Linear Algebra

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Review: Inner products on real linear space

An inner product on V is a function $\langle , \rangle : V \times V \to \mathbb{R}$ such that

1. $\langle v, v \rangle \geqslant 0$ for all $v \in V$.
2. $\langle v, v \rangle = 0$ if and only if $v = 0$.
3. $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ for all $u, v, w \in V$.
4. $\langle cu, w \rangle = c\langle u, w \rangle$ for all $u, w \in V$ and $c \in \mathbb{R}$.
5. $\langle v, w \rangle = \langle w, v \rangle$.

Review

- Suppose that V is an inner product space. For $v \in V$, we define the norm of v, denoted $\|v\|$, by $\|v\| = \sqrt{\langle v, v \rangle}$.

- Two vectors $u, v \in V$ are said to be orthogonal if $\langle u, v \rangle = 0$.

Review: Orthogonal Subspaces

Definition

Two subspaces W_1 and W_2 of the same space V are orthogonal, denoted by $W_1 \perp W_2$, if and only if each vector $w_1 \in W_1$ is orthogonal to each vector $w_2 \in W_2$:

$$\langle w_1, w_2 \rangle = 0.$$

for all w_1 and w_2 in W_1 and W_2, respectively.

Review: Orthogonal complement of a subspace

Definition

Given a subspace W in linear space V, the space of all vectors orthogonal to W is called the orthogonal complement of V. It is denoted by $W^{\perp}$.

- We emphasize that W_1 and W_2 can be orthogonal without being complements.
- $W_1 = \text{span}((1,0,0))$ and $W_2 = \text{span}((0,1,0))$.

Fundamental theorem of orthogonality

Review: Fundamental theorem of orthogonality

Let $A \in M_{mn}(\mathbb{R})$.

1. The row space is orthogonal to the nullspace (in $\mathbb{R}^n$).
2. The column space is orthogonal to the left nullspace (in $\mathbb{R}^m$).

Review: Fundamental theorem of orthogonality

Let $A \in M_{mn}(\mathbb{R})$.

1. The nullspace is the orthogonal complement of the row space in $\mathbb{R}^n$.
2. The left nullspace is the orthogonal complement of the column space in $\mathbb{R}^m$.
3. Direct Sum: $\mathbb{R}^n = N(A) \oplus N(A)^{\perp}$.
4. $\mathbb{R}^n = N(A) \oplus C(A^T)$.
5. From the row space to the column space, A is actually invertible. Every vector in the column space comes from exactly one vector in the row space.

Matrix Representation of Inner Products

- Let $B = \{v_1, \ldots, v_n\}$ be a basis for linear space V.
- Suppose that a bilinear function $\langle\, ,\, \rangle : V \times V \to \mathbb{R}$ is an inner product for $\mathbb{R}^n$.
- We want to investigate a matrix representation of this inner product.

Matrix Representation of Inner Products

- Let $B = \{v_1, \ldots, v_n\}$ be a basis for linear space V.
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- We want to investigate a matrix representation of this inner product.
- Orthonormal basis!
- Vectors $q_1, \ldots, q_n$ are orthonormal if

$$q_i^T q_j = \begin{cases} 0 \quad \text{whenever} \quad i \neq j & \text{(for orthogonality)} \\ 1 \quad \text{whenever} \quad i = j & \text{(for normality).} \end{cases}$$

Change of basis matrix for inner product space

Suppose that $B = \{v_1, \dots, v_n\}$ and $B' = \{v'_1, \dots, v'_n\}$ are two bases for an inner product V. Then for each $v \in V$, we have

$$[v]_B = P[v]_{B'}$$

such that

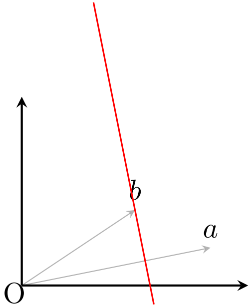
$$v'_j = \sum_{i=1}^{n} P_{ij} v_r.$$

and P is the change basis matrix.

What is the relationship between the matrix of the inner product relative to the basis B and the basis B'?

The Gram-Schmidt Process

- Suppose that a, b are independent vectors, but they are not orthogonal.
- Let $V = \text{span}(\{a, b\})$.
- So, $\{a, b\}$ is a basis for V.
- How can we find a way to make an orthogonal basis?



The Gram-Schmidt Process

- Suppose a, b, c are independent but are not orthogonal vectors.
- Let $V = \text{span}(\{a, b, c\})$.
- So, $\{a, b, c\}$ is a basis for V.
- We want to find a way to make an orthogonal basis:
- $$\begin{aligned} q_1 &= \frac{1}{\|a\|} a \\ q_2 &= \frac{1}{\|b - \langle b, q_1 \rangle q_1\|} (b - \langle b, q_1 \rangle q_1) \\ q_3 &= \frac{1}{\|c - \langle c, q_1 \rangle q_1 - \langle c, q_2 \rangle q_2\|} (c - \langle c, q_1 \rangle q_1 - \langle c, q_2 \rangle q_2) \end{aligned}$$

Example

- $a = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad c = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$
- The Gram-Schmidt Process:$q_1 = \frac{1}{\sqrt{2}} a$

$$b - \langle b, q_1 \rangle q_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad q_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$c - \langle c, q_1 \rangle q_1 - \langle c, q_2 \rangle q_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \sqrt{2} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} - \sqrt{2} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{q_3}$$

The Gram-Schmidt process

- The Gram-Schmidt process
  1. starts with independent vectors $v_1, \ldots, v_n$
  2. ends with orthonormal vectors $q_1, \ldots, q_n$.

- At step 1: $q_1 = \frac{1}{\|v_1\|} v_1$.

- At step j $(2 \leqslant j \leqslant n)$:
  1. it subtracts from a_j its components in the directions $q_1, \ldots, q_{j-1}$ that are already settled:

$$Q_j = v_j - \langle v_j, q_1 \rangle q_1 - \cdots - \langle v_j, q_{j-1} \rangle q_{j-1}.$$

  2. $q_j = \frac{1}{\|Q_j\|} Q_j$.
  3. $\text{span}(\{v_1, \ldots, v_j\}) = \text{span}(\{q_1, \ldots, q_j\}$.

Thank You!