Linear Algebra

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Inverse Matrices

- The matrix A is invertible if there exists a matrix B such that $AB = BA = I$.

Theorem

The inverse exists if and only if elimination produces n pivots.

Inverse Matrices

- The matrix A is invertible if there exists a matrix B such that $AB = BA = I$.

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The inverse exists if and only if elimination produces n pivots.

Corollary

A is invertible if and if the one and only solution t othe system equation $Ax = 0$ is $x = 0$.

Review: The Calculation of A^{-1}

The inverse of A is written A^{-1} in which $AA^{-1} = I$. Let

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$$

We want to find A^{-1} such that $AA^{-1} = I$, so consider the columns of A^{-1} as x_1, x_2, x_3, that means $A^{-1} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}$.

$$\begin{aligned} AA^{-1} &= I \\ A \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} &= \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} \\ \begin{bmatrix} Ax_1 & Ax_2 & Ax_3 \end{bmatrix} &= \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} \end{aligned}$$

So $\qquad Ax_1 = e_1 \qquad Ax_2 = e_2 \qquad Ax_3 = e_3$

A^{-1}

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 4 & -6 & 0 & 0 & 1 & 0 \\ -2 & 7 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -8 & -2 & -2 & 1 & 0 \\ 0 & 8 & 3 & 1 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -8 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

- We have

$$\left[\begin{array}{c|c} U & L^{-1} \end{array}\right]$$

A^{-1}

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -8 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] \qquad \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 2 & -1 & -1 \\ 0 & -8 & 0 & -4 & 3 & 2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 0 & \frac{12}{8} & -\frac{5}{8} & -\frac{6}{8} \\ 0 & -8 & 0 & -4 & 3 & 2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] \qquad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{12}{16} & -\frac{5}{16} & -\frac{6}{16} \\ 0 & 1 & 0 & \frac{4}{8} & -\frac{3}{8} & -\frac{2}{8} \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

The inverse of 3 by 3 Hilbert matrix

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & 0 & 0 & 1 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & \frac{1}{12} & \frac{1}{12} & -\frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{12} & \frac{4}{45} & -\frac{1}{3} & 0 & 1 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & 1 & 1 & -6 & 12 & 0 \\ 0 & 0 & \frac{1}{180} & \frac{1}{6} & -1 & 1 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & 1 & 0 & -36 & 192 & -180 \\ 0 & 0 & 1 & 30 & -180 & 180 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 0 & 0 & 9 & -36 & 30 \\ 0 & 1 & 0 & -36 & 192 & -180 \\ 0 & 0 & 1 & 30 & -180 & 180 \end{bmatrix}$$

The inverse of 3 by 3 Hilbert matrix

$$\left[\begin{array}{cccccc} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & 0 & 0 & 1 \end{array}\right]$$

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$$H^{-1} = \begin{bmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{bmatrix}$$

Hilbert matrix

The matrix H is an example of a family of matrices which are called **Hilbert** matrices. The n by n Hilbert matrix is

$$H = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \cdots & \cdots & \frac{1}{n+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{n} & \frac{1}{n+1} & \cdots & \cdots & \frac{1}{2n-1} \end{bmatrix}$$

where $H_{ij} = \frac{1}{i+j+1}$. For every n, $n \times n$ Hilbert matrix is invertible and

its inverse has integer entries.

Application of inverse matrix!

Problem. Let $AB = 3A + 4B$. Show $AB = BA$.

Solution.

- $AB - 3A - 4B = 0.$
- $(A-?)(B-?) =?.$
- $(A - 4I)(B - 3I) = 12I$
- $(A - 4I)(\frac{1}{12}B - \frac{1}{4}I) = I. \quad \Rightarrow \quad \underbrace{(A - 4I)}_{\mathcal{A}}\underbrace{(\frac{1}{12}B - \frac{1}{4}I)}_{\mathcal{A}^{-1}} = I$
- $\underbrace{(\frac{1}{12}B - \frac{1}{4}I)}_{\mathcal{A}^{-1}}\underbrace{(A - 4I)}_{\mathcal{A}} = I \Rightarrow BA = AB.$

Transpose Matrix

$$A = \begin{bmatrix} 2 & -3 & 5 \\ 0 & 9 & 10 \end{bmatrix} \qquad A^T = \begin{bmatrix} 2 & 0 \\ -3 & 9 \\ 5 & 10 \end{bmatrix}$$

In general:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \qquad A^T = \begin{bmatrix} a_{11} & \dots & a_{m1} \\ a_{12} & \dots & a_{m2} \\ \vdots & \vdots & \vdots \\ a_{1n} & \dots & a_{mn} \end{bmatrix}$$

i) $(AB)^T = B^T A^T$. ii) $(A^T)^{-1} = (A^{-1})^T$.

Symmetric Matrices

- A **symmetric matrix** is a matrix that equals its own transpose: $A^T = A$; i.e. $A_{ij} = A_{ji}$.
- A symmetric matrix A is necessarily square.
- A symmetric matrix A need not be invertible.
- A is a symmetric matrix if and only if A^{-1} a symmetric matrix. (Why?)

Symmetric Products

- Symmetric Products R^TR, RR^T, and LDL are Symmetric.
- If A is invertible, then the LDU factorization and the LU factorization are uniquely determined by A.

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  (1) Then U is the transpose of L, and the symmetric factorization becomes $A = LDL^T$.

  (2) If $D = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix}$ where $d_i \geq 0$ for each $1 \leq i \leq n$, then A is factorized $A = LL^T$ in which L is a lower triangular matrix.

Roundoff Error

Normally, we keep a fixed number of significant digits (say three, for an extremely weak computer). Then adding two numbers of different sizes raises an error:

Roundoff Error $\quad 0.456 + 0.00123 \to 0.457 \quad$ loses the digits $\underline{2}$ and $\underline{3}$.

How do all these individual errors contribute to the final error in $Ax = b$?

Two simple examples

$$A = \underbrace{\begin{bmatrix} 1.0 & 1.0 \\ 1.0 & 1.0001 \end{bmatrix}}_{\text{Nearly singular}}$$

Ill-conditioned

$$B = \underbrace{\begin{bmatrix} 0.0001 & 1.0 \\ 1.0 & 1.0 \end{bmatrix}}_{\text{far from sigular}}$$

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Well-conditioned

Consider two very close right-hand side b's:

$$\begin{array}{ccccc} u & + & v & = & 2 \\ u & + & 1.0001v & = & 2.\boxed{0000} \end{array} \qquad \begin{array}{ccccc} u & + & v & = & 2 \\ u & + & 1.0001v & = & 2.\boxed{0001} \end{array}$$

u=2 and v=0   u=1 and v=1

A change in the fifth digit of b was amplified to a change in the first digit of the solution. No numerical method can avoid this sensitivity to small perturbations.

Well-condition!

- Even a well-conditioned matrix like $B = \begin{bmatrix} 0.0001 & 1.0 \\ 1.0 & 1.0 \end{bmatrix}$ can be ruined by a poor algorithm.

- Suppose 0.0001 is accepted as the first pivot. Then 10000 times the first row is subtracted from the second. The lower right entry becomes -9999.

$$\begin{bmatrix} 0.0001 & 1.0 & 1 \\ 1.0 & 1.0 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 0.0001 & 1.0 & 1 \\ 0 & -9999 & -9998 \end{bmatrix}$$

- $v = \frac{9998}{9999} \simeq \begin{cases} 0.9999 & \Rightarrow u = 1 \\ \boxed{1} & \Rightarrow u = 0 \end{cases}$ for $\begin{cases} 0.0001\ u + v = 1 \\ u + v = 2 \end{cases}$

- B is well-conditioned but elimination is violently unstable.

Remedy Action!

- The small pivot 0.0001 brought instability.
- The remedy is clear —***exchange rows***.
- A small pivot forces a practical change in elimination. Normally we compare each pivot with all possible pivots in the same column. Exchanging rows to obtain the largest possible pivot is called ***partial pivoting***.

Special Matrices

- Consider differential equation $-\frac{d^2u}{dx^2} = f(x)$ for $0 \leq x \leq 1$ with the unknown function $u(x)$ which shows the temperature distribution in a rod with ends fixed at $0^{\cdot}C$ at each end of the interval: $u(0) = 0$ and $u(1) = 0$ (a *boundary condition*).
- We can only accept a finite amount of information about $f(x)$, say its values at n equally spaced points $x = h, x = 2h, \cdots, x = nh$.
- We compute approximate values $u_1, \ldots, u_n$ for the true solution $u(x)$ at these same points. At the ends $x = 0$ and $x = 1 = (n+1)h$, the boundary values are $u_0 = 0$ and $u_{n+1} = 0$.

Special Matrices

- How do we replace the derivative $\frac{d^2u}{dx^2}$ in $-\frac{d^2u}{dx^2} = f(x)$.
- The first derivative $\frac{du}{dx}$ can be approximated by stopping $\frac{\Delta u}{\Delta x}$ at a finite stepsize
- The difference Δu can be forward, backward:

$$\frac{\Delta u}{\Delta x} = \frac{u(x+h) - u(x)}{h} \text{ or } \frac{u(x) - u(x-h)}{h}$$

-

$$\frac{d^2u}{dx^2} \approx \frac{\Delta^2 u}{\Delta x^2} = \frac{\Delta}{\Delta x}\left(\frac{\Delta u}{\Delta x}\right) = \frac{\frac{u(x+h)-u(x)}{h} - \frac{u(x)-u(x-h)}{h}}{h}$$
$$= \frac{u(x+h) - 2u(x) + u(x-h)}{h^2}.$$

For $n = 5$, since $u_0 = u_{n+1} = 0$, we have

$$
\begin{cases}
\begin{array}{rcrcrcrcrcl}
2u_1 & - & 1u_2 & + & & & & & & = & h^2 f(h) \\
-1u_1 & + & 2u_2 & - & 1u_3 & & & & & = & h^2 f(2h) \\
 & - & 2u_2 & + & 2u_3 & - & 1u_4 & & & = & h^2 f(3h) \\
 & & & - & 1u_3 & + & 2u_4 & - & 1u_5 & = & h^2 f(4h) \\
 & & & & & - & 1u_4 & + & 2u_5 & = & h^2 f(5h)
\end{array}
\end{cases}
$$

So

$$
\underbrace{\begin{bmatrix}
2 & 1 & & & \\
-1 & 2 & -1 & & \\
 & -1 & 2 & -1 & \\
 & & -1 & 2 & -1 \\
 & & & -1 & 2
\end{bmatrix}}_{A}
\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix}
= h^2
\begin{bmatrix} f(h) \\ f(2h) \\ f(3h) \\ f(4h) \\ f(5h) \end{bmatrix}
$$

The fundamental properties of A

1) **The matrix A is tridiagonal** $\Rightarrow$ a tremendous simplification to Gaussian elimination.

$$\begin{bmatrix} 2 & 1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -1 & & & \\ 0 & \frac{3}{2} & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix} \rightarrow \dots$$

$$\begin{bmatrix} 1 & & & & \\ -\frac{1}{2} & 1 & & & \\ & -\frac{2}{3} & 1 & & \\ & & -\frac{3}{4} & 1 & \\ & & & -\frac{4}{5} & 1 \end{bmatrix} \begin{bmatrix} \frac{2}{1} & & & & \\ & \frac{3}{2} & & & \\ & & \frac{4}{3} & & \\ & & & \frac{5}{4} & \\ & & & & \frac{6}{5} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} & & & \\ & 1 & -\frac{2}{3} & & \\ & & 1 & -\frac{3}{4} & \\ & & & 1 & -\frac{4}{5} \\ & & & & 1 \end{bmatrix}$$

2) **The matrix is symmetric** $\Rightarrow A = LDL^T$.

$$\begin{bmatrix} 1 & & & & \\ -\frac{1}{2} & 1 & & & \\ & -\frac{2}{3} & 1 & & \\ & & -\frac{3}{4} & 1 & \\ & & & -\frac{4}{5} & 1 \end{bmatrix} \begin{bmatrix} \frac{2}{1} & & & & \\ & \frac{3}{2} & & & \\ & & \frac{4}{3} & & \\ & & & \frac{5}{4} & \\ & & & & \frac{6}{5} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} & & & \\ & 1 & -\frac{2}{3} & & \\ & & 1 & -\frac{3}{4} & \\ & & & 1 & -\frac{4}{5} \\ & & & & 1 \end{bmatrix}$$

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- **The matrix is positive definite**. This extra property says that the pivots are positive.

Thank You!