



SENIOR THESIS IN MATHEMATICS

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# An Exploration of Kalman Filters

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### **Abstract**

Kalman Filters create predictions for linear systems by recursively generating and correcting predictions according to incoming data. This process enables estimates to become progressively more accurate without relying on large amounts of initial data. Nonlinear forms of the Kalman Filter exist and include the Extended Kalman Filter, the Unscented Kalman Filter, and the Dual Unscented Kalman Filter. This paper explores the theory behind each of these filters and implements a few examples of the Unscented Kalman Filter.

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# Chapter 1

## Introduction

This paper explores the Kalman Filter, Extended Kalman Filter, Unscented Kalman Filter (UKF), and the Dual Unscented Kalman Filter. The Kalman Filter is a data prediction method that estimates the value of unknown parameters of linear dynamical systems. This is a recursive predictive-corrective process that allows Kalman Filters to continuously generate predictions about state variables without relying on large amounts of initial data. In the case of nonlinear systems, alternative forms of the Kalman Filter were developed, including the Extended Kalman Filter, the Unscented Kalman Filter, and the Dual Unscented Kalman Filter. The Extended Kalman filter linearizes the nonlinear system around the mean by taking the Jacobean of the nonlinear function. Navigation and signal processing are the main applications of the Extended Kalman Filter. Instead of linearizing the system around a single points, the Unscented Kalman Filter utilizes many points, known as Sigma points. The Sigma points then undergo a nonlinear transformation known as the Unscented Transform. The Dual Unscented Kalman Filter is similar to the Unscented Kalman Filter, but also estimates the value of some unknown parameter in the system. Applications of the Unscented and Dual Unscented Kalman Filters include modeling biological systems.

In addition to understanding the theory behind these algorithms, this paper also explores applications of the UKF. An example of the UKF can be applied to the Van der Pol oscillator, which is a self sustaining nonconservative oscillator.

## Chapter 2

# Kalman Filters

The Kalman Filter (KF) recursively generates predictions for linear dynamical systems [6]. The basic foundations of this algorithm include generating a prediction given some initial knowledge of the data and using actual measurements from the system to continually correct the prediction. Therefore, unlike other predictive methods like machine learning, the KF is able to begin generating estimate without large amounts of initial data. The KF begins by assuming the given data is noisy and Gaussian [5, 7]. The first predictive step assumes knowledge of initial states and the model process. Since we know the system is linear, the model process can be denoted as a matrix and the states can be expressed as a vector. Through matrix vector multiplication, this algorithm simulates how states are transformed after undergoing some process. The next step involves calculating the covariance in order to calculate the Kalman Gain, which is a measure of how much the estimate should be changed given actual measurements of the system. The corrective step utilizes the Kalman Gain, which is a measure of trust in either the measurements or the model, and the system measurements to gauge whether to depend more on the model or on the data. This process can be done recursively, allowing the model to become progressively more accurate as more data is added. Therefore, overtime, it is expected that the model will converge with the actual system measurements.

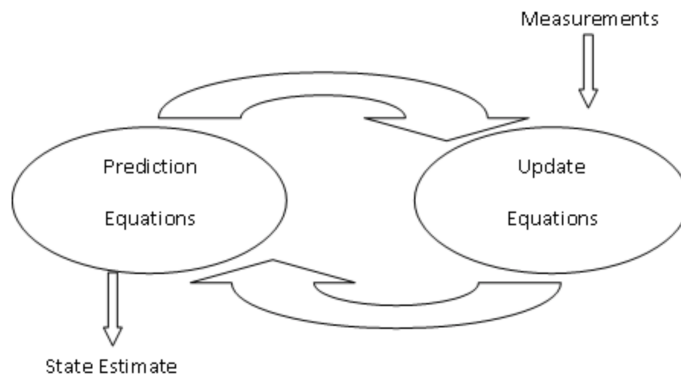


Figure 2.1: A basic diagram of the recursive nature of the Kalman Filter [3].

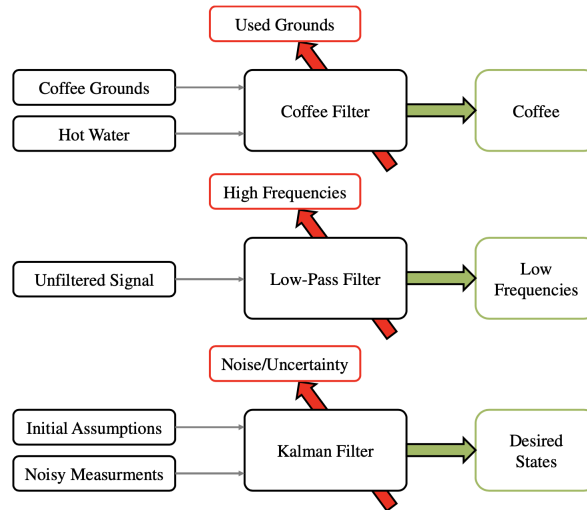


Figure 2.2: A simpler way to explore the KF is to facilitate a comparison with a coffee filter. This image is taken from a paper by Rhudy et al. comparing the Kalman Filter to a coffee filter [5].

The most common application of KFs is in navigation, image processing, and finance. A relevant example is using computer vision to monitor and track vehicles in real time. Another example is the development of the Global Positioning Systems (GPS) [?]. The KF also has many aeronautical applications, which include long-distance flight, and autopilot systems. In fact, the KF was created to aid in the development of the navigation system in the Apollo Program [?].

The Kalman Filter is named after its developer, Rudolph Emil Kalman. Kalman was born on May 19, 1930 in Budapest, Hungary. After arriving in the United States, Kalman completed his undergraduate studies and masters degree in electrical engineering from Massachusetts Institute of Technology and completed his doctoral degree in Columbia University. He would spend the next years of his life teaching. In the 1960s to 1970s he became a professor at Stanford university [?]. In the 1970s and 1990s, Kalman spent time as professor of engineering at the University of Florida. Kalman is most known for his work on the Kalman Filter, which was developed in the late 1950s. The Kalman Filter greatly aided the United States' military projects, resulting in formed President Obama to award Kalman the National Medal of Science in 2009. In addition, in 1985, Kalman was awarded the Kyoto Prize, which is the Japanese version of the Nobel Peace Prize in the united States. Kalman passed away July 2, 2016 at the age of 86 and is survived by his wife and two children [?].

## 2.1 Kalman Filter Algorithm

Before delving into the formal steps of the Kalman Filter, it is important to understand the underlying foundation of this method. The goal of the Kalman Filter is to predict the new state of a system, after it undergoes some transformation. Let this be represented by

$$\frac{dx}{dt} = f(x) + \varepsilon',$$

where  $\frac{dx}{dt}$  is the prediction of the states,  $f(x)$  is a function that transforms the states, and  $\varepsilon'$  is internal system noise. Assuming we have some understanding of the system, we can represent this transformation using matrix  $M'$  to approximate the equation above as

$$\frac{dx}{dt} \approx M'x + \varepsilon'.$$

The Kalman Filter works by generating predictions and making correction at various time steps. Let these time steps be denoted as  $k$ , such that  $k$  is a nonnegative integer and let  $x_k$  be the estimate of the state variables at time  $k$  with initial assumptions of the state variables beginning at  $x_0$  such that  $x_0 = \mathcal{N}(m_0, c_0)$ .

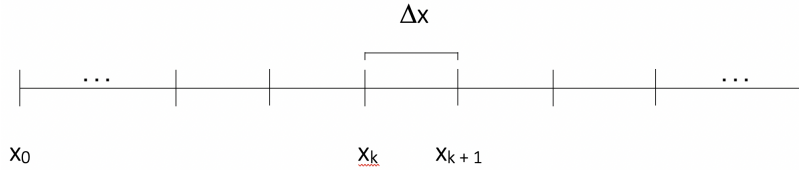


Figure 2.3: A diagram showing discretized time steps of a system. Though this diagram depicts time steps that are equal, this should not be assumed for all cases.

Next, approximate  $\frac{dx}{dt}$  using the limit definition of derivative:

$$\begin{aligned} \frac{x_{k+1} - x_k}{\Delta t} &\approx M'x_k + \varepsilon'_k \\ x_{k+1} - x_k &\approx M'x_k\Delta t + \varepsilon'_k\Delta t \\ x_{k+1} &\approx M'x_k\Delta t + \varepsilon'_k\Delta t + x_k \\ x_{k+1} &\approx (M'\Delta t + I)x_k + \varepsilon'_k\Delta t \end{aligned}$$

From here, substitute  $M$  for  $M'\Delta t + I$  and  $\varepsilon_k$  for  $\varepsilon'_k\Delta t$  to get

$$x_{k+1} \approx Mx_k + \varepsilon_k,$$

where  $\varepsilon_k = \mathcal{N}(0, c_1)$ . This general equation is significant to the generation of predictions and will continue to be used in different versions of the Kalman



Filter.

Another important aspect of the KF is the incorporation of state measurements. The estimate of the the system measurement at the next time step, call it  $y_{k+1}$ , is given by

$$y_{k+1} \approx Hx_{k+1} + v_{k+1},$$

where  $H_k$  is the observation function and  $v_k$  is measurement noise such that  $v_k = \mathcal{N}(0, c_2)$ .

1. Begin by initializing the state estimate and the initial state covariance matrix. The state estimate,  $x_0$ , is a column vector containing state variables, call them  $x_a, x_b, \dots, x_n$ , such that  $x_0 = [x_a, x_b, \dots, x_n]^T$ , where  $T$  is the transpose. The state estimate can be found by taking the expected value of the data where data is normally distributed. If the system states are finite, the expectation is denoted by

$$\mathbb{E}[x_0] = \sum_{i=a}^n x_i p_i = [x_a p_a + x_b p_b + \dots + x_n p_n]^T,$$

$p_a, p_b, \dots, p_n$  is the respective probability of getting each state variable. The state covariance matrix,  $P_0$ , is a square matrix whose contents are the covariance of the pairwise elements<sup>1</sup>, or

$$P_0 = \begin{bmatrix} \text{var}(x_a) & \dots & \text{cov}(x_a, x_n) \\ \vdots & \ddots & \vdots \\ \text{cov}(x_n, x_a) & \dots & \text{var}(x_n) \end{bmatrix}.$$

Enough should be known about the modeled system to generate these values. In general,  $\hat{x}_k$  denotes the model's estimate of the state variables. When initializing these values, we use  $x_0$ , which is the actual initial measurements of the system.

A state covariance matrix is a symmetrical positive semi-definite square matrix whose diagonals correspond to the variance of a variable at location  $i$  and elsewhere is the covariance of the pairwise elements. In practice, covariance matrices help us better understand the spread of data. For the case of the KF, calculating the state covariance is necessary for computing Kalman Gain.

2. After initializing the state estimate and state covariance, a prediction can be generated. The estimate of the system at the next time step,  $x_{k+1}$ , is given by

$$x_{k+1} = Fx_k + Gu_k + w_k,$$

---

<sup>1</sup>Recall that the covariance of a variable with itself is the variance of the variable

where  $F$  is the state matrix,  $G$  is the F and input matrix,  $u_k$  is an input vector and  $w_k$  is the process noise vector. Every state variable contained in  $x_k$  is defined by a linear differential equation. These linear differential equations can be used to generate the F and G matrices. Therefore, F should be a square matrix whose dimension is equal to the number of states variables and G can be a matrix (or in some cases a vector), depending on the dimension of input vector u.

$u_k$  is an input vector, which is a measurable value that helps define the system, but is not contained in the state vector. Depending on how the system is defined,  $u_k$  can be a constant value or it can be a value dependent on time step k.

$w_k$  is the process noise vector at time k. Process noise can be thought of as the model's accuracy. When process noise is 0, it implies that the model is perfectly accurate and does not have to correct for incoming system measurements. On the other hand, high process noise will essentially restate the system based on incoming measurements.  $w_k$  has the same dimensions as  $x_k$ , allowing us to identify whether or not to adjust the equations for the state variables. When all values of  $w_k$  equal 0, it implies that the linear differential equations we are using to define the state variables have no error.

3. Next, the state covariance matrix to calculate Kalman Gain. The state covariance matrix at time step  $k$  given the last time step, is

$$P_{k|k-1} = FP_{k-1}F^T + Q_{k-1},$$

where  $F^T$  is the transpose of  $F$ , and  $Q_k$  is the process noise covariance of  $w_k$ . The Kalman Gain at time step  $k$ , is given by

$$K_k = P_{k|k-1}H_K^T(H_K P_{k|k-1}H_K^T + R_k)^{-1},$$

where  $H$  is the observation matrix and  $R$  is measurement noise covariance.  $H$  enables the state variables to be linearly transformed to match the outputs of the system. The dimensions of  $H$  reflects which state variables have measurable values in the system. It is not assumed that every state variable is measurable, so  $H$  allows us to compare the measurable state variables to  $x_k$ . Simple applications of  $H$  include creating matrices with 0's and 1's, with 1's denoting that a state variable is measurable and a 0's representing non-measurable states. In other cases,  $H$  is an integer used as a scaling factor.

The Kalman Gain is the main component of the corrective aspect of the KF. The Kalman Gain is a measure of how much to change the model based on incoming data. Low values of the Kalman Gain imply the model is accurate while higher values indicate the model should adjust based on

the incoming data.

From this equation, one can see that balancing  $Q$  and  $R$  is critical for model performance. Larger values of  $Q$  indicate higher modeling error, which leads to a higher Kalman Gain and increased model correction. On the other hand, large values of  $R$  imply high measurement error, leading to a lower Kalman Gain and less model correction.

4. Next, calculate the transformed output vector and correct the prediction. The transformed output vector,  $\hat{y}_k$ , is given by

$$\hat{y}_k = H_k x_k + v_k,$$

where  $v_k$  is measurement noise, which is added to account for measurement error. The corrected prediction,  $x_k$ , is given by

$$x_k = x_{k-1} + K_k(y_k - \hat{y}_k),$$

where  $y_k$  is the actual measurements of the system,  $K_k$  is the Kalman Gain, and  $x_{k-1}$  are the values of the state variable at the last time step.  $\hat{y}_k$  is a way of transforming the prediction (using  $H$ ) into a format that can be compared with  $y_k$ . Subtracting the actual measurements from the predicted measurements is necessary, because  $y_k$  does not always contain measurements for all variables in state vector  $x_k$ . The quantity  $y_k - \hat{y}_k$  is also known as measurement residual or innovation.

5. The final step is to update the state covariance matrix,  $P_k$ , through the equation

$$P_k = (I - K_k H_k) P_{k|k-1},$$

where  $I$  is the identity matrix,  $K_k$  is the Kalman Gain,  $H_k$  is the observation function, and  $P_{k|k-1}$  is the state covariance at time step  $k$  given the last time step.  $P_k$  will be used in the next iteration of the filter.

Table 2.1: Description of all variables in the Kalman Filter

Variables in the Kalman Filter		
Variable	Description	Dimensions
$\mathbf{x}$	State variables	$d_x \times 1$
$\mathbf{y}$	Output vector	$d_y \times 1$
$\mathbf{u}$	System inputs	$d_u \times 1$
$\mathbf{v}$	Measurement noise	$d_y \times 1$
$\mathbf{w}$	Process noise	$d_x \times 1$
$\mathbf{F}$	State function	$d_x \times d_x$
$\mathbf{H}$	Observation function	$d_y \times d_x$
$\mathbf{G}$	Input function	$d_x \times d_u$
$\mathbf{K}$	Kalman Gain	$d_x \times d_y$
$\mathbf{Q}$	Process noise covariance	$d_x \times d_x$
$\mathbf{R}$	Measurement noise covariance	$d_y \times d_y$
$\mathbf{P}$	Covariance matrix	$d_x \times d_x$

## 2.2 Assessing Model Performance

## Chapter 3

# Extended Kalman Filters

The Extended Kalman Filter (EKF) is the non-linear version of the Kalman Filter. Though it can be used for non linear equations, it is important to note that it is not an optimal estimator. For the most part, the EKF Algorithm is nearly identical to the KF algorithm. The critical difference is in linearizing the state and observation function. The EKF uses the Jacobean to linearly approximate the non-linear function around the mean of the Gaussian distribution. Skipping this step would result in the transformed data being non-Gaussian; though taking the Jacobean enables the transformation to remain Gaussian, but it is not exact, resulting in some generalization. Linear approximation through a single point also makes the EKF inefficient when dealing with complex, higher order systems. Because of this, the model is highly subject to error, which can be somewhat reduced by setting accurate initial values. Though these flaws exist, the EKF performs strongly with applications of real time spatial fields, including navigation and positioning systems.

### 3.1 Extended Kalman Filter Algorithm

1. Begin by initializing the state estimate,  $x_0$ , and the initial covariance matrix,  $P_0$ . Similar to the KF, the initial state estimate can be found by

$$\mathbb{E}[x_0] = \sum_{i=a}^n x_i p_i = [x_a p_a + x_b p_b + \dots + x_n p_n]^T,$$

where  $x_a, x_b, \dots, x_n$  are the state variable,  $p_a, p_b, \dots, p_n$  is the probability of getting each state variable, and T is the transpose of this row vector. We also want to initialize the covariance matrix by

$$P_0 = \begin{bmatrix} \text{var}(x_a) & \dots & \text{cov}(x_a, x_n) \\ \vdots & \ddots & \vdots \\ \text{cov}(x_n, x_a) & \dots & \text{var}(x_n) \end{bmatrix}.$$

Often,  $P_0$  will be initialized as a diagonal matrix with the diagonals being the variance of each state variance and every other value being set to 0.

2. This next step deviates from the Kalman Filter because it takes a linear approximation the non-linear function by calculating the Jacobean. Recall that  $F$  is the system state function, which is a model of the system. Let  $f$  be the linearized representation of  $F$  by taking the Jacobean as shown below:

$$F = \frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \dots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}.$$

Similarly, the observation function  $H$  is a nonlinear, but a linear approximation,  $h$ , can be found by

$$H = \frac{\partial h}{\partial x} = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \dots & \frac{\partial h_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_m}{\partial x_1} & \dots & \frac{\partial h_m}{\partial x_n} \end{bmatrix}.$$

Here, both  $f$  and  $h$  have no closed form solution.  $f$  is the non-linear state function and  $F$  is the linear approximation of  $f$  with dimension  $d_x$ . Similarly,  $h$  is the non linear observation while  $H$  is linearized observation with dimension  $d_y$ .

3. A prediction can be generated by using the linear approximation,  $f$ , by

$$x_{k+1} = f(x_{k-1}, u_{k-1}),$$

where  $u_{k-1}$  are the system outputs **expand more on what this is**.

4. Now, we can use the state covariance matrix,  $P_{k|k-1}$ , to calculate Kalman Gain,  $K_k$ . Recall that the Kalman Gain is a method to determine whether to place more weight on incoming measurements or the system's model.  $P_{k|k-1}$ , can be found by

$$P_{k|k-1} = F P_{k-1} F^T + Q_{k-1},$$

where  $F$  is the nonlinear state function,  $P_{k-1}$  is the state covariance at the last time step,  $T$  is the transpose, and  $Q$  is the process noise covariance. Using this value, calculate the Kalman Gain by

$$K_k = P_{k|k-1} H_K^T (H_k P_{k|k-1} H_K^T + R_k)^{-1},$$

where  $H$  is the nonlinear observation matrix, and  $R$  is the measurement noise covariance. Recall that  $H$  and  $F$  are linear approximations of  $f$  and  $h$ . Therefore, it is assumed that there is some amount of error when calculating the Kalman Gain.

5. The correction of the prediction is similar to the Kalman Filter. Incoming measurements from the system,  $y_k$ , need to be converted in order to facilitate a comparison with our state variables. Let the converted measurement be denoted  $\hat{y}_k$  such that

$$\hat{y}_k = h(x_k + v_k),$$

$h$  is the linear approximation of the observation function and  $v_k$  is measurement noise. Using  $\hat{y}_k$ , we can calculate the residual, which is a measure of how close the estimate is the measured values. The equation can be updated by

$$x_k = x_{k-1} + K_k(y_k - \hat{y}_k).$$

6. Finally, update the covariance matrix,  $P_k$ , which will be used in the next iteration of the filter.  $P_k$  can be updated by

$$P_k = (I - K_k H_k) P_{k|k-1},$$

where  $I$  is the identity matrix,  $K_k$  is the Kalman Gain,  $P_{k|k-1}$  is the covariance matrix given the last time step, and  $H_k$  is the nonlinear observation matrix.



Variables in the Extended Kalman Filter		
Variable	Description	Dimensions
x	State variables	$d_x \times 1$
y	Output vector	$d_y \times 1$
u	System inputs	$d_u \times 1$
v	Measurement noise	$d_y \times 1$
w	Process noise	$d_x \times 1$
f	None linear state function	$d_x \times d_x$
F	State function	$d_x \times d_x$
h	Non linear observation function	$d_y \times d_x$
H	Linearized observation function	$d_y \times d_x$
K	Kalman Gain	$d_x \times d_y$
Q	Process noise covariance	$d_x \times d_x$
R	Measurement noise covariance	$d_y \times d_y$
P	Covariance matrix	$d_x \times d_x$

## Chapter 4

# Unscented Kalman Filters

The Unscented Kalman Filter (UKF) is another nonlinear version of the Kalman Filter developed to address the shortcomings of the EKF. For instance, as opposed to using the Jacobean to linearly approximate around a single point, the UKF uses the Unscented Transform (UT) to approximate around multiple points, known as sigma points. The UT is a method of approximating probability distributions that have undergone a non linear transformation using limited statistics. The UT involves using these sigma points, which are represented in a Sigma Point Matrix, to represent the normal distribution of the data. The covariance and weights of these sigma points are calculated. These sigma points undergo a non-linear transformation, resulting in a posterior distribution that is not normal [6, 8] . We are able to approximate the normal distribution of the posterior distribution using the weights and covariance that were calculated prior to the transformation. This process enables the Kalman Filter to be applied to more complex non linear problems.

Unlike the Kalman Filter and the Extended Kalman Filter, the UKF also has a set of parameters. Explanations of each parameter and their default values can be found in the chart below. For the UKF, parameters are necessary for controlling the spread of sigma points. This was not needed for the EKF, since the EKF was only linearizing around the mean. [expand here on how to tune parameters.](#)

The term 'unscented' was arbitrarily coined by the developer of the UKF, Jeffrey Uhlmann. In an interview, he shares:

"Initially I only referred to it as the "new filter." Needing a more specific name, people in my lab began referring to it as the "Uhlmann filter," which obviously isn't a name that I could use, so I had to come up with an official term. One evening everyone else in the lab was at the Royal Opera House, and as I was working I noticed someone's deodorant on a desk. The word "unscented" caught my

eye as the perfect technical term. At first people in the lab thought it was absurd—which is okay because absurdity is my guiding principle—and that it wouldn’t catch on. My claim was that people simply accept technical terms as technical terms: for example, does anyone think about why a tree is called a tree?”

## 4.1 Unscented Kalman Filter Algorithm

1. The first step is to initialize state vector,  $x_0$ , and covariance,  $P_{x_0}$ , which is exactly similar to both the KF and the EKF. Since we know that the  $x_0$  is normally distributed, the expectation can be calculated by

$$x_0 = \mathbb{E}[x_0] = \sum_{i=a}^n x_i p_i = [x_a p_a + x_b p_b + \dots + x_n p_n]^T,$$

where  $x_0 = [x_a, x_b, \dots, x_n]^T$ ,  $p_a, p_b, \dots, p_n$  is the respective probability of obtaining each state variable and  $T$  is the transpose. Next initialize the state covariance, by

$$P_{x_0} = \mathbb{E}[(x_0 - \hat{x}_0)(x_0 - \hat{x}_0)^T].$$

We need to initialize these values so the model can begin generating and correcting predictions.

It is critical to choose initial values that are close to the actual values of the data. Failure to do causes the model to converge slowly.

2. Unlike the KF and EKF, the UKF requires the calculation of sigma points, which is a way characterizing the distribution of the data. The number of sigma points is deterministic and depends on the dimensions of the system. In general, a UKF will have  $2 \cdot d_x + 1$  sigma points, where  $d_x$  represents the dimension of the state vector [6–8]. All sigma points are stored in a sigma point matrix, called  $\chi$ . To get a general idea about the distribution of the data, we use  $\lambda$ , which is a scalar value that determines how spread out the sigma points are from the mean.  $\lambda$  can be calculated by

$$\lambda = \alpha^2(d_x + \kappa) - d_x,$$

where  $\alpha$  and  $\kappa$  are both parameters that control for the spread of sigma points around the mean value of the state. The spread of the sigma points is proportional to  $\alpha$ . For both  $\alpha$  and  $\kappa$ , the smaller the values are, the closer the sigma points are to the mean.

Another parameter of the UKF is  $\beta$ , which uses information regarding state distribution to adjust sigma points.  $\beta$  has a default value of 2 if the

data is Gaussian (which is an assumption we will make throughout this paper).

Since the goal of this step is to characterize the distribution, set one of these sigma points to the mean,  $\chi_{0,k-1}$  which can be expressed as

$$\chi_{0,k-1} = x_{k-1},$$

where 0 is a row on  $\chi$  and  $k-1$  is the time step. Then allocate half of the remaining points will be smaller than the mean and the other half will be larger than the mean by

$$\chi_{i,k-1} = \hat{x}_{k-1} + \left( \sqrt{(d_x + \lambda)P_{x_{k-1}}} \right)_i \quad i = 1, \dots, d_x,$$

$$\chi_{i,k-1} = \hat{x}_{k-1} - \left( \sqrt{(d_x + \lambda)P_{x_{k-1}}} \right)_{i-n} \quad i = d_x + 1, \dots, 2d_x + 1,$$

where  $(\sqrt{(d_x + \lambda)P_{x_{k-1}}})$  is a matrix, and the  $i$  subscript is the  $i^{th}$  column of the matrix. Recall that the square root of a matrix, satisfies the following condition:  $A = B^2$ , where  $A$  and  $B$  are both matrices.

3. Next, we must calculate a weight for each sigma point. Weights are scalars used to calculate posterior sigma points after they have undergone a non-linear transformation. One set of weights,  $W^{(m)}$ , will be used to calculate the posterior mean while another set of weights  $W^{(c)}$  will be used to calculate the posterior covariance. Weights can have positive or negative values, but will ultimately sum to 1 [1].

Calculations of the weights at the initial time step follow a formula distinct from the other time steps **because.....**  $W_0^{(m)}$  can be found by

$$W_0^{(m)} = \frac{\lambda}{d_x + \lambda},$$

where the 0 is the time step. Similarly, the weight for the covariance in the initial time step,  $W_0^{(c)}$ , is given by

$$W_0^{(c)} = \frac{\lambda}{d_x + \lambda} + (1 - \alpha^2 + \beta),$$

**expand why parameters come in here.....** In later time steps,  $W_i^{(m)}$  and  $W_i^{(c)}$  follow the same equation, given by

$$W_i^{(m)} = W_i^{(c)} = \frac{\lambda}{2(d_x + \lambda)} \quad i = 1, \dots, 2d_x.$$

4. This step performs a nonlinear transformation on the sigma point matrix,  $\chi$ , in order to generate a prediction and provide an update on the covariance matrix. After  $\chi$  undergoes a nonlinear transformation,  $f$ , the result is a transformed sigma point matrix at time step  $k$  given the last time step,  $k - 1$ , which will be called  $\chi_{k|k-1}$  and is given by

$$\chi_{k|k-1} = f(\chi).$$

Though  $\chi$  has a Gaussian distribution,  $\chi_{k|k-1}$  does not because it has been transformed by the nonlinear state function  $f$ . The sum of the columns in  $\chi_{k|k-1}$  and the weights calculated in step 3 will then be used to generate a prediction of the state variables,  $x_k$ , by **expand on why this works....**

$$x_k = \sum_{i=0}^{2d_x} W_i^{(m)} \chi_{i,k|k-1}.$$

Adding the weights help approximate the Gaussian distribution of the state variables after they undergo a transformation. Next, calculate the posterior covariance matrix for the state variable, which is necessary for updating the state covariance later on by

$$P_{x,k|k-1} = \sum_{i=0}^{2d_x} W_i^{(c)} (\chi_{i,k|k-1} - x_k)(\chi_{i,k|k-1} - x_k)^T + Q,$$

where  $Q$  is process noise that provides the error in our model  $f$ ,  $T$  is the transpose, and  $W_i^{(c)}$  are the weights calculated earlier.

Steps 2 through 4 are part of a function known as the Unscented Transform.

5. Now that the prediction component of the filter is completed, we move on to the correction step, which begins by transforming our predictions into a format that can be compared with system measurements. Calculation of the posterior (also called augmented) sigma points is necessary for converting system measurements into a format that can be compared with the state variables. System measurements must undergo a non linear transformation,  $h$ , resulting in a non Gaussian distribution. Therefore, the Unscented Transform is used again. Begin by calculating sigma points,  $\chi^{(aug)}$ , by

$$\chi_{0,k|k-1}^{(aug)} = x_k$$

$$\chi_{i,k|k-1}^{(aug)} = x_k \pm \left( \sqrt{(d_x + \lambda) P_{x_k}} \right)_i \quad i = 1, \dots, 2d_x$$

Recall that  $\lambda$  was calculated in step 2. Since  $\lambda$  is not time dependent, we can use the same value used earlier.

Now we calculate a sigma point matrix that represents the transformation of the prediction so that it can be compared with the states. This is necessary, especially in cases where measurements are not being obtained for all state variables. This sigma point matrix,  $\mathcal{Y}_{k|k-1}$ , can be obtained by having the  $\chi_{k|k-1}$  undergo nonlinear transformation  $h$ , by

$$\mathcal{Y}_{k|k-1} = h(\chi_{k|k-1}^{(aug)}).$$

$\mathcal{Y}_{k|k-1}$  is a transformed sigma point matrix. While in this format, it cannot be compared with the state variables. However,  $\mathcal{Y}_{k|k-1}$  can be used to convert the system measurements into a format,  $y_k$ , which can be found by

$$y_k = \sum_{i=0}^{2d_x} W_i^{(m)} \mathcal{Y}_{i,k|k-1}.$$

Now, the prediction can be compared with actual system measurements.

6. The next step is to calculate Kalman Gain in order to determine how much to correct the model.

Unlike previous versions of the KF, in addition to calculating the covariance of the state variables, calculations is also done for the covariance of observations,  $P_y$ , and for state variables with observations,  $P_{xy}$ . When generating the covariance matrix for  $P_y$  the covariance of measurement noise is added.  $P_y$  can be found by

$$P_{y,k|k-1} = \sum_{i=0}^{2d_x} W_i^{(c)} (\mathcal{Y}_{i,k|k-1} - y_k)(\mathcal{Y}_{i,k|k-1} - y_k)^T + R,$$

where  $R$  is the covariance of measurement noise and  $W_i^{(c)}$  are the weights calculated in step 3. Next, calculate  $P_{xy}$  by

$$P_{xy,k|k-1} = \sum_{i=0}^{2d_x} W_i^{(c)} (\chi_{i,k|k-1}^{(aug)} - x_k)(\mathcal{Y}_{i,k|k-1} - y_k)^T.$$

Finally, using  $P_y$  and  $P_{xy}$ , calculate the Kalman Gain, by

$$K_k = P_{xy,k|k-1}(P_{y,k|k-1})^{-1}.$$

7. The final step includes correcting the prediction and updating the covariance matrix, which will both be used in the next iteration. Similar to the

UKF and the KF, the prediction of the state variables at the next time step,  $x_{k+1}$ , is given by

$$x_{k+1} = x_k + K_k(\hat{y}_k - y_k),$$

where  $\hat{y}_k$  is system measurements. Conclude this iteration of the filter by preparing  $P_{x,k}$  for the next iteration.  $P_{x,k}$  can be found by

$$P_{x,k} = P_{x,k|k-1} - K_k(P_{y,k|k-1})K_k^T.$$

Variables in the Unscented Kalman Filter		
Variable	Description	Dimensions
$\mathbf{x}$	State variables	$d_x \times 1$
$\mathbf{y}$	Transformed prediction	$d_y \times 1$
$\hat{\mathbf{y}}$	Sytem measurements	$d_y \times 1$
$\chi$	Sigma point matrix for states	$d_x \times (2d_x + 1)$
$\mathcal{Y}$	Sigma point matrix for obs	$d_y \times (2d_x + 1)$
$\mathbf{f}$	Nonlinear state function	$d_x \times d_x$
$\mathbf{h}$	Nonlinear observation function	$d_u \times d_x$
$P_x$	Covariance of states	$d_x \times d_x$
$P_y$	Covariance of observations	$d_y \times d_y$
$P_{xy}$	Covariance of states/obs	$d_x \times d_y$
$\mathbf{Q}$	Process noise covariance	$d_x \times d_x$
$\mathbf{R}$	Measurement noise covariance	$d_y \times d_y$
$W^{(m)}$	Weight for posterior mean	scalar
$W^{(c)}$	Weight for posterior covariance	scalar

Parameters in the Unscented Kalman Filter			
	Description	Bounds	Default
$\alpha$	Controls spread of sigma points	$0 < \alpha \leq 1$	.001
$\beta$	Adjust sigma point weight	$\beta \geq 0$	2
$\kappa$	Sigma point weighting constant	$0 \leq \kappa \leq 3^*$	0

\* Some use  $\kappa = 3 - d_x$



## 4.2 Van der Pol Example

Applying the UKF to the Van der Pol oscillator will be used as a simple example to demonstrate the impacts of measurement and process noise. The Van der Pol equation, so named after its developer Balthasar Van der Pol, describes a self sustaining oscillator that create energy at small amplitudes and remove energy from large amplitudes. The Van der Pol equation describes a nonconservative oscillator (also known as a relaxation oscillator). Applications of the Van der Pol oscillators include circuits, vacuums, and modeling biological systems [9]. The Van der Pol oscillator is represented by a nonlinear second order differential equation:

$$\frac{d^2y}{dt^2} + \mu(y^2 - 1)\frac{dy}{dt} = 0$$

where  $\mu$  is a damping coefficient,  $\frac{d^2y}{dt^2}$  is acceleration,  $\frac{dy}{dt}$  is velocity, and  $y$  is position. Therefore, for all  $\mu < 0$ , dampening occurs and the system tends to 0. The rate at which the system converges to zero is dependent on the size of  $\mu$ , with larger values taking longer to converge and smaller values converging faster. If  $\mu = 0$ , the system becomes a simple harmonic oscillator, where motion is periodic. Lastly, if  $\mu > 0$ , the system enters a limit cycle, which is an isolated closed trajectory. [2].

The UKF can be applied to this nonlinear system to determine where the system will be at a point in time. To do so, relevant state variables include position and velocity.<sup>1</sup>

$$x_k = \begin{bmatrix} y \\ v \end{bmatrix}$$

Transforming the Van der Pol equation from a second order differential equation to a first order differential equation makes it easier to define  $f$ , the state function of the system. By substituting  $v$  for  $\frac{dy}{dt}$ , and  $\frac{dv}{dt}$  for  $\frac{d^2y}{dt^2}$  one can rewrite the Van der Pol equation as

$$\frac{dv}{dt} + \mu(y^2 - 1)v + y = 0.$$

From this, we get the differential equations associated with the state variables to generate the nonlinear transformation function  $f$ . For the sake of simplicity, assuming  $\mu = 1$ , we get

$$\dot{x}_k = \begin{bmatrix} \frac{dy}{dt} \\ \frac{d^2y}{dt^2} \end{bmatrix} = \begin{bmatrix} v \\ \frac{dv}{dt} \end{bmatrix} = \begin{bmatrix} v \\ -1(y^2 - 1)v - y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 - y^2 - 1 & -y \end{bmatrix} x_k = f.$$

---

<sup>1</sup>In Matlab,  $x(1)$  and  $x(2)$  represent position and velocity, respectively

In this particular example, the only measurement received from the system is position. Therefore, this filter is continually correcting for the position state variable through measurement function  $h$ . Here, measurements of the system are simulated by adding noise to the position state variable. Even though there are only measurements for one state variable, we can still generate estimates of both state variables. In terms of parameters  $\alpha$ ,  $\kappa$ , and  $\beta$ , default values were used.

To simulate a Van der Pol Oscillator, an ordinary differential equation solver can be applied to state function  $f$  to generate true values of the system. Of course, systems do not perform perfectly; variations in model performance can be captured by randomly adding noise to the system.

All of this can be modeled on Matlab; all source code is complimentary of Matlab and can be referenced in Appendix A [?]

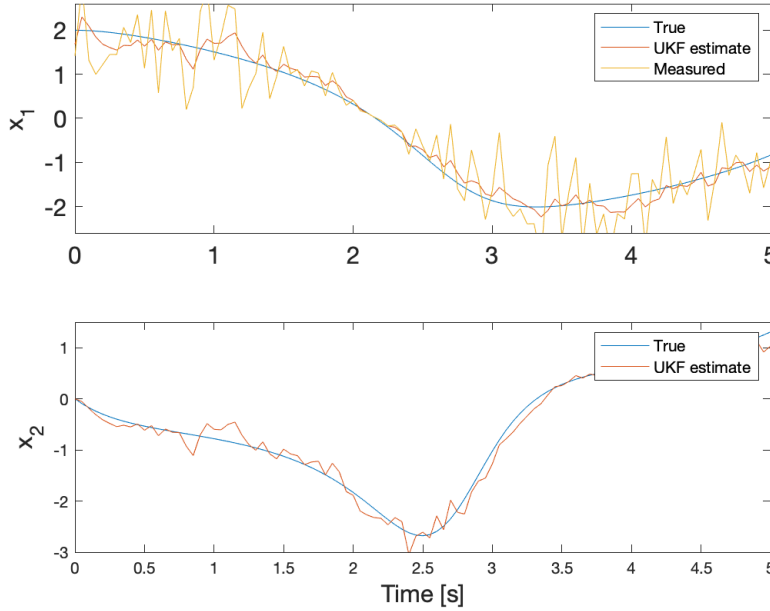


Figure 4.1: Performance of the UKF with  $R = 0.2$ , and  $Q = (0.02 \ 0.1)$ . As expected, the model converges on the true values of the system for both state variables. In this case, the only measurements we are receiving from the system are position, which is why the second state variable has no measured values.

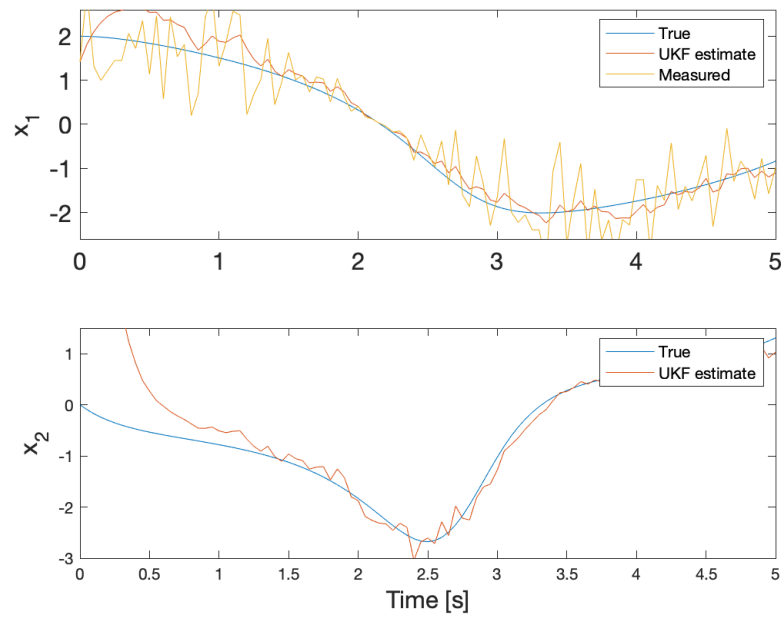
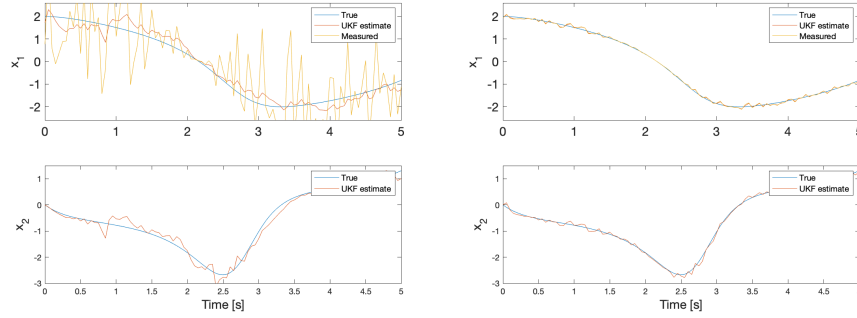
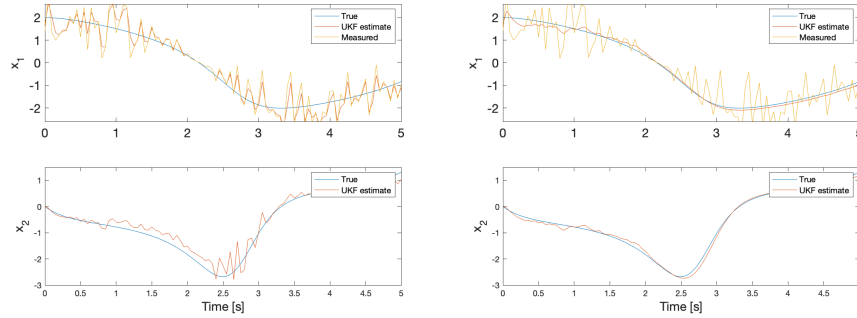


Figure 4.2: Performance of the UKF with poor initial conditions:  $(1, 7)$  instead of  $(2, 0)$ . Recall that inaccurate initial conditions cause convergence to take place more slowly. This seems to be the case, especially in the state variable that is not being corrected for.



(a) UKF with high measurement noise (0.9) (b) UKF with low measurement noise (0.002)

Figure 4.3: UKF on VDP oscillator with difference values of measurement noise. The model's behavior changes in response to the different values of measurement noise. Even when measurement noise is high, the UKF continues to perform well. Though, the rate of convergence appears to be slower. On the other hand, when measurement noise is low, the UKF seems to converge instantly with the measured values. The velocity state variable also quickly converges with the true value of the system.



(a) UKF with high process noise (0.9 0.8) (b) UKF with low process noise (0.0001 0.0001)

Figure 4.4: UKF on VDP oscillator with different values of process noise. Recall that process noise measures errors in the model. For the purpose of this example, process error was set to extreme high and low values. In theory, the Van der Pol equation should have small process error because it is a well used equation.

### 4.3 Modeling a Biological System

This example was inspired by a paper that uses an adaptation of the UKF, called the Iterative Unscented Kalman Filter (IUKF), to model the biological pathway of metabolites. Using the IUKF enabled the model converge faster by resetting the covariance to re-excite the model. By skipping this step (as is done in the UKF), the three state variables without measurements converge significantly slower. A metabolite is small bodily structure that is the byproduct of the metabolism. Data regarding metabolites is highly influenced by noise, which is a factor that makes other approaches, such as regression and annealing, fail [4].

In the paper, researchers had access to their own data sources and used an approach that was adapted from the UKF. Though we are not using their exact dataset, we will be simulating data using the same approach as the previous example. However, this example will be following UKF algorithm, as opposed to the IUKF algorithm. Ultimately, the three goals of this example is to demonstrate how the UKF works on higher dimensional and more complex systems, how the UKF can correct for multiple variables or states, and how parameters can be adjusted to fit the model.

In this example, four metabolites, or states, are explored and denoted by

$$\begin{aligned}\dot{x}_1 &= \alpha_1 x_3^{g_{13}} x_5^{g_{15}} - \beta_1 x_1^{h_{11}}, \\ \dot{x}_2 &= \alpha_2 x_1^{g_{21}} - \beta_2 x_2^{h_{22}}, \\ \dot{x}_3 &= \alpha_3 x_2^{g_{32}} - \beta_3 x_3^{h_{33}} x_4^{h_{34}}, \\ \dot{x}_4 &= \alpha_4 x_1^{g_{41}} - \beta_4 x_4^{h_{44}},\end{aligned}$$

where there are 18 unknown parameters ( $\alpha_1, \dots, \alpha_4, \beta_1, \dots, \beta_4, g_{13}, g_{15}, g_{21}, g_{32}, g_{41}, h_{11}, h_{22}, h_{33}, h_{34}, h_{44}$ ). We have a guess as to what these values might be and we will use the UKF to find parameter values that best fit model to some state. In both the original example as well as this one, sampling time will be 0.1 seconds for 5 seconds, totaling 50 UKF estimates.

Since we did not have access to the original example's dataset, data was simulated on MATLAB. Similar to the previous example, the system's true output was simulated using an ODE solver and incoming system measurements were calculated by adding noise to the true value. The model was initialized by setting the state variable to  $x_0 = [4, 1, 3, 4]^T$  and the state covariance to  $P_0 = .01I$ .

Table 4.1: True Parameter Values

True Parameter Values										
	$\alpha_i$	$g_{i1}$	$g_{i2}$	$g_{i3}$	$g_{i4}$	$\beta_i$	$h_{i1}$	$h_{i2}$	$h_{i3}$	$h_{i4}$
$x_1$	20.0	0	0	-0.8	0	10.0	0.5	0	0	0
$x_2$	8.0	.5	0	0	0	3.0	0	0.75	0	0
$x_3$	3.0	0	0.75	0	0	5.0	0	0	0.5	0.2
$x_4$	2.0	.5	0	0	0	6.0	0	0	0	0.8

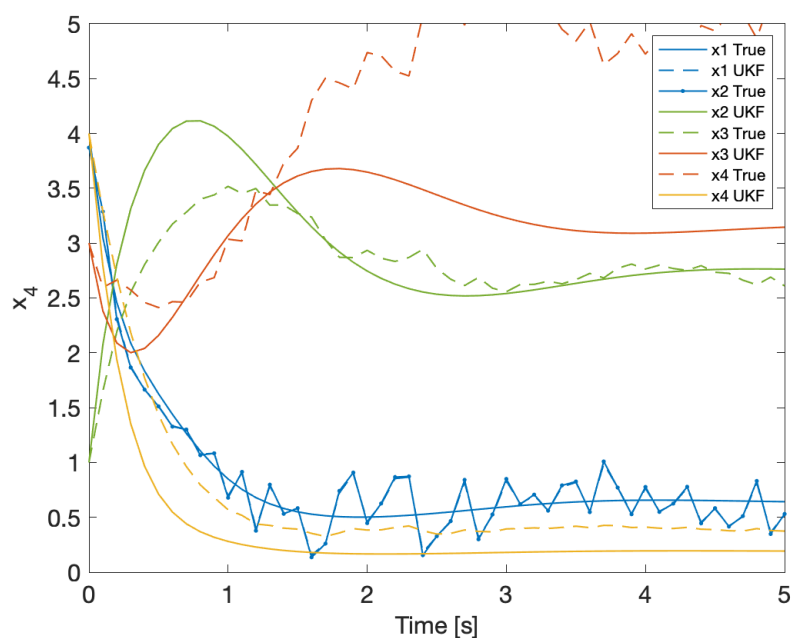


Figure 4.5: Performance of all state variables with the following conditions:  $R = 0.001$ ,  $Q = (0.02 \ 0.01 \ .03 \ .04)^T$  and initial values  $= [4, 1, 3, 4]^T$ .

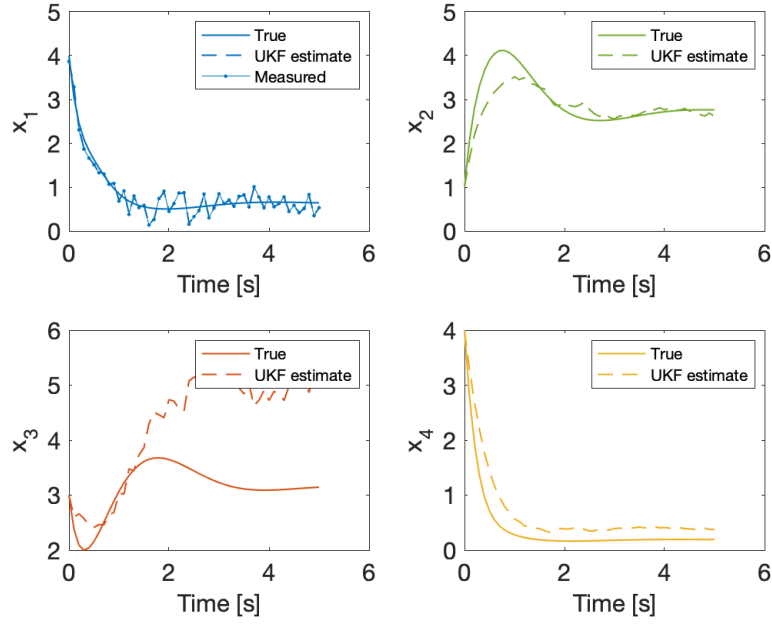


Figure 4.6: Performance of all four state variables with one corrected state. The first state variable (blue) is the only one that is receiving measurements from the system. This explains why it is more accurate and converges better with the system, as compared with the other three states.

**Expand here on tweaking parameters.** In the original example, researchers used the following parameter values:  $\epsilon = 1, \kappa = -14$ . In theory, values of  $\kappa$  can be negative, but negative values cannot be inputted into MATLAB. Therefore, the parameter values used in this example are set to **ADD PARAM VALUES HERE**.

**Expand here on correcting for 1+ states**

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- [9] Eric W Weisstein. Van der pol equation, Dec 2019.



# Appendix A

## Van der Pol Code

### A.1 Main function

```
1 % Taken from https://www.mathworks.com/help/control/ug/nonlinear-state-estimation-using-unscented-kalman-filter.html
2
3 initialStateGuess = [2;0]; % xhat[k|k-1]
4 % Construct the filter
5 ukf = unscentedKalmanFilter(...
6     @vdpStateFcn,... % State transition function
7     @vdpMeasurementNonAdditiveNoiseFcn,... % Measurement
8     function
9     initialStateGuess,...
10     'HasAdditiveMeasurementNoise',false);
11 R = 0.2; % Variance of the measurement noise v[k],
12     original
13 %R = 0.002; % little measurement noise
14 %R = .9; % high measurement noise
15 ukf.MeasurementNoise = R;
16 ukf.ProcessNoise = diag([0.02 0.1]); %original
17 %ukf.ProcessNoise = diag([0.9 0.8]); %model is corrected
18 %ukf.ProcessNoise = diag([0.0001 0.0001]); %model is
19 %corrected little
20 T = 0.05; % [s] Filter sample time
21 timeVector = 0:T:5;
22 [~,xTrue]=ode45(@vdp1,timeVector,[2;0]);
```

```

23
24 rng(1); % Fix the random number generator for
    reproducible results
25 yTrue = xTrue(:,1);
26 yMeas = yTrue .* (1+sqrt(R)*randn(size(yTrue))); % sqrt(R)
    ): Standard deviation of noise
27
28 Nsteps = numel(yMeas); % Number of time steps
29 xCorrectedUKF = zeros(Nsteps,2); % Corrected state
    estimates
30 PCorrected = zeros(Nsteps,2,2); % Corrected state
    estimation error covariances
31 e = zeros(Nsteps,1); % Residuals (or innovations)
32
33 for k=1:Nsteps
34     % Let k denote the current time.
35     %
36     % Residuals (or innovations): Measured output -
        Predicted output
37     e(k) = yMeas(k) - vdpMeasurementFcn(ukf.State); % ukf
        .State is x[k|k-1] at this point
38     % Incorporate the measurements at time k into the
        state estimates by
39     % using the "correct" command. This updates the State
        and StateCovariance
40     % properties of the filter to contain x[k|k] and P[k|
        k]. These values
41     % are also produced as the output of the "correct"
        command.
42     [xCorrectedUKF(k,:), PCorrected(k,:,:) ] = correct(ukf
        ,yMeas(k));
43     % Predict the states at next time step, k+1. This
        updates the State and
44     % StateCovariance properties of the filter to contain
        x[k+1|k] and
45     % P[k+1|k]. These will be utilized by the filter at
        the next time step.
46     predict(ukf);
47 end
48
49
50 figure();
51 subplot(2,1,1);
52 plot(timeVector, xTrue(:,1), timeVector, xCorrectedUKF(:,1),
    timeVector, yMeas(:));
53 set(gca, 'FontSize', 15);

```

```

54 e=legend('True','UKF estimate','Measured')
55 e.FontSize = 10;
56 ylim([-2.6 2.6]);
57 ylabel('x_1','FontSize',15);
58 subplot(2,1,2);%
59 plot(timeVector,xTrue(:,2),timeVector,xCorrectedUKF(:,2))
    ;
60 f=legend('True','UKF estimate')
61 f.FontSize = 10;
62 set(gca,'FontSize',10);
63 ylim([-3 1.5]);
64 xlabel('Time [s]','FontSize',15);
65 ylabel('x_2','FontSize',15);
66
67 %saveas(gcf,'\Users\lindseytam\Desktop\thesis\
    VDP_highMN_highPN.png')

```

## A.2 State function

```

1 function x = vdpStateFcn(x)
2 % vdpStateFcn Discrete-time approximation to van der Pol
    ODEs for mu = 1.
3 % Sample time is 0.05s.
4 %
5 % Example state transition function for discrete-time
    nonlinear state
6 % estimators.
7 %
8 % xk1 = vdpStateFcn(xk)
9 %
10 % Inputs:
11 %     xk - States x[k]
12 %
13 % Outputs:
14 %     xk1 - Propagated states x[k+1]
15 %
16 % See also extendedKalmanFilter, unscentedKalmanFilter
17
18 % Copyright 2016 The MathWorks, Inc.
19
20 %#codegen
21
22 % The tag %#codegen must be included if you wish to
    generate code with
23 % MATLAB Coder.
24

```

```

25 % Euler integration of continuous-time dynamics  $x'=f(x)$ 
    with sample time dt
26 dt = 0.05; % [s] Sample time
27 x = x + vdpStateFcnContinuous(x)*dt;
28 end
29
30 function dxdt = vdpStateFcnContinuous(x)
31 %vdpStateFcnContinuous Evaluate the van der Pol ODEs for
    mu = 1
32 dxdt = [x(2); (1-x(1)^2)*x(2)-x(1)];
33 end

```

### A.3 Measurement function

```

1 initialStateGuess = [4;1;3;4]; %  $\hat{x}[k|k-1]$ 
2 %initialStateGuess = [1;4;2;5]; % bad initials
3
4 % Construct the filter
5 ukf = unscentedKalmanFilter(...
6     @MeskinStateFcn,... % State transition function
7     @MeskinMeasurementNonAddFcn,... % Measurement
    function
8     initialStateGuess,...
9     'HasAdditiveMeasurementNoise',false)%,...
10    '%alpha', 0.9,...
11    '%kappa', 3,...
12    '%beta', 2);
13
14 R = .001; % Variance of the measurement noise  $v[k]$ 
15 ukf.MeasurementNoise = R;
16
17 ukf.ProcessNoise = diag([0.02 0.01 .03 .04]); %stores the
    process noise covariance
18
19 T = 0.1; % [s] Filter sample time
20 timeVector = 0:T:5;
21 [~,xTrue]=ode45(@Meskin1,timeVector,initialStateGuess);
22
23 rng(1); % Fix the random number generator for
    reproducible results
24 yTrue = xTrue(:,1);
25 yMeas = yTrue + (sqrt(R)*randn(size(yTrue)));
26 % sqrt(R): Standard deviation of noise
27 % randn(size(yTrue)): randomly sample 51 elements from
    ytrue
28 %yMeas = yTrue; Assumes no noise

```

```

29
30 for k=1:numel(yMeas)
31     % Let k denote the current time.
32
33     % Residuals (or innovations): Measured output -
34     % Predicted output
35     %e(k) = yMeas(k) - MeskinMeasurementFcn(ukf.State); %
36     % ukf.State is x[k|k-1] at this point
37
38     % Incorporate the measurements at time k into the
39     % state estimates by
40     % using the "correct" command. This updates the State
41     % and StateCovariance
42     % properties of the filter to contain x[k|k] and P[k|
43     % k]. These values
44     % are also produced as the output of the "correct"
45     % command.
46
47     [xCorrectedUKF(k,:), PCorrected(k,:,:) ] = correct(ukf
48     , yMeas(k));
49
50     % Predict the states at next time step, k+1. This
51     % updates the State and
52     % StateCovariance properties of the filter to contain
53     % x[k+1|k] and
54     % P[k+1|k]. These will be utilized by the filter at
55     % the next time step.
56     predict(ukf);
57
58 end
59
60 figure();
61 color1=[0,0.4470, 0.7410]; %blue
62 color2=[0.4660 0.6740 0.1880]; %green
63 color3=[0.8500 0.3250 0.0980]; %orange
64 color4=[0.9290 0.6940 0.1250]; %yellow
65
66 subplot(2,2,1);
67 plot(timeVector, xTrue(:,1), 'Color', color1, 'LineStyle'
68 , '-', ...
69 , 'LineWidth', 1)
70 hold on
71 plot(timeVector, xCorrectedUKF(:,1), 'Color', color1, '
72 LineStyle', '—', ...
73 , 'LineWidth', 1)

```

```

63 hold on
64 plot(timeVector,yMeas, 'Color', color1, 'Marker', '.', ...
65       'LineWidth', 1)
66
67 set(gca, 'FontSize', 15);
68 set(gcf, 'Color', 'None');
69 a = legend('True', 'UKF estimate', 'Measured')
70 a.FontSize = 10;
71 ylim([0 5]);
72 ylabel('x_1', 'FontSize', 15);
73 xlabel('Time [s]', 'FontSize', 15);
74
75 subplot(2,2,2);
76 plot(timeVector, xTrue(:,2), 'Color', color2, 'LineStyle'
77       , '-', ...
78       'LineWidth', 1)
79 hold on
80 plot(timeVector, xCorrectedUKF(:,2), 'Color', color2, '
81       LineStyle', '-', ...
82       'LineWidth', 1)
83 set(gca, 'FontSize', 15);
84 b = legend('True', 'UKF estimate')
85 b.FontSize = 10;
86 ylim([0 5]);
87 xlabel('Time [s]', 'FontSize', 15);
88 ylabel('x_2', 'FontSize', 15);
89 xlabel('Time [s]', 'FontSize', 15);
90
91 subplot(2,2,3);
92 plot(timeVector, xTrue(:,3), 'Color', color3, 'LineStyle'
93       , '-', ...
94       'LineWidth', 1)
95 hold on
96 plot(timeVector, xCorrectedUKF(:,3), 'Color', color3, '
97       LineStyle', '-', ...
98       'LineWidth', 1)
99 set(gca, 'FontSize', 15);
100 c = legend('True', 'UKF estimate')
101 c.FontSize = 10;
102 %ylim([-2.6 2.6]);
103 ylabel('x_3', 'FontSize', 15);
104 xlabel('Time [s]', 'FontSize', 15);
105
106 subplot(2,2,4);
107 plot(timeVector, xTrue(:,4), 'Color', color4, 'LineStyle'
108       , '-', ...

```

```

104         'LineWidth', 1)
105 hold on
106 plot(timeVector, xCorrectedUKF(:,4), 'Color', color4, '
        'LineStyle', '—', ...
107         'LineWidth', 1)
108 set(gca, 'FontSize', 15);
109 d = legend('True', 'UKF estimate', 'Measured')
110 d.FontSize = 10;
111 %ylim([-2.6 2.6]);
112 ylabel('x_4', 'FontSize', 15);
113 xlabel('Time [s]', 'FontSize', 15);
114
115 %saveas(gcf, '\Users\lindseytam\Desktop\thesis\
        Meskin_states_badInitial.png')
116 %saveas(gcf, '\Users\lindseytam\Desktop\thesis\
        Meskin_states.png')
117
118 subplot(1,1,1);
119
120 plot(timeVector, xTrue(:,1), 'Color', color1, 'LineStyle'
        , '—', ...
121         'LineWidth', 1)
122 hold on
123 plot(timeVector, xCorrectedUKF(:,1), 'Color', color1, '
        'LineStyle', '—', ...
124         'LineWidth', 1)
125 hold on
126 plot(timeVector, yMeas, 'Color', color1, 'Marker', '.', ...
127         'LineWidth', 1)
128 hold on
129 plot(timeVector, xTrue(:,2), 'Color', color2, 'LineStyle'
        , '—', ...
130         'LineWidth', 1)
131 hold on
132 plot(timeVector, xCorrectedUKF(:,2), 'Color', color2, '
        'LineStyle', '—', ...
133         'LineWidth', 1)
134 hold on
135 plot(timeVector, xTrue(:,3), 'Color', color3, 'LineStyle'
        , '—', ...
136         'LineWidth', 1)
137 hold on
138 plot(timeVector, xCorrectedUKF(:,3), 'Color', color3, '
        'LineStyle', '—', ...
139         'LineWidth', 1)
140 hold on

```

```

141 plot(timeVector, xTrue(:,4), 'Color', color4, 'LineStyle'
      , '—', ...
142      'LineWidth', 1)
143 hold on
144 plot(timeVector, xCorrectedUKF(:,4), 'Color', color4, '
      LineStyle', '—', ...
145      'LineWidth', 1)
146
147 set(gca, 'FontSize', 15);
148 set(gcf, 'color', 'none');
149 set(gca, 'color', 'none');
150 e = legend('x1 True', 'x1 UKF', 'x2 True', 'x2 UKF', 'x3
      True', 'x3 UKF', 'x4 True', 'x4 UKF')
151 e.FontSize = 10;
152 ylim([0 5]);
153 ylabel('x_4', 'FontSize', 15);
154 xlabel('Time [s]', 'FontSize', 15);
155
156 %saveas(gcf, '\Users\lindseytam\Desktop\thesis\
      Meskin_overall_badIntial.png')
157 saveas(gcf, '\Users\lindseytam\Desktop\thesis\
      Meskin_overall.png')

```



## Appendix B

# Meskin Code

### B.1 Main function

```
1 initialStateGuess = [4;1;3;4]; % xhat[k|k-1]
2 %initialStateGuess = [1;4;2;5]; % bad initials
3
4 % Construct the filter
5 ukf = unscentedKalmanFilter(...
6     @MeskinStateFcn,... % State transition function
7     @MeskinMeasurementNonAddFcn,... % Measurement
8     initialStateGuess,...
9     'HasAdditiveMeasurementNoise',false)%,...
10    %'alpha', 0.9,...
11    %'kappa', 3,...
12    %'beta', 2);
13
14 R = .001; % Variance of the measurement noise v[k]
15 ukf.MeasurementNoise = R;
16
17 ukf.ProcessNoise = diag([0.02 0.01 .03 .04]); %stores the
18     process noise covariance
19
20 T = 0.1; % [s] Filter sample time
21 timeVector = 0:T:5;
22
23 [%~,xTrue]=ode45(@Meskin1,timeVector,initialStateGuess);
24
25 rng(1); % Fix the random number generator for
26     reproducible results
27 yTrue = xTrue(:,1);
28 yMeas = yTrue + (sqrt(R)*randn(size(yTrue)));
```

```

26 % sqrt(R): Standard deviation of noise
27 % randn(size(yTrue)): randomly sample 51 elements from
    ytrue
28 %yMeas = yTrue; Assumes no noise
29
30 for k=1:numel(yMeas)
31     % Let k denote the current time.
32
33     % Residuals (or innovations): Measured output -
        Predicted output
34     %e(k) = yMeas(k) - MeskinMeasurementFcn(ukf.State); %
        ukf.State is x[k|k-1] at this point
35
36     % Incorporate the measurements at time k into the
        state estimates by
37     % using the "correct" command. This updates the State
        and StateCovariance
38     % properties of the filter to contain x[k|k] and P[k|
        k]. These values
39     % are also produced as the output of the "correct"
        command.
40
41     [xCorrectedUKF(k,:), PCorrected(k,:,:) ] = correct(ukf
        , yMeas(k));
42
43     % Predict the states at next time step, k+1. This
        updates the State and
44     % StateCovariance properties of the filter to contain
        x[k+1|k] and
45     % P[k+1|k]. These will be utilized by the filter at
        the next time step.
46     predict(ukf);
47
48 end
49
50
51 figure();
52 color1=[0,0.4470, 0.7410]; %blue
53 color2=[0.4660 0.6740 0.1880]; %green
54 color3=[0.8500 0.3250 0.0980]; %orange
55 color4=[0.9290 0.6940 0.1250]; %yellow
56
57 subplot(2,2,1);
58 plot(timeVector, xTrue(:,1), 'Color', color1, 'LineStyle'
    , '-', ...
59     'LineWidth', 1)

```

```

60 hold on
61 plot(timeVector,xCorrectedUKF(:,1), 'Color', color1, '
    LineStyle', '—', ...
62     'LineWidth', 1)
63 hold on
64 plot(timeVector,yMeas, 'Color', color1, 'Marker', '.', ...
65     'LineWidth', 1)
66
67 set(gca, 'FontSize', 15);
68 set(gcf, 'Color', 'None');
69 a = legend('True','UKF estimate','Measured')
70 a.FontSize = 10;
71 ylim([0 5]);
72 ylabel('x_1', 'FontSize', 15);
73 xlabel('Time [s]', 'FontSize', 15);
74
75 subplot(2,2,2);
76 plot(timeVector, xTrue(:,2), 'Color', color2, 'LineStyle'
    , '—', ...
77     'LineWidth', 1)
78 hold on
79 plot(timeVector,xCorrectedUKF(:,2), 'Color', color2, '
    LineStyle', '—', ...
80     'LineWidth', 1)
81 set(gca, 'FontSize', 15);
82 b = legend('True','UKF estimate')
83 b.FontSize = 10;
84 ylim([0 5]);
85 xlabel('Time [s]', 'FontSize', 15);
86 ylabel('x_2', 'FontSize', 15);
87 xlabel('Time [s]', 'FontSize', 15);
88
89 subplot(2,2,3);
90 plot(timeVector, xTrue(:,3), 'Color', color3, 'LineStyle'
    , '—', ...
91     'LineWidth', 1)
92 hold on
93 plot(timeVector,xCorrectedUKF(:,3), 'Color', color3, '
    LineStyle', '—', ...
94     'LineWidth', 1)
95 set(gca, 'FontSize', 15);
96 c = legend('True','UKF estimate')
97 c.FontSize = 10;
98 %ylim([-2.6 2.6]);
99 ylabel('x_3', 'FontSize', 15);
100 xlabel('Time [s]', 'FontSize', 15);

```

```

101
102 subplot(2,2,4);
103 plot(timeVector, xTrue(:,4), 'Color', color4, 'LineStyle'
      , '_-', ...
104      'LineWidth', 1)
105 hold on
106 plot(timeVector, xCorrectedUKF(:,4), 'Color', color4, '
      LineStyle', '_-', ...
107      'LineWidth', 1)
108 set(gca, 'FontSize', 15);
109 d = legend('True', 'UKF estimate', 'Measured')
110 d.FontSize = 10;
111 %ylim([-2.6 2.6]);
112 ylabel('x_4', 'FontSize', 15);
113 xlabel('Time [s]', 'FontSize', 15);
114
115 %saveas(gcf, '\Users\lindseytam\Desktop\thesis\
      Meskin_states_badInitial.png')
116 %saveas(gcf, '\Users\lindseytam\Desktop\thesis\
      Meskin_states.png')
117
118 subplot(1,1,1);
119
120 plot(timeVector, xTrue(:,1), 'Color', color1, 'LineStyle'
      , '_-', ...
121      'LineWidth', 1)
122 hold on
123 plot(timeVector, xCorrectedUKF(:,1), 'Color', color1, '
      LineStyle', '_-', ...
124      'LineWidth', 1)
125 hold on
126 plot(timeVector, yMeas, 'Color', color1, 'Marker', '.', ...
127      'LineWidth', 1)
128 hold on
129 plot(timeVector, xTrue(:,2), 'Color', color2, 'LineStyle'
      , '_-', ...
130      'LineWidth', 1)
131 hold on
132 plot(timeVector, xCorrectedUKF(:,2), 'Color', color2, '
      LineStyle', '_-', ...
133      'LineWidth', 1)
134 hold on
135 plot(timeVector, xTrue(:,3), 'Color', color3, 'LineStyle'
      , '_-', ...
136      'LineWidth', 1)
137 hold on

```

```

138 plot(timeVector,xCorrectedUKF(:,3), 'Color', color3, '
      'LineStyle', '—',...
139       'LineWidth', 1)
140 hold on
141 plot(timeVector, xTrue(:,4), 'Color', color4, 'LineStyle'
      , '—',...
142       'LineWidth', 1)
143 hold on
144 plot(timeVector,xCorrectedUKF(:,4), 'Color', color4, '
      'LineStyle', '—',...
145       'LineWidth', 1)
146
147 set(gca, 'FontSize', 15);
148 set(gcf, 'color', 'none');
149 set(gca, 'color', 'none');
150 e = legend('x1 True','x1 UKF','x2 True', 'x2 UKF', 'x3
      True','x3 UKF','x4 True', 'x4 UKF')
151 e.FontSize = 10;
152 ylim([0 5]);
153 ylabel('x_4', 'FontSize', 15);
154 xlabel('Time [s]', 'FontSize', 15);
155
156 %saveas(gcf, '\Users\lindseytam\Desktop\thesis\
      Meskin_overall_badIntial.png')
157 saveas(gcf, '\Users\lindseytam\Desktop\thesis\
      Meskin_overall.png')

```

## B.2 State function

```

1 function x = MeskinStateFcn(x)
2 % Trying Meskin in two dimensions
3 dt = 0.05;
4 t = 0; % dummy time variable for MeskinODE
5 x = x + MeskinODE(t, x)*dt;
6 end
7
8 %{
9 function dxdt = MeskinStateFcnContinuous(x)
10 parameter_values;
11
12 dxdt = [a_1 * x(3)^g_13 - b_1 * x(1)^h_11;
13         a_2 * x(1)^g_21 - b_2 * x(2)^h_22;
14         a_3 * x(2)^g_32 - b_3 * x(3)^h_33 * x(4)^h_34;
15         a_4 * x(1)^g_41 - b_4 * x(4)^h_44];
16 end
17 %}

```

### B.3 Measurement function

```
1 function yk = MeskinMeasurementNonAddFcn(xk,vk)
2 % Author: Ltam
3 % Date: November 17, 2019
4 % Summary:
5 % Inputs:   xk = states at time k, x[k]
6 %           vk = measurement noise vector at time, k v[k]
7 % Outputs:  yk = measurements at time k
8 %
9 % The measurement is the first state with multiplicative
   noise
10 yk = xk(1)+vk;
11 %yk =xk*(1+vk);
12 end
```

### B.4 Parameter Values

```
1 %This script saves the baseline parameter values for our
   model
2
3 a_1 = 20;
4 a_2 = 8;
5 a_3 = 3;
6 a_4 = 2;
7
8 b_1 = 10;
9 b_2 = 3;
10 b_3 = 5;
11 b_4 = 6;
12
13 g_11 = 0;
14 g_21 = 0.5;
15 g_31 = 0;
16 g_41 = 0.5;
17
18 g_12 = 0;
19 g_22 = 0;
20 g_32 = 0.75;
21 g_42 = 0;
22
23 g_13 = -0.8;
24 g_23 = 0;
25 g_33 = 0;
26 g_43 = 0;
```

```

27
28 g_14 = 0;
29 g_24 = 0;
30 g_34 = 0;
31 g_44 = 0;
32
33 h_11 = 0.5;
34 h_21 = 0;
35 h_31 = 0;
36 h_41 = 0;
37
38 h_12 = 0;
39 h_22 = 0.75;
40 h_32 = 0;
41 h_42 = 0;
42
43 h_13 = 0;
44 h_23 = 0;
45 h_33 = 0.5;
46 h_43 = 0;
47
48 h_14 = 0;
49 h_24 = 0;
50 h_34 = 0.2;
51 h_44 = 0.8;

```