Linear Algebra

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Review: Classification of n-alternating multilinear maps

- For an n-alternating multilinear map

$$\phi : \underbrace{V \times \cdots \times V}_{n} \to \mathbb{R}$$

we have

$$\begin{aligned}
\phi(a_1, \ldots, a_n) &= \sum_{j_1=1}^{n} \cdots \sum_{j_n=1}^{n} a_{1j_1} \cdots a_{nj_n} \phi(e_{j_1}, \ldots, e_{j_n}) \\
&= \sum_{\sigma \in S_n} \left(\prod_{i=1}^{n} a_{i\sigma(i)} \phi\left(e_{\sigma(1)}, \ldots, e_{\sigma(n)}\right) \right) \\
&= \left(\sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)} \right) \operatorname{sgn}(\sigma) \phi(e_1, \ldots, e_n) \\
&= \left(\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)} \right) \phi(e_1, \ldots, e_n)
\end{aligned}$$

Review: Determinant

- Let a_i be the i-th row of $A = [a_{ij}]$. The determinant of A is defined by

$$\det A = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}.$$

Review: Properties of the Determinant

1. The determinant changes sign when two rows are exchanged.
2. The determinant of the identity matrix is 1.
3. The determinant depends linearly on the each row.
4. If two rows of A are equal, then $\det A = 0$.
5. Subtracting a multiple of one row from another row leaves the same determinant.
6. If A has a row of zeros, then $\det A = 0$ since the map det is n-multilinear.
7. If A is triangular then $\det A = a_{11}a_{22}\cdots a_{nn}$.
8. If A is singular, then $\det A = 0$. If A is invertible, then $\det A \neq 0$.
9. The transpose of A has the same determinant as A itself: $\det A = \det A^T$.
10. The determinant of AB is the product of $\det A$ times $\det B$.
11. Let A be an invertible matrix. Then $\det A \neq 0$.

Properties of the Determinant

12. Let $A \in M_r(\mathbb{R}), B \in M_{rs}(\mathbb{R})$ and $C \in M_s(\mathbb{R})$, then

$$\det \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} = \det A \det C.$$

- Proof:

Properties of the Determinant

13. Let $A, B, C, D \in M_n(\mathbb{R})$. If $CD = DC$ then

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(AD - BC).$$

- Note that it is also true if $AC = CA$ or $AB = BA$ or $BD = DB$.
- Proof:

Properties of the Determinant

14. (**Schur formula**) Let $A \in M_n(\mathbb{R})$. and

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

where A_{11} and A_{22} are square matrices and A_{11} is invertible. Then

$$\det A = (\det A_{11}) \det(A_{22} - A_{21} A^{-1} A_{12}).$$

- Proof. The following identity is easy verified:

$$\begin{bmatrix} I & 0 \\ -A_{21}A_{11}^{-1} & A_{22} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} I & -A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{bmatrix}$$

Formulas for the Determinant

- If A is invertible, then $PA = LDU$
- $\det A = \pm \det L \times \det D \times \det U$.
- $\det L = \det U = 1$.
- $\det D = d_1 \cdots d_n$.
- $\det A = \pm d_1 \cdots d_n$.

Example

- We obtain:

$$\begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & \ddots & & \\ & & & \ddots & & \\ & & & \ddots & & -1 \\ & & & & -1 & 2 \end{bmatrix} = L \begin{bmatrix} 2 & & & & \\ & \frac{3}{2} & & & \\ & & \frac{4}{3} & & \\ & & & \ddots & \\ & & & & \frac{n+1}{n} \end{bmatrix} U.$$

- Thus,

$$\det A = 2\left(\frac{3}{2}\right)\left(\frac{4}{3}\right)\cdots\left(\frac{n+1}{n}\right) = n+1.$$

One more formula for the determinant

- Let $A \in M_n(\mathbb{R})$.
- Consider The submatrix $A(i|j)$ that is defined by throwing away row i and column j.
- Let $\phi : \underbrace{V \times \cdots \times V}_{n} \to \mathbb{R}$ be given by

$$\phi(a_1, \ldots, a_n) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A(i|j).$$

- ϕ is an n-alternating multilinear map with $\phi(I) = 1$. Then,

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A(i|j).$$

Cofactors of A

- Assume that
$$c_{ij} = (-1)^{i+j} \det A(i|j),$$
then c_{ij} is called ij-th cofactor of matrix A.
- Let
$$C = \begin{bmatrix} c_{11} & \cdots & c_{1j} & \cdots & c_{1n} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ c_{n1} & \cdots & c_{nj} & \cdots & c_{nn} \end{bmatrix}$$
- Thus, For each $1 \leqslant j \leqslant n$, inner product of the j-th column of A and the j-th column of C is equal to $\det A$.
- But inner product of the j-th column of A and the k-th column of C is equal to zero for $1 \leqslant j \neq k \leqslant n$.

Adjoint A

- We obtain

$$C^T A = \begin{bmatrix} c_{11} & \cdots & c_{n1} \\ \vdots & \dots & \vdots \\ c_{1n} & \cdots & c_{nn} \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \dots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} \det A & & \\ & \ddots & \\ & & \det A \end{bmatrix}$$

- Thus,

$$C^T A = (\det A)\, I.$$

- The matrix C^T is called the adjoint of A and is denoted by adj A. So,

$$(\operatorname{adj} A) A = (\det A) I$$

adj A

- By $(\text{adj}\, A)A = (\det A)I$, we have

  1. $(\text{adj}\, A^T)A^T = (\det A^T)I = (\det A)I$.
  2. $(\text{adj}\, A)_{ij} = (-1)^{i+j} \det A(j|i)$.

- It is easy to check that

$$(\text{adj}\, A^T) = (\text{adj}\, A)^T.$$

Computation of A^{-1}

- If $A \in M_n(\mathbb{R})$ is invertible, then

$$A\left(\frac{\operatorname{adj} A}{\det A}\right) = \left(\frac{\operatorname{adj} A}{\det A}\right) A = I.$$

- Thus,

$$A^{-1} = \left(\frac{\operatorname{adj} A}{\det A}\right).$$

Cramer's rule

- Let $A \in M_n(\mathbb{R})$ be invertible.
- The solution of $Ax = b$ is $x = A^{-1}b$: just $C^T b$ divided by $\det A$.
- **Cramer's rule:** The jth component of $x = A^{-1}b$ is the ratio

$$x_j = \frac{\det B_j}{\det A},$$

where

$$B_j = \begin{bmatrix} a_{11} & a_{12} & \cdots & b_1 & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & b_n & \cdots & a_{nn} \end{bmatrix}$$

Thank You!