**Question 1: Say whether the following is true or false and support your answer by a proof. ($\exists$m $\in$ N)($\exists$n $\in$ N)(3m+5n = 12)**

Answer: The statement ($\exists$m $\in$ N)($\exists$n $\in$ N)(3m+5n = 12) is false since no valid solutions exist.

Proof: by cases

Since both m and n are natural numbers, i.e. m ≥ 1 and n ≥ 1. This means that 3m ≥ 3 and 5n ≥ 5.

Therefore, there is no possible solution when n ≥ 2. Thus, we only have to verify cases when n = 1.

If n = 1, i.e. 5n = 5, we need 3m = 7 to make the sum 3m+5n = 12. This is not possible since 3 does not divide 7.

Therefor it is proven that there are no valid solutions to ($\exists$m $\in$ N)($\exists$n $\in$ N)(3m+5n = 12). The proof is complete.

**Question 2: Say whether the following is true or false and support your answer by a proof: The sum of any five consecutive integers is divisible by 5 (without remainder).**

Answer: The statement the sum of any five consecutive integers is divisible by 5 (without remainder) is true.

Proof:

Let $n$ be the smallest of the five integers. Then the other integers are $n+1, ..., n+4$.

The sum of the five integers thus are $n + (n +1) + (n +2) + (n +3) + (n +4)$.

This reduces to $5n + 10$ which equals $5(n + 2)$.

This proves that the sum of the five integers are divisible by 5. The proof is complete.

**Question 3: Say whether the following is true or false and support your answer by a proof: For any integer n, the number $n^2$+n+1 is odd.**

Answer: The statement for any integer n, the number $n^2$+n+1 is odd is true.

Proof: We prove this in two steps, first proving a few trivial facts, then proof by cases.

Statement 1: Even integers squared are always even.
Proof: Let there be two even numbers a and b.
Thus a = 2p and b = 2q by the definition of even numbers (divisible by two).
Then the product of the even numbers ab = (2p)(2q).
This reduces to 4pq by algebra.
Factoring out the common factor two from 4pq gives 2(2pq).
2(2pq) is in the format of even numbers 2n (where n is 2pq).
This proves that the product of any odd numbers a and b is even.

Statement 2: Odd integers squared are always odd.
Proof: Let there be two odd numbers a and b.
Thus a = 2p + 1 and b = 2q + 1 by the definition of odd numbers (not divisible by two).
Then the product of the odd numbers ab = (2p + 1)(2q + 1).
This reduces to 4pq + 2p + 2q + 1 by algebra.
Factoring out the common factor two from 4pq + 2p + 2q gives 2(2pq + p + q) + 1.
2(2pq + p + q) + 1 is in the format of odd numbers 2n + 1 (where n is 2pq + p + q).
This proves that the product of any odd numbers a and b is odd.

Statement 3: The sum of two even numbers are always even.
Proof: Let there be two even numbers a and b.
Thus, by definition of even numbers a = 2p and b = 2q.
Therefor the sum of the two integers can be written 2p + 2q.
Factoring out the common factor two gives us 2(p + q).
2(p + q) is in the format of even numbers 2n (where n is p + q).
This proves that the sum of any even numbers a and b is even.

Statement 4: The sum of two odd numbers are always even.
Proof: Let there be two odd numbers a and b.
Thus, by definition of even numbers a = 2p + 1 and b = 2q + 1.
Therefor the sum of the two integers can be written (2p +1) + (2q +1).
This is 2p + 2q + 2. Factoring out the common factor two thus gives us 2(p + q + 1).
2(p + q + 1) is in the format of even numbers 2n (where n is p + q + 1).
This proves that the sum of any odd numbers a and b is even.

Having proved the trivial facts regarding even and odd number sums and multiplications, we now prove the statement "for any integer n, the number $n^2$+n+1 is odd" via cases.

Case 1: Let n be even. Then $n^2$ is even. Thus $n^2$ + n is also even. Adding 1 makes $n^2$+n+1 odd.

Case 2: Let n be odd. Then $n^2$ is odd. Thus $n^2$ + n is even. Adding 1 makes $n^2$+n+1 odd.

Thus, we have proven that $n^2$+n+1 is always odd for any integer n**.** The proof is complete.

**Question 4: Prove that every odd natural number is of one of the forms $4n + 1$ or $4n + 3$, where n is an integer.**

Proof: We will prove that every odd number can be written as $4n + 1$ or $4n + 3$ where n is an integer.

The definition of odd numbers is $2n + 1$ and the definition of even numbers is $2n$ where n is an integer.

Thus, every odd number can be written as $2p + 1$ where p is an integer.

p can be either even or odd, i.e. be written as $p = 2n$ or $p = 2n + 1$.

Now for any n, we see that $2p + 1$ is always odd no matter if p is even or odd, i.e. $p = 2n$ or $p = 2n + 1$:

- if n is even we get $2p + 1 = 2(2n) + 1$ which reduces to $4n + 1$.
- If n is odd we get $2p + 1= 2(2n + 1) + 1$ which reduces to $4n + 2 + 1 = 4n + 3$.

Thus, we have proved that every odd number can be written as $4n + 1$ or $4n + 3$ where n is an integer. The proof is complete.

**Question 5: Prove that for any integer n, at least one of the integers n, n + 2, n + 4 is divisible by 3.**

Proof: by cases

It must be true that for any integer which we take modulo 3 we get the answer 0, 1 or 2. This means that n can be rewritten so that only one of the cases below holds.

Case 1: $n = 3k + 0$. This clearly shows that 3 divides n.

Case 2: $n = 3k + 1$. Using n, n + 2 and n + 4 (from the statement to be proven) gives us the alternatives:

$$3n + 1$$
$$3n + 2 + 1 = 3n + 3$$
$$3n + 4 + 1 = 3n + 5$$

Taking $3n + 3$ and factoring out 3 gives $3(n + 1)$ which clearly shows that 3 divides n.

Case 3: $n = 3k + 2$. Using n, n + 2 and n + 4 (from the statement to be proven) gives us the alternatives:

$$3n + 2$$
$$3n + 2 + 2 = 3n + 4$$
$$3n + 4 + 2 = 3n + 6$$

Taking $3n + 6$ and factoring out 3 gives $3(n + 2)$ which clearly shows that 3 divides n.

Since 3 divides n in all possible cases, we have proven that for any integer n, at least one of the integers n, n + 2, n + 4 is divisible by 3 holds. The proof is complete.

**Question 6: A classic unsolved problem in number theory asks if there are infinitely many pairs of 'twin primes', pairs of primes separated by 2, such as 3 and 5, 11 and 13, or 71 and 73. Prove that the only prime triple (i.e. three primes, each 2 from the next) is 3, 5, 7.**

Proof: by contradiction

Suppose that we have a prime triplet (n, n + 2, n + 4) such that n is an integer larger than 3.

It must be true that for any integer which we take modulo 3 we get the answer 0, 1 or 2. This means that n must be possible to rewrite so that one of the cases below holds.

- If n is a multiple of three (n mod 3 = 0), n is clearly not prime.
- If n is one more than a multiple of three (n mod 3 = 1), then n + 2 will be divisible by 3 and thus not prime.
- If n is two more than a multiple of three (n mod 3 = 2), then n + 4 will be divisible by 3 and thus not prime.

Thus, since each possible case shows that it is divisible by 3, we have reached a contradiction, which means that the initial assumption that there exists a prime triplet (n, n + 2, n + 4) such that n is an integer larger than 3 is invalid.

Therefore, we have proven that there is no other prime triplet apart from 3, 5, 7. The proof is complete.

**Question 7: Prove that for any natural number n, $2 + 2^2 + 2^3 + ... + 2^n = 2^{n+1} – 2$**

Proof: by direct proof

Let $L = 2 + 2^2 + 2^3 + ... + 2^n$ (the left side of the statement)

Let $R = 2^{n+1} – 2$ (the right side of the statement)

Thus, $2L = 2^2 + 2^3 + ... + 2^{n+1}$ (multiply by 2 and put in exponent)

Therefore, $2L - L = 2^{n+1} - 2$ (all terms except first and last canceling out)

Since $2L - L = L$ and thus L equals the right side of the statement R, we have proven that the statement holds for any natural number n. The proof is complete.

**Question 8: Prove (from the definition of a limit of a sequence) that if the sequence $\{a_n\}_{n=1}^{\infty}$ tends to limit L as n → ∞, then for any fixed number M > 0, the sequence $\{Ma_n\}_{n=1}^{\infty}$ tends to the limit ML.**

Proof: based on the definition of the limit of a sequence.

Definition: $(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall n \geq N)( |a_n - a| < \epsilon )$

The definition of the limit of a sequence tells us that a sequence $a_n$ tends to the real number a, if for each positive real number ϵ, there exists a natural number N such that, for every natural number n ≥ N, we have $|a_n - a| < \epsilon$.

The sequence $\{a_n\}_{n=1}^{\infty}$ tends to the limit L as n → ∞ (as described in the question). I.e. in this case we let L correspond to a in the definition.

Based on the definition of the limit of a sequence we are now going to prove that the sequence $\{Ma_n\}_{n=1}^{\infty}$ tends to the limit ML as n → ∞, for any fixed number M > 0. I.e. using the definition, we let $Ma_n$ correspond to $a_n$ and ML correspond to a.

We need to prove that the statement $(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall n \geq N)( |a_n - L| < \epsilon/M )$ is true. This means that from some point n onwards, for any ϵ all the members of the sequence $\{Ma_n\}_{n=1}^{\infty}$ are within the distance ϵ of ML. This is possible by making ϵ as small as we want to get closer and closer to ML.

Thus, given any ϵ > 0, we can find an $n \geq N \Longrightarrow |a_n - L| < \epsilon/M$

$n \geq N \Longrightarrow |Ma_n - ML| < M\epsilon/M$

Therefore $n \geq N \Longrightarrow M(|a_n - L|) < \epsilon$

This shows that $\{Ma_n\}_{n=1}^{\infty}$ tends to the limit ML as n → ∞, thus the proof is complete.

**Question 9: Given an infinite collection $A_n$, n = 1, 2, . . . of intervals of the real line, their intersection is defined to be $\bigcap_{n=1}^{\infty} A_n$ = {x | (∀n)(x ∈ $A_n$)}**

**Give an example of a family of intervals $A_n$, n = 1, 2, ..., such that $A_{n+1} \subset A_n$ for all n and $\bigcap_{n=1}^{\infty} A_n$ = ∅. Prove that your example has the stated property.**

Example: $A_n$ = (0, 1/n) where n = (1, +∞)

For intuition:

n = 1 1 / 1 = 1/1 = 1 giving the interval (0, 1)

n = 2 1 / 2 = 0.5 giving the interval (0, 0.5)

n = 3 1 / 3 = 0.3333… giving the interval (0, 1/3)

Proof: by direct proof

1/n tends to 0 as n approaches ∞.

For each n in the interval (1, +∞), $A_{n+1}$ is a smaller subset of the previous interval, i.e. $A_n$, in the sequence. i.e. (0, 1/∞) … ⊂ (0, 1/3) ⊂ (0, 1/2) ⊂ (0, 1/1).

We can also see that the intersection of these intervals, i.e. $\bigcap_{n=1}^{\infty} A_n$ is empty since as n goes to infinity we will never get any overlap between *all* intervals in the sequence. As n goes to ∞ the interval will be infinitely small and we can always find a smaller interval by letting the right point of the interval be smaller than the previous. Also, since we have an open interval 0 is not part of the interval and thus the point 0 does not overlap. This shows that the proof is complete.

**Question 10: Give an example of a family of intervals $A_n$, $n$ = 1,2,..., such that $A_{n+1} \subset A_n$ for all n and $\bigcap_{n=1}^{\infty} A_n$ consists of a single real number. Prove that your example has the stated property.**

Example: $A_n = (-1/n, 1/n)$ where n = (1, +∞)

For intuition:

n = 1 An = (-1/1, 1/1) = (-1, 1)

n = 2 An = (-1/2, 1/2)

n = 3 An = (-1/3, 1/3)

Proof: by direct proof

Both -1/n and 1/n tends to 0 as n approaches ∞.

For each n in the interval (1, +∞), $A_{n+1}$ is a smaller subset of the previous interval, i.e. $A_n$, in the sequence. i.e. (-1/+∞, 1/+∞) … ⊂ (-1/3, 1/3) ⊂ (-1/2, 1/2) ⊂ (-1, 1).

We can also see that the intersection of these intervals, i.e. $\bigcap_{n=1}^{\infty} A_n$ is 0 since as n goes to infinity we will get infinitely close to 0 from both the left (-1/n) and right side (1/n) of the interval. Thus, $\bigcap_{n=1}^{\infty} A_n$ = {0}, i.e. contains a single real number and the proof is complete.