Vector Analysis

Def: A vector is a quantity that has both magnitude (length) and direction. Geometrically vectors are "arrows" (directed line segments) from an initial point to a terminal point.

Def: A scalar is a quantity that has magnitude only. It is a number.

Equality: Two vectors are equal iff they have the same length and the same direction (they would be parallel).

⇒ Be able to represent a vector from $P(x_1, y_1)$ to $Q(x_2, y_2)$ in its component form $\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1 \rangle$

#1 Ex ⇒ Find the vector from $P(-3,5)$ to $Q(4,8)$ and represent it in its component form.

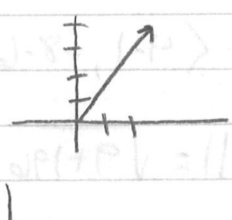
$$\overrightarrow{PQ} = \langle 4+3, 8-5 \rangle = \langle 7,3 \rangle$$

#2 Ex ⇒ Find $\overrightarrow{PQ}$ when $P(4,7)$ and $Q(-3,1)$

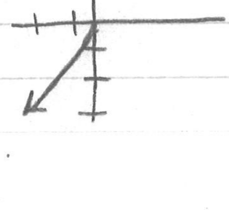
$$\overrightarrow{PQ} = \langle -3-4, 1-7 \rangle = \langle -7,-6 \rangle$$

Standard position of a vector: the initial point is at the origin and its terminal point is (x,y). Thus the vector is represented as $\langle x,y \rangle$.

Ex ⇒ Draw the vector $\langle 2,4 \rangle$.



Ex ⇒ Draw the vector $\langle -2,-3 \rangle$



<u>Length of a vector</u> $||v|| = \sqrt{x^2+y^2}$ if $v = \langle x, y \rangle$. This is also called the <u>norm</u> of the vector.

Ex ⇒ Find the component form and length of these vectors:

#1 A) From $P(3,-5)$ to $Q(4,7)$

$$\overrightarrow{PQ} = \langle 4-3, 7+5 \rangle = \langle 1, 12 \rangle$$

$$||\overrightarrow{PQ}|| = \sqrt{12^2+1^2} = \sqrt{145}$$

#2 B) From $R(7,-1)$ to $S(-3,1)$

$$\overrightarrow{RS} = \langle -3-7, 1+1 \rangle = \langle -10, 2 \rangle$$

$$||\overrightarrow{RS}|| = \sqrt{10^2+2^2} = \sqrt{104} = 2\sqrt{26}$$

#3 c) From $U(2,-9)$ to $V(-3,5)$

$$\overrightarrow{UV} = \langle -3-2, 5+9 \rangle = \langle -5, 14 \rangle$$

$$||\overrightarrow{UV}|| = \sqrt{25+196} = \sqrt{221}$$

#4 D) From $A(1,6)$ to $B(4,-8)$

$$\overrightarrow{AB} = \langle 4-1, -8-6 \rangle = \langle 3, -14 \rangle$$

$$||\overrightarrow{AB}|| = \sqrt{9+196} = \sqrt{205}$$

Two Basic Operations on Vectors : Let $\vec{u} = \langle x_1, y_1 \rangle$ and $\vec{v} = \langle x_2, y_2 \rangle$ be two vectors then,

-1) scalar multiplication $k\vec{u} = \langle kx_1, ky_1 \rangle$

-2) vector addition $\vec{u} + \vec{v} = \langle x_1 + x_2, y_1 + y_2 \rangle$

Ex ⇒ Given $\vec{u} = \langle 5, -4 \rangle$ and $\vec{v} = \langle 3, 8 \rangle$ find:

#1 A) $2\vec{v} = 2\langle 3, 8 \rangle = \langle 6, 16 \rangle$

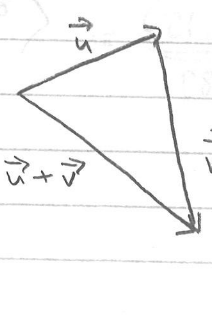
#2 B) $\frac{1}{2}\vec{u} = \frac{1}{2}\langle 5, -4 \rangle = \langle 5/2, -2 \rangle$

#3 C) $\vec{u} + \vec{v} = \langle 5+3, -4+8 \rangle = \langle 8, 4 \rangle$

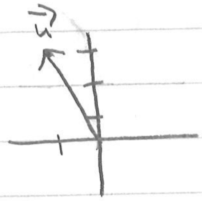
#4 D) $2\vec{u} - 3\vec{v} = 2\langle 5, -4 \rangle - 3\langle 3, 8 \rangle$
$= \langle 10, -8 \rangle + \langle -9, -24 \rangle = \langle 1, -32 \rangle$

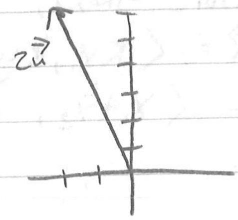
Be able to represent these operations geometrically. (A) $k\vec{u}$ would be an arrow (vector) in the same direction as $\vec{u}$ but k times as long. (B) $\vec{u} + \vec{v}$ can be represented using the tail-to-tip method.

A) 

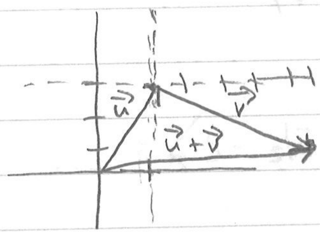
B) 

#1 Ex => If $\vec{u} = \langle -1, 3 \rangle$ represent $\vec{u}$ and $2\vec{u}$ geometrically.
$2\vec{u} \langle -2, 6 \rangle$





#2 Ex => If $\vec{u} = \langle 1, 3 \rangle$ and $\vec{v} = \langle 5, -2 \rangle$, represent $\vec{u} + \vec{v}$ geometrically.



Def: A unit vector is a vector having a length (magnitude) of one unit.

Be able to find a unit vector in the "same direction" as a given vector. This process is called normalizing a vector. Given $\vec{v} = \langle v_1, v_2 \rangle$ then a unit vector in the same direction as $\vec{v}$ would be denoted $U_{\vec{v}}$ and $U_{\vec{v}} = \dfrac{v}{\|v\|}$ or $\dfrac{\langle v_1, v_2 \rangle}{\|v\|}$. Divide a vector by its length to normalize it.

Ex => Normalize these vectors:

#1 A) $\vec{v} = \langle 5, 12 \rangle$ $\quad U_{\vec{v}} = \dfrac{\langle 5, 12 \rangle}{\sqrt{25+144}} = \dfrac{\langle 5, 12 \rangle}{\sqrt{169}} = \langle 5/13, 12/13 \rangle$

#2 B) $\vec{A} = \langle 8, 15 \rangle$ $\quad U_{\vec{A}} = \dfrac{\langle 8, 15 \rangle}{\sqrt{64+225}} = \dfrac{\langle 8, 15 \rangle}{\sqrt{289}} = \langle 8/17, 15/17 \rangle$

\#3 Ex ⇒ Find a vector of length 4 in the same direction as $\vec{v} = \langle 3,4 \rangle$.

First normalize then mult. by 4

$$u_{\vec{v}} = \frac{\langle 3,4\rangle}{\sqrt{9+16}} = \frac{\langle 3,4\rangle}{\sqrt{25}} = \langle 3/5, 4/5\rangle \times 4 = \langle 12/5, 16/5\rangle$$

There are two unit vectors in the plane which are special:
$i = \langle 1,0\rangle$ is the unit vector in the positive x direction
$j = \langle 0,1\rangle$ is the unit vector in the positive y direction

Vectors can be represented as linear combinations of i & j
... note: $\langle x,y\rangle = xi + yj$

Ex ⇒ Represent these vectors in their "i-j" notation:

\#1 A) $\vec{v} = \langle 4,8\rangle$ $4i + 8j$

\#2 B) $\vec{w} = \langle -3,12\rangle$ $-3i + 12j$

\#3 C) $\vec{z} = \langle 0,-4\rangle$ $-4j$

Ex ⇒ Represent these vectors in their component forms:

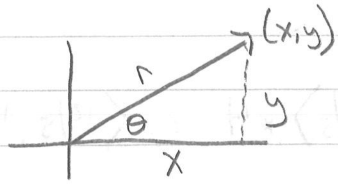
\#4 A) $7i - 3j$ $\langle 7,-3\rangle$

\#5 B) $6j$ $\langle 0,6\rangle$

\#6 C) $2i + 4j$ $\langle 2,4\rangle$

Polar Form of a vector: A vector can be described by giving its length r and its direction θ where θ is the angle the vector makes with the positive x-axis. Thus $V = (r, \theta)$.

Given



$x = r\cos\theta \qquad \theta = \tan^{-1} y/x$

$y = r\sin\theta \qquad r = \sqrt{x^2 + y^2}$

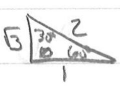
#1 Ex ⇒ Given $\vec{v} = \langle -3, 4 \rangle$ represent $\vec{v}$ in its polar form.

$r = \sqrt{9+16} = 5 \qquad \theta = \tan^{-1}(4/-3) = -53.13^\circ$ (must be +)

$\therefore \vec{v} \Rightarrow QII \Rightarrow \langle 5, 126.87^\circ \rangle$

#2 Ex ⇒ Given $\vec{w} = (10, 30^\circ)$ represent $\vec{w}$ in its rectangular form.

$x = 10\cos 30^\circ$



$10\left(\frac{\sqrt{3}}{2}\right) = 5\sqrt{3}$

$\therefore \vec{w} = \langle 5\sqrt{3}, 5 \rangle$ (QI)

$y = 10\sin 30^\circ$

$10\left(\frac{1}{2}\right) = 5$

#3 Ex ⇒ Given $\vec{u} = (4, 30^\circ)$ and $\vec{v} = (8, 150^\circ)$, find $\vec{u} + \vec{v}$ in Polar form.

Rewrite in rect. form first then transfer back to polar.

note: 150 = 30 in QII

$\vec{u} \Rightarrow x = 4\cos 30^\circ \qquad y = 4\sin 30^\circ$

$(4)(\sqrt{3}/2) \qquad = 4(1/2)$

$2\sqrt{3} \qquad = 2$

$\vec{v} \Rightarrow x = 8\cos 150^\circ \qquad y = 8\sin 150^\circ$

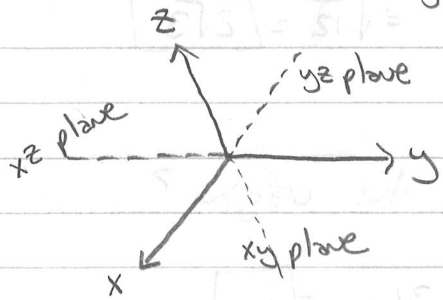
$8(-\sqrt{3}/2) \qquad 8(1/2)$

$-4\sqrt{3} \qquad 4$

$\vec{u}\langle 2\sqrt{3}, 2 \rangle + \vec{v}\langle -4\sqrt{3}, 4 \rangle = \langle -2\sqrt{3}, 6 \rangle \qquad r = \sqrt{4(3) + 36} = \sqrt{48} = 4\sqrt{3}$

$\therefore \vec{u} + \vec{v} = (4\sqrt{3}, 120^\circ) \qquad \theta = \tan^{-1}\frac{6}{-2\sqrt{3}} = -60^\circ \text{ (QII)} = 120^\circ$

Be able to plot points, identify planes, and identify axes on a 3 dimensional "right-handed" coordinate system.

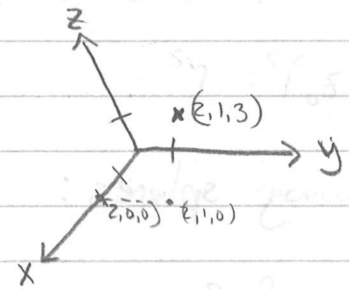


note: here are 3 mutually-perpendicular coordinate axis and each point in space can be described using 3 coordinates (x, y, z)

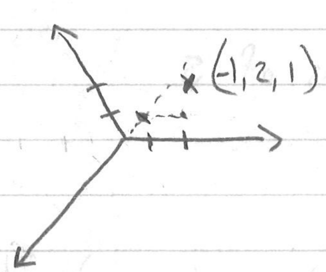
There are also 3 intersecting planes:

xy plane	has equation	$z=0$
yz plane	"	$x=0$
xz plane	"	$y=0$

* Ex => Plot the point $(2,1,3)$



* Ex => Plot the point $(-1,2,1)$



Be able to find the distance between two points in space. If $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ then

$$d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$

*1 Ex ⇒ Find the distance between $P(2,-1,7)$ and $Q(0,-3,5)$

$$D = \sqrt{(0-2)^2 + (-3+1)^2 + (5-7)^2} = \sqrt{4+4+4} = \sqrt{12} = \boxed{2\sqrt{3}}$$

*2 Ex ⇒ How far is the point $(-2,4,4)$ from the origin?

$$D = \sqrt{(-2-0)^2 + (4-0)^2 + (4-0)^2} = \sqrt{4+16+16} = \sqrt{36} = \boxed{6}$$

Def: A sphere is the locus of all points in space which are a given distance "r" from a given fixed point (x_0, y_0, z_0), its center.

Equation for sphere: $(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = r^2$

Ex ⇒ Write the equations of the following spheres:

*1 A) centered at $(0,2,5)$ with a radius of 2

$$x^2 + (y-2)^2 + (z-5)^2 = 4$$

*2 B) centered at origin with a radius of 5

$$x^2 + y^2 + z^2 = 25$$

*3 C) having a diameter from $(2,0,0)$ to $(0,6,0)$

$c = (1,3,0)$ $r = \sqrt{(2-1)^2 + (0-3)^2 + 0^2} = \sqrt{1+9} = \sqrt{10}$

$$(x-1)^2 + (y-3)^2 + z^2 = 10$$

Ex => Rewrite the following equations in their standard form (complete the square)

*4 A) $x^2+y^2+z^2-2x+6y+8z+1=0$

$$x^2-2x \quad +y^2+6y \quad +z^2+8z \quad = -1$$
$$+1 \qquad +9 \qquad +16 \qquad +1+9+16$$

$$(x-1)^2+(y+3)^2+(z+4)^2=25$$

center: $(1,-3,-4)$
radius: 5

*5 B) $4x^2+4y^2+4z^2-4x-32y+8z+33=0$

$$4x^2-4x \quad +4y^2-32y \quad +4z^2+8z \quad = -33$$
$$4(x^2-1x \quad)+4(y^2-8y \quad)+4(z^2+2z \quad)=-33$$
$$+\tfrac{1}{4} \qquad +16 \qquad +1 \qquad +1+64+4$$

$$4(x-\tfrac{1}{2})^2+4(y-4)^2+4(z+1)^2=36$$

$$(x-\tfrac{1}{2})^2+(y-4)^2+(z+1)^2=9$$

center: $(\tfrac{1}{2},4,-1)$
radius: 3

*6 Ex => Show that the points $A(0,0,0)$ and $B(2,2,1)$ and $C(2,-4,4)$ would form a right triangle "in space".

D of A to B $=\sqrt{2^2+2^2+1^2}=\sqrt{9}=3$

D of A to C $=\sqrt{2^2+4^2+4^2}=\sqrt{36}=6$

D of B to C $=\sqrt{0^2+6^2+3^2}=\sqrt{45}=3\sqrt{5}$

$3^2+6^2=3\sqrt{5}^2$

$9+36=9(5)$

$45=45$ ✓

*7) Ex => Find the equation for the locus of all points equidistant from R(6, 6, -5) and S(-3, 4, 2) in space

If $P(x, y, z)$ then $\|PR\| = \|PS\|$

$$\|PR\| = \sqrt{(x-6)^2 + (y-6)^2 + (z+5)^2} \qquad \|PS\| = \sqrt{(x+3)^2 + (y-4)^2 + (z-2)^2}$$

$$(x-6)^2 + (y-6)^2 + (z+5)^2 = (x+3)^2 + (y-4)^2 + (z-2)^2$$

Def: A vector in space is an ordered triple of coordinates
$V = \langle x, y, z \rangle$

The vector from $P(x_1, y_1, z_1)$ to $Q(x_2, y_2, z_2)$ is $\overrightarrow{PQ}\langle x_2-x_1, y_2-y_1, z_2-z_1 \rangle$

The length (or magnitude) of $V = ||V|| = \sqrt{x^2+y^2+z^2}$

The 3 special vectors are:
$i = \langle 1,0,0 \rangle$ $j = \langle 0,1,0 \rangle$ $k = \langle 0,0,1 \rangle$

Ex ⇒ Given $P(-2,3,1)$ and $Q(0,-4,4)$ find:

#1 A) $\overrightarrow{PQ}$ in component form and "i-j-k" form

$$\overrightarrow{PQ} = \langle 0+2, -4-3, 4-1 \rangle = \langle 2,-7,3 \rangle = 2i - 7j + 3k$$

#2 B) Find $||\overrightarrow{PQ}||$

$$||\overrightarrow{PQ}|| = \sqrt{2^2+7^2+3^2} = \sqrt{62}$$

Def: A unit vector in the same direction as V would be denoted $u_{\vec{v}}$ and it can be found by normalizing V thus $u_{\vec{v}} = \dfrac{V}{||V||}$

* Ex ⇒ Find $u_{\vec{v}}$ if $V = \langle -2,2,1 \rangle$

$$||V|| = \sqrt{2^2+2^2+1^2} = \sqrt{9} = 3 \qquad u_{\vec{v}} = \frac{\langle -2,2,1 \rangle}{3} = \langle -2/3, 2/3, 1/3 \rangle$$

Def: Two vectors $U = \langle u_1, u_2, u_3 \rangle$ and $W = \langle w_1, w_2, w_3 \rangle$ are equal iff $u_1 = w_1$ and $u_2 = w_2$ and $u_3 = w_3$.

Ex=> If $\overrightarrow{PQ} = \langle 3, -5, 6 \rangle$ is the vector from $P(1,2,5)$ to the terminal point $Q = (x, y, z)$ find the coordinates x, y, z of pt Q.

$$\langle x-1, y-2, z-5 \rangle = \langle 3, -5, 6 \rangle$$

$x - 1 = 3$ $\quad$ $y - 2 = -5$ $\quad$ $z - 5 = 6$

$x = 4$ $\quad$ $y = -3$ $\quad$ $z = 11$ $\quad$ $\therefore Q(4, -3, 11)$

Basic Operations on vectors "in space".

- vector addition $\vec{u} + \vec{w} = \langle u_1 + w_1, u_2 + w_2, u_3 + w_3 \rangle$
- scalar multiplication $k\vec{u} = \langle ku_1, ku_2, ku_3 \rangle$

Ex=> Given $\vec{u} = \langle 1, -2, 3 \rangle$ and $\vec{v} = \langle 3, 4, -2 \rangle$ find:

#1 A) $\vec{u} + \vec{v} = \langle 1+3, -2+4, 3-2 \rangle = \langle 4, 2, 1 \rangle$

#2 B) $2\vec{u} = 2\langle 1, -2, 3 \rangle = \langle 2, -4, 6 \rangle$

#3 C) $-3\vec{v} = -3\langle 3, 4, -2 \rangle = \langle -9, -12, 6 \rangle$

#4 D) $2\vec{u} - 3\vec{v} = \langle 2, -4, 6 \rangle + \langle -9, -12, 6 \rangle = \langle -7, -16, 12 \rangle$

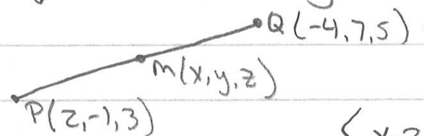
Def: Parallel vectors : two vectors in space are parallel iff $\vec{u} = k\vec{u}$ (one vector is a scalar multiple of another).

Ex⇒ Given $P(2,-1,3)$ and $Q(-4,7,5)$

#1 A) Find $\vec{PQ} = \langle -4-2,\ 7+1,\ 5-3 \rangle = \langle -6, 8, 2 \rangle$

#2 B) Is $\vec{PQ}$ parallel to $\vec{v}\langle 3,-4,-1 \rangle$, yes $\vec{PQ} = -2\vec{v}$

#3 C) Find the coordinates of the midpoint of the line segment PQ using vectors.



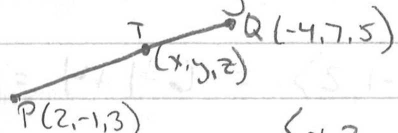
$\vec{Pm} = \frac{1}{2}\vec{PQ}$

$\langle x-2, y+1, z-3 \rangle = \frac{1}{2}\langle -6, 8, 2 \rangle \Rightarrow \langle -3, 4, 1 \rangle$

$x-2=-3$ $\quad y+1=4$ $\quad z-3=1$

$x=-1$ $\quad y=3$ $\quad z=4$ $\quad \therefore m(-1,3,4)$

#4 D) Find the coordinates of the point which is two-thirds of the way from P to Q using vectors.



$\vec{PT} = \frac{2}{3}(\vec{PQ})$

$\langle x-2, y+1, z-3 \rangle = \frac{2}{3}\langle -6, 8, 2 \rangle \Rightarrow \langle -4, \frac{16}{3}, \frac{4}{3} \rangle$

$x-2=-4$ $\quad y+1=\frac{16}{3}$ $\quad z-3=\frac{4}{3}$

$x=-2$ $\quad y=\frac{13}{3}$ $\quad z=\frac{13}{3}$ $\quad \therefore T\left(-2, \frac{13}{3}, \frac{13}{3}\right)$

Def: If $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ then the dot product (inner product) of $\vec{u}$ and $\vec{v}$ is the number $u_1v_1 + u_2v_2 + u_3v_3$

Ex ⇒ Find $u \cdot v$ for the following pairs of vectors:

#1 A) $\vec{u} = \langle 2,-1,3 \rangle \quad \vec{v} = \langle 3,-2,0 \rangle$

$$u \cdot v = (2)(3) + (-1)(-2) + (3)(0) = \boxed{8}$$

#2 B) $\vec{u} = \langle 2,3,-2 \rangle \quad \vec{v} = \langle 2,-2,-1 \rangle$

$$u \cdot v = 4 - 6 + 2 = \boxed{0}$$

#3 C) $u = i \quad v = k$

$$\langle 1,0,0 \rangle \cdot \langle 0,1,0 \rangle = 0+0+0 = \boxed{0}$$

Ex ⇒ Given $\vec{u} = \langle 2,1,2 \rangle \quad \vec{v} = \langle 3,2,-2 \rangle \quad \vec{w} = \langle 0,-3,4 \rangle$ find:

#4 A) $u \cdot (v+w) \Rightarrow \langle 2,1,2 \rangle \cdot \langle 3,-1,2 \rangle = 6 - 1 + 4 = \boxed{9}$

#5 B) $(u+v) \cdot (u-v) \Rightarrow \langle 5,3,0 \rangle \cdot \langle -1,-1,4 \rangle = -5 - 3 + 0 = \boxed{-8}$

#6 C) Prove $u \cdot u = \|u\|^2$

$$u \cdot u = 4 + 1 + 4 = 9$$

$$\|u\|^2 = \left(\sqrt{4+1+4}\right)^2 = 9$$

true

Ex ⇒ Given $\vec{u} = \langle 5, 12 \rangle$ and $\vec{v} = \langle 3, 2 \rangle$ find:

#7 A) $u \cdot v \Rightarrow 15 + 24 = \boxed{39}$

#8 B) $\|u\|^2 \Rightarrow 25 + 144 = \boxed{169}$

#9 C) $u \cdot 2v \Rightarrow \langle 5, 12 \rangle \cdot \langle 6, 4 \rangle = 30 + 48 = \boxed{78}$

#10 D) $(u \cdot v)(u + v) \Rightarrow 39 \langle 8, 14 \rangle = \boxed{\langle 312, 546 \rangle}$

Ex ⇒ Given $u = 2i - 3j + k$ and $v = i - 2j + k$ find:

#11 A) $u \cdot v \Rightarrow 2 + 6 + 1 = \boxed{9}$

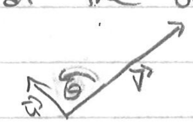
#12 B) $(u \cdot v)(v) \Rightarrow 9 \langle i - 2j + k \rangle = \boxed{9i - 18j + 9k}$

#13 C) $\|u\|^2 \Rightarrow 4 + 9 + 1 = \boxed{14}$

#14 D) $2u \cdot v \Rightarrow \langle 4, -6, 2 \rangle \cdot \langle 1, -2, 1 \rangle = 4 + 12 + 2 = \boxed{18}$

#15 E) Normalize $u \Rightarrow \dfrac{2i - 3j + k}{\sqrt{4 + 9 + 1}} = \boxed{\dfrac{2i - 3j + k}{\sqrt{14}}}$

Application of Dot Product: to find the angle between two vectors in space. Given two vectors $\vec{u}$ and $\vec{v}$ in space the angle between them is defined to be the positive angle θ ($0° \le \theta \le 180°$) between these vectors if they were in their standard positions (initial pts at the origin).



Theorem: $\cos\theta = \dfrac{u \cdot v}{\|u\| \, \|v\|}$

Two vectors are orthogonal (perpendicular) when their dot product equals zero.

#1 Ex ⇒ Find the angle between $\vec{u}=\langle 2,3,1\rangle$ and $\vec{v}=\langle 1,2,1\rangle$

$$\cos\theta = \frac{2+6+1}{\sqrt{4+9+1}\sqrt{1+4+1}} = \cos^{-1}\frac{9}{\sqrt{84}} = \boxed{10.89°}$$

#2 Ex ⇒ Which of the following pairs of vectors would be orthogonal to one another?

A) $u = j+6k$, $v = i-2j-k$

$\boxed{B)}$ $u=\langle 2,3,-1\rangle$, $v=\langle -2,1,-1\rangle$

c) $u=\langle 4,2,-6\rangle$, $v=\langle -6,-3,9\rangle$

$u\cdot v = 0-2-6=-8$

$u\cdot v = -4+3+1=0$

$u\cdot v = -24-6-54$

#3 Ex ⇒ Given $u=\langle 3,-1,2\rangle$, $v=\langle -4,0,2\rangle$, $w=\langle 1,-1,-2\rangle$, $z=\langle 2,0,-1\rangle$, find the angle between the following pairs of vectors.

A) u and v $\quad \cos\theta = \dfrac{-12+0+4}{\sqrt{9+1+4}\sqrt{16+0+4}} \Rightarrow \cos^{-1}\dfrac{-8}{\sqrt{280}} = \boxed{118.56°}$

B) u and w $\quad \cos\theta = \dfrac{3+1-4}{\sqrt{9+1+4}\sqrt{1+1+4}} \Rightarrow \cos^{-1} 0 = \boxed{90°}$

c) v and z $\quad \cos\theta = \dfrac{-8+0-2}{\sqrt{16+4}\sqrt{4+1}} \Rightarrow \cos^{-1}\dfrac{-10}{\sqrt{10}} = \boxed{180°}$

The <u>Direction of a vector "in space"</u>: In space the direction of a vector $V = \langle x, y, z \rangle$ is measured using 3 angles

$\Rightarrow \alpha$ (alpha) is the angle between V and the positive x axis (V and $i = \langle 1,0,0 \rangle$)

$\Rightarrow \beta$ (beta) " " positive y axis (V and $j = \langle 0,1,0 \rangle$)

$\Rightarrow \gamma$ (gamma) " " positive z axis (V and $k = \langle 0,0,1 \rangle$)

Def: α, β, γ are called the <u>direction angles</u> for V.

Def: $\cos\alpha, \cos\beta, \cos\gamma$ are the <u>direction cosines</u> for V.

Def: Any set of numbers proportional to the direction cosines are called a set of <u>direction numbers</u> for the vector V.

$$\cos\alpha = \frac{\langle x,y,z\rangle \cdot \langle 1,0,0\rangle}{\|v\|\,\|i\|} = \frac{x}{\sqrt{x^2+y^2+z^2}\sqrt{1^2}}$$

$$\therefore \cos\alpha = \frac{x}{\|v\|} \qquad \cos\beta = \frac{y}{\|v\|} \qquad \cos\gamma = \frac{z}{\|v\|}$$

To normalize $V \Rightarrow$ $$\frac{\langle x,y,z\rangle}{\|v\|} = \left\langle \frac{x}{\|v\|}, \frac{y}{\|v\|}, \frac{z}{\|v\|} \right\rangle = \langle \cos\alpha, \cos\beta, \cos\gamma \rangle$$

#1 Ex $\Rightarrow$ Given $v = \langle 2,2,1 \rangle$

A) Find $\|v\| \Rightarrow \sqrt{4+4+1} = \boxed{3}$

B) Normalize $V \Rightarrow \frac{\langle 2,2,1\rangle}{\|v\|} = \boxed{\langle 2/3, 2/3, 1/3 \rangle}$

* D) Find 3 sets of direction #'s for V. (any # proportional) $\Rightarrow$ 2, 2, 1

4, 4, 2

-2, -2, -1

c) Find the direction cosines for V, then the direction angles.

$\cos\alpha = 2/3 \therefore \boxed{\alpha = 48.19°}$ $\cos\beta = 2/3 \therefore \boxed{\beta = 48.19°}$ $\cos\gamma = 1/3 \therefore \gamma = \boxed{70.53°}$

Ex ⇒ Given $V = 2i + 3j + 4k$

A) Find $\|v\|$ ⇒ $\sqrt{4+9+16} = \boxed{\sqrt{29}}$

B) Normalize V ⇒ $\boxed{2/\sqrt{29}\, i + 3/\sqrt{29}\, j + 4/\sqrt{29}\, k}$

C) Find the directional cosines and angles for V.

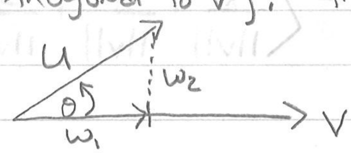
$\cos\alpha = 2/\sqrt{29} \therefore \alpha = \boxed{68.2^\circ}$ $\cos B = 3/\sqrt{29} \therefore B = \boxed{56.15^\circ}$ $\cos\gamma = 4/\sqrt{29} \therefore \gamma = \boxed{42.03^\circ}$

D) Find 3 sets of direction numbers for V.

$2, 3, 4 \rightarrow -2, -3, -4 \rightarrow 1, 3/2, 2$

Vector Projection and Vector Components

Given a vector U and another vector V be able to resolve U into two components vectors w_1 (a vector parallel to V) and w_2 (a vector orthogonal to V). Thus $U = w_1 + w_2$ where $w_1 \perp w_2$



Def: The vector component w_1 is called the vector projection of U onto V or the vector component of U along V and it is denoted $w_1 = \text{Proj}_V U$

Def: $w_2 = u - w_1$ is called the vector component of U orthogonal to V.

Theorem: $\text{Proj}_V U = \dfrac{u \cdot v}{\|v\|^2}(v) = w_1$

*\ Ex ⇒ Given $u = \langle 2,1,3 \rangle$ and $v = \langle 3,-1,4 \rangle$

A) Find $u \cdot v \Rightarrow 6 - 1 + 12 = 17$

B) Find $\|v\| \Rightarrow \sqrt{9+1+16} = \sqrt{26}$

C) Find $\text{Proj}_v u \Rightarrow \dfrac{u \cdot v}{\|v\|^2}(v) \Rightarrow \dfrac{17\langle 3,-1,4 \rangle}{26} = \left\langle 51/26, -17/26, 34/13 \right\rangle = w_1$

D) Find w_2, $w_2 = u - w_1 \therefore w_2 = \langle 2,1,3 \rangle - \langle 51/26, -17/26, 34/13 \rangle$
$w_2 = \langle 1/26, 43/26, 5/13 \rangle$

E) Find $\text{Proj}_u v \Rightarrow \dfrac{u \cdot v}{\|u\|^2}(u) \Rightarrow \dfrac{17\langle 2,1,3 \rangle}{\left(\sqrt{4+1+9}\right)^2} = \langle 17/7, 17/14, 51/14 \rangle$

*\ Ex ⇒ Given $u = \langle 0,4,1 \rangle$ and $v = \langle 0,2,3 \rangle$, find:

A) $\text{Proj}_v u = \dfrac{u \cdot v}{\|v\|^2}(v) = \dfrac{(0+8+3)\langle 0,2,3 \rangle}{\left(\sqrt{0+4+9}\right)^2} = 11/13 \langle 0,2,3 \rangle = \langle 0, 22/13, 33/13 \rangle$

B) w_2 of $\text{Proj}_v u \Rightarrow u - w_1 = \langle 0,4,1 \rangle - \langle 0, 22/13, 33/13 \rangle = \langle 0, 30/13, -20/13 \rangle$

C) $\text{Proj}_u v = \dfrac{u \cdot v}{\|u\|^2}(u) = \dfrac{11\langle 0,4,1 \rangle}{\left(\sqrt{16+1}\right)^2} = \langle 0, 44/17, 11/17 \rangle$

D) w_2 of $\text{Proj}_u v \Rightarrow v - w_1 = \langle 0,2,3 \rangle - \langle 0, 44/17, 11/17 \rangle = \langle 0, -10/17, 40/17 \rangle$

Cross Products

Def: The <u>vector product</u> or <u>cross-product</u> of two 3-dimensional vectors U and V denoted $U \times V$ is another <u>vector</u> orthogonal to the plane determined by U and V. Thus $u \times v$ is orthogonal to u and to V.

To compute this cross product we will use a determinant form which we can evaluate using minors and cofactors.

Let $u = \langle u_1, u_2, u_3 \rangle$ and $v = \langle v_1, v_2, v_3 \rangle$ then:

$$u \times v = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = i \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - j \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + k \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

$$= i(u_2v_3 - v_2u_3) - j(u_1v_3 - v_1u_3) + k(u_1v_2 - v_1u_2)$$

#1) Ex⇒ If $u = \langle 2, -1, 1 \rangle$ and $v = \langle 3, 2, -1 \rangle$ find $u \times V$.

$$u \times v = \begin{vmatrix} i & j & k \\ 2 & -1 & 1 \\ 3 & 2 & -1 \end{vmatrix} = i \begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix} - j \begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} + k \begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix}$$

$$= i(1-2) - j(-2-3) + k(4+3)$$

$$= \boxed{\langle -1, 5, 7 \rangle}$$

*2 Ex ⇒ Let $A = 3i - 2j + 4k$ and $B = i - 3j + 2k$, find $A \times B$

$$A \times B = \begin{vmatrix} i & j & k \\ 3 & -2 & 4 \\ 1 & -3 & 2 \end{vmatrix} = i \begin{vmatrix} -2 & 4 \\ -3 & 2 \end{vmatrix} - j \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} + k \begin{vmatrix} 3 & -2 \\ 1 & -3 \end{vmatrix}$$

$$= i(-4+12) - j(6-4) + k(-9+2)$$

$$= \boxed{8i - 2j - 7k}$$

*3 Ex ⇒ Given $u = \langle 1,2,-2 \rangle$ $v = \langle -1,2,3 \rangle$ $w = \langle 0,2,5 \rangle$,
prove $u \cdot (v \times w) = (u \times v) \cdot w$

$$v \times w = \begin{vmatrix} i & j & k \\ -1 & 2 & 3 \\ 0 & 2 & 5 \end{vmatrix} = i \begin{vmatrix} 2 & 3 \\ 2 & 5 \end{vmatrix} - j \begin{vmatrix} -1 & 3 \\ 0 & 5 \end{vmatrix} + k \begin{vmatrix} -1 & 2 \\ 0 & 2 \end{vmatrix}$$

$$= i(10-6) - j(-5-0) + k(-2-0) = \langle 4,5,-2 \rangle$$

$$\langle 1,2,-2 \rangle \cdot \langle 4,5,-2 \rangle = 4 + 10 + 4 = 18$$

$$u \times v = \begin{vmatrix} i & j & k \\ 1 & 2 & -2 \\ -1 & 2 & 3 \end{vmatrix} = i \begin{vmatrix} 2 & -2 \\ 2 & 3 \end{vmatrix} - j \begin{vmatrix} 1 & -2 \\ -1 & 3 \end{vmatrix} + k \begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix}$$

$$= i(6+4) - j(3-2) + k(2+2) = \langle 10,-1,4 \rangle$$

$$\langle 10,-1,4 \rangle \cdot \langle 0,2,5 \rangle = 0 - 2 + 20 = 18$$

$\therefore 18 = 18$ ✓ <u>true</u>

#4 Ex ⇒ Given $u = \langle 2,1,-3\rangle$ and $V = \langle 1,4,-2\rangle$, find a unit vector normal (orthogonal) to the plane determined by u and V.

(want $\dfrac{u \times v}{\|u \times v\|}$)

$$u \times v = \begin{vmatrix} i & j & k \\ 2 & 1 & -3 \\ 1 & 4 & -2 \end{vmatrix} = i\begin{vmatrix} 1 & -3 \\ 4 & -2 \end{vmatrix} - j\begin{vmatrix} 2 & -3 \\ 1 & -2 \end{vmatrix} + k\begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix}$$

$$= i(-2+12) - j(-4+3) + k(8-1) = \langle 10,1,7\rangle$$

$$\|u \times v\| = \sqrt{100+1+49} = \sqrt{150} = 5\sqrt{6}$$

$$\frac{u \times v}{\|u \times v\|} = \frac{\langle 10,1,7\rangle}{5\sqrt{6}} = \left\langle 2/\sqrt{6},\ 1/5\sqrt{6},\ 7/5\sqrt{6}\right\rangle$$

#5 Ex ⇒ Verify the identity $\|A \times B\|^2 = \|A\|^2\|B\|^2 - (A \cdot B)^2$ for vectors $A = \langle 1,-2,2\rangle$ and $B = \langle 6,-3,2\rangle$

$$A \times B = \begin{vmatrix} i & j & k \\ 1 & -2 & 2 \\ 6 & -3 & 2 \end{vmatrix} = i\begin{vmatrix} -2 & 2 \\ -3 & 2 \end{vmatrix} - j\begin{vmatrix} 1 & 2 \\ 6 & 2 \end{vmatrix} + k\begin{vmatrix} 1 & -2 \\ 6 & -3 \end{vmatrix}$$

$$= i(-4+6) - j(2-12) + k(-3+12) = \langle 2,10,9\rangle$$

$$\|A \times B\|^2 = \left(\sqrt{4+100+81}\right)^2 = \boxed{185}$$

$$\|A\|^2 = \left(\sqrt{1+4+4}\right)^2 = 9 \qquad \|B\|^2 = \left(\sqrt{36+9+4}\right)^2 = 49$$

$$(A \cdot B)^2 = (6+6+4)^2 = 256 \Rightarrow (9)(49) - 256 = \boxed{185}$$

185 = 185 <u>true</u>

#6 Ex Prove the theorem $\|A \times B\| = \|A\|\,\|B\| \sin\theta$ where θ is the angle between A and B

Know: $\cos\theta = \dfrac{A \cdot B}{\|A\|\,\|B\|}$ & $\|A \times B\|^2 = \|A\|^2 \|B\|^2 - (A \cdot B)^2$

If $\cos\theta = \dfrac{A \cdot B}{\|A\|\,\|B\|}$ then $A \cdot B = \cos\theta \|A\|\,\|B\|$ then $(A \cdot B)^2 = \cos^2\theta \|A\|^2 \|B\|^2$

so... $\|A \times B\|^2 = \|A\|^2 \|B\|^2 - \cos^2\theta \|A\|^2 \|B\|^2$

then $\|A \times B\|^2 = \|A\|^2 \|B\|^2 (1 - \cos^2\theta)$

then $\|A \times B\|^2 = \|A\|^2 \|B\|^2 \sin^2\theta$

$\therefore \|A \times B\| = \|A\|\,\|B\| \sin\theta$

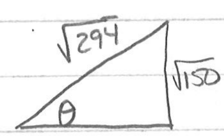
#7 Ex ⇒ Given $A = \langle 2, 1, -3 \rangle$ and $B = \langle 1, 4, -2 \rangle$, find $\|A \times B\|$

$\|A \times B\| = \sqrt{4+1+9}\,\sqrt{1+16+4}\,\sin\theta$

$= \sqrt{14}\,\sqrt{21}\,\dfrac{\sqrt{150}}{\sqrt{294}}$

$\|A \times B\| = \boxed{\sqrt{150}}$

$\cos\theta = \dfrac{A \cdot B}{\|A\|\,\|B\|} = \dfrac{2+4+6}{\sqrt{294}} = \dfrac{12}{\sqrt{294}}$



$12^2 + x^2 = (\sqrt{294})^2$

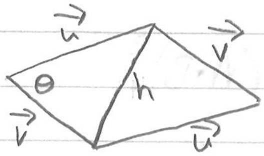
$144 + x^2 = 294$

$x^2 = 150$

$x = \sqrt{150}$

$\therefore \sin\theta = \dfrac{\sqrt{150}}{\sqrt{294}}$

<u>Area of a parallelogram</u> $= \|u \times v\|$

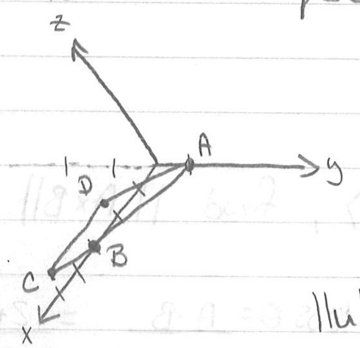


Area of parallelogram = base · height

$$\left.\begin{array}{l} \text{base} = \|v\| \\ \sin\theta = \dfrac{h}{\|u\|} \quad \therefore h = \|u\| \sin\theta \end{array}\right\} = \|v\|\,\|u\| \sin\theta \Rightarrow \|u \times v\|$$

#1) Ex⇒ Find the area of the parallelogram "in space" having vertices
$A(0,1,0)$ $B(3,0,0)$ $C(5,-2,0)$ $D(2,-1,0)$

½ area of a parallelogram = area of triangle

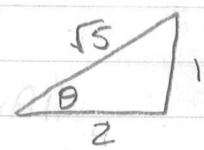


$\vec{CD} = \vec{u}$ $\quad \vec{CB} = \vec{v}$ $\quad \cos\theta = \dfrac{u \cdot v}{\|u\|\,\|v\|}$

$u = \langle -3, 1, 0 \rangle \quad v = \langle -2, 2, 0 \rangle$

$\|u\| = \sqrt{9+1} = \sqrt{10} \quad \|v\| = \sqrt{4+4} = 2\sqrt{2}$

$$\cos\theta = \frac{(6+2)}{\sqrt{10}\,2\sqrt{2}} = \frac{8}{2\sqrt{20}} = \frac{8}{4\sqrt{5}} = \frac{2}{\sqrt{5}}$$



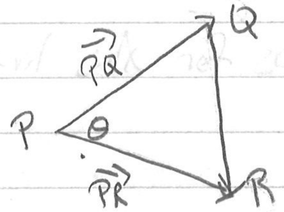
$2^2 + x^2 = \sqrt{5}^2$

$x^2 = 1 \quad x = 1$

$\sin\theta = 1/\sqrt{5}$

$$\|u \times v\| = 2\sqrt{2}\,\sqrt{10}\left(\frac{1}{\sqrt{5}}\right) = \boxed{4} \therefore \cancel{\|u \times v\| = \boxed{8}}$$

*²Ex ⇒ Find the area "in space" of the triangle PQR having vertices of $P(1,0,-1)$ $Q(2,4,5)$ $R(3,1,7)$

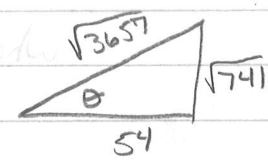


$\frac{1}{2}\|PQ \times PR\| = \|PQ\|\,\|PR\| \sin\theta$

$\vec{PQ} = \langle 1, 4, 6\rangle \quad \|PQ\| = \sqrt{1+16+36} = \sqrt{53}$

$\vec{PR} = \langle 2, 1, 8\rangle \quad \|PR\| = \sqrt{4+1+64} = \sqrt{69}$

$$\cos\theta = \frac{2+4+48}{\sqrt{53}\sqrt{69}} = \frac{54}{\sqrt{3657}}$$



$$\frac{1}{2}\|PQ \times PR\| = \sqrt{53}\sqrt{69}\,\frac{\sqrt{741}}{\sqrt{3657}} = \boxed{\frac{\sqrt{741}}{2}}$$

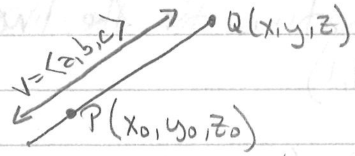
$54^2 + x^2 = \sqrt{3657}^{\,2}$

$x^2 = 741 \quad x = \sqrt{741}$

Lines in Space

What does it take to determine (graph or write the equation for) a line in the plane (in 2-dimensions)? ⇒ point & slope

In space a line is also determined if we know a point on the line and its direction, however the direction of a line is given by a vector going in the same direction (parallel to) as the line. This direction vector $v = \langle a, b, c\rangle$ has as its components a set of direction numbers a, b, c.



Line in Space

$\vec{PQ} = tv$ where t is the parameter

$$\langle x-x_0,\ y-y_0,\ z-z_0\rangle = \langle at,\ bt,\ ct\rangle$$

$x - x_0 = at \quad \therefore \quad x = x_0 + at$

$y - y_0 = bt \qquad\quad y = y_0 + bt$

$z - z_0 = ct \qquad\quad z = z_0 + ct$

- $P(x_0, y_0, z_0)$ is the pt and a, b, c are direct.#
- different values of t indicate different points on the line

#1 Ex ⇒ Given $P(5,1,3)$ is a point on the line L and that the line L is parallel to the vector $v = i + 2j - k$.

A) Write a set of parametric equations for this line.

$x = 5 + t \qquad y = 1 + 2t \qquad z = 3 - t$

B) By letting $t = -1, 1, 2$ find three other points on this line

when $t = -1 \Rightarrow (4, -1, 4)$
$t = 1 \Rightarrow (6, 3, 2)$
$t = 2 \Rightarrow (7, 5, 1)$

#2 Ex ⇒ Given $P(2,-3,0)$ is a point on the line parallel to vector $v = \langle 0, 2, 4 \rangle$, write the set of parametric equations for this line.

$x = 2 \qquad y = -3 + 2t \qquad z = 4t$

If none of the direction numbers are zero's it is possible to eliminate the parameter t and arrive at these symmetric equations for the line L. $\dfrac{x - x_0}{a} = \dfrac{y - y_0}{b} = \dfrac{z - z_0}{c}$

Ex ⇒ Find the parametric and symmetric equations for the line which passes through $P(2,4,-3)$ and $Q(3,-1,1)$

$\overrightarrow{PQ} = \langle 1, -5, 4 \rangle \therefore$ direction vector $= \langle -1, 5, -4 \rangle$ (pick a "convenient" set)

$x = 2 - t$
$y = 4 + 5t \qquad \dfrac{x-2}{-1} = \dfrac{y-4}{5} = \dfrac{z+3}{-4}$
$z = -3 - 4t$

*2 Ex ⇒ Show that the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{PQ}$ are parallel if $A(2,-1,-5)$ $B(8,8,7)$ $P(4,2,-6)$ $Q(8,8,2)$ are points connecting these vectors.

$\overrightarrow{AB} = \langle 6,9,12 \rangle$

$\overrightarrow{PQ} = \langle 4,6,8 \rangle$

yes, $\overrightarrow{PQ} = \frac{2}{3}\overrightarrow{AB}$ (scalar multiples)

*3 Ex ⇒ Show that the line through the points $A(0,1,1)$ and $B(1,-1,6)$ is orthogonal to the line which contains the points $P(-4,2,1)$ & $Q(-1,6,2)$.

$\overrightarrow{AB} = \langle 1,-2,5 \rangle$

$\overrightarrow{PQ} = \langle 3,4,1 \rangle$

$\overrightarrow{AB} \cdot \overrightarrow{PQ} = 3-8+5 = 0$ (dot product = 0)

*4 Ex ⇒ Find the parametric equations for the line which passes through the point $P(2,3,4)$ which would be parallel to the line given by $\frac{x-1}{3} = \frac{y+1}{-2} = z-3$.

$P(2,3,4)$ direction vector $\langle 3,-2,1 \rangle$ ∴ $x = 2+3t$

$y = 3-2t$

$z = 4+t$

*5 Ex ⇒ Find the symmetric equations for the line which passes through the point $P(0,2,-1)$ and which would be parallel to the line having the parametric equations $x = 1+2t$, $y = 3t$, $z = 5-7t$

$P(0,2,-1)$ direction vector $\langle 2,3,-7 \rangle$ ∴ $\frac{x}{2} = \frac{y-2}{3} = \frac{z+1}{-7}$

Lines in space which do not intersect but are also not parallel are said to be skew lines.

*1 Ex ⇒ Determine if the lines $x=-3t+1$, $y=4t+1$, $z=2t+4$ and $x=3v+1$, $y=2v+4$, $z=-v+1$ intersect and if so, find their point of intersection.

(set like terms = to each other) (solve 2 then verify in 3rd)

A: $-3t+1=3v+1$, $-3t-3v=0$

B: $4t+1=2v+4$, $4t-2v=3$

C: $2t+4=-v+1$, $2t+v=-3$

$$\begin{aligned} B:\ & 4t-2v=3 \\ 2C:\ & 4t+2v=-6 \\ \hline & 8t=-3 \\ & t=-3/8 \end{aligned}$$

$4(-3/8)-2v=3$
$-2v=3+3/2$
$-2v=9/2$
$v=-9/4$

✓ in A: $-3(-3/8)-3(-9/4)=0$
$9/8+27/4=0$
not true ∴ do not intersect

$V_1=\langle -3,4,2\rangle$ $V_2=\langle 3,2,-1\rangle$ ⇒ not parallel ∴ skew lines

*2 Ex ⇒ Determine if the lines $x=-2+3t$, $y=5-4t$, $z=1+2t$ and $x=1-v$, $y=3+2v$, $z=-4-3v$ intersect, if so find the point of intersection.

A: $-2+3t=1-v$, $3t+v=3$

B: $5-4t=3+2v$, $+4t+2v=+2$

C: $1+2t=-4-3v$, $2t+3v=-5$

$$\begin{aligned} -2A:\ & -6t-2v=-6 \\ B:\ & 4t+2v=2 \\ \hline & -2t=-4 \\ & t=2 \end{aligned}$$

$-6(2)-2v=-6$
$-2v=6$
$v=-3$

C: $2(2)+3(-3)=-5$
$-5=-5$
∴ they intersect

when $t=2$ ⇒ $(4,-3,5)$

*3 Ex ⇒ Given the lines $l_1 = \frac{x-1}{2} = y = \frac{z-1}{4}$ and $l_2 = x = \frac{y+2}{2} = \frac{z+2}{3}$

A) Find the point at which they intersect.

$l_1 \Rightarrow x = 1+2t$ $\quad l_2 \Rightarrow x = v$

$y = t$ $\quad y = -2+2v$

$z = 1+4t$ $\quad z = -2+3v$

$1+2t = v$ $\quad t = -2+2v$ $\quad 1+4t = -2+3v$

A $2t - v = -1$ $\quad$ B $t - 2v = -2$ $\quad$ C $4t - 3v = -3$

$$\begin{array}{lr} -2A & -4t + 2v = 2 \\ B & t - 2v = -2 \\ \hline & -3t = 0 \end{array} \quad \therefore t = 0$$

$0 + 2v = 2$

$v = 1$

$\checkmark$ C $0 - 3(1) = -3$

$-3 = -3$ true $\therefore$ int.

$\boxed{\text{when } t=0 \Rightarrow (1,0,1)}$

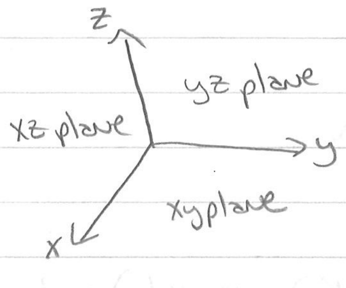
B) Find the angle between these lines at their point of intersection

$l_1 = \langle 2,1,4 \rangle \quad l_2 = \langle 1,2,3 \rangle$

$$\cos\theta = \frac{l_1 \cdot l_2}{\|l_1\|\,\|l_2\|} = \frac{2+2+12}{\sqrt{4+1+16}\sqrt{1+4+9}} = \frac{16}{\sqrt{294}} \Rightarrow \cos^{-1}(16/\sqrt{294}) = \boxed{21.07^\circ}$$

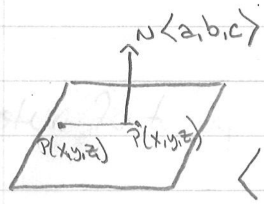
Planes in Space

What does it take to write the equation for a plane?
A point on the plane and the direction that the plane is facing (this will be the direction of a vector orthogonal to the plane) called the normal vector.



- The floor is the xy plane, $z=0$
- A plane where z is a constant will be parallel to the floor but n steps above it.

$z=4$ (4 steps above)
$z=-6$ (6 steps below)
$y=5$ (5 steps right)
$y=-2$ (2 steps left)
$x=2$ (2 steps in front)
$x=-5$ (5 steps back)



Equation for the plane
$$\langle x-x_1,\ y-y_1,\ z-z_1\rangle \cdot \langle a,b,c\rangle = 0$$

$a(x-x_1)+b(y-y_1)+c(z-z_1)=0$ standard form

$-ax_1-by_1-cz_1=d$ (all constants)

$ax-ax_1+by-by_1+cz-cz_1=0$

$ax+by+cz+d=0$ general form

#1 Ex ⇒ Write the standard and general forms for the plane which contains the point $P(1,0,-3)$ and which is normal to the line whose parametric equations are $x=1+4t$, $y=-2+t$, and $z=-3-3t$.

$N=\langle 4,1,-3\rangle$ $\quad 4(x-1)+1(y-0)-3(z+3)=0$ standard

$4x-4+y-3z-9=0$

$4x+y-3z-13=0$ general

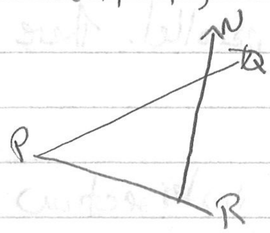
*~ Ex = Find a normal vector $n = \langle a,b,c \rangle$ to the plane

$$3x - 4y + 5z - 10 = 0$$

$$n = \langle 3,-4,5 \rangle$$

⇒ To write the equation of a plane we need a point (we'll have 3 to choose from) and a vector normal to this plane. We know that the cross product of two vectors is always orthogonal to the plane determined by these vectors.

*~ Ex ⇒ Write the equation for the plane containing the points $P(4,-3,1)$ $Q(6,-4,7)$ $R(1,2,2)$.



$$\vec{PQ} = \langle 2,-1,6 \rangle$$
$$\vec{PR} = \langle -3,5,1 \rangle$$

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} i & j & k \\ 2 & -1 & 6 \\ -3 & 5 & 1 \end{vmatrix} = i\begin{vmatrix} -1 & 6 \\ 5 & 1 \end{vmatrix} - j\begin{vmatrix} 2 & 6 \\ -3 & 1 \end{vmatrix} + k\begin{vmatrix} 2 & -1 \\ -3 & 5 \end{vmatrix}$$

$$= i(-1-30) - j(2+18) + k(10-3) \Rightarrow \langle -31,-20,7 \rangle$$

$R = (1,2,2)$ then

$$-31(x-1) - 20(y-2) + 7(z-2) = 0 \quad \text{standard}$$

$$-31x + 31 - 20y + 40 + 7x - 14 = 0$$
$$-31x - 20y + 7x + 57 = 0 \qquad \text{general}$$

*2 Ex ⇒ Write the equation of the plane containing the points $A(2,1,-3)$ $B(5,-1,4)$ $C(2,-2,4)$.

$\vec{AB} = \langle 3,-2,7 \rangle$ $\quad \vec{BC} = \langle -3,-1,0 \rangle$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ 3 & -2 & 7 \\ -3 & -1 & 0 \end{vmatrix} = i\begin{vmatrix} -2 & 7 \\ -1 & 0 \end{vmatrix} - j\begin{vmatrix} 3 & 7 \\ -3 & 0 \end{vmatrix} + k\begin{vmatrix} 3 & -2 \\ -3 & -1 \end{vmatrix}$$

$$= i(0+7) - j(0+21) + k(-3-6) = \langle 7,-21,-9 \rangle$$

$C(2,-2,4)$ so

$$7(x-2) - 21(y+2) - 9(z-4) = 0 \Rightarrow 7x - 21y - 9z - 20 = 0$$

In space, any two planes will intersect or be parallel. There are no such things as skewed planes.

Given two planes, be able to find their line of intersection if they intersect or determine if they are parallel planes.
The point $P(x,y,z)$ will be a point on this line of intersection iff it is a point on each of the planes, therefore its coordinates must satisfy the equations for each plane.

-> Given $ax + by + cz + d = 0$ and $a_1x + b_1y + c_1z + d_1 = 0$

-1) solve the system of equations for two of its unknowns (x & y) in terms of the 3rd variable (z).

-2) Let the 3rd variable be the parameter t for the desired set of parametric equations for this line (t = whatever we want).

Ex ⇒ Find the parametric equations for the lines of intersection for the following pairs of planes:

~~#1~~ A) $x + y + z = 1$ and $x - 2y + 3z = 1$

$$\begin{aligned} x + y + z &= 1 \\ -x + 2y - 3z &= -1 \\ \hline 3y - 2z &= 0 \\ 3y &= 2z \\ y &= \tfrac{2}{3}z \end{aligned}$$

$$x + \tfrac{2}{3}z + z = 1$$
$$x = 1 - \tfrac{5}{3}z$$

$$x = 1 - \tfrac{5}{3}z$$
$$y = \tfrac{2}{3}z$$
$$z = z \quad \rightarrow z = 3t \quad \therefore \quad \boxed{\begin{aligned} x &= 1 - 5t \\ y &= 2t \\ z &= 3t \end{aligned}}$$

~~#2~~ B) $2x - y + 4z = 4$ and $x + 3y - 2z = 1$

$$\begin{aligned} 6x - 3y + 12z &= 12 \\ x + 3y - 2z &= 1 \\ \hline 7x + 10z &= 13 \\ \cancel{7}x &= \frac{13 - 10z}{7} \end{aligned}$$

$$\tfrac{13}{7} - \tfrac{10}{7}z + 3y - 2z = 1$$
$$3y = \tfrac{24}{7}z - \tfrac{6}{7}$$
$$y = \tfrac{24}{21}z - \tfrac{6}{21}$$

$$x = \tfrac{13}{7} - \tfrac{10}{7}z$$
$$y = -\tfrac{6}{21} + \tfrac{24}{21}z$$
$$z = z \quad \rightarrow z = 21t \quad \therefore \quad \boxed{\begin{aligned} x &= \tfrac{39}{21} - 30t \\ y &= -\tfrac{6}{21} + 24t \\ z &= 21t \end{aligned}}$$

Given two planes be able to find the angle between them. The angle between two planes is defined to be the angle between their normal vectors: $\cos\theta = \dfrac{|N_1 \cdot N_2|}{\|N_1\|\,\|N_2\|}$

Ex ⇒ Find the angle of intersection between these planes:

#1 A) $x+z=1$ and $y+z=1$

$N_1 = \langle 1,0,1 \rangle \quad N_2 = \langle 0,1,1 \rangle$

$\cos\theta = \dfrac{0+0+1}{\sqrt{1+1}\,\sqrt{1+1}} = \cos^{-1}\dfrac{1}{2} = \boxed{60^\circ}$

#2 B) $x+y+z=1$ and $x-2y+3z=1$

$N_1 = \langle 1,1,1 \rangle \quad N_2 \langle 1,-2,3 \rangle$

$\cos\theta = \dfrac{1-2+3}{\sqrt{3}\,\sqrt{1+4+9}} \quad \cos^{-1}\dfrac{2}{\sqrt{42}} = \boxed{72.02^\circ}$

#3 c) $x+4y-3z=1$ and $-3x+6y+7z=0$

$N_1 = \langle 1,4,-3 \rangle \quad N_2 = \langle -3,6,7 \rangle$

$\cos\theta = \dfrac{-3+24-21}{\sqrt{1+16+9}\,\sqrt{9+36+49}} = \cos^{-1} 0 = \boxed{90^\circ}$

Be able to find the point of intersection between a line and a plane.

-1) substitute the coordinates for x, y, z (in terms of t) from the line's parametric equations into the general equation for the plane.

-2) solve for t

-3) substitute this value of t back into the parametric equations to find the coordinates of the intersection point.

Ex ⇒ Find the point of intersection between the following:

#1 A) the line $\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z}{3}$ and the plane $3x + 2y - z = 5$

$x = 1 + 2t$
$y = -1 - t$
$z = 3t$

$3(1+2t) + 2(-1-t) - 3t = 5$
$3 + 6t - 2 - 2t - 3t = 5$
$t = 4$

∴ pt of intersection $(9, -5, 12)$

#2 B) the line $x = 2 + 3t$ and the plane $3x - 7y + 5z + 55 = 0$
$y = -1 - 7t$
$z = 3 + 5t$

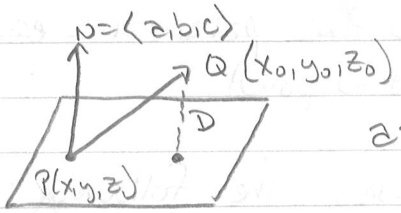
$3(2+3t) - 7(-1-7t) + 5(3+5t) + 55 = 0$
$6 + 9t + 7 + 49t + 15 + 25t + 55 = 0$
$83t = -83$
$t = -1$

∴ pt of intersection $(-1, 6, -2)$

Computing Distances in Space

Be able to find the distance between a point Q and a plane.

→ The distance between 2 points is the same as the length of the vector between them. ($\|\overrightarrow{PQ}\|$ = distance)



$ax + by + cz + d = 0$

-1) Pick a convenient point P

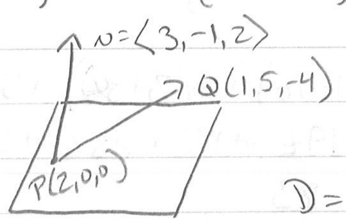
-2) Draw a vector from $\overrightarrow{PQ} = \langle x_0 - x,\ y_0 - y,\ z_0 - z \rangle$

-3) Distance = $\text{Proj}_N \overrightarrow{PQ}$ (projection of $\overrightarrow{PQ}$ onto the normal vector)

$$D = \text{Proj}_N \overrightarrow{PQ} = \frac{|\overrightarrow{PQ} \cdot N|}{\|N\|}$$

Ex⇒ Find the distance for the following: (use P(2,0,0))

#1 A) from Q(1,5,-4) to $3x - y + 2z = 6$



$\overrightarrow{PQ} = \langle -1, 5, -4 \rangle$

$$D = \text{Proj}_N \overrightarrow{PQ} = \frac{|-3-5-8|}{\sqrt{9+1+4}} = \boxed{\frac{16}{\sqrt{14}}}$$

#2 B) from Q(-1,4,6) to $2x + 3y + 6z - 4 = 0$

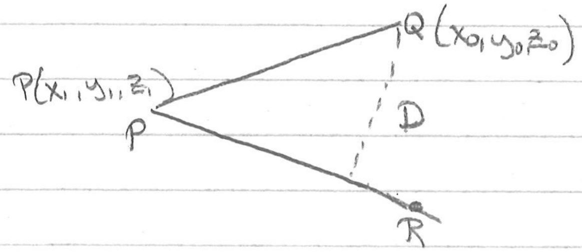
$\overrightarrow{PQ} = \langle -3, 4, 6 \rangle$

$N = \langle 2, 3, 6 \rangle$

$$D = \text{Proj}_N \overrightarrow{PQ} = \frac{|-6+12+36|}{\sqrt{4+9+36}} = \frac{42}{\sqrt{49}} = \boxed{6}$$

⇒ Be able to find the distance between a point Q and a line.

Given point $Q(x_0, y_0, z_0)$ and line $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$



-1) pick any convenient point on the line $P(x_1, y_1, z_1)$

-2) Find the vector $\vec{PQ} = \langle x_0 - x_1, y_0 - y_1, z_0 - z_1 \rangle$

-3) Find the vector $\vec{PR}$ where R is any other point on the line; you can use the direction vector for $\vec{PR} = \langle a, b, c \rangle$

-4) Find $\vec{PQ} \times \vec{PR}$

-5) we want D ⇒ know $\sin\theta = \frac{D}{\|\vec{PQ}\|}$ ∴ $D = \sin\theta \|\vec{PQ}\|$

also $\sin\theta = \frac{\|\vec{PQ} \times \vec{PR}\|}{\|\vec{PQ}\| \, \|\vec{PR}\|}$ ∴ $\boxed{D = \frac{\|\vec{PQ} \times \vec{PR}\|}{\|\vec{PR}\|}}$

#1) Ex ⇒ Find the distance from $Q(4,1,-2)$ to the line $\frac{x-2}{2} = \frac{y}{2} = z+3$

$P = (2, 0, -3)$

$\vec{PQ} = \langle 2, 1, 1 \rangle$

$\vec{PR} = \langle 2, 2, 1 \rangle$

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} i & j & k \\ 2 & 1 & 1 \\ 2 & 2 & 1 \end{vmatrix} = i \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} - j \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} + k \begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix}$$

$$= i(1-2) - j(2-2) + k(4-2) = \langle -1, 0, 2 \rangle$$

$$D = \frac{\sqrt{1+4}}{\sqrt{4+4+1}} = \boxed{\frac{\sqrt{5}}{3}}$$

#2 Ex => Find the distance from Q(3,-8,1) to the line $\frac{x-3}{3} = \frac{y+7}{-1} = \frac{z+2}{5}$

$P = (3,-7,-2)$

$\overrightarrow{PQ} = \langle 0,-1,3 \rangle$

$\overrightarrow{PR} = \langle 3,-1,5 \rangle$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ 0 & -1 & +3 \\ 3 & -1 & 5 \end{vmatrix} = i \begin{vmatrix} -1 & +3 \\ -1 & 5 \end{vmatrix} - j \begin{vmatrix} 0 & +3 \\ 3 & 5 \end{vmatrix} + k \begin{vmatrix} 0 & -1 \\ 3 & -1 \end{vmatrix}$$

$$= i(-5+3) - j(0-9) + k(0+3) = \langle -2, +9, +3 \rangle$$

$$D = \frac{\sqrt{4+81+9}}{\sqrt{9+1+25}} = \frac{\sqrt{94}}{\sqrt{35}} = \boxed{\sqrt{\frac{94}{35}}}$$