quadratic cubic (degree 3) quartic (degree 4)

c) (5 pts.) What kind of polynomial in x best describes $\det(xA)$?

constant linear quadratic cubic (degree 3) quartic (degree 4)

d) (5 pts.) What kind of polynomial in x best describes $\det(A(x))$, where

$$A(x) = \begin{bmatrix} xA_{11} & xA_{12} & xA_{13} & xA_{14} \\ A_{21}+x & A_{22}+x & A_{23}+x & A_{24}+x \\ A_{31}-x & A_{32}-x & A_{33}-x & A_{34}-x \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix}$$

constant linear quadratic cubic (degree 3) quartic (degree 4)

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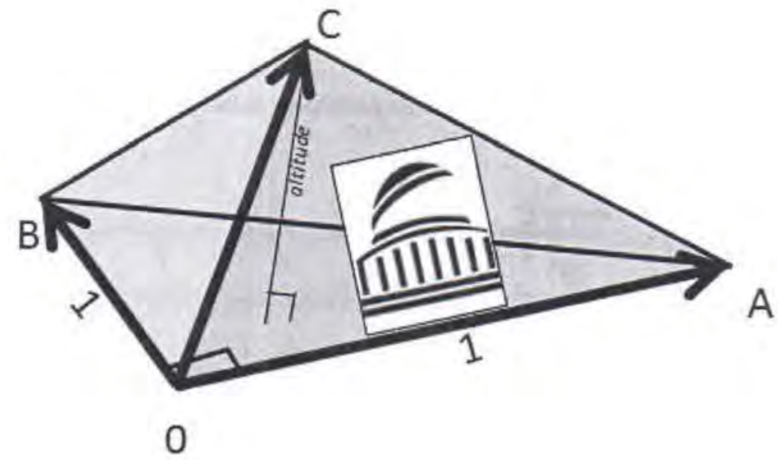

4 (20 pts.)

In R^3 an artist plans an MIT triangular pyramid artwork with one vertex at the origin. The other three vertices are at the tips of vectors A, B and C.

The triangular base of the pyramid $(0, A, B)$ is an isosceles right triangle, The vectors A and B are unit vectors orthogonal to each other.

The other vector C is not in any especially convenient position.

a) (12 pts.) Write an expression for L the length of the altitude of the top of the pyramid to the base in terms of A, B and C.

b) (8 pts.) Write an expression for the volume of the pyramid.

18.06 Professor Edelman Quiz 2 November 7, 2012

Grading

1

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Your PRINTED name is: ____________________

Please circle your recitation:

1	T 9	2-132	Andrey Grinshpun	2-349	3-7578	agrinshp
2	T 10	2-132	Rosalie Belanger-Rioux	2-331	3-5029	robr
3	T 10	2-146	Andrey Grinshpun	2-349	3-7578	agrinshp
4	T 11	2-132	Rosalie Belanger-Rioux	2-331	3-5029	robr
5	T 12	2-132	Geoffroy Horel	2-490	3-4094	ghorel
6	T 1	2-132	Tiankai Liu	2-491	3-4091	tiankai
7	T 2	2-132	Tiankai Liu	2-491	3-4091	tiankai

1 (27 pts.)

P is any $n \times n$ Projection Matrix. Compute the ranks of A,B, and C below. Your method must be visibly correct for every such P, not just one example.

a) (8 pts.) $A = (I - P)P$.

$$A = IP - P^2$$
$$= P - P = 0$$
$$\boxed{\text{rank} = 0}$$

b) (10 pts.) $B = (I - P) - P$. (Hint: Squaring B might be helpful.)

$$B^2 = (I - 2P)^2$$
$$= I^2 - 2PI - 2PI + 4P^2$$
$$I - \cancel{2P} - \cancel{2P} + \cancel{4P}$$
$$B = I \quad \boxed{\text{rank } n}$$

c) (9 pts.) $C = (I - P)^{2012} + P^{2012}$.

2 (22 pts.)

Consider a 4×4 matrix

$$A = \begin{pmatrix} 0 & x & y & z \\ x & 1 & 0 & 0 \\ y & 0 & 1 & 0 \\ z & 0 & 0 & 1 \end{pmatrix}.$$

a) (17 pts.) Compute $|A|$, the determinant of A, in simplest form.

$$-x \begin{vmatrix} x & 0 & 0 \\ y & 1 & 0 \\ z & 0 & 1 \end{vmatrix} + y \begin{vmatrix} x & 1 & 0 \\ y & 0 & 0 \\ z & 0 & 1 \end{vmatrix} - z \begin{vmatrix} x & 1 & 0 \\ y & 0 & 1 \\ z & 0 & 0 \end{vmatrix}$$

$$(-x)\,x + y(-y) + -z(-(-z))$$

$$\boxed{-x^2 - y^2 - z^2 = \det(A)}$$

b) (5 pts.) For what values of x, y, z is A singular?

when $x^2 + y^2 + z^2 = 0$

$x + y + z = 0$

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3 (22 pts.)

The 3×3 matrix $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ has QR decomposition

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = Q \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{pmatrix}.$$

a) (7 pts.) What is r_{11} in terms of the variables $a, b, c, d, e, f, g, h, i$? (but not any of the elements of Q.)

a) (15 pts.) Solve for x in the equation,

$$Q^T x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

expressing your answer possibly in terms of r_{11}, r_{22}, r_{33} and the variables $a, b, c, d, e, f, g, h, i$, (but not any of the elements of Q.)

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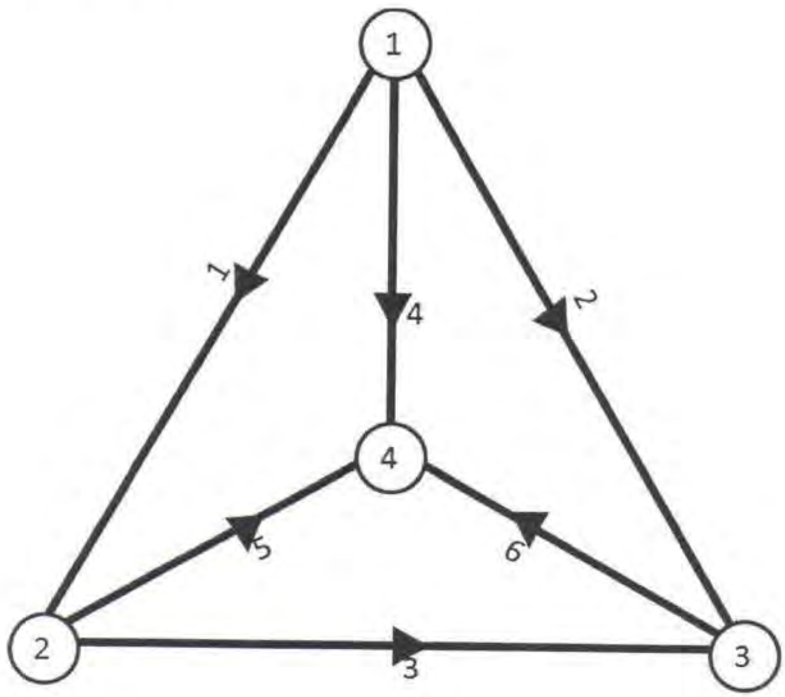
4 **(29 pts.)**



a) (15 pts.) Use loops or otherwise to find a basis for the left nullspace of the incidence matrix A for the graph above. We will start you off, one basis vector is $\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$.

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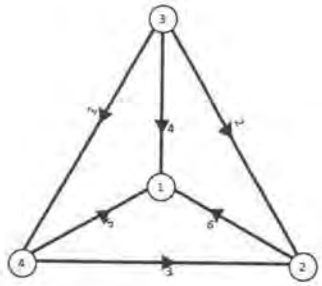

There are 24 ways to relabel the four nodes in the graph in part(a). Edge labels remain unchanged. One of the 24 ways is pictured above. This produces 24 incidence matrices A.

b) (7 pts.) Is the row space of A independent of the labeling? Argue convincingly either way.

c) (7 pts.) Is the column space of A independent of the labeling? Argue convincingly either way.

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18.06 Quiz 2 Professor Strang April 10, 2013

Please PRINT your name Fernando Trujano

1.
2.
3.

Please Circle your Recitation:

r1	T	10	36-156	Russell Hewett	r7	T	1	36-144	Vinoth Nandakumar
r2	T	11	36-153	Russell Hewett	r8	T	1	24-307	Aaron Potechin
r3	T	11	24-407	John Lesieutre	r9	T	2	24-307	Aaron Potechin
r4	T	12	36-153	Stephen Curran	r10	T	2	36-144	Vinoth Nandakumar
r5	T	12	24-407	John Lesieutre	r11	T	3	36-144	Jennifer Park
r6	T	1	36-153	Stephen Curran					

(1) **(40 pts)**

(a) If P projects every vector b in $\mathbb{R}^5$ to the nearest point in the subspace spanned by $a_1 = (1, 0, 1, 0, 4)$ and $a_2 = (2, 0, 0, 0, 4)$, what is the rank of P and **why?**

(b) If these two vectors are the columns of the 5 by 2 matrix A, which of the four fundamental subspaces for A is the nullspace of P?

(c) By Gram-Schmidt find an orthonormal basis for the column space of A (spanned by a_1 and a_2).

$A = a_1$

$B = a_2 - \frac{A^T a_2}{A^T A} A$

$= a_2 - \frac{18}{18} A$

$B = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 4 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}$

$q_1 = \frac{A}{\|A\|} = \frac{1}{3\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 4 \end{bmatrix}$

$q_2 = \frac{B}{\|B\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 4 \end{bmatrix}$

q_1 and q_2 form basis

(d) If P is any (symmetric) projection matrix, show that $Q = I - 2P$ is an orthogonal matrix.

Orthogonal $\Rightarrow QQ^T = I$

$(I - 2P)(I - 2P)^T =$

$(I - 2P)(I^T - 2P^T)$

$II^T - 2IP^T - 2PI^T + 4PP^T$

$P^T = P$
$P^2 = P$ > Projection Matrix
$I^T = I$

$I - 2P - 2P + 4P$

$= I$

$\boxed{I = I}$ ✓

(2) **(30 pts.)**

(a) Find the determinant of the matrix A

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & 2 & 3 & 0 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 4 \end{bmatrix}$$

$$-3(8) = \boxed{-24}$$

(b) The absolute value of det A tells you the volume of a box in $\mathbb{R}^4$. Describe that box (2 points – describe a different box with the same volume).

A box whose corners lie in $\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 4 \\ 4 \end{pmatrix}$ or $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 4 \end{pmatrix}$

(c) Suppose you remove row 3 and column 4 of an invertible 5 by 5 matrix A. If that reduced matrix is not invertible, what fact does that tell you about A^{-1}?

$$\begin{bmatrix} x & x & x & x & x \\ x & x & x & x & x \\ x & x & x & x & x \\ x & x & x & x & x \\ x & x & x & x & x \end{bmatrix}$$

Reduced matrix is a cofactor matrix whose determinant is 0

since

$$(A^{-1})_{ij} = \frac{C_{ji}}{\det(A)}$$

$$A^{-1}_{43} = 0$$

(3) (30 pts.) This 4 by 4 Hadmard matrix is an orthogonal matrix. Its columns are orthogonal unit vectors.

$$Q = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} q_1 & q_2 & q_3 & q_4 \end{bmatrix}$$

(a) What projection matrix P_4 (give numbers) will project every b in $\mathbb{R}^4$ onto the line through q_4? $P = A(A^TA)^{-1}A^T$

$A = q_4 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$

$(A^TA) = \begin{bmatrix} 1 & -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} = 4$

$(A^TA)^{-1} = \frac{1}{4}$

$P = \frac{1}{4}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & -1 & 1 \end{bmatrix}$

$$\boxed{P = \frac{1}{4}\begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}}$$

(b) What projection matrix P_{123} will project every b in $\mathbb{R}^4$ onto the subspace spanned by q_1, q_2, and q_3? Remember that those columns are orthogonal.

$P = A(A^TA)^{-1}A^T$

$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{bmatrix}$ $\qquad (A^TA)^{-1} = \frac{1}{4}I$ $\qquad P = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix}$

$(A^TA) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = 4I$

$$\boxed{P = \frac{1}{4}\begin{bmatrix} 3 & 1 & 1 & -1 \\ 1 & 3 & -1 & 1 \\ 1 & -1 & 3 & 1 \\ -1 & 1 & 1 & 3 \end{bmatrix}}$$

(c) Suppose A is the 4 by 3 matrix whose columns are q_1, q_2, q_3. Find the least-squares solution $\widehat{x}$ to the four equations

$$Ax = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = b.$$

What is the error vector e? $\qquad Ax = b$

$A^TA\hat{x} = A^Tb$

$A^TA = I$ because A orthogonal

$x = A^Tb = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 10 \\ -2 \\ -4 \end{bmatrix}$

$$\boxed{x = \begin{bmatrix} 10 \\ -2 \\ -4 \end{bmatrix} \quad e = [0]}$$

$e = \cancel{Ax}\ b - p = b - Ax = 0$

18.06 Notes

ocw

Lecture 21

Eigen values

Eiganvectors

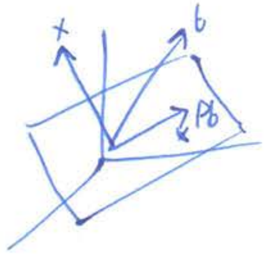
Eigan vectors

Ax parallel to x

$$\underset{\uparrow}{Ax} = \lambda x \quad \text{Eiganvalue}$$

Eiganvectors

If a is singular, $\lambda = 0$ is eiganvalue



What are the x and λ For a projection matrix?

Any x in plane $\to$ eigenvector

$Px = x$ $\quad \lambda = 1$

$Px = \lambda x$

Any x perpendicular to plane

$Px = 0$ $\quad \lambda = 0$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad Ax = x \quad \lambda = 1$

$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\quad Ax = -x \quad \lambda = -1$

$$\text{sum of eiganvalues } \lambda = \underbrace{\text{sum of diagonal } A}_{\text{trace}}$$

determinant = product of eiganvalues

*

How to solve $Ax = \lambda x$

rewrite

$$\underbrace{(A - \lambda I)}_{\text{must be singular}} x = 0 \quad \to \quad \det(A - \lambda I) = 0$$

$$* \quad \boxed{|A - \lambda I| = 0}$$

Find λ first!

$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 - 1 = 0$$

trace

$$\lambda^2 - 6\lambda + 8 = 0$$

determinant

$$(\lambda - 4)(\lambda - 2)$$

$$\lambda_1 = 4$$
$$\lambda_2 = 2$$

Find eiganvector

$$A - 4I = \begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix}$$

should be singular

$$(A - 4I)x_1 = 0$$

$$x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A - 2I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad (A - 2I)x_2 = 0$$

$$x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

If $Ax = \lambda x$

$$(A + 3I)x = \lambda x + 3x = (\lambda + 3)x$$

stays the same

λ changes

Example

$$Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

90° rotation

trace $= 0+0 = \lambda_1 + \lambda_2$

det $= 1 = \lambda_1 \lambda_2$

no solution?

$$\det(Q - \lambda I) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0$$

$\lambda_1 = i$

$\lambda_2 = -i$

imaginary λ

$Q^T = -Q$

$\Longrightarrow$ Pure immaginary λ

complex conjugate pair
- change sign of imaginary part.

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$$

Triangular matrix $\to$ Eigan values on diagonal

Proof

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 1 \\ 0 & 3-\lambda \end{vmatrix} = (3-\lambda)(3-\lambda)$$

$\lambda_1 = 3$

$\lambda_2 = 3$

Eiganvectors

$$(A - \lambda I)x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \end{bmatrix} = 0$$

$x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Degenerate Matrix

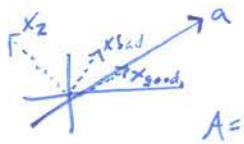
Shortage of λ

$x_2 =$ No second independent eigan vectors

3/31/13 18.06 Lecture

Eigenvalues

$$\overset{\text{Eigenvalues}}{} Ax = \lambda x \leftarrow \text{Eigenvector}$$



A= projection onto line $\qquad A = \dfrac{aa^T}{a^T a}$

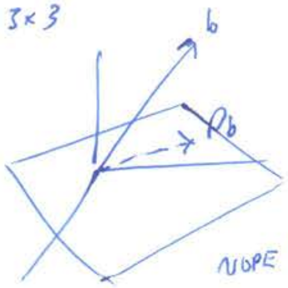
$Ax_1 = \lambda_1 x_1$

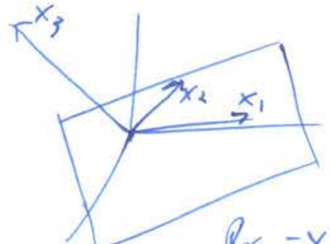
$\lambda_1 = 1$

$Ax_2 = 0x_2$

$\lambda_2 = 0$

3×3





$Px_1 = x_1 \qquad Px_2 = x_2$

$Px_3 = 0x_3$

Singular A has $\lambda = 0$ as 1 eigenvalue

$Ax = 0x$

has nonzero solutions

$Ax = \lambda x$

-Can't use elimination :(

eig(A) = all eiganvalues

$Ax = \lambda x$

$A^2x = A\lambda x = \lambda A x = \lambda^2 x$

A^2 has the same aigan vector as A

$Ax = \lambda x$

$(A+I)x = Ax + Ix$

$= \lambda x + x$

$(A+I)x = (\lambda+1)x$

shifs all eigan by 1

Note: I fell asleep on tho Lecture

-watch OCW!

Q orthogonal matrix

$$Q^{-1} = Q^T$$
$$|QQ^T| = |I|$$

- Determinant of a product is product of the determinant

$$|Q||Q^T| = 1$$
$$|Q^T| = |Q|$$
$$|Q|^2 = 1$$
$$\boxed{|Q| = \pm 1}$$

$$|-A| = ?\,|A|$$

$$-A = (-I)A$$
$$|-A| = |-I||A|$$
$$= \begin{pmatrix} -1 & & 0 \\ & \ddots & \\ 0 & & \ddots \end{pmatrix}_{m \times m} |A|$$

$m \times m \leftarrow$ dimension

$$|-A| = (-1)^m |A| \qquad (-1)^m = \begin{cases} 1 & m \text{ even} \\ -1 & m \text{ odd} \end{cases}$$

∴ Dependent on even or odd!

$$|4A|$$
$$= |(4I)A| = |4I| \cdot |A|$$
$$\left| \begin{pmatrix} 4 & & 0 \\ & \ddots & \\ 0 & & 4 \end{pmatrix}_{m \times m} \right|$$

Determinant of diagonal matrix = diag ^{m}

$$4^m |A|$$

- dependent on dimension

Problem 6 ← on HW

$$\left| I + vv^T \right| = ?$$

$v = \begin{bmatrix} a \\ b \\ c \end{bmatrix}_{1\times 3}$

$v^T = \begin{bmatrix} a & b & c \end{bmatrix}_{3\times 1}$

$$vv^T = \begin{pmatrix} a^2 & ab & ac \\ ba & b^2 & bc \\ ca & cb & c^2 \end{pmatrix}$$

$$\begin{vmatrix} 1+a^2 & ab & ac \\ ba & 1+b^2 & bc \\ ca & cb & 1+c^2 \end{vmatrix} = ?$$

- Cofactor method

row column

$$= (1+a^2)(-1)^{1+1} \begin{vmatrix} 1+b^2 & bc \\ cb & 1+c^2 \end{vmatrix}$$

$$+ (ab)(-1)^{1+2} \begin{vmatrix} ba & bc \\ ca & 1+c^2 \end{vmatrix}$$

$$+ ac(-1)^4 \begin{vmatrix} ba & 1+b^2 \\ ca & cb \end{vmatrix}$$

$$= \boxed{1 + a^2 + b^2 + c^2}$$

Eigenvalues and Eigenvectors

$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ - Find Eigen $\substack{\text{values} \\ \text{vectors}}$

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x - y \\ -x + 2y \end{pmatrix}$$

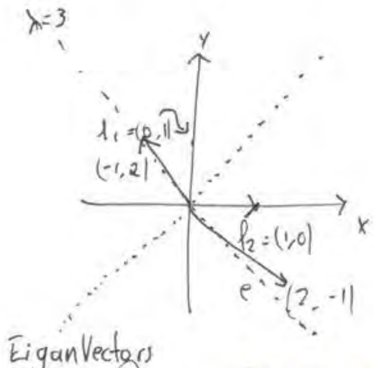
$\mathbb{R}^2 \to \mathbb{R}^2$

$(x,y) \to (2x-y, \ -x+2y)$

$(0,0) \to (0,0)$

$(1,0) \to (2,-1)$

$(0,1) \to (-1,2)$



$\lambda = 1 \quad (x,x) \to (x,x) = \lambda(x,x)$

λ ↓

$\lambda = 3 \quad (-y,y) \to (-2y-y, \ y+2y) = 3(-y,y)$

Eigan vector - linear subspaces of the plane

Find line passing through origin

★ Eiganvalues of Vector : $|A - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = 0$$

can also use cofactor method $|5| = 5$

$$(2-\lambda)^2 - 1 = 0$$

solutions = Eiganvals

$\lambda = 1, \ \lambda = 3$

$\lambda = 1$

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 1 & 0 \end{array}\right] \xrightarrow{l_2 \to l_2 + l_1} \left[\begin{array}{ccc} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \begin{cases} x - y = 0 \\ 0 = 0 \end{cases}$$

① Solutions $= \{ x, x \mid x \in \mathbb{R} \}$

$= \langle 1, 1 \rangle$

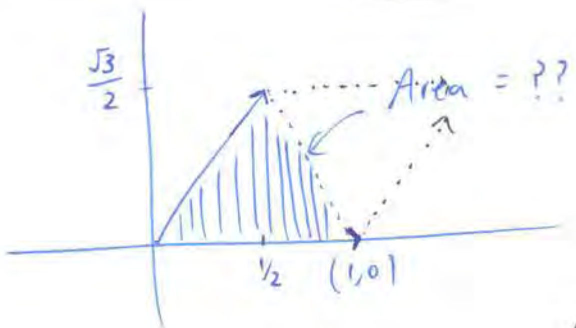
Use other eigan value

$\lambda = 3$

$$\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left[\begin{array}{cc|c} -1 & -1 & 0 \\ -1 & -1 & 0 \end{array}\right] \xrightarrow{\ell_2 \to \ell_2 - \ell_1} \begin{bmatrix} -1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{cases} -x-y=0 \\ 0=0 \end{cases}$$

$$\text{Solutions} = \{ (-y, y) \mid y \in \mathbb{R} \}$$



determinant = area of parallelogram

$$\text{Area} = \frac{1}{2} \begin{vmatrix} 1 & 1/2 \\ 0 & \sqrt{3}/2 \end{vmatrix}$$

$$\boxed{= \frac{\sqrt{3}}{4}}$$

4/4

18.06 Lecture 22 OCW

Diagonalization and Powers of A

- Suppose n independent Eigan vectors of A ← or else can't diagonalize
  - Put them in the columns of S

$$AS = A\begin{bmatrix} | & | & | \\ x_1 & x_2 & x_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} \lambda x_1 & \cdots & \lambda_n x_n \end{bmatrix} = \begin{bmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_n \\ | & | & & | \end{bmatrix}\begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} = S\Lambda$$

Diagonal Eigenvalue matrix Λ

$$AS = S\Lambda$$

$$\boxed{S^{-1}AS = \Lambda}$$

$$\boxed{A = S\Lambda S^{-1}}$$

If $Ax = \lambda x$

$A^2x = \lambda Ax$

$A^2x = \lambda^2 x$ ← same eiganvectors

squared eiganvalues

OR

$A^2 = S\Lambda S^{-1}S\Lambda S^{-1} = S\Lambda^2 S^{-1}$

squared

same

$$A^k = S\Lambda^k S^{-1}$$

Theorem

$A^k \to 0$ as $k \to \infty$

if all $|\lambda_i| < 1$

A is sure to have n indep evectors (and be diagonalizable) if all the λ's are different!

eig(A) - matlab command

Repeated λ's $\Rightarrow$ may or may not have n indep evectors

Triangular

$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

$$\det(A-\lambda I) = \left|\begin{bmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{bmatrix}\right| = (2-\lambda)^2 = 0$$

$\lambda = 2$

$\lambda = 2$

$$A - 2I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- Not diagonalizable

Equation

$$U_{k+1} = AU_k$$

start with U_0

$U_1 = AU_0$, $\quad U_k = A^k U_0$

$U_2 = A^2 U$

To really solve

write $U_0 = C_1 \overset{\text{evectors}}{x_1} + C_2 x_2 + \dots + C_n x_n$

$$AU_0 = C_1 \lambda_1 x_1 + C_2 \lambda_2 x_2 + C_n \lambda_n x_n = S_c$$

so

$$A^{100} U_0 = C_1 \lambda_1^{100} x_1 + C_2 \lambda_2^{100} x_2 + C_n \lambda_n^{100} x_n$$

stays same

$$= \Lambda^{100} S_c$$

Fibonacci example: 0, 1, 1, 2, 3, 5, 8, 13 ... $F_{100} = ?$

how fast are they growing

second order

$$F_{k+2} = F_{k+1} + F_k$$

$F_{k+1} = F_{k+1}$ $\quad$ Trick $\quad U_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} \Rightarrow U_{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} U_k$

$\begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$

First order

$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ $\lambda_1 + \lambda_2 = 1$

$\lambda_1 \lambda_2 = \det(A) = -1$

$F_{k+2} - F_{k+1} - F_k$

Find eigenvalue and eigan vectors

$$|A - \lambda I| = \left| \begin{bmatrix} 1-\lambda & 1 \\ 1 & 0-\lambda \end{bmatrix} \right| = \lambda^2 - \lambda - 1 = 0$$

$$\lambda = \frac{1 \pm \sqrt{1+4}}{2}$$

$$\lambda_1 = \tfrac{1}{2}(1+\sqrt{5}) \approx 1.618$$

$$\lambda_2 = \tfrac{1}{2}(1-\sqrt{5}) \approx -.618$$

Diagonalizable ✓

How fast are they growing?

$$F_{100} \approx c_1 \lambda_1^{100} + c_2 \lambda_2^{100}$$

$x_1 =$ Trick

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix} \begin{bmatrix} \lambda \\ 1 \end{bmatrix} = 0$$

$$x_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} \quad x_2 \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$$

$$U_0 = \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$c_1 x_1 + c_2 x_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

3/3

$$A = \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix}$$

diagonals
2+2+2

det(A)

$$|A-\lambda I| = \begin{vmatrix} 2-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{vmatrix} = (-\lambda)^3 + 6\lambda^2 + \_\_\_\lambda + \underset{\text{constant}}{4}$$

Trace → sum of λ

constant → product of λ's

Characteristic Polynomial

$y'' + 4y' + 3y = 0$ Look for $y = e^{\lambda t}$

Plug in

$$\lambda^2 \cancel{e^{\lambda t}} + 4\lambda \cancel{e^{\lambda t}} + 3\cancel{e^{\lambda t}} = 0$$

$$\lambda^2 + 4\lambda + 3$$

$$(\lambda+3)(\lambda+1) = 0$$

$$\lambda = -3, -1$$

1st order Matrix

$y' = z$

$z' = -4z - 3y$

$$\begin{bmatrix} y \\ z \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix}$$

Eiganvalues = $-3, -1$

$$\begin{vmatrix} 0-\lambda & 1 \\ -3 & -4-\lambda \end{vmatrix} = \lambda^2 + 4\lambda + 3 = 0$$

Sum of λ's = trace of A = sum down diagonal

product of λ's = det of A

$A = S\Lambda S^{-1}$

or $AS = S\Lambda$

eigan vector

$$S = \begin{bmatrix} | & | & | \\ x_1 & x_2 & x_n \\ | & | & | \end{bmatrix}$$

Eiganvector Matrix

$$AS = A\begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} = \begin{bmatrix} | & | & \\ \lambda_1 x_1 & \lambda_2 x_2 & \cdots \\ | & | & \end{bmatrix}$$

Beauty

$$= \begin{bmatrix} | & | & & \\ x_1 & x_2 & \cdots & x_n \\ | & | & & \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & & 0 \\ 0 & \lambda_2 & & \\ 0 & 0 & 0 & \lambda_n \end{bmatrix} = S\Lambda$$

$$A^2 = S\overset{\Lambda}{A}\underbrace{S^{-1} S}_{I} \overset{\text{not } A}{A} S^{-1} = S\Lambda^2 S^{-1}$$

$$A^n = SA^n S^{-1} = S\begin{bmatrix} \lambda_1^n & & \\ & \ddots & \\ & & \end{bmatrix} S^{-1}$$

① Find c's
② Mult by λ^n
③ Add up $c_1 d_1^n x_1 + \ldots$

Want $A^n v$

① Write $v = c_1 x_1 + c_2 x_2$
② $A^n v = c_1 \lambda_1^n x_1 + c_2 \lambda_2^n x_2$

Eigenvalue and eigenvectors, Markov matrix, Diagonalization

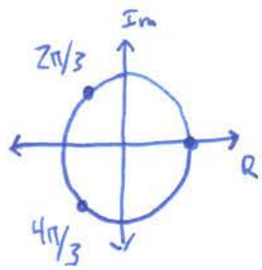
① Eigenvalue and eigenvectors

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$ Quick determinant trick.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & 0 & -\lambda \end{vmatrix} = (-\lambda)^3 + 1 = 0$$

$$\lambda^3 = 1$$



$\lambda_1 = 1$

$\lambda_2 = \frac{-1 + \sqrt{3}i}{2}$

$\lambda_3 = \frac{-1 - \sqrt{3}i}{2}$

Eigenvalues

$$Av = \lambda v \iff (A - \lambda I)v = 0$$

nullspace of $A - \lambda I$

λ_1: $(A - I)v = 0$

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} v = 0$$

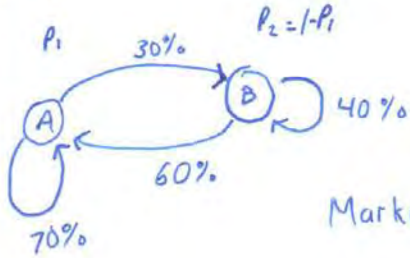
$$v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Repeat for λ_2, λ_3

②

Markov Matrix

Topic 2, #1



Markov Chain

Column → transition probability

$$\begin{array}{cc} & \text{From} \\ & \begin{array}{cc} A & B \end{array} \\ M = & \begin{bmatrix} .7 & .6 \\ .3 & .4 \end{bmatrix} \end{array} \begin{array}{l} A \\ B \end{array} \; \text{To}$$

$$\underset{t=0}{\begin{pmatrix} P_1 \\ P_2 \end{pmatrix}} \qquad \underset{t=1}{M\begin{pmatrix} P_1 \\ P_2 \end{pmatrix}} \qquad \underset{t=2}{M^2\begin{pmatrix} P_1 \\ P_2 \end{pmatrix}} \qquad \cdots\cdots \qquad \underset{t=n}{M^n\begin{pmatrix} P_1 \\ P_2 \end{pmatrix}}$$

$$M\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$$

↖For a stable state

$$(M - I)\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = 0 \qquad \begin{bmatrix} -3 & .6 \\ .3 & -6 \end{bmatrix}\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = 0$$

$$P_1 + P_2 = 1$$

$P_1 = 2/3$
$P_2 = 1/3$

③ Diagonalization

$$M = \begin{bmatrix} .7 & .6 \\ .3 & .4 \end{bmatrix}$$

Diagonalize
↳ split M into product of 3 matrices

$$M = S \cdot \Lambda \cdot S^{-1}$$

$$\begin{pmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{pmatrix} \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_3 \end{pmatrix}$$

Eiganvectors

diagonal Eiganvalues

$$M = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & .1 \end{bmatrix} \frac{1}{3} \begin{bmatrix} +1 & +1 \\ 1 & -2 \end{bmatrix}$$

check

$$M\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$M^n = (S\Lambda S^{-1})^n = (S\Lambda \cancel{S^{-1}})(\cancel{S}\Lambda \cancel{S^{-1}})(\cancel{S}\Lambda \cancel{S^{-1}}) \dots = S\Lambda^n S^{-1}$$

Diagonalizing makes it easy to compute high powers

$n \to \infty$

$$\lim_{n\to\infty} M^n = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1^\infty & 0 \\ 0 & .1^\infty \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 2/3 & 2/3 \\ 1/3 & 1/3 \end{bmatrix}$$

$P_2 = 1 - P_1$

$$\lim_{n\to\infty} M^n \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{bmatrix} 2/3 & 2/3 \\ 1/3 & 1/3 \end{bmatrix} \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{matrix} P_1 \frac{2}{3} + P_2 \frac{2}{3} = \\ P_1 \frac{1}{3} + P_2 \frac{1}{3} = \end{matrix} \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}$$

04/08/2014 Recitation

Topic 1: Eigenvalue and eigenvectors

1. Complex eigenvalues

   Permutation matrix P=[0 1 0; 0 0 1; 1 0 0]. Find eigenvalues.
   (Answer: eigenvalue $\lambda_1 = 1, \lambda_2 = \frac{-1+\sqrt{3}i}{2}, \lambda_3 = \frac{-1-\sqrt{3}i}{2}$. If want more: eigenvectors $v_1 = (1,1,1), v_2 = (\frac{-1-\sqrt{3}i}{2}, \frac{-1+\sqrt{3}i}{2}, 1), v_3 = (\frac{-1+\sqrt{3}i}{2}, \frac{-1-\sqrt{3}i}{2}, 1)$.

Topic 2: Markov matrix

1. Suppose you have two coins A and B. The probability of getting head from A is 70%, and probability to get a head from B is 60%. Each time you flip one of the coins. When you flip A, if you get head, next time you'll flip A again; otherwise if you get tail, then flip B for the next time. Same for B: if you get head then continue with B. Write out a markov matrix to describe it.
   (Hint: matrix M=[0.7 0.6; 0.3 0.4].)

2. And if you play this game for a long time, what's the long-term probability to get A (and B)?

   (Hint: Long-term probability $M * (pA, pB) = (pA, pB)$, get $(pA, pB) = \cancel{(1/3, 2/3)}$ note pA+pB=1)
   (2/3, 1/3)

Topic 3: Diagonalization

1. Compute the diagonalization of markov matrix M above.
   (Hint: $\Lambda = [0.1, 0; 0, 1]$, $S = [1, 2; -1, 1]/\sqrt{3}$).

2. Compute M^n. What happens when $n \to \infty$?
   (Hint: $S\Lambda^n S^{-1} = [1, 2; -1, 1] * [0.1^n, 0; 0, 1] * [1, -2; 1, 1]/3$. When n goes to infinity, $M^n = [2, 2; 1, 1]/3$.

18.06 Lecture 23 αw

Markov Matrices

① All entries ≥ 0

② All colums add to 1

steady state: $\lambda = 1$

$$A = \begin{bmatrix} .1 & .01 & .3 \\ .2 & .99 & .3 \\ .7 & 0 & .4 \end{bmatrix}$$

1. $\lambda = 1$ <u>is</u> an evalue

2. All other evals $|\lambda_c| \leq 1$

$$U_k = A^k u_0 = c_1 \underbrace{\lambda_1}_{=1}{}^k x_1 + c_2 \underbrace{\lambda_2}_{<1}{}^k x_2 \rightarrow 0$$

$\lambda_2^{\infty} = 0$

steadystate

$\rightarrow c_1 x_1$

1

$$A^{-1} = \frac{C^T}{\det(A)}$$

$$Q^T Q = I$$

$$P = A(A^T A)^{-1} A^T \qquad P^T = P \quad P^2 = P$$

$$A^T A x = A^T b$$

Gram Schmidt

$$A = a$$

$$B = b - \frac{A^T b}{A^T A} A$$

$$C = c - \frac{A^T c}{A^T A} A - \frac{B^T c}{B^T B} B$$

Differential Equations $\frac{du}{dt} = Au$

Exponential e^{At} of a matrix

Example

$$\frac{du_1}{dt} = -u_1 + 2u_2 \qquad u(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\frac{du_2}{dt} = u_1 - 2u_2$$

$$A = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \qquad |A - \lambda I| = \begin{vmatrix} -1-\lambda & 2 \\ 1 & -2-\lambda \end{vmatrix}$$

$$= -\lambda^2 + 3\lambda = 0$$

$$\lambda(\lambda+3)$$

Eigenvals

$\lambda = 0, -3$

Eigenvectors

$\lambda_1 = 0$

$$A = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

$$x_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad Ax_1 = 0x_1$$

$\lambda_2 = -3$

$$A = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_2 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

$$x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad Ax_2 = -3x_2$$

Solution:

$$u(t) = \underbrace{c_1 e^{\lambda_1 t}}_{\text{Pure solution}} x_1 + \underbrace{c_2 e^{\lambda_2 t} x_2}_{\text{Pure solution}}$$

Check: $\frac{du}{dt} = Au$ Plug in

$$\lambda_1 \cancel{e^{\lambda t}} x_1 = A \cancel{e^{\lambda t}} x_1$$

$$\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

S ← eigenvector matrix

$$u(t) = c_1 \cdot 1 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

use initial condition $u(0)$

At $t=0$

$$c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$c_1 = 1/3 \qquad c_2 = 1/3$$

$$u(t) = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{1}{3} e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Steadystate $u(\infty) = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

① Stability $u(t) \to 0$

$e^{\lambda t} \to 0 \Rightarrow \lambda < 0$

only the real part b/c $e^{xit} = 1$

② Steady state

$\lambda_1 = 0$ and other $\mathrm{Re}\,\lambda < 0$

③ Blow up

if any $\mathrm{Re}\,\lambda > 0$

2 x 2 stability

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\mathrm{Re}\,\lambda_1 < 0$, $\mathrm{Re}\,\lambda_2 < 0$

Trace $(a+d) = \lambda_1 + \lambda_2 < 0$

$\det = \lambda_1 \lambda_2 > 0$

$\frac{du}{dt} = Au$

Set $u = Sv$

$S \frac{dv}{dt} = ASv$

$\frac{dv}{dt} = S^{-1}ASv = \Lambda v$ $\frac{dv_1}{dt} = \lambda_1 v_1$

$v(t) = e^{\Lambda t}(v(0))$

☆ $u(t) = \underbrace{S e^{\Lambda t} S^{-1}}_{e^{At}} u(0)$

Taylor Series

Matrix Exponentials e^{At} ← power series $e^x = \sum \frac{x^n}{n!}$

$e^{At} = I + At + \frac{(At)^2}{2} + \frac{(At)^3}{6} + \ldots + \frac{(At)^n}{n!}$

$= \underset{SS^{-1}}{I} + S\Lambda S^{-1}t + \frac{S\Lambda \cancel{S^{-1}S}\Lambda S^{-1}t^2}{2} \ldots$

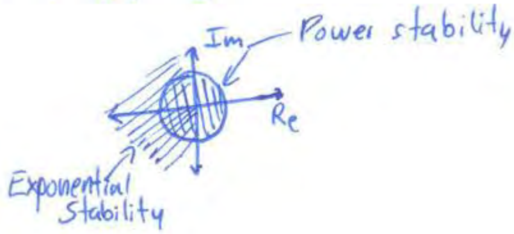
$= Se^{\Lambda t}S^{-1}$ – works if A is diagonalizable

$e^{\Lambda t}$

$$\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \qquad e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix}$$

e^{At} goes to 0 when $\to 0 \Rightarrow R\lambda < 0$

$$e^{At} = S e^{\Lambda t} S^{-1}$$



$$y'' + by' + ky = 0$$

Take 1 2nd order equation $\Rightarrow$ 2 x 2 1st order

$$u = \begin{bmatrix} y' \\ y \end{bmatrix} \qquad u' = \begin{bmatrix} y'' \\ y' \end{bmatrix}$$

$$u' = \begin{bmatrix} -b & -k \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y' \\ y \end{bmatrix}$$

4/9/14

18.06 Lecture

Differential Equations

$$y' = Ay$$

Solutions $y(t) = e^{\lambda t} x$

Stability $\operatorname{Re}\lambda < 0$

Matrix Exponential $\underset{y(t)}{\,} e^{At} y(0)$

Example

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad \text{Given } \begin{matrix} y_1(0) \\ y_2(0) \end{matrix}$$

① Find Eiganstuff

$$\begin{vmatrix} 4-\lambda & 3 \\ 1 & 2-\lambda \end{vmatrix} = \lambda^2 - \underset{\text{Trace}}{6}\lambda + \underset{\text{det}}{5} = 0$$

$$(\lambda-5)(\lambda-1)$$

$$\lambda_1 = 5$$

$$\lambda_2 = 1$$

Eiganvector

$$\begin{bmatrix} -1 & 3 \\ 1 & -3 \end{bmatrix} x_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$y' = \lambda \cancel{e^{\lambda t}} x = A \cancel{e^{\lambda t}} x$$

$$\lambda x = Ax$$

$$\begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = c_1 e^{\overset{\lambda_1 t}{5t}} \overset{x_1}{\begin{bmatrix} 3 \\ 1 \end{bmatrix}} + c_2 e^{\overset{\lambda_2 t}{\cancel{t}}} \overset{x_2}{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}$$

when $t = 0$

$$\begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = c_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$c_1\begin{bmatrix}3\\1\end{bmatrix} + c_2\begin{bmatrix}1\\-1\end{bmatrix} \Rightarrow \underbrace{\begin{bmatrix}3 & 1\\1 & -1\end{bmatrix}}_{S}\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}y_1(0)\\y_2(0)\end{bmatrix} = y(0)$$

Eiganvector matrix

① $\begin{bmatrix}c_1\\c_2\end{bmatrix} = S^{-1}\begin{bmatrix}y_1(0)\\y_2(0)\end{bmatrix}$

Find coefficent by using initial values

② Multiply c_1 by $e^{\lambda t}$
c_2 by $e^{\lambda_2 t}$

③ Add the 2 solutions
$y(t) = c_1 e^{5t} x_1 + c_2 e^{t} x_2$

Stability

Both eiganvalue → negative real part

Stable matrices

2×2 $A = \begin{bmatrix}a & b\\c & d\end{bmatrix}$

want $Re\,\lambda < 0$ $Re\,\lambda_2 < 0$

Real part

negative

- Trace $= a + d < 0$
$= \lambda_1 + \lambda_2$

also applies

If λ's are complex, but A real.

complex conjugates

$\lambda_1 = \sigma + ib$ $\lambda_2 = \sigma - ib$

$A = \begin{bmatrix}-4 & 3\\1 & 2\end{bmatrix}$

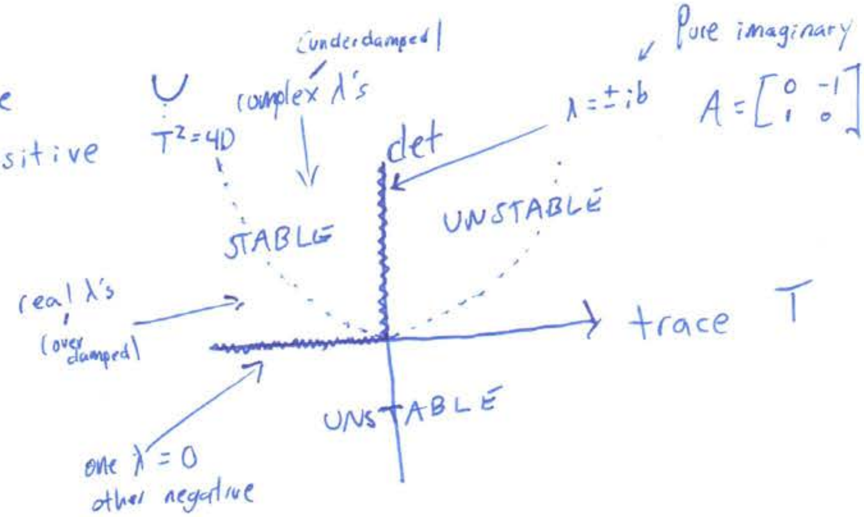
trace $= -2$ ✓

det $= -11$ ✗ bad, should be positive

$= \lambda_1\lambda_2$

2/13

- Trace is negative
- Determinant is positive
  $= \lambda_1 \lambda_2$



trick

$$\det(A - \lambda I) = \lambda^2 - T\lambda + D = 0$$

damping coefficient

stiffness coefficient

$$x'' + bx' + kx$$

1 second order equation $\rightarrow$ $\boxed{18.03}$ $\rightarrow$ 2 first order equation

$$y_1 = x \qquad y_2 = x'$$

$$y_1' = y_2$$

$$y_2' = -b y_2 - k y_1$$

$$= \begin{bmatrix} 0 & 1 \\ -k & -b \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

companion

Eiganstuff

$$\det \begin{pmatrix} 0-\lambda & 1 \\ -k & -b-\lambda \end{pmatrix} = \lambda^2 + b\lambda + k = 0$$

next time

$$e^{At} = I + At + \frac{1}{2!}(At)^2 + \frac{1}{3!}(At)^3 + \ldots$$

$$= S e^{\Lambda t} S^{-1}$$

18.06 Lecture #25 ocw

Symmetric matrices

Eigenvalues/Eigenvectors

Positive Definite Matrices

$A = A^T$ ← Real

① The eigenvalues are <u>Real</u>

② The eigenvectors are <u>perpendicular</u>

↳ columns of Q

usual case: $A = S\Lambda S^{-1}$

* symmetric case $A = Q\Lambda Q^{-1} \qquad Q^{-1} = Q^T$

$= Q\Lambda Q^T$

why are the eigenvalues real

complex conjugate $\qquad \overline{a+ib} = a - ib$

$Ax = \lambda x \implies \bar{A}\bar{x} = \bar{\lambda}\bar{x}$

$\bar{x}^T A x = \lambda \bar{x}^T x$

$\Rightarrow \bar{x}^T \underbrace{A^T}_{A} = \bar{x}^T \bar{\lambda}$

$\bar{x}^T A x = \bar{x}^T \bar{\lambda} x$

$\lambda \bar{x}^T x = \bar{\lambda} \bar{x}^T x$

$\lambda = \bar{\lambda}$

$A = Q\Lambda Q^T$

$= \begin{bmatrix} | & | & \\ q_1 & q_2 & \cdots \\ | & | & \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \lambda_n \end{bmatrix} \begin{bmatrix} q_1 & — \\ q_2 & — \\ & — \end{bmatrix} = \lambda_1 \underbrace{q_1 q_1^T}_{\text{Projection matrix}} + \lambda_2 q_2 q_2^T + \cdots$

⇒ Every symmetric Matrix is a combination of perpendicular projection matrices

$A = A^T$

# of signs (pos, neg) of the pivots = # of signs (pos, neg) of λ's

Positive definite symmetric matrix

- All eigenvalues are positive
- All pivots are positive
- All sub determinant are positive

$$\begin{bmatrix} 5 & 2 \\ 2 & 3 \end{bmatrix}$$

determinant

Pivots: $5, \frac{11}{5}$

$\lambda_1 = 4 \pm \sqrt{5}$

18.06 Lecture

$y' = Ay \quad y(0)$ given

$y = c_1 e^{\lambda_1 t} x_1 + \dots + c_n e^{\lambda_n t} x_n$

ok if

$Ax_1 = \lambda_1 x_1$

$= e^{At} y(0)$

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots$$

$$\frac{d}{dx} e^x = 1 + x + \frac{1}{2}x^2 + \dots = e^x$$

$$e^x e^t = e^{x+t} \qquad e^{At} = I + At + \frac{1}{2!}(At)^2 + \dots$$

$$\frac{d}{dt} e^{At} = A + A^2 t + \frac{A^3 t^2}{2}$$

$$= A\left(I + At + \frac{A^2 t^2}{2!}\right) = Ae^{At}$$

$$\frac{d}{dt}\left[e^{At} y(0)\right] = Ae^{At} y(0)$$

$$e^A e^B \neq e^{A+B}$$

not always

$$e^{At} e^{Ax} = e^{A(t+x)}$$

$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

skew-symmetric

$A^T = -A$

$(e^A)^{-1} = A^{-A} = (e^A)^T$

always invertible. no zero λ

$e^A e^{-A}$

$$\left(I + A + \frac{A^2}{2}\right)\left(I - A + \frac{A^2}{2} \dots\right) = I$$

If $A^T = -A$

then $e^A =$ orthogonal matrix

$e^{i\theta} \to$ orthog matrix

$r \to$ symmetric matrix

Every matrix $A = S\,Q$

some symmetric (S), some orthogonal (Q)

degrees of freedom.. n^2

$$\underset{n^2}{A} = \underset{\frac{n(n+1)}{2}}{S}\;\underset{\frac{n(n-1)}{2}}{Q}$$

$$e^{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}t} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}t + -\frac{t^2}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{t^3}{3!}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 - t^2/2 + \dots & t - t^3/6 \\ -t + t^3/6 + \dots & 1 - t^2/2 \end{bmatrix}$$

> Apparently this means sin cos because of Euler, or something like that :)

$$= \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

Defective Matrix

$$A = \begin{bmatrix} 4 & 16 \\ -1 & -4 \end{bmatrix}$$

repeated $\lambda = 0,0$
only 1 evector

trace = 0
det = 0

$Ax = 0x$

$$x \underset{c}{=} \begin{bmatrix} -4 \\ 1 \end{bmatrix}$$

the only eigen vector

$$e^{At}x = \left(I + At + \tfrac{1}{2}A^2t^2 \dots\right) x$$
$$= \left(1 + \lambda t + \tfrac{1}{2}\lambda^2 t^2 \dots\right) x$$
$$e^{At}x = e^{\lambda t}x$$

always

Real
Symmetric matrix $S = S^T$

① Eigan values are real
② Eigan vectors are orthogonal

①
$Sx = \lambda x \rightarrow x^T S^T = \lambda x^T = x^T S$

$S\bar{x} = \bar{\lambda}\bar{x}$

Show that λ is real

- Complex conjugate is the same

$x^T S \bar{x} = \bar{\lambda} x^T \bar{x}$

$x^T S \bar{x} = \lambda x^T \bar{x}$

Therefore: λ is real.

18.06 Lecture #27 ocw

Positive Definite Matrix (Tests)

Tests For Minimum $x^TAx > 0$

Ellipsoids in R^n

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Positive Definite Tests

① $\lambda_1 > 0 \quad \lambda_2 > 0$

② $a > 0 \quad ac - b^2 > 0$

③ Pivots $a > 0 \quad \frac{ac-b^2}{a} > 0$ ← second pivot

* ④ $x^TAx > 0$

Examples

$\begin{bmatrix} 2 & 6 \\ 6 & x \end{bmatrix}$ Make Positive Definite

$x > 18$

if $x = 18 \rightarrow$ Positive semidefinite

$\lambda = 0, 20$

Pivots

2, None

$$[x_1\ x_2]\begin{bmatrix} 2 & 6 \\ 6 & 18 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [x_1\ x_2]\begin{bmatrix} 2x_1 & 6x_2 \\ 6x_1 & 18x_2 \end{bmatrix} = 2x_1^2 + 12x_1x_2 + 18x_2^2 \overset{?}{>} 0$$

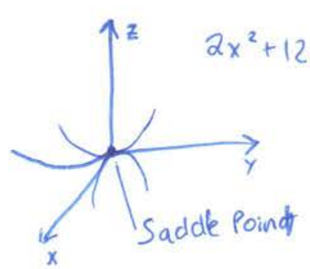
$$ax^2 + 2bxy + cy^2$$

Graphs of $f(x,y) = \vec{x}^TA\vec{x}$

$= ax^2 + 2bxy + cy^2$

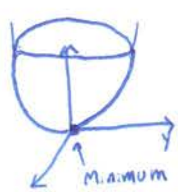
use $\begin{bmatrix} 2 & 6 \\ 6 & 7 \end{bmatrix}$ ↗ not positive definite

$2x^2 + 12xy + 7y^2$



$\begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix}$

$x^TAx > 0$ except $x = 0$



$f(x,y) = 2x^2 + 12xy + 20y^2$

$= 2(x + 3y)^2 + 2y^2$ → completing the square

↳ all squares

↳ has to be positive!

< 1st deriv = 0

< 2nd deriv > 0 calc

Calculus: Min ~ $\frac{d^2u}{dx^2} > 0$

$\frac{du}{dx} = 0$

$$\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

$f_{xy} = f_{yx}$ Always b/c calculus

18.06: Min ~ Matrix of 2nd derivs

$f(x_1 x_2 \ldots x_n)$ is positive definite

Elimination

$$\underset{A}{\begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix}} \rightarrow \underset{U}{\begin{bmatrix} 2 & 6 \\ 0 & 2 \end{bmatrix}}$$

$$L = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

* Completing the square is elimination

3×3 Example

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Dets: 2, 3, 4

Pivots: 2, $3/2$, $4/3$

Eigenvalues: $2-\sqrt{2}$, 2, $2+\sqrt{2}$

$$x^T A x = 2x_1^2 + 2x_2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3 > 0$$

/12

Diagonalization $A = S\Lambda S^{-1}$

symmetric $A = Q\Lambda Q^T$

2^{nd} order

$y'' + Ay = 0$

Positive Definite Matrix

- symmetric Matrix with positive eiganvals

$A = A^T$
symmetric

① λ's are real

② x's (evectors) are orthonormal

Eiganvector matrix $S = \begin{bmatrix} | & | & | \\ x_1 & x_2 & x_3 \\ | & | & | \end{bmatrix}$

← orthogonal matrix

$Q^{-1} = Q^T$

$A = Q\Lambda Q^T$

$\frac{dy}{dt} = \lambda y$

$y' = Ay$ Solutions: $e^{\lambda t}x \rightarrow Ax = \lambda x$

$y'' = -Ay$

$y'' + Ay = 0$

1x1 matrix

$y'' + w^2 y = 0 \longrightarrow y = b \underset{y'(0)/w}{\sin wt}, \underset{y(0)}{a\cos wt}$

$y(0)$ $y'(0)$

symmetric matrix w/ positive eiganvals

$y'' + Ay = 0$

$y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$ Given $y(0)$ n conditions
and $y'(0)$ n conditions

→ want 2n constants in $y(t)$

$y(t) = a_1 \cos w_1(t) x_1 + b_1 \sin w_1(t) x_1$

Example

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}'' + \underbrace{\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}}_{A} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Find Eiganvals

$$\begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = (2-\lambda^2)(\cancel{1}) = \lambda^2 - 4\lambda + 3 = 0$$

$$(\lambda - 3)(\lambda - 1)$$

$$\lambda_1 = \cancel{3}\ 1$$
$$\lambda_2 = 3$$

$$\left.\begin{matrix} w_1 = \sqrt{1} \\ w_2 = \sqrt{3} \end{matrix}\right\} \text{because 2 derivatives}$$

Find Eigan vectors

$$x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Orthogonal ← b/c matrix is symmetric

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = a_1 \cos(t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + b_1 \sin(t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + a_2 \cos(\sqrt{3}t) \begin{bmatrix} 1 \\ -1 \end{bmatrix} + b_2 \sin(\sqrt{3}t) \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Positive Definite Matrices

- Always symmetric

Test #1: All $\lambda_i > 0$

#2: All pivots > 0

#3: All upper left det > 0

#4

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 2 & 5 \end{bmatrix}$$

dets = 2, 5, 18

2/12

(2)

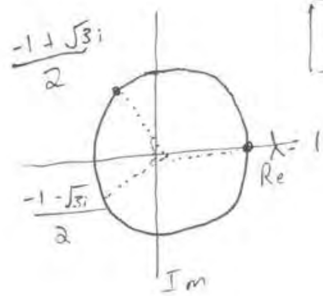
$$P_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Find Eigenvals

$= \det(P_1 - \lambda I) = 0 \rightarrow \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & 0 & -\lambda \end{vmatrix} = -\lambda \begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} - 1 \begin{vmatrix} 0 & 1 \\ 1 & -\lambda \end{vmatrix}$

$= -\lambda^3 + 1 = 0$

$\boxed{\lambda^3 = 1}$



Eiganvals can be complex

$$A = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}$$

a) Diagonalize A

b) Compute e^{At}

a) $A = S \Lambda S^{-1}$

Any Eiganvals

$$\Lambda = \begin{pmatrix} & 0 \\ 0 & \end{pmatrix}$$

$\begin{vmatrix} -\lambda & 1 \\ -2 & 3-\lambda \end{vmatrix} = 0 \iff -\lambda(3-\lambda) + 2 = 0$

$\lambda^2 - 3\lambda + 2 = 0$

$\lambda_1 = 1$

$\lambda_2 = 2$

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

$\lambda_1 = 1 \quad \left[\begin{array}{cc|c} -1 & 1 & 0 \\ -2 & 2 & 0 \end{array}\right] \rightarrow \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$x - y = 0$

$x = y$

$\{(x, x) \mid x \in \mathbb{R}\}$

$x = 1 \rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$S = \begin{pmatrix} | & | \\ | & | \end{pmatrix}$$

Eiganvector matrix $\lambda_{1\,ev}$ $\lambda_{2\,ev}$

$\lambda_2 = 2 \quad \begin{bmatrix} -2 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$S = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\boxed{A = \overbrace{\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}}^{S} \underbrace{\overbrace{\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}}^{\Lambda}}_{\text{Diagonal}} \overbrace{\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}}^{S^{-1}}}$$

b) Compute Exponential e^{At}

$$e^{At} = I + At + \frac{1}{2}(At)^2 + \dots\dots$$

$$= I + (S\Lambda S^{-1})t + \frac{1}{2}\underbrace{((S\Lambda S^{-1})t)^2}_{(S\Lambda \cancel{S^{-1}})(\cancel{S}\Lambda S^{-1})}$$

$$= \underbrace{S(I + \Lambda t + \frac{1}{2}(\Lambda t)^2 + \dots) S^{-1}}_{\text{by definition}}$$

$$e^{\Lambda t} = \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix} \qquad \Lambda^n = \begin{bmatrix} 1^n & 0 \\ 0 & 2^n \end{bmatrix}$$

$$A = \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}}_{S} \underbrace{\begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}}_{\cancel{e^{\Lambda}}\ e^{At}} \underbrace{\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}}_{S^{-1}}$$

$$\boxed{= \begin{bmatrix} 2e^t - e^{2t} & -e^t + e^{2t} \\ 2e^t - 2e^{2t} & ee^t + 2e^{2t} \end{bmatrix}}$$

Compute Exponential e^{At}

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Repeated eiganvalues $\xrightarrow{\text{not always}}$ not diagonalizable $\to$ use definition

By definition:

$$e^{At} = I + At + \frac{1}{2}(At)^2 + \dots$$

$$A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow A^n = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ if } n \geq 2$$

$$e^{At} = I + At + \underbrace{\frac{1}{2}(At)^2 + \dots}_{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}$$

$$= I + At$$

$$= \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

3 on HW

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

a) Diagonalize A

b) compute $V\Lambda^k V^{-1}$

c) Prove $A^k = \frac{1}{2}\begin{bmatrix} 1+3^k & 1-3^k \\ 1-3^k & 1+3^k \end{bmatrix}$

a) Eiganvalues

$$\begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 1 = 4 - 4\lambda + \lambda^2 - 1 = 0$$

$$\lambda^2 - 4\lambda + 3 = 0$$

$$(\lambda-3)(\lambda-1)$$

$$\lambda_1 = 3$$
$$\lambda_2 = 1$$

$$\begin{bmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{bmatrix}$$

$$\lambda = 1 \quad \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0 \quad \text{ev}$$

$$\lambda = 3 \quad \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0 \quad \text{ev}$$

$$A = V\Lambda V^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

b) $V\Lambda^k V^{-1}$

$$\Lambda^k = \begin{bmatrix} 1^k & 0 \\ 0 & 3^k \end{bmatrix}$$

$$V\Lambda^k V^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3^k \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1+3^k & 1-3^k \\ 1-3^k & 1+3^k \end{bmatrix}$$

c) $A^k = (V\Lambda V^{-1})^k$

$$= (V\Lambda \cancel{V^{-1}})(\cancel{V}\Lambda \cancel{V^{-1}})(\cancel{V}\Lambda V^{-1}) \dots\dots$$

$$= V \Lambda^k V^{-1}$$

↳ matches b)

4/4

4/16/14

18.06 Lecture

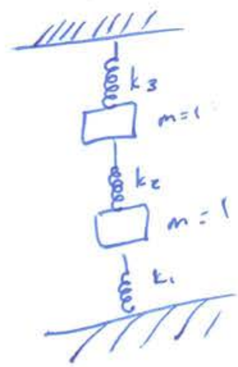
Positive definite M
- All $\lambda_i > 0$
- All pivots > 0
- All upper left dets > 0
- $M = A^TA$, independent columns in A

$\star$ Energy ≥ 0

Energy
↳ Definition of pos. def

For every vector $x \neq 0$, $x^TMx > 0$

Example:
$k_1 = k_2 = k_3 = 1$



$$M = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$\lambda = 3, 1$ ✓
pivots $= 2, \frac{3}{2}$ ✓
dets $= 2, 3$ ✓

↳ $A^TA = M$

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$x^TMx > 0$

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} =$$

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} 2x_1 - x_2 \\ -x_1 + 2x_2 \end{bmatrix} = 2x_1^2 + 2x_2^2 - 2x_1x_2$$

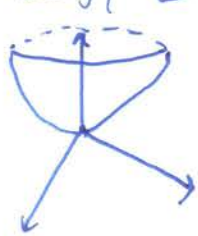
Shortcut

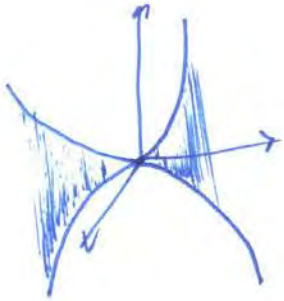
How many x_1^2, x_2^2

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

how many $x_1 x_2$

Positive energy $E = ax_1^2 + 2bx_1x_2 + cx_2^2$





SADDLE

$x^2 - y^2$ or $x^2 - x_2{}^2$

$$M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$z = f(x, y)$

$\frac{df}{dx} = 0 \quad \frac{df}{dy} = 0$

$$\left(\frac{d^2f}{dx^2}\right)\left(\frac{d^2f}{dy^2}\right) > \left(\frac{d^2f}{dxdy}\right)$$

- See OCW

18.06 Lecture #28 ocw

Similar Matrices and Jordan Form

A^TA is positive definite
similar Matrix A, B
$B = M^{-1}AM$
Jordan Form

Positive definite matrix means

$x^TAx >$ except for $x = 0$

Eiganvalues λ of inverse $= \frac{1}{\lambda}$ $\Rightarrow$ inverse of positive definite is pos def.

If A, B are pos def
$A + B$ pos def? yes
$x^T(A+B)x > 0$

Now A m by n
A^TA square and symmetric and posinite definite
$x^TA^TAx > 0$ rank n so only 0 in nullspace
$= (Ax)^T(Ax) = \|Ax\|^2 > 0$

Similar Matrices

$\overset{n\times n}{A}$ and B are similar
means: For some M
$B = M^{-1}AM$

Example: A is similar to Λ diagonal Matrix
$S^{-1}AS = \Lambda$

$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ $\Lambda = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

$$\underbrace{\begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}}_{M^{-1}}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\underbrace{\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}}_{M} = \underbrace{\begin{bmatrix} -2 & -15 \\ 1 & 6 \end{bmatrix}}_{B}$$

★ Similar Matrices have the same eigen values ★

Reminder:

Trace = sum of λ_r

Det = prod of λ_s

$$B = M^{-1}AM$$

$$Ax = \lambda x \qquad \lambda \text{ eigenval of } A$$

$$\underbrace{M^{-1}AM}_{B}M^{-1}x = \lambda M^{-1}x$$

$$BM^{-1}x = \lambda M^{-1}x \qquad \lambda \text{ eigenval of } B$$

eigenvector of B is different
$= M^{-1}(EV(A))$

<u>Bad Case</u>: $\lambda_1 = \lambda_2$ Matrix might not be diagonalizable

$\lambda_1 = \lambda_2 = 4$

$$\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix} \leftarrow \text{Jordan form}$$

More members of Family

$$\begin{bmatrix} 5 & 1 \\ -1 & 3 \end{bmatrix}, \begin{bmatrix} 4 & 0 \\ 17 & 4 \end{bmatrix}, \begin{bmatrix} a & \sim \\ \sim & 8-a \end{bmatrix}$$

Trace = 8
Det = 16

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{not similar to}} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \leftarrow \text{Jordan Block}$$

$\lambda = 0,0,0,0$

1 eigen vector

$$J_i = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \lambda_i & 1 & \\ & & \lambda_i & 1 \\ 0 & & & \lambda_i \end{bmatrix}$$

Every square matrix A is similar to J

$$J = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_d \end{bmatrix}$$

blocks = # eigenvectors

Good Case:
J is Λ

18.06 Recitation

Fernando Trujano

Q: A symmetric matrix can't be similar to a non symmetric

A symmetric, P invertible $\overset{?}{\Rightarrow}$ $P^{-1}AP$ is symmetric?

$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$P = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ $\quad P^{-1}AP = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}}_{\text{Not symmetric}}$

$P^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Q: An invertible matrix can't be similar to a singular matrix?

A invertible, P invertible $\overset{?}{\Rightarrow}$ $P^{-1}AP$ is invertible?

Assume that $P^{-1}AP$ is singular → get contradiction

$\Rightarrow \exists\ v \neq 0$ such that $(P^{-1}AP)v = 0$

$\Downarrow$

$APv = 0$

$A(\underbrace{Pv}_{\neq 0}) = 0$ null space of A

A is invertible. $N(A) = 0$

Q: A can't be similar to -A unless A = 0 ?

$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ } Same eigenvalues 1, -1

$-A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

Find P invertible such that $P^{-1}AP = -A$

P that interchanges colums

$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad P^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$P^{-1}AP$

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

- They are similar

(2) B invertible.

Prove that AB is similar to BA

Find P invertible such that $P^{-1}(AB)P = BA$

Solution $P = B^{-1}$

Singular Value Decomposition

$$A = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} \qquad \text{Compute } A = U\Sigma V^T$$

(1) Compute $A^TA = \begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$

(2) Find eigenstuff of A^TA

*Eigenvectors of symmetric matrices are orthogonal

$$\begin{vmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix} = 0 \quad (5-\lambda)^2 - 3^2 = 0$$

$$(5-\lambda-3)(5-\lambda-3) = 0$$

$$\lambda_1 = 8$$
$$\lambda_2 = 2$$

$\lambda_1 = 8$

$$\left[\begin{array}{cc|c} -3 & 3 & 0 \\ 3 & -3 & 0 \end{array}\right] \Leftrightarrow -3x + 3y = 0 \Leftrightarrow x = y$$

$$\rightsquigarrow \{(x,x) \mid x \in \mathbb{R}\}$$

dimension = 1

$$= \underbrace{\langle (1,1) \rangle \tfrac{1}{\sqrt{2}}}_{v_1}$$

$$\lambda_2 = 2 \left[\begin{array}{cc|c} 3 & 3 & 0 \\ 3 & 3 & 0 \end{array}\right] \Leftrightarrow 3x + 3y = 0 \Leftrightarrow x = -y$$

$$\rightsquigarrow \{(-x,x) \mid x \in \mathbb{R}\}$$

dim

$$= \underbrace{\langle (-1,1) \rangle \tfrac{1}{\sqrt{2}}}_{v_2}$$

orthonormal

$$\longrightarrow V = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$\rightarrow$

$$V^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

③ Find U $\qquad u_1 = \frac{Av_1}{\|Av_1\|}$

- do same as ② with AA^T

OR

$$Av_1 = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 2\sqrt{2} \\ 0 \end{bmatrix}$$

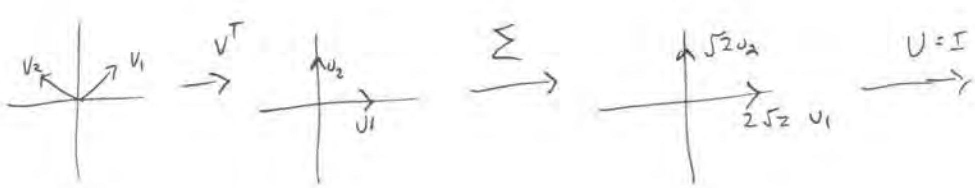
$$Av_2 = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{2} \end{bmatrix}$$

$$\begin{bmatrix} 2\sqrt{2} \\ 0 \end{bmatrix} = \overbrace{2\sqrt{2}}^{\Sigma_1\ \sigma_1}\underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{u_1}$$

$$\begin{bmatrix} 0 \\ \sqrt{2} \end{bmatrix} = \overbrace{\sqrt{2}}^{\Sigma_2}\underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{u_2} \qquad U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A = U \Sigma V^T$$

$$\boxed{A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}$$



3/3

4/18/14 18.06 Lecture

Positive definite A+B
Matrices in Engineering
Similar matrices $B = M^{-1}AM^{+1}$

If A, B are pos def symm: Show A+B is too

Proof: $x^TAx > 0$ all $x \neq 0$ $\quad x^TBx > 0 \quad$ ADD $\quad x^T(A+B)x > 0$

$$\begin{bmatrix} 3 & 1 & 0 \\ 1 & 0 & 4 \\ 0 & 4 & 10 \end{bmatrix}$$

- not positive definite

$$\begin{bmatrix} & & \\ & & \\ & & 0 \end{bmatrix}$$

not positive definite

$$\underset{1\times5}{[0\,0\,\ldots\,1]}\ \underset{5\times5}{\begin{bmatrix} & & \\ & & \\ & & 0 \end{bmatrix}}\ \underset{5\times1}{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}} = 0\,x_5^2 = 0$$

Matrices in Engineering

$$\frac{d\vec{y}}{dt} = A\vec{y}$$

$$y(t) = e^{At}y(0)$$

Euler's Method

$$\frac{y(t+\Delta t) - y(t)}{\Delta t} = Ay(t)$$

$$y(t+\Delta t) = (I + \Delta t A)\,y(t)$$

$$y_{n+1} = (I + \Delta t A)\,y_n$$

SOLUTION $\quad y_n = (I + \Delta t A)^n y_0$

0 $\quad t = T$

n steps

$$\Delta t = T/n$$

$$= \left(I + \frac{TA}{n}\right)^n \rightarrow e^{At}$$

$n \to \infty$

$n\,\Delta t = t$

error $\approx \frac{1}{n}$

Similar Matrices $B = M^{-1} A M^{+1}$

$$\begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -5 \\ 0 & 1 \end{bmatrix} =$$

$M \qquad \lambda = 3, 1 \qquad M^{-1}$

$$\begin{bmatrix} -3 & 9 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -5 \\ 0 & 1 \end{bmatrix} =$$

4/23/14 18.06 Lecture Fernando Trujano

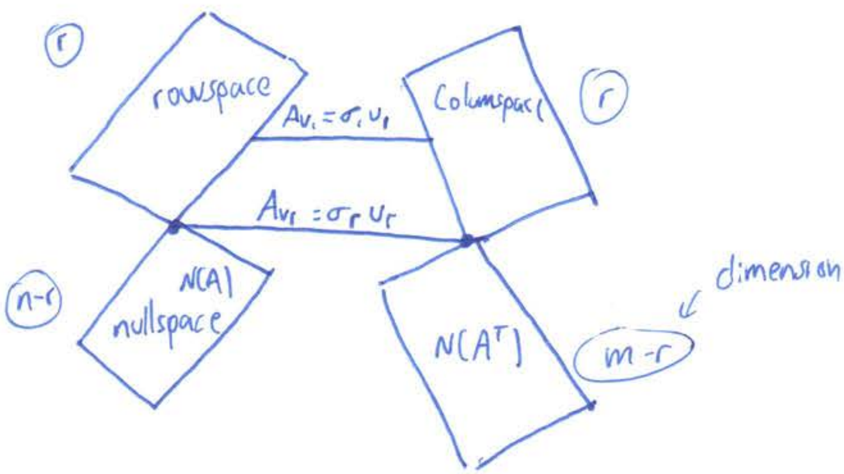
SVD = Singular Value Decomposition

Any Matrix

$$\underset{m\times n}{A} = \underset{m\times m}{U}\ \underset{m\times n}{\Sigma}\ \underset{n\times n}{V^T}$$

diagonal

orthogonal



$$\underset{m\times n}{A}\ \underset{n\times 1}{v_1} = \sigma_1 u_1$$

Limitations of $Ax = \lambda x$

- Square
- x's not orthogonal (usually)
- sometimes not full set of x's

$$A = U\Sigma V^T$$

$$= \begin{bmatrix} u_1 & u_r & u_m \end{bmatrix} \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_r \end{bmatrix} \begin{bmatrix} v_1 & \cdots & v_r & \cdots & v_n \end{bmatrix}^T$$

compare $A = U\Sigma V^T$ when same as ??? $S\Lambda S^{-1}$

$U = S$

★ $\Sigma = \Lambda$

$V^T = S^{-1}$

Symmetric:
Positive Definite $A = Q\Lambda Q^T$
↪ $U = V = Q$

unit v's eiganvectors of a symmetric matrix $= A^TAv_i = \lambda v_i$

λ's eiganvalues of A^TA

A^TA is positive semidefinite $\rightarrow \lambda \geq 0$

v's are orthonormal b/c A^TA is symmetric

u's will be $\frac{Av\text{'s}}{\sigma}$ $\frac{Av\text{'s}}{\|Av\|}$

¡¿¿ Why are the Av's orthogonal ??! $(Av_i)^T(Av_j)$

$= v_i^T(A^TAv_j)$

$= v_i^T \lambda_j v_j$

$= \lambda_j \underbrace{v_i^T v_j}_{v\text{'s are orthogonal}}$

$= 0$

I have $\overset{\text{NEW}}{AV = U\Sigma}$ $\qquad$ $AS^{\text{OLD}} = S\Lambda$

$A\left[v_1 \cdots\right] = \left[u \cdots\right]\left[\sigma \cdots\right]\left[\quad\right]^T$

Example

$$A = \begin{bmatrix} -1 & 2 \\ 2 & -1 \\ 2 & 2 \end{bmatrix}$$

① Form A^TA

② Find evectors and evals $\lambda = \sigma^2$

$$\overset{A^T}{\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \end{bmatrix}} \overset{A}{\begin{bmatrix} -1 & 2 \\ 2 & -1 \\ 2 & 2 \end{bmatrix}} = \overset{A^TA}{\begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}}$$

$\lambda = 9, 9$ $\sigma = \sqrt{\lambda}$

$\sigma = 3, 3$

eigenvectors

$N(A^T)$

$$\begin{bmatrix} -1 & 2 \\ 2 & -1 \\ 2 & 2 \end{bmatrix} = \underbrace{\begin{bmatrix} -1/3 & 2/3 & \\ 2/3 & -1/3 & \\ 2/3 & 2/3 & \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 3 & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^T}_{V}$$

$Av_1 = \sigma_1 u_1$

$\frac{Av_1}{\sigma_1} = u_1$

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 5 \end{bmatrix}$$

18.06 Lecture #26 ocw

Complex Matrices:
Fast Fourier Transformation (FFT)

$$z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ x_n \end{bmatrix} \text{ in } \mathbb{C}^n$$

<u>length</u>

$z^T z$ no good

$$\bar{z}_1 z_1 = |z_1|^2$$

so

$$\bar{z}^T z = \text{length}^2$$

$$\begin{bmatrix} 1 & -i \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = 1 + 1 = 2$$

length $= \sqrt{2}$

$$\bar{z}^T z = z^H z$$

Hernitian

Inner product no longer $y^T x$
instead $y^H x$

<u>Symmetric</u> $A^T = A$ no good if A complex
instead $\bar{A}^T = A$

$$A = \begin{bmatrix} 2 & 3+i \\ 3-i & 5 \end{bmatrix}$$

Hermetian Matrix

$$A^H = A$$

<u>Perpendicular</u>

$Q^T Q = I$ ← no good when complex

$q_1, q_2, q_n \ldots$

$$\bar{q}_i^T q_J = \begin{cases} 0 & i \neq J \\ 1 & i = J \end{cases}$$

$$Q = \begin{bmatrix} | & | & & | \\ q_1 & q_2 & \cdots & q_n \\ | & | & & | \end{bmatrix}$$

$$\bar{Q}^T Q = I = Q^H Q$$

Unitary Matrix

Fourier Matrix

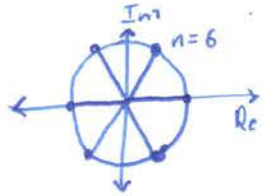
$$F_n = \begin{bmatrix} 1 & 1 & 1 & & 1 \\ 1 & w & w^2 & & w^{n-1} \\ 1 & w^2 & w^4 & & w^{2(n-1)} \\ 1 & w^3 & w^6 & & \vdots \\ \vdots & \vdots & \vdots & & \\ 1 & w^{n-1} & w^{2(n-1)} & & w^{(n-1)^2} \end{bmatrix}$$

$$F_{n(ij)} = w^{ij}$$

What is w

$$w^n = 1$$

$$\boxed{w = e^{i2\pi/n}} = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$$



$n=4 \Rightarrow w^4 = 1$

$w = e^{2\pi i/4} = i$

$i,\ i^2 = -1,\ i^3 = -i,\ i^4 = 1$

$$F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$$

- Orthogonal columns — makes inverse easy

inner product of colums = 0

orthonormal if times $\frac{1}{2}$

$$F_4^H F_4 = I$$

$$(W_{64})^2 = W_{32}$$

$$\begin{bmatrix} F_{64} \end{bmatrix} = \begin{bmatrix} I & D \\ I & -D \end{bmatrix} \begin{bmatrix} F_{32} & 0 \\ 0 & F_{32} \end{bmatrix} \begin{bmatrix} 1 & & 1 & \\ & 1 & & 1 \\ 1 & & 1 & \\ & 1 & & 1 \end{bmatrix}$$

Permutation matrix

$\rightarrow 64^2 \qquad 2(32)^2 + \overset{32}{\text{Fix}}$

$$D = \begin{bmatrix} 1 & & & & \\ & w & & & \\ & & w^2 & & \\ & & & \ddots & \\ & & & & w^{31} \end{bmatrix}$$

FFT. Multiplies in $n\log_2 n$ steps instead n^2

2/2

18.06 Lecture #29

Singular Value Decomposition SVD

$$A = U \Sigma V^T$$

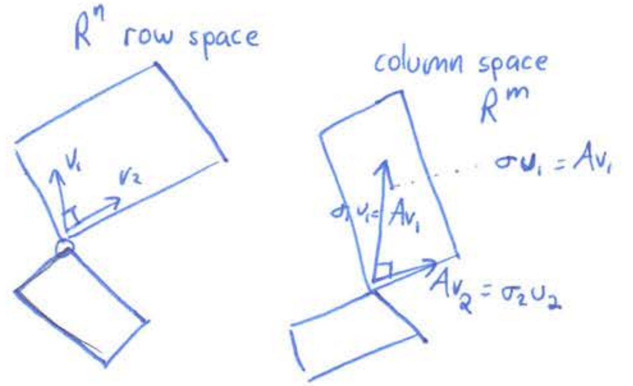
Σ: diagonal

U, V^T: orthogonal

if symmetric positive definite

$$A = Q \Lambda Q^T$$

$$\cancel{A = S \Lambda S^{-1}}$$



$$A \begin{bmatrix} v_1 & v_2 & \cdots & v_r \end{bmatrix} = \begin{bmatrix} u_1 & u_2 & \cdots & u_r \end{bmatrix} \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_r \end{bmatrix}$$

Eigenvectors of a symmetric matrix are orthogonal

$$AV = U\Sigma$$

U: orthonormal basis in columnspace

V: orthonormal basis in rowspace

- Diagonalize the Matrix

$$A = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}$$

$v_1\ v_2$ in rowspace $\mathbb{R}^2$
$u_1\ u_2$ in columspace $\mathbb{R}^2$

$\sigma > 0 \quad \sigma_2 > 0$

$$Av_1 = \sigma_1 u_1$$
$$Av_2 = \sigma_2 u_2$$

$$A = U \Sigma V^{-1}$$
$$= U \Sigma V^T$$

symmetric positive definite

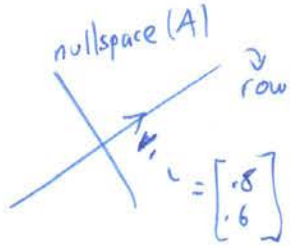
$$A^T A = V \Sigma^T \underbrace{U^T U}_{I} \Sigma V = V \begin{bmatrix} \sigma_1^2 & & & \\ & \sigma_2^2 & & \\ & & \ddots & \\ & & & \sigma_n^2 \end{bmatrix} V^T$$

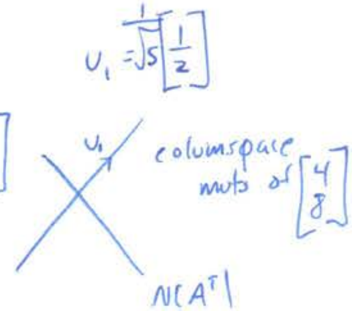
* V's eigenvectors of $A^T A$ *
* U's eigenvectors of AA^T *

Example 2

$$A = \begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix}$$

rank 1





$$\underset{A}{\begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix}} = \underset{U}{\begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & -1/\sqrt{5} \end{bmatrix}} \underset{\Sigma}{\begin{bmatrix} \sqrt{125} & 0 \\ 0 & 0 \end{bmatrix}} \underset{V^T}{\begin{bmatrix} .8 & .6 \\ .6 & -.8 \end{bmatrix}} \quad \underline{I} -$$

1

$$A^TA = \begin{bmatrix} 4 & 8 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix} = \begin{bmatrix} 80 & 60 \\ 60 & 45 \end{bmatrix}$$

rank 1

Eigenvals: 0
125

$v_1, \dots v_r$	orthonormal	basis	row space
$u_1 \dots u_R$	"	"	column space
$v_{r+1} \dots v_n$	"	"	null space (A)
$u_{r+1} \dots u_m$	"	"	$n(A^T)$

$$Av_1 = \sigma_1 u_1$$

Example

$$A^TA = \begin{bmatrix} 4 & -3 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} = \begin{bmatrix} 25 & 7 \\ 7 & 25 \end{bmatrix}$$

normalized Eigenvectors

$$\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \overset{\sigma_1}{32}\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} = 18\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$$\overset{A}{\begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}} = \overset{U}{\begin{bmatrix} \quad & \quad \\ & \end{bmatrix}}\overset{\Sigma}{\begin{bmatrix} \sqrt{32} & 0 \\ 0 & \sqrt{18} \end{bmatrix}}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

Now find U ... u_1, u_2

$$AA^T = U\Sigma\overbrace{V^TV}^{I}\Sigma^TU$$

$$= U\Sigma\Sigma^TU^T$$

$$AA^T = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} 4 & -3 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 32 & 0 \\ 0 & 18 \end{bmatrix}$$

same sigmas

Eigenvector $AA^T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 32\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$AA^T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 18\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

same eigenvalues

Eigenvalues of AB = eval of BA

$$U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \sqrt{32} & 0 \\ 0 & \sqrt{18} \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

↳ watch out! signs are wrong!

18.06 Lecture #32

- 6.1 - 6.2 λ and x
- 6.3 $\frac{du}{dt} = Au$ and e^{At}
- 6.4 $A = A^T$
  - Eigenvals real
  - Always enough eigenvectors $A = Q \Lambda Q^T$
- 6.5 Positive Definite
- 6.6 Similar $B = M^{-1} A \cancel{m} M$ ← same eigenvalues $B^k = M^{-1} A^k M$
- 6.7 SVD
  - $A = U \Sigma V^T$

Example 1

$$\frac{du}{dt} = Au = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} u$$

$$u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2 + c_3 e^{\lambda_3 t} x_3$$

$\lambda_1 = 0$ ← because singular

$$|A - \lambda I| = \begin{vmatrix} -\lambda & -1 & 0 \\ 1 & -\lambda & -1 \\ 0 & 1 & -\lambda \end{vmatrix} = -\lambda^3 - 2\lambda = 0$$

$$\lambda^2 + 2\lambda = 0$$

$$\lambda(\lambda^2 + 2) = 0$$

$\lambda_1 = 0$ $\quad$ $\lambda^2 = -2$

$\lambda_2 = \sqrt{2} i$

$\lambda_3 = -\sqrt{2} i$

$$u(t) = c_1 e^{0t} x_1 + c_2 e^{\sqrt{2} i t} x_2 + c_3 e^{-\sqrt{2} i t} x_3$$

Find Period T

$\sqrt{2} \cancel{i} T = 2\pi \cancel{i}$

$T = \pi \sqrt{2}$

* Eigenvectors of skew-symetric are also orthogonal

Orthogonal Evectors $AA^T = A^T A$ | symmetric, skew-symetric, orthogonal

- Find e^{At}

$$u(t) = e^{At} u(0)$$

$$e^{At} = S \left(e^{\Lambda t}\right) S^{-1} \qquad \begin{bmatrix} e^{\lambda_1 t} & & \\ & e^{\lambda_2 t} & \\ & & \ddots e^{\lambda_n t} \end{bmatrix}$$

↳ if A is diagonalizable

Next Question

$\lambda_1 = 0 \quad \lambda_2 = c \quad \lambda_3 = 2$

$x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad x_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad x_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$

a) For which c:

a) Is matrix diagonalizable

↳ need three independent eigenvectors

↳ Eigenvectors don't really matter

$\boxed{\text{YES}}$ For all c

b) Symmetric

↳ Orthogonal eigenvectors

→ Real eigenvalues

$\boxed{\text{All real } c}$

c) Positive Definite

No. $\lambda_1 = 0$

but if $c \geq 0 \rightarrow$ positive semidefinite

d) Markov

↳ one eigenval = 1

↳ all others $\lambda < 1$

e) $\frac{A}{2}$ projection?

Eigenvals of projection are 0 and 1

b/c $P^2 = P$

$\lambda^2 = \lambda$

Need: $\boxed{\begin{matrix} c = 0 \\ c = 2 \end{matrix}}$

SVD

every

$A = U \Sigma V^T$

$$\underbrace{A^TA}_{\substack{\text{symmetric} \\ U=V}} = (V\Sigma^T \cancel{U^T})(\cancel{U}\Sigma V^T) = V(\Sigma^T\Sigma)V^T$$

$\Downarrow$

V is eigenvector matrix for A^TA

singular values: $\sigma_1, \sigma_2 \ldots \sigma_n$

$$\sigma_1^2 = \lambda_1(A^TA)$$

$$Av_1 = \sigma_1 u_1$$

$$AV = U\Sigma$$

Next Question

Given: A is symmetric and orthogonal

never singular

don't change lengths. $\|\lambda\| = 1$

$Qx = \lambda x$

$\|x\| = |\lambda| \|x\|$

① Eigenvalues ~~of~~ can be 1 and -1

* All symmetric matrices
  All orthogonal matrices > can be diagonalized

Show that $\frac{1}{2}(A + I)$ is a projection matrix

symmetric

$P^2 = P$

Square it

$$\frac{1}{4}(\underbrace{A^2}_{} + 2A + I) = \frac{1}{2}(A + I)$$

$A = A^T = A^{-1}$

so $A^2 = AA^{-1} = I$

Eigen vals of $\frac{1}{2}(\underbrace{A + I}_{\lambda = 0, 2})$

$\lambda = 0, 1$

Projection!

5/5/14

18.06 Lecture

Exam 3 Topics: WALKER

Eigenvalues
Evectors $A = S \Lambda S^{-1}$

$\frac{du}{dt} = Au$

symmetric positive definite
SVD

Markov

1) $\frac{du}{dt} = \begin{bmatrix} 1 & 3 \\ -3 & -9 \end{bmatrix} u \qquad u(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = 0, -8$ ← Trace

① Find Eigen stuff

$\lambda = 0$
$(A - 0I)x_1 = 0$

$x_1 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

$\lambda_2 = -8$
$(A + 8I)x_2 = 0$

$\begin{bmatrix} 9 & 3 \\ -3 & -1 \end{bmatrix} \underbrace{\begin{bmatrix} 1 \\ -3 \end{bmatrix}}_{x_2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

② General solution

$u(t) = c_1 e^{0t} x_1 + c_2 e^{-8t} x_2$

$\xrightarrow{t \to \infty} c_1 x_1$ ← steady state

$e^{tA} = \begin{bmatrix} 3 & 1 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} 1 & \\ & e^{-8t} \end{bmatrix} \frac{1}{-8} \begin{bmatrix} -3 & -1 \\ 1 & 3 \end{bmatrix}$

Similar to A
$\begin{bmatrix} 0 & 0 \\ 0 & -8 \end{bmatrix}$

Markov

2)

$$A = \begin{bmatrix} .5 & .2 & .2 \\ .1 & .5 & .5 \\ .4 & .3 & .3 \end{bmatrix} \qquad \lambda = 1, 0, .3$$

Cayley-Hamilton

$\det(A - \lambda I) =$ characteristic polynomial $p(\lambda)$

from 1)

$\lambda^2 + 8\lambda -$

$A^2 + 8A =$ zero matrix

$S\Lambda S^{-1} S\Lambda S^{-1} + 8 S\Lambda S^{-1}$

$S(\Lambda^2 + 8A) S^{-1} =$ zero matrix

18.06 Todo

- ~~General Solution $\frac{du}{dt} = Au$~~
- Limits
- ~~SVD~~
- ~~Singular Values?~~
- U_{k+1}

Requirements for similar

- ~~Same eigenvalues~~ ✓
- ~~Diagonalizable?~~ ✓

Markov - ~~same~~ eig = 1, all eig = < 1

Powers A^k ~~ST~~ $A^k = S\Lambda^k S^{-1}$

- Steady State

$x < 1$

18.06 Quiz 3 ① Professor Strang May 10, 2013

Please PRINT your name Fernando Trujano

1.

2.

3.

Please Circle Your Recitation:

r1	T	10	36-156	Russell Hewett	r7	T	1	36-144	Vinoth Nandakumar
r2	T	11	36-153	Russell Hewett	r8	T	1	24-307	Aaron Potechin
r3	T	11	24-407	John Lesieutre	r9	T	2	24-307	Aaron Potechin
r4	T	12	36-153	Stephen Curran	r10	T	2	36-144	Vinoth Nandakumar
r5	T	12	24-407	John Lesieutre	r11	T	3	36-144	Jennifer Park
r6	T	1	36-153	Stephen Curran					

(1) **(40 pts)**

In all of this problem, the 3 by 3 matrix A has eigenvalues $\lambda_1, \lambda_2, \lambda_3$ with independent eigenvectors x_1, x_2, x_3.

(a) What are the trace of A and the determinant of A?

Trace $= \lambda_1 + \lambda_2 + \lambda_3$

Determinant $= \lambda_1 \lambda_2 \lambda_3$

(b) Suppose: $\lambda_1 = \lambda_2$. Choose the true statement from 1, 2, 3:

1. A can be diagonalized. Why?
2. A can not be diagonalized. Why?
3. I need more information to decide. Why?

Eigenvectors are independent

(c) From the eigenvalues and eigenvectors, how could you find the matrix A? Give a formula for A and explain each part carefully.

$A = S\Lambda S^{-1}$

$$A = \begin{bmatrix} | & | & | \\ x_1 & x_2 & x_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} | & | & | \\ x_1 & x_2 & x_3 \\ | & | & | \end{bmatrix}^{-1}$$

(d) Suppose $\lambda_1 = 2$ and $\lambda_2 = 5$ and $x_1 = (1, 1, 1)$ and $x_2 = (1, \text{-}2, 1)$. Choose λ_3 and x_3 so that A is symmetric positive semidefinite but not positive definite.

semidefinite $\rightarrow \lambda_3 = 0$

symmetric $\rightarrow x_3 = x_1 \times x_2$

$$(1,1,1) \times (1,-2,1) = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & -2 & 1 \end{vmatrix}$$

$= \langle 3, 0, -3 \rangle$

$$\boxed{\begin{array}{l} \lambda_3 = 0 \\ x_3 = \langle 3, 0, -3 \rangle \end{array}}$$

(2) **(30 pts.)**

Suppose A has eigenvalues $1, \frac{1}{3}, \frac{1}{2}$ and its eigenvectors are the columns of S:

$$S = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{with} \quad S^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

(a) What are the eigenvalues and eigenvectors of A^{-1}?

Eiganvals = $\lambda_1 = 1$
$\lambda_2 = 3$
$\lambda_3 = 2$

$x_1 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ $x_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ $x_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Same eigen vectors

(b) What is the general solution (with 3 arbitrary constants c_1, c_2, c_3) to the differential equation $du/dt = Au$? Not enough to write e^{At}. Use the c's.

$$u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2 + c_3 e^{\lambda_3 t} x_3$$

$$u(t) = c_1 e^{t} \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + c_2 e^{t/3} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_3 e^{t/2} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

(c) Start with the vector $u = (1, 4, 3)$ from adding up the three eigenvectors: $u = x_1 + x_2 + x_3$. Think about the vector $v = A^k u$ for VERY large powers k. What is the limit of v as $k \to \infty$?

(3) **(30 pts.)**

(a) For a really large number N, will this matrix be positive definite? Show why or why not.

Upper-Left Dets: 2, $2N-16$

$$A = \begin{bmatrix} 2 & 4 & 3 \\ 4 & N & 1 \\ 3 & 1 & 4 \end{bmatrix}.$$

$2(\underbrace{4N-1}_{8N}) - 4(16-3) + 3(\underbrace{4-3N}_{-9N})$

$= \sim -N$

negative

$\boxed{\text{NO}}$

(b) Suppose: A is positive definite symmetric

Q is orthogonal (same size as A)

B is $Q^TAQ = Q^{-1}AQ$

Show that: 1. B is also symmetric.

2. B is also positive definite.

1) $B = B^T$

$B^T = (Q^TAQ)^T$

$= Q^T A^T Q^{TT}$ ← sym

$= Q^T A Q = B$

2) Since $B = Q^TAQ$

A and B are similar $\rightarrow$ same λ's

$\rightarrow$ both positive def

(c) If the SVD of A is $U\Sigma V^T$, how do you find the orthogonal V and the diagonal Σ from the matrix A?

1. Find A^TA
2. Find Eigenvectors and Eigenvalues of A^TA
3. Columns of V = normalized eigenvectors
   ↳ for non 0 λ's
   - IF repeated λ pick the orthogonal evectors
4. Diagonal entries of $\Sigma = \sqrt{\text{evals}(A^TA)}$

18.06 Professor Edelman Quiz 3 December 4, 2013

(2)

Your PRINTED name is: Fernando Trujano

Grading

1

2

3

Please circle your recitation:

1	T 9	Dan Harris	E17-401G	3-7775	dmh
2	T 10	Dan Harris	E17-401G	3-7775	dmh
3	T 10	Tanya Khovanova	E18-420	4-1459	tanya
4	T 11	Tanya Khovanova	E18-420	4-1459	tanya
5	T 12	Saul Glasman	E18-301H	3-4091	sglasman
6	T 1	Alex Dubbs	32-G580	3-6770	dubbs
7	T 2	Alex Dubbs	32-G580	3-6770	dubbs

1 (32 pts.) (2 points each)

There are sixteen 2×2 matrices whose entries are either 0 or 1. For each of the sixteen, write down the two singular values. Time saving hint: if you really understand singular values, then there is really no need to compute AA^T or A^TA, but it is okay if you must.

2 (30 pts.) (3 points each: Please circle true or false, and either way, explain briefly.)

a) If A and B are invertible, then so is $(A+B)/2$. True? $\boxed{\text{False?}}$ (Explain briefly).

invertible: no $\lambda=0$

but $\lambda_1(A)=2$

$\lambda_1(B)=-2$

so $\lambda_1(A+B)=0$ X

b) If A and B are Markov, then so is $(A+B)/2$. $\boxed{\text{True?}}$ False? (Explain briefly).

c) If A and B are positive definite, then so is $(A+B)/2$. $\boxed{\text{True?}}$ False? (Explain briefly).

Eigenvalues remain positive

OR

$$x^T\left(\frac{A+B}{2}\right)x = \frac{\overbrace{x^TAx}^{>0} + \overbrace{x^TBx}^{>0}}{2} > 0$$

because positive definite

d) If A and B are diagonalizable, then so is $(A+B)/2$. True? $\boxed{\text{False?}}$ (Explain briefly).

e) If A and B are rank 1, then so is $(A+B)/2$. True? $\boxed{\text{False?}}$ (Explain briefly).

f) If A is symmetric then so is e^A. $\boxed{\text{True?}}$ False? (Explain briefly).

$$e^A = \underbrace{I + A}_{\text{sym}} + \underbrace{\frac{A^2}{2!}}_{\text{sym}}$$

g) If A is Markov then so is e^A. True? $\boxed{\text{False?}}$ (Explain briefly).

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad e^A = \begin{bmatrix} e & 0 \\ 0 & e \end{bmatrix}$$

not Markov

h) If A is symmetric, then e^A is positive definite. $\boxed{\text{True?}}$ False? (Explain briefly).

Real λ's

$e^{\lambda_{real}} > 0$

i) If A is singular, then so is e^A. True? $\boxed{\text{False?}}$ (Explain briefly).

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad e^A = \left[I + \cancel{A + A^2} \right]$$

singular   not singular

j) If A is orthogonal, then so is e^A. True? $\boxed{\text{False?}}$ (Explain briefly).

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad e^A = \begin{bmatrix} e & 0 \\ 0 & e \end{bmatrix}$$

3 (38 pts.)

Let $A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$.

a) (10 pts.) Find a nonzero solution $y(t)$ in R^2 to $dy/dt = Ay$ that is independent of t, in other words, $y(t)$ is a constant vector in R^2. (Hint: why would a vector in the nullspace of A have this property?)

b) (10 pts.) Show that e^{At} is Markov for every value of $t \geq 0$.

c) (10 pts.) What is the limit of e^{At} as $t \to \infty$?

d) (8 pts.) What is the steady state vector of the Markov matrix e^{A} ?

18.06 Professor Strang Quiz 3 May 7th, 2012

3

Grading

Your PRINTED name is: Fernando Trujano

1

2

3

Please circle your recitation:

r01	T 11	4-159	Ailsa Keating	ailsa
r02	T 11	36-153	Rune Haugseng	haugseng
r03	T 12	4-159	Jennifer Park	jmypark
r04	T 12	36-153	Rune Haugseng	haugseng
r05	T 1	4-153	Dimiter Ostrev	ostrev
r06	T 1	4-159	Uhi Rinn Suh	ursuh
r07	T 1	66-144	Ailsa Keating	ailsa
r08	T 2	66-144	Niels Martin Moller	moller
r09	T 2	4-153	Dimiter Ostrev	ostrev
r10	ESG		Gabrielle Stoy	gstoy

1 (33 pts.)

Suppose an $n \times n$ matrix A has n independent eigenvectors $x_1, \ldots, x_n$. Then you could write the solution to $\frac{du}{dt} = Au$ in three ways:

$$u(t) = e^{At}u(0), \quad \text{or}$$
$$u(t) = Se^{\Lambda t}S^{-1}u(0), \quad \text{or}$$
$$u(t) = c_1 e^{\lambda_1 t}x_1 + \ldots + c_n e^{\lambda_n t}x_n.$$

Here, $S = [x_1 \,|\, x_2 \,|\, \ldots \,|\, x_n]$.

(a) From the definition of the exponential of a matrix, show why e^{At} is the same as $Se^{\Lambda t}S^{-1}$.

$A = S\Lambda S^{-1}$

$$e^{At} = I + At + \frac{1}{2!}(At)^2 + \ldots\ldots$$
$$= I + (S\Lambda S^{-1})t + \frac{1}{2!}S\Lambda \cancel{S^{-1}S}\Lambda S^{-1}t^2 + \ldots\ldots$$
$$= I + (S\Lambda S^{-1})t + \frac{1}{2!}(S\Lambda^2 S^{-1})t^2 + \ldots$$
$$S(\underbrace{I + \Lambda t + \frac{1}{2!}\Lambda t^2 + \ldots}_{e^{\Lambda t}})S^{-1}$$
$$= Se^{\Lambda t}S^{-1}$$

(b) How do you find $c_1, \ldots, c_n$ from $u(0)$ and S?

$$u(t) = c_1 e^{\lambda_1 t}x_1 + c_2 e^{\lambda_2 t}x_2 + .. c_n e^{\lambda_n t}x_n$$
$$u(0) = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$$

Columns of eigenvector matrix S

$$u(0) = S\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$
$$\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = S^{-1}u(0)$$

(c) For this specific equation, write $u(t)$ in any one of the three forms, using *numbers* not symbols: You can choose which form.

$$\frac{du}{dt} = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} u, \quad \text{starting from} \quad u(0) = \begin{bmatrix} 4 \\ 3 \end{bmatrix}.$$

$$\frac{du}{dt} = Au$$

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$$

Eigenvalues: $\begin{vmatrix} 1-\lambda & 2 \\ -1 & 4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda) + 2 = 0$

$$4 - 5\lambda + \lambda^2 + 2 = 0$$

$$\lambda^2 - 5\lambda + 6 = 0$$

$$(\lambda-3)(\lambda-2) = 0$$

$$\lambda_1 = 2$$

$$\lambda_2 = 3$$

Eigenvectors: $\lambda_1 = 2$

$$\begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \end{bmatrix} = 0 \qquad x_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$\lambda_2 = 3$

$$\begin{bmatrix} -2 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_2 \end{bmatrix} = 0 \qquad x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$S = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \quad \Lambda = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \quad S^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

$$u(t) = Se^{\Lambda t}S^{-1}u(0)$$

$$\boxed{u(t) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \end{bmatrix}}$$

or

$$u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = S^{-1}u(0) = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\boxed{u(t) = e^{2t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

2 (30 pts.)

This question is about the real matrix

$$A = \begin{bmatrix} 1 & c \\ 1 & -1 \end{bmatrix}, \quad \text{for} \quad c \in \mathbb{R}.$$

(a) - Find the eigenvalues of A, depending on c.

- For which values of c does A have real eigenvalues?

$A = \begin{bmatrix} 1 & c \\ 1 & -1 \end{bmatrix}$

$\begin{vmatrix} 1-\lambda & c \\ 1 & -1-\lambda \end{vmatrix} = (1-\lambda)(-1-\lambda) - c = 0$

$-1 \cancel{-\lambda+\lambda} + \lambda^2 - c = 0$

$\lambda^2 = 1 + c$

$\boxed{\lambda = \pm\sqrt{1+c}}$

$\boxed{c \geq 1}$

(b) - For one particular value of c, convince me that A is similar to both the matrix

$$B = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix},$$

and to the matrix

$$C = \begin{bmatrix} 2 & 2 \\ 0 & -2 \end{bmatrix}.$$

- Don't forget to say which value c this happens for.

$B = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$ $\lambda_1 = 2$ $\lambda_2 = -2$

$\cancel{A}C = \begin{bmatrix} 2 & 2 \\ 0 & -2 \end{bmatrix}$ $\lambda_1 = 2$ $\lambda_2 = -2$

$\underline{A}$ $\lambda = \pm\sqrt{c+1}$

so $\boxed{c = 3}$

Same eigenvalues that are different from each other $\to$ all diagonalizable $\to$ all similar!

(c) For one particular value of c, convince me that A cannot be diagonalized. It is not similar to a diagonal matrix Λ, when c has that value.

- Which value c?

- Why not?

Repeated λ $\rightarrow$ Same eigenvector $\rightarrow$ Not diagonalizable
b/c not n independent eigenvectors

$A_{\lambda = \sqrt{1+c}}$

$\boxed{c = -1}$

3 (37 pts.)

(a) Suppose A is an $n \times n$ symmetric matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$.

- What is the largest number real number c that can be subtracted from the diagonal entries of A, so that $A - cI$ is positive semidefinite?

- Why?

Diagonal entries of A = trace(A)

Eigenvalues of $A - cI$ are

$$\lambda_1 - c \leq \lambda_2 - c \leq \lambda_n - c$$

$$\boxed{c = \lambda_1}$$

Makes an eigen value 0 but the others are positive $\longrightarrow$ positive definite

(b) Suppose B is a matrix with independent columns.

- What is the nullspace $N(B)$?

- Show that $A = B^T B$ is positive definite. Start by saying what that means about $x^T Ax$.

Independent columns

$N(B) =$ zero vector

Positive definite:

$$x^T A x > 0$$

$$A = B^T B$$

$$x^T B^T B x > 0$$

$$(Bx)^T B x > 0$$

$$\|Bx\|^2 > 0$$

$\Rightarrow$ positive definite

$x = 0$

(c) This matrix A has rank $r = 1$:

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}.$$

- Find its largest singular value σ from $A^T A$.

- From its column space and row space, respectively, find unit vectors u and v so that

$$Av = \sigma u, \quad \text{and} \quad A = u\sigma v^T.$$

- From the nullspaces of A and A^T put numbers into the full SVD (Singular Value Decomposition) of A:

$$A = \begin{bmatrix} | & & | \\ u & & \dots \\ | & & | \end{bmatrix} \begin{bmatrix} \sigma & & 0 \\ & & \\ 0 & & \dots \end{bmatrix} \begin{bmatrix} | & & | \\ v & & \dots \\ | & & | \end{bmatrix}^T.$$

18.06 Quiz 3 4 Professor Strang May 6, 2011

Your PRINTED name is Fernando Trujano 1.

Your Recitation Instructor (and time) is ______ 2.

Instructors: (Hezari)(Pires)(Sheridan)(Yoo) 3.

Please show enough work so we can see your method and give due credit.

1. (a) Find two eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}.$$

$\boxed{\begin{matrix}\lambda_1 = 2 \\ \lambda_2 = 5\end{matrix}}$

$\lambda_1 = 2$

$\begin{bmatrix} 0 & 3 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\boxed{x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}}$

$\lambda_2 = 5$

$\begin{bmatrix} -3 & 3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\boxed{x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$

(b) Express any vector $u_0 = \begin{bmatrix} a \\ b \end{bmatrix}$ as a combination of the eigenvectors.

$u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2$

$u(0) = c_1 x_1 + c_2 x_2$

$u_0 = S \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = S^{-1} u_0 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a-b \\ b \end{bmatrix}$

$\boxed{u_0 = (a-b)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$

(c) What is the solution $u(t)$ to $\frac{du}{dt} = Au$ starting from $u(0) = u_0$?

$u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2$

$\boxed{u(t) = (a-b)e^{2t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + (b)e^{5t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$

(d) Find a formula $u_k =$ ______ for the solution to $u_{k+1} = Au_k$ which starts from that vector u_0. **Set $k = -1$ to find $A^{-1}u_0$.**

2. This problem is about the matrix

$$A = \begin{bmatrix} \sqrt{2} & 1 \\ 0 & \sqrt{2} \end{bmatrix}.$$

(a) Find all eigenvectors of A. Exactly why is it impossible to diagonalize A in the form $A = S\Lambda S^{-1}$? $\lambda_1 = \sqrt{2}$
$\lambda_2 = \sqrt{2}$

impossible b/c: repeated λ's $\rightarrow$ not enough independent eigenvectors to form invertible matrix S

(b) Find the matrices U, Σ, V^T in the Singular Value Decomposition $A = U\,\Sigma\,V^T$. Tell me *two orthogonal vectors* v_1, v_2 in the plane so that Av_1 and Av_2 are also orthogonal.

(c) Find a matrix B that is similar to A (but different from A). Show that A and B meet the requirement to be similar *(what is it?)*.

3. Suppose A is a real m by n matrix.

(a) Prove that the symmetric matrix $A^T A$ has the property $x^T(A^T A)x \geq 0$ for every vector x in R^n. Explain each step in your reason.

(b) According to part (a), the matrix $A^T A$ is positive semidefinite at least — and possibly positive definite. Under what condition on A is $A^T A$ positive definite?

(c) If $m < n$ prove that $A^T A$ is *not* positive definite.

18.06 Lecture #30

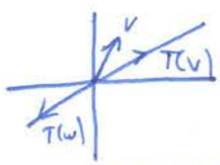
Linear Transformations T

w/o coordinates : no matrix

w/ coordinates : MATRIX

Example 1: Projection

$T: \mathbb{R}^2 \to \mathbb{R}^2$



← no axis required

Linear Transformation!

Rules:

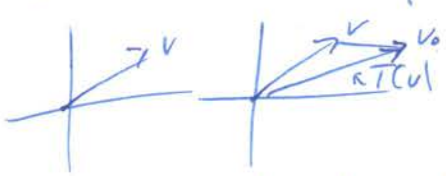
$T(v+w) = T(v) + T(w)$

$T(cv) = c\,T(v)$

$T(cv+dw) = cT(v) + dT(w)$

$T(0) = 0$

Example 2: Shift whole plane by v_0



not a linear transformation

Example 3:

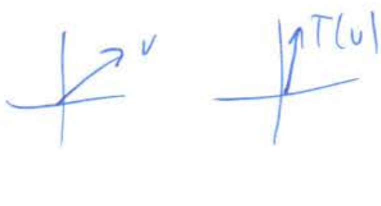
$T(v) = \|v\|$

$T: \mathbb{R}^3 \to \mathbb{R}^1$ ← $T(-v) = \|v\|$!!

NOT LINEAR

Example 4 Rotation by 45°

$T: \mathbb{R}^2 = \mathbb{R}^2$



Example 3 Matrix

$T(v) = Av$

$A(v+w) = Av + Aw$ LINEAR!

Start: $T: \mathbb{R}^3 \to \mathbb{R}^2$ ← linear transformation

Examples: $T(v) = Av$

output in $\mathbb{R}^2$; A ← 2 × 3 matrix ; input in $\mathbb{R}^3$

Information needed to know $T(v)$ for all inputs

$T(v_1), T(v_2), \ldots, T(v_n)$ for any basis $v_1, \ldots v_n$

because every $v = c_1 v_1 + \ldots + c_n v_n$

so $T(v) = c_1 T(v_1) + \ldots + c_n T(v_n)$

Coordinates come from a basis.

" of v $= c_1 v_1 + \ldots + c_n v_n$

$$v = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Standard Basis xyz

Construct Matrix A that represents linear Transformation T

$T: \mathbb{R}^n \to \mathbb{R}^m$

Choose basis $v_1, \ldots v_n$ for inputs $\mathbb{R}^n$

" " $w_1 \ldots w_n$ for outputs $\mathbb{R}^m$

want matrix A

5/2/14 18.06 Lecture

- Review similar matrix
- Change of basis
- Computer graphics

M invertible

$\checkmark$ = no change
X = change

A	MA	MAM^{-1} (similar to A)	
rank	$\checkmark$	$\cancel{X}$ $\checkmark$	
column space	X	X	
null space	$\checkmark$	X	
determinant	X	$\checkmark$	
evalues	X	$\checkmark$	
evectors	X	X	QAQ^{-1}
$A^T=A$ symmetry	NO	NO	YES
pos. definite	NO	NO	YES

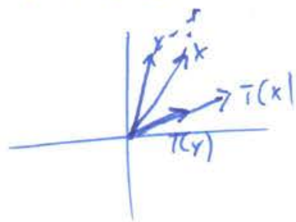
4/30/14 18.06 Lecture

Linear Transformation $T(x)$

$$T(cx+dy) = cT(x) + dT(y)$$

Examples: Rotation $T(x)$

Ex 2: Projection



Proj of $x+y$
$= proj(x) + proj(y)$

Before today: $T(x) = A(x)$

$$A(x+y) = Ax + Ay = T(x) + T(y)$$

Can't have constants in Linear Transformation

$$T(x) = \underbrace{2x}_{\text{OKAY}} + \underbrace{3}_{\text{BAD}}$$

Suppose $x_1, x_2, \ldots x_n$ are a basis for input space.

If we know $T(x_1), T(x_2) \ldots T(x_n)$ — n outputs

then we know all $T(x)$ outputs

step 1 $x = c_1 x_1 + c_2 x_2 + c_n x_n$

step 2 $T(x) = c_1 T(x_1) + c_2 T(x_2) + \ldots$

Given $T(x)$

Given input basis vectors $x_1 \ldots x_n$

Any output space basis $v_1 \ldots v_m$

Then there is a matrix A that represents $T(x)$

A: coeffs of input $x \longrightarrow$ coeffs of output $T(x)$

Idea:

Find matrix A, using input, output bases,

any input $x = c_1 x_1 + \dots + c_n x_n$

$$A \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} d_1 \\ \vdots \\ d_m \end{bmatrix}$$

and d's are coeffs of output $T(x)$ in output basis

$$T(x) = d_1 v_1 + d_2 v_2 + \dots + d_m v_m$$

5/13/14 # 18.06 Recitation

Fernando Trujano

① Find basis for all 4 subspaces of $R = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{3\times 4}$

$R^T = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 3 & 0 \end{bmatrix}$

Linearly independent

Column space: $\{(1,0,0), (0,1,0)\}$ ← can write all columns as linear combination of basis

Row space: $\{(1,2,0,4), (0,0,1,3)\}$

Nullspace of R^T: $\{(0,0,1)\}$

$x = 0$
$y = 0$
$4x + 3y + 0z = 0$
$0z = 0$
z can be anything!

Nullspace: $\{(-2,1,0,0), (-4,0,-3,1)\}$

$x + 2y + 0z + 4w = 0$
$z - 3w = 0$
$0 = 0$

② A Symmetric + Orthogonal

a) How is A^{-1} related to A?

$A^T = A$ (symmetric) $\quad A^{-1} = A^T$ (orthogonal)

$A^{-1} = A$

b) What can the eigenvalues of A be?

symmetric ⇒ eigenvalues are real
orthogonal ⇒ eigenvalues have norm 1
⇒ Eigenvalues of A are ± 1

c) $A = \begin{bmatrix} .5 & -.5 & -.1 & -.7 \\ -.5 & +.5 & -.1 & -.7 \\ -.1 & -.1 & +.98 & -.14 \\ -.7 & -.7 & -.14 & +.02 \end{bmatrix}_{4\times 4}$ Find the eigenvalues.

$A^T = A$
Also Orthogonal

$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$

From b) $\rightarrow \lambda_i = \pm 1$ $\quad = (1, 1, 1, -1)$

$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = trace(A) = 2$

③ q_1, q_2, q_3 real orthonormal column vectors

Show that the matrix $A = q_1 q_1^T + 2 q_2 q_2^T + 5 q_3 q_3^T$ has eigenvalues 1, 2, 5

$Av = \lambda v$

$$A q_1 = (q_1 q_1^T + 2 q_2 q_2^T + 5 q_3 q_3^T) q_1 = q_1 \underbrace{q_1^T q_1}_{1} + 2 q_2 \underbrace{q_2^T q_1}_{0} + 5 q_3 \underbrace{q_3^T q_1}_{0}$$

$$= q_1 = 1 q_1$$

$A q_2 = 2 q_2$

$A q_3 = 5 q_3$

④ Compute $\left|\begin{bmatrix} 1 & b & 0 & 0 \\ b & 1 & b & 0 \\ 0 & b & 1 & b \\ 0 & 0 & b & 1 \end{bmatrix}\right|$ using the cofactors of row 1.

$$= 1 \begin{vmatrix} 1 & b & 0 \\ b & 1 & b \\ 0 & b & 1 \end{vmatrix} - b \begin{vmatrix} b & b & 0 \\ 0 & 1 & b \\ 0 & b & 1 \end{vmatrix} = b^4 - 3b^2 + 1$$

$$\left(1 \times \begin{vmatrix} 1 & b \\ b & 1 \end{vmatrix} - b \begin{vmatrix} b & b \\ 0 & 1 \end{vmatrix}\right) \qquad \left(b \begin{vmatrix} 1 & b \\ b & 1 \end{vmatrix} - b \begin{vmatrix} 0 & b \\ 0 & 1 \end{vmatrix}\right)$$

⑤ $A_{3\times3}$ $B_{3\times3}$

$\text{rank}(A) = 1$
$\text{rank}(B) = 2$

a) What are the possible ranks of $A+B$

$0 \leq \text{rank}(A+B) \leq 3$

Assume that $\text{rank}(A+B) = 0 \Rightarrow A+B = 0 \Rightarrow \text{rank}(A) = \text{rank}(-B)$

$\text{rank}(A) = \text{rank}(B)$
$1 \neq 2$
cannot be 0

b) Give an example for the remaining number

$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\text{rank}(A+B) = 1$

$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\text{rank}(A+B) = 2$

$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\text{rank}(A+B) = 3$

Is the Following Function linear?

a) $\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto 3x - 2y + 1$

Conditions:

$\mathbb{R}^n \xrightarrow{F} \mathbb{R}^m$

(i) $f(\lambda \vec{v}) = \lambda f(\vec{v})$

(ii) $f(\vec{v}_1 + \vec{v}_2) = f(\vec{v}_1) + f(\vec{v}_2)$

(iii) zero vector $\to$ zero

a) $\mathbb{R}^2 \to \mathbb{R}$

0 vector $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \to 1 \neq 0$

NOT LINEAR!

b) $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \underset{1\times 2}{\begin{pmatrix} x & y \end{pmatrix}} \underset{2\times 2}{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} \underset{2\times 1}{\begin{pmatrix} x \\ y \end{pmatrix}}$ $\mathbb{R}^2 \to \mathbb{R}$

$\begin{pmatrix} x \\ y \end{pmatrix} \to 2xy$

$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \to 0$

$\lambda \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lambda x \\ \lambda y \end{pmatrix} \to \lambda^2 2xy$

NOT LINEAR!

c) Counterclockwise rotation of $\mathbb{R}^2$ by $\pi/2$

(i) Write the matrix of c) in the canonical basis $\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{e_1} \mapsto \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\quad \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{e_2} \to \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

$\boxed{\Rightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}$

(ii) Write the matrix of c) in the basis $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \to \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $\quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

$\boxed{\Rightarrow \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}}$

$\begin{pmatrix} -1 \\ 1 \end{pmatrix} = a \begin{pmatrix} 1 \\ 1 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$a = -1$
$b = 2$

$\begin{pmatrix} -1 \\ 0 \end{pmatrix} = c \begin{pmatrix} 1 \\ 1 \end{pmatrix} + d \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$c = -1$
$d = 1$

Linear combination of basis

(iii) Compute the matrix change of basis From $\{\vec{e}_1, \vec{e}_2\}$ to $\left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$

- Do nothing $\Rightarrow$ then change basis

$$\begin{pmatrix}1\\0\end{pmatrix} = a\begin{pmatrix}1\\1\end{pmatrix} + b\begin{pmatrix}0\\1\end{pmatrix} \qquad \begin{pmatrix}0\\1\end{pmatrix} = c\begin{pmatrix}1\\1\end{pmatrix} + d\begin{pmatrix}0\\1\end{pmatrix}$$

$a=1 \quad b=-1 \qquad\qquad c=0 \quad d=1$

$$\boxed{\Longrightarrow \begin{pmatrix}1 & 0\\-1 & 1\end{pmatrix}}$$

(iv) Compute the matrix change of basis From $\left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$ to $\{\vec{e}_1, \vec{e}_2\}$

$$\begin{pmatrix}1\\1\end{pmatrix} = a\begin{pmatrix}1\\0\end{pmatrix} + b\begin{pmatrix}0\\1\end{pmatrix} \qquad \begin{pmatrix}0\\1\end{pmatrix} = c\begin{pmatrix}1\\0\end{pmatrix} + d\begin{pmatrix}0\\1\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}1 & 0\\1 & 1\end{pmatrix}$$

* inverse of iii)

(v) Compute (ii) using (i), (iii), (iv)

(ii): non trivial basis; (i): canonical basis; (iii), (iv): Change of basis

matrix from to

$$\begin{tikzcd}
{}_{B}\,\mathbb{R}^{\cancel{m}n} \arrow[r, "M(F,B,B_1)", "F"'] \arrow[d, bend right, "{M(I,B,B')}"'] & \mathbb{R}^n_{B_1} \\
{}_{B'}\,\mathbb{R}^m \arrow[r, "F"'] & \mathbb{R}^m_{B_2'} \arrow[u, bend right, "{M(I,B_2',B_1)}"']
\end{tikzcd}$$

$M(F, B', B_2')$

in this case
$B' = B_2'$

$$M(F,B,B_1) = M(I, B_2', B')\ M(F, B', B_2')\ M(I, B, B')$$

$$\begin{bmatrix}-1 & -1\\2 & 1\end{bmatrix} = \begin{bmatrix}1 & 0\\-1 & 1\end{bmatrix}\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}$$

SVD

Singular Value Decomposition

Fo sho on exam!

$$A = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad A = U \Sigma V^T$$

(i) $A^T A$

$A^T = A$

$\to A^2 = A$

(ii) Eigen stuff of $A^T A$

$$A^T A = \underbrace{\frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{v} = \frac{1}{3}\begin{bmatrix} x+y+z \\ x+y+z \\ x+y+z \end{bmatrix}$$

$$Av = \lambda v \implies x = y = z$$

$(1,1,1)$

$v_1 = \frac{1}{\sqrt{3}}(1,1,1)$

so

$$V^T = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$$

$$Av_1 = \sigma_1 u_1$$

$\quad \shortparallel$

$1 \cdot v_1 \implies \sigma_1 = 1 \quad u_1 = v_1$

$$\underbrace{\frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}}_{A} = \underbrace{\begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}}_{U_{3\times 1}} \underbrace{\begin{bmatrix} 1 \end{bmatrix}}_{\Sigma_{1\times 1}} \underbrace{\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}}_{V^T_{1\times 3}}$$

18.06 Final Exam Review

symmetric $\rightarrow A^T = A$. Real λ

thogonal $\rightarrow A^T = A^{-1}$. $|\lambda| = 1$

$\Sigma \lambda_i$ = trace

$Av = \lambda v$

Incidence Matrix:

m = # edges

n = # nodes

$\text{rank}(A) = n - \dim(N(A))$

$\dim(N(A^T)) = m - \text{rank}(A)$

$0 \leq \text{rank}(AB) \leq \min(\text{rank}(A), \text{rank}(B))$

$0 \leq \text{rank}(A+B) \leq \text{rank}(A) + \text{Rank}(B)$

$\det(A)$ = product of pivots of A

$C(P)$ = subspace P projects on

unit vector v: $v^T v = \|v\|^2 = 1$

$A = U \Sigma V^T$

rank A = rank Σ

$U, V \rightarrow$ rotations and reflections

$\Sigma \rightarrow$ streaches

$V \rightarrow \underset{r \text{ cols}}{C(A^T)}, \underset{n-r \text{ cols}}{N(A)}$

$U \rightarrow \underset{r \text{ cols}}{C(A)}, \underset{m-r}{N(A^T)}$

18.06 Final Exam ① Professor Strang May 22, 2013

Please PRINT your name Fernando Trujano

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Please Circle Your Recitation

r1	T	10	36-156	Russell Hewett	r7	T	1	36-144	Vinoth Nandakumar
r2	T	11	36-153	Russell Hewett	r8	T	1	24-307	Aaron Potechin
r3	T	11	24-407	John Lesieutre	r9	T	2	24-307	Aaron Potechin
r4	T	12	36-153	Stephen Curran	r10	T	2	36-144	Vinoth Nandakumar
r5	T	12	24-407	John Lesieutre	r11	T	3	36-144	Jennifer Park
r6	T	1	36-153	Stephen Curran					

(1) **(7+7 pts)**

(a) Suppose the nullspace of a square matrix A is spanned by the vector $v = (4, 2, 2, 0)$. Find the reduced echelon form $R = \text{rref}(A)$.

(b) Suppose S and T are subspaces of $\mathbf{R}^5$ and Y and Z are subspaces of $\mathbf{R}^3$. When can they be the four fundamental subspaces of a 3 by 5 matrix B? Find any required conditions to have $S = C(B^T)$, $T = N(B)$, $Y = C(B)$, and $Z = N(B^T)$.

(2) **(6+6 pts.)**

(a) Find bases for all four fundamental subspaces of this R.

$$R = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) Find U, Σ, V in the Singular Value Decomposition $A = U\Sigma V^T$:

$$A = \begin{bmatrix} 2 & 2 \\ -1 & 2 \\ 2 & -1 \end{bmatrix}.$$

(3) **(5+5 pts.)**

Suppose $q_1, \ldots q_5$ are orthonormal vectors in $\mathbf{R}^5$.

The 5 by 3 matrix A has columns q_1, q_2, q_3.

(a) If $b = q_1 + 2q_2 + 3q_3 + 4q_4 + 5q_5$, find the best least squares solution $\widehat{x}$ to $Ax = b$.

(b) If terms of q_1, q_2, q_3 find the projection matrix P onto the column space of A.

(4) **(3+4+3 pts.)**

The matrix A is symmetric and also orthogonal.

(a) How is A^{-1} related to A?

$A^T = A$ symmetric $\quad A^{-1} = A^T$ orthogonal

$$A^{-1} = A^T = A$$

(b) What number(s) can be eigenvalues of A and why?

only Real b/c symmetric

orthogonal $\rightarrow |\lambda| = 1$

so $\lambda = \pm 1$

(c) Here is an example of A. What are the eigenvalues of this matrix? I don't recommend computing with $\det(A - \lambda I)$! Find a way to use part (b).

$$A = \begin{bmatrix} .5 & -.5 & -.1 & -.7 \\ -.5 & .5 & -.1 & -.7 \\ -.1 & -.1 & .98 & -.14 \\ -.7 & -.7 & -.14 & .02 \end{bmatrix}.$$

Trace = 2

so

$\lambda = (1, 1, 1, -1)$

(5) **(4+5+3 pts.)**

Suppose the real column vectors q_1 and q_2 and q_3 are orthonormal.

(a) Show that the matrix $A = q_1 q_1^T + 2q_2 q_2^T + 5q_3 q_3^T$ has the eigenvalues $\lambda = 1, 2, 5$.

Definition: $Aq_1 = \lambda q_1$

$$q_1 \underbrace{q_1^T q_1}_{I} + 2q_2 \underbrace{q_2^T q_1}_{0} + 5q_3 \underbrace{q_3^T q_1}_{0 \leftrightarrow \text{orthogonal}} = \lambda q_1$$

$q_1 = \lambda q_1$ $\lambda = 1$ Repeat For $\lambda_2 = 2$ $\lambda_3 = 5$

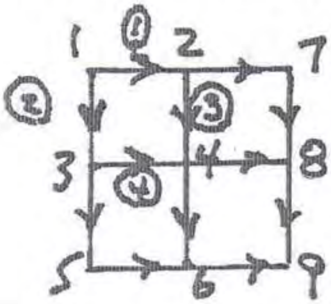
(b) Solve the differential equation $du/dt = Au$ starting at any vector $u(0)$. Your answer can involve the matrix Q with columns $q_1, q_2, q_3,$.

(c) Solve the differential equation $du/dt = Au$ starting from $u(0) = q_1 - q_3$.

(6) (4+3+3 pts.)

This graph has $m = 12$ edges and $n = 9$ nodes. Its 12 by 9 incidence matrix A has a single -1 and $+1$ in every row, to show the start and end nodes of the corresponding edge in the graph.

(a) Write down the 4 by 4 submatrix S of A that comes from the 4-node graph (a loop) in the corner. Find a vector x in the nullspace $N(S)$ and a vector y in $N(S^T)$.



$$S = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \end{array}$$

$S^T[x_2] = 0$

$x_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \in$ in $N(S^T)$

$S[x] = 0$

$x = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \in$ in $N(S)$

(b) For the whole matrix A, find a vector Y in $N(A^T)$. You won't need to write A or to know more edge numbers.

$$Y = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

(c) The all-ones vector $(1, 1, \ldots, 1)$ spans $N(A)$. Find the dimension of the left nullspace $N(A^T)$ (give a number).

$\text{Rank}(A) = 9 - \dim(N(A)) = 8$

$\text{Dim}(N(A^T)) = 12 - \text{rank}(A) = 4$

(7) **(3+3+3+3 pts.)**

The equation $y_{n+2} + By_{n+1} + Cy_n = 0$ has the solution $y_n = \lambda^n$ if $\lambda^2 + B\lambda + C = 0$. In most cases this will give two roots λ_1, λ_2 and the complete solution is $y_n = c_1\,\lambda_1^n + c_2\,\lambda_2^n$.

Now solve the same problem the matrix way (slower). Create this vector unknown and vector equation $u_{n+1} = A\,u_n$.

$$u_n = \begin{bmatrix} y_n \\ y_{n+1} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} y_{n+1} \\ y_{n+2} \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{bmatrix} y_n \\ y_{n+1} \end{bmatrix}.$$

(a) What is the matrix A in that equation?

(b) What equation gives the eigenvalues λ_1 and λ_2 of A?

(c) If λ_1 is an eigenvalue, show directly that

$$A \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} \quad \text{so we have the eigenvector.}$$

(d) If $\lambda_1 \neq \lambda_2$, what is now the complete solution u_n (including constants c_1 and c_2) to our equation $u_{n+1} = A\,u_n$? ((Then y_n is the first component of u_n.))

(8) **(5+5 pts.)**

(a) Find the determinant of this matrix A, using the cofactors of row 1.

$$A = \begin{bmatrix} 1 & b & 0 & 0 \\ b & 1 & b & 0 \\ 0 & b & 1 & b \\ 0 & 0 & b & 1 \end{bmatrix}.$$

$$1\begin{vmatrix} 1 & b & 0 \\ b & 1 & b \\ 0 & b & 1 \end{vmatrix} - b\begin{vmatrix} b & b & 0 \\ 0 & 1 & b \\ 0 & b & 1 \end{vmatrix}$$

$$\det(A) = 1 - 3b^2 + \cancel{4}b^4$$

(b) Find the determinant of A by the BIG formula with 24 terms. This means to find all the nonzero terms in that formula with their correct signs.

(9) **(6+4 pts.)**

(a) Suppose $\mathbf{v} = (v_1, v_2, v_3)$ is a column vector, so $A = \mathbf{v}\mathbf{v}^T$ is a symmetric matrix. Show that A is positive semidefinite, using one of these tests:

1. The eigenvalue test
2. The determinant test
3. The energy test on $x^T A x$.

$A = vv^T$

$x^T A x$

$x^T v v^T x$

$(v^T x)^T v^T x$

$\| v^T x \|^2 \geq 0$

(b) Suppose A is m by n of rank r. What conditions on m and n and r guarantee that $A^T A$ is positive definite? If those conditions fail, prove that $A^T A$ **will not be positive definite.**

18.06 Professor Strang Final Exam May 23rd, 2012

Your PRINTED name is: Fernando Trujano

Grading

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r03	T 12	4-159	Jennifer Park	jmypark
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r09	T 2	4-153	Dimiter Ostrev	ostrev
r10	ESG		Gabrielle Stoy	gstoy

?

(c) If you solve $\frac{d\mathbf{u}}{dt} = -A\mathbf{u}$ (notice the minus sign), with $\mathbf{u}(0)$ a given vector, then as $t \to \infty$ the solution $\mathbf{u}(t)$ will always approach a multiple of a certain vector $\mathbf{w}$.

- Find this steady-state vector $\mathbf{w}$.

steady-state @ x_1 b/c $\lambda_1 = 0$

$$\boxed{w = \begin{bmatrix} 1 \\ -15 \\ 3 \end{bmatrix}}$$

2 (12 pts.)

Suppose A has rank 1, and B has rank 2 (A and B are both 3×3 matrices).

(a) - What are the possible ranks of $A + B$?

if $\text{rank}(A+B) = 0$
$\rightarrow A+B = 0 \rightarrow A = -B$
$\rightarrow \text{rank}(A) = \text{rank}(B)$
FALSE $\Rightarrow \text{rank}(A+B) \neq 0$

$0 \leq \text{rank}(A+B) \leq \text{rank}(A) + \text{rank}(B)$

$\boxed{1, 2, 3}$

(b) - Give an example of each possibility you had in (a).

$\text{rank}(A+B) = 1$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\text{rank}(A+B) = 2$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\text{rank}(A+B) = 3$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(c) - What are the possible ranks of AB?

$$0 \leq \text{rank}(AB) \leq \cancel{ma}\ \min(\text{rank}(A), \text{rank}(B))$$

$$0, 1$$

- Give an example of each possibility.

$\text{rank}(AB) = 0$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\text{rank}(AB) = 1$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

3 (12 pts.)

(a) - Find the three pivots and the determinant of A.

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} \boxed{1} & 0 & -1 \\ 0 & \boxed{1} & 1 \\ 0 & 1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{1} & 0 & -1 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & \boxed{-2} \end{bmatrix}$$

$$\boxed{\text{Pivots} = 1, 1, -2}$$

$$\det(A) = (1)(1)(-2)$$
$$\boxed{= -2}$$

(b) - The rank of $A - I$ is ___2___, so that $\lambda =$ ___1___ is an eigenvalue.

- The remaining two eigenvalues of A are $\lambda =$ ___-1 , 0___.

- These eigenvalues are all ___real___, because $A^T = A$.

symmetric

$$A - I = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & -1 \end{bmatrix}$$

(c) The unit eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ will be orthonormal.

- Prove that:

$$A = \lambda_1 \mathbf{x}_1 \mathbf{x}_1^T + \lambda_2 \mathbf{x}_2 \mathbf{x}_2^T + \lambda_3 \mathbf{x}_3 \mathbf{x}_3^T.$$

You may compute the $\mathbf{x}_i$'s and use numbers. Or, without numbers, you may show that the right side has the correct eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ with eigenvalues $\lambda_1, \lambda_2, \lambda_3$.

$Av = \lambda v$

$Ax_1 = \lambda x_1$

$$\lambda_1 x_1 \underbrace{x_1^T x_1}_{I} + \lambda_2 x_2 \underbrace{x_2^T x_1}_{0} + \lambda_3 x_3 \underbrace{x_3^T x_1}_{0}$$

$\lambda x_1 = \lambda x_1$ ✓

Same for $Ax_2 = \lambda x_2$
and $Ax_3 = \lambda x_3$

4 (12 pts.)

This problem is about $x + 2y + 2z = 0$, which is the equation of a plane through $\mathbf{0}$ in $\mathbb{R}^3$.

(a) - That plane is the nullspace of what matrix A?

$$\begin{bmatrix} 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$A = \begin{bmatrix} 1 & 2 & 2 \end{bmatrix}$

- Find an orthonormal basis for that nullspace (that plane).

← why 2 of them ?

orthogonal

$$N(A) = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ -1 \end{bmatrix}$$

$$q_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \qquad q_2 = \frac{1}{3\sqrt{2}} \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}$$

(b) That plane is the column space of many matrices B.

- Give two examples of B.

$$B_1 = \begin{bmatrix} 0 & -4 \\ 1 & -1 \\ -1 & -1 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} 0 & -4 & 0 \\ 1 & -1 & 0 \\ -1 & -1 & 0 \end{bmatrix}$$

(c) - How would you compute the projection matrix P onto that plane? (A formula is enough)

- What is the rank of P?

$$P = B(B^TB)^{-1}B^T$$

P

Projects into 2 d plane = $C(P)$

$\text{rank}(P) = \dim_C(P) = 2$

5 (12 pts.)

Suppose $\mathbf{v}$ is any unit vector in $\mathbb{R}^3$. This question is about the matrix H.

$$H = I - 2\mathbf{v}\mathbf{v}^T.$$

(a) - Multiply H times H to show that $H^2 = I$.

vv^Tvv^T $v^Tv = \|v\|$

$(I - 2vv^T)(I - 2vv^T)$

$\cancel{I^2 - 4vv^T + 4v\underbrace{v^Tv}_{I}v^T}$

$\cancel{H^2 = -4vv^T + 4vv^T + I}$

$H^2 = I$

$= I^2 + 4(vv^T)^2 - 4vv^T$

$= I + 4vv^T - 4vv^T$

$H^2 = I$

(b) - Show that H passes the tests for being a symmetric matrix <u>and</u> an orthogonal matrix.

symmetric $H^T = H$

$I^T = I$

$2vv^T$ always symmetric

$(2vv^T)^T = v^{TT}v^T = vv^T$

Orthogonal

$HH^T = H^2 = I$

symmetry

(c) - What are the eigenvalues of H?

You have enough information to answer for any unit vector $\mathbf{v}$, but you can choose one $\mathbf{v}$ and compute the λ's.

$$Av = \lambda v$$

$$H = I - 2vv^T \qquad A^TA = \|A\|^2$$

$$Hv = \lambda v$$

unit

$$v - 2v(v^Tv) \qquad \|v\| = 1$$

$$v - 2v(1) = \lambda v$$

$$-v = \lambda v$$

$$\boxed{\lambda = -1}$$

6 (12 pts.)

(a) - Find the closest straight line $y = Ct + D$ to the 5 points:

$$(t, y) = (-2, 0),\ (-1, 0),\ (0, 1),\ (1, 1),\ (2, 1).$$

① $y = Ct + D$

(t,y)	
$(-2,0)$	$0 = -2C + D$
$(-1,0)$	$0 = -C + D$
$(0,1)$	$1 = D$
$(1,1)$	$1 = C + D$
$(2,1)$	$1 = 2C + D$

② $Ax = b$

$$\begin{bmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

③ $A^T A x = A^T b$

$$\begin{bmatrix} -2 & -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} -2 & -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

④ $$\begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix}^{-1} = \frac{1}{50} \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix} = \begin{bmatrix} \cancel{5}^{1}/\cancel{50} & 0 \\ 0 & \cancel{10}/\cancel{50} \end{bmatrix} = \begin{bmatrix} 1/10 & 0 \\ 0 & 1/5 \end{bmatrix}$$

$$\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1/10 & 0 \\ 0 & 1/5 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 3/10 \\ 3/5 \end{bmatrix}$$

So $y = Ct + D$

$$\boxed{y = \frac{3}{10}t + \frac{3}{5} D}$$

(b) - The word "closest" means that you minimized which quantity to find your line?

The error squared

$$\|Ax-b\|^2$$

(c) - If $A^T A$ is invertible, what do you know about its eigenvalues and eigenvectors? (Technical point: Assume that the eigenvalues are distinct – no eigenvalues are repeated).

$A^T A$ is always symmetric so all eigenvalues are real

and $x(A^T A x) = \|Ax\|^2 \geq 0$

Positive semi-definite

$\rightarrow$ all positive eigenvalues

Eigenvectors of symmetric matrices are orthogonal

7 (12 pts.)

This symmetric Hadamard matrix has orthogonal columns:

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}, \quad \text{and} \quad H^2 = 4I.$$

(a) What is the determinant of H?

$$\rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & -2 & -2 \\ 0 & -2 & -2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{1} & 1 & 1 & 1 \\ 0 & \boxed{-2} & 0 & -2 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & -2 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{1} & 1 & 1 & 1 \\ 0 & \boxed{-2} & 0 & -2 \\ 0 & 0 & \boxed{-2} & -2 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

Pivots = 1, -2, -2, 4

$$\boxed{\det(H) = 16}$$

(b) What are the eigenvalues of H? (Use $H^2 = 4I$ and the trace of H).

trace(H) = 0

$H = \lambda I$

$H^2 = 4I$

$\lambda = \pm 2$

Two λ's = 2

Two λ's = -2

(c) What are the singular values of H?

$\sqrt{\text{Eigenvalues of } H^T H}$

$H^T = H$

$H^2 = 4I$

$\text{Eig}(4I) = 4$

$\sqrt{4} = 2$

$$\boxed{\begin{aligned} \sigma_1 &= 2 \\ \sigma_2 &= 2 \\ \sigma_3 &= 2 \\ \sigma_4 &= 2 \end{aligned}}$$

8 (16 pts.)

In this TRUE/FALSE problem, you should *circle* your answer to each question.

(a) Suppose you have 101 vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{101} \in \mathbb{R}^{100}$.

- Each v_i is a combination of the other 100 vectors: TRUE – $\boxed{\text{FALSE}}$

- Three of the v_i's are in the same 2-dimensional plane: TRUE – $\boxed{\text{FALSE}}$

(b) Suppose a matrix A has repeated eigenvalues 7, 7, 7, so $\det(A - \lambda I) = (7 - \lambda)^3$.

- Then A certainly cannot be diagonalized ($A = S\Lambda S^{-1}$): TRUE – $\boxed{\text{FALSE}}$

- The Jordan form of A must be $\mathcal{J} = \begin{bmatrix} 7 & 1 & 0 \\ 0 & 7 & 1 \\ 0 & 0 & 7 \end{bmatrix}$: TRUE – $\boxed{\text{FALSE}}$

(c) Suppose A and B are 3×5.

- Then $\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B)$: $\boxed{\text{TRUE}}$ – FALSE

(d) Suppose A and B are 4×4.

- Then $\det(A + B) \leq \det(A) + \det(B)$: TRUE – $\boxed{\text{FALSE}}$

(e) Suppose $\mathbf{u}$ and $\mathbf{v}$ are orthonormal, and call the vector $\mathbf{b} = 3\mathbf{u} + \mathbf{v}$. Take V to be the line of all multiples of $\mathbf{u} + \mathbf{v}$.

- The orthogonal projection of $\mathbf{b}$ onto V is $2\mathbf{u} + 2\mathbf{v}$: $\boxed{\text{TRUE}}$ – FALSE

(f) Consider the transformation $T(x) = \int_{-x}^{x} f(t)dt$, for a fixed function f. The input is x, the output is $T(x)$.

- Then T is always a linear transformation: TRUE – $\boxed{\text{FALSE}}$

Gram Schmidt

$A = a$

$B = b - \frac{A^T b}{A^T A} A$

$C = c - \frac{A^T c}{A^T A} A - \frac{B^T c}{B^T B} B$

$q_1 = A / \|A\|$

$q_2 = B / \|B\|$

$q_3 = C / \|C\|$

$\frac{dU}{dt} = Au$

$u(t) = e^{At} u(0)$

$u(t) = S \Lambda^{\Lambda t} S^{-1} u(0)$

$u(t) = c_1 e^{\lambda_1 t} x_1 + \ldots + c_n e^{\lambda_n t} x_n$

$\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = S^{-1} u(0)$

$P = A(A^T A)^{-1} A^T$

---

dim of:

$C(A) = r$

$C(A^T) = r$

$N(A) = n - r$

$N(A^T) = m - r$

Least Squares:

$A^T A x = A^T b$

18.06 Test Review

Trace $= \lambda_1 + \lambda_2 + \lambda_3$
Det $= \lambda_1 \lambda_2 \lambda_3$

$A = S\Lambda S^{-1} \quad A^k = S\Lambda^k S^{-1}$

evector cols; λ diagonal

Semidefinite: $|\lambda = 0$
Symmetric: Orthogonal evectors

$\text{Evec}(A) = \text{Evec}(A^{-1})$
$\frac{1}{\cancel{E}}\text{val}(A) = \text{Eval}(A^{-1})$

Similar Matrices $\rightarrow$ same λ's
$(Q^TAQ)^T = Q^TA^TQ^{TT}$

$\text{Evec}(A^2) = \text{Evec}(A)$

Positive Def $x^TAx > 0$

$e^{At} = I + At + \frac{1}{2!}(At)^2 = Se^{\Lambda t}S^{-1}$

$\frac{du}{dt} = Au$:
$u(t) = e^{At}u(0)$
$u(t) = Se^{\Lambda t}S^{-1}u(0)$
$u(t) = c_1e^{\lambda_1 t}x_1 + \dots + c_ne^{\lambda_n t}x_n$

$\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = S^{-1}u(0)$ $\quad e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & \\ & e^{\lambda_2 t} & \\ & & \ddots e^{\lambda_n t} \end{bmatrix}$

Similar matrices: Same eigenvalues + diagonalizable $B = M^{-1}AM$
Repeated $\lambda \rightarrow$ not always same eigenvector $\rightarrow$ not diagonalizable
$A^TA = \|A\|^2$
$(Ax)^T = x^TA^T$

SVD
$A^TA \rightarrow \text{Estuff}(A^TA) \rightarrow$ norm evect (V) $\rightarrow$ Find U

$A = U\Sigma V^T$
U: eve AA^T; V: evec A^TA

$Av_1 = \sigma_1 u_1 \longrightarrow u_1 = \frac{Av_1}{\sigma_1}$
$\sigma_1 = \|Av_1\|$

Diagonal entries of Σ (σ_i) $= \sqrt{\text{eig}(A^TA)}$

Singular values: $\sigma_1, \sigma_2 \dots \sigma_n$
↳ Cannot be negative

Row/column swaps don't change SV's

Diagonalizable:
- n independent eigenvectors
- same λ's can be okay
- all different λ's guarantee.

Symmetric:
- Orthogonal eigenvectors
- Real eigenvalues
- Always diagonalizable

Positive Definite:
- λ's are positive
- Positive upper left determinants
- $x^TAx > 0$

Markov
- 1 eigenval $= 1$
- ~~all~~ others < 1

Projection:
Eigenvalues $= 0, 1$
$P^2 = P$

Orthogonal:
- Does not change lengths
- never singular
- Always diagonalizable
- $|\lambda| = 1$

18.06 Final Review

rank r = # of pivot columns

Free variables = $n-r$

Nullspace A: $N(A)$ $Ax=0$

1) Elimination $\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}$
2) Assign anything to free cols
3) get x_1 ← special solutions
4) Repeat 2-3, get x_2
5) $N(A) = cx_1 + dx_2$

Complete solution $Ax=b$

1) Particular Solution
   - Set free vars to 0
   - Solve, x_p
2) Find Nullspace
3) Complete Solution = $x_p + N(A)$

if: $r=n$: $N(A)=0$ → unique sol
$r=m$: can always solve
$r=m=n$: 1 solution
$r=m<n$: ∞ solutions
$r=n<m$: 0 or 1 sol
$r<m$ $r<n$: 0 or ∞ sol

Basis for space:
Independent + span the space
(combinations not 0)

Dimension = # of vectors in basis
$\dim(C(A)) = r$ $\dim(N(A)) = n-r$

perp
$C(A)$: basis = pivot cols $\dim = r$
$C(A^T)$ basis = $\dim = r$
$N(A)$ basis = special solutions $\dim = n-r$
$N(A^T)$ basis = $\dim = m-r$

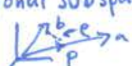
$\text{rref}(A_{\text{invertible}}) = I$

Elimination does not change rowspace

$v\cdot w = v^Tw$ ⇒ $v^Tw = 0$ ⇒ Orthogonal vectors

$Q^TQ = I$ ↔ Orthogonal subspaces

Projections: $Pb = p$ ← Along a



Projection matrix $e = b - p$

Along line:
$p = a\frac{a^Tb}{a^Ta}$ $P = \frac{aa^T}{a^Ta}$

General:
$P = A(A^TA)^{-1}A^T$
$P^2 = P$ $P^T = P$
$A^TA\hat{x} = A^Tb$

Least Squares: (see sample problem)

1) Plug points into equation
2) Get $Ax=b$
3) $A^TAx = A^Tb$
4) Solve for x
5) Plug back in!

Want to minimize $\underbrace{\|Ax-b\|^2}_{e}$

Gram-Schmidt $a,b,c \to q_1, q_2, q_3$

$A = a$ $q_1 = A/\|A\|$

$B = b - \frac{A^Tb}{A^TA}A$ $q_2 = B/\|B\|$

$C = c - \frac{A^Tc}{A^TA}A - \frac{B^Tc}{B^TB}B$ $q_3 = C/\|C\|$

$A = QR$ Factorization
1) Do Gram Schmidt
2) $R = Q^TA$. Find R
3) Pass the test :)

Determinants:

$\det I = 1$
Row exchange → switch signs
Elimination does not change det.
$\det(A)$ = product of pivots
$\det(A)$ = ? Use cofactor method
$A^{-1} = C^T/\det(A)$
(C ← Cofactor Matrix)

Signs:
$\begin{matrix}+&-&+&-\\-&+&-&+\\+&-&+&-\\-&+&-&+\end{matrix}$

2 equal rows → det = 0

Eigenvalues
$|A - \lambda I| = 0$ (det; λ ← eigenvalue)
sum of λ's = trace (diagonal sum)
product of λ's = det

Eigenvectors
1) Find λ's
2) For each $A - \lambda I$ Find nullspace (eigenvectors)

$Ax = \lambda x$ (x: Eigenvectors, λ: Eigenvalues)

Diagonalization $\Lambda = \begin{bmatrix}\lambda_1 & & 0\\ & \lambda_2 & \\ 0 & & \lambda_n\end{bmatrix}$

$A = S\Lambda S^{-1}$
(S: Eigenvector matrix, Λ: diagonal eigenval matrix)

→ Must have n independent eigenvectors!

Easier to calculate powers.

$A^k = S\Lambda^kS^{-1}$

If all different λ's → diagonalizable

if not

$\Lambda^k = \begin{bmatrix}\lambda_1^k & & 0\\ 0 & \ddots & \lambda_n^k\end{bmatrix}$

Markow Matrices.
All entries > 0
One $\lambda = 1$, others ≤ 1
Steadystate ⇒ $\lambda = 1$

Differential Equations $\frac{du}{dt} = Au$

$u(t) = e^{At}u(0)$
$e^{At} = I + At + \frac{1}{2!}(At)^2 + \ldots$
$u(t) = Se^{\Lambda t}S^{-1}u(0)$
$u(t) = c_1e^{\lambda_1 t}x_1 + \ldots + c_ne^{\lambda_n t}x_n$

$\begin{bmatrix}c_1\\ \vdots \\ c_n\end{bmatrix} = S^{-1}u(0)$

$e^{\Lambda t} = \begin{bmatrix}e^{\lambda_1 t} & 0\\ 0 & \ddots\ e^{\lambda_n t}\end{bmatrix}$

1) Find eigenvalues of A
2) Find eigenvectors of A
3) Use above formulas to solve. (if necessary, for c)
4) Write $u(t)$

Stability: $e^{\lambda t}$
only real component of λ counts
Stable: $\lambda < 0$
Steady State: $\lambda = 0$ and other $\lambda < 0$
Blow up: Any $\lambda > 0$

Symmetric Matrices

$A^T = A$
- Eigenvalues are real
- Eigenvectors: orthogonal cols of Q
  $Q^{-1} = Q^T$

$A = S\Lambda S^{-1} \to A = Q\Lambda Q^{-1}$

Always Diagonalizable

Positive Definite Matrices

All $\lambda_i > 0$
All pivots > 0
All upper left determinants > 0
$x^TAx > 0$
symmetric

Similar Matrices
A & B are similar if $B = M^{-1}AM$ (Any matrix, invertible)
Have the same eigenvalues ↑ does not change eigenvalues
eigenvectors multiplied by M^{-1}

Singular Value Decomposition SVD

$A = U\Sigma V^T$
(U: Orthogonal, Eigenvector matrix of AA^T; Σ: diagonal; V: Orthogonal, Eigenvector matrix of A^TA)

1) Compute A^TA
2) Find eigenvalues + eigenvectors
3) V = normalized eigenvector
4) Find Σ
   - Diagonal entries $\begin{bmatrix}\sigma_1 & & 0\\ 0 & \ddots & \sigma_n\end{bmatrix}$ (singular value) $= \sqrt{\text{eigv}(A^TA)}$
5) Find U
   $u_1 = \frac{Av_1}{\sigma_1}$ $A = U\Sigma V^T$ (U: column space, V^T: rowspace)

Complex Matrices