MATH 18.02 LECTURE PLAN, FALL 2013

The plan for each lecture is given below. Each item has the date of the lecture, the main topics covered, and the assigned reading. The reading should be done before lecture and reviewed after the lecture.

If the reading is in the main text (Edwards & Penney), then just the section number is given below. Some of the reading will be in the online notes; in this case, the reading starts with a letter.

- 9/5: Vectors and the dot product, 12.1 and 12.2.
- 9/6: Cross products, 12.3 and D.
- 9/10: Matrices, M.1.
- 9/12: Systems of linear equations, M.2 and M.4. Homework 1 due.
- 9/13: Parametric curves, 10.4 to page 647 and 12.4.
- 9/17: Velocity and acceleration, 12.5 to page 808, 12.6 to (7) on page 818, K.
- 9/19: Functions of several variables and partial derivatives, 13.2, 13.4, TA. Homework 2 due.
- 9/20: MIT student holiday - no lecture.
- 9/24: Review.
- 9/26: Midterm 1. Homework 3 due.
- 9/27: Min-max problems and the least squares approximation, 13.5 and LS.
- 10/1: Second derivative test, 13.10 and SD.
- 10/3: The chain rule, 13.7. Homework 4 due.
- 10/4: Gradients and directional derivatives, 13.8. **MIT add date.**
- 10/8: Lagrange multipliers,13.9.
- 10/10: Non-independent variables, N. Homework 5 due.
- 10/11: Partial differential equations; double integrals, 14.1, 14.2, 14.3, P and I.1.
- 10/15: MIT student holiday - no lecture.
- 10/17: Polar coordinates and integration, 14.4, 14.5 and I.2. Homework 6 due.
- 10/18: Review.
- 10/22: Midterm 2.
- 10/24: Change of variables, 14.9 and CV. Homework 7 due.
- 10/25: Vector fields, 15.2 and V1.
- 10/29: Line integrals, 15.3.
- 10/31: Potential functions and the curl of a vector field, 15.3 and V2. Homework 8 due.
- 11/1: Greens theorem, 15.4 to top of page 1043.
- 11/5: Flux of a vector field, 15.3 and 15.4.
- 11/7: Simply connected regions; triple integrals, 12.8 to page 841, 14.6 (the 3 examples), 14.7 page 988 and 989, V5 and I3. Homework 9 due.
- 11/8: Review.

Date: September 3, 2013.

/5/13

18.02 Lecture

- Review topics before class
- HW due Thursday by 12:45 pm Rm. E17-131
- Change recitation on Stellar
- Can go to any TA's office hours

9/5/13

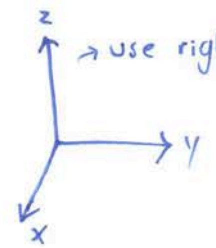
Vectors

A vector $v \in R^3$ is an ordered triple of real #'s

$\underline{i} \quad \underline{j} \quad \underline{k}$

$v = (v_1, v_2, v_3)$

Has direction and length
↳ magnitude



→ use right hand rule

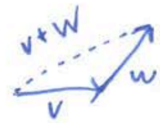
↓ Apply Pythagorean to solve

$|v| = \sqrt{v_1^2 + v_2^2 + v_3^2}$

Unit Vector = length one

Addition

Add components of vectors



Scalar Multiplication

Take scalar and vector and multiply components

$r \in \mathbb{R}$

$rv = (rv_1, rv_2, rv_3)$

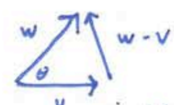
$2(1,2,1) = (2,4,2)$

- Changes length but not direction

↓

Parallel Vectors
↳ Point in same or opposite direction

Dot Product



$w \cdot v = |w|^2 + |v|^2 - 2\, v \cdot w$

$|w - v|^2 = |v|^2 + |w|^2 - 2|v|\,|w|\cos\theta$

Law of cosines

$v \cdot w = (v_1 w_1 + v_2 w_2 + v_3 w_3)$

$= |v|\,|w| \cos\theta$

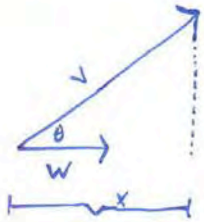
length of v ↑

Eg $(1,2,1) \cdot (0,3,-1) = 0 + 6 - 1 = 5$

$\perp$ when dot product $\overset{v \cdot w}{=} 0$

Orthogonal = Perpendicular

Projection



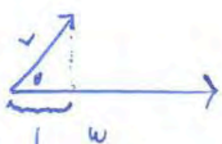
for x need length and direction
└ same as w

Projection of v onto w

$= |v| \cos\theta$ length $=$ $\text{comp}_w v = v \cdot \frac{w}{|w|}$
└ length = 1

$= v \cdot \frac{w}{|w|}$

2



└ length $= |v| \cos\theta$

$v = (1,2,1)$

$w = (1,0,1)$

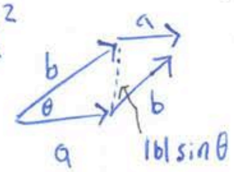
$|v| \cos\theta = \frac{v \cdot w}{|w|} = \frac{(1,2,1)\cdot(1,0,1)}{} = \frac{2}{\sqrt{2}} = \sqrt{2}$

$|v| \cos\theta = v \cdot \frac{w}{|w|}$

9/6/13

18.02 - Matrix, Cross Product, Determinant Fernando Trujano
12.3

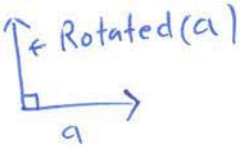
$\mathbb{R}^2$



Deriving Area $= |a||b|\sin\theta$
of Parallelogram

Rotate by $\frac{\pi}{2}$

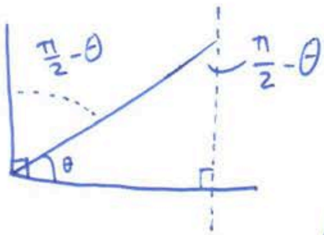
$a = (a_1, a_2)$



Rotate(a) $= (-a_2, a_1)$

$(a_1, a_2) \cdot (-a_2, a_1) = 0$

same length



$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$

$\downarrow$

$A = |a||b|\cos\left(\frac{\pi}{2} - \theta\right) = \text{Rotate}(a) \cdot b = (-a_2, a_1), (b_1, b_2)$

Area $= a_1 b_2 - a_2 b_2$

Determinants

$m \times n$ matrix A

rows ↗ m, Columns ↑ n

Square Matrix if $m = n$

$$A = \begin{array}{c} \text{3 colums} \\ \begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \end{array} \text{4 colums}$$

$\uparrow$ 4×3 matrix

a_{ij} entry

Ex $a_{12} = 2$

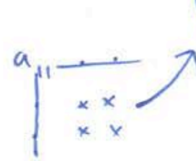
Determinant of a Square Matrix:

1×1 $\det(a) = a$

2×2 $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$ → Area of parallelogram

3×3

$$\det\begin{pmatrix} \overset{+}{a_{11}} & \overset{-}{a_{12}} & \overset{+}{a_{13}} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11} \det\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - \dots$$



→

$\mathbb{R}^2$ = all real #

18.02

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$$= a_{11} \det\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{12} \det\begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} + a_{13} \det\begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

- Keep breaking up into smaller dets

Properties

1) Switching rows (or cols) → mults det by -1
2) Row (or col) of 0 (all will eventually be multiplied by 0) → det is 0
3) Mult a row (or col) by r → det mults by r
4) Add a multiple of a row to another row → det does not change

↳ Just stretches out parall. Area remains same

Ex of 4)

$$\det\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = (1\cdot 4)-(2\cdot 3) = -2$$

Add 2x of last to 1st

multiple of a row

$$\det\begin{pmatrix} 7 & 10 \\ 3 & 4 \end{pmatrix} = 28-30 = -2$$

$$\det\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} = 1\cdot\det\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = 1$$

OR

Subtract 2nd from 1st

Subtract 2 from 3

$$\det\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} = \det\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \det\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

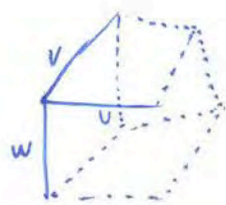
↳ Identity Matrix = 1

Cross Products

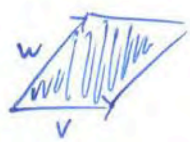
$$v \times w = \det\begin{pmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix} = i\left(v_2 w_3 - v_3 w_2\right) - j\left(v_1 w_3 - w_1 v_3\right) + k\left(v_1 w_2 - v_2 w_1\right)$$

u, v, w

$$u \cdot (v \times w) = \det\begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix} = \text{volume}$$



$|v \times w|$ = Area of



$v \times w \perp$ to v, w

Matrix A is an $m \times n$ array of numbers $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$

3x2 Matrix

$m \times 1$ matrix is a column vector ↰ a_{ij} = i'th row, j'th column

└ Coordinates in space

3x1 matrix $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$a_{12} = 2$

$a_{21} = 3$

$1 \times n$ matrix = row vector

$(1, 2\ 3)$

Matrix Operations

1) Scalar multiplications $7\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 7 & 14 \\ 21 & 28 \end{pmatrix}$

2) Matrix Addition
   - Same size $m \times n$

   A, B both $m \times n$ matrices
   - Add each entry

$C = A + B$ $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 3 & 5 \end{pmatrix}$

$c_{ij} = A_{ij} + b_{ij}$

3) Transpose
   - switch x and y

   row ↔ column

   $m \times n \to n \times m$

   ↓

   Changes size of matrix (unless square $m \times n$

$A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$ $B = A^T$

$A^T = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ $B_{ij} = A_{ji}$

A is symmetric if $A^T = A$

$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ is symetric $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ NOT

Skew-symetric $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $A^T = -A$

4) Matrix Multiplications

$A = m \times n$

$B = n \times r$

$C = AB$ is $m \times r$

$$C_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1\cdot1+2\cdot0 & 1\cdot-1+2\cdot1 \\ 3\cdot1+4\cdot0 & 3\cdot-1+4\cdot1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix}$$

$$\begin{array}{cc} 3\times3 & 3\times2 \\ \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 3 \end{bmatrix} & \begin{bmatrix} 2 & -1 \\ 1 & 2 \\ 3 & -2 \end{bmatrix} \end{array} = \begin{bmatrix} 5 & -3 \\ 4 & 3 \\ 10 & -4 \end{bmatrix}$$

$C = AB$

Dot Product

$C_{ij} = [1^{th} \text{ row of } A] \bullet [j^{th} \text{ column of } B]$

row1 · column1
+ row1 · column 2
+ row2 · column 1
. . . .

Matrix Multiplication not commutative — Except with identity matrix

Order does ~~not~~ matter

$A \quad 3\times2$
$B \quad 2\times2$ $\Rightarrow$ AB is defined, BA is NOT

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$$

} Not equal

Rules:

1) Distributive $(A+B)C = (A\cdot C) + (B\cdot C)$

$C(A+B) = (C\cdot A) + CB$

$r \in R \quad r(A+B) = rA + rB$

2) Associative $(AB)C = A(BC)$

Same order

$r \in R \quad (rA)B = r(AB) = A(rB)$

3) Identity matrix

I_m $m\times n$ identity matrix

$$I_m = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

Ex $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$A \; m\times n$

$I_m A = A$

$A I_n = A$

4) Determinents

$\det(AB)$

$\det(A)\det(B)$

Ex

A is nilpotent

if $A^n = 0$ for some $n > 0$

$A^2 = AA$

$A^3 = AAA = A^2A = AA^2$ ← same by assoc

↓ Can define powers of matrices

$$\underset{\substack{\| \\ 0}}{\det(A^n)} = [\det(A)]^n$$

det of identity is 1

A $\underset{\text{square}}{n \times n}$

A^{-1} is matrix so that $\begin{Bmatrix} AA^{-1} = I_n \\ A^{-1}A = I_n \end{Bmatrix}$

Caveat A^{-1} might not exist

⇓

if A is nilpotent = no inverse

and more...

Computing Inverse.

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Swap 7 and 1

Make minus on other diagonal

* See book.

Linear maps

$L = \mathbb{R}^n \to \mathbb{R}^n$

is linear if

$L(v+w) = L(v) + L(w)$

$L(rv) = rL(v)$

r in $\mathbb{R}$

3/4

A $n \times n$ matrix

V $n \times 1$ column vector

$L(v) = A v$

rule $L(v+w) = A(v+w)$

$= A(v) + A(w)$

Matrix Multiplication gives Linear map

$L(a\hat{i} + b\hat{j}) = aL(i) + bL(j)$

linear

if you know what linear map does to basis vectors you know what it does to every vector.

$L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$ rotation by $\frac{\pi}{2}$

$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

6/6/13 # 18.02 Recitation Fernando Trujano

An $n \times n$ matrix is nilpotent if $\exists k > 0$ s.t. $A^k = 0$
then $A^{k+1} = 0$

$\exists$ exists; s.t. such that

$\mathbb{R}$ = The set of real #'s

$\mathbb{R}^2$ = {ordered pairs of real #'s = pts (or vectors) in plane

Dot Product

$\vec{x}, \vec{y} \in \mathbb{R}^3$ set

$\in$ is in

$\vec{x} = (x_1, x_2, x_3)$

$\vec{y} = (y_1, y_2, y_3)$

$\vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2 + x_3 y_3$

A real #

iff $\vec{x} \cdot \vec{y} = = 0$

x is perpendicular to y

$\vec{x} \cdot \vec{y} = |x||y| \cos\theta$

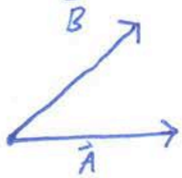
θ Angle b/w x and y

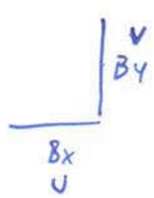
$\vec{x} \cdot \vec{x} = |x|^2$

Ex

Let $\vec{a}, \vec{b}$ in $\mathbb{R}^3$ be non-parallel

Express $\vec{b}$ as a sum of a vector $\parallel$ to $\vec{a}$ & one $\perp$ to $\vec{a}$





$\vec{u} \parallel a$

$|\vec{u}| = \dfrac{\overbrace{\vec{a} \cdot \vec{b}}^{|a||b|\cos\theta}}{|\vec{a}|} = |\vec{b}| \cos\theta$

$\vec{u} = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}|} \dfrac{1}{|\vec{a}|} \vec{a} = \dfrac{\vec{a} \cdot \vec{b}}{\vec{a} \cdot \vec{a}} \vec{a}$

$\vec{V} = \vec{b} - \vec{u}$

$\vec{V} \cdot \vec{a} = \vec{b} \cdot \vec{a} - \dfrac{a \cdot b}{\cancel{a \cdot a}} (\cancel{a \cdot a}) = 0$

Perpendicular

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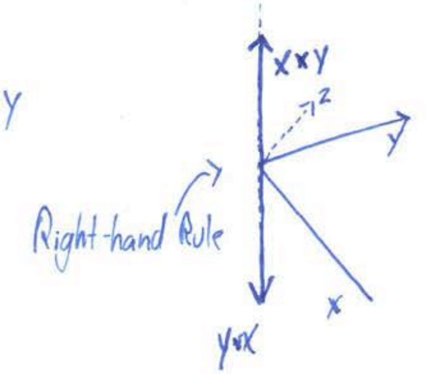
Cross Product

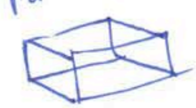
$\vec{x}, \vec{y} \in \mathbb{R}^3, \quad \vec{x} \times y \in \mathbb{R}^3$

$\underbrace{|\vec{x} \times \vec{y}|}_{p}$ = Area of Parallelogram formed by x and y

$\vec{p} \perp \vec{x} \; . \; p \perp y$



$\vec{z} \cdot (\vec{x} \times \vec{y})$ / Volume of Parallelopiped



$$\text{Volume} = z \cdot (x \times y)$$

$$= \begin{vmatrix} z_1 & z_2 & z_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

$$|z \cdot (x \times y)| = |(z \times y) \cdot x| = |\vec{y} \cdot (x \times z)|$$

Mixed Product

Example

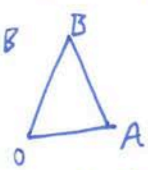
Look at tetrahedron w/ vertices $\overset{a_1}{(1,1,1)}, \overset{a_2}{(1,-1,-1)}, \overset{a_3}{(-1,1,-1)}, \overset{a_4}{(-1,-1,1)}$

a) Find length of all edges

b) Find coordinates of C, determine bonding angle

c) Find area of each face

A, B



a) $\langle 1,1,1\rangle - \langle 1,-1,-1\rangle = \langle 0,2,2\rangle = 2\sqrt{2}$

b) $\frac{\vec{A}+\vec{B}}{2}$ $\quad \underbrace{\vec{OC}}_{1} = \frac{1}{4}(\Sigma \text{ vertecies})$ $\quad \cos = \frac{-1}{3}$

$|AB| = |\vec{B} - \vec{A}|$

$C = \frac{1}{4}(\Sigma \text{vertecies})$

c) $^{\text{Subtract \&}}$ Cross Product of any $= 2\sqrt{3}$ $\quad$ Area $= \frac{|(a_2 - a_1) \times (a_3 - a_1)|}{2}$

9/11/13 # 18.02 Recitation Fernando Trujano

Linear Maps

$A \cdot B$

if $n \times m$ $m \times k$ $\to n \times k$

Example $k = 1$

$$B_{m \times 1} \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} = \vec{b} \in \mathbb{R}^m$$

column vector

↳ Can think of it as a vector in $\mathbb{R}^m$

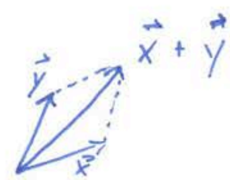
For a given A there is an assignment

Fix A $n \times m$

$$\underset{m\text{-vector}}{\vec{x} \in \mathbb{R}^m} \longrightarrow \underset{n \times 1}{A\vec{x}} \in \mathbb{R}^n$$

A map of sets from $\mathbb{R}^m \to \mathbb{R}^n$

Definition



$f: \mathbb{R}^m \to \mathbb{R}^n$ is linear

if $f(\vec{x} + \vec{y}) = f(\vec{x}) + f(\vec{y})$ True for all $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^m$

& $f(\underset{\text{scalar}}{\alpha} \vec{x}) = \alpha f(\vec{x})$ $\forall \alpha \in \mathbb{R}, \vec{x} \in \mathbb{R}^m$

Claim: ① The map defined by A is linear

② Any linear map has a matrix

A s.t. $f(\vec{x}) = A\vec{x}$

$$\underset{m \cdot n}{AB} + \overbrace{AC}^{n \times k} = \overset{f}{A}(\overset{\vec{x}}{B} + \overset{\vec{y}}{C})$$

$A(\alpha B) = \alpha (AB)$

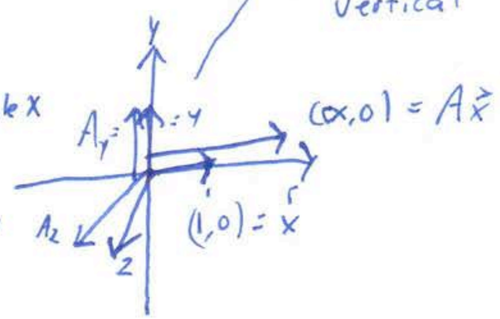
$A = (a_{ij})$, $\alpha A = (\alpha a_{ij})$

Example

$$A = \begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2} \qquad A : \mathbb{R}^2 \to \mathbb{R}^2$$

Map from $\mathbb{R}^2$ to $\mathbb{R}^2$

does not change vertical



Scaling

$$A\vec{x} = \underbrace{\begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix}}_{m\times n}\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{n\times k} = \begin{bmatrix} \alpha \cdot x_1 \\ x_2 \end{bmatrix}$$

only scale x

y stays same

$$\begin{bmatrix} \alpha x_1 \cdot 0x_2 \\ 0x_1 \cdot 1x_2 \end{bmatrix} = \begin{bmatrix} \alpha x_1 \\ x_2 \end{bmatrix}$$

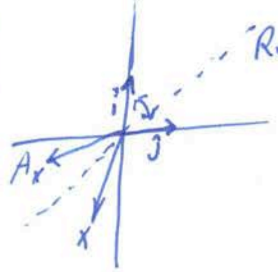
Multiply Matrix

$(i^{th} \text{ row}) \bullet (j^{th} \text{ column}) = (i, j)$ entry

---

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Reflect about $y = x$



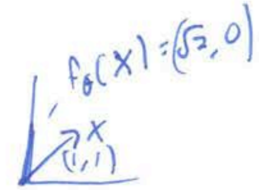
$$A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$$

To prove:

Show perpendicular

θ is an angle

if $\theta = 45°$



$f_\theta : \mathbb{R}^2 \to \mathbb{R}^2$

$\vec{x} \sim f(\vec{x}) =$ the result of rotating $\vec{x}$ by θ ccw

1. Show R is linear
2. Find Matrix A_θ representing f_θ
3. Find Inverse of A_θ

Determinants

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

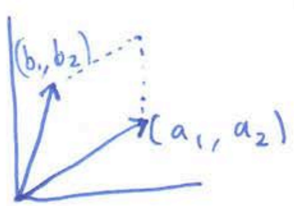
$$\det\begin{array}{c} \begin{matrix} + & - & + \end{matrix} \\ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \end{array}$$

$$= a_{11} \det\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{12} \det\begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} + a_{13} \det\begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

$$\begin{array}{l|l} a_{11} & \\ \hline & \begin{matrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{matrix} \end{array}$$

det(a)

- Multiplied by -1 if switch two rows or columns
- = 0 if one row or column is 0
- Multiplied by c if every entry of a row or column is ×c



$$\det\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} = \text{Area of parallelogram}$$

* Works with x y z too!

Matrix Algebra

$m \times n$ matrix

rows (m), columns (n)

Row vectors: $1 \times n$ matrix $\begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix}$

Column vectors: $m \times 1$ matrix $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$

Matrix Operations

1) Scalar Multiplication
   - Multiply each entry
2) Matrix Addition
   - Add corresponding entries
3) Transposition
   - Switch rows and colums A^T

Example

$$A = \begin{bmatrix} 2 & -3 \\ 0 & 1 \\ -1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 5 \\ -2 & 3 \\ -1 & 0 \end{bmatrix}$$

$$A + B = \begin{bmatrix} (2+1) & (-3+5) \\ (0-2) & (3+1) \\ (-1+-1) & (2+0) \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -2 & 4 \\ -2 & 2 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 2 & 0 & -1 \\ -3 & 1 & 2 \end{bmatrix}$$

$$2A + -3B = \begin{bmatrix} (2\cdot 2) & (-3\cdot 2) \\ (0\cdot 2) & (1\cdot 2) \\ (-1\cdot 2) & (2\cdot 2) \end{bmatrix} + \begin{bmatrix} (-3\cdot 1) & (5\cdot -3) \\ (-2\cdot -3) & (3\cdot -3) \\ (-1\cdot -3) & (0\cdot -3) \end{bmatrix} = \begin{bmatrix} 4 & -6 \\ 0 & 2 \\ -2 & 4 \end{bmatrix} + \begin{bmatrix} -3 & -15 \\ 6 & -9 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -21 \\ 6 & -7 \\ 1 & 4 \end{bmatrix}$$

Matrix Multiplication

$$\underset{m \times n}{A} \cdot \underset{n \times p}{B} = \underset{m \times p}{C}$$

Example

$$\underset{3 \times 1}{\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}} \cdot \underset{1 \times 2}{\begin{bmatrix} 4 & 5 \end{bmatrix}} = \underset{3 \cdot 2}{?}$$

$$\begin{bmatrix} (1 \cdot 4) & (1 \cdot 5) \\ (2 \cdot 4) & (2 \cdot 5) \\ (-1 \cdot 4) & (-1 \cdot 5) \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ 8 & 10 \\ -4 & -5 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & -2 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 1 \\ -1 & 0 & 2 \end{bmatrix}$$

row 1 • column 1

$(2\ 0\ 1)$ $(1\ 0\ -1)$

$(1 \cdot 2) + (0 \cdot 0) + (1 \cdot -1) = 1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & -2 & -6 \\ 0 & 2 & 2 \end{bmatrix}$$

row 1 • column 2
row 1 • column 3
row 2 • column 1
. . . .

Laws

$A(B+C) = AB + AC$ Distributive

$A(BC) = (AB)C$ Associative

Identy Matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $AI = A$
does not affect

$$\det(AB) = \det(A)\det(B)$$

Inverse of Matrix

$$A^{-1} = \frac{1}{\det|A|} \underset{A^T}{\underline{\operatorname{adj} A}} = \frac{1}{\det|A|} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}^T$$

9/12/13 # 18.02 Lecture Fernando Trujano

Inverse Matrix
Linear systems
Solving by Algebra

Geometry
General theory

Inverse Matrix

A $n \times n$ matrix

Theorem: A is invertible iff $\det(A) \neq 0$

$\Leftrightarrow$ -both ways

Ex $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ det $= -2 \Rightarrow$ invertible

Prove $\Rightarrow$

$AA^{-1} = I_n$ ← Identity

$\underbrace{\det(A)}_{\text{Cannot be } 0} \det(A^{-1}) = \det(I_n) = 1$

$\Leftarrow$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $ad - bc \neq 0$

$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

determinant

with a, d
add negs

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} ad-bc & 0 \\ 0 & -bc+ad \end{pmatrix}$$

Lineal Systems

(square)

unknowns = equations

$n=1 \quad \underbrace{a\,x = b}_{\text{given \#'s}}$ Solve x

$n=2 \quad a_{11}x_1 + a_2 x_2 = b_1$

$\qquad a_{21}x_1 + a_{22}x_2 = b_2$

2eq and 2unknowns

* You can't always solve this

Ex

$x_1 + x_2 = 2$

$x_1 - x_2 = -2$

$\Updownarrow$

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$x_1 + 2x_2 = 1$

$2x_1 + 3x_2 = 2$

$\Updownarrow$

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Matrix equation - Algebra

$Ax = b$

IF A is invertible $\Rightarrow x = A^{-1}b$

$A(A^{-1}b) = (AA^{-1})b = I_b = b$

determinent

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} = \frac{1}{-2}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{-1}{2}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ -2 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \quad \begin{matrix} x_1 = 0 \\ x_2 = 2 \end{matrix}$$

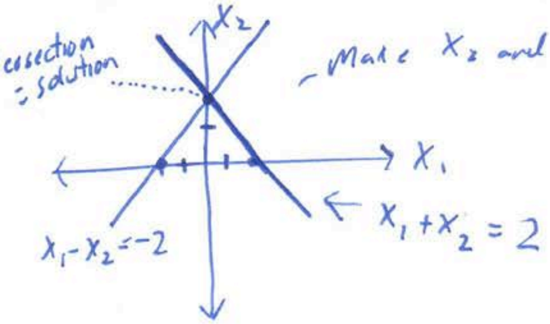
$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}^{-1} = \frac{1}{-1}\begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}$$

$$\begin{matrix} x_1 \\ x_2 \end{matrix} = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Geometry

$x_1 + x_2 = 2$

$x_1 - x_2 = -2$

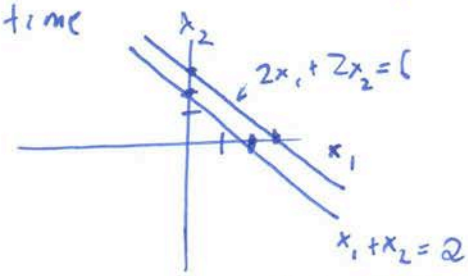


Make x_2 and $x_1 = 0$ to find 2 points

Each equation cuts down the dimentional space of solution by 1.
does not work all the time

$x_1 + x_2 = 2$

$2x_1 + 2x_2 = 6$



NO SOLUTION

$A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

$\det(A) = 0$

if $\det = 0$

* if same line ie $x_1 + x_2 = 2$, $2x_1 + 2x_2 = 4$
then infinately many solutions

General Theory

$Ax = b$
A $n \times n$

1) If $\det(A) \neq 0$ then $\exists!$ solution
exists only one

$x = A^{-1}b$

$A(A^{-1}b) = (AA^{-1})b = b$
Satisfies the equation

Any solution

$Ax = b$
mult both by A^{-1}

$A^{-1}(Ax) = A^{-1}b$

$(A^{-1}A)x =$

Ix

x

Only one solution

no A inverse

2) $\det(A) = 0$

~~Either~~ (1) $Ax = 0$ homogenious
go through the origin
L always has non-zero solutions

(2) $Ax = b$ inhomogenous if $b \neq 0$
L might not have any solutions
L might have infinately many

0 ∞

inverse 3×3

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

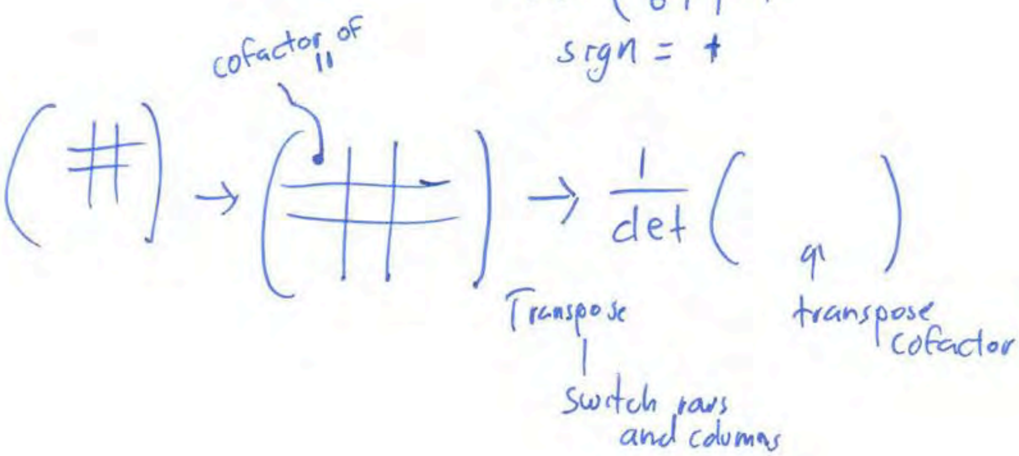
cofactor

Ex $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

22 cofactor

$\det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$

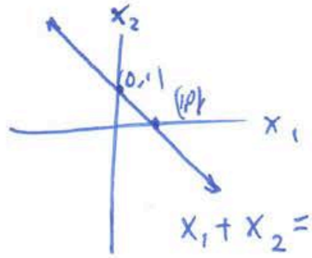
sign = +



Lines (Parametrics), Planes in Space

Lines

2 space (x,y) $\mathbb{R}^2$ 3 space (x,y,z) $\mathbb{R}^3$



$x_1 + x_2 = 1$

Parametrize the line

$(0,1) + t(\text{direction})$ — See Vector of line

$(0,1) + t((1,0) - (0,1))$

point + t(direction)

$(0,1) + t(1,-1)$

as t varies this traces line

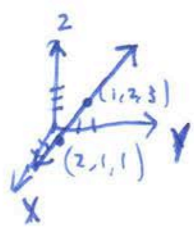
Each point happens at a specific t

OR

$(1,0) + t(9,-9)$ → also param's L — same direction as $(1,-1)$

infinately many because any speed works

$\mathbb{R}^3$



L $(1,2,3)\ (2,1,1) \in L$

(Pt in L) + t(Dir of L) — vector b/w two points

$(1,2,3) + t((2,1,1) - (1,2,3))$

$(1,2,3) + t((1,-1,-2))$

$$\begin{cases} x = 1 + t \\ y = 2 - t \\ z = 3 - 2t \end{cases}$$

L_1 L_2 lines in space

Possibility

(1) $L_1 = L_2$

(2) L_1 L_2 parallel — one is a scalar multiple of the other

Example $(1,1,-1)$ and $(2,2,-2)$ are parallel

(3) $L_1 \cap L_2 =$ one pt

intercept

(4) Not parallel nor touch ("skew")

Example

$\begin{cases} x = 1+3t \\ y = 2-t \\ z = 3 \end{cases}$ $(1+3t, 2-t, 3)$ $(2t, -1+t, 1+t)$

1) Find Direction $(3,-1,0)$ (t coefficient) · any mult. works too

$(2,1,1)$

not parallel aka not multiples

(3) or (4)?

Can't usually solve. Get lucky!

Look for a point in both lines.

s, t var's — same t not needed. - see if they passed same point if $t=s$ then = crash

$$\begin{cases} 1+3t = 2s \\ 2-t = -1+s \\ 3 = 1+s \end{cases}$$

↳ Look for s and t that satisfy

1st line · Second line

$1+3t = 2s$
$2-t = -1+s$
$3 = 1+s \Rightarrow s = 2$

$\left.\begin{matrix} 1+3t = 4 \\ 2-t = 1 \end{matrix}\right\} \Rightarrow t = 1$

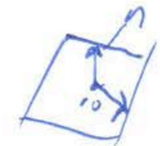
Both go through $(4,1,3)$

2/14

Planes in Space

Plane $\mathbb{P}$ determined by pt $r_0 \in \mathbb{P}$ — dot product 0, orthogonal

normal vector n — perpendicular to plane



$$r \in \mathbb{P} \iff (r - r_0) \cdot n = 0$$

Example

$\mathbb{P}$ throu $(1,0,0)$ normal vector $(1,1,2)$

$$(x-1, y, z) \cdot (1,1,2) = 0$$

$$x - 1 + y + 2z = 0$$

$$x + y + 2z = 1 \quad \text{- equation for } \mathbb{P}$$

A 2D plane satisfy a $\mathbb{R}^3$ equation

$\overbrace{(1,0,0)}^{A}, \overset{B}{(0,1,0)}, \overset{C}{(0,0,1)}$ in $\mathbb{P}$

- 3 points, find plane through them

$x + y + z = 1 \Rightarrow$ should be answer

normal = cross product

Do not point in same direction

$$\underset{(1,-1,0)}{(A - B)} \times \underset{(1,0,-1)}{(A - C)} = \text{vector perp to both}$$

are parallel to $\mathbb{P}$

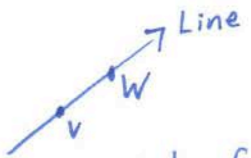
$$\det \begin{pmatrix} i & j & k \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} = (1,1,1)$$

normal vector

$$0 = (x-1, y, z) \cdot (1,1,1) = x - 1 + y + z$$

$$x + y + z = 1 \quad ✓$$

3 / 4



vector from v to w points in direction of line

vector from v to w $= w - v$
parallel to L

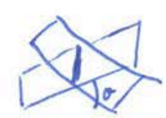
2 planes in $\mathbb{R}^3$

Possibilities

1) $P_1 = P_2$

2) P_1, P_2 parallel $\left\{ \begin{array}{l} x = 1 \\ x = 2 \end{array} \right\}$ never cross ... normals are parallel
aka normals are multiples of the other

3) P_1, P_2 cross in a line
angle between normals



take dot pro b/w normals
find $\cos\theta$

normals
n_1, n_2

$$\cos\theta = \frac{n_1 \cdot n_2}{|n_1|\,|n_2|}$$

Examples

$x + y + z = 2$
$x + y + z = 3$
not equal

Parallel

$x + y + z = 1$ norm $(1,1,1)$
$x + y + 2z = 1$ $(1,1,2)$

not parallel $\Rightarrow$ cross in line

$$\cos\theta = \frac{(1,1,1)\cdot(1,1,2)}{(\sqrt{3}\cdot\sqrt{6})} = \frac{4}{\sqrt{18}}$$

4/4

9/16/13

18.02 Recitation

Fernando Trujano

Office Hours W 5-7pm @ 66-156

① $\det A \neq 0$ A^{-1} exists

$\vec{x} = A^{-1}A\vec{x} = A^{-1}b \implies$ unique solution

② $\det A = 0$

A not invertible $\iff \exists\ \vec{x}_1 \neq x_2$ such that $A\vec{x}_1 = A\vec{x}_2$

onto?

one-to-one

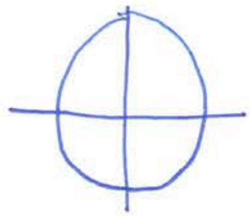
not a function

Map means function

x map to z

$Ax \to z$

x is mapped to z



$x^2+y^2=1$

$\vec{r}_1 = \cos t\,\hat{i} + \sin t\,\hat{j}$

26)

$\vec{r} = (e^t \sin t,\ e^t \cos t)$

Find tangential w/ normal component of a

$\vec{r}(t)$ position

$v(t) = r'(t)$

$\hat{T}(t) = \dfrac{v(t)}{\|v(t)\|}$

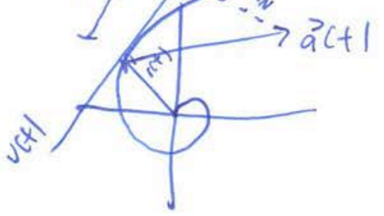
$\vec{a}(t) = \vec{v}\,'(t)$

$\vec{a}_{TAN}$ = (components of acceleration along $\hat{T}$) $\cdot\ \hat{T}$

$= (\vec{a} \cdot \hat{T})\ \hat{T}$ ... direction

$= \dfrac{\vec{a} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}\ \vec{v}$

Tangent line at pt $\vec{r}(t_0)$ of curve $\vec{r}$ passes through $\vec{r}(t_0)$ and is parallel to $\vec{v}(t_0)$



$$\begin{cases} \vec{a}_T + \vec{a}_N = \vec{a} \\ a_N \perp \vec{v} \end{cases}$$

In 3D

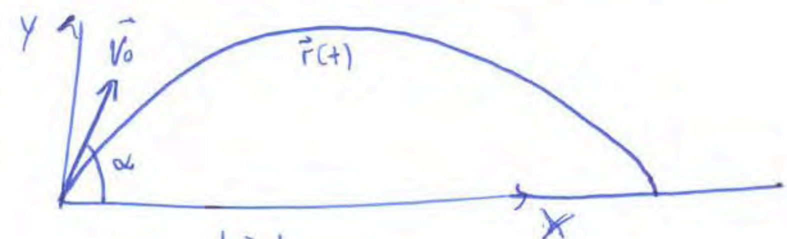
$$\hat{N}(t) = \frac{\vec{r}(t) \times \vec{v}(t)}{|\vec{r}(t) \times \vec{v}(t)|}$$

$$\vec{a}_N = (\vec{a} \cdot \hat{N})\, \hat{N}$$

$$\hat{N}(t) = \frac{\vec{r}(t) \times \vec{v}(t)}{|\vec{r}(t) \times \vec{v}(t)|}$$

Projectile motion w/ initial speed V_0, starting from $(0,0)$

Find angle α that maximizes range



$V_0 = |\vec{V}_0|$

17/9/13

18.02 Lecture

Fernando Trujano

-Pset solutions online

Multivariable Calculus

Curves, Derivatives, Velocity and accel, Kepler, Examples

Curves

Ex: Line

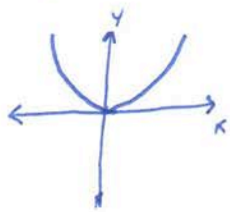
$P = (P_x, P_y, P_z)$

$P + tv$

both vectors

$x = p_x + tv_x$
$y = p_y + tv_y$
$z = p_z + tv_z$

More generally



$x = t$
$y = t^2$

A general curve

$x = f(t)$
$y = g(t)$
$z = h(t)$

or one vector equation

3 components

$r(t)$

Derivatives

Take derivative of each component

$$r'(t) = \lim_{\Delta t \to 0} \frac{r(t+\Delta t) - r(t)}{\Delta t}$$

If $r(t) = (f(t), g(t), h(t))$

then $r'(t) = (f'(t), g'(t), h'(t))$

Single Variable Calculus

Same rules:

<u>Linear</u> $\quad \left(r(t)+s(t)\right)'$
$$= r'(t) + s'(t)$$
(scalars too)

<u>Product Rule</u>

$$\left(r(t)\cdot s(t)\right)' = r'(t)\cdot s(t) + r(t)\cdot s'(t)$$

Velocity and Acceleration

$r(t)$ is our curve

velocity $\quad v(t) = r'(t)$

vector

speed $= |v(t)| = \sqrt{v(t)\cdot v(t)}$

$\cdots$ norm, magnitude, length

acceleration $\quad a(t) = v'(t) = r''(t)$

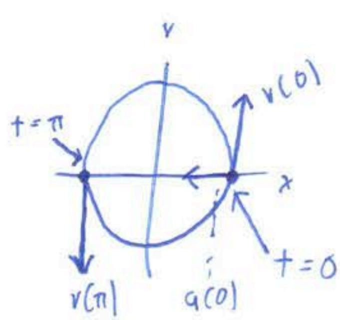
Example

$r(t) = (\cos t, \sin t, 0)$

$v(t) = (-\sin t, \cos t, 0)$

$|v(t)| = 1 \quad \because \sin^2 + \cos^2 = 1$

$a(t) = (-\cos t, \sin t, 0)$



Length of Curve

$$L = \int |v(t)|\, dt$$

Fundamental Theorem of Calculus $\Rightarrow$ $L' =$ speed

Example

circle

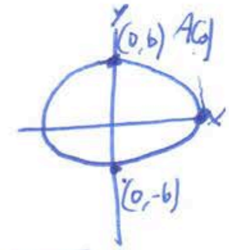
$$= \int_0^{2\pi} 1\, dt = 2\pi$$

Elipse

$r(t) = (a\cos t, b\sin t, 0)$

$v(t) = (-a\sin t, b\cos t, 0)$

$|v(t)| = \sqrt{a^2\sin^2 t + b^2\cos^2 t}$



Kepler's Laws
of Planetary Motion

1) Orbit of planet is an elipse in a plane
2) Covers same area
3) Relates Period to Revolution

Line from the sun to the planet sweeps area at constant ~~speed~~ rate

$\Downarrow$

acceleration points towards the sun
Force

$\Downarrow$ ~~Proof~~ $\Downarrow$ (by observation

O = position of Sun

$r(t)$ = position of planet at time t

Area $\approx$ triangle by $r(t)$, $r(t+\Delta t)$

$$\text{Area} \approx \frac{1}{2}\left| r(t) \times r(t+\Delta t) \right|$$

$$\approx \frac{1}{2}\left| r(t) \times \left[r(t) + r'(t)\Delta t \right] \right|$$

$$= \frac{1}{2}\left| \underbrace{\cancel{r(t) \times r(t)}}_{=0} + r(t) \times r'(t)\Delta t \right|$$

$$\approx \frac{1}{2}\Delta t \left| r(t) \times r'(t) \right|$$

$$\frac{d}{dt}\text{Area} = \lim_{\Delta t \to 0} \frac{\text{Area}}{\Delta T} = \frac{1}{2}\left| r(t) \times r'(t) \right|$$

This is constant by K③

$r(t) \times r'(t)$ has constant direction too!
($\perp$ to plane of motion)

$r(t) \times r'(t)$ is constant

so

$$\left(r(t) \times r'(t) \right)' = 0$$

$$0 = \underbrace{\cancel{r'(t) \times r'(t)}}_{=0} + r(t) \times r''(t)$$

acceleration

$$0 = r(t) \times a(t) = 0$$

Parallel

9/23/12

18.02 Lecture
Exam Review

Fernando Trejano

⋮... No partial derivatives

Review: Vector Products, Inverse Matrices, Planes and lines

Vector Products

- Different ways to multiply vectors

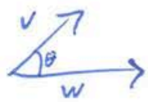
<u>Dot Product</u>

$\vec{v} \cdot \vec{w} = \underbrace{v_1 w_1 + v_2 w_2 + v_3 w_3}$ ↗ scalar

$|\vec{v}| = \sqrt{v \cdot v}$

↳ length

if $\cdot = 0$ $\perp$



$\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos\theta$

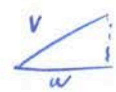
<u>Cross Product</u>

$\vec{v} \times \vec{W} =$ ↗ vector

$\vec{v} \times \vec{w} = \det \begin{pmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix} = \cancel{v_2 w_3 - v_3 w_2, \; - v_1 w_3}$

$\begin{array}{ccc} i & j & k \\ (v_2 w_3 - v_3 w_2, & w_1 v_3 - v_1 w_3, & v_1 w_2 - v_2 w_1) \end{array}$

$|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}| \sin\theta$



Ex Area (triangle $\vec{v}, \vec{w}$) $= \frac{1}{2} |\vec{v} \times \vec{w}|$

Direction of $v \times w$ is $\perp$ to both v and w

Use RHR to choose + or -

Ex Typical Problem

Find Vector $\perp$ to both $(1,2,3)$ and $(1,1,-1)$

$\det \begin{pmatrix} i & j & k \\ 1 & 2 & 3 \\ 1 & 1 & -1 \end{pmatrix} = \begin{array}{ccc} i & j & k \\ (-5, & 4, & -1) \end{array}$

Inverse matrices

* Know nilpotent
* inverse of 3×3 ?
* Parametric Curves

A $n \times n$ matrix

Want to Find A^{-1}

*? $$A^{-1}A = AA^{-1} = I_n$$

A^{-1} exists iff $\det A \neq 0$

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

$$\det\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a\left(\det\begin{pmatrix} e & f \\ h & i \end{pmatrix}\right) - b\det\begin{pmatrix} d & f \\ g & i \end{pmatrix} + c\det\begin{pmatrix} d & e \\ g & h \end{pmatrix}$$

What is inverse good for?

Solve Linear System of equations

Solve $A\vec{x} = \vec{b}$ $\quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

If A^{-1} exists

↳ ie $\det A \neq 0$

then solution is $\vec{x} = A^{-1}\vec{b}$

$$\det\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 1 & 2 \end{pmatrix}\begin{matrix} -① \\ -② \\ ③ \end{matrix}$$ if ① + ② = ③ then $\det = 0$

Planes and Lines

Plane $\mathbb{P}$ in $\mathbb{R}^3$

$$ax + by + cz = d$$

normal $= (a, b, c)$

d says whether $0 \in P$
origin

To determine a plane $\mathbb{P}$

- 3 pts not in a line
- A line + a point not in line
- A point and a normal
  P_0 $\vec{n}$

$$\hookrightarrow (P - P_0) \cdot \vec{n} = 0$$

$$\underbrace{P \cdot n}_{ax+by+cz} = \overbrace{P_0 \cdot n}^{d}$$

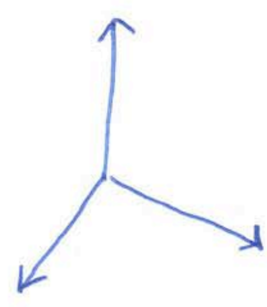
Example

$x + y + 2z = 1$

Normal is $(1, 1, 2)$

└ Any multiple is also a normal
ie $(2, 2, 4)$

How far from Origin to P
in $\mathbb{R}^3$



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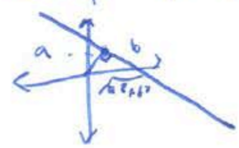
Baby Problem

Closest pt to line in $\mathbb{R}^2$



Line connecting origin to closest point is perpendicular to line!

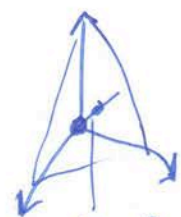
(same in $\mathbb{R}^3$

Back to problem ...

$x + y + 2z = 1$

normal $(1,1,2)$

How far from O^{rigin} to P



line ℓ

$(t, t, 2t)$

need pt $(0,0,0)$ and direction

Question

Where does $(t, t, 2t)$

hit $x + y + 2z = 1$

$1 = t + t + 2(2t) = 6t$

$t = 1/6$ - intersection

$(1/6, 1/6, 1/3)$ is our pt

$\sqrt{\left(\frac{1}{6}\right)^2 + \left(\frac{1}{6}\right)^2 + \left(\frac{1}{3}\right)^2} = \frac{\sqrt{6}}{6}$

Shortest distance from origin to plane

Find P containing $(3, -4, -1) \leftarrow P_0$

and line $\begin{cases} 2x - y + z = 3 \\ 3x + y + z = 1 \end{cases}$] - not linear multiples

↓

not parallel

↓

must intersect in a line

$(3, -4, -1)$ is not in line

Now we have a line and a point (not in line)

Now

Find normal

Find two points in the line

plane $x = 0$ in the line $\begin{cases} -y + z = 3 \\ y + z = 1 \end{cases}$

:: Can chose any x ::

add

$2z = 4$

$z = 2$

$y = -1$

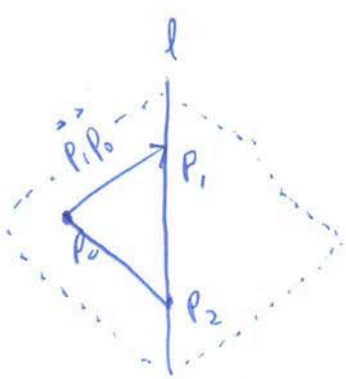
$= (0, -1, 2) \leftarrow P_1$ is in the line ✓

plane $x = 2$ $\begin{cases} 4 - y + z = 3 \\ 6 + y + z = 1 \end{cases}$

$z = -3$

$y = -2$

$= (2, -2, -3) \leftarrow P_2$ is also in line



Find Normal

↳ need 2 vectors parallel to plane

$(14, 13, 3)$

$\llcorner$ normal direction

$\Downarrow$

$14x + 13y + 3z = ??$

$$14x + 13y + 3z = -7$$

18.02 Exam Review

Fernando Trujano

Inverse of 3by3 Matrix

① Matrix A → Compute cofactor matrix

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

ij^{th} entry

2×2 determinant of ij^{th} minor

② Sign change according to checkerboard

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

③ Transpose

↳ Switch rows with columns

④ Divide by $\det A$

Example

$$A = \begin{vmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ -1 & -3 & 0 \end{vmatrix}$$

$\det(A) =$ [illegible] $- 2$

↳ $\det(A) \neq 0$

can take inverse

11 minor $= \det \begin{vmatrix} 3 & 1 \\ -3 & 0 \end{vmatrix}$

$$A = \begin{vmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ -1 & -3 & 0 \end{vmatrix} \xrightarrow[\text{determinants of minors}]{\text{cofactor}} \begin{vmatrix} 3 & 1 & 3 \\ -3 & -1 & -1 \\ 5 & 1 & 3 \end{vmatrix}$$

33 minor $= \det \begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix}$

Sign change

$$\begin{vmatrix} 3 & -1 & 3 \\ 3 & -1 & 1 \\ 5 & -1 & 3 \end{vmatrix} \xrightarrow{\text{Transpose}} \begin{vmatrix} 3 & 3 & 5 \\ -1 & -1 & -1 \\ 3 & 1 & 3 \end{vmatrix}$$

divide by det A

$$\begin{vmatrix} -\frac{3}{2} & -\frac{3}{2} & -\frac{5}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & -\frac{1}{2} & -\frac{3}{2} \end{vmatrix}$$

Check work

$$AA^{-1} = I_n$$

should equal

$$\begin{vmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ -1 & -3 & 0 \end{vmatrix} \begin{vmatrix} -3/2 & -3/2 & -5/2 \\ 1/2 & 1/2 & 1/2 \\ -3/2 & -1/2 & -3/2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

To multiply dot row with column

$1(-3/2) + 2(1/2) + 1(3/2)$

2^{nd} practice exam.

$$M = \begin{vmatrix} 1 & 1 & c \\ 1 & 2 & 0 \\ -1 & 0 & 4 \end{vmatrix}$$

Find a value of c so that $M\vec{x} = 0$ has non trivial solutions

$\downarrow$

$\det M = 0$

many solutions

if $\det M \neq 0$ guarantees a unique solution

$$\det M = 1(8) - 1(4) + c(2)$$
$$\det M = 4 + 2c$$
$$0 = 4 + 2c$$
$$c = -2$$

Find $(x_1, x_2, x_3) \rightarrow$ a non trivial solution to $M\vec{x} = 0$ for $c = -2$

$$\begin{pmatrix} \text{Row 1} \\ \text{Row 2} \\ \text{Row 3} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \text{Row 1} \cdot x \\ \text{Row 2} \cdot x \\ \text{Row 3} \cdot x \end{pmatrix}$$

$$1 \cdot x_1 + 1 \cdot x_2 - 2 \cdot x_3 = 0$$
$$x_1 + 2x_2 = 0$$
$$-x_1 + 4x_3 = 0$$

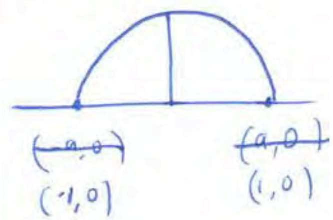
Solve. infinately many

$(4, -2, 1)$

Check:

$(4, -2, 1)M$

Ladybug Problem



bug moves at constant speed along the car.

speed of car = 10 units $\rightarrow$

$t = \pi$

the bug hits back of car

If the car was stationary

↳ position of bug relative to car

position $(\cos\theta, \sin\theta)$

or $(\cos t, \sin t)$

position of the center of the car

is $\underbrace{(0,0)}_{\text{starting position}} + \underbrace{(10t,0)}_{\text{change at end.}} = (10t, 0)$

$p(t) = (10t + \cos t, \sin t)$

9/19/13 18.02 Lecture Fernando Trejano

21 Partial Derivatives

$$f(x,y)$$

vary in x variable $\frac{df}{dx}(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - F(x,y)}{h}$

$$\frac{df}{dy}(x,y) = \lim_{h\to 0} \frac{F(x,y+h) - f(x,y)}{h}$$ - Assume x is constant

Ex

$F(x,y) = x^2+y^2$

$\frac{df}{dx}(x,y) = 2x$

$\frac{df}{dy}(x,y) = 2y$

$g(x,y) = xy$

$\frac{dg}{dx}(x,y) = y$

$\frac{dg}{dy}(x,y) = x$

- Pretend it is a function of one variable
- Do one variable at a time

$f(x,y) = xe^y$

$\frac{df}{dx} = e^y$

$\frac{df}{dy} = xe^y$

$f(x,y,z) = xyz^2$

$\frac{df}{dz} = 2xyz$

$\frac{df}{dx} = yz^2$ $\frac{df}{dy} = xz^2$

Functions of several variables

$f(x, y)$

ex

$$f(x,y) = \sqrt{25 - x^2 - y^2}$$

Domain : $25 - x^2 - y^2 \geq 0$

$$f(x,y) = \frac{y}{\sqrt{x - y^2}}$$

Domain

$x - y^2 \geq 0$

Max

$x = y^2$

$f(x,y) = 1$ when

$y^2 = x - y^2$

$2y^2 = x$

$\therefore$ Multiple Solutions. Any in $x = 2y^2$

4/23/13 18.02 Recitation Fernando Trujano

① $f(x,y) = z$

Tangent Plane $(x_0, y_0, f(x_0,y_0))$ $\vec{n} = (f_x(x_0,y_0), f_y(x_0,y_0), -1)$

② $g(x,y,z) = 0$ $g_x = \frac{dg}{dx}$

$= f(x,y) - z$ Partial derivative

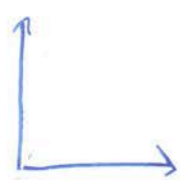
(x_0, y_0, z_0) $\vec{n} = g_x(x_0,y_0,z_0), g_y(x_0,y_0,z_0), g_z(x_0,y_0,z_0)$

$g(x_0,y_0,z_0)$

$\frac{dg}{dx} = \frac{df}{dx} - \cancel{\frac{dz}{x}} = 0$ Components of normal vector $\Rightarrow$ Partial derivative

$f(x,y) = x^2 - y^2$

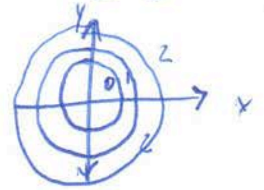
$L_c = \{(x,y) \mid f(x,y) = c$



$L_1 : x^2 + y^2 - z^2 = 1$

$x^2 + y^2 = z^2 + 1$

when $z = 0$



9/25/13

13.4 Notes: Partial Derivatives

Fernando Troyano

Partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$

- $\frac{\partial f}{\partial x}$: Derivative with respect to x. Treat y as a constant
- $\frac{\partial f}{\partial y}$: Derivative with respect to y. Treat x as a constant.

Ex

$$f(x,y) = x^2 + 2xy^2 - y^3$$

$$\frac{\partial f}{\partial x} = 2x + 2y^2$$

$$\frac{\partial f}{\partial y} = 4xy - 3y^2$$

* Applies with more variables too!

Ex

$$g(x,y,u,v) = e^{ux}\sin vy$$

x variable everything else is constant

$$g_x = ve^{ux}\sin vy$$

$$g_y = ve^{ux}\cos vy$$

$$g_u = xe^{ux}\sin vy$$

$$g_v = ye^{ux}\cos vy$$

Planes Tangent to surface

surface $z = f(x, y, z)$

point $P = (a, b, f(a,b))$

$-z$

* $$z - F(a,b) = f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

partial derivative with respect to x

Example

Surface $z = 5 - 2x^2 - y^2$

Point $(1, 1, 2)$ ← P

$f_x(x,y) = -4x$

$f_y(x,y) = -2y$

$f_x(1,1) = -4$

$f_y(1,1) = -2$

$$z - 2 = -4(x-1) - 2(y-1)$$

$$z = 8 - 4x - 2y$$

↳ Plane tangent to surface z at point P

2/3

Higher-Order Partial derivatives

$f_{xy} \Rightarrow$ 2nd order partial derivative
(1) Partial derivative with respect to x
(2) Partial derivative of (1) with respect to y

$$\underbrace{(f_x)_x}_{\text{Partial derivative w/r x}} = f_{xx} = \underbrace{\frac{\partial f_x}{\partial x}}_{\text{Partial deriv of the partial deriv}} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

Example

if $f_x(x,y) = 2x + 2y^2$

and $f_y(x,y) = 4xy - 3y^2$

$f_{xx}(x,y) = 2$ $\quad f_{xy}(x,y) = 4y$ $\quad f_{yy}(x,y) = 4x - 6y$

$f_{xxx}(x,y) = 0$ $\quad f_{xxy}(x,y) = 0$ $\quad f_{xyy}(x,y) = 4$ $\quad f_{yyy}(x,y) = -6$

* Take previous partial derivative and take another partial derivative.

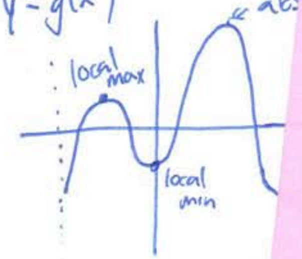
INCEPTION

18.02 Lecture

Fernando Trujano

Critical Points
Least Squares

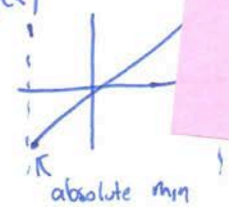
$y = g(x)$



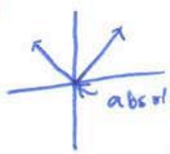
Review

s

$y = g(x)$



$y=|x|$



Derivative not defined

How do we find these using calculus

→ absolute min/max
- point where function is greates/smallest

→ local min

18.100 : Closed region R and a continious function $\Big] \Rightarrow \exists$ global max and global min

R : BOO region that includes the boundry

on a closed region R

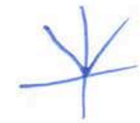
Criteria: At any local max, 3 possibilities

(1) Max achieved at interior critical pt

↳ where $f_x = f_y = 0$ — Partial derivatives

derivative is 0

(2) Achieved at interior point where not diff.

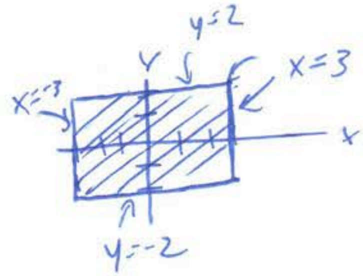


(3) Achieved at boundry point.

Example — Find global min

$f(x,y) = x^2 + 3x + xy + y^2 + 1$

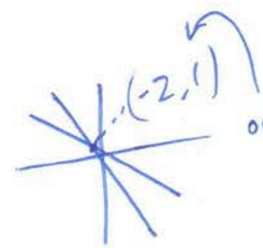
$R = |x| \le 3$ and $|y| \le 2$



$f_x = 2x + 3 + y$

$f_y = x + 2y$

$f_x = 0 \Rightarrow y = -2x - 3$



only critical point

might be a min or a max!

$f_y = 0 \Rightarrow y = -\frac{1}{2}x$

$f(-2,1) = 4 - 6 - 2 + 1 + 1 = -2$

If we're in case (1) then min $= -2$

Case (3) - check 4 boundries — complete the square — $\left(\frac{5}{2}\right)^2$

Top: $f(x,2) = x^2 + 3x + 2x + 4 + 1 = x^2 + 5x + 5 = \left(x^2 + \frac{5}{2}\right)^2 + 5 - \frac{25}{4} > \frac{-5}{4} >$

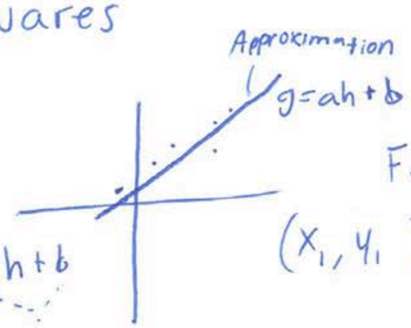
Bot: $f(x,-2) = x^2 + 3x - 2x + 4 + 1 = x^2 + x + 5 = \left(x + \frac{1}{2}\right)^2 + 5 - \frac{1}{4} > -2$

Right: $f(3,y) = 9 + 9 + 3y + y^2 + 1 = y^2 + 3y + 19 > -2$

Left: $f(-3,y) = 9 - 9 - 3y + y^2 + 1 = y^2 - 3y + 1 = \left(y - \frac{3}{2}\right)^2 + 1 - \frac{9}{4} > \frac{-5}{4} > -$

On the top it's always bigger than -2

* Least Squares



Find relationship

(x_i, y_i) data points

$g = ah + b$

Constants that we don't know.

Deviation

actual data

$$D(a,b) = \sum_{i=1}^{n} \left(y_i - (ax_i + b) \right)^2$$

so no negatives
errors add up

our guess

Choose a and b to make D as small as possible

Want global min of D

chain rule

$$D_a = 2\sum_i \left(y_i - (ax_i + b) \right) (-x_i)$$

chain rule derivative with respect to a

$$D_b = 2\sum_i \left(y_i - (ax_i + b) \right)(-1)$$

$D_a = 0 \quad \sum y_i x_i - a\sum x_i^2 - b\sum x_i = 0$

$D_b = 0 \quad \sum y_i - a\sum x_i - \underbrace{\sum b}_{= bn} = 0$

x_i	y_i
1	1
5	5/2
4	3

$\sum x_i y_i = 18$

$\sum x_i^2 = 21$

$\sum x_i = 7$

$\sum x_i = 13/2$

Solve
└ Gives least approximation

$18 - 21a - 7b = 0$

$\Downarrow \quad \frac{13}{2} - 7a - 3b = 0$

$\therefore$ One solution

a and b

$\therefore$

Least Squares approximations

9/30/13

18.02 Recitation

Fernando Trujano

~~Restrict f to~~

$f(x,y) =$

Review

① Find critic ... are in the interior of

$\frac{\partial f}{\partial x}$

$\frac{\partial f}{\partial y} =$

$-6y = 0$

$y = 0$

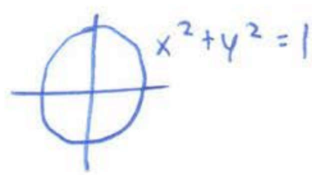
$\Downarrow$

$x = 0$

Critical Point $C_1 = \{(0,0)\}$

$f(0,0) = 10$

② Restrict F to boundry of disk



if $x^2+y^2=1$

~~then~~ $x = \cos t$

~~$f(x,y) =$~~ $y = \sin t$

$t \in [0, 2\pi]$

$f(x,y) = g(t) = 10 - \cos^2 t - 4\sin^2 t + 2\sin t \cos t$

$= 9 - 3\sin^2 t + \sin(2t)$

g has critical points where $g' = 0$

$$\frac{dg}{dt} = -6 \sin t \cos t + 2 \cos 2t = 0$$

$$2\cos(2t) = 3\sin(2t)$$

$$\tan(2t) = 2/3$$

$$t = \frac{1}{2}\arctan\left(\frac{2}{3}\right)$$

3 When $2\cos(2t) = 3\sin(2t)$

$$g(t) = 9 - \sin^2 t + \sin 2t$$

Evaluate f on pts in C_1 and C_2

Smallest is abs min

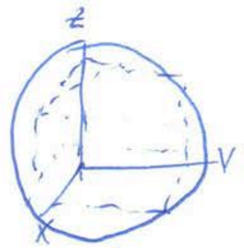
Largest is abs max

Finally, max is 10 @ $(0,0)$

and min is $g(t_0) = g(t_0 + \pi)$

at $(\cancel{\sin}\cos t_0, \cancel{\cos}\sin t_0)$ and

$(-\cancel{\sin}\cos t_0, -\cancel{\cos}\sin t_0)$



hemisphere of radius R

Find the max possible volume of a rectangular box inscribed in the hemisphere. Assuming one face lies on the planes part of hemisphr

$$x^2 + y^2 + z^2 = r^2$$

Hint: Given a corner $(x, y, 0)$ for the box, express V of box with max volume and that corner point.

(x, y) such that $x^2 + y^2 \leq R^2$

Maximize $V(x,y) = 2x \cdot 2y \underbrace{\sqrt{R^2 - x^2 - y^2}}_{z}$

$$\frac{\partial V}{\partial x} = 4y\sqrt{R^2-x^2-y^2} + \left(2x \cdot 2y \cdot \tfrac{1}{2}(-2x)\right) \Big/ \sqrt{R^2-x^2-y^2} = 0$$

can't be 0

~~$4y = 4x^2y$~~

$$\boxed{R^2 - x^2 - y^2 = x^2}$$

$$\frac{\partial V}{\partial y} = 0 \rightsquigarrow \boxed{R^2 - x^2 - y^2 = y^2}$$

$$\boxed{x^2 = y^2}$$

$x = y$
or $x = -y$

Also
$x^2 + y^2 \leq R^2$
$|x| \leq R/\sqrt{2}$

$V(x) = 4x^2 \sqrt{R^2 - 2x^2}$

$$\frac{dV(x)}{dx} = 8\cancel{x}\sqrt{R^2 - 2x^2} + 4x^2 \frac{(-4\cancel{x})\ 1}{\sqrt{R^2 - 2x^2}} = 0$$

x will not be 0

$R^2 - 2x^2 = \cancel{2}x^2$

$R^2 = 3x^2$

$x = \frac{R\sqrt{3}}{3} = y \ , \quad z = R\frac{\sqrt{3}}{3}$

4/4

10/1/13

18.02 Notes

13.10 and SD

Second Derivative Test

Critical Points of Functions with two Variables

To have local min or max, must have critical point

$f_x(a,b) = 0 = f_y(a,b)$

$A = f_{xx}(a,b)$ $B = f_{xy}(a,b)$ $C = f_{yy}(a,b)$

$\Delta = \underbrace{AC - B^2}_{\text{discriminant}}$

if

$A > 0$ and $\Delta > 0$

$f(a,b) \to$ local minimum

$A < 0$ $\Delta > 0$

$f(a,b) \to$ local maximum

$\Delta < 0$

$\underbrace{f(a,b)}_{\text{saddle point}} \to$ nothing

If discriminant is positive, f has either min or max

$\Delta = 0$

Second derivative test fails.
Anything could happen

1/4

Example:

Locate and classify critical points of

$$f(x,y) = 3x - x^3 - 3xy^2$$

$$f_x(x,y) = 3 - 3x^2 - 3y^2$$

$$f_y(x,y) = -6xy$$

$3 - 3x^2 - 3y^2 = 0$ if $x = 0$, $y = \pm 1$; if $y = 0$, $x = \pm 1$

$-6xy = 0$

$x = 0$ or $y = 0$

Critical Points

$(0,1)$ $(0,-1)$

$(1,0)$ $(-1,0)$

$A = f_{xx}(x,y) = -6x$ $\quad B = f_{xy}(x,y) = -6y$ $\quad C = f_{yy}(x,y) = -6x$

$\Delta = AC - B^2$

Critical Point	A	B	C	Δ
$(1,0)$	-6	0	-6	$36 \rightarrow$ Local max
$(-1,0)$	6	0	6	$36 \rightarrow$ Local min
$(0,1)$	0	-6	0	$-36 \rightarrow$ Saddle Points
$(0,-1)$	0	6	0	$-36 \rightarrow$ Saddle Points

$$\begin{aligned}\Delta &= AC - B^2 \\ &= (-6x)(-6x) - (-6y)^2 \\ &= 36x^2 - 36y^2 \\ &= 36(x^2 - y^2)\end{aligned}$$

2/14

SD

The Second Derivative Test

In one variable $f(x)$

Find local min max

① Critical Points $f'(x) = 0$

② $f''(x) > 0 \rightarrow$ local min $\smile$ cc↓

$f''(x) < 0 \rightarrow$ local max $\frown$ cc↑

In two variables

$A = f_{xx}(x,y) \quad B = f_{xy}(x,y) = f_{yx}(x,y) \quad C = f_{yy}(x,y)$

$D = AC - B^2$

* See conditions on previous page

Example:

Find critical points of $w = 12x^2 + y^3 - 12xy$ and determine their type

$w_x = 24x - 12y$

$w_y = 3y^2 - 12x$

$A = w_{xx}(x,y) = 24 \quad B = w_{xy} = -12 \quad C = w_{yy} = 6y$

Find Critical Points

$w_x = 24x - 12y = 0$

$12(2x - y) = 0$

$y = 2x$

sub

$w_y = 3y^2 - 12x = 0$

$3y^2 = 12x$

$y^2 = 4x$

$(2x)^2 = 4x$

$4x^2 = 4x$

$x^2 = x$

$x = 1$

$x = 0$

plug into $y = 2x$

$(1, 2)$

$(0, 0)$

$w_{xx}(\cancel{1,2}) = 24 = A$

$w_{xy}(\cancel{1,2}) = -12 = \cancel{A}\,B$

$w_{yy}(\cancel{1,2}) = 6y = \cancel{12} = C$

$\Delta = AC - B^2$

$\cancel{\Delta = (24)(12) - (-12)^2}$

$\Delta = (24)(6y) - (-12)^2$

$= 144y - 144$

at $(0,0)$

$\Delta = -144 \rightarrow$ saddle point

at $(1,2)$

$\Delta = (144)(2) - 144 = 144 \rightarrow$ local minimum

$A > 0$

10/1/13

18.02 Lecture

Fernando Trujano

Calc BC 2nd Derivative Test, Saddle Points + Quad Poly's 2nd Deriv Test (The Discrim.)

Critical Point → where derivative is 0
↳ could be min or max

Calc BC 2nd Derivative Test

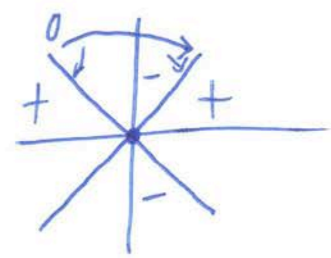
$f(x)$ function

Critical point x_0, $f'(x_0)=0$

2nd deriv $f''(x_0) > 0$ ∪ → local min
$f''(x_0) < 0$ ∩ → local max
$f''(x_0) = 0$ inconclusive

Saddle Points + Quadratic Polynomials

level set



New Critical Point $f(x,y) = x^2 - y^2$

$(0,0)$ is crit point
∴ not a max or min
∴ Saddle Point

depending on restrictions could be a max or a min!

Constants

$Q(x,y) = Ax^2 + 2Bxy + Cy^2$

$Q_x = 2Ax + 2By$ Critical Point $(0,0)$
$Q_y = 2Bx + 2Cy$

x^2+y^2 → Paraboloid → local min Parabola revolved around z	$-x^2-y^2$ local min	x^2-y^2 saddle

Write general polynomial as square

$$Q(x,y) = Ax^2 + 2Bxy + Cy^2$$

$$\frac{1}{A}\left(A^2x^2 + 2AB\,xy + ACy^2\right)$$

$$= \frac{1}{A}\left((Ax+By)^2 - B^2y^2 + ACy^2\right)$$

$$= \frac{1}{A}\left[(Ax+By)^2 + (AC-B^2)y^2\right]$$

x^2+y^2

$A > 0$ and $AC - B^2 > 0 \rightarrow$ local min $\cup$

$A < 0$ and $AC - B^2 > 0 \rightarrow$ local max $\cap$

$AC - B^2 < 0 \Rightarrow$ Saddle Point

if $\underbrace{AC - B^2}_{\text{Discriminant} = \Delta} = 0 \Rightarrow$ Ignore Case

Delta or triangle, if you are less sophisticated. :P

2nd Derivative Test

$f(x,y)$ x_0, y_0 crit pt.

$A = f_{xx}(x_0, y_0)$

$B = f_{xy}(x_0, y_0)$

$C = f_{yy}(x_0, y_0)$

$\Delta = AC - B^2$

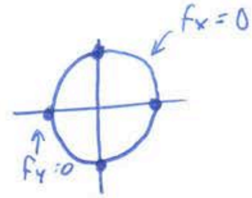
1) $A > 0$ $\Delta > 0 \rightarrow$ local min
2) $A < 0$ $\Delta > 0 \rightarrow$ local max
3) $\Delta < 0 \rightarrow$ Saddle Point
4) if $\Delta = 0 \rightarrow$ inconclusive

Example p298

$$f(x,y) = 3x - x^3 - 3xy^2$$

1) Find Critical Points
Compute Partial Derivatives

$$f_x = 3 - 3x^2 - 3y^2$$
$$f_y = -6xy$$



crit pt	A	B	C	Δ	
(1,0)	-6	0	-6	36	local max
(0,1)	0	-6	0	-36	saddle point
(-1,0)	6	0	6	36	local min
(0,-1)	0	6	0	-36	saddle point

$$f_{xx} = A = -6x$$
$$f_{xy} = B = -6y$$
$$f_{yy} = C = -6x$$

Equality of mixed Partials $\Rightarrow$ Order does not matter
$f_{xy} = f_{yx}$ $\quad$ $f_{xxxy} = f_{xyxx}$

Example

$$f(x,y) = 2x^2 + 2xy + y^2 + 4x - 2y + 1$$

$$f_x = 4x + 2y + 4$$
$$f_y = 2x + 2y - 2$$

intersect at $(-3,4) \Rightarrow$ critical pt.

$A = f_{xx} = 4$
$B = f_{xy} = 2$
$C = f_{yy} = 2$

$\Delta = AC - B^2$
$= (4)(2) - 2^2 = 4 > 0$
$\Rightarrow$ local min

3/4

Taylor Series

$f(x)$

$f(0) + f'(0)\,x$

$f(0) + f'(0)\,x + \frac{1}{2} f(0)\,x^2$

Can know what type of critical point.

18.02 Recitation

Fernando Trujano

$F(x,y) = \sin x \sin y$

1) Find critical points and determine type

$f_x(x,y) = \cos x \sin y + \cancel{\sin x(0)}$

$f_y(x,y) = \cos y \sin x + 0$

$\cos x \sin y = 0$ $\qquad$ $\cos y = 0$

$\cos x = 0$ $\qquad$ $\sin x = 0$

$\sin y = 0$ $\qquad$ $x = 0$

$y = 0$ $\qquad$ $y = \frac{\pi}{2}$

or $x = \frac{\pi}{2}$

$\cancel{(0, \frac{\pi}{2})}$

$\cancel{(\frac{\pi}{2}, 0)}$

$(0,0)$

$(\frac{\pi}{2}, \frac{\pi}{2})$

$f_{xx} = A = -\sin x \sin y$

$f_{xy} = B = \cos x \cos y$

$f_{yy} = C = -\sin y \sin x$

$\Delta = AC - B^2$

	A	B	C	Δ	
$\cancel{(0, \frac{\pi}{2})}$	0	0	0	0	$(0, \pi)$
$\cancel{(\frac{\pi}{2}, 0)}$	0	0	0	0	
$(0,0)$	0	1	0	-1	$(0,0) \to$ saddle and all mults of $x\pi$ $(0,0)$ $x \neq 0$
$(\frac{\pi}{2}, \frac{\pi}{2})$	-1	0	-1	1	$\to$ max or min '' $\left\|\frac{\pi}{2}, \frac{\pi}{2}\right\|$

The Chain Rule 13.7

Chain rule

$$f(g(t))' = f'g(t) \cdot g'(t)$$

$w = f(x,y)$
$x = g(t)$
$y = h(t)$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt}$$

$$w' = f(g(t), h(t))' = f_x(g(t), h(t)) \cdot g'(t) + f_y(g(t), h(t)) \cdot h'(t))$$

Example

$w = e^{xy}$
$x = t^2$
$y = t^3$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt}$$

$$\frac{dw}{dt} = (ye^{xy})(2t) + (xe^{xy})(3t^2)$$

$$\frac{dw}{dt} = (t^3 e^{t^5})(2t) + (t^2 e^{t^5})(3t^2)$$

$$= 2t^4 e^{t^5} + 3t^4 e^{t^5}$$

$$= 5t^4 e^{t^5}$$

Example

$w = x^2 + ze^y + \sin xz$
$x = t$
$y = t^2$
$z = t^3$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$(2x + z\cos xz)(1) + (ze^4)(2y) + (e^y)(x\cos zx)(3t^2)$$

$$2x + z\cos xz + 2yze^4 + 3t^2 e^y x\cos zx$$

$$2t + t^3\cos t^4 + 2t^2 t^3 e^4 + 3t^2 e^{t^2} + \cos t^4$$

Several Independent Variables

$$w = f(x, y, z)$$

$$x = g(u, v)$$
$$y = h(u, v)$$
$$z = k(u, v)$$

$$w = f(g(u,v), h(u,v), k(u,v))$$

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}$$

General Chain Rule

Example

$$z = f(u, v)$$
$$u = 2x + y$$
$$v = 3x - 2y$$

Given

$$\left.\begin{matrix}\frac{\partial z}{\partial u} = 3 \\ \frac{\partial z}{\partial v} = -2\end{matrix}\right\} \text{at } (u,v) \to (3,1)$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

$$(3)(2) + (-2)(3) = 0$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

$$(3)(1) + (-2)(-2) = +7$$

2/2

9/3/13

18.02 Lecture

Fernando Trujano

BC Chain Rule $\xrightarrow{}$ Review, Differentials, Chain Rule $\xrightarrow{}$ multivariable

Polar coords

Friday is tomorrooooooooooooow

!!

BC Chain Rule

f, g

Review

$f \circ g(x) = f(g(x))$

composed

I'm sooooooooooorryyy

eg

Fo

$(2\sin x + 1)(\cos x)$

$2\sin x \cos x + \cos x$

Chain Rule

$$(f \circ g)'(x) = f'(g(x))\, g'(x)$$

Differentials

linear aproximation

$$f \circ g(x + \Delta x) \approx f \circ g(x) + (f \circ g)'(x)\, \Delta x$$

$$f(g(x+\Delta x)) \approx f(g(x) + g'(x)\Delta x) \approx f(g(x)) + f'(g(x))g'(x)\Delta x$$

Prove chain rule by using linear approximation twice

$$h(y + \Delta y) = h(y) + h'(y)\Delta y$$

$h = f \circ g$

$h = f$

$h = g$

NOTATION
$x = x(t)$
$y = y(t)$
$z = z(t)$

Chain Rule in 18.02

$f(x,y,z)$

Composition

$$w(t) = f(x(t), y(t), z(t))$$

What is $w'(t)$

Chain Rule:

$$w'(t) = f_x(x'(t)) + f_y(y'(t) + f_z(z'(t)$$

Example

$$f(x,y) = x^2 + y^2$$

Unit Circle

Curve
$x(t) = \cos(t)$
$y(t) = \sin(t)$

composition constant so deriv $= 0$

$$w(t) = 1 \Rightarrow w'(t) = 0$$

Chain Rule

$$w' = f_x x' + f_y y'$$
$$= (2\cos t)(-\sin t) + (2\sin t)(\cos t)$$
$$= 0$$

2^{nd} example

$$f(x,y,z) = xyz + z^2 \qquad w = f(x(t), y(t), z(t))$$

$x(t) = t^2$
$y(t) = 3/t$
$z(t) = \sin t$

$x' = 2t$

$y' = -3/t^2$

$z' = \cos t$

$f_x = yz, \quad f_y = xz \quad f_z = xy + 2z$

$w' = f_x x' + f_y y' + f_z z'$

$$= \frac{3}{t}\sin t\,(2t) + t^2 \sin t\left(\overset{y'}{\frac{-3}{t^2}}\right) + \left(t^2\left(\frac{3}{t}\right) + 2\sin t\right)\cos t$$

$$6\sin t - 3\sin t + 3t - \cos t + 2\sin t\cos t$$

$\sim$

$w(t) = t^2\,\frac{3}{t}\sin t + \sin^2 t = 3t\sin t + \sin^2 t = 3t\sin t + \sin^2 t$

Example

#33 on pg 905

Ice  Melting

$H(t)$ $E(t)$ square base

when $H = 1$ and $E = 2$

then $H' = -1/6$

$E' = -1/4$

What is the rate of change of volume at this time?

$\sim$

$V(H, E) = HE^2$

chain Rule

$(V(H(t), E(t)))' = V_H H' + V_E E'$

$(4)(-1/6) + 4(-1/4)$

$= -2/3 - 1 = -5/3$

Rate of change

3/6

$V(H,E) = HE^2$

t deriv $\quad H'E^2 + H(E^2)'$

$\quad = H'E^2 + 2HEE'$

Chain Rule for Partial Derivatives

$f(x,y) \qquad x = x(r,\theta)$

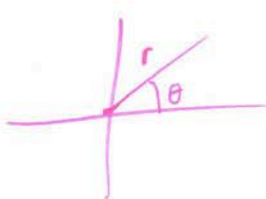
$\qquad\qquad\quad y = y(r,\theta)$

$$f_r = ? \quad \left(f_r = \frac{\partial f}{\partial r}\right)$$

$$f_r = f_x x_r + f_y y_r$$

$$f_\theta = f_x x_\theta + f_y y_\theta$$

Polar Coordinates



$$(x,y) = (r\cos\theta,\ r\sin\theta)$$

$x_r = \cos\theta$

$y_r = \sin\theta$

$x_\theta = -r\sin\theta$

$y_\theta = r\cos\theta$

$$f(x,y) = x^2 + y^2$$

$$w(r,\theta) = f(x(r,\theta), y(r,\theta))$$

$$= (r\cos\theta)^2 + (r\sin\theta)^2 = r^2$$

$w_r = 2r$

$w_\theta = 0$

4/6

Chain Rule

$$w_r = f_x x_r + f_y y_r$$

$$= 2x\cos\theta + 2y\sin\theta$$

$$= 2r\cos^2\theta + 2r\sin^2\theta = 2r$$

$$w_\theta = f_x x_\theta + f_y y_\theta$$

$$= (2r\cos\theta)(-r\sin\theta) + (2r\sin\theta)(r\cos\theta)$$

$$= 0$$

Example

$f(x,y)$ = Elevation at (x,y)

↖ latitude and longitude

path $x(t)$
$y(t)$

$E(t)$

elevation at time t

$$E'(t) = f_x x'(t) + f_y y'(t)$$

saddle at origin

$$f(x,y) = x^2 - y^2$$

$$x(t) = \cos(t)$$
$$y(t) = \sin(t)$$

$$E(t) = \cos^2 t - \sin^2 t$$

$$E'(t) = 2\cos t\sin t - 2\sin t\cos t$$

$$= -4\cos t\sin t$$

5/6

Cham Rule

$E' = f_x x' + f_y y'$

$= 2x x' - 2y y'$

$= 2\cos t(-\sin t) - 2\sin(t)\cos(t)$

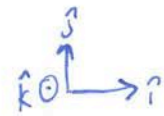
$= -4\sin(t)\cos(t)$

works!

Gradients and Directional Derivatives

$D_u f(x)$ = Derivative of f in the direction of u

so



$f_x(x,y) = D_{\hat{i}}(f,x)$

f_x points in the direction of $\hat{i}$

$D_u f(x) = \nabla F(x) \cdot u$

∇ = Gradient

$\nabla F(x) = (f_x, f_y, f_z)$

Example

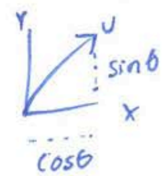
$u = \langle a, b \rangle \quad z = f(x,y)$

$D_{\langle a,b \rangle} f(x,y) = \nabla f(x,y) \cdot \langle a, b \rangle$

$= (f_x, f_y) \langle a, b \rangle$

$= a f_x(x,y) + b f_y(x,y)$

Now assume u makes angle θ to x axis



$u = \langle \cos\theta, \sin\theta \rangle$

$D_u f(x,y) = f_x \cos\theta + f_y \sin\theta$

↳ should be unit vector

$u = \frac{v}{|v|}$

The Gradient Vector

$$\nabla f(x,y) = [f_x, f_y]$$

Maximum directional derivative is $|\nabla f(P)|$

↳ Length of gradient vector

∇f

Gradient vector points in the direction where the function increases most rapidly

18.02 Lecture

Fernando Tr...

Directional Derivatives, Gradients

Normal to level sets

$F(x)$ function

$$F(F^{-1}(x)) = x$$

$$F^{-1}(F(x)) = x$$

$$f(x) = e^x = \exp(x)$$

$$f^{-1}(x) = \ln x$$

$\exp(\ln(x)) = x$ Chain Rule $\Rightarrow$ actually usefull

$(\exp \circ \ln)'(x) = \exp'(\ln(x))\ln'(x)$

$= \exp(\ln(x))\ln'(x)$

$= x\ln'(x)$

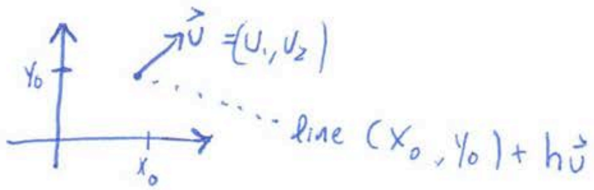
Review

Last

Page

Directional Deri...

$F(x,y)$ ... $(x_0, y_0) = \lim_{h \to 0} \dfrac{F(x_0+h, y) - F(x_0, y_0)}{h}$



$$D_{\vec{u}} F(x_0, y_0) = \lim_{h \to 0} \frac{F((x_0, y_0) + h\vec{u}) - F(x_0, y_0)}{h}$$

when $\vec{u} = (1,0) \Rightarrow D_{\vec{u}} = F_x$

Fernando Trejano

Computing $D_{\vec{u}}f$

$$w(h) = f(x_0 + hu_1,\ y_0 + hu_2)$$

$$D_{\vec{u}} f(x_0, y_0) = w'(0)$$

Chain rule =>

$$w' = f_x u_1 + f_y u_2$$

$$w'(0) = \lim_{h \to 0} \frac{w(h) - w(0)}{h} = \lim_{h \to 0} \frac{f(x_0 + hu_1,\ y_0 hu_2) - f(x_0, y_0)}{h}$$

How did you do that?

$$w' = f_x x' + f_y y'$$

$$x(t) = x_0 + tu_1$$
$$y(t) = y_0 + tu_2$$

$$\star = f_x u_1 + f_y u_2$$

Gradients

$\star$ is a dot product!

Can write directional derivative as dot product

$$D_{\vec{u}} f = \underbrace{(f_x, f_y)}_{\text{Gradient of } f} \cdot \vec{u}$$

$$\nabla f = (f_x, f_y)$$

$$g(x, y, z)$$
$$\nabla g = (g_x, g_y, g_z)$$

Example

$$F(x,y,z) = xy + z^2$$

$$\nabla F = (f_x, f_y, f_z)$$

$$= (y, x, 2z)$$

Direction of gradient?

$$D_{\vec{u}} f = \nabla F \cdot \vec{u} = |\nabla f||\vec{u}| \cos\theta$$ angle between

Now assume $|\vec{u}| = 1$

$$D_{\vec{u}} f = |\nabla f| \cos\theta$$

$D_{\vec{u}} f$ ranges between

$|\nabla f|$ when $\theta = 0$

$-|\nabla f|$ when $\theta = \pi$

0 when $\theta = \pi/2$

Level Sets

$F(x,y,z)$ Level set $F(x,y,z) = 0$

∇F is normal
aka $\perp$ to the level set

$\vec{r}(t)$ is a curve in the level set

$F \circ \vec{r}(t)$ Chain Rule $\nabla F \cdot \vec{r}(t) = 0$

Gradient perpendicular to every curve in level set

3/5

Example

Find tangent plane to $\overset{F(x)}{\overset{\shortparallel}{xyz - 1 = 0}}$

need point and normal

at $(1, 2, 1/2)$

└ Gradient

$$\nabla F = (yz, xz, xy)$$

$$= (1, 1/2, 2)$$

Plane

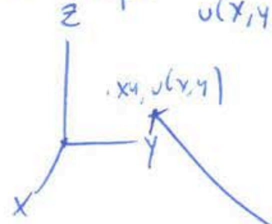
$$x + \frac{1}{2}y + 2z = d$$

Choose d so that 1, 2, 1/2 is in the plane

$d = 3$

$$x + \frac{1}{2}y + 2z = 3$$

Normal to Graph $u(x,y)$



A week ago

$(-u_x, -u_y, 1)$ was normal

Graph is level set

$$F(x, y, z) = -u(x, y) + z$$

$$\nabla F = (-u_x, -u_y, 1)$$

The tangents to graph

$\cancel{(x_0+t)}$

$\vec{r}(t) = x_0+t, y_0, u(x_0+t, y_0))$

$\vec{v}(t) = \vec{r}'(t) = (1, 0, u_x)$ ← Tangent to graph

$\vec{R}(t) = x_0, y_0+t, u(x_0, y_0+t))$

$\vec{R}'(t) = (0, 1, u_y)$ ← Point in different directions

$$\begin{vmatrix} i & j & k \\ 1 & 0 & u_x \\ 0 & 1 & u_y \end{vmatrix} = (-u_x, -u_y, 1)$$

5/5

10/7/13

18.02 Recitation

Fernando Trujano

= blue

Let $f(x,y,z)$ look at level surfaces

$\nabla f(a,b,c)$ is perp to Level surf. $f(x,y,z) = f(a,b,c)$

At P in direction of $\hat{u}$ $D\hat{u} = \nabla f\ (a,b,c) \cdot \hat{u}$

$= \text{comp}_{\hat{u}} \nabla f$

Fastest increase happens in direction $\perp$ to level curve

normal level surfaces

$$D\hat{u} = \nabla f \cdot \frac{\nabla f}{|\nabla f|} = |\nabla f|$$

Example

$$f(x,y,z) = \frac{(x^2-y^2)\sin z}{\cancel{xy\sin z}} = 0$$

At Point $= (-1, 1, \frac{\pi}{2})$

~~In which direction~~

Determine directional derivative in direction of greatest increase.

$\nabla f = (f_x, f_y, f_z)$ $\quad f_x = 2x\sin z$

$\cancel{2x, 2y7}$ $\quad f_y = 2y\sin z$

$\quad f_z = (x^2-y^2)\cos z$

$$\nabla f = \left(2x\sin z,\ 2y\sin z,\ (x^2-y^2)\cos z\right)$$

$$(-2, -2, 0)$$

$$\sqrt{2^2+2^2} = \sqrt{8}$$

$$\boxed{\text{Greatest } \underbrace{\text{directional derivative}}_{\text{dot product}} \text{ at } (a,b,c) \text{ is } |\nabla f(a,b,c)|}$$

Delta 1/2

Suppose we define z by

$$x^3+y^3+z^3 = 5xyz$$

When would $\frac{dz}{dx}$ exists? and What it does.
for which x,y
Compute it.

Ex

y defined as $x^2+y^2=4$ $\quad \frac{dy}{dx}=?$

Implicit differentiation

$$2x+2yy'=0$$

if $(y==0)\{y' \text{ DNE}\}$

else $\{y' \cancel{\text{exist}} = -\frac{x}{y}\}$

Tangent plane at $(2,1,1)$

$$\cancel{3x^2+3y^2y'+3z^2z'} = 5yz+5xyz'$$

$$\frac{dz}{dx} = \frac{3x^2-5yz}{5xy-3z^2} = z'$$

$$\frac{dz}{dy} =$$

$$\left(\frac{dz}{dx}, \frac{dz}{dy}, -1\right)$$

$$x-y-z=0$$

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Constrained Optimization, Lagrange multipliers
More constraints

Constrained Optimization

Maximize $f(x,y)$
such that $g(x,y)=0$

Example

$f(x,y) = xy$
$g(x,y) = x^2+y^2-1 = 0$ — unit circle

Parametrize unit circle $\vec{r}(t) = (\cos t, \sin t)$

$w(t) \equiv f \circ \vec{r} = f(\vec{r}(t))$ — composition

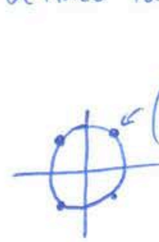
Answer: Find max of w.

Find critical points for w

chain rule
$w' = \nabla f \cdot \vec{r}\,'(t) = (y, x)\cdot(-\sin t, \cos t)$
$= (\sin t, \cos t)\cdot(-\sin t, \cos t)$
$= -\sin^2 + \cos^2 t$

- Defined for all t

$w' = 0$ if $\sin^2 = \cos^2$



$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ and $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

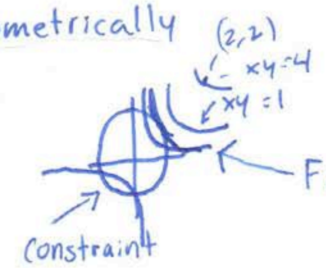
At these points $f(x,y) = xy$

$f = -1/2$

$f = 1/2$

Max occurs at 
value is $1/2$

Geometrically



Find last level set that touches g!

↳ will be tangent to circle

At the point we want

∇f and ∇g are parallell

$$\Rightarrow \nabla F = \underset{\substack{\uparrow \\ \text{scalar}}}{\lambda} (\nabla g)$$

How to prove this?

Given any ~~k~~ curve $r(t)$ param part of level set $g(x,y)=0$

looking for $\nabla f \cdot r'(t) = 0$

∇f: Perpendicular to curve

$r'(t)$: curve in the constraints g

Lagrange Multiplier Theorm

$F(x,y)$, $g(x,y)$ ctsly (continiously) diff function

If max of $f(x,y)$ such that $g(x,y)=0$ occurs at (x_0, y_0) and $\nabla g(x_0, y_0) \neq 0$

then $\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$ for some λ

Example 1st problem

$\nabla f = (y, x)$

$\nabla g = (2x, 2y)$

Solve for

$$y = 2\lambda x$$
$$x = 2\lambda y$$

CONSTRAINT

$$x^2 + y^2 = 1$$

$$y = 2\lambda(2\lambda y) = 4\lambda^2 y$$
$$x = 4\lambda^2 x$$
$$4\lambda^2 = 1$$
$$2\lambda = \pm 1$$

if $2\lambda = 1 \quad y = x \implies \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ and $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

$2\lambda = -1 \quad y = -x \quad \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ and $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

---

3 Variable example

Maximize $f(x,y,z) = x - y + z$ ← linear level sets plane

such that

$$g(x,y,z) = x^2 + y^2 + z^2 - 3 = 0$$

↑ constraint sphere

- Which is last level set that touches sphere?

$\nabla f = (1, -1, 1)$

$\nabla g = (2x, 2y, 2z)$

$$\left[\begin{aligned} 1 &= 2\lambda x \\ -1 &= 2\lambda y \\ 1 &= 2\lambda z \\ &\text{CONSTRAINT} \\ x^2 + y^2 &+ z^2 = 3 \end{aligned}\right.$$

Solve.

$$3\left(\frac{1}{2\lambda}\right)^2 = 3$$
$$\left(\frac{1}{2\lambda}\right)^2 = 1$$
$$4\lambda^2 = 1$$
$$2\lambda = \pm 1$$

if $2\lambda = 1 \quad (1, -1, 1) \leftarrow f = x - y + z = +3$

$2\lambda = -1 \quad (-1, 1, -1) \leftarrow f = -3$

MAX

Multiple Constraints

Maximize function subject to multiple constraints

$$f(x,y,z)$$

such that $\begin{cases} g_1(x,y,z) = 0 \\ g_2(x,y,z) = 0 \end{cases}$

Example

Max of $xyz + x^2 z$

on curve $\begin{cases} x^2+y^2=1 & \text{cylinder} \\ x+z+1=0 & \text{plane} \end{cases}$ Elipse inside cylinder

Lagrange for multiple constraints

Look for (x,y,z) and λ_1, λ_2 such that

$$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2$$

$$\left.\begin{array}{l} g_1 = 0 \\ g_2\ 0 \end{array}\right\} \text{Constraints}$$

$$\nabla f = (yz + 2xz,\ xz,\ xy + x^2)$$

$$\nabla g_1 = (2x, 2y, 0)$$

$$\nabla g_2 = (1,0,1)$$

$$\left.\begin{array}{l} yz + 2xz = 2\lambda_1 x + \lambda_2 \\ xy = 2\lambda_1 y \\ xy + x^2 = 0 + \lambda_2 \\ \text{constraints} \\ x^2+y^2 = 1 \\ x + z = 1 \end{array}\right] \text{SOLVE}$$

Lagrange multipliers

13.9

value of $f(x,y)$ subject to constraint

$g(x,y)=0$

if $\nabla g(P) \neq 0$

$\nabla f(P) = \underbrace{\lambda}_{\text{constant}} \nabla g(P)$

Example

Find points of $\underbrace{xy=1}_{g}$ that are closest to the origin

1) Minimize distance from origin to point P

$f \quad \sqrt{x^2+y^2}$

$f = x^2+y^2 \rightarrow$ easier to minimize

constraint

$g(x,y)=xy-1$

$\nabla f = 2x, 2y$

$\nabla g = y, x$

$\nabla f = \lambda \nabla g$

$2x = \lambda y$

$2y = \lambda x$

$yx-1=0$

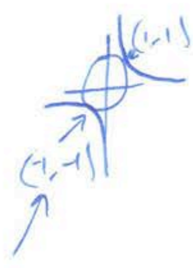
$2xy = \lambda x^2 = \lambda y^2$

$xy-1=0$

same sign

$x=y$

$x=y=\pm 1$



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Example:

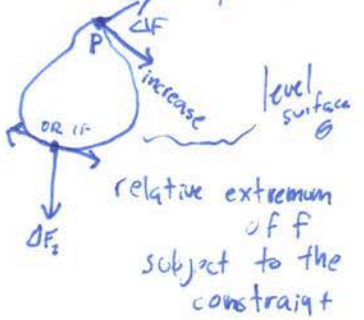
Find pts on surface $z = xy + 1$ which are closest to the origin.

Hints: $f_1(x,y,z) = \sqrt{x^2+y^2+z^2}$ ← distance to origin

$f(x,y,z) = x^2+y^2+z^2$ → minimize subject to constraint $g(x,y,z) = z-xy-1$

easier to minimize $\xrightarrow{\text{but}}$ same min

$\nabla F(P)$ points in direction of greatest increase of F.



$\nabla F = \lambda \nabla g$

$\nabla f = (2x, 2y, 2z)$

$2x = -y\lambda$

$2y = -x\lambda$

$2z = \lambda$

$z = xy + 1$

① $\frac{x}{y} = -\frac{\lambda}{2} = -\frac{2}{\lambda}$ $\lambda^2 = 4$

if $y \neq 0$ $\lambda = \pm 2$

$x = \pm y$

$z = \pm x^2 + 1$

$x^2 + 1 = 1$

$x = 0$

contradiction

or

$-x^2 + 1 = -1$

$x^2 = 2$

$\pm\sqrt{2}, \pm\sqrt{2}$

$f(\sqrt{2}, \sqrt{2}) > 1$

② if $y = 0$

$z = 1$

$x = \frac{1}{2}$

$(0,0,1)$

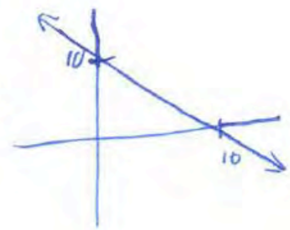
$\boxed{f = 1}$

min

1

② Find pts on ellipse $\cancel{4}x^2 + \overset{2}{\cancel{6}}y^2 = 1\cancel{0}$

closest to line $x + y = 10$



Hard part: Find the function to minimize

③ Let $f(x,y)$ for a function, &

$x = r\cos\theta$

$y = r\sin\theta$

Express $f_{rr}, f_{r\theta}, f_{\theta r}$ in terms of

$f_x, f_y, f_{xx}, f_{xy}, f_{yy}, r, \theta$

UNDERSTAND
ABOVE THIS

10/10/13 18.02 Notes Fernando Trujano

Non-independent variables N.

$w = x^2 + y^2 + z^2$

$\frac{\partial w}{\partial x} = ??$

$\underbrace{z = x^2 + y^2}_{\text{Not independent variables}}$

x, y

1) - Substitution

$w = x^2 + y^2 + (x^2+y^2)^2$

$\frac{\partial w}{\partial x} = 2x + 4x^3 + 4xy^2$

2) - chain rule

$\frac{dw}{dx} = 2x + 2z\frac{\partial z}{\partial x} = 2x + 2z \cdot 2x$

$\frac{\partial w}{\partial x} = 2x + 2(x^2+y^2)\,2x$

3) substitution

x, z independent

$y^2 = z - x^2$

$w = x^2 + (x - x^2)^2 + x z^2$

$\frac{\partial w}{\partial x} = 0$

$\frac{\partial w}{\partial x} \rightarrow$ no definite meaning

~

Calculate as if x, y, z were independent

$dw = f_x dx + f_y dy + f_z dz$

Formal Partial derivatives

Example

Find $\frac{\partial w}{\partial y}_{x,t}$ $\qquad w = x^3 y - z^2 t$

$\qquad xy = zt$

1) Chain rule

$$\frac{\partial w}{\partial y} = x^3 - 2zt \frac{\partial z}{\partial t}_{xt}$$

$$= x^3 - 2z\cancel{t}\frac{x}{\cancel{t}} = x^3 - 2zx$$

2) differentials of both sides

$$dw = 3x^2y\,dx + x^3dy - 2zt\,dz - z^2dt$$

$$y\,dx + x\,dy = z\,dt + t\,dz$$

- z not independent → eliminate dz

$$dw = (3x^2y - 2zy)dx + (x^3 - 2zx)dy + z^2dt$$

$$\frac{\partial w}{\partial x}_{y,t} = 3x^2y - 2zy \qquad \frac{\partial w}{\partial y}_{x,t} = x^3 - 2zx \qquad \frac{\partial w}{\partial t}_{x,y} = z^2$$

10/10/13

18.02 Lecture

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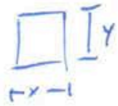
Constraints → independent vs dependent variables
Derivatives with constraints.

Example



$g(x,y)$
$= 2x+2y-40=0$

If I change x, how does the area change
Not f_x

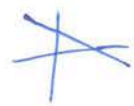
$A = f(x,y) = xy$

Constraint

x, y constraint $g(x,y)=0$

Can't vary them both freely.

$\cdot\ x+1-1=0$



$\cdot\ xy-1=0$

The if I change one the other will change as well.

$x+y+z=0$ (A plane)
$x^2+y^2+z^2-1=0$ (A sphere)

2 constraints

$$\begin{cases} x+y+z=0 \\ x=0 \end{cases}$$

∴ Cross in a line

↳ Each constraint reduce the # of free parameter by 1

*Some constraints might be redundant
otherwise

Dimention = # variables − # constraints

Derivatives and constraints

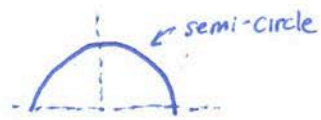
$$f(x,y) = xy$$
$$g(x,y) = x^2+y^2-1$$

What's the rate of change of f as I vary x?

- More than one way to solve!

1) get rid of a variable

$$y = \sqrt{1-x^2}$$



$$f(x) = x\sqrt{1-x^2}$$

$$f' = \sqrt{1-x^2} + x\,\frac{1}{2}\frac{(-2x)}{\sqrt{1-x^2}} = \sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}}$$

$$f_x = y$$
$$f_y = x$$

2) $x(t)$, $y(t)$

vary such that

$$g(x(t), y(t)) = 0$$
for all t

Chain rule

$$g_x g' + g_y y' = 0 = \star$$

Chain rule again $\frac{df}{dt}$

$$\Rightarrow f' = f_x x' + f_y y'$$

By $\star$ $y' = \frac{-g_x}{g_y} x'$

$$f' = f_x x_1 + \left(\frac{-g_x}{g_y} x'\right) f_y$$

2/15

$$f' = x'\left[f_x - \frac{g_x}{g_y} f_y\right]$$

$$\cancel{f'} : \frac{df}{dx} = f_x - \frac{g_x}{g_y} f_y$$

From example

$f_x = y$ $\quad g_x = 2x$ $\quad \frac{df}{dx} = y - \frac{2x}{2y}x = y - \frac{x^2}{y}$

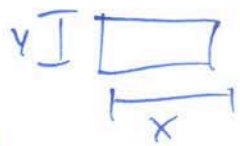
$f_y = x$ $\quad g_y = 2y$

Same Answer

0 when $y = \pm x$

Example

$g(x,y) = 2x + 2y - 40 = 0$



Area= $F(x,y) = xy$

$y = x$ $\square$

$f_x = y$

$f_y = x$

$g_x = 2$

$g_y = 2$

$$\frac{df}{dx} = f_x - \frac{g_x}{g_y} f_y = y - \frac{2}{2}x = y - x$$

Other method - get rid of y

$y = 20 - x$

$f = x(20 - x)$

$\frac{df}{dx} = 20 - 2x$

$y - x$

3/5

3 variables

$F(x,y,z)$

$g(x,y,z)=0$

can move two variables freely. Third one is fixed.

Example

$f = xy + z^2$

$g = x + y - z$

p

$\left(\frac{\partial f}{dx}\right)_y$ ← Fix y, z is determined by x and y

vary x

~~can vary y~~

$(x(t), y, z(t))$

constraint $\Rightarrow$ $g_x x' + g_z z' = 0$

$f_x = y$

$f_z = 2z$

$g_x = 1$

$g_z = -1$

$z' = \frac{-g_x}{g_z} x'$

$f' = f_x x' + f_z z'$

$= f_x x' + f_z((-g_x/g_z)x')$

$= x'\left[f_x - \frac{g_x}{g_z} f_z\right]$

$\left(\frac{df}{dx}\right)_y = f_x - \frac{g_x}{g_z} f_z$

$\left(\frac{df}{dx}\right)_y = y - \frac{1}{-1}2z = y + 2z$

4/15

~~Another example~~:

$$\left(\frac{df}{dx}\right|_z = f_x - \frac{g_x}{g_y} f_y \underset{\text{in our example}}{=} y - \frac{1}{1} x = y - x$$

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Exam Review

Gradient, Max/min (2^{nd} derivative test), λ Multiple, Non-independent var

Gradient

$f(x,y,z)$ Chain rule $f(x(t), y(t), z(t))$

$\nabla f = (f_x, f_y, f_z)$ $f' = \nabla f \cdot (x', y', z')$

Level set - where f is constant

∇f is normal to level set of f

Application

Tangent plane to $f = 3x^2 + 5y^2 + 3z^2 = 11$ at $(1,1,1)$

normal

$\downarrow$

$\nabla f = (6x, 10y, 6z)$

$\nabla f = \langle 6, 10, 6 \rangle$

$6x + 10y + 6z = d$

$6(1) + 10(1) + 6(1) = 22$

$6x + 10y + 6z = 22$

$3x + 5y + 3z = 11$

Application 2

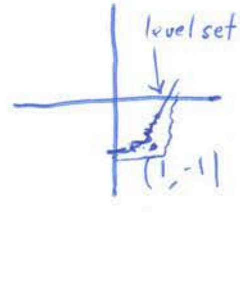
Linear Approximation

$f = x^3 - x + 2y^2 = 1.9$

Approximate coordinates of closest point in level set to point $(1,-1)$

$f(1,-1) = 2$

↳ not in level set



★ Fastest way to decrease f is to move in $-\nabla f$

" increase " ∇f

$\nabla f = (3x^2 - 1, 4y)$ $-\nabla f = (-2, 4)$

$_{(1,-1)} = (2, -4)$

Choose $(\Delta x, \Delta y)$

to minimize $|(\Delta x, \Delta y)|$

and decrease f by $.1$

$(\Delta x, \Delta y) = -h(2,-4) \quad h>0$

$$f(1+\Delta x, -1+\Delta y) \simeq f(1,-1) + \nabla f \cdot (\Delta x, \Delta y)$$
$$= 2 + (2,-4)\cdot(-h)(2,-4)$$
$$= 2 - h20$$
$$20h = .1$$
$$h = 1/200$$
$$h(\Delta x, \Delta y) = \frac{1}{200}(2,-4)$$
$$= \frac{(1,-2)}{100}$$
$$(.99, -.98)$$

$f(x,y,z) = 2x + 3y + z$

$g(x,y,z) = x^2 y^3 z - 1$

Always $x,y,z \geq 0$

Minimize f

on level set $g=0$

Method 1: Eliminate z

$$z = 1/x^2y^3 \Rightarrow f(x,y) = 2x + 3y + \frac{1}{x^2y^3}$$

$$f_x = 2 - \frac{2}{x^3y^3} = 0 \quad f_y = 3 - \frac{3}{x^2y^4} = 0$$

$$x^3y^3 = 1 \qquad x^3y^4 = 1$$

$x = 1 = y$ ← Critical Point $(1,1)$ $\quad f(1,1) = 6$

2/4

(1,1)

↳ what type of critical point?

$\Rightarrow$ Second derivative test

$$f_{xx} = A = 6/x^4y^3 = 6$$
$$f_{xy} = B = 6$$
$$f_{yy} = C = 12$$

$$\Delta = AC - B^2$$
$$= (6)(12) - (6)^2 > 0$$

positive

local min

Method 2

Lagrange Multipliers

$$\nabla f = \lambda \nabla g \qquad 2 = \lambda 2xy^3z$$
$$g = 0 \qquad 3 = \lambda 3x^2y^2z$$
$$1 = \lambda x^2y^3$$

solve $\Rightarrow$ $x=1$, $y=1$, $z=1$

Non-Independent variables

If I keep $g=0$ change x but fix y

how does f change?

$$\left(\frac{\partial f}{\partial x}\right)_y$$

Differential method

$$0 = dg = 2xy^3z + 3x^2y^2z\,dy + x^2y^3dz$$

divide by x^2y^3

$$dz = \frac{-2z}{x}dx - \frac{3z}{y}dy$$

$$df = 2dx + 3dy + dz \quad (\star)$$

Use $\star$

$$= 2dx + 3dy - \frac{2z}{x}dx - \frac{3z}{y}dy$$

$$\underbrace{\left(2 - \frac{2z}{x}\right)}_{\left(\frac{\partial f}{\partial x}\right)_y} dx + \underbrace{\left(3 - \frac{3z}{y}\right)}_{\left(\frac{\partial f}{\partial y}\right)_x} dy$$

critical point

$$\left.\begin{aligned} 2 - \frac{2z}{x} &= 0 \\ 3 - \frac{3z}{y} &= 0 \\ \underbrace{x^2 y^3 z = 1}_{\text{constraint}} \end{aligned}\right|_{\text{solve}} \quad (1,1,1)$$

1. (20pts) Let p be the point on the curve $x^2 + x^3 - y^2 = 3.1$ which is closest to $(2, 3)$. Use the gradient to estimate the coordinates of p.

$f(x,y) = x^2 + x^3 - y^2$

$f(2,3) = 3$

WANT TO GO UP! → move in direction of gradient

$\nabla f = (2x + 3x^2, -2y)$

@$(2,3)$

$= (16, -6)$

$= 2(8, -3)$

$u = \frac{\nabla f}{|\nabla f|} = \frac{\cancel{2}(8,-3)}{\cancel{2}\sqrt{73}}$

$h\nabla f = .1$

$h = .1/|\nabla f|$

$h = \frac{1}{20\sqrt{73}}$

$h\vec{u} = h\nabla f$

$(2,3) + h\vec{u}$

$(2,3) + \left(\frac{1}{20\sqrt{73}}\right)\left(\frac{(8,-3)}{\sqrt{73}}\right)$

$(2,3) + \frac{8, -3}{(20 \cdot 73)}$

$\left(2 + 8/20 \cdot 73,\ 3 - \frac{3}{(20 \cdot 73)}\right)$

2. (10pts) Find the equation of the tangent plane to the surface

$$x^2 + 3y^2 + 2z^2 = 12 \qquad \text{at the point} \qquad (1, -1, 2).$$

$f(x,y,z) = x^2 + 3y^2 + 2z^2 - 12$

$\nabla f = (2x, 6y, 4z)$

$\nabla f(1,-1,2) = \langle 2, -6, 8 \rangle$

Normal vector $\Rightarrow \langle 1, -3, 4 \rangle$

Tangent plane

Normal · x-point $= 0$

$\langle 1, -3, 4 \rangle \cdot \langle x-1, y+1, z-2 \rangle = 0$

$1(x-1) - 3(y+1) + 4(z-2) = 0$

$x - 1 - 3y - 3 + 4z - 8 = 0$

$\boxed{x - 3y + 4z = 12}$

3. (20pts) (i) Find the critical points of

$$w = f(x, y) = 5x^2 - 2xy + 2y^2 - 8x - 2y + 7,$$

and determine their type.

$f_x = 10x - 2y - 8$

$f_y = -2x + 4y - 2$

$2(10x - 2y - 8 = 0)$

$-2x + 4y - 2 = 0$

$18x - 18 = 0$

$18x = 18$

$x = 1$

if $x = 1$

$10(1) - 2y - 8 = 0$

$-2y = -2$

$y = 1 \Rightarrow (1, 1)$ Critical Point

$A = f_{xx} = 10$

$B = f_{xy} = -2$

$C = f_{yy} = 4$

$\Delta = AC - B^2 > 0$

$A > 0$ ← local min

(ii) Find where $f(x, y)$ is smallest in the first quadrant, $x \geq 0$ and $y \geq 0$. Justify your answer.

Smallest at $(1,1)$ b/c it is the local min and only critical point!

4. (20pts) Using Lagrange multipliers, find the points on the ellipse $x^2 + 2y^2 = 1$ where the function $f(x, y) = xy$ has a maximum and a minimum.

(i) Write down the equations satisfied by the Lagrange multiplier.

$g(x,y) = x^2+2y^2-1$

$f(x,y) = xy$

$\nabla f = \lambda \nabla g$

$\nabla f = (y, x)$

$\nabla g = (2x, 4y)$

$$\boxed{\begin{array}{l} y = \lambda 2x \\ x = \lambda 4y \\ x^2+2y^2=1 \end{array}}$$

(ii) Solve these equations and find the global maximum and minimum.

$xy = \lambda 2x^2$

$xy = \lambda 4y^2$

$\cancel{\lambda} \cancel{2}x^2 = \cancel{\lambda}\cancel{4}^{2}y^2$

$x^2 = 2y^2$

$2y^2+2y^2=1$

$4y^2=1$

$y = \frac{1}{\sqrt{4}} = \pm\frac{1}{2}$

if $y = \frac{1}{2}$

$x^2+2\left(\frac{1}{4}\right)=1$

$x^2 = \frac{1}{2}$

$x = \pm\sqrt{\frac{1}{2}} = \pm\frac{1}{\sqrt{2}}$

$f\left(\frac{1}{\sqrt{2}}, \frac{1}{2}\right) = \frac{\sqrt{2}}{4} \to max$

$f\left(\frac{1}{\sqrt{2}}, -\frac{1}{2}\right) = -\frac{\sqrt{2}}{4} \to min$

$f\left(-\frac{1}{\sqrt{2}}, \frac{1}{2}\right) = \frac{\sqrt{2}}{4} \to max$

$f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = -\frac{\sqrt{2}}{4} \to min$

5. (15pts) Given that the variables w, x, y and z satisfy $w = xyz$ and $w^2 + z^2 = 13$, find

$$\left(\frac{\partial w}{\partial x}\right)_y,$$

when $w = 3$, $x = 3$, $y = 1/2$ and $z = 2$.

$dw = yz\,dx + \cancel{xz\,dy} + xy\,\underline{dz}$

$2w\,dw + 2z\,\underline{dz} = 0$

get rid of

$2w\,dw = -2z\,dz$

$3\,dw = -2\,dz$

$dz = -\frac{3}{2}\,dw$

$dw = dx + \frac{3}{2}\,dz$

$dw = dx + \left(\frac{3}{2}\right)\left(\frac{-3}{2}\right)dw$

$dw = dx - \frac{9}{4}\,dw$

$dw + \frac{9}{4}dw = dx$

$\frac{13}{4}\,dw = dx$

$dw = \frac{4}{13}\,dx$

$$\boxed{\left(\frac{\partial w}{\partial x}\right)_y = \frac{4}{13}}$$

6. (15pts) The two surfaces $x^4 - y^3 + z^2 = 1$ and $x^2y^2 + 3z^4 = 4$ intersect along a curve for which y is a function of x. Find

$$\frac{dy}{dx} \quad \text{at} \quad (x_0, y_0, z_0) = (1, 1, 1).$$

$4x^3dx - 3y^2dy + 2z\,dz = 0$

$2x\,dx\,y^2 + 2yx^2dy + 12z^3dz = 0$

$x = 1$
$y = 1$
$z = 1$

$2dx + 2dy + 12dz = 0$

$4dx - 3dy + 2dz = 0$

Get rid of dz

$dz = \dfrac{3dy - 4dx}{2}$

$2dx + 2dy + 6(3dy - 4dx) = 0$

$2dx + 2dy + 18dy - 24dk = 0$

$-22dx + 20dy = 0$

$-11dx + 10dy = 0$

$+11dx = 10dy$

$\dfrac{+11}{10} = \dfrac{dy}{dx}$

2

1. (15pts) Two sides of a triangle are a and b and θ is the angle between them. The third side is c.
(i) Write down an approximation for Δc in terms of a, b, c, θ, Δa, Δb and $\Delta\theta$.

(ii) If $a = 1$, $b = 2$, $\theta = \pi/3$, is c more sensitive to changes in Δa or Δb?

2. (20pts) A particle is moving in the xy-plane so that relative to the origin it is rotating clockwise at a rate of 3 radians per second while its distance to the origin is increasing at a rate of 7 metres per second. At a time when the particle is at $(5, -12)$, what is

$$\frac{dx}{dt}?$$

3. (20pts) Find the global maximum and minimum of

$$f(x, y) = x^2 - xy + y^2 + 4$$

over the triangle with sides $x = 0$, $y = 4$ and $y = x$.

4. (15pts) Let $u = x^2 - y^2$, $v = 2xy$ and $w = w(u, v)$.
(i) Express the partial derivatives w_x and w_y in terms of w_u and w_v (and x and y).

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x}$$

$$w_x = 2x\,w_u + 2y\,w_v$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u}\frac{du}{\partial y} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y}$$

$$w_y = -2y\,w_u + w_v\,2x$$

(ii) Assume that w solves the differential equation $uw_u + vw_v = 1$. Find $xw_x + yw_y$.

$$uw_u + vw_v = 1 \qquad\qquad 2xy\,w_v = A$$

$$(x^2-y^2)w_u + (2xy)w_v = 1 \qquad (x^2-y^2)w_u = B$$

$$xw_x + yw_y$$

$$x(2xw_u + 2yw_v) + y(-2yw_u + w_v 2x)$$

$$2x^2w_u + 2yxw_v + -2y^2w_u + 2xyw_v$$

$$2x^2w_u + A - 2y^2w_u + A$$

$$2A + 2w_u(x^2-y^2)$$

$$A + B = 1$$

$$2A + 2B = 2$$

$$xw_x + yw_y = 2$$

5. (15pts) (i) Write down the equations for the Lagrange multiplier for the point of the surface

$$x^4 + y^4 + z^4 + yz + xz + xy = 6$$

with the largest x-value.

$g(x,y,z) = x^4 + y^4 + z^4 + yz + xz + xy - 6$

$f(x,y,z) = x$

$$\nabla f = \lambda \nabla g$$

$\nabla f = (1,0,0)$

$\nabla g = (4x^3 + z + y,\ 4y^3 + x + z,\ 4z^3 + y + x)$

$1 = \lambda(4x^3 + z + y)$

$0 = \lambda(4y^3 + x + z) \qquad 6 = x^4 + y^4 + z^4 + yz + xz + xy$

$0 = \lambda(4z^3 + y + x)$

(ii) If the point with the largest x-value is (x_0, y_0, z_0) find the equation of the tangent plane to the surface at this point.

$P = (x_0, y_0, z_0)$

$\nabla f = \langle 1,0,0 \rangle$

normal

$\cancel{\langle 1,0,0 \rangle \cdot \langle x_0, y_0, z_0 \rangle = 0}$

$\cancel{1x_0 + 0 + 0}$ $\quad \langle 1,0,0 \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$

$x - x_0 = 0$

$x = x_0$

6. (15pts) Suppose that $x^2y + xz^2 = 1$ and let $w = x^3y$. Express

$$\frac{\partial w}{\partial z}_{y},$$

as a function of x, y and z and determine its value when $(x, y, z) = (1, 1, 2)$.

$2x\,dx\,y + \cancel{x^2 dy} + dx\,z^2 + 2z\,dz\,x = 0$

$dw = 3x^2 dx\,y + \cancel{x^3 dy}$

-Get rid of dx

$x=1$
$y=1$
$z=2$

(1) $2dx + 4dx + 4dz = 0$

$dw = 3dx$

$dx = \frac{1}{3}dw$

(1) $6dx + 4dz = 0$

$6\left(\frac{1}{3}dw\right) + 4dz = 0$

$2dw + 4dz = 0$

$2dw = -4dz$

$dw = -2dz$

$\left(\frac{\partial w}{\partial z}\right)_y = -2$

10/17/13 # 18.02 Midterm Review
Fernando Trujano

Partial Derivatives

- Pretend its a function of one variable

$f(x,y) = x^2 y$

$$\frac{\partial f}{\partial x} = 2yx \qquad \frac{\partial f}{\partial y} = x^2 y$$

Derivative w/ respect to x. Treat y as a constant.

Planes Tangent to Surface

surface $z = F(x,y,z)$

point $P = (a, b, F(a,b))$

$\nabla f = (a,b,c)$

$P = (P_1, P_2, P_3)$

$\langle a,b,c \rangle \cdot \langle x-P_1,\ y-P_2,\ z-P_3 \rangle = 0$

$$z - F(a,b) = F_x(a,b)(x-a) + F_y(a,b)(y-b)$$

Example

Surface $z = 5 - 2x^2 - y^2$

Point $P = (1,1,2)$

$$F_x = -4x \qquad F_y = -2y$$

$$z - F(1,1) = F_x(1,1)(x-1) + F_y(1,1)(y-1)$$

$$z - 2 = -4x + 4 + -2y + 2$$

$z = 8 - 4x - 2y \rightarrow$ Plane tangent to z at P

Higher Order Partial Derivatives

- Partial derivative of the partial derivative

Example:

$f(x,y) = 2x^3 + y^2$

$F_x = 6x^2 \qquad F_{xx} = 12x \qquad F_{xy} = 0$

$F_y = 2y \qquad F_{yy} = 2 \qquad f_{yx} = 0$

Critical Points

$f_x = 0 = f_y$

$A = f_{xx} \quad B = f_{xy} \quad C = f_{yy}$

$\Delta = AC - B^2$

$\cup$ $A > 0$ and $\Delta > 0 \rightarrow$ local min at $f(a,b)$
$\cap$ $A < 0$ and $\Delta > 0 \rightarrow$ local max
$A \qquad \Delta < 0 \rightarrow$ Saddle Point

Example

$f(x,y) = 3x - x^3 - 3xy^2$

$f_x = 3 - 3x^2 - 3y^2$
$f_y = -6xy$

$3 - 3x^2 - 3y^2 = 0$
$1 - x^2 = y^2$
$1 - y^2 = x^2$

if $x = 0$: $y^2 = 1$, $y = \pm 1$ → $(0,1)$, $(0,-1)$
if $y = 0$: $x^2 = 1$, $x = \pm 1$ → $(1,0)$, $(-1,0)$

$-6xy = 0$
$x = 0$ OR $y = 0$ } Doesn't help much!

$A = f_{xx} = -6x \quad B = f_{xy} = -6y \quad C = f_{yy} = -6x$

$\Delta = AC - B^2 = (-6x)(-6x) - (-6y)^2 = 36(x^2 - y^2)$

	A	B	C	Δ	
$(1,0)$	-6	0	-6	36	→ Local max
$(-1,0)$	6	0	6	36	→ Local min
$(0,1)$	0	-6	0	-36	→ Saddle Point
$(0,-1)$	0	6	0	-36	→ Saddle Point

Example

$f(x,y) = 2x^2 + 2xy + y^2 + 4x - 2y + 1$

$f_x = 4x + 2y + 4$
$f_y = 2x + 2y - 2$

$f_x = 0 = f_y$

$4x + 2y + 4 = 2x + 2y - 2$
$2x = -6$
$x = -3$

Critical Point
$(-3, 4)$

$A = f_{xx} = 4$
$B = f_{xy} = 2$
$C = f_{yy} = 2$

$\Delta = AC - B^2 = 16 - 4 > 0$ $\cup$

Local min

The Chain Rule

$w = f(x,y,z)$

$x = g(u,v) \quad y = h(u,v) \quad z = k(u,v)$

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}$$

Example

$z = f(u,v)$

$u = 2x + y$

$v = 3x - 2y$

Given

$\frac{\partial z}{\partial u} = 3$, $\frac{\partial z}{\partial v} = -2$ at $(3,1)$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

$$(3)(2) + (-2)(3) = 0$$

Directional Derivatives

$$D_u f(x) = \nabla f(x) \cdot u$$

vector

Derivative of f in the direction of $u = \frac{v}{|v|}$

Gradient

$$\nabla f(x) = (f_x, f_y, f_z)$$

Gradient Vector ∇f

Maximum directional derivative = Length of gradient vector $|\nabla f(P)|$

∇f Points in direction where function increases most rapidly

Example

$f(x,y,z) = xy + z^2$

$\nabla f = (y, x, 2z)$

$D_u f = \nabla f \cdot \vec{u} = |\nabla f||\vec{u}|\cos\theta$

Level Sets

∇f is normal/perpendicular to the level set

every curve in

Example: Find tangent plane to $xyz - 1 = 0$ at $(1, 2, 1/2)$

need point and normal

$\nabla f = (yz, xz, xy)$

$= (1, 1/2, 2)$

$x + \frac{1}{2}y + 2z = d$

plug in point to find $d = 3$

$x + \frac{1}{2}y + 2z = 3$

Lagrange Multipliers

$\nabla f = \lambda \nabla g$

$\quad\quad\quad \uparrow$ Constraint

Example:

Maximize $f(x,y,z) = x - y + z$ such that $\underset{\text{constraint}}{g(x,y,z)} = x^2 + y^2 + z^2 - 3 = 0$

$\nabla f = (1,-1,1)$ $\nabla g = (2x, 2y, 2z)$ → aka what is the last level set that touches sphere g

$\nabla f = \lambda \nabla g$

$1 = \lambda 2x$ $\quad -1 = \lambda 2y$ $\quad 1 = \lambda 2z$ $\quad \underset{\text{constraint}}{x^2 + y^2 + z^2 = 3}$

$x = \frac{1}{2\lambda}$ $\quad y = -\frac{1}{2\lambda}$ $\quad z = \frac{1}{2\lambda} \rightarrow 3\left(\frac{1}{2\lambda}\right)^2 = 3$

$4\lambda^2 = 1$

$2\lambda = \pm 1$

if $2\lambda = 1 \rightarrow (1,-1,1)$

MAX $= 3$

if $2\lambda = -1 \rightarrow (-1,1,-1)$

MIN $= -3$

Multiple Constraints

$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2$

→ Same idea, but harder to solve!

Example:

Find points on surface $z = xy + 1$ closest to origin

Function = distance to origin = $\sqrt{x^2 + y^2 + z^2}$

use $f(x,y,z) = x^2 + y^2 + z^2$ to make it easier

Now minimize f using $g(x,y,z) = xy + 1 - z$ as constraint $\quad \nabla f = \nabla g \lambda$

Partial Differential Equations, Double Integrals

14.1, 14.2, 14.3
P, I.1

Double Integrals - 14.1

$f(x,y)$ continious

$R = [a,b] \times [c,d]$ ← rectangle

x constant
integral with respect to y

$$\iint_R F(x,y)\,dA = \int_a^b \left(\int_c^d F(x,y)\,dy \right) dx$$

$$= \int_c^c \left(\int_a^b F(x,y)\,dx \right) dy$$

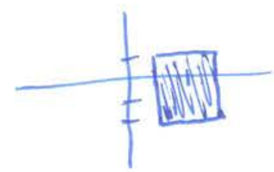
Example

$f(x,y) = 4x^3 + 6xy^2$

$R = [1,3] \times [-2,1] \Rightarrow$

$x=3$, $x=1$

$y=1$, $y=-2$



$$\int_1^3 \left(\int_{-2}^1 4x^3 + 6xy^2\,dy \right) dx$$

$$\Downarrow$$

$$\left(4x^3y + 2xy^3\right)\Big|_{-2}^{1}$$

$$\int_1^3 (4x^3 + 2x) - (-8x^3 - 16x)\,dx$$

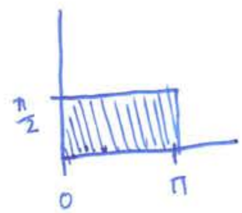
$$x^4 + x^2 + 2x^4 + 8x^2$$

$$3x^4 + 9x^2 \Big|_1^3$$

$$= 312$$

Example

$$\int_0^{\pi}\int_0^{\pi/2} \cos x \cos y \, dy\, dx$$



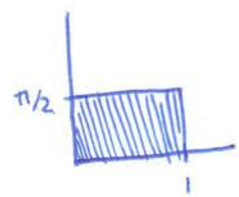
$$+\cos x \sin y \Big|_{y=0}^{y=\pi/2}$$

$0 \le x \le \pi$
$0 \le y \le \pi/2$

$$\int_0^{\pi} +\cos x \, dx = \sin x \Big|_0^{\pi} = 0$$

Example

$$\int_0^1 \int_0^{\pi/2} \left(e^y + \sin x\right) dx\, dy$$



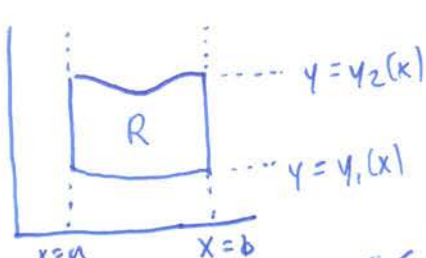
$$e^y x + -\cos x \Big|_{x=0}^{x=\pi/2}$$

$$\int_0^1 \frac{\pi}{2} e^y + 1 \, dy$$

$$\frac{\pi}{2} e^y + y \Big|_0^1 = \frac{\pi}{2} e + 1 - \left(\frac{\pi}{2}\right)$$

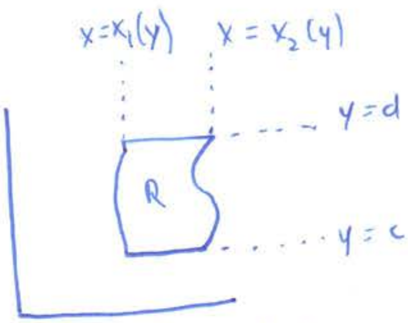
Double Integrals over more General Regions - 14.2

R no longer a nice rectangle



Vertically Simple

$a \le x \le b$
$y_1(x) \le y \le y_2(x)$



Horizontaly simple

$c \le y \le d$
$x_1(y) \le x \le x_2(y)$

Solving Vertically Simple

$$\iint_R F(x,y)\,dA = \int_a^b \int_{y_1(x)}^{y_2(x)} F(x,y)\,dy\,dx$$

Solving Horizontally Simple

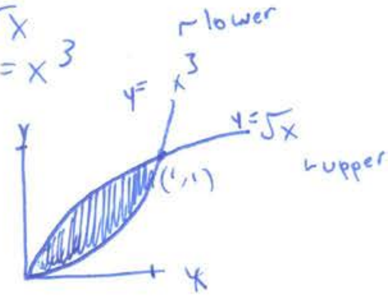
$$\iint_R F(x,y)\,dA = \int_d^c \int_{x_1(y)}^{x_2(y)} F(x,y)\,dx\,dy$$

Example

$$\iint_R xy^2\,dA$$

$R =$ 1st Quadrant $y = \sqrt{x}$, $y = x^3$

1) Sketch R



* R is vertically and horizontally simple

$$\int_0^1 \int_{x^3}^{\sqrt{x}} xy^2\,dy\,dx$$

$$= \frac{xy^3}{3}\Bigg|_{x^3}^{\sqrt{x}}$$

$$= \int_0^1 \frac{x^{5/2}}{3} - \frac{x^{10}}{3}\,dx$$

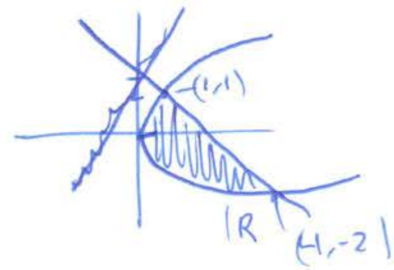
$$= \frac{2}{21} - \frac{1}{33} = 5/77$$

3/4

Example

$$\iint_R (6x+2y^2)\,dA$$

R $\quad x = y^2$
$\quad\;\; x + y = 2$



~~$-x+2=$~~

$2-y = y^2$

$2 = y^2 + y$

$2 = y(1+y)$

$y = 1$

$y = -2$

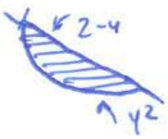
intersections

$(4,-2)$

$(1,1)$

$y = -x + 2$

Horizontal Rectangles top
Left → Right ⇒ bottom
" ~~Right-Left~~



$$= \int_{-2}^{1} \int_{y^2}^{2-y} 6x + 2y^2 \, dx \, dy$$

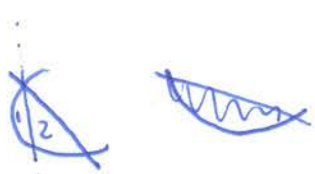
in the ~~x~~ y-direction

$$3x^2 + 2y^2 x \Big|_{x=y^2}^{x=2-y}$$

$$\int_{-2}^{1} 3(2-y)^2 + 2y^2(2-y) - (3y^4 + 2y^2y^2)$$

$$= \int_{-2}^{1} 12 - 12y + 7y^2 - 2y^3 - 5y^4 \, dy$$

$$= 12y - 6y^2 + \frac{7y^3}{3} - \frac{y^4}{2} - y^5 \Big|_{-2}^{1}$$



$$= 99/2$$

* Going top
Vertical Rectangles would require two integrals

4/4 $\iint_R f(x,y)\,dA = \int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} (6x + 2y^2)\,dy\,dx + \int_1^4 \int_{-\sqrt{x}}^{2-x} (6x + 2y^2)\,dy\,dx$

PDE's, Double Integrals, Funky Domains

PDE

└ Partial differential equation

ODE
└ ordinary de

$F(t)$ ← one variable

$f' = af$ ← constant

$f(t) = Ce^{at}$
↑ need an initial condition $f(0) = C$

PDE

many variables

equation for a function and its partials.

Example

1) $f_x = f_y$
2) $xf_x = yf_y$ } First order PDE
3) $f_{xx} + f_{yy} = 0$ ← Second order (Laplace)
4) $f_{xx} - f_{yy} = 0$ (wave)
3) $f_t = f_{xx} + f_{yy}$ (heat equation)

solution

$f(x,y) = x+y$

$f(x,y) = xy$

$f(x,y) = x^2 - y^2$, $f(x,y) = xy$

$f(x,y) = g(x-y)$

$f(t,x,y) = e^{-a^2 t}\cos(ax)$

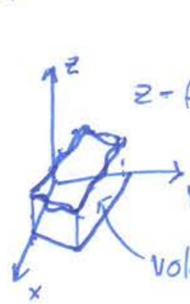
~~1) f(x,y) = x+y~~ 4) $f(x,y) = g(x-y)$ ← $g(t)$ one var

$f_x = g'(x-y)$ $f_{xx} = g''(x-y)$

$f_y = -g(x-y)$ $f_{yy} = g''(x-y)$

Double Integral

$f(x,y)$



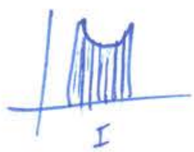
$z = f(x,y)$

$\iint_R f(x,y)\,dA$

volume under graph

← Chop off in cubes

Calc BC Reiman Sums



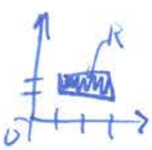
$\int_I f(x)\,dx$ = Area under curve

Chop off in squares

Compute

$$f = 2x + 3y - 1$$

$$R = \{1 \le x \le 3 \quad 1 \le y \le 2$$



$$\int_{x=1}^{x=3}\int_{y=1}^{y=2} (2x + 3y - 1)\,dy\,dx$$

↳ do inside then outside

fix x

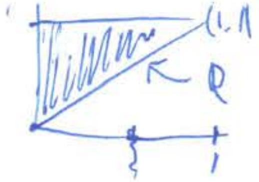
$$= \int_{x=1}^{x=3} \left[2xy + \frac{3}{2}y^2 - 1\right]_{y=1}^{y=2} dx$$

$$= \int_{x=1}^{x=3} \left[4x + 6 - 2 - (2x + \frac{3}{2} - 1)\right] dx = \int_{x=1}^{x=3} 2x + \frac{7}{2}x\,dx$$

$$= x^2 + \frac{7}{2}x \Big]_1^3 = 9 + \frac{21}{2} - (1 + \frac{7}{2}) = 15$$

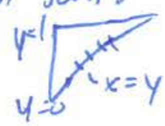
2/4

Intersecting Region



$f = x + y$

Tricky! ← Think about Reiman sums



$$\int_{y=0}^{y=1} \int_{x=0}^{x=y} (x+y)\,dx\,dy$$

Y is constant

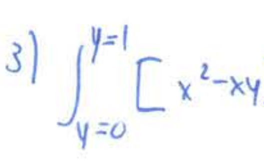
$$= \int_{y=0}^{y=1} \frac{1}{2}x^2 + xy \Big|_{x=0}^{x=y} dy = \int_{y=0}^{y=1} \frac{3}{2} y^2\,dy$$

$$= \frac{1}{2} y^3 \Big|_{y=0}^{y=1} = \frac{1}{2}$$

$R = \begin{cases} \text{above } y = x \\ \text{below } x = y^2 \end{cases}$

$f = (2x - y)$

1) Sketch R
2) Set up Integral
3) Evaluate
   i) Inner
   ii) Outer

1) 

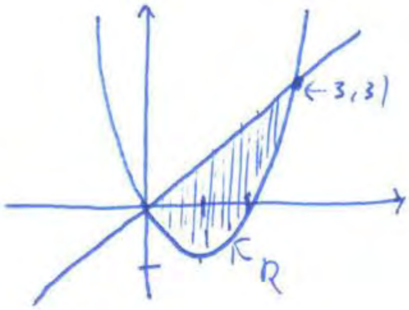
2) $$\int_{y=0}^{y=1} \int_{x=y^2}^{x=y} (2x - y)\,dx\,dy$$

3) $$\int_{y=0}^{y=1} \left[x^2 - xy\right]\Big|_{x=y^2}^{x=y} = \int_{y=0}^{y=1} 0 - (y^4 - y^3)\,dy = \frac{1}{4}y^4 - \frac{1}{5}y^5 \Big]_{y=0}^{y=1} = \frac{1}{20}$$

3/4

Example

Region bounded by $y = x^2 - 2x$
$y = x$



$$\int_{x=0}^{x=3} \int_{y=x^2-2x}^{y=x} 1 \, dy \, dx$$

$f(x,y)$

4 '4

18.02 Notes

14.4, 14.5

Polar Integrals

$x = r\cos\theta$

$y = r\sin\theta$

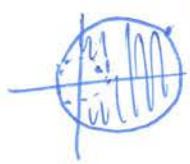
$dA = r\,dr\,d\theta$

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_{r_1(\theta)\ \text{inner}}^{r_2(\theta)\ \text{outer}} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

Example

$r_{inner} = 1$

$r_{outer} = 2+\cos\theta$



$2+\cos\theta$

$$A = \int_\alpha^\beta \int_{r_{inner}}^{outer} r\,dr\,d\theta$$

$$= 2\int_0^\pi \int_1^{2+\cos\theta} r\,dr\,d\theta$$

$$\frac{1}{2}r^2\Big|_1^{2+\cos\theta}$$

$$= 2\int_0^\pi \frac{1}{2}(2+\cos\theta)^2 - \frac{1}{2}(1^2)\,d\theta$$

$$= 2\int_0^\pi \frac{1}{2}\left((2+\cos\theta)^2 - 1\right)d\theta$$

$$\cancel{2}\int_0^\pi \cancel{\tfrac{1}{2}}\left(4 + 4\cos\theta + \cos^2\theta - 1\right.$$

$$\int_0^\pi 3 + 4\cos\theta + \cos^2\theta\ d\theta$$

11/6/13

Polar Integrals

OCW

$\iint_R f(x,y)\,dA$

$= \iint f(x,y)\,dy\,dx$

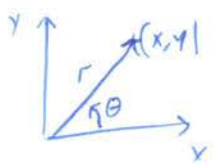
Example

$x^2+y^2 \le 1$

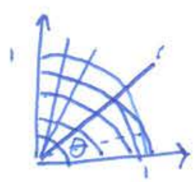
$x, y \ge 0$

use x and y would be hard.

→ we polar instead

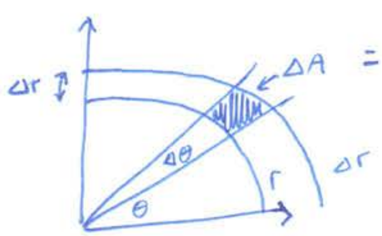


$x = r\cos\theta$
$y = r\sin\theta$



$\int_0^{\pi/2}\int_0^1 \sim\sim\, dr\,d\theta$

r starts at 0
stops at 1



$\Delta A = \Delta r\, r\Delta\theta$

Arc length

$\Delta A = r\,\Delta r\,\Delta\theta$

$\boxed{dA = r\,dr\,d\theta}$

☆

$f = 1 - x^2 - y^2$

- Switch to polar

$x = r\cos\theta$
$y = r\sin\theta$
$r = x^2+y^2$

$f = 1-(x^2+y^2)$

$f = 1 - r^2$

$\int_0^{\pi/2}\int_0^1 (1-r^2)\,r\,dr\,d\theta$

$$\int_0^{\pi/2} \left[\frac{r^2}{2} - \frac{r^4}{4}\right]_0^1 d\theta$$

$$\int_0^{\pi/4} \frac{1}{4}\, d\theta = \frac{1}{4}\frac{\pi}{2} = \frac{\pi}{8}$$

Applications of Double Integral

1) Find area of region R

$$A = \iint_R 1\, dA$$

Mass of a flat object with varying density δ = mass per unit area

$$\Delta m = \delta\, \Delta A$$

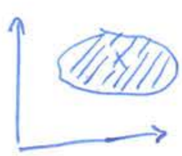
$$\text{Mass} = \iint_R \delta\, dA$$

2) Average value of f in R

$$\text{Average of } f = \bar{f} = \frac{1}{\text{Area}(R)} \iint_R f\, dA$$

$$\text{Weighted average of } f \text{ with density } \delta = \frac{1}{\text{Mass}(R)} \iint_R f \underbrace{\delta\, dA}_{\text{mass}}$$

2a b) Center of mass of a flat (planar) object with δ



$(\bar{x}, \bar{y})$ average values of x and y

$$\bar{x} = \frac{1}{\text{Mass}} \iint x\, \delta\, dA$$

$$\bar{y} = \frac{1}{\text{Mass}} \iint y\, \delta\, dA$$

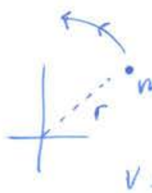
2/4

3) Moment of Inertia

↳ How hard it is to spin something

<u>Idea</u> Kinetic Energy

$\overset{\longrightarrow}{m \quad \vec{v}}$ $\quad \frac{1}{2}\underset{\text{mass}}{m}v^2$



angular velocity w, $\frac{d\theta}{dt}$

$v = rw$

$$KE = \frac{1}{2}mv^2 = \frac{1}{2}\underbrace{mr^2}_{\text{moment of inertia for point mass}}w^2$$

For a solid with density δ

$$\Delta m = \delta \Delta A$$

has moment of Inertia $\Delta m \cdot r^2$

$$= r^2 \delta \Delta A$$

$$I_0 = \text{Moment of Inertia about origin} = \iint_R \underset{x^2+y^2}{r^2}\,\delta\, dA$$

... How hard it is to spin object around origin

$$\text{Rotational } KE = \frac{1}{2} I_0 w^2$$

What about rotation around x axis?

distance to x axis $= |y|$

$|\ |^2 = y^2$

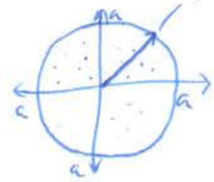
distance to x-axis

$$I_x = \iint y^2 \delta\, dA$$

— sum square of distance to the axis of rotation

3/4

Example



r is b/w 0 and a

disk of radius a

$\delta = 1$

Rotate around center

$$I_0 = \iint r^2 \delta \, dA$$

Polar

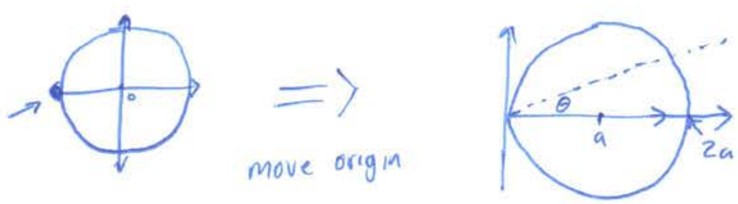
$$= \int_0^{2\pi}\int_0^a r^2 \, r \, dr \, d\theta$$

$$= \int_0^{2\pi} \frac{r^4}{4}\Big|_0^a d\theta = \int_0^{2\pi} \frac{a^4}{4} d\theta$$

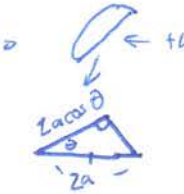
$$= \cancel{\frac{a^5}{20}\Big|_0^{2\pi}} = \frac{a^4}{4}\theta \Big|_0^{2\pi}$$

$$= \frac{2\pi a^4}{4} - 0 = \frac{\pi a^4}{2}$$

Now rotate around point on the circumference



r goes from 0 to



← that

$$I_0 = \iint r^2 dA$$

$$= \iint r^2 r \, dr \, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2a\cos\theta} r^2 r \, dr \, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\pi/2} \frac{r^4}{4}\Big|_0^{2a\cos\theta} d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4a\cos^4\theta \, d\theta$$

$$= \frac{3\pi a^4}{2}$$

3 times harder to spin from circumference than center

~~10/18/94~~
10/17/13

18.02 Lecture

Fernando Trujano

Integrals in Polar Coordinates
Center of mass
Moment of Inertia

NOT on Midterm
- Curvature
- Partial Differential Eq

WILL BE
- non-independent variables

Integral in Polar Coordinates

Area of circle $\{ x^2+y^2 \leq 1$

$$4\int_{x=0}^{x=1}\int_{y=0}^{y=\sqrt{1-x^2}} 1\,dy\,dx = 4\int_{x=0}^{x=1}\sqrt{1-x^2}\,dx$$

Ewww

In Polar Coordinates

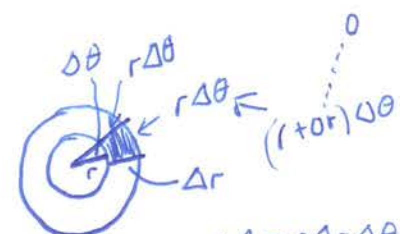
r, θ

$x^2+y^2=r^2$

$x = r\cos\theta$

$y = r\sin\theta$

$\tan\theta = \frac{y}{x}$



$$\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1} dA$$

Area element

what does it look like in Polar

$\Delta A = r\Delta r\Delta\theta$

$dA = r\,dr\,d\theta$

$$\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1} r\,dr\,d\theta$$

$$\int_0^{2\pi}\frac{1}{2}r^2\Big|_0^1\,d\theta = \pi$$

Area of unit circle

Example

$$\iint_R x^2 dA = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} r^2 \cos^2\theta \, r \, dr \, \theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \frac{1}{4} \cos^2\theta \, d\theta = \frac{\pi}{4}$$

Reminder $\int_0^{2\pi} \cos^2\theta \, d\theta = \pi = \int_0^{2\pi} \sin^2\theta$

$\cos^2\theta + \sin^2\theta = 1$

The Gaussian

$$\int_{-\infty}^{\infty} e^{-x^2} dx = I$$

Reminder
$e^{-a} e^{-b} = e^{-a-b}$

$$I = \int_{-\infty}^{\infty} e^{-y^2} dy$$

$$\Rightarrow I^2 = \int_{x=-\infty}^{x=\infty} \int_{y=-\infty}^{y=\infty} e^{-x^2} e^{-y^2} dy \, dx$$

in Polar

$$I^2 = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=\infty} e^{-r^2} r \, dr \, d\theta$$

Substitution $u = r^2$
$du = 2r dr$

$$\int_{\theta=0}^{2\pi} \int_{u=0}^{\infty} \left[\frac{1}{2} e^{-u} du\right] d\theta = \frac{1}{2} \int_{\theta=0}^{2\pi} d\theta = \pi$$

$I = \sqrt{\pi}$

Center of Mass

Planar lamina R in plane

density $\delta(x,y)$

$$\text{Mass} = M = \iint_R \delta(x,y)\, dA$$

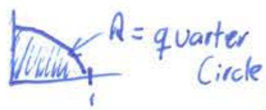
$$\bar{x} = \text{Avg of } x$$

$$= \frac{1}{M} \iint_R x\, \delta\, dA$$

$$\bar{y} = \frac{1}{M} \iint_R y\, \delta\, dA$$

Easy Example

$\delta = 1$



$M = \frac{\pi}{4}$

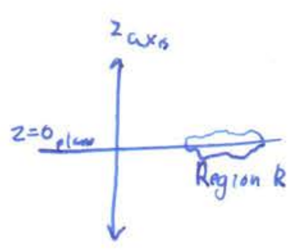
$$\left(\frac{\pi}{4}\right)\bar{x} = \int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{r=0}^{r=1} r\cos\theta \, r\, dr\, d\theta$$

$$= \int_0^{\pi/2} \frac{1}{3}\cos\theta\, d\theta = \frac{1}{3}\sin\theta \Big|_0^{\frac{\pi}{2}}$$

$$\bar{x} = \frac{4}{3\pi}$$

same thing for $\bar{y}$

$$\bar{y} = \frac{4}{3\pi}$$



Moment of Inertia

∴ how hard it is to rotate

$$\underset{\substack{\uparrow \\ z\text{-axis}}}{I_0} = \iint_R (x^2+y^2)\, \delta\, dA$$

Line L

moment of inertia w/ respect to L

$$\iint_R [\text{distance to } L]^2\, \delta\, dA$$

Example

$R =$ disk of radius a

$\delta = 1$

$$I_{Oma} = \int_{\theta=0}^{\pi} \int_{r=0}^{a} r^2 \, r \, dr \, \theta$$

$$= \int_{\theta=0}^{\pi} \frac{a^4}{4} \, d\theta = \frac{\pi a^4}{2}$$

Example

14.4 #5

Area inside $r = 1$

and $r = 2\sin\theta \longleftrightarrow r^2 = 2r\sin\theta$

$x^2 + y^2 = 2y$

$x^2 + (y-1)^2 = 1$

$\leftarrow \theta = \frac{\pi}{6}$

$A = ??$

$$A = \left[\int_{\theta=0}^{\pi/6} \int_{r=0}^{r=2\sin\theta} r \, dr \, d\theta + \int_{\theta=\pi/6}^{x=\pi/2} \int_{r=0}^{r=1} r \, dr \, d\theta \right]$$

$x^3+y^3+z^3=5xyz$ Find tangent at $(2,1,1)$

↳ can't solve for var

Write as level surface

$f(x,y,z) = x^3+y^3+z^3-5xyz$

Normal is $\nabla F = (3x^2-5yz,\ 3y^2-5xz,\ 3z^2-5xy)$

@ $(2,1,1)$ $\nabla f(7,-7,-7) \rightarrow (1,-1,-1)$

$\nabla f \cdot (x-2, y-1, z-1) = 0$

$(1,-1,-1)\cdot(x-2, y-1, z-1) = 0$

$x-2-y+1-z+1=0$

$\boxed{x=y+z}$

Non-independent variables

(1) $w = x^3y - z^2t$

(2) $xy = zt$

Find $\left(\frac{\partial w}{\partial z}\right)_{x,y}$

x, y, z are independent

w, t are dependent

1) Differentials

(1) $dw = \underbrace{3x^2y\,dx + x^3dy}_{\text{Product rule}} - \underbrace{2zt\,dz - z^2dt}$

(2) $x\,dy + y\,dx = z\,dt + t\,dz$

Fixing x, y gets

(1) $dw = \cancel{3x^2y\,dx} + \cancel{x^3 dy} - 2zt\,dz - z^2 dt$

(2) $\cancel{x\,dy} + \cancel{y\,dx} = z\,dt + t\,dz$

(1) $dw = -2zt\,dz - z^2 dt$

(2) $0 = z\,dt + t\,dz$ $\quad z\,dt = -t\,dz$

Express dw in terms of dz

(1) $dw = -2zt\,dz - z(z\,dt)$

$= -2zt\,dz + zt\,dz$

$= -zt\,dz$

$$\left(\frac{\partial w}{\partial z}\right)_{x,y} = -zt = -xy$$

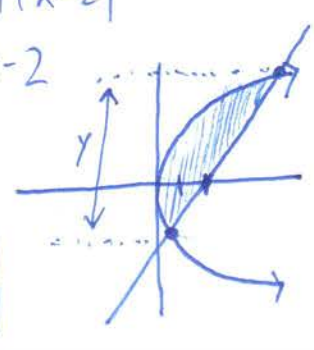
Express $\iint_R dx\,dy \qquad \iint_R dy\,dx$

where R is the finite region bounded by

$x = y^2$ and line through $(2,0)$ of slope 1

$y - 0 = 1(x-2)$

$y = x - 2$

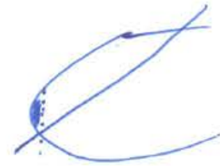


Intersection pts $x = y + 2 = y^2$

$y = 2 \to (4,2)$

$y = -1 \to (1,-1)$

$$\int_{-1}^{2}\int_{y^2}^{y+2} dx\,dy =$$



 + 

$$\int_{0}^{1}\int_{-\sqrt{x}}^{\sqrt{x}} dy\,dx + \int_{1}^{4}\int_{x-2}^{\sqrt{x}} dy\,dx$$

mmm

In Polar coordinates

- NUM

why

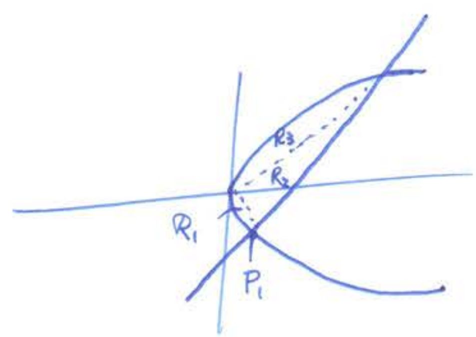
$$\begin{cases} y = x-2 \\ x = y^2 \end{cases}$$

$x = r\cos\theta$
$y = r\sin\theta$

line

$$r(\cos\theta - \sin\theta) = 2$$

$$r\cos\theta = r^2\sin^2\theta$$



$(0,0) \in R$

& bonded by $\begin{cases} r\sin^2\theta = \cos\theta \\ r(\cos\theta - \sin\theta) = 2 \end{cases}$

$P_1 = 1,-1$

The inside of

$$r \cdot \underbrace{\sin^2\theta}_{1-\cos^2\theta} \geq \cos\theta$$

$$1-\cos^2\theta + \cos\theta - r \leq 0$$

$$\frac{-1 \pm \sqrt{1-4r^2}}{2r}$$

.....

waaaaaa...

10/23/13 18.02 Recitation Fernando Trujano

Compute area and center of mass.

1. Region inside circle w/ center at origin and radius 2 and left of vertical line through (-1,0)

2. Circle of radius 1 & center at 0,1

3. Region inside cardioid $r = 1 - \cos\theta$ and outside circle of radius $\frac{3}{2}$ and center at origin

4. Finite region banded by x-axis, line $y=a$ and quarter of circle of radians a and center $(a,0)$

Change of variables

14.9

The Jacobian

$$J_T(u,v) = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

$$\iint_R F(x,y)\,dx\,dy = \iint_S F(T(u,v))\,|J_T(u,v)|\,du\,dv$$

Example

$x = r\cos\theta \quad y = r\sin\theta$

$$J_T = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta - -(r\sin^2\theta)$$

$$J_T = r(\cos^2\theta + \sin^2\theta)$$

$$\iint_R F(x,y)\,dx\,dy = F(T(u,v))\,|J_T(u,v)|\,dr\,d\theta$$

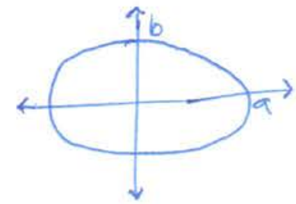
$$= F(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

18.02 OCW

Change of variables

Change of variables in double $\iint$
aka substitution

Example
Area of ellipse with semaxes a, b



$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

$$\iint_{\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 < 1} dx\,dy = \iint_{u^2+v^2<1}$$

set $\frac{x}{a} = u \rightarrow du = \frac{1}{a}dx$

$\frac{y}{b} = v \rightarrow dv = \frac{1}{b}dy$

$$du\,dv = \frac{1}{ab}dx\,dy$$

$$dx\,dy = ab\,du\,db$$

$$= \iint_{u^2+v^2<1} ab\,du\,dv = ab \overset{\text{area of unit disk}}{\iint_{u^2+v^2<1}} du\,dv$$

$$= ab\pi$$

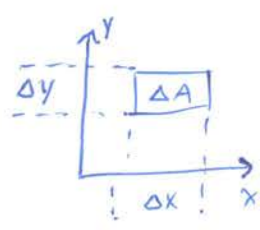
In general find the scaling factor dxdy vs dudv
↳ to simplify bounds

Example 2

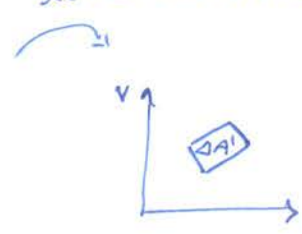
$u = 3x - 2y$

$v = x + y$

Understand relation between $\overset{dA}{dx\,dy}$ and $\overset{dA'}{du\,dv}$



see how they are related

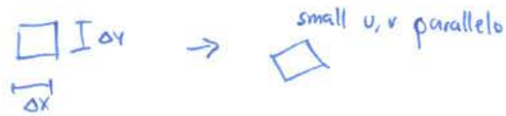


General case

$$u = u(x,y) \qquad \Delta u = u_x \Delta x + u_y \Delta y$$
$$v = v(x,y) \qquad \Delta v \approx v_x \Delta x + v_y \Delta y$$

scaling factor

$$\frac{\Delta u}{\Delta v} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \begin{vmatrix} \Delta x \\ \Delta y \end{vmatrix}$$



Jacobian: $J = \dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$

★ $\cancel{|J|}\, du\,dv = \cancel{|J|}\, dx\,dy\,|J|$ ← Jacobian in terms of u, v goes with x, y and viceversa

Example

Polar coordinates

$$x = r\cos\theta \qquad J_T = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$
$$y = r\sin\theta$$

$$r\cos^2\theta + r\sin^2\theta$$
$$= r$$

$$dx\,dy = |J|\,du\,du$$
$$dx\,dy = |J|\,dr\,d\theta$$
$$dx\,dy = r\,dr\,d\theta$$

inverses

$$\frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(u,v)} = 1$$

Compute whichever one we want, ie easiest

$$du\,dv = \frac{1}{J}\,dx\,dy$$

Example

compute $\int_0^1 \int_0^1 x^2 y \; dx\,dy$

by changing

$u = x$
$v = xy$

① Find Area Element ← easier to solve U_x than X_u so just put J on other side

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ y & x \end{vmatrix} = x$$

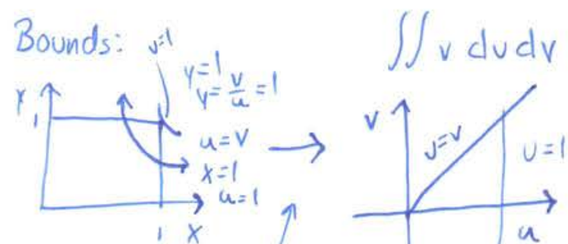
$$du\,dv = x\,dx\,dy$$

② Integrand in terms of u, v

$$x^2 y\,dx\,dy = \frac{1}{x}\,du\,dv = xy\,du\,dv$$
$$= v\,du\,dv$$

$$\iint_{??} v\,du\,dv \quad \text{or} \quad dv\,du\,?$$

③ Bounds:



switch to u, v

$$\int_0^1 \int_v^1 v\,du\,dv$$

└ u changes as v = constant
xy = constant

Go through each side and see what happens

$y = 1 \rightarrow$
$x = 0 \rightarrow$
$x = 1 \rightarrow$
$y = 0 \rightarrow$

10/24/13 18.02 Lecture Fernando Trejano

Integral in diff coords, Substitution and COV
3 dims and COV, Brutal Example

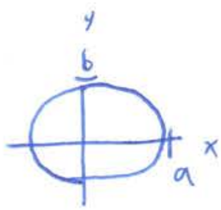
Integrals in BC

$$\int_0^1 e^{x^2} x\,dx \qquad u = x^2 \quad du = 2x\,dx$$

- change limits of integration

$$\int_0^1 e^{u} \tfrac{1}{2}\,du = \tfrac{1}{2}\left| e^{u} \right|_0^1 = \tfrac{1}{2}(e-1)$$

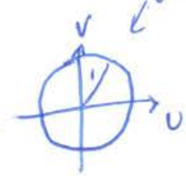
Ellipse



$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

Area = ??

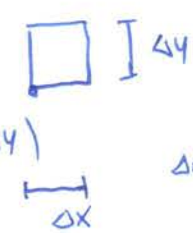
New vars $u = \frac{x}{a}$, $v = \frac{y}{b}$ $\Rightarrow$



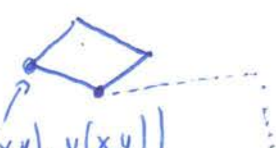
Change of variable Formula in 2 dimensions

$u = u(x,y)$
$v = v(x,y)$

xy space $\Rightarrow$ uv space



$\Delta A = \Delta x \Delta y$



$(u(x,y), v(x,y))$

$(u(x+\Delta x), y, v(x+\Delta x, y)$
is Linear Approx
$(u + \Delta x\, u_x, v + \Delta x\, v_x)$

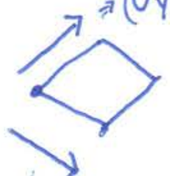
How?

Linear Approximation

Any fn $F(x,y)$

$$F(x+h, y) \approx F(x,y) + hF_x(x,y)$$

$(U_y, V_y)\,\Delta Y$



Subtract Point

$(U_x, V_x)\,\Delta X$

$\Delta A = \Delta X \Delta Y$

$(U_y, V_y)\Delta Y$

(U_x, V_x)

$\Delta A =$ Cross Product!!

$$\Delta A = \begin{vmatrix} i & j & k \\ U_x & V_x & 0 \\ U_y & V_y & 0 \end{vmatrix} = \underbrace{|U_x V_y - V_x U_y|}_{\text{The Jacobian } J} \Delta X \Delta Y$$

Now take limits

$$\Delta u \, \Delta v \approx |J| \, \Delta x \Delta y$$

u-v space area

★★

$$|J|\,dx\,dy = du\,dv$$

COV Formula ★

$$du\,dv = |J|\,dx\,dy$$

└ Change of Variables Formula

~ Back to the elipse problem

$u = \frac{x}{a} \quad v = \frac{y}{b}$

$J = (u_x v_x - u_y v_y) = \frac{1}{a}\frac{1}{b} - (0)(0) = \frac{1}{ab} \Rightarrow$ $dudv = \frac{1}{ab}\, dx\,dy$

$dx\,dy = ab\,du\,dv$

Area of ellipse $= \iint_E dx\,dy$

Change limits of integration $= \iint_{\text{circle}} ab\,du\,dv = ab$ Area (circle) $= \pi ab$

Polar Coordinates

$dx\,dy = r\,dr\,d\theta$

$x = r\cos\theta$
$y = r\sin\theta$

Jacobian

$$dx\,dy = dr\,d\theta \left| x_r y_\theta - x_\theta y_r \right|$$

$$= dr\,d\theta \left| \cos\theta(r\cos\theta) - (-r\sin\theta)(\sin\theta) \right|$$

$r\cos^2\theta + r\sin^2\theta = r(\cos^2\theta + \sin^2\theta)$, $\cos^2\theta + \sin^2\theta = 1$

$$= r\,dr\,d\theta$$

Change of Variables in 3 dimensions

$u = u(x,y,z)$
$v = v(x,y,z)$
$w = w(x,y,z)$

$$du\,dv\,dw = |J_T|\, dx\,dy\,dz$$

$$J_T = \det \begin{vmatrix} u_x & v_x & w_x \\ u_y & v_y & w_y \\ u_z & v_z & w_z \end{vmatrix}$$

$(u_z\ v_z\ w_z)\Delta z$
$(u_y, v_y, w_y)\Delta y$
$(u_x, v_x, w_x)\Delta x$
det

3/4

Brutal Example

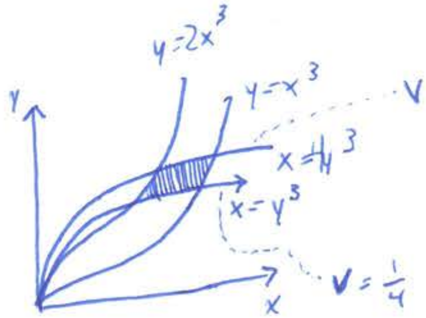
14.9 #11

$y = x^3$
$y = 2x^3$

$x = y^3$
$x = 4y^3$

1^{st} Quadrant

1) Draw

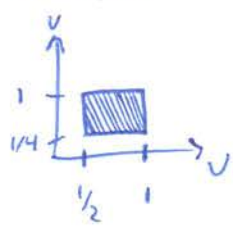


$x = \frac{4y^3}{x}$

$u = x^3/y$

$v = \frac{y^3}{x}$

In uv space



$$du\,dv = \left| u_x v_y - u_y v_x \right| dx\,dy$$

$$= \left| \frac{3x^2}{y} \frac{3y^2}{x} - \left(\frac{-x^3}{y^2}\right)\left(\frac{-y^3}{x^2}\right) \right|$$

$$= 8xy$$

Jacobian

$$= 8xy\,dx\,dy \qquad dx\,dy = \frac{du\,dv}{8xy}$$

$$\text{Area} = \int_{u=\frac{1}{2}}^{1} \int_{v=1/4}^{1} \frac{dv\,du}{8xy} = \frac{1}{8}\int_{\frac{1}{2}}^{1}\int_{1/4}^{1} \frac{dv\,du}{\sqrt{uv}}$$

$$\int_{1/4}^{1} \frac{dv}{\sqrt{v}} = 2\sqrt{v}\Big|_{1/4}^{1} = 2\left(1-\frac{1}{2}\right) = 1$$

$uv = x^2y^2$

$xy = \sqrt{uv}$

$$= \frac{1}{8}\int_{1/2}^{1} \frac{1}{\sqrt{u}}\,du = \frac{1}{4}\sqrt{u}\Big|_{1/2}^{1} = \frac{1}{4}\left(1-\frac{\sqrt{2}}{2}\right) = \frac{2-\sqrt{2}}{8}$$

Vector Fields

and Line integrals in plane

$\vec{F} = M\hat{i} + N\hat{j}$

M and N are functions of x and y

At each point (x,y), $\vec{F}$ a vector that depends on (x,y)

Examples:

velocity in a fluid

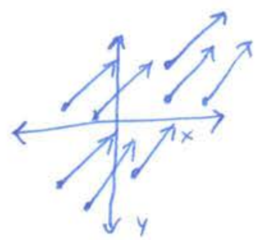
Force field $\vec{f}$

↳ force depends on the point

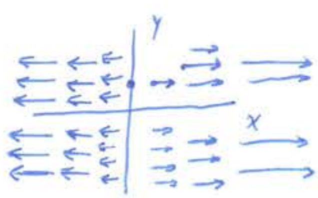
Drawing vector field

Example

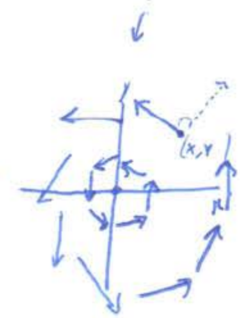
$\vec{F} = 2\hat{i} + \hat{j}$



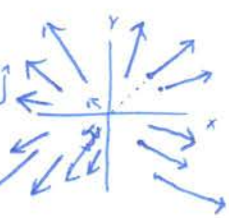
$\vec{F} = x\hat{i}$



$\vec{F} = -y\hat{i} + x\hat{j}$

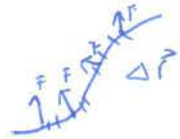


$\vec{F} = x\hat{i} + y\hat{j}$



Work done by vector field → Line Integral

$W = \vec{F} \cdot \Delta\vec{r}$



Along a trajectory C, work adds up to $W = \int_C \vec{F} \cdot d\vec{r}$

$W = \int_C \vec{F} \cdot dr$

$= \lim_{\Delta r_i \to 0} \sum_i \vec{F} \cdot \Delta \vec{r}_i = \lim_{\Delta t \to 0} \sum_i \vec{F} \cdot \left(\frac{\Delta \vec{r}}{\Delta t} \Delta t \right)$

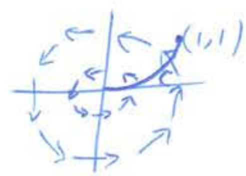
velocity vector $\frac{d\vec{r}}{dt}$

$$= \int_{t_1}^{t_2} \vec{F} \cdot \frac{dr}{dt}\, dt$$

Example

$\vec{F} = -y\hat{i} + x\hat{j}$

Find work



$C_1 \quad \begin{array}{l} x = t \\ y = t^2 \end{array} \quad 0 \le t \le 1$

$F = \langle -y, x \rangle$

$$\int_C F \cdot dr = \int_0^1 F \cdot \frac{dr}{dt}\, dt = \int_0^1 \langle -t^2, t \rangle \cdot \langle \overset{x'}{1}, \overset{y'}{2t} \rangle \, dt$$

$\frac{dx}{dt} = 1$

$\frac{dy}{dt} = 2t$

$$= \int_0^1 (-t^2 + 2t^2)\, dt$$

$$= \int_0^1 t^2 dt = \frac{1}{3}$$

Another way

$\vec{F} = \langle M, N \rangle$

$d\vec{r} = \langle dx, dy \rangle$

$\vec{F} \cdot d\vec{r} = M dx + N dy$

$$\int_C \vec{F} \cdot d\vec{r} = \int_C M dx + N dy$$

Express x, y in terms of single variable and substitute

Example (again)

$$\vec{F} = -y\,\hat{\imath} + x\,\hat{\jmath} \qquad \begin{matrix} x = t \\ y = t^2 \end{matrix}$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C -y\,dx + x\,dy$$

Express everything in terms of t

$$\begin{matrix} x = t & dx = dt \\ y = t^2 & dy = 2t\,dt \end{matrix}$$

$\therefore \frac{dy}{dt} = 2t$

$$= \int_C \underbrace{-t^2 dt + t\,2t\,dt}_{-t^2+2t^2=t^2}$$

$$= \int_0^1 t^2\,dt = \frac{1}{3}$$

Note:

$\int_C \vec{F}\cdot d\vec{r}$ depends on trayectory C

but not on parametrization

Vector Field, Force and Velocity Fields, Line Integrals

What is a vector field

$f(x,y)$

$\nabla f(x,y) = (f_x, f_y)$

at each point (x,y) we get a vector => Vector Field

General Vector Field

$$v(x,y) = (v_1(x,y), v_2(x,y))$$

Example

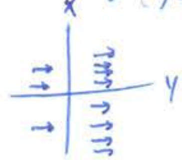
$$(x^2, x+y)$$

How to describe a Vector Field

- Can't graph in $\mathbb{R}^4$

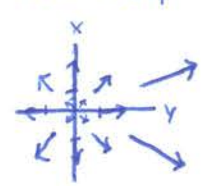
Instead: Pictures with arrows

Example $V = (1,0)$



Radial VF

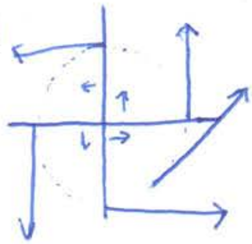
$V(x,y) = (x,y)$



Rotational

$v(x,y) = (-y, x)$

Tangent to circle

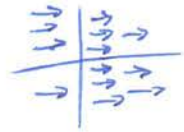


Force + Velocity Fields

$v(x,y)$ = velocity at (x,y)

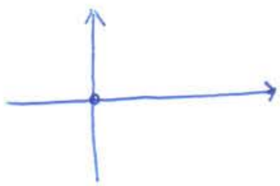
Example

velocity of current in a river



Force Field

$v(x,y)$ = Force at (x,y)



$$v(x,y) = \left(\frac{-x}{(x^2+y^2)^{3/2}}, \frac{y}{(x^2+y^2)^{3/2}} \right)$$

attractive force. Points toward origin

$$|v| = \frac{1}{r^2}$$

$$\rightarrow |v|^2 = \frac{x^2+y^2}{(x^2+y^2)^3} = \frac{1}{(x^2+y^2)^2} = \frac{1}{r^4}$$

Line Integrals

Curve C $\begin{cases} x(t) \\ y(t) \end{cases}$ $a \le t \le b$

Parametrized

Example

$C =$  $x(t) = \cos t$, $y(t) = \sin(t)$, $a = 0$, $b = \frac{\pi}{2}$

$f(x,y)$ = height of fence at (x,y)

$$\text{Area} = \int_C f = ??$$

Integrating w/ respect to arclength

$$\int_C f = \int_{t=a}^{t=b} f(x(t), y(t))\, ds$$

$$ds = \sqrt{(x')^2 + (y')^2}\, dt$$

$f(x,y) = x$

$x' = -\sin t$

$y' = \cos t$

$$\int_C f = \int_0^{\pi/2} \cos t \sqrt{\cos^2 t + \sin^2 t}\, dt = \int_{t=0}^{t=\pi/2} \cos t\, dt$$

$= \sin t \Big|_0^{\pi/2} = 1$

Second type of line integrals

$\int_C f\, dx$

$x(t) = \cos(t)$
$y(t) = \sin(t)$

~~Integrating~~
Integrating dx $\cancel{x'(t)dt}$ $x'(t)dt$

$dx = -\sin t\, dt$

$f = \cos t$

$-\int_0^{\pi/2} \cos t \sin t\, dt$

$\cancel{u = \sin t}$
$\cancel{du = \cos t\, dt}$

$du = -\sin t\, dt$
$u = \cos t$

$= \int_1^0 u\, du$

$\frac{1}{2} u^2 \Big|_1^0$

$= -\frac{1}{2}$

15.2 #6

$f(x,y) = xy$

Curve C $\begin{cases} x = 3t \\ y = t^4 \end{cases}$ $0 \le t \le 1$

$x' = 3$
$y' = 4t^3$

$\int f_C\, ds$

$ds = \sqrt{9 + 16t^6}\, dt$

$\int_0^1 3t^5 \sqrt{9+16t^6}\, dt$

$u = 9 + 16t^6$
$du = 96t^5\, dt$

$= \int_9^{25} \sqrt{u}\, \frac{du}{32} = \frac{2}{3 \cdot 32} u^{3/2} \Big|_9^{25} = \frac{1}{48}\, 125 - 27 = \frac{49}{24}$

$$\int_C f\,dy = \int_0^1 3t^5 (4t^3)\,dt$$
$$= 12\int_0^1 t^8\,dt$$
$$\frac{12}{9} t^9 \Big|_0^1 = \frac{4}{3}$$

Work:

$F =$ Force Field $\qquad F = (u, v)$

Move along curve C $\quad \vec{r}(t)$

$$\text{work} = \int_C \vec{F} \cdot \vec{r}\,'(t)\,dt$$

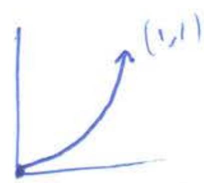
$$\vec{r}\,'(t) = (x'(t), y'(t))$$

$$\int_C u \underbrace{x'(t)\,dt}_{dx} + \int_C v \underbrace{y'(t)\,dt}_{dy}$$

$F = (-y, x)$

$C = \begin{cases} x = t \\ y = t^2 \end{cases} \quad 0 \le t \le 1$

How much work?



$x' = 1$
$y' = 2t$

$$\text{Work} = \int_0^1 (-t^2, t)\cdot(1, 2t)\,dt = \int_0^1 t^2\,dt = \frac{1}{3}t^3 \Big|_0^1 = \frac{1}{3}$$

11/6/13

18.02 ocw

Gradient Fields and Potential Functions

if $\vec{F} = \nabla F$ gradient field

then $\int_C \vec{F} \cdot d\vec{r} = F(P_1) - F(P_0)$

not dependent on path

conservative:

if closed C $\int_C \vec{F} \cdot d\vec{r} = 0$

Testing whether $\vec{F} = \langle M, N \rangle$ is gradient field

If $\vec{F} = \nabla f$

$M = f_x$ then $f_{xy} = f_{yx}$ Equality of mixed partials

$N = f_y$

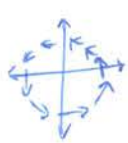
★ $M_y = N_x$

Conversly if $\vec{F} = \langle M, N \rangle$ defined, diferentiable everywhere and $M_y = N_x$

then $\vec{F}$ is a gradient field

Example

$\vec{F} = \underbrace{-y\hat{\imath}}_{M} + \underbrace{x\hat{\jmath}}_{N}$



$\left.\begin{array}{l} \dfrac{\partial M}{\partial y} = -1 \\ \dfrac{\partial N}{\partial x} = 1 \end{array}\right\} \neq$

$\therefore$ Not a Gradient field

Example

$\vec{F} = \overbrace{(4x^2 + axy)}^{M}\hat{\imath} + \overbrace{(3y^2 + 4x^2)}^{N}\hat{\jmath}$

For which a would $\vec{F} = \nabla F$

$M_y = ax$

$N_x = 8x$ $a = 8$

Potential

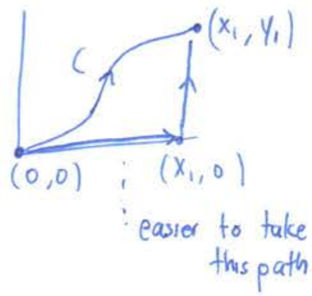
$$\vec{F} = (4x^2 + 8xy)\,\hat{\imath} + (3y^2 + 4x^2)\,\hat{\jmath}$$

Potential: what function $f(x,y)$ has $\vec{F}$ as Gradient

↳ can only do if $N_x = M_y$

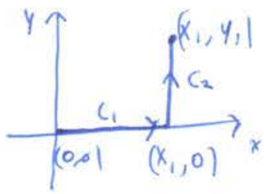
Method

1) Compute Line Integrals



$$\int_C \vec{F} \cdot d\vec{r} = f(x_1, y_1) - f(0,0)$$

$$f(x_1, y_1) = \int_C \vec{F} \cdot d\vec{r} + \underbrace{f(0,0)}_{\text{constant}}$$



$$\int_C \vec{F} \cdot d\vec{r} = \int_C 4x^2 + 8xy\,dx + 3y^2 + 4x^2\,dy$$

On C_1

x from 0 to x_1

$y = 0 \quad dy = 0$

$$\Rightarrow \int_{C_1} \vec{F} \cdot d\vec{r} = \int_0^{x_1} 4x^2\,dx = \frac{4}{3}x_1^3$$

On C_2

y from 0 y_1

$x = x_1 \quad dx = 0$

constant

$$\Rightarrow \int_{C_2} \vec{F} \cdot d\vec{r} = \int_0^{y_1} 3y^2 + 4x_1^2\,dy = y^3 + 4x_1^2 y\Big|_0^{y_1} = y_1^3 + 4x_1^2 y_1$$

So $f(x_1, y_1) = \frac{4}{3}x_1^3 + y_1^3 + 4x_1^2 y_1 + C$

$$f(x,y) = \frac{4}{3}x^3 + y^3 + 4x^2 y + C$$

$$\nabla f = (4x^2 + 8xy,\ 3y^2 + 4x^2)$$

2/14

Method
2) Anti-derivatives

Want to solve $\begin{cases} f_x = 4x^2 + 8xy & (1) \\ f_y = 3y^2 + 4x^2 & (2) \end{cases}$

constant that depends on y

$(1) \Rightarrow f = \frac{4}{3}x^3 + 4x^2y + g(y) \quad (3)$

integrate w/ respect to x

$\cancel{(2)} \Rightarrow \cancel{y^3 + 4x^2y + g(x)}$

$f_y = 4x^2 + g'(y)$

match w/ equation (2)

$4x^2 + g'(y) = 4x^2 + 3y^2$

$g'(y) = 3y^2$

Regular constant

$g(y) = y^3 + C$

(3)

$f = \frac{4}{3}x^3 + 4x^2y + y^3 + C$

Recap

$\vec{F} = \langle M, N \rangle$ is a

gradient field in a region of the plane

$\Updownarrow$

conservative

$\oint_C \vec{F} \cdot d\vec{r} = 0$

closed curve

if F is defined in entire plane.

$\Uparrow \quad \Downarrow$

$N_x = M_y$

curl(F) = 0

Curl

Definition $\quad \operatorname{curl}(\vec{F}) = \overset{P}{N}_x - \overset{Q}{M}_y$

test for conservativeness $\quad \operatorname{curl}(\vec{F}) = 0$

For a velocity field

Curl measures rotation component of motion

 $F = \langle a, b \rangle$ constant $\quad$ curl $= 0$: no rotation

$F = \langle -y, x \rangle$

$$\operatorname{curl} F = \frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y) = 2$$

curl measures (2x) angular velocity of rotation component of velocity field

10/31/13 — 18.02 Lecture — Fernando Trujano

- Review Line Integrals
- Path Independence
- Path ind $\Rightarrow \nabla$ v/F
- Curl + Potential Functions

Review Line Integrals

Curve C — From A to B

Parametrized $\vec{r}(t)$

Vector Field $\vec{v}$

Line Integral

$$\int_C \vec{v} \cdot d\vec{r} = \int_C \vec{v} \cdot r'(t)\, dt$$

Anti-derivative — not used

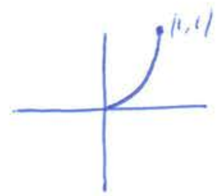
FTC for Line Integrals:

$$\int_C \nabla F \cdot d\vec{r} = F(B) - F(A)$$ ← line integral is the antiderivative

Example

$\vec{v} = (x, -y)$

$f = \frac{1}{2}x^2 - \frac{1}{2}y^2$



$C \begin{cases} x(t) = t \\ y(t) = t^3 \end{cases}$ $0 \le t \le 1$

$$\int_C \vec{v} \cdot dr = \int_0^1 \underbrace{(t, -t^3)}_{\text{vector field along curve.}} \cdot \underbrace{(1, 3t^2)}_{\vec{r}'(t)}\, dt$$

$$\int_C \vec{v} \cdot dr = \int_0^1 t - 3t^5\, dt = \frac{1}{2}t^2 - \frac{1}{2}t^6 \Big|_0^1 = 0$$

Kinds of line integrals

1) $\int_C f \, ds$ ← arclength

Function

$ds = \sqrt{x'(t)^2 + (y')^2} \; dt$

Gives you
* Area

2) $\int_C \vec{V} \cdot d\vec{r}$

$\vec{V} = (P, Q)$

Function

$= \int (P, Q) \cdot (x', y') \, dt$

$\int P \underbrace{x' dt}_{dx} + \int Q \underbrace{y' dt}_{dy}$

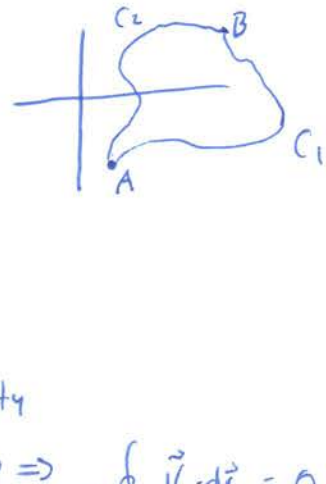
Gives you
* Antiderivative

Path Independence

A line integral of $\vec{V}$ is path independent if the value is the same for any

C_1 or C_2 for any A and B

two curves



I_P

$$\int_{C_1} \vec{V} \cdot d\vec{r} = \int_{C_2} \vec{V} \cdot dr$$

is Path Independent if true

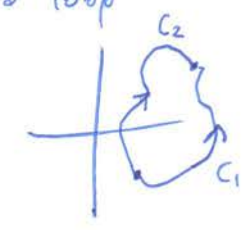
FTC

$\vec{V} = \nabla f \Rightarrow$ Path independence

Closed Loop Property

Path Indep $\Rightarrow \oint_C \vec{V} \cdot d\vec{r} = 0$ any closed loop

around closed loop



$C =$ Follow $C_1 \to$ go backwards on C_2

$\underbrace{C = C_1 - C_2}_{\text{Closed!}}$

$$\oint_{C_1 - C_2} \vec{V} \cdot d\vec{r} = \int_{C_1} \vec{V} \cdot dr - \int_{C_2} \vec{V} \cdot dr = 0$$ ← By path independence

Comes from the opposite orientation
ie -> going back

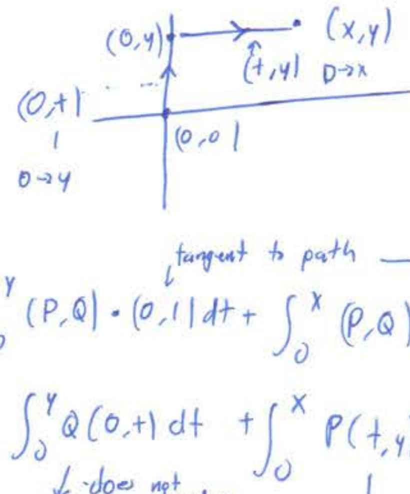
Path independence $\Rightarrow$ V is a gradient vector field

- Find the potential function such that $\vec{V} = \nabla F$

Define $f(x,y) = \int_{(0,0)}^{(x,y)} \vec{V} \cdot dr$ $\quad$ $\vec{r}$ connects $(0,0)$ to (x,y)
curve

Must show that $\nabla f = \vec{V}$
$= (P, Q)$ → suppose

Let's show that $f_x = P$ choose easy path!



tangent to path

$$f(x,y) = \int_0^y (P,Q) \cdot (0,1)\, dt + \int_0^x (P,Q) \cdot (1,0)\, dt$$

$$f(x,y) = \int_0^y Q(0,t)\, dt + \int_0^x P(t,y)\, dt$$

does not depend on x

$$f_x = 0 \quad + \quad f_x = P(x,y)$$

$$0 + P(x,y) = P(x,y)$$

Curl

$\vec{V} = (P, Q)$

$\mathrm{curl}(\vec{V}) \overset{def}{=} Q_x - P_y$ ← gradient

when $\vec{V} = (f_x, f_y)$

then $\mathrm{curl}\, V = (f_{yx} - f_{xy}) = 0$ ← Equality of mixed partials $\Rightarrow$ Curl is 0 on any gradient

Theorem

$\text{curl}(\vec{v}) = 0$ on R^2

on any point

$\vec{V}$ is a gradient

angular motion of vector field

Ex

$$\vec{v} = \left(4x^2 + 8xy \,,\, 3y^2 + 4x^2\right)$$

$$\text{curl}(\vec{v}) = 8x - (8x) = 0$$

$\Rightarrow \vec{v}$ is a gradient

$$\vec{v} = \Delta f$$

11/6/13 18.02 OCW

Green's Theorem

$\text{curl}(\vec{F}) = N_x - M_y$

$\vec{F} = \langle M, N \rangle$

compute line integral on closed curve when curl $\neq 0$

$$\oint_C \vec{F} \cdot d\vec{r} = ?$$

Green's Theorem

If C counter clock wise is closed curve

if clock wise → take negative

ONLY FOR closed curve!

enclosing region R

$\vec{F}$ vector Field defined and differentiable in R

then

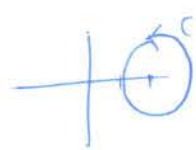
$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R \text{curl}\, \vec{F} \, dA$$

$$\oint_C M\,dx + N\,dy = \iint_R (N_x - M_y)\, dA$$

Line integral → double integral

Example

Let C = circle of $r = 1$ at $(2, 0)$ ccw



$$\oint_C \underbrace{y e^{-x}}_{M} dx + \underbrace{\left(\tfrac{1}{2}x^2 - e^{-x}\right)}_{N} dy$$

Green's Theorem → double integral

$$\iint_R \underbrace{\text{curl}\, \vec{F}}_{N_x - M_y} dA = \iint_R (x + e^{-x}) - e^{-x} \, dA$$

$$= \iint_R x \, dA$$

Shortcut!

$$\iint_R x\,dA = \underbrace{Area(R)}_{\pi}\ \underbrace{\bar{x}}_{2} = 2\pi$$ ← avg value x

$\bar{x} = \frac{1}{A}\iint x\,dA$

$\frac{1}{Mass}\iint x\,\delta\,dA$

Special Case

If curl $\vec{F} = 0$

then

$\vec{F}$ conservative

Proof - if curl $\vec{F} = 0$ then $\oint_C \vec{F}\cdot d\vec{r} = 0$

Green's $\oint_C \vec{F}\cdot d\vec{r} = \iint_R \text{curl } F\,dA$

if curl $F = 0$

$= 0$

Proof of Green's Theorem

$$\oint_C M\,dx + N\,dy = \iint_R (N_x - M_y)\,dA$$

... is probably really complicated

1) Finding the potential
2) Green's Theorem
3) Explanation for $\circlearrowleft$
4) Example

Finding the Potential

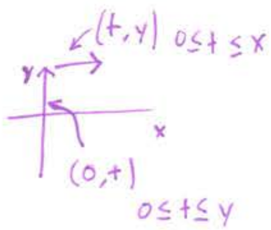
$$\vec{V} = \left(4x^2 + 8xy \, , \, 3y^2 + 4x^2\right)$$

$$\text{curl } \vec{V} = Q_x - P_y = 8x - 8x = 0 \implies V \text{ is gradient vector field}$$

$\vec{V} = \nabla f$ ← some f ↳ The Potential

Method 1: Line Integral

$$f(x,y) = \int_{(0,0)}^{(x,y)} \vec{V} \cdot d\vec{r}$$



$$= \int_{t=0}^{y} \underbrace{(0, 3t^2) \cdot (0,1)\, dt}_{(0,t)} + \int_{t=0}^{x} \underbrace{(4t^2 + 8ty \, , \, 3y^2 + 4t^2) \cdot (1,0)\, dt}_{(t,y)}$$

$$\int_0^y 3t^2 \, dt + \int_0^x 4t^2 + 8ty \, dt$$

$$\implies f(x,y) = t^3 \Big|_0^y + \frac{4}{3}t^3 + 4t^2 y \Big|_0^x$$

$$= y^3 + \frac{4}{3}x^3 + 4x^2 y$$

↖ Potential

Double check

$f_x = 4x^2 + 8xy$
$f_y = 3y^2 + 4x^2$ } matches original $\vec{V}$ ✓

Method 2

$f_x = 4x^2 + 8xy$ ... something that depends only on y

$f = \frac{4}{3}x^3 + 4x^2y + g(y)$

integrate w/ respect to x

$f_y = 4x^2 + g'(y)$

↳ supposed to be ↓ given

$= 4x^2 + 3y^2 \Rightarrow$

$g'(y) = 3y^2$

$g(y) = y^3$

$f = \frac{4}{3}x^3 + 4x^2y + y^3$ ✓

Green's Theorem

↳ Relates integral on boundary to integral on inside

Positively Oriented (ccw)

C closed curve

enclosed region R

Greens Theorem

C encloses R positively, $\vec{v}$ v.f

$$\Rightarrow \oint_C \vec{v} \cdot d\vec{r} = \iint_R \text{curl}(\vec{v})\, dA$$

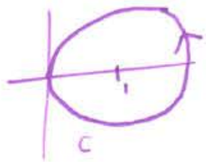
Corollary: $\text{curl}(\vec{v}) = 0$ on $\mathbb{R}^2$

$\Rightarrow \vec{v} = \Delta F$

$\text{curl}(\vec{v}) = 0$

and Green $\Rightarrow$ Path Indep $\Rightarrow \nabla$ V.F

Example:



C

$$\oint_C x\,dy \qquad \vec{v} = (0, x)$$

$$\vec{v} = (P, Q)$$
$$\Downarrow$$
$$\int_C \vec{v} \cdot d\vec{r} = \int_C P\,dx + \int Q\,dy$$

$x'\,dt = dx$
$y'\,dt = dy$

$$\oint_C x\,dy = \oint_C x\,y'(t)\,dt$$
$$= \oint (?, ?) \cdot (x', y')\,dt$$

$(0, x)$

$\vec{v} = (0, x)$

$P = 0$
$Q = x$

$\mathrm{curl}(v) = Q_x - P_y$
$= 1$

$x = 1 + \cos t$
$y = \sin t$

How?

$\mathrm{curl}(\vec{v}) = 1$

$$\oint_C x\,dy = \iint_R 1\,dA = \pi$$

Green's theorem

Without Green

$$\oint_C x\,dy = \int_0^{2\pi} (1 + \cos t)(\underbrace{\cos t}_{y'})\,dt = \int_0^{2\pi} \cos^2 t\,dt = \pi$$

(any closed curve (pos. oriented)

$$\oint_C x\,dy = \text{Area}(R) \qquad \int_C \tfrac{1}{2} x\,dy - \tfrac{1}{2} y\,dx = \text{Area}(R)$$

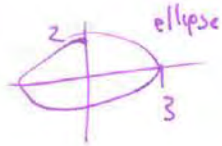
Also $\oint_C -y\,dx = \text{Area}(R)$

$\vec{v} = (-y, 0)$

Some other stuff

$$\int_C \vec{v}\,d\vec{r} = \int_C (P, Q) \cdot (x', y')\,dt = \int P\underbrace{x'\,dt}_{dx} + \int_C Q\underbrace{y'\,dt}_{dy}$$

Example 15.4 # 9



$$\oint_C \underbrace{y^2}_{P} dx + \underbrace{xy}_{Q} dy$$

$$\vec{V} = (y^2, xy)$$

$$\text{curl}(\vec{V}) = (xy)_x - (y^2)_y = y - 2y = -y$$

$$\oint_C y^2 dx + xy dx \underset{\text{Greens}}{=} \iint_R (-y) dA$$

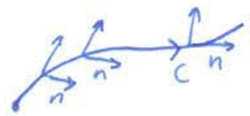
Flux
↳ Another kind of line integral

C plane curve
$\vec{F}$ vector field

Flux of $\vec{F}$ accross C =

$$\star \quad \int_C \vec{F} \cdot \hat{n} \, ds$$

unit normal to curve



$\hat{n}$ = unit normal to curve C
90° clockwise from $\hat{T}$

If break C into small pieces Δs

$$\text{Flux} = \lim_{\Delta s \to 0} \left(\sum \vec{F} \cdot \Delta s \right)$$

Recall
work $\int_C F \cdot d\vec{r} = \int_C F \cdot \hat{T} \, ds$

summing the <u>tangental</u> componen- of $\vec{F}$

$$\text{Flux} = \int_C \vec{F} \cdot \hat{n} \, ds$$

summing the <u>normal</u> component of $\vec{F}$

Interpretation
For $\vec{F}$ a velocity field

Flux measures how much fluid passes through C per unit time

Area = base * height
normal of F

$= \Delta S \, (\vec{F} \cdot n)$

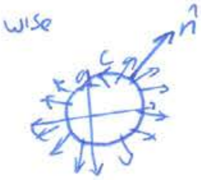
What flows accross C $\overrightarrow{\text{left}}$ positively
$\leftarrow$ negatively

Example

C = circle of radius a at origin

Counter clock wise

$\vec{F} = x\hat{\imath} + y\hat{\jmath}$



Along C $\vec{F}$ is parallel to $\hat{n}$

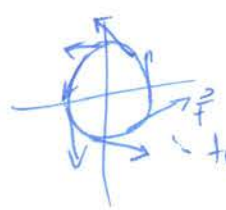
$$\vec{F}\cdot\hat{n} = |\vec{F}|\,\underbrace{|\hat{n}|}_{1} = a$$

$$\int_C \vec{F}\cdot\hat{n}\,ds = \int_C \underbrace{a}_{\text{constant}}\,ds = a \overbrace{\text{length } C}^{2\pi a} = 2\pi a^2$$

Example

Same C

$\vec{F} = \langle -y, x\rangle$



tangent to C

$$\vec{F}\cdot\vec{n} = 0$$

so Flux $= 0$

Calculation using components

Remember $d\vec{r} = \vec{T}\,ds = \langle dx, dy\rangle$

$\hat{n}$ is $\vec{T}$ rotated $90°$ clock wis

$$\hat{n}\,ds = \langle dy, -dx\rangle$$

★

So if $\vec{F} = \langle P, Q\rangle$ then

$$\int_C \vec{F}\cdot\hat{n}\,ds = \int_C \langle P, Q\rangle \cdot \langle dy, -dx\rangle$$

★ $$= \int_C -Q\,dx + P\,dy$$

x and y are related → express everything in terms of a single variable. then solve

Green's Theorem for flux
in normal form

If C encloses a region R ccw

and $\vec{F}$ defined and differentiable in R

$\vec{F} = \langle P, Q \rangle$

then $\oint_C \vec{F} \cdot \hat{n} \, ds = \iint_R \underset{\substack{\uparrow \\ \text{divergence of } F}}{\operatorname{div}} \vec{F} \, dA$

★

Divergence

$$\operatorname{div} \langle P, Q \rangle = P_x + Q_y$$

Interpretation of div $\vec{F}$

1) Measures how much the flow is expanding
2) Source rate = amount of fluid added to the system

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1) Green Review
2) Flux
3) $\vec{T}\,ds$, (dx, dy), $d\vec{r}$ etc
4) Greens Theorem for Flux

1) Green Review

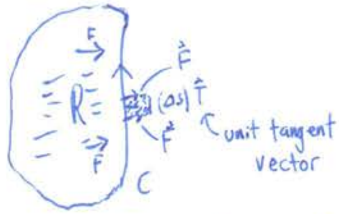
$\vec{F} = (P, Q)$

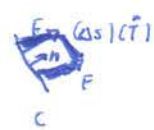
$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R \operatorname{curl}(\vec{F})\, dA$$

$$\operatorname{curl}(\vec{F}) = Q_x - P_y$$

2) Flux

A measure of flow accross boundry





ΔFlow = Area (Parallelo

$= \Delta S \cdot (\vec{F} \cdot \vec{n})$

base (ΔS); height ($\vec{F} \cdot \vec{n}$); unit normal to curve ($\vec{n}$)

$x(t) = \cos t$
$y(t) = \sin t$
$(x', y') = (-\sin t, \cos t)$
$= -y, x$
$\hat{n} = (x, y)$ or $(-x, -y)$

Total Flow

$$= \oint \vec{F} \cdot \hat{n}\, ds$$

New Line Integral

1) If $\vec{F} \perp \vec{n} \rightarrow$ Flux $= 0$

C = unit circle
$\vec{F}$ = rotation vector field $= (-y, x)$

2)

C = unit circle
$\vec{F}$ = radial VF $= (x, y)$
$\rightarrow \vec{F} = \hat{n}$ along C

Total Flux $= \int 1 = 2\pi > 0$

3) Line Integrals $\vec{T}ds$, (dx, dy), $d\vec{r}$ etc

$\vec{r}(t) = (x(t), y(t))$ ← Parametrized curve C

$d\vec{r} = \vec{r}\,'(t)\,dt = (x'(t), y'(t))\,dt$

$dx = x'(t)\,dt$
$dy = y'(t)\,dt$

$\Downarrow$

$d\vec{r} = (dx, dy)$

$\vec{T}$ = unit tangent

$= \dfrac{(x'(t), y'(t))}{\sqrt{x'(t)^2 + y'(t)^2}}$

$ds = \sqrt{x'^2(t) + y'^2(t)}\,dt$

$\Rightarrow \vec{T}ds = (x'(t), y'(t))\,dt$

$= d\vec{r}$

$= (dx, dy)$

* All the same thing

$\vec{F} = (P, Q)$

$\vec{F} \cdot d\vec{r} = P\,dx + Q\,dy$

$\vec{F} = (dx, dy)$

$= \vec{F} \cdot \vec{T}\,ds$

4) Green's Theorem for Flux

$\oint_C \underbrace{\vec{F} \cdot \hat{n}\,ds}$

Re-write so Green's Theorem applies

- write $\hat{n}$ in terms of t x', y'

unit normal

A: Rotate tangent vector by 90°

$\vec{n} = \dfrac{y'(t),\ -x'(t)}{\sqrt{x'(t)^2 + y'(t)^2}}$

2/14

$$\oint_C \vec{F}\cdot\vec{n}\,ds$$

$$\vec{F}=(P,Q)$$

$$\vec{F}\cdot\vec{n}\,ds = \frac{(P,Q)\cdot(y',-x')}{\sqrt{(x')^2+(y')^2}}\underbrace{\sqrt{(x')^2+(y')^2}\,dt}_{ds}$$

$$= P y'\,dt - Q x'\,dt$$

★ $$= P\,dy - Q\,dx$$

$$= (-Q,P)\cdot\underbrace{\vec{T}\,ds}_{(dx,dy)}$$

→ Greens Theorem for Flux

$$\oint_C \vec{F}\cdot\vec{n}\,ds = \oint (-Q,P)\cdot d\vec{r} \underset{\text{Greens Theorem}}{=} \iint_R \operatorname{curl}(-Q,P)\,dA$$

$$= \iint_R (P_x+Q_y)\,dA$$

Example

C = unit circle

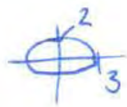
$F=(x,y)$

$$2\pi \underset{\text{did this already}}{=} \oint_C \vec{F}\cdot\vec{n}\,ds = \iint_R (x_x+y_y)\,dA = 2\overbrace{\text{Area}(R)}^{\pi} = 2\pi$$

nabla

Example

$\vec{F} = (y, x)$

$C =$ ellipse 

$\oint_C \vec{F} \cdot \vec{n}\, ds = \iint_R 0\, dA = 0$

Bounded Integral = Double Integral

↑

Greens Theorem

$\vec{F} = (P, Q)$

Divergence of $\vec{F} = P_x + Q_y$

$\parallel$

$\mathrm{div}(\vec{F})$

$\parallel$

$\vec{\nabla} \cdot \vec{F} = (\partial_x, \partial_y) \cdot (P, Q)$

4/4

11/8/13

18.02 Lecture
Midterm 3 Review

1) Double Integrals
2) Polar Integrals
3) Line Integrals
4) Green's Theorem
5) Flux
6) Change of variables

1) Double Integrals

$$\int_{x=0}^{1}\int_{y=0}^{x^2} x\,dy\,dx$$

- Solve inner integral first
- Plug into outer and solve

Inner: $\int_{y=0}^{x^2} x\,dy = xy\Big|_{y=0}^{x^2} = x^3$

Outer: $\int_{x=0}^{1} x^3\,dx = \frac{1}{4}x^4\Big|_0^1 = 1/4$

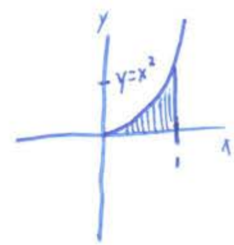
Change the order of integration

$$\int_{x=0}^{1}\int_{y=0}^{x^2} x\,dy\,dx$$

$\vdots$

$$\_\_\_\ \_\ dx\,dy$$

1) Draw Picture



y also $0 \to 1$

x $\sqrt{y} \to 1$

$$\int_{y=0}^{1}\int_{x=\sqrt{y}}^{1} x\,dx\,dy$$

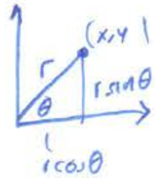
Inner: $\int_{x=\sqrt{y}}^{1} x\,dx = \frac{1}{2}x^2\Big|_{\sqrt{y}}^{1} = \frac{1}{2} - \frac{y}{2}$

Outer: $\int_{y=0}^{1} \frac{1}{2} - \frac{y}{2}\,dy = \frac{y}{2} - \frac{y^2}{4}\Big|_0^1 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

2) Polar Integrals

Polar coordinates

$x = r\cos\theta$
$y = r\sin\theta$



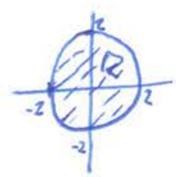
★ $$dA = dx\,dy = r\,dr\,d\theta$$

Example.

Metal plate $\begin{smallmatrix}\text{circle}\\ \text{radius}\end{smallmatrix} = 2$

density $\delta = 3\sqrt{x^2+y^2} + 2$

Find total mass. $\cdots\;\; r^2 = x^2 + y^2$



$$\iint_R 3\sqrt{x^2+y^2} + 2 \; dx\,dy$$

. Easier to solve in polar

$$M = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{2} (3r+2)\, \underbrace{r\,dr\,d\theta}_{dA}$$

Inner: $\displaystyle\int_{r=0}^{2} 3r + 2r\,dr = r^3 + r^2 \Big|_0^2 = 12$

Outer: $\displaystyle\int_{\theta=0}^{2\pi} 12\,d\theta = 24\,d\theta$

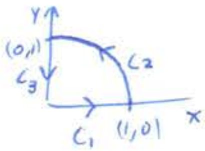
3) Line Integrals

$\vec{F} = -x\hat{\imath} + y\hat{\jmath} = \langle -x, y \rangle$

t goes from 1 to 0

when y=1 t=0

when y=0 t=1

C 

Parametrize curves

C_1 $(t, 0)$ $\quad 0 \le t \le 1$

C_2 $(\cos t, \sin t)$ $\quad 0 \le t \le \frac{\pi}{2}$

C_3 $(0, 1-t)$ $\quad 0 \le t \le 1$

$\oint_C \vec{F} \cdot d\vec{r}$

Recall

$d\vec{r} = (dx, dy) = \vec{T}\,ds$

unit tangent

Solving line integral directly

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_0^1 \underset{(-x,\,y)}{(-t, 0)} \cdot \overset{(dx,\,dy)}{(1,0)}\,dt = \frac{-t^2}{2}\Big|_0^1 = -\frac{1}{2}$$

$$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_0^{\pi/2} (-\cos t, \sin t) \cdot (-\sin t, \cos t)\,dt = 2\int_0^{\pi/2} \underbrace{\sin t}_{u}\,\overbrace{\cos t\,dt}^{du}$$

$$= \sin^2 t \Big|_0^{\pi/2} = 1$$

$$\int_{C_3} F \cdot d\vec{r} = \int_0^1 (0, \underset{t-1}{1-t}) \cdot (0, -1)\,dt = \frac{1}{2}t^2 - t \Big|_0^1 = -\frac{1}{2}$$

Add all three together

$= 0$

4) Green's Theorem

★ $$\oint \vec{F} \cdot d\vec{r} = \iint_R \text{curl}(\vec{F})\, dA$$

$$\text{curl}(P, Q) = Q_x - P_y$$

Same example as 3)

$P = -x$
$Q = y$ $\quad \text{curl}(\vec{F}) = (y)|_x - (-x)|_y = 0$

$$\oint \vec{F} \cdot dr = \iint_R 0\, dA = 0 \checkmark$$

Conservative Vector Field

$$\vec{F} = \nabla \underbrace{F}_{\text{potential}}$$

$\vec{F}$ conservative if $\text{curl}(\vec{F}) = 0$ everywhere
or on a simply connected region

$$(-x, y) = \nabla\left(\tfrac{1}{2}y^2 - \tfrac{1}{2}x^2\right)$$

5) Flux

$\vec{F} = (P, Q)$

Curve C

unit normal $\hat{n}$ (outward)

Positively oriented

arc length

★ Total Flux $= \oint_C \vec{F} \cdot \vec{n}\, ds = \oint_C \vec{F} \cdot (dy, -dx) = \oint_C P\,dy - Q\,dx$

$\hat{n}\,ds = (dy, -dx)$ ---- Proof

$ds = \sqrt{(x')^2 + (y')^2}\, dt$

$\vec{T} = \dfrac{(x', y')}{\sqrt{x'^2 + y'^2}}$ ← tangent, make it unit $\ell = 1$ $\Rightarrow \vec{n} = \dfrac{(y', -x')}{\sqrt{x'^2 + y'^2}}$

so $\vec{n}\,ds = (y'dt, -x'dt)$
$= (dy, -dx)$

$$\oint_C F \cdot \vec{n}\, ds = \oint_C P\,dy - Q\,dx$$

Green's Theorem

$$\oint_C P\,dy = \oint_C (0,P) \cdot \underbrace{(dx, dy)}_{dr} = \iint_R P_x - 0_y \, dA = \iint_R P_x \, dA$$

$$\oint_C Q\,dx = \oint_C (Q,0) \cdot dr = \iint_R -Q_y \, dA$$

$$\text{Flux} = \iint_R \underbrace{(P_x + Q_y)}_{\text{divergence}} dA$$

$$\text{div}(\vec{F}) = \vec{\nabla} \cdot \vec{F}$$

Example

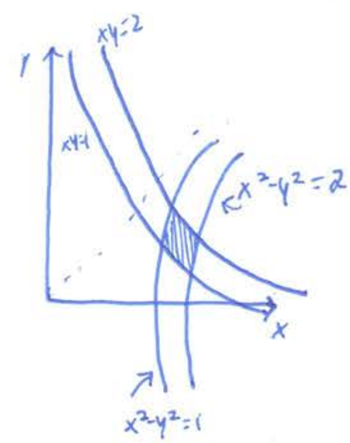
C unit circle

$\vec{F} = (x, -y)$

$\text{div}(\vec{F}) = (x)_x + (-y)_y = 0 \implies \oint_C F \cdot dr = \iint_R 0 \, dA = 0$

6) Change of Variables

R bounded by $xy = 1$, $xy = 2$, $x^2 - y^2 = 1$, $x^2 - y^2 = 2$ In Q1



Find Polar Moment of Inertia

$$I_0 = \iint_R (x^2 + y^2) \underbrace{dA}_{dx\,dy}$$

- Brutal limits of integration
- Fuck that so change xy coordinates to uv coordinates →

$u = xy$

$v = x^2 - y^2$ $\quad R \,\Big|\, \begin{array}{l} 1 \le u \le 2 \\ 1 \le v \le 2 \end{array}$

What happens to dA?

Change of variables

$$\bigstar \quad du\,dv = \left| \det \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \right| dx\,dy$$

$$= \left| \det \begin{pmatrix} y & x \\ 2x & -2y \end{pmatrix} \right| dx\,dy$$

$$du\,dv = \left(2(x^2+y^2)\right) dx\,dy =$$

$$I_0 = \int_{u=1}^{2} \int_{v=1}^{2} \underbrace{(x^2+y^2)\,dx\,dy}_{= \frac{1}{2}\,du\,dv}$$

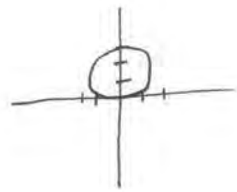
$$= \frac{1}{2} \int_{u=1}^{2} \int_{v=1}^{2} \underbrace{du\,dv}_{\text{area of box } = 1} = \frac{1}{2}$$

18.02 P-set Practice

Polar Coordinates and Integration

14.4

2) Area bounded by circle $r = 3\sin\theta$



$$A = \int_{\theta=0}^{\pi}\int_0^{3\sin\theta} r\,dr\,d\theta$$

$$= \int_0^{\pi} \frac{1}{2}r^2\Big|_0^{3\sin\theta} d\theta = \int_0^{\pi} \frac{1}{2}3^2\sin^2\theta\,d\theta = \int_0^{\pi}\frac{9}{2}\sin^2\theta\,d\theta$$

$$\boxed{\sin^2\theta = \frac{1-\cos 2\theta}{2} \qquad \cos^2\theta = \frac{1+\cos 2\theta}{2}}$$

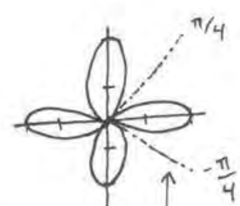
$$= \int_0^{\pi}\frac{9}{2}\left(\frac{1+\cos 2\theta}{2}\right)d\theta$$

$$\frac{9}{4}\int_0^{\pi} 1+\cos 2\theta\,d\theta = \frac{9}{4}\theta - \frac{1}{2}\sin 2\theta\Big|_0^{\pi}$$

$u = 2\theta d\theta$
$du = 2$

$$\boxed{= \frac{9}{4}\pi} \checkmark$$

4) Area by one loop of $r = 2\cos 2\theta$



Find intersections $r=0$
so $0 = \cos 2\theta$
$\cos 2\frac{\pi}{4} = \cos\frac{\pi}{2} = 0$

$$A = \int_{\theta=-\frac{\pi}{4}}^{\pi/4}\int_{r=0}^{2\cos 2\theta} r\,dr\,d\theta$$

$$\int_{-\pi/4}^{\pi/4}\frac{1}{2}r^2\Big|_0^{2\cos 2\theta} d\theta$$

$$= \int_{-\pi/4}^{\pi/4}\frac{1}{2}(2\cos 2\theta)^2\,d\theta = \int_{-\pi/4}^{\pi/4} 2\cos^2 2\theta\,d\theta$$

$$= \int_{-\pi/4}^{\pi/4}\cancel{2}\left(\frac{1+\cos 4\theta}{\cancel{2}}\right)d\theta = \int_{-\pi/4}^{\pi/4} 1+\cos 4\theta\,d\theta = \theta + \frac{1}{4}\sin 4\theta\Big|_{-\pi/4}^{\pi/4}$$

$$= \frac{\pi}{2} \checkmark$$

14.4

16) Convert to Polar Coordinates

$$\int_0^1 \int_x^1 x^2 \, dy \, dx$$

Polar

$x = r\cos\theta$

$y = r\sin\theta$

$r^2 = x^2 + y^2$

$$x^2 = r^2\cos^2\theta$$

$$\iint r^2\cos^2\theta \; r\,dr\,d\theta = \iint r^3\cos^2\theta \, dr\,d\theta$$

7

- How do you get boundries for polar integratr

SCRATCH

$\frac{2\cancel{4}\sqrt{2}}{\cancel{4}\ \cancel{2}2}$

$$\frac{\pi}{4} - \frac{\sqrt{2}}{2} - \left(-\frac{\pi}{4} + \frac{\sqrt{2}}{2}\right)$$

$$\frac{\pi}{4} - \frac{\sqrt{2}}{2} + \frac{\pi}{4} - \frac{\sqrt{2}}{2}$$

$$\frac{2\pi}{4}$$

HOMEWORK 7 SOLUTIONS

- 14.4: 2, 4, 16, 22, 24, 30.
- 14.5: 2, 6, 14, 24.

14.4

2:

$$\int_0^\pi d\theta \int_0^{3\sin(\theta)} r dr = \int_0^\pi \frac{9}{2}\sin^2(\theta)) = \frac{9}{2}\int_0^\pi \frac{1-\cos(2\theta)}{2} d\theta = \frac{9}{4}\pi$$

4: One loop of $r = 2\cos(2\theta)$ has 4 petals of equal area. Let us compute the area of the petal corresponding to $-\pi/4 < \theta < \pi/4$.

$$\int_{-\pi/4}^{\pi/4} d\theta \int_0^{2\cos(2\theta)} r dr = \int_{-\pi/4}^{\pi/4} 2\cos^2(2\theta) d\theta = \int_{-\pi/4}^{\pi/4} (1+\cos(4\theta)) d\theta = \frac{\pi}{2}.$$

16: In this problem we are integrating over the triangle with vertices $(0,0)$, $(0,1)$, $(1,1)$.

$$\int_0^1 \int_x^1 x^2 dy dx = \int_{\pi/4}^{\pi/2} d\theta \int_0^{1/\sin(\theta)} r^3 \cos^2(\theta) dr = \frac{1}{4}\int_{\pi/4}^{\pi/2} \frac{\cos^2(\theta)}{\sin^4(\theta)} d\theta$$

In the last integral we make a change of variables $u = \cot(\theta)$, $du = -\frac{1}{\sin^2(\theta)} d\theta$ and get

$$\frac{1}{4}\int_0^1 u^2 du = \frac{1}{12}.$$

22: The volume is given by the following integral in polar coordinates:

$$\int_0^{2\pi} d\theta \int_0^{1+\cos(\theta)} (1 + r\cos(\theta)) r dr = \int_0^{2\pi} \left(\frac{r^2}{2} + \frac{r^3}{3}\cos(\theta)\right)\Big|_0^{1+\cos(\theta)} d\theta$$

$$= \int_0^{2\pi} \left(\frac{(1+\cos(\theta))^2}{2} + \frac{(1+\cos(\theta))^3}{3}\cos(\theta)\right) d\theta$$

$$= \int_0^{2\pi} \frac{1}{2} d\theta + \int_0^{2\pi} \frac{4}{3}\cos(\theta) d\theta + \int_0^{2\pi} \frac{3}{2}\cos^2(\theta) d\theta + \int_0^{2\pi} \cos^3(\theta) d\theta + \int_0^{2\pi} \frac{\cos^4(\theta)}{3} d\theta.$$

Out of the above integrals, the second and forth ones vanish. Thus, we get

$$\pi + \frac{3}{2}\pi + \int_0^{2\pi} \frac{1 + 2\cos(2\theta) + \cos^2(2\theta)}{12} d\theta = \frac{5}{2}\pi + \left(\frac{1}{6}\pi + 0 + \frac{1}{12}\pi\right) = \frac{11}{4}\pi$$

24: Two paraboloids intersect by the curve

$$z = 12 - 2x^2 - y^2 = x^2 + 2y^2.$$

The x coordinate of the centroid is

$$\frac{1}{M}\int_{-3}^{3}dy\int_{0}^{9-y^2}x^3dx=\frac{1}{4M}\int_{-3}^{3}(9-y^2)^4dy=\frac{1}{4M}\int_{-3}^{3}(9^4-4*9^3y^2+6*9^2y^4-4*9y^6+y^8)dy$$

$$=\frac{1}{4M}(6*9^4-2*4*9^3*3^3/3+2*6*9^2*3^5/5-2*4*9*3^7/7+2*3^9/9)\approx\frac{15996.34}{4*666.51}\approx 6.$$

In fact, if you do all the computations carefully, then you will find that the answer is precisely 6

The y coordinate of the centroid is 0 because of the symmetry of the domain.

24:The intersection of bounding curves is given by $y=x^2=2x+3$, which has solutions $(-1,1)$ and $(3,9)$. Thus, the mass is

$$M=\int_{-1}^{3}\int_{x^2}^{2x+3}x^2\,dydx=\int_{-1}^{3}x^2(2x+3-x^2)dxdx$$

$$=(x^4/2+x^3-x^5/5)\Big|_{-1}^{3}=(3^4/2+3^3-3^5/5-1/2+1-1/5)=19.2$$

The x coordinate of centroid is

$$\frac{1}{M}\int_{-1}^{3}\int_{x^2}^{2x+3}x^3\,dydx=\frac{1}{M}\int_{-1}^{3}x^3(2x+3-x^2)dx$$

$$=\frac{1}{M}(2x^5/5+3x^4/4-x^6/6)\Big|_{-1}^{3}=\frac{2*3^5/5+3*3^4/4-3^6/6+2/5-3/4+1/6}{19.2}\approx 1.888$$

The y coordinate of centroid is

$$\frac{1}{M}\int_{-1}^{3}\int_{x^2}^{2x+3}x^2y\,dydx=\frac{1}{2M}\int_{-1}^{3}x^2((2x+3)^2-x^4)dx$$

$$=\frac{1}{2M}\int_{-1}^{3}(-x^6+4x^4+12x^3+9x^2)dx=\frac{1}{2M}(-x^7/7+4x^5/5+3x^4+3x^3)\Big|_{-1}^{3}$$

$$=\frac{1}{2M}(-3^7/7+4*3^5/5+3*3^4+3*3^3-1/7+4/5-3+3)\approx 5.38$$

18.02 P-set Review

14.9 Change of variables

4) Solve for x and y in terms of u, v and find Jacobian

$u = 2(x^2+y^2)$

$v = 2(x^2-y^2)$

$u = 2x^2+2y^2$

$\frac{u-2y^2}{2} = x^2 \qquad v = 2\left(\frac{u-2y^2}{2} - y^2\right)$

$v = u - 2y^2 - 2y^2$

$v = u - 4y^2$

$y = \sqrt{\frac{u-v}{4}}$ ✓

$\frac{u-2x^2}{2} = y^2$

$v = 2\left(x^2 - \left(\frac{u-2x^2}{2}\right)\right)$

$v = 2\left(x^2 - \frac{u+2x^2}{2}\right)$

$v = 2x^2 - u + 2x^2 = 4x^2 - u$

$x = \sqrt{\frac{v+u}{4}}$

$J_T = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$

$x = \frac{\sqrt{v+u}}{2}$

$x_u = \frac{(u+v)^{1/2}}{2} = \frac{1(u+v)^{-1/2}}{4} = \frac{1}{4\sqrt{u+v}}$

$x_v = 1/4\sqrt{u+v}$

$y = \frac{\sqrt{u-v}}{2}$

$y_u = 1/4\sqrt{u-v}$

$y_v = -1/4\sqrt{u-v}$

$J_T = \begin{vmatrix} \frac{1}{4\sqrt{u+v}} & \frac{1}{4\sqrt{u+v}} \\ \frac{1}{4\sqrt{u-v}} & -\frac{1}{4\sqrt{u-v}} \end{vmatrix}$

$= \left(\frac{1}{4\sqrt{u+v}}\right)\left(-\frac{1}{4\sqrt{u-v}}\right) - \left(\frac{1}{4\sqrt{u+v}}\right)\left(\frac{1}{4\sqrt{u-v}}\right)$

$\frac{-1}{16\sqrt{}}$ $\downarrow$??

$= \frac{1}{8\sqrt{u^2-v^2}}$

14.9

8)

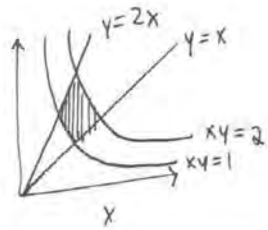
$u = xy$

$v = \frac{y}{x}$

$x = \frac{y}{v}$ $\qquad u = \frac{y\,y}{v}$

$u = \frac{y^2}{v}$

$\boxed{\sqrt{uv} = y}$ ✓

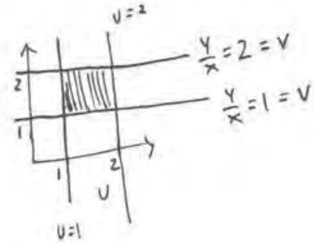


$y = vx$

$u = xvx$

$u = x^2 v$

$\boxed{x = \frac{\sqrt{u}}{\sqrt{v}}}$ ✓



$$A = \iint_R dx\,dy = \int_1^2 du \int_1^2 J_T \,\cancel{du}\,dv$$

$$J_T = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

$x = \frac{\sqrt{u}}{\sqrt{v}}$

$x_u = \frac{1}{2\sqrt{u}\sqrt{v}}$

$x_v = -\frac{1\sqrt{u}}{2\,v^{3/2}}$

$\uparrow$

$x = \sqrt{u}\,v^{-1/2}$

$y = \sqrt{uv} = u^{1/2}v^{1/2}$

$y_u = \frac{1\sqrt{v}}{2\sqrt{u}}$

$y_v = \frac{1\sqrt{u}}{2\sqrt{v}}$

$$J_T = \left(\frac{1}{2\cancel{\sqrt{u}}\sqrt{v}}\right)\left(\frac{1\cancel{\sqrt{u}}}{2\sqrt{v}}\right) - \left(-\frac{1}{2}\frac{\cancel{\sqrt{u}}}{v^{3/2}}\right)\left(\frac{1\sqrt{v}}{2\cancel{\sqrt{u}}}\right)$$

$$= \frac{1}{4v} + \frac{v^{1/2}}{2v^{3/2}}$$

$$= \frac{1}{4v} + \frac{1}{2v^{1/2}}$$

$\downarrow$??

$1/2v$

$$A = \int_1^2 du \int_1^2 \frac{1}{2v}\,\cancel{du}\,dv$$

$$= (2-1)\int_1^2 \frac{1}{2v}\,dv = \frac{1}{2}\int_1^2 \frac{1}{v}\,dv = \frac{1}{2}\left(\ln v\right)\Big|_1^2$$

$$\frac{\ln 2}{2} - \cancel{\frac{\ln 1}{2}}$$

$$= \frac{\ln 2}{2}$$ ✓

18.02 Homework ♯8 Solution

14.9 ♯4

$$x=\sqrt{u+v}/2,\quad y=\sqrt{u-v}/2,\quad J_T=\det\begin{pmatrix} 1/4\sqrt{u+v} & 1/4\sqrt{u+v} \\ 1/4\sqrt{u-v} & -1/4\sqrt{u-v}\end{pmatrix}=\frac{1}{8\sqrt{u^2-v^2}}$$

14.9♯8

$$u=xy, v=y/x,\quad\Rightarrow\quad x=\sqrt{u/v}, y=\sqrt{uv}$$

$$J_T=\det\begin{pmatrix} \frac{1}{2}u^{-1/2}v^{-1/2} & -\frac{1}{2}u^{1/2}v^{-3/2} \\ \frac{1}{2}u^{-1/2}v^{1/2} & \frac{1}{2}u^{1/2}v^{-1/2}\end{pmatrix}=\frac{1}{2v}$$

$$A=\iint_R dxdy=\int_1^2 du\int_1^2 dv|J_T|=\int_1^2 du\int_1^2 dv\frac{1}{2v}=\int_1^2 dv\frac{1}{2v}=\frac{\ln v}{2}|_1^2=\frac{\ln 2}{2}$$

14.9♯12

$$u=\frac{2x}{x^2+y^2}, v=\frac{2y}{x^2+y^2},\quad\Rightarrow\quad x=\frac{2u}{u^2+v^2}, y=\frac{2v}{u^2+v^2}$$

$$J_T=\det\begin{pmatrix} 2(v^2-u^2)/(u^2+v^2)^2 & -4uv/(u^2+v^2)^2 \\ -4uv/(u^2+v^2)^2 & 2(u^2-v^2)/(u^2+v^2)^2\end{pmatrix}=\frac{-4}{(u^2+v^2)^2}$$

$$\begin{aligned}&\iint_R \frac{1}{(x^2+y^2)^2}dxdy\\ =\ &\int_{1/3}^1 du\int_{1/4}^1 dv\left(\frac{u^2+v^2}{4}\right)^2|J_T|=\int_{1/3}^1 du\int_{1/4}^1 dv\frac{1}{4}=\frac{1}{8}\end{aligned}$$

14.9♯14

$$x=au, y=bv, z=cw,\quad J_T=abc$$

$$V(R)=\iiint_R dxdydz=\iiint_D dudvdw|J_T|=abcV(D)=\frac{4\pi}{3}abc$$

where D is the unit ball in $\mathbb{R}^3$ whose volume is $4\pi/3$.

4A♯2

$$a)\quad \nabla\omega=a\mathbf{i}+b\mathbf{j},\quad b)\quad \nabla\omega=\frac{x\mathbf{i}+y\mathbf{j}}{r^2},\quad c)\quad \nabla\omega=f'(r)\frac{x\mathbf{i}+y\mathbf{j}}{r}$$

4A♯3

$$a)\quad \mathbf{F}(x,y)=\mathbf{i}+2\mathbf{j},\quad b)\quad \mathbf{F}(x,y)=-r(x\mathbf{i}+y\mathbf{j}),\quad c)\quad \mathbf{F}(x,y)=\frac{y\mathbf{i}-x\mathbf{j}}{r^3},\quad d)\quad \mathbf{F}(x,y)=\rho(x,y)(\mathbf{i}+\mathbf{j})$$

4A♯4 The direction of the force is tangent to the circle counterclockwise.

$$\mathbf{F}(x,y)=\frac{k}{r^2}(-y\mathbf{i}+x\mathbf{j})$$

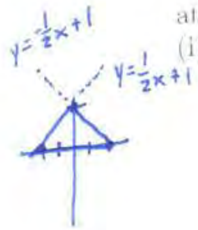
15.2♯4

$$\int_C f(x,y)ds=\int_0^{\pi/2}(2\sin t-\cos t)dt=(-2\cos t-\sin t)|_0^{\pi/2}=2-1=1$$

$$\int_C f(x,y)dx=\int_0^{\pi/2}(2\sin t-\cos t)\cos t dt=\int_0^{\pi/2}(\sin 2t-\frac{\cos 2t}{2}-\frac{1}{2})dt=(\frac{-\cos 2t}{2}-\frac{\sin 2t}{4}-\frac{t}{2})|_0^{\pi/2}=1-\frac{\pi}{4}$$

1. (15pts) Let $(\bar{x}, \bar{y})$ be the center of mass of the triangle with vertices at $(-2,0)$, $(0,1)$ and $(2,0)$ (assume uniform density $\delta = 1$).

(i) Express $\bar{y}$ in terms of an integral.



$$M = A\delta = \left(\tfrac{1}{2}\right)(4)(1) = 2$$

$$\bar{y} = \frac{1}{M}\iint y\,\delta\,dA$$

$$\frac{1}{2}\int_0^1 \int_{2y-2}^{-2y+2} y \quad d\cancel{y}\,dx$$

(ii) Find $\bar{x}$.

$\bar{x} = 0$

By symmetry

2. (15pts) Find the moment of inertia about the origin of the half disk $x^2 + y^2 < a^2$, $x > 0$, where the density $\delta = x^2$.

$$I_0 = \iint_R r^2 \delta \, dA$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^a r^2 x^2 \, dA = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^a r^5 \cos^2\theta \, dr \, d\theta$$

$$= \frac{r^6}{6} \cos^2\theta \Big|_0^a = \frac{a^6}{6} \cos^2\theta$$

$$\int_{-\frac{\pi}{2}}^{\pi/2} \frac{a^6}{6} \cos^2\theta \, d\theta$$

3. (10pts) For which value of a is the vector field

$$\vec{F} = \left(\underbrace{axy - \frac{1}{x}}_{M}\right)\hat{\imath} + \left(\underbrace{x^2 - \frac{1}{y}}_{N}\right)\hat{\jmath}$$

a gradient vector field?

$\text{Curl} = N_x - M_y = 0$

$2x - ax = 0$

$\boxed{a = 2}$

4. (20pts) Let $\vec{F} = 2y\hat{\imath} - x\hat{\jmath}$. Let C be the curve $y = x^2$ starting at $(0,0)$ and ending at $(1,1)$.

(a) Compute the work done on a particle that moves along C.

$\vec{F} = \langle 2y, -x \rangle$

$r = \langle dx, dy \rangle$

$\text{Work} = \int_C F \cdot dr$

$y = x^2$

$dy = 2x\,dx$

$= 2y\,dx - x\,dy$

$\int_C 2x^2 dx - x\,2x\,dx$

$= \int_C 0\,dx = 0$ $\boxed{0}$

(b) Compute the flux of $\vec{F}$ across C.

$\text{Flux} = \int_C \vec{F} \cdot \hat{n}\,ds$

$F = \langle 2y, -x \rangle$

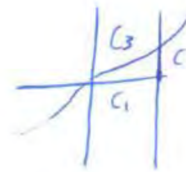
$\hat{n}\,ds = \langle dy, -dx \rangle$

$\int_C 2y\,dy + x\,dx$

$\int_0^1 2y\,dy + \int_0^1 x\,dx$

$y^2 \Big|_0^1 + \frac{1}{2}x^2 \Big|_0^1$

$= 1 + \frac{1}{2} = \boxed{\frac{3}{2}}$



6. (20pts) Consider the region R enclosed by the x-axis, $x = 1$ and $y = x^3$. Let $\vec{F} = (1 + y^2)\hat{j}$.

(i) Use the normal form of Green's theorem to find the flux of $\vec{F}$ out of R.

$F = \langle 0, 1 + y^2 \rangle$
$\quad\; P \quad\;\; Q$

$$\text{Flux} = \oint_C F \cdot \hat{n}\, ds = \iint_R \operatorname{div}(F)\, dA$$

$$= \iint_R 2y\, dA = \int_0^1 \int_0^{y=x^3} 2y\, dy\, dx \quad y^2\Big|_0^{x^3} \quad \int_0^1 x^6 dx = \frac{x^7}{7}\Big|_0^1 = \frac{1}{7}$$

(ii) Find the flux across the horizontal side C_1 of R and the vertical side C_2 of R.

$\hat{n}\, ds = \langle dy, -dx \rangle$

$\underline{C_1}$

$x = t \quad y = 0$
$dx = dt \quad dy = 0$

$$\text{Flux} = \int_{C_1} F \cdot \hat{n}\, ds$$
$$= \int_{C_1} 0\, dy - (1 + y^2)\, dx$$
$$\int_0^1 -1\, dt = -t\Big|_0^1 = -1$$

$\underline{C2}$

$x = 1 \quad y = t$
$dx = 0 \quad dy = dt$

$$\text{Flux} = \int_{C2} F \cdot \vec{n}\, ds$$
$$= \int_{C2} 0\, dy (-1 - y^2) dx$$
$$= \int_{C_3} 0 = 0$$

(iii) Find the flux across the third side C_3.

$\underline{C3}$

$x = t \quad y = t^3$
$dx = dt \quad dy = 3t^2$

$$\text{Flux} = \int_{C3} F \cdot \hat{n}\, ds = \int_{C3} 0\, dy - (1 + y^2)\, dx$$

$$\int_0^1 -1 - t^6\, dt$$

$$-t - \frac{t^7}{7}\Big|_1^0 = 1 - \frac{1}{7}$$

$$= \frac{8}{7}$$

CHECK

~~$\frac{1}{7} + -1 + 0 = 0$~~ closed loop
$C_3 \quad C_1 \quad C_2$

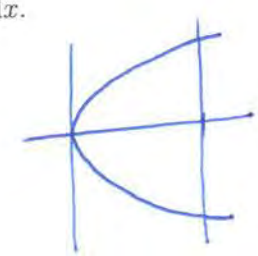
$\frac{1}{7} = C_1 + C_2 + C_3$
$\qquad\; -1 \quad 0 \quad 8/7$
Total F

$\frac{1}{7} = \frac{1}{7}$ ✓

18.02 Practice Midterm #3

1. (15pts) Evaluate

$$\int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} \frac{1}{1-y}\,\mathrm{d}y\,\mathrm{d}x.$$



$$\int_{-1}^{1}\int_{x=y^2}^{1} \frac{1}{1-y}\,dx\,dy$$

$$\int_{-1}^{1} \frac{x}{1-y}\Big|_{y^2}^{1}\,dy$$

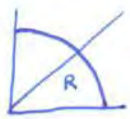
$$\frac{1}{1-y}-\frac{y^2}{1-y} = \frac{1-y^2}{1-y} = \frac{\cancel{(1-y)}(1+y)}{\cancel{(1-y)}} = 1+y$$

$$\int_{-1}^{1} 1+y\,dy = y+\frac{1}{2}y^2\Big|_{-1}^{1}$$

$$1+\frac{1}{2}-\left(-1+\frac{1}{2}\right)$$

$$1+\cancel{\tfrac{1}{2}}+1\cancel{-\tfrac{1}{2}} = 2$$

2. (10pts) Find the x-coordinate $\bar{x}$ of the centre of mass of the portion of the unit disk in the first quadrant, bounded by the x-axis and the line $y = x$ (assume the density $\delta = 1$).



$$M = \iint_R \delta \, dA$$

$$= \int_0^{\pi/4} \int_0^1 r \, dr \, d\theta$$

$$\frac{r^2}{2}\Big|_0^1 = \frac{1}{2}$$

$$\int_0^{\pi/4} \frac{1}{2} \, d\theta = \frac{1}{2}\theta \Big|_0^{\pi/4} = \frac{\pi}{8}$$

$$M = \frac{\pi}{8}$$

$$\bar{x} = \frac{1}{M} \iint x \, \delta \, dA$$

$$= \frac{1}{\pi/8} \iint r\cos\theta \, r \, dr \, d\theta = \frac{8}{\pi} \int_0^{\pi/4} \int_0^1 r^2 \cos\theta \, dr \, d\theta$$

$$\frac{r^3}{3}\cos\theta \Big|_0^1 = \frac{1}{3}\cos\theta$$

$$= \frac{8}{\pi} \int_0^{\pi/4} \frac{1}{3}\cos\theta \, d\theta$$

$$= \frac{8}{\pi}\left(\frac{1}{3}\sin\theta\right)\Big|_0^{\pi/4}$$

$$= \frac{8}{\pi}\left(\frac{1}{3}\frac{\sqrt{2}}{2}\right)$$

$$\frac{8\sqrt{2}}{6\pi} = \frac{8}{3\pi\sqrt{2}}$$

3. (20pts) (i) Find a such that

$$\vec{F} = \underbrace{(x^2 - 14xy)}_{P}\hat{\imath} + \underbrace{(ax^2 - 2y)}_{Q}\hat{\jmath}.$$

is conservative.

$P_y = Q_x$

$-14x = 2ax$

$\boxed{a = -7}$

(ii) Find a potential function $f(x, y)$ for $\vec{F}$ for this value of a.

$f_x = x^2 - 14xy$

$f_y = ax^2 - 2y$

$f = \frac{1}{3}x^3 - \frac{14x^2}{2}y + g(y)$

$f_y = -7x^2 + g'(y)$

$g'(y) = -2y$

$g'(y) = -y^2$

$\boxed{f = \frac{1}{3}x^3 - 7x^2y + -y^2}$

(iii) Calculate the line integral

$$\int \vec{F} \cdot \mathrm{d}\vec{r},$$

for the curve C given by $x = 3\sin t$, $y = \cos t$, $0 \leq t \leq \pi$, where C has the orientation which starts at $(0, 1)$.

5. (20pts) Let C be the positively oriented closed curve formed by the parabola $y = x^2$ running from $(-1, 1)$ to $(1, 1)$ and by a horizontal line segment running from $(1, 1)$ back to $(-1, 1)$. Evaluate

$$\oint_C y^2\,\mathrm{d}x + 4xy\,\mathrm{d}y,$$

$\vec{F} = \langle y^2, 4xy \rangle$

(i) directly, and

C_1

$x: -1 \to 1$ $\quad y = x^2$ $\quad dy = 2x$

$\int_{C_1} x^4 dx + 4x(x^2)(2x)dx$

$= \int_{C_1} x^4 dx + 8x^4 dx$

$= \int_{C_1} 9x^4 dx = \int_{-1}^{1} 9x^4 dx$

$= \frac{9}{5}x^5 \Big|_{-1}^{1} \quad \frac{9}{5} - -\frac{9}{5} = \frac{18}{5}$

C_2

$y = 1 \quad dy = 0$

$x: -1 \to 1$

$\int_{C_2} dx + 4x$

$\int_{1}^{-1} 1 dx = -1 - 1 = -2$

$C_1 + C_2 = \frac{18}{5} - 2 = \frac{18}{5} - \frac{10}{5} = \frac{8}{5} \quad \boxed{\frac{8}{5}}$

(ii) by using Green's theorem.

$\oint_C F \cdot d\vec{r} = \iint_R \operatorname{curl}(\vec{F})\, dA \qquad \vec{F} = \langle \underline{y^2}_{P}, \underline{4xy}_{Q} \rangle$

$\operatorname{curl}(\vec{F}) = Q_x - P_y$

$= 4y - 2y = 2y$

$\iint 2y\, dA = \int_{-1}^{1}\int_{y=x^2}^{1} 2y\, dy\, dx$

$y^2 \Big|_{x^2}^{1} = 1 - x^4$

$\int_{-1}^{1} 1 - x^4 dx = x - \frac{x^5}{5} \Big|_{-1}^{1}$

$\left(1 - \frac{1}{5}\right) - \left(-1 + \frac{1}{5}\right)$

$1 - \frac{1}{5} + 1 - \frac{1}{5}$

$2 - \frac{2}{5} = \frac{10}{5} - \frac{2}{5}$

$= \boxed{\frac{8}{5}}$

11/11/13

18.02 Midterm Review #3

Fernando Trujano

$dA = dx\,dy = r\,dr\,d\theta$

$\sin^2\theta = \frac{1-\cos 2\theta}{2}$

$\cos^2\theta = \frac{1+\cos 2\theta}{2}$

$\text{Mass} = \iint_R \delta\, dA$

Avg. Val of $F = \bar{f} = \frac{1}{\text{Area}(R)} \iint_R F\, dA$

Weighted Avg $= \frac{1}{\text{Mass}(R)} \iint_R F\,\delta\, dA$

Center of mass: $\bar{x} = \frac{1}{\text{Mass}} \iint x\delta\, dA$

$\bar{y} = \frac{1}{\text{Mass}} \iint y\,\delta\, dA$

Moment of Inertia about origin $= I_0 = \iint_R r^2 \delta\, dA$ ($r^2 = x^2+y^2$)

about x $\quad I_x = \iint y^2 \delta\, dA$

$\vec{F} \langle M, N \rangle$ is ∇F if $M_y = N_x \Rightarrow \text{curl} = 0$

gradient field $\Rightarrow$ conservative

Potential: 1) Line Integrals - $C_1 + C_2 + C_3 \ldots$ 2) Antiderivatives - f_x, f_y

$\text{curl}(\vec{F}) = N_x - M_y$

- - - - - - - - - - - -

Green's Theorem

$\oint_C \vec{F} \cdot d\vec{r} = \iint_R \text{curl}\,\vec{F}\, dA$

$\oint_C M\,dx + N\,dy = \iint_R (N_x - M_y)\, dA$

Line int $\rightarrow$ Double int

Flux $= \int_C \vec{F} \cdot \hat{n}\, ds$ (unit normal to C)

$\hat{n}\, ds = \langle dy, -dx \rangle$

$F = \langle P, Q \rangle$

Flux $= \int_C -Q\,dx + P\,dy$

- - - - - - - - - -

Normal Green

$\oint_C \vec{F} \cdot \hat{n}\, ds = \iint_R \underbrace{(P_x + Q_y)}_{\text{div} F}\, dA$

Change of Variables

$J_T = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$

$du\,dv = |J_T|\, dx\,dy$

$J_T = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$

$|J_T|\, du\,dv = dx\,dy$

Line Integrals

work $= \int_C \vec{F} \cdot d\vec{r}$

$\vec{F} = \langle M, N \rangle$

$d\vec{r} = \langle dx, dy \rangle$

$\int_C F \cdot dr = \int_C M\,dx + N\,dy$ ($x'(t)\,dt$, $y'(t)\,dt$)

Express xy in terms of single var and substitute

- - - - - - - - - - - -

$\int_C f\,ds = \int_a^b f(x(t), y(t))\, ds$

$ds = \sqrt{(x')^2 + (y')^2}\, dt$

*Area

11/7/13

18.02 Lecture

1) A puzzle
2) Simply Connected Regions
3) Triple Integrals

1) A puzzle

$$\vec{V} = \left\langle \overset{P}{\frac{-y}{x^2+y^2}}, \overset{Q}{\frac{x}{x^2+y^2}} \right\rangle$$

C = unit circle

$x(t) = \cos t$

$y(t) = \sin t$

$$\oint_C \vec{v} \cdot d\vec{r} = \int_0^{2\pi} (-\sin t, \cos t) \cdot (\overset{x'}{-\sin t}, \overset{y'}{\cos t})\, dt$$

$$= \int_0^{2\pi} 1\, dt = 2\pi$$

$$\text{curl}(\vec{v}) = Q_x - P_y$$

$$= \left(\frac{x}{x^2+y^2}\right)_x - \left(\frac{-y}{x^2+y^2}\right)_y$$

Quotient Rule

$$= \frac{1}{x^2+y^2} - \frac{x(2x)}{(x^2+y^2)^2} + \frac{1}{x^2+y^2} - \frac{y(2y)}{(x^2+y^2)^2} = \frac{2}{x^2+y^2} - \frac{2(x^2+y^2)}{(x^2+y^2)^2} = 0$$

Green's Theorem

$$\oint_C \vec{V} \cdot dr = \iint_R \text{curl}(v)\, dA = 0$$

$$2\pi \neq 0$$

WUT??

Problem? – vector field not defined at $(0,0)$

$$\vec{v} = \nabla \underset{\uparrow}{\tan^{-1}} \left(\frac{y}{x}\right)$$

not defined at $(0,0)$

2) Simply Connected Regions (in plane)

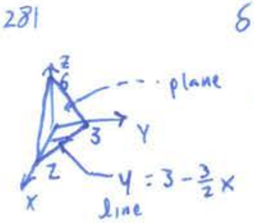
annulus

Region R is simply-connected if it doesn't have a hole
↳ for any closed curve $C \subset R$ inside
then inside of C in R

Theorem
curl $(\vec{v}) = 0$ on a simply connected ~~se~~ R, then $\vec{v} = \nabla f$
some f on R

3) Triple Integrals

Example #2 pg 281 $\delta(x,y,z) = z$

Region 

Point and normal
$= z = 6 - 3x - 2y$

Compute Mass

Compute $\iiint_R \delta \, dV$

$dV = dx\,dy\,dz$ 

x goes from 0 to 2
y goes from 0 to $3 - \frac{3}{2}x$
z goes from 0 to $6 - 3x - 2y$

$$M = \int_{x=0}^{2} \int_{y=0}^{3-\frac{3}{2}x} \int_{z=0}^{6-3x-2y} \underbrace{z}_{\delta} \, dz\,dy\,dx$$

$$M = \int_{x=0}^{2} \int_{y=0}^{3-\frac{3}{2}x} \int_{z=0}^{6-3x-2y} z \, dz\, dy\, dx$$

Same as double integral

$$\int_{x=0}^{2} \int_{y=0}^{3-\frac{3}{2}x} \frac{1}{2} z^2 \Big|_0^{6-3x-2y} = \int_{x=0}^{2} \int_{y=0}^{3-\frac{3}{2}x} \frac{1}{2}(6-3x-2y)^2 \, dy\, dz$$

$u = 6-3x-2y$

$du = -2\,dy$

$\frac{1}{2}\int u^2 \left(\frac{-1}{2}\right) du$

$= \frac{-1}{12} u^3$

$$\cancel{\int_{x=0}^{2} \frac{1}{2} u^2 \left(\frac{-1}{2}\right) du}$$

$$= \frac{-1}{12} u^3$$

$$= \frac{-1}{12}(6-3x-2y)^3 \Big|_{y=0}^{\frac{6-3x}{2}}$$

$$= \frac{-1}{12}\left[0 - (6-3x)^3\right] = \frac{1}{12}(6-3x)^3$$

$$\int_0^2 \frac{1}{12}(6-3x)^3 \, dx$$

$u = 6-3x$

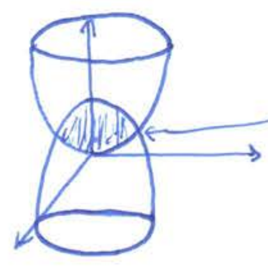
$du = -3\,dx$

$$= \frac{1}{2}\int_0^2 u^3 \; \frac{-1}{3} du = \frac{-1}{144} u^4 \Big|_{u=6}^{u=0}$$

$$= \frac{6^4}{144} = 9$$

Another Example

$z = x^2 + y^2$
$z = 4 - x^2 - y^2$
} Volume b/w them



$x^2+y^2 = 4-x^2-y^2$

$x^2+y^2 = 2$

at height $z=2$

z goes from $0 \to 4$

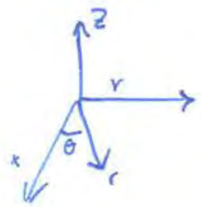
$$\iiint dz\,dy\,dx$$

x	$-\sqrt{2}$ to $\sqrt{2}$
y	$-\sqrt{2-x^2}$ to $\sqrt{2-x^2}$
z	x^2+y^2 to $4-x^2-y^2$

too hard

$$V = \int_{x=-\sqrt{2}}^{\sqrt{2}} \int_{y=-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{x^2+y^2}^{4-x^2-y} 1 \, dz\,dy\,dx$$

Cylindrical Coordinates



(r, θ, z)

- like polar coordinates w/ height

$$dV = \underbrace{r\,dr\,d\theta}_{dA \text{ in polar}}\,dz$$

θ	0 to 2π
r	0 to $\sqrt{2}$
z	r^2 to $4-r^2$

$$\int_{\theta=0}^{2\pi}\int_{r=0}^{\sqrt{2}}\int_{z=r^2}^{4-r^2} r\,dz\,dr\,d\theta$$

18.02 Recitation

Whirlwind overview

- Cylindrical coordinates and spherical coordinates
  - Why would we care?

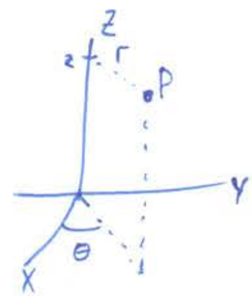
    We want to compute integrals ← hard

    ↳ explort symmetry to make it easier

    ↑

    Rectangular coord too hard

  ↳ useful when symmetry about z axis



(r, θ, z) ← cylindrical coordinates

$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \\ r^2 &= x^2 + y^2 \\ \tan\theta &= y/x \end{aligned}}$$

Spherical Coordinates

- symmetry about a point

(ρ, ϕ, θ)

↙ Angle from x axis in xy plane

↑ angle from z axis

↑ distance from origin

$$\boxed{\begin{aligned} x &= \rho\sin\phi\cos\theta \\ y &= \rho\sin\phi\cos\theta \\ z &= \rho\cos\phi \\ \rho^2 &= x^2 + y^2 + z^2 \end{aligned}}$$

Triple Integralls

T: Region in 3-space

$F(x,y,z)$ ← continious

$$\iiint_T f(x,y,z)\,dV$$

↳ compute as iterated integral ↴

$$\iiint_T f(x,y,z)\,dV = \iint_R \left(\int F\,dz \right) dA$$

Change of Coordinates

don't forget the r.

$$\iiint_T f\,dV = \iint_U \left(f(r\cos\theta, r\sin\theta, z)\,dz \right) r\,dr\,d\theta$$

↖ Polar coor corresponding to R

$$\iiint_T f\,dV = \iiint_S \left(f(r\cos\theta, r\sin\theta, z \right) r\,dz\,dr\,d\theta$$

↖ cylindrical coordinate description of T

→ Formula for mass, volume, centroid, and I_0 ← In Book

Triple Integralls

T: Region in 3-space

$F(x,y,z)$ ← continious

$$\iiint_T f(x,y,z)\,dV$$

↳ compute as iterated integral ↴

$$\iiint_T f(x,y,z)\,dV = \iint_R \left(\int F\,dz\right) dA$$

Change of Coordinates

don't forget the r.

★ $$\iiint_T f\,dV = \iint_U \left(f(r\cos\theta, r\sin\theta, z)\,dz\right) r\,dr\,d\theta$$

↑ Polar coor corresponding to R

$$\iiint_T f\,dV = \iiint_S \left(f(r\cos\theta, r\sin\theta, z\right) r\,dz\,dr\,d\theta$$

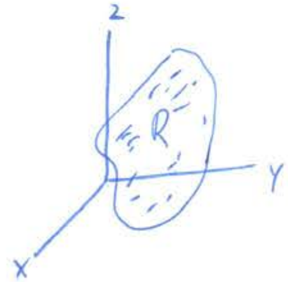
← cylindrical coordinate description of T

→ Formula for mass, volume, centroid, and I_0 ← In Book

Fernando Trujano

Triple Integrals in Rectangular / Cylindrical Coordinates

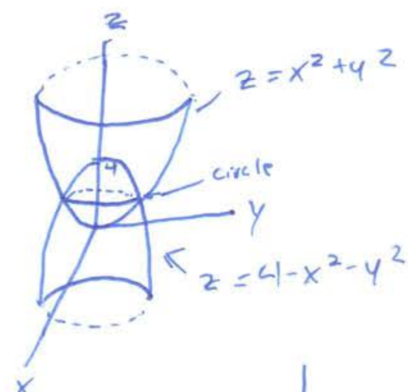
Triple Integrals



$$\iiint_R f\, dV$$ volume element $= dz\,dA$ $= dx\,dy\,dz$

Example

region b/w paraboloids $R \begin{cases} z = x^2+y^2 \\ z = 4-x^2-y^2 \end{cases}$

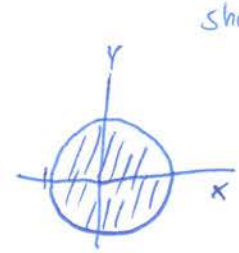


$$\text{Volume} = \iiint 1\, dV$$ volume of region

1) Choose order of integration

$$\int\int\int_{z=x^2+y^2}^{z=4-x^2-y^2} 1\; dz\, dy\, dx$$

all (x, y) in shadow



Find intersection of paraboloids

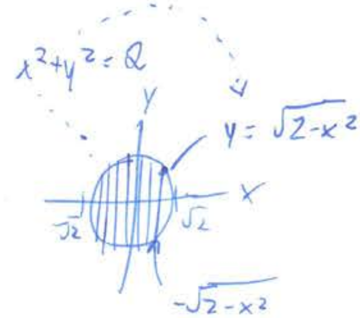
Find shadow in xy plane

wherever $z_{bottom} < z_{top}$

$$x^2+y^2 < 4-x^2-y^2$$

$$x^2+y^2 < 2$$

disk of radius $\sqrt{2}$



$$\int_{-\sqrt{2}}^{\sqrt{2}}\int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}}\int_{x^2+y^2}^{4-x^2-y^2} dz\,dy\,dx$$

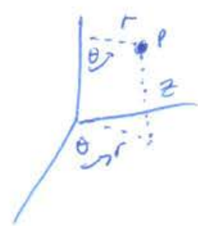
compute inner $= 4-2x^2-2y^2$

gets ugly after

Better: Use polar instead of xy

$$\int_0^{2\pi}\int_0^{\sqrt{2}}\int_{r^2}^{4-r^2} dz\; r\,dr\,d\theta$$ ← Easy to evaluate

Cylindrical Coordinates



(r,θ,z)

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

Note $r = a$

all points a distance a from z axis.

$$dV = r\,dr\,d\theta\,dz$$

Applications:

- Mass: density $\delta = \frac{\Delta m}{\Delta V}$

$$dm = \delta\,dV$$

$$\text{Mass} = \iiint_R \delta\,dV$$

Average value of $f(x,y,z)$ in R

$$\bar{f} = \frac{1}{\text{Volume}(R)} \iiint_R f \, dV$$

Weighted average $= \frac{1}{\text{Mass}(R)} \iiint_R f \delta \, dV$

Center of mass $(\bar{x}, \bar{y}, \bar{z})$

$$\bar{x} = \frac{1}{\text{Mass}(R)} \iiint x \delta \, dV$$

Moment of Inertia / an axis $\iiint_R (\text{distance to axis})^2 \delta \, dV$

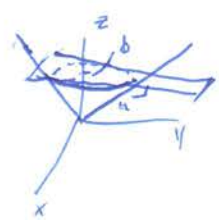
$$I_z = \iiint_R r^2 \delta \, dV = \iiint_R (x^2+y^2) \delta \, dV$$

$$I_x = \iiint_R y^2+z^2 \delta \, dV$$

$$I_y = \iiint_R x^2+z^2 \delta \, dV$$

Example

I_z of solid cone $\delta = 1$
b/w $z = ar$
$z = b$



$$I_z = \iiint r^2 \, r \, dr \, d\theta \, dz$$

- Fix z and slice by horizontal plane $z = ar$

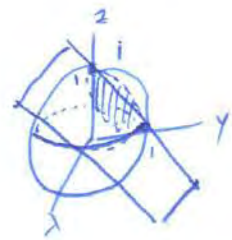
$r = \frac{z}{a}$

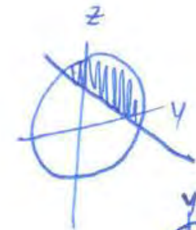
$$\int_0^b \underbrace{\int_0^{2\pi} \int_{r=0}^{r=z/a} r^2 \, r \, dr \, d\theta}_{\text{Just polar over disk}} \, dz = \frac{\pi b^5}{10 a^4}$$

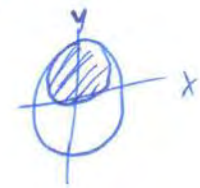
Example

setup $\iiint$ for region $z > 1-y$ inside unit ball origin

$x^2+y^2+z^2 < 1$





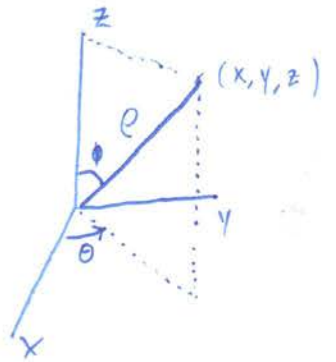


In Rectangular

$$\int_0^1 \int_{-\sqrt{2y-2y^2}}^{\sqrt{2y-2y^2}} \int_{1-y}^{\sqrt{1-x^2-y^2}} dz\, dx\, dy$$

↖ ewww

Spherical coordinates



ρ = rho = distance from origin

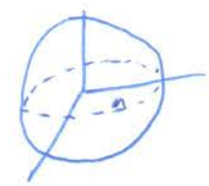
ϕ = phi = Angle down from positive z-axis $0 \to \pi$

θ = Angle CCW from x axis after projection to xy plane

If fix ρ you move in a sphere

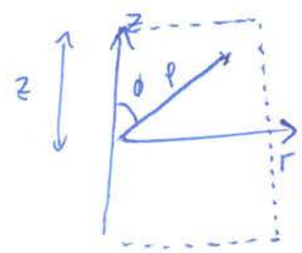
Sphere centered at origin

$\rho = a$



$\phi \longleftrightarrow$ latitude N S

$\theta \longleftrightarrow$ longitude WE



Spherical $\Rightarrow$ Polar in the r-z plane

$z = \rho\cos\phi$

$r = \rho\sin\phi$

$x = r\cos\theta = \rho\sin\phi\cos\theta$

$y = r\sin\theta = \rho\sin\phi\sin\theta$

$z = \rho\cos\phi$

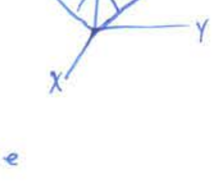
$$\rho = \sqrt{r^2+z^2} = \sqrt{x^2+y^2+z^2}$$

$\rho = \sqrt{x^2+y^2+z^2}$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$\phi = \cos^{-1}\left(\frac{z}{\rho}\right)$

- if $\rho = a \Rightarrow$ sphere of radius a at origin
- if $\phi = \frac{\pi}{4} \Rightarrow$ Cone

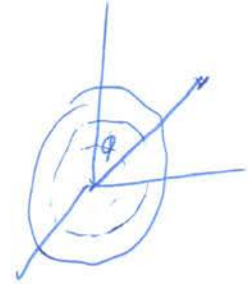


$z = r$

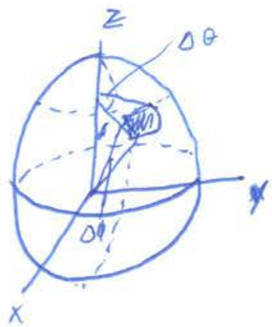
- if $\phi = \frac{\pi}{2} \Rightarrow$ x-y plane

Triple Integrals in spherical coordinates

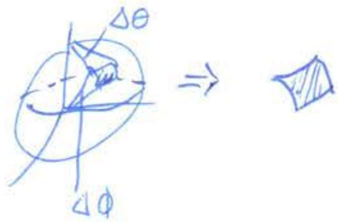
$dV = ??\ d\rho\, d\phi\, d\theta$



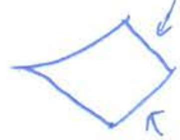
$\Delta V = \_\_\ \Delta\rho\,\Delta\phi\,\Delta\theta$



Surface area on sphere of radius a



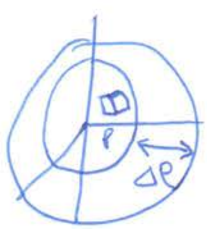
piece of ◯ of radius a
length $= a\Delta\phi$



piece of circle of radius r
$r = a\sin\phi$
length $=$ $a\sin\phi\,\Delta\theta$

$$\Delta S \approx (a\sin\phi\,\Delta\theta)(a\,\Delta\phi) = a^2 \sin\phi\,\Delta\theta\,\Delta\phi$$

$$dS = a^2 \sin\phi\, d\phi\, d\theta$$



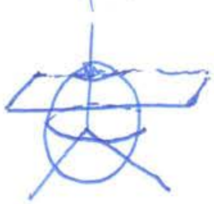
$$\Delta V = \Delta\rho \cdot \Delta S$$

$$= \rho^2 \sin\phi\,\Delta\rho\,\Delta\phi\,\Delta\theta$$

☆ $$\boxed{dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta}$$

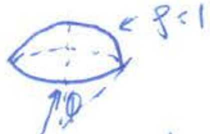
Example

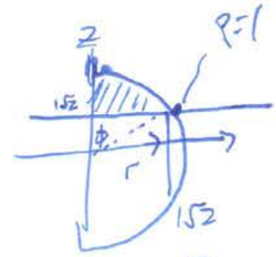
volume of portion of unit sphere above $z = \frac{1}{\sqrt{2}}$



2/4

$$\iiint 1 \underbrace{\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}_{dV}$$





ϕ

Plane = $z = \frac{1}{\sqrt{2}}$

ρ enters at
exits at 1

Lower bound => Plane
Upper bound => Sphere

$\rho \cos\phi = \frac{1}{\sqrt{2}}$

$\rho = \frac{1}{\sqrt{2}\cos\phi}$

$\rho : \frac{1}{\sqrt{2}} \cos\phi \to 1$

$\phi : 0 \to \pi/4$

$\theta : 0 \to 2\pi$

$\phi = \frac{\pi}{4}$

OR

$\rho \cos\phi = \frac{1}{\sqrt{2}}$

$\cos\phi = \frac{1}{\sqrt{2}}$

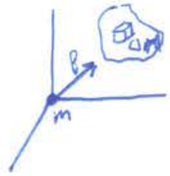
$\phi = \pi/4$

$$\int_0^{2\pi} \int_0^{\pi/4} \int_{\frac{1}{\sqrt{2}}\sec\phi}^{1} 1 \, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \frac{2\pi}{3} - \frac{5\pi}{6\sqrt{2}}$$

Applications

Gravitational attraction



Gravitational Force
by ΔM at (x, y, z)
on mass m at Origin

$$|\vec{F}| = \frac{G \, \Delta M m}{\rho^2} \; \text{dir } F$$

Direction towards $\langle x, y, z \rangle$

$\text{dir } \vec{F} = \frac{\langle x, y, z \rangle}{\rho}$, unit vector

$$F = \frac{G \Delta M m}{\rho^3} \langle x, y, z \rangle$$

$$\Delta M = \delta \Delta V$$

☆ $$\boxed{\vec{F} = \iiint \frac{G m \langle x, y, z \rangle}{\rho^3} \delta \, dV}$$

Setup · place solid so z axis is axis of symmetry

Then $\vec{F} = \langle 0, 0, F \rangle$

z-component

$$= Gm \iiint \frac{z}{\rho^3} \delta \, dV$$

- Use spherical coordinates

$$Gm \iiint \frac{\rho \cos\phi}{\rho^3} \delta \cdot \underbrace{\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}_{dV}$$

$$\text{So } \underset{\text{z component}}{F_z} = Gm \iiint \delta \cos\phi \sin\phi \, d\rho \, d\phi \, d\theta$$

1) Spherical Coordinates
2) Gravitation

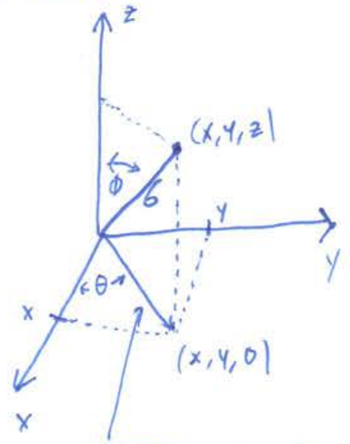
Spherical Coordinates

ϕ, θ, ρ

Angle to z axis (ϕ)

θ in plane

distance to origin (ρ)

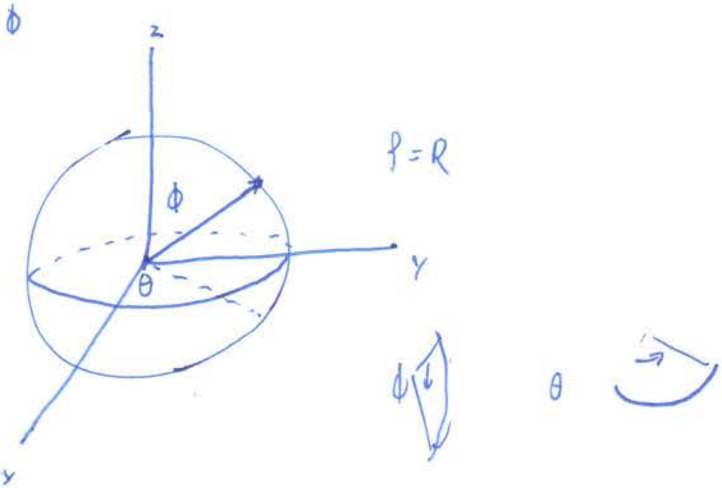


$\sqrt{x^2+y^2} = \rho \sin\phi$

$z = \rho\cos\phi$
$x = \rho \sin\phi \cos\theta$
$y = \rho \sin\phi \sin\theta$

Cylindrical

$z = \rho\cos\phi$
$r = \rho \sin\phi$
$\theta = \theta$



What is dV in spherical coordinates
- Use change of variables

$$dV = dx\,dy\,dz = \left| \det\begin{pmatrix} x_\rho & x_\phi & x_\theta \\ y_\rho & y_\phi & y_\theta \\ z_\rho & z_\phi & z_\theta \end{pmatrix} \right| d\rho\, d\phi\, d\theta$$

$$\det \begin{pmatrix} x_\rho & x_\phi & x_\theta \\ y_\rho & y_\phi & y_\theta \\ z_\rho & z_\phi & z_\theta \end{pmatrix} = \det \begin{pmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{pmatrix}$$

★ $$\boxed{dV = \rho^2 \sin\phi \; d\rho\, d\phi\, d\theta}$$

Proof
$$= \sin\phi\cos\theta\,(\rho^2\sin^2\phi\cos\theta) - \rho\cos\phi\cos\theta\,(-\rho\sin\phi\cos\phi\cos\theta)$$
$$-\rho\sin\phi\sin\theta\,(-\rho\sin^2\phi - \rho\cos^2\phi\sin\theta)$$
$$= \rho^2\sin\phi \; d\rho\, d\phi\, d\theta$$

Example
Volume inside ball of radius a in $\mathbb{R}^2$

1) Find limits of integration
$\rho: 0 \to a$
$\theta: 0 \to 2\pi$
$\phi: 0 \to \pi$

$$\text{Vol} = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} \int_0^a \rho^2 \sin\phi \; d\rho\, d\phi\, d\theta$$

Inside $\displaystyle\int_{\rho=0}^{a} \rho^2 \sin\phi \, d\rho = \frac{\rho^3}{3}\Big|_0^a = \frac{a^3}{3}\sin\phi$

Next $\displaystyle\frac{a^3}{3}\int_{\phi=0}^{\pi} \cancel{\frac{a^3}{3}} \sin\phi \, d\phi = \frac{a^3}{3}(-\cos\phi)\Big|_0^{\pi} = \frac{2a^3}{3}$

Last $\displaystyle\int_0^{2\pi} \frac{2}{3}a^3 \, d\theta = \frac{4\pi}{3}a^3$

2) Gravitation

2 point masses m, M at (x, y, z)

m at origin

What is the gravitational force

$$|\vec{F}| = \frac{GmM}{\underbrace{x^2+y^2+z^2}_{r^2}}$$

$$|F| = GmM\frac{(x,y,z)}{\rho^3}$$

what if M is not a point mass, but instead has a density

M density δ

Mass $= \iiint_R \delta \, dV$

$$F = \iiint_R \frac{Gm}{\rho^3}(x, y, z)\, \delta \, dV$$

$$F_z = \iiint_R \frac{Gm}{\cancel{\rho^3}} \overbrace{\cancel{\rho}\cos\phi}^{z}\, \delta \underbrace{\cancel{\rho^2}\sin\phi \, d\rho\, d\phi\, d\theta}_{dV}$$

☆ $$F_z = \iiint_R Gm\cos\phi\, \delta \sin\phi \, d\rho\, d\phi\, d\theta$$

Example

hemisphere radius a

$\delta = 1$

$$F_z = \int_0^a \int_0^{2\pi} \int_0^{\pi/2} Gm\cos\phi\sin\phi \, d\rho\, d\theta\, d\phi$$

$$= 2\pi a Gm \int_{\phi=0}^{\pi/2} \cos\phi\sin\phi \, d\phi = \pi a Gm$$

$u = \cos\phi$
$du = \sin\phi \, d\phi$

Example

$\theta : 0 \to 2\pi$

$\phi : 0 \to \pi/2$

$$x^2+y^2+(z-a)^2=a^2$$

$$x^2+y^2+z^2-2az+\cancel{a^2}=\cancel{a^2}$$

$$* \rho^2 = 2a\rho\cos\phi$$

$$\rho = 2a\cos\phi$$

$\rho : 0 \to 2a\cos\phi$

$$F = 6m\int_0^{\pi/2}\int_0^{2\pi}\int_0^{2a\cos\phi} \cos\phi\sin\phi \, d\rho\, d\theta\, d\phi$$

5/13 18.02 OCW

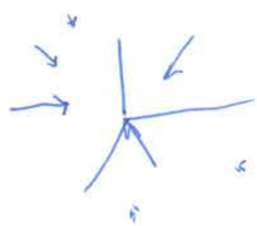
Vector Fields in 3D
Surface Integrals
Flux

Vector fields in Space

At every point $\langle P, Q, R \rangle$
function of x, y, z

Examples

Force Fields
↳ Gravitational Attraction of solid mass at origin on mass M at (x,y,z)
$\vec{F}$ directed towards origin. Magnitude $\sim \frac{c}{\rho^2}$ ← ρ distance from origin



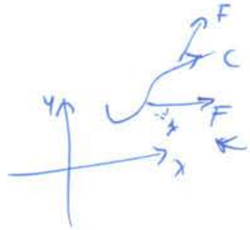
$$\vec{F} = \frac{-c \langle x, y, z \rangle}{\rho^3}$$

Electric Fields
Magnetic Fields

Velocity Fields $\vec{v}$
Gradient Fields $u = \langle x, y, z \rangle$ $\nabla u = \langle u_x, u_y, u_z \rangle$

Flux

recall in 2D

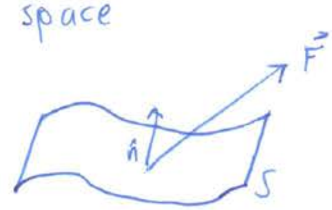


← look at normal n

$$\text{Flux} = \int_C \vec{F} \cdot n \, ds$$

In 3D, Flux measured through a surface.
↓
Surface integral
↳ not line.

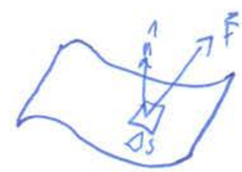
$\vec{F}$ vector field, S surface in space



$\hat{n}$ = unit normal vector

-Choose one side of surface $\uparrow$ or $\downarrow$ $\Rightarrow$ orientation

★ $$\text{Flux} = \iint_S F \cdot \hat{n} \underbrace{dS}_{\text{surface area element}}$$

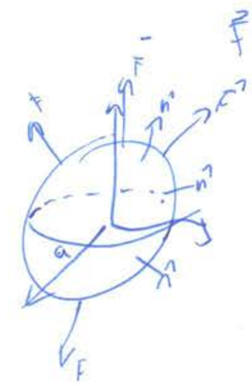


$$d\vec{S} = \hat{n}\, dS$$

Examples

1) Flux of vector field

$\vec{F} = \langle x, y, z \rangle$ through sphere of radius a at origin



$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S F \cdot \hat{n}\, dS$$

$$\hat{n} = \frac{1}{a} \langle x, y, z \rangle$$

unit $\Rightarrow$ $a = \sqrt{x^2+y^2+z^2}$

F and $\hat{n}$ are parallel so $\vec{F} \cdot \hat{n} = |\vec{F}|$
$= a$

$$\iint_S F \cdot \hat{n}\, dS = \iint_S a\, dS = a \underbrace{\iint dS}_{\text{total Area of } S} = 4\pi a^2 \Rightarrow 4\pi a^3$$

sphere

2.14

21 Same sphere

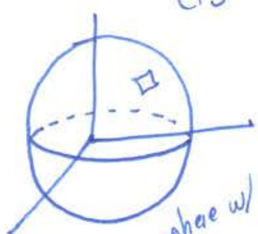
$\vec{H} = z\hat{k}$

$\hat{n} = \frac{\langle x,y,z\rangle}{a} \Rightarrow H\cdot\hat{n} = \frac{\langle 0,0,z\rangle \cdot \langle x,y,z\rangle}{a} = \frac{z^2}{a}$

$\iint_S \vec{H}\cdot\hat{n}\,dS = \iint_S \frac{z^2}{a}\,dS$

constant

$dS = \_\_\_\_\_\_\_\_ d\phi\, d\theta$



Parametrizing sphere w/ ϕ,θ

$\Delta S \approx (a\sin\phi\,\Delta\theta)(a\,\Delta\phi)$

$dS = a^2\sin\phi\, d\phi\, d\theta$

★ $dS = a^2\sin\phi\, d\phi\, d\theta$

$z = a\cos\phi \rightsquigarrow$ 

$\iint_S \frac{z^2}{a}\,dS = \int_0^{2\pi}\int_0^{\pi} \underbrace{\frac{a^2\cos^2\phi}{a}}_{z^2/a} \cdot \underbrace{a^2\sin\phi\,d\phi\,d\theta}_{dS}$

$= \frac{4}{3}\pi a^3$

Conclusion

- Use Geometry 1)
  or set up $\iint_S F\cdot\hat{n}\,dS$ 2)

0) S = Horizontal Plane $z = a$

$\hat{n} = \pm \hat{k}$

↑ your choice!

$dS = dx\,dy$

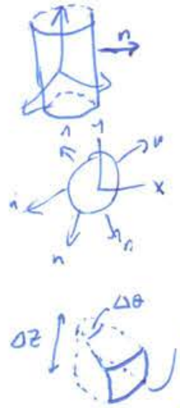
1) Sphere of radius a at origin

$\hat{n} = \dfrac{\langle x, y, z \rangle}{a}$

★ $dS = a^2 \sin\phi \, d\phi \, d\theta$

- Do dot product, then substitute ϕ and θ

2) Cylinder of radius a centered on z axis



$\hat{n} = \frac{1}{a} \langle x, y, 0 \rangle$

★ $dS = \underline{\quad} \, dz \, d\theta$

how high you are

$\Delta S = a \Delta\theta \Delta z$

$dS = a \, d\theta \, dz$

★ 3) Graph of $z = f(x,y)$

$\hat{n}\,dS = \underbrace{\langle -f_x, -f_y, 1 \rangle}_{\text{NOT } \hat{n}} \underbrace{dx\,dy}_{\text{not } dS}$

To setup bounds $\iint \ldots\ldots dx\,dy$

look at shadow of S in x,y plane

11/15/17

18.02 Lecture

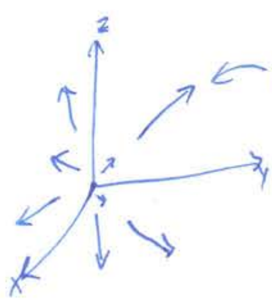
Fernando Trujano

- Vector Fields in Space
- Flux
- Spheres and cylinders
- Surface Integrals

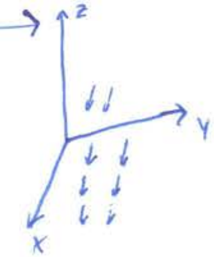
Vector Fields in Space

$$\vec{F}(x,y,z) = \left(M(x,y,z), N(x,y,z), P(x,y,z) \right)$$

$\hat{i} = (1,0,0)$



← Radial V.F (x,y,z)



Gravitational V.F

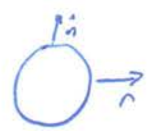
$\vec{F} = \frac{(x,y,z)}{\rho^3}$

Flux in $\mathbb{R}^3$

VF $\vec{F}$

Surface S

unit normal $\vec{n}$



$$\iint_S F \cdot \hat{n}\, \underbrace{dS}_{\text{Surface Area measure}}$$

Example

$\vec{F} = (x,y,z)$

S: Sphere of radius a

$\overset{\text{unit vector}}{\vec{n}} = \frac{(x,y,z)}{a}$ — Proof that is normal

$x^2+y^2+z^2 = a^2$

$\nabla = (2x, 2y, 2z)$

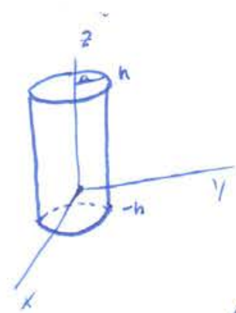
$2(x,y,z)$ ✓

$$\vec{F} \cdot \vec{n} = \frac{(x,y,z)(x,y,z)}{a} = \frac{a^2}{a} = a$$

$$\iint \vec{F} \cdot \vec{n}\, dS = a\, \text{Area}(S) = 4\pi a^3$$

5) Spheres and Cylinders

Cylinder radius a



coords θ, z
Angle
height

Area $= 2h(2\pi a)$
height
length of circle

What is $\iint z\, dS = ??$

$$\int_{z=-h}^{h}\int_{\theta=0}^{2\pi} z\, dS$$

$\Delta S = (\text{height})(\text{width})$
$= (\Delta z)(a\Delta\theta)$

★ $dS = a\, dz\, d\theta$

$$\int_{z=-h}^{h}\int_{0}^{2\pi} z a\ d\theta\, dz = 0$$

by Symmetry

Sphere radius a

$\phi\ \theta$

1) Parametrize
2) Find ~~Volume~~ SA element $\to dS$

ϕ, θ ← $0 \to 2\pi$ (for θ)
← 0 to π (for ϕ)

★ $dS = a^2 \sin\phi\, d\phi\, d\theta$

↳ Proof: $\text{Vol} = \text{Area (Base)} \times \text{Height}$

from last time: $a^2 \sin\phi\, \Delta\phi\, \Delta\theta\, \Delta\rho$; ΔA ; $\Delta\rho$

$\vec{F} = (0, 0, z)$

z in sphere

$$\iint_S \vec{F}\cdot\vec{n}\, dS = \int_{\phi=0}^{\pi}\int_0^{2\pi} (0, 0, a\cos\phi)\cdot\left(\frac{a\sin\phi\cos\theta,\ a\sin\phi\sin\theta,\ a\cos\phi}{a}\right)$$

$\hookrightarrow a^2 \sin\phi\, d\theta\, d\phi$

5) Write out integral
substitute x, y, z in terms of parameters

$$\int_{\phi=0}^{\pi}\int_{\theta=0}^{2\pi} a\cos^2\phi \; a^2 \sin\phi \, d\theta \, d\phi = a^3 2\pi \int_{\phi=0}^{\pi} \overbrace{\cos^2\phi}^{u^2} \; \overbrace{\sin\phi \, d\phi}^{-du}$$

$u = \cos\phi$
$du = -\sin\phi \, d\phi$

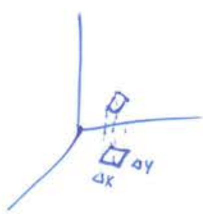
$$= -\frac{a^3 2\pi}{3} \cos^3\phi \Big|_0^{\pi}$$

$$= \frac{4\pi}{3} a^3$$

4) Surface Integrals

x and y are parameters

$z = f(x,y)$



2 sides 1) $(x, y, f(x,y))$ to $(x+\Delta x, y, f(x+\Delta x, y))$

2) $(x, y, f(x,y))$ to $(x, y+\Delta y, f(x+\cancel{y}+\Delta y))$

Get 2 sides as vectors → take cross products = Area

2) $(0, \Delta y, \underbrace{f(x, y+\Delta y) - f(x,y)}_{dy \Delta y})$

2 vectors

$(0, 1, f_y)\Delta y$
$(1, 0, f_x)\Delta x$

x-product → vector that is normal and whose magnitude is Area.

$$\begin{vmatrix} i & j & k \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} \Delta x \Delta y$$

$$\vec{n}\, dS = (-f_x, -f_y, 1)\, dx\, dy$$

11/19/13 — 18.02 Lecture — Fernando Trejo

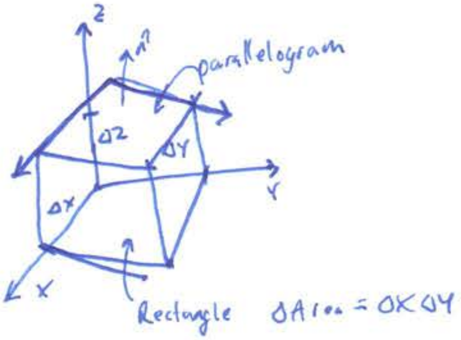
1) $\vec{n}\,dS$ on graphs
2) Computing Flux
3) Divergence Theorem

1) $\vec{n}\,dS$ on graphs

S a surface, $\vec{n}$ unit normal, dS surface area

$z = f(x,y)$

$$\vec{n}\,dS = \left(-f_x, -f_y, 1\right) dx\,dy \quad \bigstar$$



2 sides of Parallelo: x-side, y-side

$(x\text{-side}) \times (y\text{-side}) = \vec{n}\,\Delta S$ — normal, A of Parallel.

x-side: connect $(x, y, f(x,y)) \to (x+\Delta x, y, f(x,y) + f_x\Delta x)$ — linear approxim.

$(\Delta x, 0, f_x \Delta x)$

y-side: $(x, y, f(x,y)) \to (x, y+\Delta y, f(x,y) + f_y \Delta y)$

$(0, \Delta y, f_y \Delta y)$

$(x\text{-side}) \times (y\text{-side})$

= ~ is normal

its magnitude = Area of Parallelogram $= \vec{n}\,\Delta S$

$\Rightarrow$ Taking limits (i.e. writing d instead of Δ)

$$\vec{n}\,dS = (dx, 0, f_x\,dx) \times (0, dy, f_y\,dy)$$

$$\begin{vmatrix} i & j & k \\ dx & 0 & f_x dx \\ 0 & dy & f_y dy \end{vmatrix} = (-f_x\,dx\,dy, -f_y\,dx\,dy, dx\,dy)$$

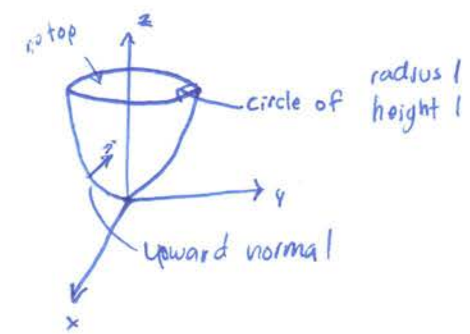
$$= dx\,dy(-f_x, -f_y, 1) = \vec{n}\,dS \quad \checkmark$$

2) Computing Flux

$\vec{F} = (0,0,z)$

S graph of $z = x^2+y^2$

$x^2+y^2 \le 1$



$f = x^2+y^2$

$f_x = 2x \quad f_y = 2y \quad \Rightarrow \vec{n}\,dS = (-2x, -2y, 1)\,dx\,dy$

$$\text{Flux} = \iint_S \vec{F}\cdot\vec{n}\,dS = \iint_{x^2+y^2\le 1} (0,0,x^2+y^2)\cdot(-2x, -2y, 1)\,dx\,dy$$

$$= \iint_{x^2+y^2\le 1} (x^2+y^2)\,dx\,dy = \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{1} r^2\, r\,dr\,d\theta = \int_0^{2\pi} \frac{1}{4}\,d\theta = \frac{\pi}{2}$$

Note

$$\underbrace{\left(\frac{-f_x, -f_y, 1}{\sqrt{1+f_x^2+f_y^2}}\right)}_{\vec{n}}\underbrace{\left(\sqrt{1+f_x^2+f_y^2}\,dx\,dy\right)}_{dS} = \vec{n}\,dS$$

3) Divergence Theorem

Surface S closed and bounding D

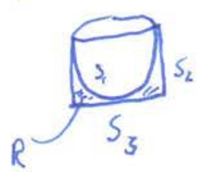
$\vec{n}$ outward normal

$\vec{F}$ = Vector Field , defined and differentiable everywhere in Region

★ $$\iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_R \mathrm{div}(\vec{F})\,dV$$

$$\mathrm{div}(N,M,P) = N_x + N_y + P_z$$

Example



$S = S_1 \cup S_2 \cup S_3$ $\quad F = (0,0,z)$

S_1 = paraboloid $z = x^2+y^2,\ x^2+z^2 \le 1$

S_2 = side of can

S_3 = bottom $\quad \mathrm{div}(F) = 1$

$$\iiint_R \mathrm{div}(F) = \mathrm{Volume}(R)$$

$$\iint_{S_1} \vec{F}\cdot\vec{n}\,dS = \frac{\pi}{2}$$ ← From last example

$$\iint_{S_2} \vec{F}\cdot\vec{n}\,dS = 0$$ ← Vector field is vertical. x,y components are 0.

$$\iint_{S_3} \vec{F}\cdot\vec{n}\,dS = 0$$ ← $\vec{F}=0$ on S_3

$$\mathrm{Volume}(R) = \frac{\pi}{2}$$

Example

$S = \{x^2+y^2+z^2 = a^2\}$

$\vec{F} = (0,0,z)$

Last class

$$\iint_S \vec{F}\cdot\vec{n}\,dS = \frac{4}{3}\pi a^3 = \text{Volume inside}$$

$u(x,y,z)$

$\Delta u = u_{xx} + u_{yy} + u_{zz} = \operatorname{div}(\nabla u)$

u is harmonic if $\Delta u = 0$

S is closed surface bounding R

$$\begin{cases} \Delta u = 0 & \text{inside } R \\ u = 0 & \text{on } S \end{cases}$$

This $\Rightarrow$ $u = 0$ inside

Proof

$$\operatorname{div}(u\nabla u) = \operatorname{div}(uu_x, uu_y, uu_z) = u_x^2 + \underbrace{uu_{xx}} + u_y^2 \underbrace{uu_{yy}}_{\Delta u = 0} + u_z^2 \underbrace{uu_{zz}}$$

$$= u_x^2 + u_y^2 + u_z^2$$

$F = u\nabla u$ is 0 on S

$$\text{Div} \Rightarrow 0 = \iiint_R \operatorname{div}(u\nabla u)$$

$$= \iiint_R u_x^2 + u_y^2 + u_z^2 = 0$$

u is constant

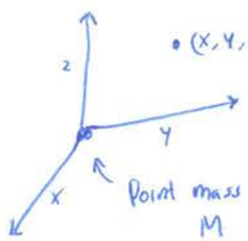
11/22/13 — 18.02 Lecture — Fernando Trujano

1) Divergance and Gravitation

2) Line integrals

3) Curl in $\mathbb{R}^3$

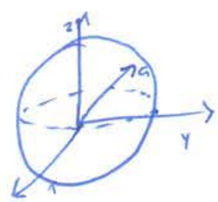
Divergance and Gravitation



$$\vec{F}(x,y,z) = \frac{GM}{\rho^2}\left[\frac{-(x,y,z)}{\rho}\right]$$

$\frac{-(x,y,z)}{\rho}$ ← direction

Gravitational Flux



$$\text{Flux} = \iint_S \vec{F}\cdot\vec{n}\, dS$$

$\vec{n} = \frac{(x,y,z)}{\rho}$

So
$$\vec{F}\cdot\vec{n} = \frac{GM}{\rho^2}\left(\frac{-(x,y,z)}{\rho}\cdot\frac{(x,y,z)}{\rho}\right)$$

$$= -GM/\rho$$

$$\text{So Flux} = \frac{-GM}{a^2}\,\text{Area}(S) = -GM\,4\pi\,\frac{\cancel{a^2}}{\cancel{a^2}}$$

↑ not dependent on A

$$R = \{a^2 \le \rho^2 \le b^2\}$$

$$\iiint_R \operatorname{div}(\vec{F}) = \iint_{S_b} \vec{F}\cdot\vec{n}\,dS - \iint_{S_a} \vec{F}\cdot\vec{n}\,dS$$

$= 0$

$$\operatorname{div}\left(\frac{(x,y,z)}{\rho^3}\right) = \left(\left(\frac{x}{\rho^3}\right)_x, \left(\frac{y}{\rho^3}\right)_y, \left(\frac{z}{\rho^2}\right)_z\right)$$

$$\rho^2 = x^2+y^2+z^2 \longrightarrow \rho_x = \frac{x}{\rho}$$

Product Chain Rule

$$\rho\rho = x^2+y^2+z^2 \qquad \rho_y = y/\rho$$

$$2\rho\rho_x = 2x \qquad \rho_z = z/\rho$$

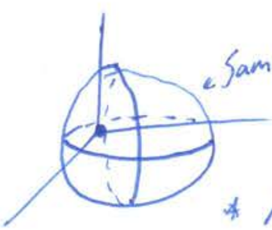
$\rho\rho_x + \rho_x\rho$

Product Rule

$$= \left(\frac{1}{\rho^3} - \frac{3x\rho_x}{\rho^4}\right) + \left(\frac{1}{\rho^3} - \frac{3y\rho_y}{\rho^4}\right) + \frac{1}{\rho^3} - \frac{3z\rho_z}{\rho^4}$$

$$\frac{3}{\rho^3} - \frac{3}{\rho^4}\left[x\frac{x}{\rho} + y\frac{y}{\rho} + z\frac{z}{\rho}\right] = 0 \checkmark$$

$x^2+y^2+z^2 = \rho^2$



Same thing.

Gauss's Law

* Any Sphere containing the mass has the same gravitational Flux.

- Vector field not defined at origin

Line Integrals

curve C in $\mathbb{R}^3$

$\vec{r}(t) = (t, t^2, t^3)$ Gradient Field of (x,y,z)

$\vec{F} = (yz, xz, xy)$ from $t=0$ to $t=1$

Line integral $\int_C \vec{F}\cdot d\vec{r} = \int_{t=0} (\underbrace{t^5}_{yz}, \underbrace{t^4}_{xz}, t^3)\cdot$ →

2

$$\int_C \vec{F}\cdot d\vec{r} = \int_{t=0}^{1} (t^5, t^4, t^3)\cdot(1, 2t, 3t^2)\,dt$$

$$= \int_0^1 6t^5\,dt$$

$$= 1$$

★ Fundamental Theorem of Calculus ∇ Line integrals

$$\int_C \nabla f \cdot d\vec{r} = f(B) - f(A)$$

from A to B

Previous example

$$= (x,y,z)(1,1,1) - (x,y,z)(0,0,0) = 1 \checkmark$$

Curl $\quad F = (P, Q, R)$

★ $\operatorname{curl}(\vec{F}) = \vec{\nabla} \times \vec{F} = \begin{vmatrix} i & j & k \\ dx & dy & dz \\ P & Q & R \end{vmatrix} = (R_y - Q_z,\ P_z - R_x,\ Q_x - P_y)$

$$\vec{F} = \nabla f = (f_x, f_y, f_z)$$

$$\vec{\nabla} \times \nabla f = \begin{vmatrix} i & j & k \\ dx & dy & dz \\ f_x & f_y & f_z \end{vmatrix} = (\overset{0}{f_{zy} - f_{yz}},\ \overset{0}{f_{xz} - f_{zx}},\ \overset{0}{f_{yx} - f_{xy}})$$

$$= 0$$ by equality of mixed partials

★ ★ Curl of a gradient vector field $= 0$

3

Example

Find a, b such that $\vec{F} = (axy, x^2+z^3, byz^2-4z^3)$

is a ∇ V.F

$$\text{curl}(\vec{F}) = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ axy & x^2+z^3 & byz^2-4z^3 \end{vmatrix} = (bz^2-3z^2,\ 0,\ 2x-ax)$$

$b=3$
$a=2$

$\nabla f = (2xy,\ x^2+z^3,\ 3yz^2-4z^3)$

Find Potential

$f_z = 3yz^2-4z^3$ Integrate w/ respect to z

$f = yz^3 - z^4 + g(x,y)$

$f_y = z^3 + g_y$

$f_y = x^2+z^3 \Longrightarrow g_y = x^2 \Rightarrow g = x^2y + h(x)$

$\cancel{f_x =}$

$f = yz^3 - z^4 + x^2y + h(x)$

$f_x = 2xy + h'(x)$

$h'(x) = 0$

$h = \text{Konstant}$

so

$$\boxed{f = yz^3 - z^4 + x^2y} + C$$

Rotation: $(-y, x, 0)$

$$\Downarrow$$

$(-ay, ax, 0)$

$\mathrm{curl}(-ay, ax, 0)$

$$\begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ -ay & ax & 0 \end{vmatrix} = (0, 0, 2a)$$

direction => Vertical direction axis

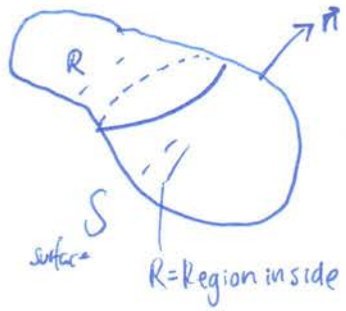
magnitude => $2a$

18.02 Lecture

Fernando Trujano

1) Divergence Theorem (again)
2) Computing an example
3) Divergance and Gravitation
   ↳ Gaussos Law

Divergence Theorem



Flux

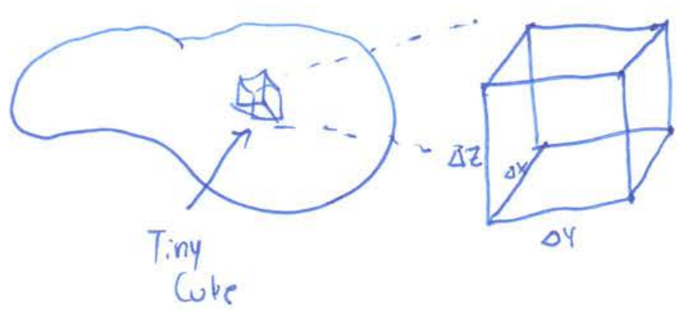
$$\underbrace{\iint_S \vec{F}\cdot\vec{n}\,d\vec{S}}_{\text{Flux}} = \iiint_R \operatorname{div}(\vec{F})\,dV$$

Surface

Total Flux out of the region

Imagine: $\vec{F}$ = velocity of a flow
Left side = Rate of flow out of R

Total Net amout leaving = Net amount created inside



How much created inside here?

$\vec{F} = (M, N, P)$

Net amount (this cube)

(Out of top) - (in bottom)
(out right) - (in left)
(out front) - (in back)

$$\underbrace{P(x,y,z+\Delta z)}_{\text{Amount Flowing up/Area}}\ \underbrace{\Delta x \Delta y}_{\text{Area}} - P(x,y,z)\,\Delta x \Delta y \approx \underbrace{P_z \Delta z\, \Delta x \Delta y}_{\text{Linear Approximation}}$$

$$N(x,y+\Delta y,z)\,\Delta x \Delta z - N(x,y,z)\,\Delta x \Delta z \approx N_y \Delta y\, \Delta x \Delta z$$

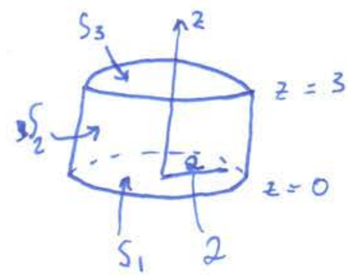
$$\approx M_x\, \Delta x\, \Delta y \Delta z$$

$$\Rightarrow \text{Total} = \underbrace{(M_x + N_y + P_z)}_{\text{Divergance!!}}\ \Delta x \Delta y \Delta z$$

Example

$\vec{F} = (yz,\ x^2+2,\ 5)$ Note $\operatorname{div}(F) = 0$

$\iint_R \operatorname{div}(\vec{F}) = 0$ since $\operatorname{div}(\vec{F}) = 0$



On S_1

$r: 0 \to 2$
$\theta: 0 \to 2\pi$
$z = 0$

outward normal

$$\text{Flux} = \iint_{S_1} (0,\ r^2\cos^2\theta + 2,\ \underset{z=0}{5}) \cdot \underset{\text{outward normal}}{(0,0,-1)}\, r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 -5\,r\,dr\,d\theta = -20\pi$$

On S^2 θ, z

Parametrizing cylinder

$\vec{r}(\theta, z) = (2\cos\theta, 2\sin\theta, z)$

$\theta: 0 \to 2\pi$
$z: 0 \to 3$

$$\vec{n}\,dS = \pm\, \vec{r}_\theta \times \vec{r}_z \,|\, d\theta\,dz$$

$$= \begin{vmatrix} i & j & k \\ -2\sin\theta & 2\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} d\theta\,dz$$

$$= \pm(2\cos\theta, 2\sin\theta, 0)\,d\theta\,dz$$

On S_3 $z = 3$

$$\int_0^{2\pi}\int_0^2 (\overbrace{3r\sin\theta}^{3y},\ r^2\cos^2\theta + 2,\ 5) \cdot (0,0,1)\, r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 5r\,dr\,d\theta = 20\pi$$

$\vec{n}\,dS = (2\cos\theta, 2\sin\theta, 0)\,d\theta\,dz$

$$\int_0^3\int_0^{2\pi} (2\sin\theta\, z,\ 4\cos^2\theta + 2,\ 5) \cdot (2\cos\theta \Rightarrow$$

$$\int_{z=0}^{3}\int_{0}^{2\pi} (2\sin\theta z,\ 4\cos^2\theta + 2,\ 5)\cdot(2\cos\theta,\ 2\sin\theta,\ 0)\, d\theta dz$$

$$= \int_0^3\int_0^{2\pi} (4\sin\overset{u}{\cos}\overset{du}{\theta} z + 8\overset{u^2}{\cos^2\theta}\ \overset{-du}{\sin\theta} + 4\sin\theta)\, d\theta\, dz$$

$$= 0$$

Total $= S_1 + S_2 + S_3$

$= -20\pi + 0 + 20\pi = 0$ ✓

3) Divergance and Gravitation

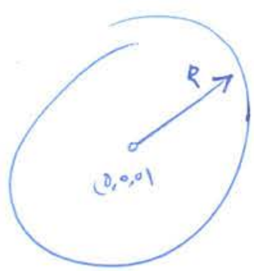
$$\vec{F} = \frac{-GM}{\rho^2}\ \underbrace{\frac{(x,y,z)}{\rho}}$$ pt mass M at $(0,0,0) \Rightarrow \vec{F}$ at (x,y,z)

towards origin

outward normal to sphere centered at origin.

Level set of $x^2+y^2+z^2$

$\nabla = (2x, 2y, 2z)$



Flux accross sphere $= \iint_{\text{Sphere}} \frac{-GM}{\rho^2}\, dS = \underbrace{\frac{-GM\ 4\pi R^2}{R^2}}_{\text{does not depend on } R}$

$\rho \to R$

18.02 Recitation

Fernando Trujano

★ Spherical: $dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

$u = u(x,y,z)$
$v = v(x,y,z)$
$w = w(x,y,z)$

$$dV = \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| du\,dv\,dw$$

$$\text{Jacobian} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

Gravitational Attraction

V: solid of mass M
sym about z axis

Gravitational force about z-axis $\hat{k}$

$$F_z = G \iiint \frac{\cos\phi}{\rho^2} \delta \, dV$$

$$= G \iiint \delta \cos\phi \sin\phi \, d\rho \, d\phi \, d\theta$$

★

Vector Fields in 3-D

$$\vec{F}(x,y,z) = M(x,y,z)\,\hat{i} + N(x,y,z)\,\hat{j} + P(x,y,z)\,\hat{k}$$

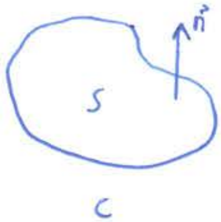
11/26/13 — 18.02 Lecture — Fernando Trujano

1) Stoke's Theorem
2) Irrational Vector Fields

Stoke's Theorem

$$\vec{F} = (P, Q, R)$$

$$\text{curl}(F) = \vec{\nabla} \times \vec{F} \quad \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{vmatrix} = (R_y - Q_z,\ P_z - R_x,\ Q_x - P_y)$$



Stokes theorem

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \underbrace{\text{curl}(F)}_{\vec{\nabla} \times \vec{F}} \cdot \hat{n}\, dS$$

If $S \subset$ Plane $z = 0$
→ in xy plane

$n = (0, 0, 1)$

$$(\nabla \times F) \cdot \hat{n} = \underbrace{Q_x - P_y}$$

Green's Theorem in Plane!

Example

$\vec{F} = (x, y, z)$
C = unit circle in $z = 0$
S = Northern hemisphere

Line integral

$\vec{r}(t) = (\cos t, \sin t, 0)$

$$\oint_C \vec{F} \cdot dr = \int_{t=0}^{2\pi} \oint (\cos t, \sin t, 0) \cdot dr$$

$$\oint F \cdot dr = \int_{t=0}^{2\pi} (\cos t, \sin t, 0) \cdot (-\sin t, \cos t, 0) = 0$$

Makes sense b/c

- It is a gradient Vector Field
- Irrotational Vector curl = 0

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ x & y & z \end{vmatrix} = (0,0,0)$$

Stokes Theorem:

$$= \iint_S \overbrace{\text{curl}(F)}^{0} \cdot n \, dS = 0$$

Example

Same C and S

$\vec{F} = (x, x-y, zx)$

$$\text{curl}(F) = \begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ x & x-y & zx \end{vmatrix} = (0, -z, 1)$$

$r(t) = (\cos t, \sin t, 0)$

Line Integral:

$r'(t)$

$$\oint_C \vec{F} \cdot d\vec{r} = \int_{t=0}^{t=2\pi} (\cos t, \cos t - \sin t, 0) \cdot (-\sin t, \cos t, 0) \, dt$$

$$= \int_0^{2\pi} -2\sin t \cos t + \cos^2 t \quad dt = \pi$$

$$\iint_S (\nabla \times F) \cdot n \, dS$$

Spherical coordinates $\rho = 1$ unit sphere

Parametrization $\longrightarrow r(\phi, \theta) = (\sin\phi \cos\theta, \sin\phi \sin\theta, \cos\phi)$

$\phi : 0 \to \frac{\pi}{2}$

$\theta : 0 \to 2\pi$

also $= \vec{n}$

$dS = \cancel{\rho^2} \sin\phi \, d\phi \, d\theta$

$\longrightarrow$

normal

$$\iint_S (\vec{\nabla}\times\vec{F})\cdot \hat{n}\, dS = \int_{\phi=0}^{\pi/2}\int_{\theta=0}^{2\pi} (0, -\cos\phi, 1)\cdot \underbrace{(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)}_{\hat{n}} \underbrace{\sin\phi\, d\phi\, d\theta}_{dS}$$

How to get $\vec{n}dS$? another way

$r_\phi = (\cos\phi\cos\theta, \cos\phi\sin\theta, -\sin\phi)$

$r_\theta = (-\sin\phi\sin\theta, \sin\phi\cos\theta, 0)$ theta

$r_\phi \times r_\theta \to \pm \vec{n}\, dS$

$$\int_{\phi=0}^{\pi/2}\int_{\theta=0}^{2\pi} \left[-\cos\phi\sin^2\phi\sin\theta + \underset{du}{\cos\phi}\,\underset{u}{\sin\phi} \right] d\theta\, d\phi$$

$$\int_{\phi=0}^{\pi/2} (0 + 2\pi \underset{du}{\cos\phi}\,\underset{u}{\sin\phi})\, d\phi = 2\pi \frac{1}{2}\sin^2\phi \Big]_0^{\pi/2} = \pi \checkmark$$

Cool Story Bro. Tell it again ..... but with different S!

S = unit circle

$\theta: 0 \to 2\pi$

$r: 0 \to 1$

$(r\cos\theta, r\sin\theta, 0)$

$\hat{n} = (0, 0, 1)$

$dS = r\, dr\, d\theta$

$$\iint_{circle} (\vec{\nabla}\times\vec{F}\cdot\hat{n}\, dS = \int_{r=0}^{1}\int_{\theta=0}^{2\pi} (0,0,1)\cdot(0,0,1)\, r\, d\theta\, dr = \pi$$

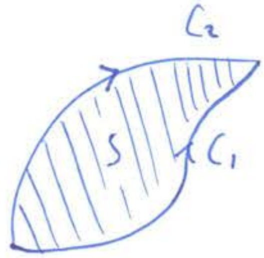
2| Irrotational VF

$\vec{F}$ is irrotational if $\vec{\nabla} \times \vec{F} = 0$

curl measures how much circulation around C

Theorem: if $\vec{F}$ is irrotational everywhere
$\Rightarrow \vec{F}$ is a gradient V.F

Can find potential using line integral.



$C = C_1 - C_2 \longrightarrow$ Closed curve

$$\underbrace{\oint_C \vec{F} \cdot d\vec{r} = \iint_S}_{\text{Stoke's Theorem}} \underbrace{\overset{(\vec{\nabla} \times \vec{F})}{\operatorname{curl}(\vec{F})}}_{\text{zero}} \cdot n \, dS$$

$$\int_{C1} F \cdot dr - \int_{C2} F \cdot dr = 0$$

$$\int_{C_1} F \cdot dr = \int_{C_2} F \cdot dr$$

$\therefore$ line integral is path independent

12/1/13 — 18.02 Midterm 4 Review — Fernando Trujano

Cylindrical Coordinates

$dV = r\,dr\,d\theta\,dz$

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

Spherical Coordinates

$dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$

$x = \rho\sin\phi\cos\theta$

$y = \rho\sin\phi\sin\theta$

$z = \cancel{\rho\cos\theta} = \rho\cos\phi$

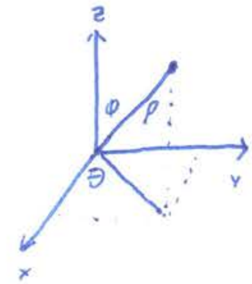
$\rho = \sqrt{x^2+y^2+z^2}$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$\phi = \cos^{-1}\left(\frac{z}{\rho}\right)$

$\phi: 0 \to \pi$

$dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$



Mass $= \iiint_R \delta\, dV$

Avg. val of f in R

$$\bar{f} = \frac{1}{\mathrm{Vol}(R)} \iiint_R f\, dV$$

Weighted Avg.

$$\frac{1}{\mathrm{Mass}(R)} \iiint_R f\,\delta\, dV$$

Centroid/Center of Mass $(\bar{x}, \bar{y}, \bar{z})$

$$\bar{x} = \frac{1}{\mathrm{Mass}(R)} \iiint_R x\,\delta\, dV$$

Moment of Inertia

$$I = \iiint_R (\text{distance to axis})^2 \delta\, dV$$

$$I_z = \iiint_R (x^2+y^2)\delta\, dV$$

$$I_x = \iiint_R (y^2+z^2)\delta\, dV$$

$$I_y = \iiint_R (x^2+z^2)\delta\, dV$$

Fundamental Theorem of Calculus ∇ Line Integrals

$$\int_C \nabla F \cdot d\vec{r} = F(B) - F(A)$$

from $A \to B$

Gravitation

$$F_z = Gm \iiint \delta \cos\phi \sin\phi\, d\rho\, d\phi\, d\theta$$

$$\vec{F} = Gm \iiint_R \frac{x,y,z}{\rho^3}\,\delta\, dV$$

Flux $= \iint_S F \cdot \hat{n}\, dS$

Sphere $r = a$ at origin

$$\hat{n} = \frac{(x,y,z)}{a} \qquad dS = a^2 \sin\phi\, d\phi\, d\theta$$

Cylinder $r = a$ centered on z axis

$$\hat{n} = \frac{1}{a}(x,y,0) \qquad dS = a\, d\theta\, dz$$

Graph of $z = f(x,y)$

$$\hat{n}\, dS = (-f_x, -f_y, 1)\, dx\, dy$$

Divergence Theorem

$$\iint_S \vec{F} \cdot \hat{n}\, dS = \iiint_R \mathrm{div}(\vec{F})\, dV$$

$$\mathrm{div}(A,B,C) = A_x + B_y + C_z$$

$$\mathrm{Curl}(\vec{F}) = \vec{\nabla} \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ P & Q & R \end{vmatrix}$$

$\vec{F} = (P,Q,R)$

$$= (R_y - Q_z,\ P_z - R_x,\ Q_x - P_y)$$

* Curl of Gradient Vector Field $= 0$

1. (20pts) Set up a triple integral in cylindrical coordinates for the mass of the region of space bounded below by the paraboloid $z = x^2 + y^2$ and above by the plane $z = 4$. Assume the density $\delta = 3x$.

$$\text{Mass} = \iiint_R \delta \, dV$$

$$z = \underbrace{x^2 + y^2}_{r^2}$$

$$z = 4 = r^2$$

$$r = 2$$

$$z: \underbrace{x^2+y^2}_{} \to 4$$

$$r^2 \to 4$$

$$r: 0 \to 2$$

$$\theta: 0 \to 2\pi$$

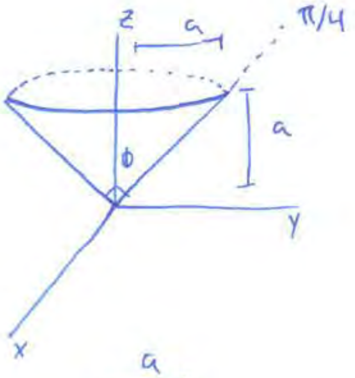
$$x = r\cos\theta$$

$$\int_{\theta=0}^{2\pi} \int_{r=0}^{2} \int_{z=r^2}^{4} 3x \underbrace{r \, dz \, dr \, d\theta}_{dV}$$

$$\boxed{\text{Mass} = \int_0^{2\pi}\int_0^2\int_{r^2}^4 3r^2\cos\theta \, dz \, dr \, d\theta}$$

2. (20pts) Set up an iterated integral, in both cylindrical and spherical coordinates, giving the average distance from the origin to the portion of the unit cylinder $x^2 + y^2 < 1$ which lies between $z = 0$ and $z = 1$.

3. (20pts) A solid D has the shape of a right circular cone, with axis along the z-axis, and a flat base. The base radius and the height are a. Set up an integral in spherical coordinates which gives the gravitational attraction on a unit mass placed at the vertex. Assume the density δ is one.



$\delta = 1$

$m = 1$

$F_x = F_y = 0$ by symmetry

$F_z = Gm \iiint \delta \sin\phi\cos\phi \, d\rho \, d\phi \, d\theta$

$\rho\cos\phi = a$

$\rho = \frac{a}{\cos\phi} = a\sec\phi$

$\rho : 0 \to a\sec\phi$

$\phi : 0 \to \pi/4$

$\theta : 0 \to 2\pi$

$$\boxed{F_z = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/4} \int_{\rho=0}^{a\sec\phi} \sin\phi\cos\phi \, d\rho \, d\phi \, d\theta}$$

4. (20pts) Let S be the surface formed by the part of the paraboloid $z = 1 - x^2 - y^2$ lying above the xy-plane. Orient S so that the normal vector is pointing upwards. Let $\vec{F} = x\hat{\imath} + y\hat{\jmath} + 2(1-z)\hat{k}$.

(i) Find the flux of $\vec{F}$ across S directly. $F = \langle x, y, 2(1-z) \rangle$

$\text{Flux} = \iint_S F \cdot \hat{n}\, dS$

$\hat{n}\, dS = (-f_x, -f_y, 1)\, dx\,dy$

$= \iint_S \langle x, y, 2(1-z) \rangle \cdot \langle 2x, 2y, 1 \rangle\, dx\,dy$

$= \iint_S 2x^2 + 2y^2 + 2\underbrace{(1-z)}_{-z}\, dx\,dy$

$= \iint_S 2(x^2 + y^2 + \cancel{1} - 1 + x^2 + y^2)\, dx\,dy$

$= \iint_S 2(2x^2 + 2y^2)\, dx\,dy$

$= \iint_S 4r^2\, r\, dr\, d\theta$

$= \iint_S 4r^3\, dr\, d\theta$

$\boxed{= 2\pi}$

(ii) By computing the flux across a simpler surface and using the divergence theorem.

5. (20pts) Let

$$\vec{F} = (y+z)\hat{\imath} - x\hat{\jmath} + (7x+5)\hat{k},$$

be a vector field and let S be the part of the surface $z = 9 - x^2 - y^2$ that lies above the xy-plane. Orient S by using the outward normal vector. Find the outward flux of $\vec{F}$ across S.

(2) 18.02 Midterm #4

1. (20pts) Let D be the domain in the first octant cut off by the plane $3x + 2y + z = 1$. Assume the density $\delta = z$.

(a) Set up an iterated integral in rectangular coordinates for the total mass of D.

$z = 1 - 2y - 3x$

$z: 0 \to 1 - 2y - 3x$

$y: 0 \to \frac{1}{2} - \frac{3}{2}x$

$x: 0 \to 1/3$

$2y = 1 - 3x$

$y = \frac{1}{2} - \frac{3}{2}x$

$x = 1/3$

$$\int_{x=0}^{1/3} \int_{y=0}^{1/2 - 3/2x} \int_{z=0}^{1-2y-3x} z \, dz \, dy \, dx$$

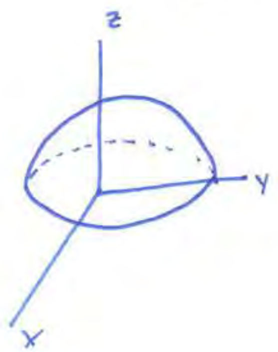
(b) Evaluate only the inner integral

$$\int_0^{1-2y-3x} z \, dz$$

$$\frac{z^2}{2} \Big|_0^{1-2y-3x}$$

$$= \frac{(1-2y-3x)^2}{2}$$

2. (20pts) A solid hemisphere has radius a and density 1 and it is placed so its flat side is on the xy-plane, with the centre at $(0,0)$. Set up and evaluate a triple integral in spherical coordinates which gives its gravitational attraction on a unit point mass at the origin $(0,0,0)$.



$F_x = F_y = 0$
By Symmetry

$F_z = G m \iiint \delta \sin\phi \cos\phi \, d\phi \, d\rho \, d\theta$

$\rho : 0 \to a$
$\phi : 0 \to \pi/2$
$\theta : 0 \to 2\pi$

$= \int_0^{2\pi} \int_0^{\pi/2} \int_0^{a} \sin\phi \cos\phi \, d\rho \, d\phi \, d\theta$

$\int_0^{\pi/2} a \sin\phi \cos\phi \, d\phi$

$u = \sin\phi$
$du = \cos\phi \, d\phi$

$\int a u \, du = \frac{a u^2}{2}$

$= \frac{a \sin^2 \phi}{2} \Big|_0^{\pi/2}$

$= \frac{a}{2} - 0$

$G \int_0^{2\pi} \frac{a}{2} \, d\theta = \frac{G \cancel{2} \pi a}{\cancel{2}}$

$\boxed{F_z = G \pi a}$

3. (20pts) Consider the surface S given by the equation

$$z = (x^2 + y^2 + z^2)^2.$$

(a) Show that S lies in the upper half space $z \geq 0$.

$(x^2+y^2+z^2)^2 \geq 0$ for any x, y or z

because the final quantity is squared

(b) Write out the equation for this surface in spherical coordinates.

$z = \rho\cos\phi$

$\rho^2 = x^2+y^2+z^2$

$\rho\cos\phi = \rho^4$

$\boxed{\cos\phi = \rho^3}$

(c) Write down an iterated integral for the volume of the region inside S.

Because $z \geq 0$ $\phi: 0 \to \pi/2$

$\rho: 0 \to (\cos\phi)^{1/3}$

$\theta: 0 \to 2\pi$

$$\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/2}\int_{\rho=0}^{(\cos\phi)^{1/3}} \underbrace{\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}_{dV}$$

$$\boxed{\int_0^{2\pi}\int_0^{\pi/2}\int_0^{(\cos\phi)^{1/3}} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}$$

4. (20pts) Let S be the portion of the ellipsoid $x^2 + y^2 + 3z^2 = 1$ that lies above the xy-plane. Let

$$\vec{F} = (x + y^3)\hat{\imath} + (2y - e^x)\hat{\jmath} - (3z + 1)\hat{k}.$$

Compute the flux of $\vec{F}$ through S (orient S upwards).

5. (20pts) Let S be the part of the surface $z = xy$ where $x^2 + y^2 < 1$. Compute the flux of

$$\vec{F} = y\hat{\imath} + x\hat{\jmath} + z\hat{k},$$

upward across S.

$z = xy$

$F = (y, x, z)$

$\text{Flux} = \iint_S F \cdot \hat{n}\, dS$

$\hat{n}\, dS = (-f_x, -f_y, 1)$

$(-y, -x, 1)$

$\iint_S F \cdot \hat{n}\, dS = \iint_S (y, x, z) \cdot (-y, -x, 1)$

$= \iint_S -y^2 - x^2 + \underbrace{z}_{xy}$

$= \iint (-r^2 + r^2 \cos\theta \sin\theta)\, r\, dr\, d\theta$

$= \int_0^{2\pi} \int_0^1 r^3 (\cos\theta \sin\theta - 1)\, dr\, d\theta$

$\dfrac{r^4 \cos\theta \sin\theta - 1}{4} \Big|_0^1 = \dfrac{\cos\theta \sin\theta - 1}{4}$

$\int_0^{2\pi} \dfrac{\cos\theta \sin\theta - 1}{4} = \frac{1}{4}\left(\frac{1}{2}\sin^2\theta - \theta\right)\Big|_0^{2\pi}$

$u = \sin\theta$

$du = \cos\theta\, d\theta$

$= \dfrac{-2\pi}{4} = \boxed{-\dfrac{\pi}{2}}$

③ 18.02 Midterm #4

18.02 Practice Exam 4B

Problem 1. (10 points)

Let C be the portion of the cylinder $x^2 + y^2 \leq 1$ lying in the first octant ($x \geq 0$, $y \geq 0$, $z \geq 0$) and below the plane $z = 1$. Set up a triple integral in *cylindrical coordinates* which gives the moment of inertia of C about the z-axis; assume the density to be $\delta = 1$.

(Give integrand and limits of integration, but *do not evaluate.*)

Problem 2. (20 points: 5, 5, 10)

a) A solid sphere S of radius a is placed above the xy-plane so it is tangent at the origin and its diameter lies along the z-axis. Give its equation in *spherical coordinates.*

b) Give the equation of the horizontal plane $z = a$ in spherical coordinates.

c) Set up a triple integral in spherical coordinates which gives the volume of the portion of the sphere S lying *above* the plane $z = a$. (Give integrand and limits of integration, but *do not evaluate.*)

Problem 3. (20 points: 5, 15)

Let $\vec{F} = (2xy + z^3)\,\hat{\imath} + (x^2 + 2yz)\,\hat{\jmath} + (y^2 + 3xz^2 - 1)\,\hat{\mathrm{k}}$.

a) Show that $\vec{F}$ is conservative.

b) Using a systematic method, find a potential function $f(x, y, z)$ such that $\vec{F} = \vec{\nabla} f$. Show your work, even if you can do it mentally.

Problem 4. (25 points: 15, 10)

Let S be the surface formed by the part of the paraboloid $z = 1 - x^2 - y^2$ lying above the xy-plane, and let $\vec{F} = x\,\hat{\imath} + y\,\hat{\jmath} + 2(1 - z)\,\hat{\mathrm{k}}$.

Calculate the flux of $\vec{F}$ across S, taking the upward direction as the one for which the flux is positive. Do this in two ways:

a) by direct calculation of $\iint_S \vec{F} \cdot \hat{\mathrm{n}}\, dS$;

b) by computing the flux of $\vec{F}$ across a simpler surface and using the divergence theorem.

Problem 5. (25 points: 10, 8, 7)

Let $\vec{F} = -2xz\,\hat{\imath} + y^2\,\hat{\mathrm{k}}$.

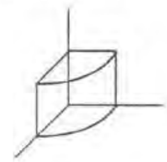
a) Calculate $\operatorname{curl} \vec{F}$.

b) Show that $\iint_R \operatorname{curl} \vec{F} \cdot \hat{\mathrm{n}}\, dS = 0$ for any finite portion R of the unit sphere $x^2 + y^2 + z^2 = 1$. (take the normal vector $\hat{\mathrm{n}}$ pointing outward).

c) Show that $\oint_C \vec{F} \cdot d\vec{r} = 0$ for any simple closed curve C on the unit sphere $x^2 + y^2 + z^2 = 1$.

18.02 Practice Exam 4B – Solutions

Problem 1.



$$\int_0^{\pi/2}\int_0^1\int_0^1 r^2\, r\, dz\, dr\, d\theta.$$

Problem 2.

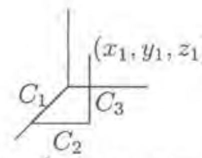
a) sphere: $\rho = 2a\cos\phi$. b) plane: $\rho = a\sec\phi$.

c) $\displaystyle\int_0^{2\pi}\int_0^{\pi/4}\int_{a\sec\phi}^{2a\cos\phi} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta.$

Problem 3.

a) $\frac{\partial}{\partial y}(2xy+z^3) = 2x = \frac{\partial}{\partial x}(x^2+2yz)$; $\frac{\partial}{\partial z}(2xy+z^3) = 3z^2 = \frac{\partial}{\partial x}(y^2+3xz^2-1)$;
$\frac{\partial}{\partial z}(x^2+2yz) = 2y = \frac{\partial}{\partial y}(y^2+3xz^2-1)$; so $\vec F$ is conservative.

b) <u>Method 1:</u> $f(x,y,z) = \int_{C_1+C_2+C_3} \vec F\cdot d\vec r$;



$\int_{C_1} \vec F\cdot d\vec r = \int_0^{x_1}(2xy+z^3)\,dx = \int_0^{x_1} 0\,dx = 0$ $(y=0,\ z=0)$
$\int_{C_2} \vec F\cdot d\vec r = \int_0^{y_1}(x^2+2yz)\,dy = \int_0^{y_1} x_1^2\,dy = x_1^2 y_1$ $(x=x_1,\ z=0)$
$\int_{C_3} \vec F\cdot d\vec r = \int_0^{z_1}(y^2+3xz^2-1)\,dz = \int_0^{z_1}(y_1^2+3x_1z^2-1)\,dz = y_1^2 z_1 + x_1 z_1^3 - z_1$ $(x=x_1,\ y=y_1)$
So $f(x,y,z) = x^2y + y^2z + xz^3 - z + c$.

<u>Method 2:</u> $\frac{\partial f}{\partial x} = 2xy+z^3$, so $f(x,y,z) = x^2y + xz^3 + g(y,z)$.
$\frac{\partial f}{\partial y} = x^2 + \frac{\partial g}{\partial y} = x^2 + 2yz$, so $\frac{\partial g}{\partial y} = 2yz$.
Therefore $g(y,z) = y^2z + h(z)$, and $f(x,y,z) = x^2y + xz^3 + y^2z + h(z)$.
$\frac{\partial f}{\partial z} = 3xz^2 + y^2 + h'(z) = y^2 + 3xz^2 - 1$, so $h'(z) = -1$.
Therefore $h(z) = -z + c$, and $f(x,y,z) = x^2y + xz^3 + y^2z - z + c$.

Problem 4.

a) S is the graph of $z = f(x,y) = 1 - x^2 - y^2$, so $\hat{n}\,dS = \langle -f_x, -f_y, 1\rangle\,dA = \langle 2x, 2y, 1\rangle\,dA$.
Therefore $\iint_S \vec F\cdot\hat n\,dS = \iint_S \langle x, y, 2(1-z)\rangle\cdot\langle 2x,2y,1\rangle\,dA = \iint_S 2x^2+2y^2+2(1-z)\,dA = \iint_S 4x^2 + 4y^2\,dA$ (since $z = 1-x^2-y^2$).
Shadow = unit disc $x^2+y^2\le 1$; switching to polar coordinates, we have
$\iint_S \vec F\cdot\hat n\,dS = \int_0^{2\pi}\int_0^1 4r^2\,r\,dr\,d\theta = \int_0^{2\pi}\left[r^4\right]_0^1 d\theta = 2\pi$.

b) Let $T =$ unit disc in the xy-plane, with normal vector pointing down ($\hat n = -\hat k$). Then
$\iint_T \vec F\cdot\hat n\,dS = \iint_T \langle x,y,2\rangle\cdot(-\hat k)\,dS = \iint_T -2\,dS = -2\,\text{Area} = -2\pi$. By divergence theorem,
$\iint_{S+T}\vec F\cdot\hat n\,dS = \iiint_D \operatorname{div}\vec F\,dV = 0$, since $\operatorname{div}\vec F = 1+1-2 = 0$. Therefore $\iint_S = -\iint_T = +2\pi$.

Problem 5.

a) $\operatorname{curl}\vec F = \begin{vmatrix} \hat\imath & \hat\jmath & \hat k \\ \partial_x & \partial_y & \partial_z \\ -2xz & 0 & y^2 \end{vmatrix} = 2y\hat\imath - 2x\hat\jmath.$

b) On the unit sphere, $\hat n = x\hat\imath + y\hat\jmath + z\hat k$, so $\operatorname{curl}\vec F\cdot\hat n = \langle 2y, -2x, 0\rangle\cdot\langle x,y,z\rangle = 2xy - 2xy = 0$; therefore $\iint_R \operatorname{curl}\vec F\cdot\hat n\,dS = 0$.

c) By Stokes, $\oint_C \vec F\cdot d\vec r = \iint_R \operatorname{curl}\vec F\cdot\hat n\,dS$, where R is the region delimited by C on the unit sphere. Using the result of b), we get $\oint_C \vec F\cdot d\vec r = \iint_R \operatorname{curl}\vec F\cdot\hat n\,dS = 0$.

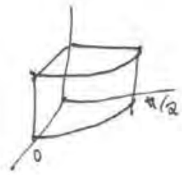
12/2/13 18.02 Midterm #4 Practice 3 Fernando Trejo

1)

$z: 0 \to 1$

$\theta: 0 \to \cancel{2\pi}$ ~~then~~ $\pi/2$

$r: 0 \to 1$

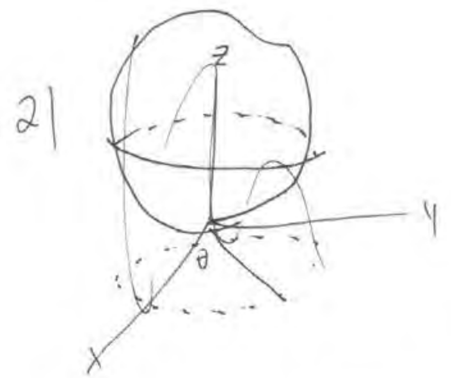


$$I_z = \iiint_R (x^2+y^2)\,\delta\, dV$$

$$= \iiint r^2 r \, dr\, d\theta\, dz$$

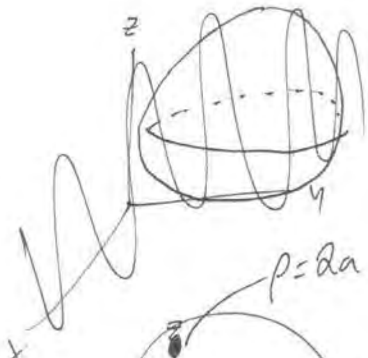
$$\boxed{\int_0^{\pi/2} \int_0^1 \int_0^1 r^3 \, dr\, dz\, d\theta}$$

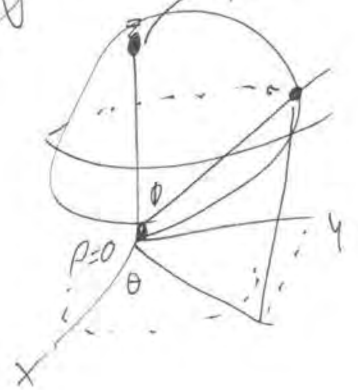
2)



θ

$\rho =$





$$\boxed{\rho = 2a\underbrace{\cos\phi}}$$ ✓

$\cos 0 = 1$
so $\rho = 2a$ ✓

b) $z = a$
~~ρ a cos φ = a~~ $z = \rho\cos\phi$
$a = z = \rho\cos\phi$
$\rho = a/\cos\phi$ $\boxed{\rho = a\sec\phi}$ ✓

c)

$$\boxed{\int_0^{2\pi}\int_0^{\pi/4}\int_{a\sec\phi}^{2a\cos\phi} \overbrace{\rho^2\sin\phi\,d\rho\,d\phi\,d\theta}^{dV}}$$ ✓

3) $\vec{F} = \langle \overbrace{2xy + z^3}^{P},\ \overbrace{x^2 + 2yz}^{Q},\ \overbrace{y^2 + 3xz^2 - 1}^{R} \rangle$

a) Conservative if curl $F = 0$

$$\mathrm{Curl}(F) = \begin{vmatrix} i & j & k \\ dx & dy & dz \\ P & Q & R \end{vmatrix} = (R_y - Q_z) - (R_x - P_z) + (Q_x - P_y)$$

$$= (2y - 2y) - (3z^2 - 3z^2) + (2x - 2x)$$

$$= 0 \quad ✓✓$$

b) $f_y = x^2 + 2yz$

$f = x^2y + y^2z + g(x, z)$

$f_z = y^2 + g_z$

$f_z = y^2 + 3xz^2 - 1$

$g_z = 3xz^2 - 1$

$g = xz^3 - z + h(z)$

$$\boxed{f = x^2y + y^2z + xz^3 - z \cancel{+ h(z)}} \quad ✓$$

$f_x = 2xy + z^3 \Rightarrow h(z) = \text{Konstant}$

4) $z = 1 - x^2 - y^2$

$\vec{F} = \langle x, y, 2(1-z) \rangle$

a) $\text{Flux} = \iint_S F \cdot \hat{n}\, dS$

$\hat{n}\, dS = (-f_x, -f_y, 1)$

$= (2x, 2y, 1)$

$(2x, 2y, 1) \cdot \langle x, y, 2(1-z) \rangle$

$2x^2 + 2y^2 + 2(1-z)$

$2(x^2 + y^2 + \cancel{1 - 1} + x^2 + y^2)$

$\iint_S 4(x^2 + y^2)$

$\int_0^{2\pi} \int_0^1 4r^2\, r\, dr\, d\theta$

$\int_0^{2\pi} \int_0^1 4r^3\, dr\, d\theta$

$\boxed{= 2\pi}$ ✓

b)

5)

$F = \cancel{\ldots}$ $\cancel{-2xz + y^2}$

a) $\langle -2xz, \underset{B}{0}, \underset{C}{y^2} \rangle$ with $A = -2xz$

$$\text{curl}(F) = \begin{vmatrix} i & j & k \\ dx & dy & dz \\ A & B & C \end{vmatrix} = (C_y - B_z) - (C_x - A_z) + (B_x - A_y)$$

$$(2y - 0), (0 + 2x) + (0 - 0)$$

$$= (2y, -2x) \checkmark$$

b) $\iint_R \text{curl } F \cdot \vec{n} \, dS = 0$

$$(2y, -2x, 0) \cdot (x, y, z)$$

$$\cancel{2yx - 2yx} - 0 = 0 \checkmark$$

c) $\oint F \cdot d\vec{r} = 0$

$$\oint_C F \cdot dr = \iint_R \text{curl } F \cdot n \, dS = 0 \checkmark$$

Stokes

④ 18.02 Midterm #4

18.02 – Practice Exam 4A

Problem 1. (15)

a) Show that the vector field

$$\mathbf{F} = \langle e^x yz, e^x z + 2yz, e^x y + y^2 + 1\rangle$$

is conservative.

b) By a systematic method, find a potential for $\mathbf{F}$.

c) Show that the vector field $\mathbf{G} = \langle y, x, y\rangle$ is not conservative.

Problem 2. (20) Let S be the part of the spherical surface $x^2 + y^2 + z^2 = 4$, lying in $x^2 + y^2 > 1$, which is to say outside the cylinder of radius one with axis the z-axis.

a) Compute the flux outward through S of the vector field $\mathbf{F} = y\mathbf{i} - x\mathbf{j} + z\mathbf{k}$.

b) Show that the flux of this vector field through any part of the cylindrical surface is zero.

c) Using the divergence theorem applied to $\mathbf{F}$, compute the volume of the region between S and the cylinder.

Problem 3. (20) Let S be the part of the spherical surface $x^2 + y^2 + z^2 = 2$ lying in $z > 1$. Orient S upwards and give its bounding circle, C, lying in $z = 1$ the compatible orientation.

a) Parametrize C and use the parametrization to evaluate the line integral

$$I = \oint_C xzdx + ydy + ydz.$$

b) Compute the curl of the vector field $\mathbf{F} = xz\mathbf{i} + y\mathbf{j} + y\mathbf{k}$.

c) Write down a flux integral through S which can be computed using the value of I.

Problem 4. (15) Use the divergence theorem to compute the flux of $\mathbf{F} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ outwards across the closed surface $x^4 + y^4 + z^4 = 1$.

Problem 5. (15) Consider the surface S given by the equation

$$z = (x^2 + y^2 + z^2)^2.$$

a) Show that S lies in the upper half space ($z \geq 0$).

b) Write out the equation for the surface in spherical polar coordinates.

c) Using the equation obtained in part b), give an iterated integral, with explicit integrand and limits of integration, which gives the volume of the region inside this surface. Do not evaluate the integral.

Problem 6. (15) Let S be the part of the surface $z = xy$ where $x^2 + y^2 < 1$. Compute the flux of $\mathbf{F} = y\mathbf{i} + x\mathbf{j} + z\mathbf{k}$ upward across S by reducing the surface integral to a double integral over the disk $x^2 + y^2 < 1$.

18.02 Practice Exam 4A - Solutions

1a)

$$M_y = e^x z = N_x$$
$$M_z = e^x y = P_x$$
$$N_z = e^x + 2y = P_y$$

1b) We begin with

$$\begin{cases} f_x = e^x yz \\ f_y = e^x z + 2yz \\ f_z = e^x y + y^2 + 1 \end{cases}$$

Integrating f_x we get $f = e^x yz + g(y, z)$. Differentiating and comparing with the above equations we get

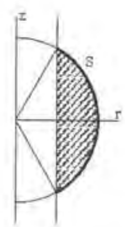
$$\begin{cases} f_y = e^x z + g_y \\ f_z = e^x y + g_z \end{cases} \rightarrow \begin{cases} g_y = 2yz \\ g_z = y^2 + 1 \end{cases}$$

Integrating g_y we get $g = y^2 z + h(z)$. Then $g_z = y^2 + h'(z)$ so comparing with the second equation above we get $h'(z) = 1$. Hence $h = z + C$. Putting everything together we get

$$f = e^x yz + y^2 z + z + C$$

1c) $N_z = 0$ and $P_y = 1$ hence the field is not conservative.

2a) Consider the figure



$\vec{n} = \frac{1}{2}(x, y, z)$ hence

$$\vec{F} \cdot \vec{n} = (y, -x, z) \cdot \frac{(x, y, z)}{2} = \frac{z^2}{2}$$

$z = 2\cos\phi$ and $dS = 2^2 \sin\phi \, d\phi \, d\theta$ hence we get

$$\int_0^{2\pi} \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{4\cos^2\phi}{2} 4 \sin\phi \, d\phi \, d\theta = 16\pi \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \cos^2\phi \sin\phi d\phi = -16\pi \left[\frac{\cos^3\phi}{3}\right]_{\frac{\pi}{6}}^{\frac{5\pi}{6}} = 4\sqrt{3}\pi$$

2b) $\vec{n} = \pm(x, y, 0)$ hence $\vec{F} \cdot \vec{n} = 0$. So the flux is 0.

2c) $div\vec{F} = 1$ hence

$$Vol(R) = \iiint_R 1 \, dV = \iiint_R div\vec{F} \, dV = \iint_S \vec{F} \cdot \vec{n} \, dS + \iint_{\text{Cylinder}} \vec{F} \cdot \vec{n} \, dS = 4\sqrt{3}\pi$$

3a) C is given by the equations $x^2 + y^2 + z^2 = 2$ and $z = 1$. So $x^2 + y^2 = 1$. Parametrization:

$$x = \cos t \qquad y = \sin t \qquad z = 1$$
$$dx = -\sin t \, dt \qquad dy = \cos t \, dt \qquad dz = 0$$

(5) 18.02 Midterm #4

1. (20pts) Let R be the region in space which lies above the xy-plane and below the paraboloid $z = 1 - x^2 - y^2$. Calculate the moment of inertia about the z-axis; assume the density $\delta = 1$.

Solution:

$$I_z = \iiint_R x^2 + y^2 \, \mathrm{d}V = \int_0^{2\pi} \int_0^1 \int_0^{1-r^2} r^3 \, \mathrm{d}z \, \mathrm{d}r \, \mathrm{d}\theta.$$

The inner integral is

$$\int_0^{1-r^2} r^3 \, \mathrm{d}z = \left[r^3 z \right]_0^{1-r^2} = r^3(1 - r^2).$$

The middle integral is

$$\int_0^1 r^3 - r^5 \, \mathrm{d}r = \left[\frac{r^4}{4} - \frac{r^6}{6} \right]_0^1 = \frac{1}{12}.$$

The outer integral is

$$\int_0^{2\pi} \frac{1}{12} \, \mathrm{d}\theta = \frac{\pi}{6}.$$

2. (20pts) (i) A solid sphere of radius a is placed above the xy-plane so it is tangent at the origin and so the z-axis is a diameter. Give its equation in spherical coordinates.

Solution:
Using symmetry in θ, we might as well assume that $\theta = 0$ and we may work in the xz-plane. So we have a circle, centred at $(0, a)$ of radius a;

$$x^2 + (z-a)^2 = a^2 \qquad \text{so that} \qquad x^2 + z^2 = 2za.$$

As $x^2 + z^2 = \rho^2$ and $z = \rho\cos\phi$, the equation of the sphere is

$$\rho = 2a\cos\phi.$$

(ii) Give the equation of the horizontal plane $z = a$ in spherical coordinates.

Solution:

$$\rho\cos\phi = a.$$

(iii) Set up a triple integral in spherical coordinates which gives the volume of the portion of the sphere S lying above the plane $z = a$.

Solution: Let's figure out the limits for ϕ. ϕ starts at zero. We go down all the way to a point on the plane $\rho\cos\phi = a$ and on the sphere $\rho = 2a\cos\phi$. So $2\cos^2\phi = 1$, that is, $\phi = \pi/4$ (one can also check this by drawing a picture).

$$\iiint_S 1\,\mathrm{d}V = \int_0^{2\pi}\int_0^{\pi/4}\int_{a\sec\phi}^{2a\cos\phi} \rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta.$$

3. (20pts) Let D be the disk described by $x^2 + y^2 \leq 4$ and $z = 2$. Let T be the right circular cone formed by joining every point of D to the origin, so that $(0,0,0)$ is the vertex of the cone. Assume that T has constant density $\delta = 1$. Set up an iterated integral that gives the magnitude of the gravitational force acting on a unit mass at the origin.

Solution: If

$$\vec{F} = \langle F_x, F_y, F_z \rangle,$$

is the force due to gravity then $F_x = F_y = 0$ by symmetry. We have

$$|\vec{F}| = F_z = \iiint_T \frac{Gz}{\rho^3}\, \mathrm{d}V = G \int_0^{2\pi} \int_0^{\pi/4} \int_0^{2\sec\phi} \cos\phi \sin\phi \, \mathrm{d}\rho \, \mathrm{d}\phi \, \mathrm{d}\theta.$$

4. (20pts) Find the flux of the vector field

$$\vec{F} = y^4\hat{\imath} - x^3\hat{\jmath} + z\hat{k},$$

coming out of the unit sphere centred at the origin.

Solution: We apply the divergence theorem. Let S be the surface of the unit sphere, oriented outwards and let V the solid enclosed by S. Then the flux coming out of the sphere is

$$\oiint_S \vec{F} \cdot \mathrm{d}\vec{S} = \iiint_V \operatorname{div} \vec{F} \, \mathrm{d}V = \iiint_V 1 \, \mathrm{d}V = \frac{4\pi}{3}.$$

5. (20pts) Let

$$\vec{F} = (y+z)\hat{\imath} - x\hat{\jmath} + (7x+5)\hat{k},$$

be a vector field and let S be the part of the surface $z = 9 - x^2 - y^2$ that lies above the xy-plane. Orient S by using the outward normal vector. Find the outward flux of $\vec{F}$ across S.

Solution: Let S' be the surface $x^2 + y^2 < 9$, $z = 0$, oriented upwards. By the divergence theorem

$$\oiint_{S-S'} \vec{F} \cdot \mathrm{d}\vec{S} = \iiint_V \operatorname{div} \vec{F} \, \mathrm{d}V = \iiint_V 0 \, \mathrm{d}V = 0.$$

So

$$\iint_S \vec{F} \cdot \mathrm{d}\vec{S} = \iint_{S'} \vec{F} \cdot \mathrm{d}\vec{S}.$$

The unit normal to S' is $\hat{k}$. So

$$\vec{F} \cdot \hat{k} = 7x + 5.$$

We have

$$\iint_{S'} \vec{F} \cdot \mathrm{d}\vec{S} = \iint_{S'} 7x + 5 \, \mathrm{d}A = \iint_{S'} 5 \, \mathrm{d}A = 45\pi,$$

since x is skew-symmetric about the y-axis.

12/2/13

18.02 Recitation

Midterm Review

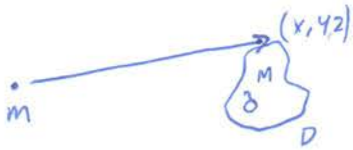
Fernando Trujano

- 3D Integration
- Triple Integrals
- Surface Integrals
- Divergence Theorem
- Physics Applications

$|F| = G\frac{Mm}{|R|^2}\hat{R}$ ← unit $\frac{\vec{R}}{|\vec{R}|}$

$\Downarrow$

Gravitational Force



$$\vec{F} = Gm\iiint_D \frac{(x,y,z)}{\rho^3}\,\delta\, dV$$

$\delta\, dV$ → M

$$\frac{\vec{R}}{|\vec{R}|^3}$$

So

$$\boxed{\vec{F} = Gm\iiint_D \frac{x,y,z}{\rho^3}\,\delta\, dV}$$

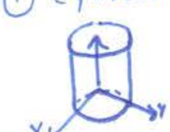
z component

$$F_z = Gm\iiint_D \frac{z}{\rho^3}\,\delta\, dV$$

Surface Integrals

How to get $d\vec{S} = \hat{n}\, dS$ ← $dV = r\,dr\,d\theta\, dz = dS\, dr$

① Cylinder centered at z-axis



$\hat{n} = \frac{(x,y,0)}{r}$ ← $\sqrt{x^2+y^2}$

$dS = r\,d\theta\, dz \Rightarrow \boxed{d\vec{S} = (x,y,0)\,d\theta\, dz}$

② Sphere centered at origin

$\hat{n} = \frac{(x,y,z)}{\rho}$ $\boxed{d\vec{S} = (x,y,z)\rho\sin\phi\, d\phi\, d\theta}$

$dS = \rho\sin\phi\, d\phi\, d\theta$

③ Plane

- case 1: $z = \text{const}$ $\boxed{d\vec{S} = (0,0,1)\,dx\,dy}$
- $x = \text{const}$ $\boxed{d\vec{S} = (1,0,0)\,dy\,dz}$
- $y = \text{const}$ $\boxed{d\vec{S} = (0,1,0)\,dx\,dz}$

④ Level Surface $F(x,y,z) = \text{const}$ $\hat{n} = \frac{\nabla f}{|\nabla f|}$

$$\boxed{d\vec{S} = \pm\frac{\nabla F}{\partial F/\partial z}\,dx\,dy \quad = \frac{\nabla F}{\partial F/\partial x}\,dy\,dz}$$

Plane

$f = ax + by + cz = \text{constant}$

$\nabla f = (a, b, c)$

$d\vec{S} = \pm\left(\frac{a}{c}, \frac{b}{c}, \overset{1}{\frac{c}{c}}\right) dx\,dy \quad \text{if } c \neq 0$

or $\pm\left(\frac{a}{b}, 1, \frac{c}{b}\right) dx\,dz \quad \text{if } b \neq 0$

or $\pm\left(1, \frac{b}{a}, \frac{c}{a}\right) dy\,dz \quad \text{if if } a \neq 0$

12/5/13 18.02 Lecture

1) div, ∇, curl
2) Topology
3) Conservative V.F
4) More Topology (optional)

1) Div, ∇, curl

- Involve taking one derivative

div: VF → function

∇: function → VF

curl: VF → VF

$$f \to \nabla F \nearrow \operatorname{div}(\nabla F) \leftarrow \overbrace{F_{xx} + F_{yy} + F_{zz}}^{\text{Laplacian}} = \Delta F$$

$$f \to \nabla F \searrow \operatorname{curl}(\nabla F) = 0_{\text{always}}$$

$$\begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ f_x & f_y & f_z \end{vmatrix} = (f_{zy} - f_{yz},\ f_{xz} - f_{zx},\ f_{yx} - f_{xy}) = (0,0,0)$$

$\nabla \operatorname{div}(\vec{F}) =$ junk $\qquad \operatorname{curl}(\operatorname{curl}(\vec{F})) =$ junk

$\operatorname{div}(\operatorname{curl}(F)) = 0$

Proof $\qquad F = P, Q, R$

$$\operatorname{curl}(F) = \begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ P & Q & R \end{vmatrix} = \langle R_y - Q_z,\ P_z - R_x,\ Q_x - P_y \rangle$$

$$\operatorname{div}(\operatorname{curl}(F)) = R_{yx} - Q_{zx} + P_{zy} - R_{xy} + Q_{xz} - P_{yz} = 0$$

if div = 0 → incompressible

if curl = 0 → irrotational

2) Topology (in $\mathbb{R}^3$)

∇f has curl zero

Is it true that $\text{curl}(\tilde{F}) = 0$
$\Rightarrow \vec{F}$ is ∇f for some f

Not Always!

But it is true on a $\underline{\text{simply connected region}}$ R

Every closed curve C in R bounds a surface in R

take away

$R^3 / z\text{-axis}$ => not simply connected $\oint$

$R^3 \setminus \text{circle}$ => not simply connected

R^3 / point => simply connected

3)

suppose R is simply connected and $\text{curl}(F) = 0$

line integral

must show $\int_C \vec{F} \cdot d\vec{r} = 0$ for every closed C in R

Stoke's Theorem

$$\int_C F \cdot dr = \iint_S \text{curl}(\vec{F}) \cdot \hat{n} \, dS$$

MOAR

4) TOPOLOGY

1-dimentional curves

closed

finate, no boundry

Surfaces

closed

oriented

Same Topology

Donut

Not Simply Conneded

called "genus"
= # of holes you can stick
your finger through

Review I

- Linear Algebra
- Derivatives (∇, max-min, Lagrange)

2 lines in R^3

$\vec{r}_1(t) = (-4+5t, 1+t, -2-t)$

Direction = $(5, 1, -1)$

$$\begin{cases} 2x-y-z=6 & \text{normal} = (2,-1,-1) \\ x+y-2z=3 & \text{normal} = (1,1,-2) \end{cases}$$

∴ not the same or not multiples ⇒ not parallel

intersection of two planes

Find distance b/w two lines.

Approach: Parametrize 2^{nd} line – $\vec{r}_2(s)$
and then minimized $F(s,t) = |r_1(t) - r_2(t)|^2$ distane b/w lines.

Finding $\vec{r}_2(s)$ ∴ Find 2 pts

$2x-y-z=6$
$x+y-2z=3$

$z=0$ … choose any equation and get lucky.

Solve: $2x-y=6$, $x+y=3$ … $3x=9$, $x=3$, $y=0$, $z=0$

First Point $(3,0,0)$

now find $x=0$ solve: ~~2x~~ $-y-z=6$, $y-2z=3$

$-3z=9$, $z=-3$, $y=-3$, $x=0$

$(0,-3,-3)$

Direction = $(3,3,3) \Rightarrow (1,1,1)$ subtract points

$\vec{r}_2(s) = (3+s, s, s)$ subtract r_1 and r_2

$f = (-4+5t-(3+s))^2 + (1+t-s)^2 + (-2-t-s)^2$

chain rule

$f_s = 2(-7+5t-s)(-1) + 2(1+t-s)(-1) + 2(-2-t-s)(-1)$

$= 6s - 10t + 16$

$f_t = 2(-7+5t-s)5 + 2(1+t-s) + 2(-2-t-s)(-1)$

$= -64 - 10s + 54t$

Set $f_s = f_t = 0$

$f_s = 3s - 5t = -8$

$-5s + 27t = 32$

now $t=1$ $s=-1$ $\Rightarrow$ critical pt is at $t=1$ and $s=-1$

$(1,2,-3)$ $(2,-1,-1)$

$\text{dist} = \sqrt{1+9+4}$

$= \sqrt{14}$

$$\begin{pmatrix} 3 & -5 \\ -5 & 27 \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} -8 \\ 32 \end{pmatrix}$$

Check what happens at infinity.

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $A\vec{x} = b$

If A^{-1} exists then $\vec{x} = A^{-1}\vec{b}$

Test for invertibility: $\det(A) \neq 0$

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$\det(A) = 3 \cdot 27 - (5)(-5) = 56 \neq 0$ $A^{-1} = \frac{1}{56} \begin{pmatrix} 27 & 5 \\ 5 & 3 \end{pmatrix}$

2/3

Solution

$$\begin{pmatrix} s \\ t \end{pmatrix} = \frac{1}{56}\begin{pmatrix} 27 & 5 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} -8 \\ 32 \end{pmatrix} = \frac{1}{56}\begin{pmatrix} 27(-8) + 5(32) \\ \cancel{-40} + 96 \end{pmatrix}$$

-40

$$\begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} -1 \\ t \end{pmatrix} \qquad s = -1, \quad t = 1$$

$4x^2 + 4y^2 - z^2$

Find tangent plane at $(1,0,1)$

normal to plane is $\nabla(2x^2 + 4y^2 - z^2)$

$= (4x, 8y, -2z)$ at $(1,0,1)$

$(4,0,-2)$

$4x - 2z = d$

stick in $(1,0,1)$

$d = 2$

Find closest pt on unit sphere to $(1,2,3)$

Minimize $F(x,y,z) = (x-1)^2 + (y-2)^2 + (z-3)^2$

constraint $g(x,y,z) = x^2 + y^2 + z^2 - 1 = 0$

$f_x = \lambda g_x$

$f_y = \lambda g_y$

$f_z = \lambda g_z$

constraint $x^2 + y^2 + z^2 = 1$

$\cancel{2}(x-1) = \cancel{2}\lambda x$

$\cancel{2}(y-2) = \cancel{2}\lambda y$

$\cancel{2}(z-3) = \cancel{2}\lambda z$

$x^2 + y^2 + z^2 = 1$

$(1-\lambda)x = 1$

$(1-\lambda)y = 2$

$(1-\lambda)z = 3$

$x^2 + y^2 + z^2 = 1$

Solve Order

See 2nd Deriv Test

$(1-\lambda)^2(x^2+y^2+z^2) = 1 + 4 + 9 = 14$

$(1-\lambda) = \pm\sqrt{14}$

$x = 1/\sqrt{14}$ $\quad z = 3/\sqrt{14}$

$y = 2/\sqrt{14}$

or opposite

2/9/13

18.02 Recitation

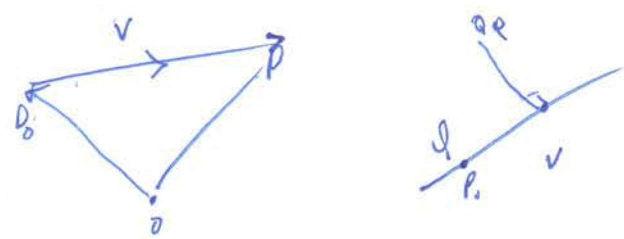
Distances

Min will occur along orthogonal line.

a) pt & line

line ℓ through P_0 & direction $\vec{v}$

$$\underset{(x,y,z)}{P} \in \ell \Rightarrow P - P_0 = t\vec{v}$$



$QR \perp \ell$

$$\text{Dist}(\ell, Q) = \min \text{ dist}(P \in \ell, Q) = \text{dist}(\underset{\ell}{R}, Q)$$

Example $(x,y,z) - \underbrace{(1,0,3)}_{P_0} = t\underbrace{(1,2,3)}_{v}$

$Q = (1, 0, -1)$

Review II Integration

- Line Integrals { Green / Flux ($d\vec{r}$ vs $n d\vec{s}$) }
- Change of variables
- Order of integration
- Surface Integrals and Flux - divergence theorem
- Curl and Stokes

Line Integrals

$\vec{F}(P,Q)$ in $\mathbb{R}^2$

curve C parametrized by $\vec{r}(t) = (x(t), y(t))$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot \underset{\substack{\uparrow \\ \text{unit} \\ \text{tangent}}}{\vec{T}}\, ds = \int_C F\cdot(dx, dy) = \int_C P\,dx + Q\,dy$$

Example

$\vec{F} = (x, y)$

$\vec{r}(t) = (\cos t, \sin t)$

$dr = (-\sin t, \cos t)\, dt$

$$\oint_C F\cdot dr = \int_{t=0}^{2\pi} (\overset{x}{\cos t}, \overset{y}{\sin t}) \cdot (-\sin t, \cos t)\, dt$$

$= 0$

Green: $\oint_C \vec{F}\cdot d\vec{r} = \iint_R \operatorname{curl}(\vec{F})\, dx\, dy$

In $\mathbb{R}^2$

$\operatorname{curl}(P,Q) = Q_x - P_y$ so $\operatorname{curl}(x,y) = \frac{dy}{dx} - \frac{dx}{dy} = 0$

$$\oint_C P\,dx + Q\,dy \overset{\text{Green}}{=} \iint_R (Q_x - P_y)\, dx\, dy$$

Flux integrals on planar curves

$\vec{r}(t)$ curve

$\vec{n}$ = unit normal $\qquad \int_C \vec{F}\cdot\vec{n}\, ds$

$$\vec{n}\, dS = \pm(-dy, dx)$$

$\vec{T}ds = (dx, dy)$

Rotate 90°

Example

$\vec{F} = (x, y)$

$\vec{r}(t) = (\cos t, \sin t)$

$\vec{n}$ = outward normal $\Rightarrow \vec{n}\, dS = (+dy, -dx)$

$$\oint_C \vec{F}\cdot\vec{n}\, dS = \int_C \vec{F}\cdot(dy, -dx) = \oint_C P\,dy - Q\,dx \overset{\text{normal Greens}}{=} \iint_R \overbrace{(P_x + Q_y)}^{\text{div}}\, dx\, dy$$

$$= 2\pi$$

Change of Variables

Area of region $\quad (2x+y)^2 + (x-y)^2 \le 5$

set $\quad u = 2x+y \qquad u^2 + v^2 \le 5$
$\quad v = x-y$

Jacobian J
relates $dx\,dy$ and $du\,dv$

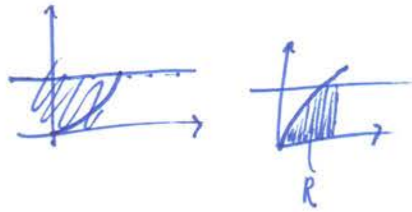
$$J = \frac{\partial(u,v)}{\partial(x,y)} = \det\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = -2-1 = -3$$

$$du\,dv = |J|\,dx\,dy = 3\,dx\,dy$$

$$\iint_{(2x+y)^2+(x-y)^2 \le 5} dx\,dy = \iint_{u^2+v^2 \le 5} \tfrac{1}{3}\, du\,dv = \tfrac{1}{3}(5\pi)$$

Order of integration

$$\int_0^1 \int_{y^3}^1 \frac{6y^2}{x^2+2}\, dx\, dy$$ ... ewww => Switch order of integration



$x: 0 \to 1$
$y: 0 \to x^{1/3}$

$$\int_{x=0}^{1} \int_{y=0}^{x^{1/3}} \frac{6y^2}{x^2+2}\, dy\, dx$$

$$= \int_{x=0}^{1} \frac{2x}{x^2+2}\, dx$$

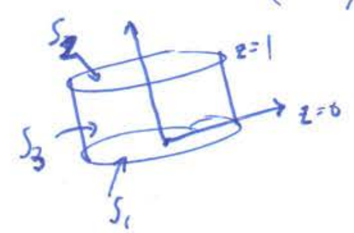
$u = x^2+2$
$du = 2dx$ $\quad = \int \frac{du}{u}$

$$= \ln|x^2+2| \Big|_0^1$$

$$= \ln\frac{3}{2}$$

Surface Integrals

$$\vec{F} = (xz^2,\ yz^2,\ z^3)$$



Find Outward Flux

Surface Int $\iint_{S_1} F\cdot \hat{n}\, dS = 0$

$$\iint_{S_2} \vec{F}\cdot\hat{n}\, dS = \iint_2 (x,y,1)\cdot \overset{\text{outward normal}}{(0,0,1)}\, dS = \pi$$

$$\iint_{S_3} (\cos\theta\, z^2,\ \sin\theta\, z^2,\ z^3)\cdot(\cos\theta,\ \sin\theta,\ 0)\, d\theta\, dz$$

$$\int_{\theta=0}^{2\pi}\int_{z=0}^{1} z^2\, dz\, d\theta = \frac{2\pi}{3}$$

Parametrize

$\vec{r}(\theta, z) = (\cos\theta, \sin\theta, z)$

$$n\,dS = \vec{r}_\theta \times \vec{r}_z = \begin{vmatrix} i & j & k \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \overset{\text{outward pointing}}{(\cos\theta,\ \sin\theta,\ 0)}\, d\theta\, dz$$

Add all $0 + \frac{2\pi}{3} + \pi = \frac{5\pi}{3}$

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18.02 Final Review

Matrix Multiplication

row1 · col1   row1 · col 2

row2 · col 1   . . . . .

Inverse of 3×3 Matrix

$$A = \begin{vmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ -1 & -3 & 0 \end{vmatrix} \xRightarrow{\text{cofactor}} \begin{vmatrix} 3 & 1 & 3 \\ -3 & -1 & -1 \\ 5 & 1 & 3 \end{vmatrix} \xrightarrow{\text{sign change}} \begin{vmatrix} 3 & -1 & 3 \\ 3 & -1 & 1 \\ 5 & -1 & 3 \end{vmatrix} \xrightarrow{\text{Transpose}} \begin{vmatrix} 3 & 3 & 5 \\ -1 & -1 & -1 \\ 3 & 1 & 3 \end{vmatrix}$$

det $\begin{vmatrix} 3 & 1 \\ -3 & 0 \end{vmatrix}$

$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$

$\det(A) \neq 0$ ∴ Can take inverse

$= -2$

divide by $\det(A)$

$$\begin{vmatrix} -3/2 & -3/2 & -5/2 \\ 1/2 & +1/2 & 1/2 \\ -3/2 & -1/2 & -3/2 \end{vmatrix}$$

Line Parametrization

(Point) + t(direction)

Directional Derivatives

$D_u f = \nabla \cdot u$

$\nabla f = \langle f_x, f_y, f_z \rangle$

Second Derivative Test

$A = f_{xx}$ $B = f_{xy}$ $C = f_{yy}$ for each critical point

$\Delta = AC - B^2$

if $A > 0$ $\Delta > 0 \to \vee$ min

$A < 0$ $\Delta > 0 \to \wedge$ max

$\Delta < 0 \to$ saddle

Lagrange Multipliers

$\nabla f = \lambda \nabla g$

+ constraint $\to g(x,y)$

Chain Rule

$w = f(x, y, z)$ $f_x x' + f_y y' + f_z z'$

$x = g(u,v)$

$y = h(u,v)$ $\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}$

$z = k(u,v)$ $f' = \nabla f \cdot \langle x', y', z' \rangle$

Planes Tangent to surface

$\nabla f = \langle a, b, c \rangle$

$P = (P_1, P_2, P_3)$ $0 = \nabla f \cdot \langle x - P_1, y - P_2, z - P_3 \rangle$

Change of Variables

$$J_T = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} U_x & U_y \\ V_x & V_y \end{vmatrix} \Rightarrow du\,dv = J_T\,dx\,dy$$

OR

$$J_T = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} X_u & X_v \\ Y_u & Y_v \end{vmatrix} \Rightarrow J_T\,du\,dv = dx\,dy$$

1. (15pts) A rectangular box lies in the corner of the first octant, with one vertex at the origin and the diagonally opposite vertex D at $(2,3,1)$. Three other vertices are $A = (2,0,1)$, $B = (0,3,1)$ and $P = (2,3,0)$.

(i) Express the vectors $\overrightarrow{PA}$ and $\overrightarrow{PB}$ in terms of $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$.

$$\vec{PA} = A - P = \langle 0, -3, 1 \rangle \Rightarrow -3\hat{j} + \hat{k}$$

$$\vec{PB} = B - P = \langle -2, 0, 1 \rangle \Rightarrow -2\hat{i} + \hat{k}$$

(ii) Find the cosine of the angle APB.

$$\vec{PA} \cdot \vec{PB} = |PA||PB| \cos\theta$$

$$\cos\theta = \frac{PA \cdot PB}{|PA||PB|} = \frac{1}{\sqrt{10}\sqrt{5}} = \frac{1}{5\sqrt{2}}$$

(iii) Find the equation of the plane through P, A and B.

$$\begin{vmatrix} i & j & k \\ 0 & -3 & 1 \\ -2 & 0 & 1 \end{vmatrix} = \langle -3, -2, -6 \rangle$$

$$-3(x-2) + -2(y-0) + -6(z-1) = 0$$

$$-3x + 6 - 2y - 6z + 6 = 0$$

$$-3x - 2y - 6z = -12$$

2. (10pts) What are the closest points on the plane $2x - y + z = 1$ to the line given parametrically by

$$\vec{r}(t) = \langle 1+t, 2+3t, 2+t \rangle?$$

3. (10pts) Suppose a particle moves according to

$$\vec{r}(t) = \langle \cos t, \sin t, \sin 2t \rangle.$$

Find the speed and acceleration vector $\vec{a}(t)$ at time t.

$$v(t) = \langle -\sin t, \cos t, 2\cos(2t) \rangle$$

$$|v(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2 + (2\cos(2t))^2}$$

$$= \sqrt{1 + 4\cos^2(2t)}$$

$$a(t) = \langle -\cos t, -\sin t, -4\sin(2t) \rangle$$

4. (15pts) Let

$$A = \begin{pmatrix} -2 & 1 & 1 \\ a & -1 & -2 \\ -4 & 1 & a \end{pmatrix}.$$

det(A) = 0

(i) For which values of a is A not invertible?

$$\det(A) = -2(-a+2) - (a\,1(a^2 - 8)) + 1(a - 4)$$

$$2a - 4 - a^2 + 8 + a - 4 = 0$$

$$-a^2 - 3a = 0$$

$$a = 0$$
$$a = 3$$

(ii) Let B be the matrix obtained by replacing a by 1. Find b and c if

$$B^{-1} = \frac{1}{2}\begin{pmatrix} 1 & b & -1 \\ 7 & 2 & -3 \\ -3 & c & 1 \end{pmatrix}.$$

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -1 & -2 \\ -4 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & b & -1 \\ 7 & 2 & -3 \\ -3 & c & 1 \end{pmatrix} = 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(iii) Solve the equation $B\vec{x} = \vec{b}$ for $\vec{x}$:

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -1 & -2 \\ -4 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix}.$$

5. (15pts) Let

$$f(x,y) = \frac{x^2}{y} + xy^2 - 3y.$$

(i) Find the gradient of f at $(2,1)$.

$$\nabla f = \left\langle \frac{2x}{y} + y^2 \,,\; \frac{-x^2}{y^2} + 2yx - 3 \right\rangle$$

$$= \langle 5 \,, -3 \rangle$$

(ii) Use linear approximation to estimate the value of $f(2.01, 0.99)$.

(iii) Use the chain rule to find the rate of change of f,

$$\frac{df}{dt},$$

along the parametric curve C, $x(t) = 2t^2$, $y(t) = t^3$ at time $t = 1$.

$$r(t) = (dx, dy) = (4t, 3t^2)$$

$$r(1) = (4, 3)$$

$$\nabla \cdot r(t) = \frac{\partial f}{dx}$$

$$\langle 5, -3 \rangle \cdot \langle 4, 3 \rangle = \frac{\partial f}{\partial x}$$

$$= 11$$

6. (15pts) (i) Find the point P on the surface

$$z^2 = xy + x + 1$$

closest to the origin, by writing the square of the distance to the origin as a function $f(x, y)$ of only x and y.

$x^2+y^2+z^2$

$x^2+y^2+xy+x+1$

(ii) Use the 2nd derivative test to check that P is a minimum of $f(x, y)$.

$f_x = 2x+y+1$

$f_y = 2y+x$

$2x+y+1=0$

$(-2)(x+2y)=0$

$-3y+1=0$

$y=1/3$

$2x = -\frac{4}{3}$

$x = -\frac{4}{6} = -\frac{2}{3}$

$A = f_{xx} = 2$

$B = f_{xy} = 1$

$C = f_{yy} = 2$

$\Delta = AC - B^2$

$= 3$

$\Delta > 0$

$A > 0$ $\cup \leftarrow$ min

7. (15pts) (i) Let $f(x, y, z)$ be a function of three variables. The equation

$$f(r\cos\theta, r\sin\theta, z) = 1,$$

implicitly defines a function $z = g(r, \theta)$. Express the partial derivative

$$\frac{\partial z}{\partial r}$$

in terms of the partials of f.

(ii) Suppose that f and g are functions of three variables with

$$(\nabla f)_{(1,2,-3)} = \langle -2, 1, -1\rangle \qquad \text{and} \qquad (\nabla g)_{(1,2,-3)} = \langle 1, -2, 2\rangle.$$

If $g(1, 2, -3) = 0$ and the points (x, y, z) are constrained to lie on the surface $g = 0$, what is

$$\left(\frac{\partial f}{\partial x}\right)_y$$

at $(1, 2, -3)$?

9. (15pts) Two circles of radius a have their centres at $(0, 0)$ and $(a, 0)$. R is the region outside the first circle inside the second circle. Set up an iterated integral in polar coordinates to find the moment of inertia of R about the origin, where the density $\delta = x$.

10. (15pts) Consider the integral

$$\oint_C (1-y)\,\mathrm{d}x + x\,\mathrm{d}y,$$

over the curve consisting of the upper half unit circle and the line segment from $(-1,0)$ to $(1,0)$. Calculate the integral in two ways:
(i) directly.

(ii) by relating it to a double integral.

11. (15pts) The solid D is bounded below by a right angled cone with vertex at the origin, central axis the z-axis and bounded above by the sphere of radius 1. Find the gravitational attraction on a unit mass placed at the origin. Assume the density $\delta = 1$.

12. (15pts) Let

$$\vec{F} = (x^2 - xy)\hat{\imath} + 2y\hat{\jmath},$$

and let C be the ellipse

$$(2x - y)^2 + (5x + y)^2 = 3.$$

Find the flux of $\vec{F}$ out of the region R bounded by C.

13. (15pts) Let

$$\vec{F} = z^2\hat{\imath} + z \sin y \hat{\jmath} + (2z + axz + b\cos y)\hat{k}.$$

(i) Find the values of a and b such that $\vec{F}$ is conservative.

(ii) For those values of a and b find a potential function f for $\vec{F}$ in a systematic way.

(iii) For the same values of a and b, calculate the work done to move a particle in the force field $\vec{F}$ along the parametrised curve

$$x = t^3 \qquad y = 1 - t^2 \qquad z = t \qquad \text{for} \qquad -1 \le t \le 1.$$

14. (15pts) Let R be the region

$$0 \leq z \leq a \qquad \text{and} \qquad x^2 + y^2 \leq 1.$$

(i) Set up the volume of R as a triple integral in cylindrical coordinates

(ii) Set up the volume of R as the sum of two triple integrals in spherical coordinates.

15. (20pts) Calculate the flux of $\vec{F} = (1 - z^2)\hat{k}$ out of the solid hemisphere

$$x^2 + y^2 + z^2 < 1 \qquad \text{and} \qquad z > 0,$$

(i) directly,

(ii) using the divergence theorem.

16. (20pts) Let $\vec{F}$ be the vector field $y\hat{\imath} + xz\hat{\jmath} + y\hat{k}$. Let C be the boundary of the half circular cylinder S, $x^2 + y^2 = 1$, $y \geq 0$, $0 \leq z \leq 1$, with corners at $(1,0,0)$, $(-1,0,0)$, $(-1,0,1)$ and $(1,0,1)$, oriented in that order.

(i) What is curl$\vec{F}$?

(ii) Calculate the work done going along C using Stokes' theorem.

1

1. (15pts) (i) Let $A = (1,2,3)$, $B = (4,-1,4)$ and $C = (2,4,6)$. Find the angle between $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

(ii) Let $P = (0,1,1)$, $Q = (2,1,0)$ and $R = (1,3,2)$. Find the cross product of $\overrightarrow{PQ}$ and $\overrightarrow{PR}$.

(iii) Find the equation of the plane containing P, Q and R.

2. (10pts) At what point does the line through $(1, 0, -1)$ and $(2, 1, 2)$ intersect the plane $x + y - z = 3$?

3. (10pts) Give parametric equations for the line given as the intersection of the two planes $2x - y - 4z = 0$ and $5x - 2z = 1$.

4. (10pts) A moon M revolves around a planet P in a circular orbit of radius one in the xy-plane, so that in one year it completes two revolutions. Meanwhile the planet revolves around a star O in a circular orbit of radius five, one revolution per year. The star is always at the origin and at time $t = 0$ the planet and the moon are on the x-axis, the planet to the right of the sun and the moon to the right of the planet. Find the position of the moon as a function of the number of years t.

5. (15pts) Let S be the surface defined by the equation

$$z = x^2y + xy^2 - 3y^2.$$

(i) Find the tangent plane to S at the point $P = (2, 1, 3)$.

(ii) Give a formula approximating the change Δz in z if x and y change by small amounts Δx and Δy.

(iii) Approximate the value of z at the point $(x, y) = (2.01, 1.01)$.

6. (20pts) A rectangular box lies in the first quadrant. One vertex is at the origin and the diagonally opposite vertex P is on the plane $2x+y+z=2$. We want the coordinates of the point P which maximises the volume of the box.
(i) Show that this lead to maximising the function

$$f(x,y) = xy(2-2x-y).$$

Find the critical points of $f(x,y)$.

(ii) Determine the type of the critical point in the first quadrant.

(iii) Now solve this problem using the method of Lagrange multipliers.

7. (15pts) Find the point on the surface

$$z^2 = xy + x + 1$$

closest to the origin, using the method of Lagrange multipliers.

8. (15pts) Let $w(x, y, z) = x^4 + 2xy^2 - z^3$.
(i) Find the equation of the tangent plane to the surface $w = 2$ at $(1, 1, 1)$.

(ii) Assume that x, y and z are constrained by the equation $w(x, y, z) = 2$. Find the value of

$$\left(\frac{\partial x}{\partial z}\right)_y$$

at $(1, 1, 1)$.

9. (15pts) Let R be the plane triangle with vertices $(0,0)$, $(1,-1)$ and $(1,1)$. Set up an iterated integral which gives the average distance of a point from the origin,
(i) in rectangular coordinates

(ii) in polar coordinates.

10. (15pts) Let C_1 be the line segment from $(0,0)$ to $(1,0)$, C_2 the arc of the unit circle running from $(1,0)$ to $(0,1)$ and let C_3 be the line segment $(0,1)$ to $(0,0)$. Let C be the simple closed curve formed by C_1, C_2 and C_3 and let

$$\vec{F} = x^3\hat{\imath} + x^2 y\hat{\jmath}.$$

Calculate

$$\oint_C \vec{F} \cdot \mathrm{d}\vec{r},$$

(i) directly.

(ii) using Green's theorem.

11. (15pts) (i) Calculate the flux of $\vec{F} = x\hat{\imath}$ out of each side, S_1, S_2, S_3 and S_4 of the square $-1 \leq x \leq 1$, and $-1 \leq y \leq 1$. Label the sides so that S_1 and S_3 are horizontal, S_1 below S_3, and S_2 and S_4 are vertical, S_2 to the right of S_4.

(ii) Explain why the total flux out of any square of sidelength 2 is the same, regardless of its location or how its sides are tilted.

12. (15pts) Find the area of the region R bounded by the curves $xy = 2$, $xy = 5$, $y = x^2$ and $y = 4x^2$.

13. (15pts) Let

$$\vec{F} = (y - z)\hat{\imath} + (x + y)\hat{\jmath} + (1 - x)\hat{k}$$

(i) Find a potential function f for $\vec{F}$.

(ii) Let C be the parametric curve

$$x = 3\cos^3 t \qquad y = 3\sin^3 t \qquad z = t \qquad \text{for} \qquad 0 \le t \le 2\pi.$$

Find

$$\int_C \vec{F} \cdot \mathrm{d}\vec{r}.$$

14. (15pts) Let D be the portion of the solid sphere

$$x^2 + y^2 + z^2 < 1,$$

lying above the plane

$$z = \frac{\sqrt{2}}{2}.$$

The surface bounding D consists of two parts, a curved part S and flat part T. Orient both surfaces so that the normal vector points upwards. Let

$$\vec{F} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}.$$

(i) Calculate the flux of $\vec{F}$ across S.

(ii) Calculate the flux of $\vec{F}$ across T.

(iii) Find the volume of D using the divergence theorem.

15. (20pts) Calculate the flux of

$$\vec{F} = x\hat{\imath} + y\hat{\jmath} + (1 - 2z)\hat{k}$$

out of the solid bounded by the xy-plane and the paraboloid $z = 4 - x^2 - y^2$.

(i) directly,

(ii) using the divergence theorem.

16. (20pts) Let $\vec{F} = -y\hat{\imath} + x\hat{\jmath}$ and let S be the surface of the hemisphere

$$x^2 + y^2 + (z-1)^2 = 1 \qquad \text{and} \qquad z \geq 1,$$

oriented updwards.

(i) Calculate the flux of $\vec{F}$ across S.

(ii) Find the curl of $\vec{F}$.

(iii) Calculate the flux of curl$\vec{F}$ across S using Stokes' theorem.

18.02 Practice Final Exam ①

1)

$A = (1, 2, 3)$
$B = (4, -1, 4)$
$C = (2, 4, 6)$

b) i) ~~$AB = (3, -3, 1)$~~

$\vec{AB} = \vec{B} - \vec{A} = (3, -3, 1)$

$\vec{AC} = (2, 4, 6)$

$\cos\theta = \dfrac{\vec{AB} \cdot \vec{AC}}{|AB||AC|}$

ii) $PQ = (2, 0, -1)$

$PR \ (1 \ 2 \ 1)$

$$\begin{vmatrix} i & j & k \\ 2 & 0 & -1 \\ 1 & 2 & 1 \end{vmatrix} = 2\hat{i} + -3\hat{j} + 4\hat{k}$$

$\boxed{\langle 2, -3, 4 \rangle}$ ✓

iii)

$2(x-0) + -3(y-1) + 4(z-1) = 0$

$2x - 3y + 3 + 4z - 4$

$\boxed{2x - 3y + 4z = 1}$ ✓

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4)

5) $\cancel{} = x^2y + xy^2 - 3y^2 - z$

i) $\nabla z = \langle \cancel{2xy+1, y} \left(2xy + y^2 , x^2 + 2yx - 6y , -1 \right)$

at $(2,1,3)$

$\nabla z \cdot (x-2, y-1, z-3)$

$\cancel{(2xy+y^2)(x+2)} +$

$\nabla z = \langle 4+1 , 4+4-6 , -1 \rangle$

$= \langle 5, 2, -1 \rangle$

$5x - 10 + 2y - 2 - z + 3 = 0$

$$\boxed{5x + 2y - z = 9}$$

ii) ← Approximation

iii)

6

i) $f(x,y) = xy(2-2x-y)$

$= 2xy - 2x^2y - xy^2$

$f_x = 2y - 4xy - y^2$

$f_y = 2x - 2x^2 - 2yx$

$1 \quad -y^2 - 4xy + 2y = 0$

$-2(-2x^2 - 2xy + 2x = 0)$

$-y^2 + 4x^2 + 2y - 4x = 0$

$y(2-y) + 4x(x-1) = 0 \qquad (0,2)$

$x=0$
$y=0$
$x=1$
$y=2$

ii) $f_{xx} = -4y = A$

$f_{xy} = B = 2 - 4x - 2y$

$f_{yy} = -2x$

	A	B	C	$\Delta = AC - B^2$	
(0,2)	-4	-2	0	-4	$\Rightarrow$ Saddle Point

iii) Lagrange $\quad f(x,y) = xy(2-2x-y)$

$g(x,y,z) = xyz \qquad \nabla g = \langle yz, xz, yx \rangle$

$\nabla f = \lambda \nabla g$

$f(x,y,z) = xy + x + 1 - z^2$

$g(x,y,z) = x^2 + y^2 + z^2 \leftarrow$ min

$\nabla f = \langle y+1, x, -2z \rangle$

$\nabla g = \langle 2x, 2y, 2z \rangle$

$y + 1 = \lambda 2x$

$x = \lambda 2y$

$-2\cancel{z} = \lambda \cancel{2}\cancel{z}$

$\boxed{\lambda = -1}$

$y + 1 = -2x$

$x = -2y$

$-2x = +4y$

$y + 1 = 4y$

$1 = 3y$

$\boxed{y = 1/3}$

$\frac{4}{3}(-2) = x$

$\boxed{-\frac{2}{3} = x}$

$\cancel{x^2+y^2+z^2=0}$

$xy + x + 1 - z^2 = 0$

... ? ?

g) $w(x,y,z) = x^4 + 2xy^2 - z^3$

$w = 2$ @ $(1,1,1)$

$\nabla w = \left\langle 4x^3 + 2y^2 \,,\; 4xy \,,\; -3z^2 \right\rangle$

i.) $w = x^4 + 2xy^2 - z^3$

$w = 2$ at $(1,1,1)$

$dw = 4x^3dx + 2y^2dx + \underline{4xy\,dy} - 3z^2dz$

$dw = 0$

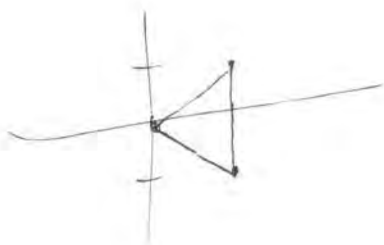
$(4x^3 + 2y^2)dx = 3z^2dz$

$6dx = 3dz$

$2dx = dz$

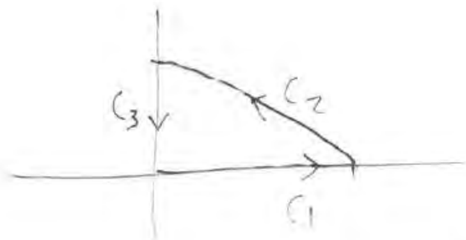
$$\boxed{\left(\frac{dx}{dz}\right)_z = \frac{1}{2}}$$ ✓

9)



NO?

10)



$$F = \langle x^3, x^2y \rangle$$

$$\oint \vec{F} \cdot d\vec{r}$$

on C_1
$x: 0 \to 1$
$y = 0 \quad dy = 0$

$$\int_0^1 x^3\,dx = \frac{x^4}{4}\Big|_0^1 = \frac{1}{4}$$

on
C_2

$x(t) = \cos\theta$
$y(t) = \sin\theta$

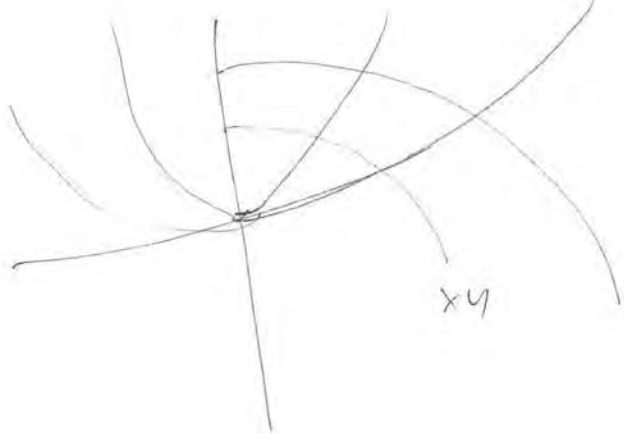
$\int$

11)

$$\oint F \cdot dr = \iint_R \operatorname{curl} F \, dA$$

11)

12)



$U = xy$

$v = \frac{y}{x^2}$ $\qquad U: 2 \to 5$

$v: 1 \to 4$

$$J_T = \begin{vmatrix} U_x & U_y \\ V_x & V_y \end{vmatrix} = \begin{vmatrix} y & x \\ -\frac{2y}{x^3} & \frac{1}{x^2} \end{vmatrix}$$

$$J_T = \frac{y}{x^2} - \frac{-2y\cancel{x}}{x^{\cancel{3}2}}$$

$$= v - 2v$$

$$J_T = 3v$$

$$du\,dv = J_T\,dx\,dy$$

$$\int_1^4 \int_2^5 \frac{1}{v}\,du\,dv$$

$$\int_1^4 \frac{3}{v}\,dv$$

$$\boxed{3 \ln \left|\frac{4}{1}\right|}$$

13)

$$\bar{F} = (y-z), (x+y), (1-x)$$

$f_x = y-z$

$f_y = x+y$

$f_z = 1-x$

$f = yx - zx + g(x,y)$

$f_y = x + g'(x,y)$

$= y$

$\frac{y^2}{2}$

$$\boxed{f = yx - zx + \frac{y^2}{2}}$$

ii)

$x = 3\cos^3 t$

$y = 3\sin 3t$

$z = t$

FOURTH MIDTERM
MATH 18.02, MIT, FALL 2013

You have 50 minutes. This test is closed book, closed notes, no calculators.

There are 5 problems, and the total number of points is 100. Show all your work. *Please make your work as clear and easy to follow as possible.*

Name: Fernando Trujano

Signature: [signature]

Student ID #: 928972346

Recitation instructor: Stojanoska

Recitation Number+Time: r13 MW1

Problem	Points	Score
1	20	20
2	20	19
3	20	20
4	20	19
5	20	20
Total	100	98

1. (20pts) Let R be the region in space which lies above the xy-plane and below the paraboloid $z = 2 - x^2 - y^2$. Calculate the moment of inertia about the z-axis; assume the density $\delta = 1$.

$z=0$

$2-x^2-y^2=0$

$2=x^2+y^2$

$r=\sqrt{2}$

$z: 0 \to 2-x^2-y^2$

$I_z = \iiint \underbrace{x^2+y^2}_{r^2} \delta \, dV$

$= \iiint r^2 \, r \, dr \, d\theta \, dz$

$\int_0^{2\pi} \int_0^{\sqrt{2}} \int_0^{2-r^2} r^3 \, dz \, dr \, d\theta$

$r: 0 \to \sqrt{2}$

$\theta: 0 \to 2\pi$

$z: 0 \to 2-r^2$

$\cancel{\int_0^{\sqrt{2}} r^3(2-r) \, dr}$

$\cancel{\int 2r^3 - r^4 \, dr}$

$\cancel{\frac{2r^4}{4} - \frac{r^5}{5}\Big|_0^{\sqrt{2}} = \frac{4}{2} - \frac{2^{5/2}}{5}}$

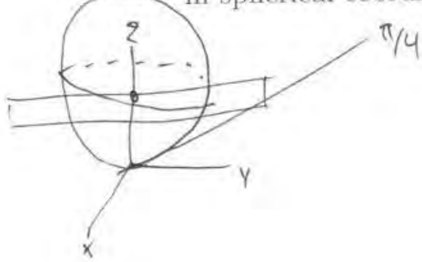
$\cancel{= 2\pi\left(2 - \frac{2^{5/2}}{5}\right)}$

$\int_0^{\sqrt{2}} 2r^3 - r^5 \, dr$

$\frac{2r^4}{4} - \frac{r^6}{6}\Big|_0^{\sqrt{2}} = \frac{4}{2} - \frac{8}{6} = \frac{6}{3} - \frac{4}{3} = \frac{2}{3}$

$I_z = \frac{2\pi 2}{3} = \boxed{\frac{4\pi}{3}}$

2. (20pts) (i) A sphere of radius 1 is placed above the xy-plane so it is tangent at the origin and so the z-axis is a diameter. Give its equation in spherical coordinates (ρ, ϕ, θ).



$2\rho\cos\phi$

$\rho = \boxed{2\cos\phi}$

why?

− 1 pt

(ii) Give the equation of the horizontal plane $z = 1$ in spherical coordinates.

$z = 1$

$\boxed{\rho\cos\phi = 1}$

$\rho = \sec\phi$

(iii) Set up a triple integral in spherical coordinates which gives the volume of the region inside of the sphere from part (i) and lying above the plane $z = 1$.

$\rho : 2\cos\phi \rightleftarrows \sec\phi$

$\phi : 0 \to \pi/4$

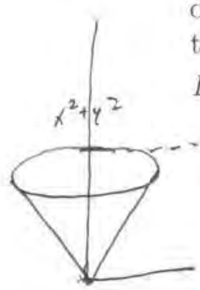
$\theta : 0 \to 2\pi$

$$\boxed{\int_0^{2\pi}\int_0^{\pi/4}\int_{\sec\phi}^{2\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}$$

$$\boxed{\int_0^{2\pi}\int_0^{\pi/4}\int_{\sec\phi}^{2\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}$$

✓3. (20pts) Let D be the disk described by $x^2 + y^2 \leq 1$ and $z = 1$. Let T be the cone formed by joining every point of D to the origin, so that $(0,0,0)$ is the vertex of the cone. Assume that T has constant density $\delta = 1$. Set up an iterated integral that gives the magnitude of the gravitational force acting on a unit mass at the origin.

Do not evaluate the integral, just set it up.



$\delta = 1$

$F = (F_x, F_y, F_z)$

$F = \frac{z}{\rho^3}\, dV$

$F_x = F_y = 0$ by symmetry ✓

AH

$\cos\phi = \frac{1}{\rho}$, $\rho\cos\phi = 1$, $\rho = \sec\phi$

$F_z = Gm \iiint \sin\phi\cos\phi \, d\phi \, d\rho \, d\theta$

$\phi: 0 \to \pi/4$

$\theta: 0 \to 2\pi$

$\rho: 0 \to \sec\phi$

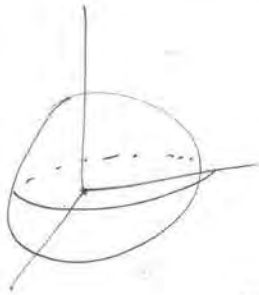
$$\boxed{F_z = G \int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sec\phi} \sin\phi\cos\phi \, d\rho \, d\phi \, d\theta}$$ ✓

4. (20pts) Find the flux of the vector field

$$\vec{F} = (y^4 + \cos z)\,\hat{\imath} - x^2\hat{\jmath} + (z - \log(x^2+1))\hat{k},$$

coming out of the unit sphere centered at the origin.

$\hat{n} = \frac{(x, y, z)}{a}$ $dS = a^2 \sin\phi \, d\phi \, d\theta$



$F = \left(y^4 + \cos z, -x^2, z - \log(x^2+1)\right)$

$\text{Flux} = \iint_S F \cdot \hat{n} \, dS$

$\cancel{(y^4x + x\cos z + -x^2y + z^2 - z\log(x^2+1)) a \sin\phi\, d\phi\, d\theta}$

$\text{Flux} = \iint_S F \cdot \hat{n} \, dS \underset{\text{divergence Theorem}}{=} \iiint_R \operatorname{div}(F)\, dV$

19.

$\operatorname{div}(F) = 0 + 0 + 1 = 1$

$= \iiint_R 1 \, dV$

$= \int_0^{2\pi}\int_0^{\pi}\int_0^{1} 1 \, \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$

$\boxed{\frac{4}{3}\pi}$ $\frac{\rho^3}{3}$ $\frac{1}{3}\,\pi\, 2\pi$ $(+)$

$\frac{\rho^3}{3}\Big|_0^1 = \frac{1}{3} \sin\phi$ $-\frac{1}{3}\cos\phi\Big|_0^{\pi}$ $\frac{1}{3} = \frac{2\pi}{3}$

$\boxed{\frac{2}{3}\pi}$

5. (20pts) Let

$$\vec{F} = (y + z^2)\hat{\imath} - x\hat{\jmath} + (2x + 3)\hat{k},$$

$z = 4 - r^2$

be a vector field and let S be the part of the surface $z = 4 - x^2 - y^2$ that lies above the xy-plane. Orient S by using the outward normal vector. Find the outward flux of $\vec{F}$ across S.

$F = y + z^2, -x, 2x + 3$

$\downarrow$

$z = 4 - x^2 - y^2$

$0 = 4 - x^2 - y^2$
$4 - r^2$
$4 = r^2$
$r = 2$
$\cancel{x^2 + y^2 = 2}$

$\hat{n}\,dS = (-f_x, -f_y, 1)$
$= (2x, 2y, 1)$

$F \cdot n\,dS =$
$(y + z^2, -x, 2x + 3) \cdot (2x, 2y, 1)$

$r: 0 \to 2$
$\theta: 0 \to 2\pi$

$\cancel{2xy} + 2xz^2 \cancel{- 2xy} + 2x + 3$

$(a+b)(a+b)$

$2xz^2 + 2x + 3$

$(2r\cos\theta(4 - r^2)^2 + 2r\cos\theta + 3)\, r\, dr\, d\theta$

$2r\cos\theta\left((4 - r^2)^2 + 3\right) r\, dr\, d\theta$

$\int\int_0^1 2r^2\cos\theta\,(16 - 8r^2 + r^4 + 3)\, dr\, d\theta$
$(16 - 8r^2 + r^4 + 3)$

$\mathrm{div}(F) = 0 + 0 + 0 = 0$ $\quad \pi r^2$

$\iint_S F = \iint_{S'} F$ $\qquad \iint F \cdot \hat{n}\, dS$

$\hat{n} = (0, 0, 1)$ $\qquad \iint 2x + 3\, dS$
$= \boxed{12\pi}$

THIRD MIDTERM
MATH 18.02, MIT, FALL 2013

You have 50 minutes. This test is closed book, closed notes, no calculators.

There are 6 problems, and the total number of points is 100. Show all your work. *Please make your work as clear and easy to follow as possible.*

Name: Fernando Trujano

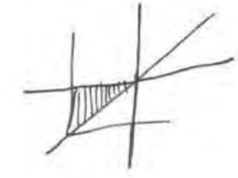
Signature: Fernando Trujano

Student ID #: 928972346

Recitation instructor: Stojanoska

Recitation Number+Time: R13 MW1

Problem	Points	Score
1	15	13
2	15	15
3	15	6
4	15	15
5	20	20
6	20	20
Total	100	89



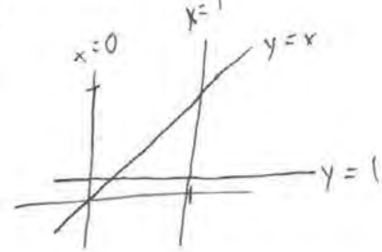
$y: 0 \to 1$
$x: 0 \to y$

7 1. (15pts) (i) Change the order of integration in

$$\int_0^1 \int_x^1 \frac{x}{1+y^3}\, dy\, dx.$$

$y=x$
$y=1$
$x=0$
$x=1$

$\to dx\,dy$



$$\boxed{\int_{y=0}^{y=1} \int_{x=0}^{x=y} \frac{x}{1+y^3}\, dx\, dy}$$

$$\cancel{\int_0^1 \int_y^1 \frac{x}{1+y^3}\, dx\, dy}$$

$$\cancel{\frac{x^2}{2(1+y^3)}\Big|_y^1 = \frac{1}{2+2y^3} - \frac{y^2}{2+2y^3} = \frac{1-y^2}{2+2y^3}}$$

$$\cancel{\int_0^1 \frac{1-y^2}{2+2y^3}\, dy}$$

6 (ii) Evaluate the integral.

$$\frac{x^2}{2(1+y^3)}\Big|_0^y = \frac{y^2}{2(1+y^3)} = \frac{y^2}{2+2y^3}$$

$$\int_0^1 \frac{y^2}{2+2y^3}\, dy = \int_0^1 \frac{y^2}{2(1+y^3)}\, dy$$

$$u = y^3$$
$$du = 2y^2 dy$$

$$\frac{1}{2}\int_0^1 \frac{du}{2(1+u)} \qquad 2(1+u) \neq 2+u$$

$$= \frac{1}{2}\int_0^1 \frac{1}{2+u}\, du$$

$$\frac{1}{2}\left(\ln|2+u| \right.$$

$$\frac{1}{2}\left|\ln|2+y^3|\right|_0^1$$

$$= \frac{1}{2}\left(\ln|3| - \ln|2| \right)$$

$$\boxed{= \frac{1}{2}\ln\frac{3}{2}}$$

15

✓2. (15pts) Find the mass of the upper half (i.e., $y \geq 0$) of the annulus $4 < x^2 + y^2 < 16$ with density

$$\delta = \frac{y}{x^2 + y^2}.$$

Mass $= \iint_R \delta \, dA$

$r^2: 4 \to 16$

$r: 2 \to 4$

$\theta: 0 \to \pi$

$$= \int_0^\pi \int_2^4 \frac{y}{(x^2+y^2)} r \, dr \, d\theta = \int_0^\pi \int_2^4 \frac{\cancel{r^2} \sin\theta}{\cancel{r^2}} \, dr \, d\theta$$

$$\sin\theta \, r \Big|_2^4 = 4\sin\theta - 2\sin\theta = 2\sin\theta$$

$$\int_0^\pi 2\sin\theta \, d\theta$$

$$-2\cos\theta \Big|_0^\pi$$

~~$-2(-1) - 2(1)$~~

~~$2 - 2$~~

$$(-2)(-1) - (-2)(1)$$

$$2 + 2$$

$$= \boxed{4}$$

3. (15pts) (i) Let C be a simple closed curve going counterclockwise around a region R. Let $M = M(x, y)$ be a function. Express

$$\oint_C M\,dx,$$

as a double integral over R (of M or a derivative of M, hint, hint).

$\oint_C M \cdot dr = \oint_C P\,dx + \cancel{Q\,dy}$ $\qquad$ $\text{curl}(F) = Q_x - P_y$

$dr = (dx, dy)$ $\qquad$ $\oint M \cdot dr = \iint_R \text{curl}(F)\,dA$

$dy = 0$

$$\boxed{= \iint_R \cancel{M_x}\,dx}$$

-3

(ii) Find a particular function M such that

$$\oint_C M\,dx,$$

is the mass of R, when the density $\delta = (x + y)^2$.

$\text{Mass} = \iint_R (x + y)^2\,dA$

$M = (P, Q)$

$\bar{F} \quad (P, Q)$

$\oint_C M \cdot dr = \iint_R \text{curl}(M)\,dA$ $\qquad$ $\int_C F \cdot dr = P\,dx + \cancel{Q\,dy}$

$dr = (dx, dy)$ $\qquad$ $\text{curl}\, M = Q_x - P_y$

$dy = 0$

$\oint_C M\,dx = \iint_R (x + y)^2\,dA = \iint M_x\,dx$ $\qquad$ -6

$dA = dy\,dx$ $\qquad$ $\iint (x+y)^2\,dx\,dy = \iint M_x\,dx = \oint_C M\,dx$

$x^2 + 2xy + y^2$

$M_x = 2\cancel{x} + 2y$ $\qquad$ $M = 2x + 2y$

4. (15pts) (i) Show that the vector field

$$\vec{F} = (2xy - 2y^2)\hat{\imath} + (x^2 - 4xy)\hat{\jmath}$$

is conservative. ie $\text{curl}(\vec{F}) = 0$

$$\vec{F} = \langle \underbrace{2xy - 2y^2}_{P}, \underbrace{x^2 - 4xy}_{Q} \rangle$$

$$\text{curl}(F) = Q_x - P_y$$

$$2x - 4y - (2x - 4y) = 0$$

7 .

(ii) Let $\vec{F}$ be the vector field $\vec{F} = xy\hat{\imath} + y\hat{\jmath}$ and find

$$\int_C \vec{F} \cdot d\vec{r},$$

over the curve $y = x^2$ from $(0,0)$ to $(1,1)$.

$$\int_C F \cdot dr \qquad \vec{F} = \langle xy, y \rangle \qquad dr = \langle dx, dy \rangle$$

$$\int_C F \cdot dr = \int_C \langle xy, y \rangle \cdot \langle dx, dy \rangle = \int_C xy\,dx + y\,dy$$

$$y = x^2$$
$$dy = 2x\,dx$$

$$= \int_C \cancel{(x)(2x)dx + x^2 2x\,dx}$$

$$= \int (x)(x^2)\,dx + x^2\,2x\,dx$$

8 .

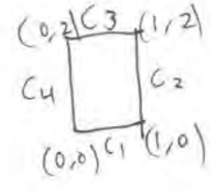
$$= \int_C x^3 + 2x^3\,dx = \int_C 3x^3\,dx$$

$$\int_0^1 3x^3\,dx$$

$$\frac{3}{4}x^4 \Big|_0^1 = \boxed{\frac{3}{4}}$$

✓ 5. (20pts) Consider the rectangle R with vertices $(0,0)$, $(1,0)$, $(1,2)$ and $(0,2)$. Let $C = C_1 + C_2 + C_3 + C_4$ be the boundary of R, starting at $(0,0)$ and going around counterclockwise, so that C_1 and C_3 are horizontal and C_2 and C_4 are vertical. Let

$$\vec{F} = (xy + \overset{P}{\sin} x \cos y)\hat{\imath} - (\overset{Q}{\cos} x \sin y)\hat{\jmath}.$$



✓(i) Find the flux of $\vec{F}$ across C.

10

$\vec{F} = \langle xy + \sin x \cos y,\ -\cos x \sin y \rangle$

$$\text{Flux} = \oint_C F \cdot \hat{n} = \iint_R (\text{div})(F)\, dA$$

$$\text{div}(F) = P_x + Q_y$$

$$= y + \cancel{\cos x \cos y} + \cancel{-\cos x \cos y}$$

$$= \int_0^1\int_0^2 y \, dy \, dx$$

$$\frac{y^2}{2}\Big|_0^2 = \frac{4}{2} = 2 \qquad \int_0^1 2\, dx = 2$$

✓(ii) Is the total flux out of R across C_1, C_2 and C_3 more than or less than across C?

C_4

10 $\quad C = C_1 + C_2 + C_3 + C_4$

$$n = (dy, -dx)$$

$\underline{C_4}$

when $x = 0$
$dx = 0$

when $t = 0$
$y = 2$ $\quad y = 2 - t$
when $t = 2$ $\quad t: 0 \to 2$
$y = 0$

$$\int_{C_4} F \cdot n = \int_{C_4} xy + \sin x \cos y \, dy$$

$$= \int (0) + \sin(0) \cos y \, dy$$

$$\int_{C_4} 0 \, dy = 0$$

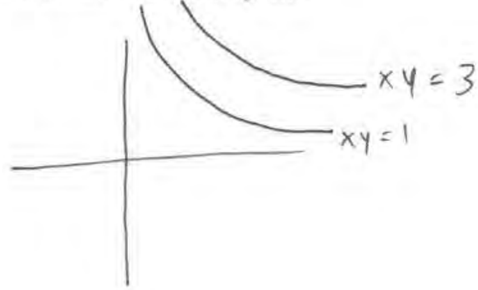
Because $C_4 = 0$

$$C_1 + C_2 + C_3 = C$$

√√ 6. (20pts) Compute the area of the region in the plane bounded by the curves $xy = 1$, $xy = 3$, $xy^3 = 2$ and $xy^3 = 4$.

$u = xy \quad u: 1 \to 3$

$v = xy^3 \quad v: 2 \to 4$



$$J_T = \begin{vmatrix} U_x & U_y \\ V_x & V_y \end{vmatrix}$$

$U_xV_y - U_yV_x$

$(y)(3xy^2) - (x)(y^3)$

$3xy^3 - xy^3 = 2xy^3$

$= 2v$

$du\,dv = 2v\,dx\,dy$

$dx\,dy = \frac{1}{2v}\,du\,dv$

$$\iint_R dx\,dy = \int_2^4 \int_1^3 \frac{1}{2v}\,du\,dv$$

$\left(\frac{1}{2v}\right)u \Big|_1^3 = 3\left(\frac{1}{2v}\right) - \left(\frac{1}{2v}\right) = \frac{\cancel{2}}{\cancel{2}v} = \frac{1}{v}$

$\int_2^4 \frac{1}{v}\,dv$

$\ln|v| \Big|_2^4 = \ln|4| - \ln|2|$

$= \ln\left|\frac{4}{2}\right| = \ln|2|$

$\boxed{A = \ln|2|}$

SECOND MIDTERM
MATH 18.02, MIT, FALL 2013

You have 50 minutes. This test is closed book, closed notes, no calculators.

There are 6 problems, and the total number of points is 100. Show all your work. *Please make your work as clear and easy to follow as possible.*

Name: Fernando Trujano

Signature: Fernando Trujano.

Student ID #: 928972346

Recitation instructor: Stojanoska

Recitation Number+Time: 14W (

Problem	Points	Score
1	20	15
2	15	12
3	20	07
4	15	12
5	15	15
6	15	11
Total	100	72

~~15 15 15 15 20~~

15
15
20
5
15
15
85 :/

1. (20pts) Let $f(x,y) = 3xy^2 - x - y$.

(i) Find ∇f at $(1,1)$.

$$\nabla f = (3y^2 - 1,\ 6xy - 1)$$

$$\nabla f = (2,\ 6)$$

-1

(ii) Find the equation of the tangent plane to the graph of $f(x,y)$ at the point $(1,1,1)$.

$$\langle 2, 6, 1\rangle \cdot \langle x-1, y-1, z-1\rangle = 0$$

$$2(x-1) + 6(y-1) + 1(z-1) = 0$$

$$2x - 2 + 6y - 6 + z - 1 = 0$$

$$2x + 6y + z = 9$$

-1

(iii) Find the directional derivative of f at $(1,1)$ in the direction of $\frac{\hat{\imath}}{\sqrt{2}} - \frac{\hat{\jmath}}{\sqrt{2}}$.

$$\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \qquad D_u f = \nabla f \cdot \vec{u}$$

$$\langle 2, 6\rangle \cdot \langle \tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}\rangle$$

$$\frac{2}{\sqrt{2}} - \frac{6}{\sqrt{2}} = -\frac{4}{\sqrt{2}}$$

✓

(iv) Use linear approximation to estimate the value of $f(1.1, 0.9)$.

$h\hat{u} = hf$

$$u = \frac{\nabla f}{|\nabla f|} = \frac{(2,6)}{\sqrt{40}} = \frac{(2,6)}{2\sqrt{10}}$$

$$h = .1/\langle 2,6\rangle = 1/10|\langle 2,6\rangle| = 1/10 \cdot 2\sqrt{10} = \frac{1}{20}$$

$$hu = \left(\frac{1}{20}\right)\left(\frac{(2,6)}{2\sqrt{10}}\right)$$

$$(1,1) + \left(\frac{2}{40\sqrt{10}}, \frac{6}{40\sqrt{10}}\right)$$

-3

$$\left(1 + \frac{2}{40\sqrt{10}},\ 1 - \frac{6}{40\sqrt{10}}\right)$$

$$\boxed{\left(1 + \frac{1}{20\sqrt{10}},\ 1 - \frac{3}{20\sqrt{10}}\right)}$$

2. (15pts) Let p be the point on the curve $x^2 - y^3 = 2.9$ which is closest to $(2, 1)$. Use the gradient to estimate the coordinates of p.

$f(x,y) = x^2 - y^3$

$f(2,1) = 4-1 = 3$ ↑ - move in the direction of ∇f

$\nabla f = (2x, -3y^2)$

$\nabla f = \langle 4, -3 \rangle$

$u = \frac{\nabla f}{|\nabla f|} = \frac{\langle 4,-3 \rangle}{\sqrt{25}} = \frac{\langle 4,-3 \rangle}{5}$

$h = .1 / \langle 4, -3 \rangle$

$h\vec{u} = h \nabla f$

$h = \frac{1}{10|\langle 4,-3 \rangle|} = \frac{1}{10\sqrt{25}} = \frac{1}{50}$

$(2,1) + hu$

$hu = \left(\frac{1}{50}\right)\left\langle \frac{\langle 4,-3 \rangle}{5} \right\rangle$

$(2,1) + \left(\frac{4}{250}, \frac{-3}{250}\right)$

$\boxed{\left(2 + \frac{4}{250},\ 1 - \frac{3}{250}\right)}$

-3

6

3. (20pts) (i) Find the critical points of $w = \frac{1}{3}x^3 - x^2 + 2xy + y^2$, and determine the type of the critical point closest to the origin.

$w_x = x^2 - 2x + 2y$

$w_y = 2x + 2y$

$x^2 - 2x + 2y = 0$

$2x + 2y = 0$

$x^2 + 4y = 0$

$x^2 = -4y$

$x^2 - 2x + \cancel{2}y = 2x + \cancel{2}y$

$x^2 = 4x$

$4x = -4y$

$x = -y$

TURN OVER

$\boxed{(0,0) \to \text{saddle point}}$

critical points.

~~(4,4) → min~~

~~(4,-4) → min~~

~~(-4,-4) → max~~

~~(-4,4) → max~~

only (4,-4) is crit

$w_{xx} = A = 2x - 2$

$w_{xy} = B = 2$

$w_{yy} = C = 2$

$\Delta = AC - B^2$

$\cancel{(-2)(2) - 4 = 0}$

$(2)(4) - 2 = 6 \quad A > 0$

$A < 0$

0

(ii) Find the point of the region $y \geq 0$, $y \leq x$ at which w is the smallest. Justify your answer.

Check boundaries

$(4,4)$ is the local min in the region $y \geq 0$, $y \leq x$

if $y = 0$
$x = 0$

$w(0,0) = 0$

~~if y = x~~

$w(4,4) = \left(\frac{1}{3}\right)4^3 - \cancel{16} + 32 + \cancel{16}$

$\frac{64}{3} + 32 = \frac{160}{3}$

$\boxed{(0,0) \text{ is when } w \text{ is smallest}}$

check boundary!

$16 \times 4 = 64$, $\frac{64}{3}$

$32 \times 3 = 96$

$64 + 96 = 160$

$3\overline{)160}$

10

if $y = x$

$(0,0)$

$x \geq 0$

$x = 0$

$(0,0)$

Lowest

4. (15pts) (i) Write down the equations to find the point on the surface $(x+y)z^2 = 1$ in the first octant, closest to the origin, using the method of Lagrange multipliers.

$$g(x,y,z) = (x+y)z^2 - 1 = z^2x + z^2y - 1$$
$$f(x,y,z) = x^2+y^2+z^2$$

$$\nabla f = \lambda \nabla g$$
$$\nabla f = (2x, 2y, 2z)$$
$$\nabla g = (z^2, z^2, 2zx + 2zy)$$

$$\boxed{\begin{aligned} 2x &= \lambda z^2 \\ 2y &= \lambda z^2 \\ 2z &= \lambda(2zx+2zy) \\ (x+y)z^2 &= 1 \end{aligned}}$$

(ii) Solve these equations to find the closest point.

12

$$2x = 2y$$
$$x = y$$

$$2yz = \lambda z^3$$
$$2zy = \lambda(2zxy + 2zy^2)$$

$$(x+y) \quad (2x)\left(\frac{x}{2}\right)^2 = 1$$
$$2x\left(\frac{x^2}{4}\right)$$

$$\cancel{\lambda} z^3 = \cancel{\lambda}(2zy^2 + 2zy^2)$$
$$z^3 = 4zy^2$$
$$\cancel{z^3 = z2y2y}$$

$$x = z = y$$

$$x = y \quad 2x^3 = 4$$
$$x^3 = 2$$
$$\boxed{x = \sqrt[3]{2}}$$

$$\lambda z^3 = \lambda\, 2zy^2 + 2zy^2)$$
$$z^3 = 4zy^2$$

$$(2+2)\left(\frac{2}{2}\right)^2$$
$$4(1) =$$
$$(1+1)\left(\frac{1}{2}\right)^2$$

$$z^2x + z^2y = 1$$
$$2z^2x = 1$$

$$2z^2 = \lambda(2z^2x + 2z^2x)$$
$$2z^2 = \lambda(4z^2x)$$
$$z^2 = \lambda(2z^2x)$$

$$x = y = 1$$
$$x = \frac{z}{2}$$

$$2y = \lambda(\lambda 2z^2y)$$

Turn over
$\rightarrow$

✓ 5. (15pts) Let $w = f(u, v)$ where $u = xy$ and $v = x/y$. Using the chain rule, express

$$\frac{\partial w}{\partial x} \quad \text{and} \quad \frac{\partial w}{\partial y}$$

in terms of x, y, f_u and f_v.

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x}$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y}$$

$$\frac{\partial w}{\partial x} = f_u\, y + f_v\left(\frac{1}{y}\right)$$ ✓

$$\frac{\partial w}{\partial y} = f_u\, x + f_v\left(-x/y^2\right)$$ ✓

√6. (15pts) The two surfaces $x^2 + y^3 - z^4 = 1$ and $z^3 + zx + xy = 3$ intersect along a curve for which y is a function of x. Find

$$\frac{dy}{dx} \quad \text{at} \quad (x_0, y_0, z_0) = (1, 1, 1).$$

$$\left. y(x) \right|_z$$

$$2x\,dx + 3y^2 dy - 4z^3 dz = 0$$

$$3z^2 dz + dz\,x + dx\,z + dx\,y + dy\,x = 0$$

$$2dx + 3dy \overset{-4dz}{\cancel{=4dz}} = 0$$

$$\underbrace{\cancel{3dz + dz}}_{4dz} + dx + dx + dy = 0$$

$$\cancel{3}2dx + 3dy - 4dz = 0$$

$$4dz + 2dx + dy = 0$$

$$dz = \frac{-2dx - dy}{4}$$

$$\cancel{3}2dx + 3dy - \cancel{4}\left(\frac{-2dx - dy}{\cancel{4}}\right) = 0$$

11/15

$$\cancel{3}2dx + 3dy + 2dx + dy = 0$$

$$\cancel{5}4\,dx + 4dy = 0$$

$$\cancel{5}4\,dx = -4dy$$

$$\frac{\cancel{5}4}{-4} = \frac{dy}{dx}$$

$$\boxed{\frac{dy}{dx} = -\frac{\cancel{5}4}{4}}$$

FIRST MIDTERM
MATH 18.02, MIT, FALL 2013

You have 50 minutes. This test is closed book, closed notes, no calculators.

There are 5 problems, and the total number of points is 100. Show all your work. *Please make your work as clear and easy to follow as possible.*

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Student ID #: 928972346

Recitation instructor: Stojanoska

Recitation Number+Time: r13 MW1

Problem	Points	Score
1	20	18
2	20	10
3	20	09
4	20	2
5	20	10
Total	100	49

1.

(i) (10pts) Let $P = (1, 2, 1)$, $Q = (-1, 1, 1)$ and $R = (1, 1, -1)$. What is the cosine of the angle between $\overrightarrow{PQ}$ and $\overrightarrow{PR}$?

$\vec{PQ} = (-2, -1, 0)$

$\vec{PR} = (0, -1, -2)$

$\vec{PQ} \cdot PR = (-2)(0) + (-1)(-1) + \cancel{0}(-2) = 1$

$|PQ| = \sqrt{2^2 + 1^2 + 0^2} = \sqrt{5}$

$|PR| = \sqrt{0^2 + 1^2 + 2^2} = \sqrt{5}$

$|PQ||PR| \cos\theta = \vec{PQ} \cdot \vec{PR}$

$(\sqrt{5})(\sqrt{5}) \cos\theta = 1$

$5 \cos\theta = 1$

$\boxed{\cos\theta = 1/5}$

$\cos^{-1}\left(\frac{1}{5}\right) = \theta$

(ii) (10pts) If $\vec{r}(t) = (-3\cos 2t, 3\sin 2t, 5t)$ is a parameterized curve, then what is the speed at time t?

$\vec{r}(t) = (-3\cos 2t, 3\sin 2t, 5t)$

$r'(t) = v(t) = (3\sin(2t)(2), 3\cos(2t)(2), 5)$

speed $= \sqrt{(6\sin(2t))^2 + (6\cos(2t))^2 + 5^2}$

$\sqrt{6((\sin(2t))^2 + (\cos(2t))^2) + 5^2}$ $= ?$

$\cancel{\sqrt{6(1) + 25}} = \cancel{\sqrt{31}}$

simplify

8

2. Let

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 2 & -1 & 1 \\ 1 & -1 & 1 \end{pmatrix}$$

cofactor ↓ sign ↓ transpose ↓ $\frac{}{\det}$

(i) (10pts) A is invertible and the inverse matrix is of the form

$$A^{-1} = \begin{pmatrix} 0 & 1 & a \\ 1 & 1 & b \\ 1 & 0 & -1 \end{pmatrix}.$$

Find a and b.

$AA^{-1} = I_n$

$$A = \begin{vmatrix} 1 & -1 & 2 \\ 2 & -1 & 1 \\ 1 & -1 & 1 \end{vmatrix} \to \begin{vmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ 1 & -3 & 1 \end{vmatrix} \to \begin{vmatrix} 0 & -1 & -1 \\ -1 & -1 & 0 \\ 1 & +3 & 1 \end{vmatrix} \to \begin{vmatrix} 0 & -1 & 1 \\ -1 & -1 & 3 \\ -1 & 0 & 1 \end{vmatrix} \to \begin{vmatrix} 0 & 1 & -1 \\ 1 & 1 & -3 \\ 1 & 0 & -1 \end{vmatrix}$$

10

$d \quad 1 = 1(0) - 1(1) + 2(-1)$

$= 1 - 2 = -1$

$$\boxed{\begin{aligned} a &= -1 \\ b &= -3 \end{aligned}}$$

(iii) (10pts) Solve the system $A\vec{x} = \vec{b}$, where

$$\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{and} \quad \vec{b} = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}.$$

0

$b^{-1}A \qquad b^{-1}x = A$

$$\begin{vmatrix} x \\ y \\ z \end{vmatrix} \begin{vmatrix} a_1 & a_2 & a_3 \end{vmatrix} = \begin{vmatrix} 0 \\ -2 \\ 1 \end{vmatrix}$$

$$xa_1 + ya_2 + za_3 = \begin{vmatrix} 0 \\ -2 \\ 1 \end{vmatrix}$$

$0 \qquad -2 \qquad 1$

$xa_1 = 0 \quad ya_2 = -2 \quad za_3 = 1$

$a_1 = 0 \quad a_2 = -2/y \quad a_3 = 1/z$

$A = \left[0, -2/y, 1/z \right]$

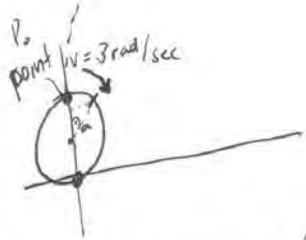
$$\begin{vmatrix} 0 \\ -2 \\ 1 \end{vmatrix} \Big|$$

10

$\sin^2\theta - \cos^2\theta = 1$

$x^2+y^2=a^2$

SOH CAH TOA



$\frac{\pi}{2}$ rad

P_0 point $v = 3$ rad/sec

$\sin^2\theta = 0$ $\cos^2\theta = 1$ $\sin^2\theta = 1$ $\cos^2\theta = 0$

5. A wheel of radius a is rolling along the ground, the x-axis, in the xy-plane. The wheel is rotating at 3 radians per second clockwise. At time $t = 0$, the bottom of the wheel is at the origin.

~~$a + a\sin(\theta)i + a\cos(\theta)j$~~

$a + a\cos(\theta)i + a\sin(\theta)j$

(i) (10pts) Find the position vector $\vec{r}(t)$ of the point on the rim which is at the top of the wheel at time $t = 0$. $t=0$ $\langle 2a, 0\rangle$

$x^2+y^2=a^2$

6

~~$\vec{r}(t) = \frac{\pi}{2} - 3(t)$~~ xy plane...

~~$\vec{r}(t) = \langle a + a\sin(3t), a\cos(3t)\rangle$~~

$\vec{r}(t) = \langle a + a\cos(3t),\ \cancel{a\cos(}\ a\sin(3t)\rangle$

$$\boxed{\vec{r}(t) = \langle a + a\cos(3t),\ a\sin(3t)\rangle}$$

3t

(ii)(5 pts) What is the speed of this point at time t?

4

$\vec{r}'(t) = \vec{v}(t)$

$\vec{v}(t) = \langle -a\sin(3t)(3),\ 3a\cos(3t)\rangle$

$\vec{v}(t) = \langle -3a\sin(3t),\ 3a\cos(3t)\rangle$

1?

Speed $= \sqrt{(-3a\sin(3t))^2 + (3a\cos(3t))^2}$

$\sqrt{9a^2\sin(3t)^2 + 9a^2}$

(iii)(5 pts) What is the distance traveled by the point in one revolution?

0

$d =$ circumference

$= 2\pi r = 2\pi a$