Linear Algebra

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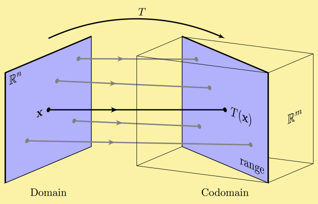
Linear functions

- Suppose that $A \in M_{mn}(\mathbb{R})$. Let $x \in \mathbb{R}^n$:
- Consider the mapping $x \longmapsto Ax$

$$\begin{cases} T : \mathbb{R}^n \to \mathbb{R}^m \\ T(x) = Ax \end{cases}$$



- $\{Ax \mid x \in \mathbb{R}^n\} = \{x_1A_1 + \cdots + x_nA_n \mid x \in \mathbb{R}^n\} = C(A) \subseteq \mathbb{R}^m$.
- $\{x \in \mathbb{R}^n \mid Ax = 0\} = N(A) \subseteq \mathbb{R}^n$.

Linear functions

- A function

$$\begin{cases} T : \mathbb{R}^n \to \mathbb{R}^m \\ T(x) = Ax \end{cases}$$

has two properties:

1. $T(x+y) = T(x) + T(y)$, for each $x, y \in \mathbb{R}^n$.
2. $T(cx) = cT(x)$, for each $x \in \mathbb{R}^n$ and $c \in \mathbb{R}$.

Linear function

Let V and W be two linear spaces. Every function $T : V \to W$ that meets two below requirements is a linear function (transformation):

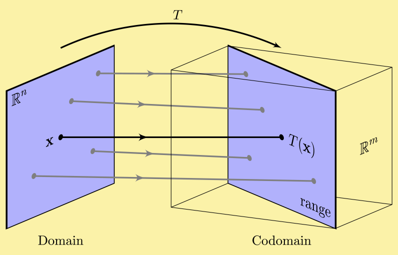
1. $T(x + y) = T(x) + T(y)$, for each $x, y \in V$.
2. $T(cx) = cT(x)$, for each $x \in V$ and $c \in \mathbb{R}$.

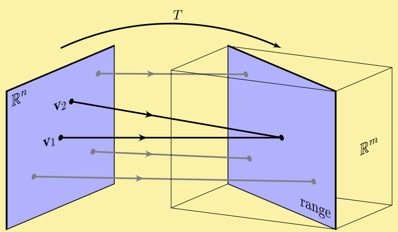
Linear function

- Let $T : V \rightarrow W$ be a linear function.
- $Im(T) = \{T(x) \mid x \in V\}$ is a linear subspace of W and is called image of T.
- $N(T) = \{x \in V \mid T(x) = 0\}$ is a linear subspace of V and is called nullspace of T.
- The dimension of $Im(T)$ is called **rank** of T and denoted by $\text{rank}(T)$.

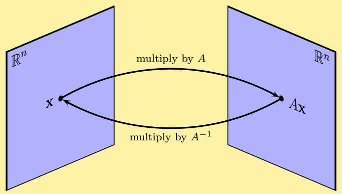
Injective and surjective linear functions

$$\begin{cases} T : \mathbb{R}^n \to \mathbb{R}^m \\ T(x) = Ax \end{cases}$$





T not injective



A^{-1} transforms $A\mathbf{x}$ back to $\mathbf{x}$.

Linear functions

- $T : V \to W$ is injective if and only if $N(T) = \{0\}$.

Example

What linear function takes e_1 and e_2 to Ae_1 and Ae_2, respectively?

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow Te_1 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \qquad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow Te_2 = \begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix}$$

Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ with the above conditions. Thus $x = x_1e_1 + x_2e_2$ for $x \in \mathbb{R}^2$.

$$T(x) = x_1T(e_1) + x_2T(e_2) = x_1 \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & 4 \\ 3 & 6 \\ 4 & 8 \end{bmatrix}}_{A} x = Ax.$$

Linear functions

- Suppose that V is a finite linear space with dimension n. Let $\{v_1, \ldots, v_n\}$ be a basis for V and $\{w_1, \ldots, w_n\} \subseteq W$. Then there is a unique linear function $T : V \to W$ such that

$$T(v_i) = w_i$$

Linear functions

- If V is a finite dimensional linear space and W is a linear space and

$$T : V \to W$$

then

$$\dim(ImT) + \dim(N(T)) = n$$

Does every linear function lead to a matrix?

Linear functions Represented by Matrices

- Assume $\dim(V) = n$ and $T : V \to W$.

- Let $\{v_1, \ldots, v_n\}$ be a basis for V.

- If $v \in V$, then there are $c_i \in \mathbb{R}$ such that $v = c_1 v_1 + \cdots + c_n v_n$.

- By Linearity,

$$\begin{aligned} T(v) &= c_1 T(v_1) + \cdots + c_n T(v_n) \\ &= \begin{bmatrix} T(v_1) & \cdots & T(v_n) \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \end{aligned}$$

- $A = \begin{bmatrix} T(v_1) & \cdots & T(v_n) \end{bmatrix}$ is a matrix representation of T.

Example (the operation of **differentiation**)

- Suppose that
$$P_n = \{p(t) = a_0 + a_1 t + \cdots + a_n t^n \mid a_i \in \mathbb{R}, 0 \leqslant i \leqslant n\}.$$
- $\dim(P_n) = n + 1$.
- Consider
$$\begin{aligned} &\frac{d}{dt} : P_3(x) \to P_3(x) \\ &\frac{d}{dt}(a_3x^3 + a_2x^2 + a_1x + a_0) = 3a_3x^2 + 2a_2x + a_1 \end{aligned}$$
- This is a linear function.
- It is surjective since $Im(\frac{d}{dt}) = P_2$.
- It is not injective since $N(\frac{d}{dt}) = \mathbb{R} \neq \{0\}$.
- $$\dim(Im(\tfrac{d}{dt})) + \dim(N(\tfrac{d}{dt})) = 3 + 1 = 4.$$

Representation matrix of differentiation

$$\begin{cases} \frac{d}{dt} : P_3 \to P_3 \\ \\ \frac{d}{dt}(a_3x^3 + a_2x^2 + a_1x + a_0) = 3a_3x^2 + 2a_2x + a_1 \end{cases}$$

- Take that $\{1, t, t^2, t^3\}$ as a basis for P_3.
- The representation matrix for $\frac{d}{dt}$ is:

$$A_{\text{diff}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Example (the operation of **integration**)

- Consider $\int_0^t : P_n \to P_{n+1}$ with

$$\int_0^t (a_0 + a_1 t + \cdots + a_n t^n)dt = a_0 t + \frac{a_1}{2} t^2 + \cdots + \frac{a_n}{n+1} t^{n+1}$$

- This is a linear function.
- It is not surjective since $c \in \mathbb{R} \subseteq P_{n+1}$ but $c \notin Im(\int_0^t)$.
- It is injective since $N(\int_0^t) = \{0\}$

Representation matrix of integration

$$\begin{cases} \int_0^t : P_3 \rightarrow P_4 \\ \\ \int_0^t (a_0 + a_1 t + a_2 t^2 + a_3 t^3) dt = a_0 t + \frac{a_1}{2} t^2 + \frac{a_2}{3} t^3 + \frac{a_3}{4} t^4 \end{cases}$$

- Take $B = \{1, t, t^2, t^3\}$ as a basis for $P_3 (\dim P_3 = 4)$.
- Take $B' = \{1, t, t^2, t^3, t^4\}$ as a basis for P_4 $(\dim P_4 = 5)$.
- The representation matrix for $\int_0^t$ is:

$$A_{\text{int}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}_{5 \times 4}$$

Composition

Function Composition

Suppose that $T : V \to W$ is a linear function with representation matrix A and $S : W \to U$ representation matrix B , then $S \circ T : V \to U$ is a linear function with representation matrix BA

$$
\begin{tikzcd}
V \arrow[r, "T"] \arrow[dr, "S \circ T"'] & W \arrow[d, "S"] \\
& U
\end{tikzcd}
\qquad
\begin{tikzcd}
V \arrow[r, "A"] \arrow[dr, "BA"'] & W \arrow[d, "B"] \\
& U
\end{tikzcd}
$$

Thank You!