Linear Algebra

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Review: Linear functions

Linear function

Let V and W be two linear spaces. Every function $T : V \to W$ that meets two below requirements is a linear function (transformation):

1. $T(x + y) = T(x) + T(y)$, for each $x, y \in V$.
2. $T(cx) = cT(x)$, for each $x \in V$ and $c \in \mathbb{R}$.

Question

Additive Closure

Let V and W be two linear spaces. Is Additive closure is enough for linearity of the function $T : V \to W$. That means we can figure out the property

$$T(cx) = cT(x),$$

for each $x \in V$ and $c \in \mathbb{R}$. form the additive property

$$T(x + y) = T(x) + T(y)$$

for each $x, y \in V$.

Answewr: No

Let $T : \mathbb{Q}[\sqrt{2}] \to \mathbb{Q}[\sqrt{2}]$ such that $T(a + b\sqrt{2}) = a + b\sqrt{2}$

1. $T(2) = 2.$
2. $T(2) = T(\sqrt{2}\sqrt{2}) = \sqrt{2}(1 + \sqrt{2}).$

Review: Linear functions Represented by Matrices

- Assume $\dim(V) = n$ and $T : V \to W$.
- Let $B = \{v_1, \ldots, v_n\}$ be a basis for V.
- If $v \in V$, then there are $c_i \in \mathbb{R}$ such that $v = c_1 v_1 + \cdots + c_n v_n$.
- By Linearity, $T(v) = c_1 T(v_1) + \cdots + c_n T(v_n)$.
- Let $B' = \{w_1, \ldots, w_n\}$ be a basis for W.

$$[T(v)]_{B'} = \begin{bmatrix} [T(v_1)]_{B'} & \cdots & [T(v_n)]_{B'} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

- $A = \begin{bmatrix} [T(v_1)]_{B'} & \cdots & [T(v_n)]_{B'} \end{bmatrix}$ is a matrix representation of T respect to basses B and B'.

Representation matrix of differentiation

$$\begin{cases} \frac{d}{dx} : P_3 \to P_3 \\ \\ \frac{d}{dx}(a_3x^3 + a_2x^2 + a_1x + a_0) = 3a_3x^2 + 2a_2x + a_1 \end{cases}$$

- Take that $\{1, x, x^2, x^3\}$ as a basis for P_3.
- The representation matrix for $\frac{d}{dx}$ is:

$$A_{\text{diff}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Representation matrix of differentiation

$$\begin{cases} \frac{d}{dx} : P_3 \to P_2 \\ \\ \frac{d}{dx}(a_3x^3 + a_2x^2 + a_1x + a_0) = 3a_3x^2 + 2a_2x + a_1 \end{cases}$$

- Take that $B = \{1, x, x^2, x^3\}$ as a basis for P_3.
- Take that $B' = \{1, x, x^2\}$ as a basis for P_2.
- The representation matrix for $\frac{d}{dx}$ is:

$$A_{\text{diff}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Review: Change of basis

- Suppose that V be a linear space with finite dimension.
- Let $B = \{v_1, \ldots, v_n\}$ be a ordered basis for V.
- The coordinate representation of $v \in V$ is denoted by $[v]_B$.
- Let $B' = \{v'_1, \ldots, v'_n\}$ be another ordered basis for V.
- What is relation between $[v]_B$ and $[v]'_B$ for any vector $v \in V$?

Representation Matrix and change basis

- Suppose that V be a linear space with finite dimension.
- Let $B = \{v_1, \ldots, v_n\}$ be a ordered basis for V.
- Consider $T : V \to V$ be a linear function. The representation matrix of T with respect to B is denoted by $[T]_B$.
- Let $B' = \{v'_1, \ldots, v'_n\}$ be another ordered basis for V.
- What is relation between $[T]_B$ and $[T]_{B'}$?

Example

- $V = P_2(x)$.
- Order bases $B = \{1, x, x^2\}$ and $B' = \{1, x, x^2 - \frac{1}{3}\}$ for V.
- Let $T : V \longrightarrow V$ given by

$$T\big(f(x)\big) = f(x) + \frac{d}{dx}f(x) + \frac{d^2}{dx^2}f(x).$$

- Find the matrix representation T in bases B and B'.

Solution

$$[T]_B = \Big[[T\big(1\big)]_B \quad [T\big(x\big)]_B \quad [T\big(x^2\big)]_B\Big]$$

$$T\big(1\big) = 1 + \frac{d}{dx}1 + \frac{d^2}{dx^2}1 = 1 \times 1 + 0 \times x + 0 \times x^2 \Rightarrow [T\big(1\big)]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$T\big(x\big) = x + \frac{d}{dx}x + \frac{d^2}{dx^2}x = 1 \times 1 + 1 \times x + 0 \times x^2 \Rightarrow [T\big(x\big)]_B = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Solution

$$[T]_B = \Big[[T(1)]_B \quad [T(x)]_B \quad [T(x^2)]_B\Big]$$

$$T(x^2) = x^2 + \frac{d}{dx}x^2 + \frac{d^2}{dx^2}x^2 = 2 \times 1 + 2 \times x + 1 \times x^2$$

$$\Rightarrow [T(x^2)]_B = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

$$[T]_B = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution

$$[T]_{B'} = \Big[[T\big(1\big)]_{B'} \quad [T\big(x\big)]_{B'} \quad [T\big((x^2 - \tfrac{1}{3})\big)]_{B'} \Big]$$

$$T\big(1\big) = 1 + \frac{d}{dx}1 + \frac{d^2}{dx^2}1 = 1 \times 1 + 0 \times x + 0 \times (x^2 - \frac{1}{3})$$

$$\Rightarrow [T\big(1\big)]_{B'} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Solution

$$[T]_{B'} = \begin{bmatrix} 1 & & \\ 0 & [T(x)]_{B'} & [T((x^2 - \frac{1}{3}))]_{B'} \\ 0 & & \end{bmatrix}$$

$$T(x) = x + \frac{d}{dx}x + \frac{d^2}{dx^2}x = 1 \times 1 + 1 \times x + 0 \times (x^2 - \frac{1}{3})$$

$$\Rightarrow [T(x)]_{B'} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Solution

$$[T]_{B'} = \begin{bmatrix} 1 & 1 & \\ 0 & 1 & [T\big((x^2 - \tfrac{1}{3})\big)]_{B'} \\ 0 & 0 & \end{bmatrix}$$

$$T\big((x^2 - \frac{1}{3})\big) = (x^2 - \frac{1}{3}) + \frac{d}{dx}(x^2 - \frac{1}{3}) + \frac{d^2}{dx^2}(x^2 - \frac{1}{3}) =$$

$$2 \times 1 + 2 \times x + 1 \times (x^2 - \frac{1}{3})$$

$$\Rightarrow [T\big(x^2 - \frac{1}{3})]_{B'} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

Representation matrices

$$[T]_B = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[T]_{B'} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Representation Matrix and change basis

- What is relation between $[T]_B$ and $[T]_{B'}$?

Invertible Matrices

Definition

Let V and W be two linear spaces and $T : V \rightarrow W$ be a linear function. If there is a linear function $U : W \rightarrow V$ such that $UT = I_W$ and $TU = I_V$ where I_V and I_W are identical function on V and W, then T is called invertible.

Existence of Inverses

Definition

For $A \in M_{mn}(\mathbb{R})$, if there is a $C \in M_{nm}(\mathbb{R})$ such that $AC = I$, then C is a right-inverse for A.

Definition

For $A \in M_{mn}(\mathbb{R})$, if there is a $B \in M_{nm}(\mathbb{R})$ such that $BA = I$, then B is a left-inverse for A.

Fact

Only a square matrix can have a two-sided inverse.

Right-inverse

- Suppose that $A \in M_{mn}$ has a right inverse. That means there is a matrix $C \in M_{nm}(\mathbb{R})$ such that $AC = I_m$.
- Let C_i be the i-th column of C.
- We have

$$AC = A\left[C_1 \cdots C_m\right] = \left[AC_1 \cdots AC_m\right] = I_m = \left[e_1 \cdots e_m\right]$$

- Thus, $AC_i = e_i$ for each $1 \leqslant i \leqslant m$.
- For every $b \in \mathbb{R}^m$, we have $b = b_1 AC_1 + \ldots + b_m AC_m$.
- Consequently, $\dim\left(\underbrace{C(A)}_{\text{Column space of } A}\right) = m$.
- As a result, $\text{rank}(A) = r = m$, Full row rank.

Left-inverse

- Suppose that $A \in M_{mn}$ has a left inverse. That means there is a matrix $B \in M_{nm}(\mathbb{R})$ such that $BA = I_n$.
- Let B_i be the i-th row of B.
- $BA = \begin{bmatrix} B_1A \\ \vdots \\ B_nA \end{bmatrix} = I = \begin{bmatrix} e_1^T \\ \vdots \\ e_n^T \end{bmatrix} \Rightarrow A^TB_i^T = e_i$, for each $1 \leqslant i \leqslant n$.
- For every $x \in \mathbb{R}^n$,, we have $x = x_1A^TB_1^T + \ldots + x_nA^TB_n^T$.
- Consequently, $\dim \left(\underbrace{C(A^T)}_{\text{Row space of } A} \right) = n$.
- As a result, $\operatorname{rank}(A) = r = n$, Full column rank.

When a matrix has a left-inverse (right-inverse)?

Right-inverse

A matrix A has a right-inverse if and only if $r = m$, full row rank.

Left-inverse

A matrix A has a left-inverse if and only if $r = n$, full column rank.

The condition for invertibility is full rank: $r = m = n$.

Corollary

Only a square matrix can have a two-sided inverse.

Example

- Let $A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix}$.

- $\text{rank}(A) = r = m = 2$ shows that A has a right-inverse.

- $AC = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \begin{cases} c_{11} = \frac{1}{4} & c_{12} = 0 \\ c_{21} = 0 & c_{22} = \frac{1}{5} \end{cases}$

- There are many right-inverses because the last row of C is **completely arbitrary**.

- This is a case of **existence** but not **uniqueness**.

Example

- Let $A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix}$.
- $C = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{5} \\ c_{31} & c_{32} \end{bmatrix}$ is a right-inverse for A for every $c_{31}, c_{32} \in \mathbb{R}$.
- $AA^T = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 5 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 16 & 0 \\ 0 & 25 \end{bmatrix}$ and $(AA^T)^{-1} = \begin{bmatrix} \frac{1}{16} & 0 \\ 0 & \frac{1}{25} \end{bmatrix}$
- $A^T(AA^T)^{-1} = \begin{bmatrix} 4 & 0 \\ 0 & 5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{16} & 0 \\ 0 & \frac{1}{25} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{5} \\ 0 & 0 \end{bmatrix} = C$, pseudo-inverse.

Example

- The transpose of A yields an example with infinitely many left-inverses:

$$BA^T = \begin{bmatrix} \frac{1}{4} & 0 & b_{13} \\ 0 & \frac{1}{5} & b_{23} \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 5 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Now, it is the last column of B that is completely arbitrary.

- The pseudo-inverse: $b_{13} = b_{23} = 0$. That means

$$\begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{5} & 0 \end{bmatrix} = \left(A^T(AA^T)^{-1}\right)^T$$

Review: Two-sided inverse

- The matrix A is invertible if there exists a matrix B such that $AB = BA = I$.

- Not all matrices have inverses.
- If $AB = I$ and $CA = I$, then $B = C$ (prove!). Therefore inverse matrix is unique. We denote it by A^{-1}.

- The matrix A is invertible if and only if $AX = b$ has one and only solution for a given b.

- The matrix A is invertible if and only if $A = LU$ where LU is a triangular factorization of A with no zeros on the diagonal of U.

When does a square matrix have inverse?

Each of these conditions is a necessary and sufficient test:

1. The columns span $\mathbb{R}^n$, so $Ax = b$ has at least one solution for every b.
2. The columns are independent, so $Ax = 0$ has only the solution $x = 0$.
3. The rows of A span $\mathbb{R}^n$.
4. The rows are linearly independent.
5. Elimination can be completed: $A = LDU$, with all n pivots.
6. (In Future) The determinant of A is not zero.
7. (In Future) Zero is not an eigenvalue of A.
8. (In Future) $A^T A$ is positive definite.

θ rotation

$$\begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$$\begin{bmatrix} \cos\theta & ? \\ \sin\theta & ? \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$$\begin{bmatrix} \cos\theta & ? \\ \sin\theta & ? \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{Q_\theta} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

Notes

- Does the inverse of Q_θ equal $Q_{-\theta}$ (rotation backward through θ)?
- Yes. $Q_\theta^{-1} = Q_{-\theta}$

$$Q_\theta Q_{-\theta} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Does the square of Q_θ equal $Q_{2\theta}$ (rotation through a double angle)? Yes.

$$Q_\theta^2 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}$$

- Does the product of Q_θ and Q_ϕ equal $Q_{\theta+\phi}$ (rotation through θ then ϕ)? Yes. $Q_\theta Q_\phi = Q_{\theta+\phi}$

Projections onto the x-axis

- Projections onto the x-axis $\begin{bmatrix} c \\ s \end{bmatrix}$

$$\begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \begin{bmatrix} c \\ s \end{bmatrix} = \begin{bmatrix} c \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c \\ s \end{bmatrix} = \begin{bmatrix} c \\ 0 \end{bmatrix}$$

Projections onto the θ-lines

$$\begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos^2\theta \\ \cos\theta\sin\theta \end{bmatrix}$$

$$\begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta\sin\theta \\ \sin^2\theta \end{bmatrix}$$

$$P = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$$

Projections onto the θ-lines

- The linear function has no inverse. (Why?)
- Points on the θ-line are projected to themselves.
- Projecting twice is the same as projecting once, and $P^2 = P$

Reflections through the 45° line

- Reflections through the 45° line.

$$\begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \begin{bmatrix} c \\ s \end{bmatrix} = \begin{bmatrix} s \\ c \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c \\ s \end{bmatrix} = \begin{bmatrix} c \\ s \end{bmatrix}$$

Reflection in the θ-line

- The reflection of $\begin{bmatrix} x \\ y \end{bmatrix}$ in the θ-lines.

$$H = \begin{bmatrix} 2\cos^2\theta - 1 & 2\cos\theta\sin\theta \\ 2\cos\theta\sin\theta & 2\sin^2\theta^2 - 1 \end{bmatrix}$$

$$H = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

Reflection in the θ-line

- $H^2 = I$: Two reflections bring back the original.
- $H^2 = I \Rightarrow H^{-1} = H$.
- Two reflections bring back the original which is clear from the geometry but less clear from the matrix.
- To show that $H^2 = I$, we use $P^2 = P$:

  We have $H = 2P - I$, thus

$$H^2 = (2P - I)^2 = 4P^2 - 4P + I = I$$

Thank You!