Lecture11

Monday, October 25, 2021 4:23 PM

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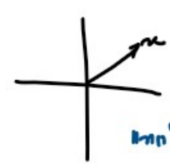
Linear Algebra

Samira Hossein Ghorban
s.hosseinghorban@ipm.ir

Fall, 2021

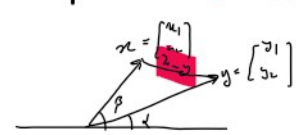
Orthogonality

- Generalization of the linear structure (addition and scalar multiplication) of $\mathbb{R}^2$ and $\mathbb{R}^2$ leads the definition of a linear space.
- How to generalize the concepts of length and angle?
- These concepts are embedded in the concept we now investigate, **inner products**



$\|x\| = \sqrt{x_1^2 + \dots + x_n^2}$

$\|x\|^2 + \|y\|^2 = \|x-y\|^2$



$\sin\alpha = \frac{y_2}{\|y\|}$ $\quad \cos\alpha = \frac{y_1}{\|y\|}$

$\sin\beta = \frac{x_2}{\|x\|}$ $\quad \cos\beta = \frac{x_1}{\|x\|}$

$\theta = ?$ $\quad \theta = \beta - \alpha$

$\cos\theta = \cos(\beta - \alpha) = \cos\beta\cos\alpha + \sin\beta\sin\alpha$

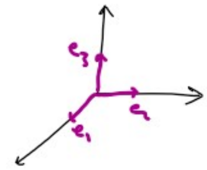
$= \frac{x_1 y_1 + x_2 y_2}{\|x\|\|y\|}$

$\cos\theta = \frac{x_1 y_1 + x_2 y_2}{\|x\|\|y\|} = \frac{\langle x, y \rangle}{\|x\|\|y\|}$

$\langle x, y \rangle = \underline{0}$

Coordinate axes

- Every time we think of the $x - y$ plane or three-dimensional space or $\mathbb{R}^n$, the axes are there.
- They are usually perpendicular!
- The coordinate axes that the imagination constructs are practically always orthogonal.
- In choosing a basis, we tend to choose an orthogonal basis.



Inner products on real linear space

فضای برداری

Definition:

An inner product on V is a function $\langle , \rangle : V \times V \to \mathbb{R}$ such that

$\|v\| = \sqrt{\langle v, v \rangle}$

1. $\langle v, v \rangle \geqslant 0$ for all $v \in V$. مثبت بودن
2. $\langle v, v \rangle = 0$ if and only if $v = 0$. صفر بودن

   $\langle -, w \rangle : V \longrightarrow \mathbb{R}$
3. $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ for all $u, v, w \in V$. خطی
4. $\langle cu, w \rangle = c\langle u, w \rangle$ for all $u, w \in V$ and $c \in \mathbb{R}$.
5. $\langle v, w \rangle = \langle w, v \rangle$. تقارن

$\langle u, v \rangle \in \mathbb{R}$

$\langle v, 0 \rangle = \langle 0, v \rangle = 0$

$\langle 0, v \rangle = \langle 0, v \rangle + \langle 0, v \rangle$

خطی $\langle u, - \rangle : V \longrightarrow \mathbb{R}$

$\Longrightarrow \langle u, w + v \rangle = \langle u, w \rangle + \langle u, v \rangle$

$\langle u, cw \rangle = c\langle u, w \rangle$

$\langle w + v, u \rangle = \langle w, u \rangle + \langle v, u \rangle$

Example

- The Euclidean inner product on $\mathbb{R}^n$:

$$\begin{cases} \langle\,,\,\rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \\ \langle x, y\rangle = y^T x = y_1 x_1 + \cdots + y_n x_n. \end{cases}$$

where $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$.

$\langle x, x\rangle = x_1^2 + \cdots + x_n^2 \geq 0$

$\langle x, x\rangle = 0 \iff x = 0$

$\langle x+z, y\rangle = y^T(x+z) = y^T x + y^T z$

$= \langle x, y\rangle + \langle z, y\rangle$

$\langle cx, y\rangle = y^T(cx) = c y^T x$

$\langle x, y\rangle = y^T x = \sum x_i y_i = x^T y$

$= \langle y, x\rangle$

Example

- The linear space $P_n(x)$ of all polynomials with coefficients in $\mathbb{R}$ and degree at most n:

$$\begin{cases} \langle\,,\,\rangle : P_n(x) \times P_n(x) \to \mathbb{R} \\ \langle f, g\rangle = \int_0^1 \underline{f(x)g(x)dx} \end{cases}$$

1- $\int_0^1 f(x)^2 dx \geq 0$

2- $\int_0^1 f(x)^2 = 0 \iff f(x) = 0$

3- $\int_0^1 (f(x) + h(x))g(x)\,dx = \int_0^1 fg\,dx + \int hg\,dx$

$\langle f+h, g\rangle$

In an inner-product space:

1. In an inner-product space, we have additivity and the homogeneity in the second slot, as well as the first slot:

$$\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle .$$

$$\langle u, cv \rangle = c \langle u, v \rangle .$$

2. For each fixed $u \in V$ the function

$$\langle u, - \rangle : V \to \mathbb{R}$$

is a linear map from $V \to \mathbb{R}$.

3. For each fixed $u \in V$,

$$\langle u, 0 \rangle = \langle 0, u \rangle = 0$$

Bilinear function

$\langle \, , \rangle : V \times V \longrightarrow \mathbb{R}$

خطی $\langle u, - \rangle : V \longrightarrow \mathbb{R}$

$\langle -, u \rangle : V \longrightarrow \mathbb{R}$

- Consider a linear function $T : \mathbb{R}^n \to \mathbb{R}$. Assuming the standard basis for $\mathbb{R}^n$, we have

$$T(x) = x_1 T(e_1) + \cdots + x_n T(e_n).$$

$y \in \mathbb{R}^n$

Let $y_i = T(e_i)$ for $1 \leqslant i \leqslant n$. Thus

$$T(x) = x_1 y_1 + \cdots + x_n y_n = y^T x.$$

$T(e_1) = y_1$

$\vdots$

$T(e_n) = y_n$

- So each the linear function $T : \mathbb{R}^n \to \mathbb{R}$ is determined by a fixed vector $y \in \mathbb{R}^n$.

$T : \mathbb{R}^n \longrightarrow \mathbb{R}$ $\Rightarrow$ $B(-, y) : \mathbb{R}^n \longrightarrow \mathbb{R}$

$T(x) = y^T x = B(x, y)$

- Naturally, we can consider

$$\boxed{\begin{cases} B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \\ B(x, y) = y^T x \end{cases}}$$

$\underbrace{B(-, y) : \mathbb{R}^n \longrightarrow \mathbb{R}}_{T}$

which is linear with respect to each of its variables.

$\ast$ $\langle v, v \rangle \geq 0$

$\ast$ $\langle v, v \rangle = 0 \iff v = 0$

— خطی بودن

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— تقارن

Inner product as a bilinear functions

$B : \mathbb{R}^n$

$\langle -, - \rangle : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$

- The Euclidean inner product on $\mathbb{R}^n$, $\langle x, y \rangle$ is naturally defined as a bilinear function:

Inner product as a bilinear functions

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$B : \mathbb{R}^n$

$\langle -, - \rangle : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$

Inner product spaces

Definition

An inner product space is a linear space with a binary operation called an **inner product**.

- The Eucildean inner product on $\mathbb{R}^n$ with the inner product $\langle x, y \rangle = y^T x$ for each $x, y \in \mathbb{R}^n$.
- The linear space $P_n(x)$ of all polynomials with coefficients in $\mathbb{R}$ and degree at most n with an inner product

$$\begin{cases} \langle\, , \,\rangle : P_n(x) \times P_n(x) \to \mathbb{R} \\ \langle f, g\rangle = \int_0^1 f(x)g(x)dx \end{cases}$$

is an inner product space.

$\langle \ \rangle : V \times V \longrightarrow \mathbb{R}$

$\det : \underbrace{V \times \cdots \times V}_{n} \longrightarrow \mathbb{R}$

$T : V \longrightarrow W$

$T : V \longrightarrow \mathbb{R}$

Norms

Definition

For $v \in V$, we define the norm of v, denoted $\|v\|$, by

$$\|v\| = \sqrt{\langle v, v\rangle}.$$

Examples

Example. With the Euclidean inner product:

$$\|x\| = \sqrt{x_1^2 + \ldots + x_n^2}.$$

where $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$.

Example. With inner product on P_n:

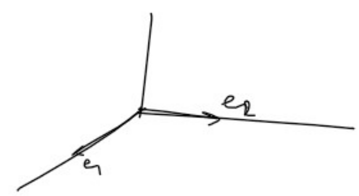
$$\|f(x)\| = \sqrt{\int_0^1 f(x)^2 dx}$$

Orthonormal vectors in V

Definition

Two vectors $u, v \in V$ are said to be orthogonal if $\langle u, v\rangle = 0$.

$\dim V = n$ $\mathbb{R}^n$ $\{e_1, \dots, e_n\}$



Useful fact

If nonzero vectors $v_1, \dots, v_n$ are mutually orthogonal (every vector is perpendicular to every other), then those vectors are linearly independent.

فرض : $\langle v_i, v_j\rangle = 0$, $1 \leq i < j \leq n$

Why?

حکم : استقلال خطی

$c_1v_1 + \dots + c_nv_n = 0 \Rightarrow c_1 = \dots = c_n = 0$

$c_i = 0$

$0 = \langle c_1v_1 + \dots + c_nv_n, v_i\rangle = \sum_{j=1}^{n} c_j \langle v_j, v_i\rangle$ $\quad j \neq i \quad \langle v_j, v_i\rangle = 0$

$0 = \underset{\neq 0}{c_i \langle v_i, v_i\rangle}$, $\underset{v_i \neq 0}{\langle v_i, v_i\rangle \neq 0} \Rightarrow c_i = 0 \quad \forall i$

$\|u\|^2 = \|p\|^2 = \frac{\langle u,v\rangle^2}{\|v\|^2} \Rightarrow \|u\|^2\|v\|^2 = \langle u,v\rangle^2$

Cauchy-Schwarz Inequality

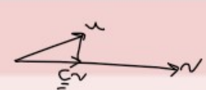
$\|u\|^2 = \|p\|^2 + \|u-p\|^2 \geq \|p\|^2$

$p = \frac{\langle u,v\rangle}{\|v\|^2} v$

$\Rightarrow \|p\|^2 = \frac{\langle u,v\rangle^2}{\|v\|^4}\|v\|^2 = \frac{\langle u,v\rangle^2}{\|v\|^2}$

Proposition

V



Let V be a linear space and $u, v \in V$. Then

$$|\langle u, v\rangle| \leq \|u\|\|v\|$$

$\langle u,v\rangle - c\langle v,v\rangle = 0$

$\langle u,v\rangle = c\langle v,v\rangle$

$\langle u - cv, v\rangle = 0 \quad c = \frac{\langle u,v\rangle}{\langle v,v\rangle}$

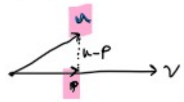
$v \neq 0$

$\Rightarrow \|u\|^2 \geq \frac{\langle u,v\rangle^2}{}$

$p = ?$ $\quad p = cv$

$$\langle u - cv, v \rangle = 0 \qquad c = \frac{\langle u, v \rangle}{\langle v, v \rangle}$$

$v \neq 0$



$p = ?$ $\quad p = cv \quad \langle u - p, v \rangle = 0$

$$0 = \langle u-p, v \rangle = v^T(u-p) = v^Tu - v^Tp = v^Tu - \underbrace{v^T(cv)}_{c v^T v} = v^Tu - c\|v\|^2$$

$$\Rightarrow v^Tu - c\|v\|^2 = 0 \Rightarrow c = \frac{v^Tu}{\|v\|^2} \Rightarrow p = \frac{v^Tu}{\|v\|^2}v = \frac{\langle u, v\rangle}{\|v\|^2}v$$

$u = p + (u - p)$ $\quad p, u-p$ برهم عمودند

$$\|u\|^2 = \langle u, u \rangle = \langle p + (u-p), p + (u-p) \rangle = \langle p, p \rangle + \cancel{\langle p, u-p \rangle} + \cancel{\langle u-p, p \rangle} + \langle u-p, u-p \rangle$$

$$= \|p\|^2 + \|u-p\|^2$$

$$\Rightarrow \|u\|^2 \geq \frac{\langle u, v \rangle^2}{\|v\|^2}$$

$v \neq 0 \Rightarrow$ حکم ثابت می‌شود

$$u = \overbrace{\frac{\langle u, v \rangle}{\|v\|^2}v}^{p} + \left(u - \frac{\langle u, v \rangle}{\|v\|^2}v\right)$$

$v, u \in V \rightarrow$ ضرب داخلی

$$0 = \left\langle \frac{\langle u, v \rangle}{\|v\|^2}v, \; u - \frac{\langle u, v \rangle}{\|v\|^2}v \right\rangle$$

Triangle Inequality

Proposition

Let V be a linear space. If $u, v \in V$, then

$$\|u + v\| \leqslant \|u\| + \|v\|$$

$$\langle u, v \rangle = \overline{\langle v, u \rangle}$$

$$|\langle u, v \rangle| \leqslant \|u\| \, \|v\|$$

$$\|u+v\|^2 = \langle u+v, u+v \rangle$$

$$= \langle u, u \rangle + \langle u, v \rangle + \langle v, u \rangle + \langle v, v \rangle$$

تقارن $= \|u\|^2 + 2\langle u, v \rangle + \|v\|^2$

نامساوی شوارتز $\leqslant \|u\|^2 + 2\|u\|\,\|v\| + \|v\|^2 = (\|u\| + \|v\|)^2$

Thank You!