Linear Algebra

Samira Hossein Ghorban
s.hosseinghorban@ipm.ir

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Inner products on real linear space

An inner product on V is a function $\langle , \rangle : V \times V \to \mathbb{R}$ such that

1. $\langle v, v \rangle \geqslant 0$ for all $v \in V$.
2. $\langle v, v \rangle = 0$ if and only if $v = 0$.
3. $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ for all $u, v, w \in V$.
4. $\langle cu, w \rangle = c\langle u, w \rangle$ for all $u, w \in V$ and $c \in \mathbb{R}$.
5. $\langle v, w \rangle = \langle w, v \rangle$.

Inner products on linear space

- The definition of the above inner product is not useful for complex vector spaces V.
- Let $0 \neq u \in V$ and $i \in \mathbb{C}$.

$$\langle iu, iu \rangle = i^2 \langle u, u \rangle < 0.$$

Inner products on complex linear space

An inner product on V is a function $\langle , \rangle : V \times V \rightarrow \mathbb{R}$ such that

1. $\langle v, v \rangle \geqslant 0$ for all $v \in V$.
2. $\langle v, v \rangle = 0$ if and only if $v = 0$.
3. $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ for all $u, v, w \in V$.
4. $\langle cu, w \rangle = c \langle u, w \rangle$ for all $u, w \in V$ and $c \in \mathbb{R}$.
5. $\langle v, w \rangle = \overline{\langle w, v \rangle}$.

Notes

Let V be an inner product, Then

1. $\langle w, u + v \rangle = \langle w, u \rangle + \langle w, v \rangle$ for all $u, v, w \in V$.

2. $\langle u, cw \rangle = \bar{c} \langle u, w \rangle$ for all $u, w \in V$ and $c \in \mathbb{R}$.

Symmetric Matrices

Let $A \in M_n(\mathbb{R})$, then there is a matrix $B \in M_n(\mathbb{R})$ such that $\langle Ax, y\rangle = \langle x, By\rangle$ for each $x, y \in \mathbb{R}^n$.

Definition

A symmetric matrix is a square matrix that is equal to its transpose.

Hermitian Matrices

Let $A \in M_n(\mathbb{C})$, then there is a matrix $B \in M_n(\mathbb{C})$ such that $\langle Ax, y \rangle = \langle x, By \rangle$ for each $x, y \in \mathbb{C}^n$.

Definition

A hermitian matrix is a square matrix, which is equal to its conjugate transpose matrix.

Self-adjoint Matrices

Definition

A matrix $A \in \mathbb{F}$ is self-adjoint if $A^* = A$.

Definition

A matrix $A \in \mathbb{R}$ is symmetric if $A^T = A$.

Definition

A matrix $A \in \mathbb{C}$ is Hermitian if $A^H = A$.

Unitary Matrices

Definition

A matrix $U \in \mathbb{F}$ is unitary if $U^*U = UU^* = I$.

- For each $x, y \in \mathbb{F}^n$,

$$\langle Ux, Uy \rangle = \langle x, U^*Uy \rangle .$$

- If U is unitary, then

$$\langle Ux, Uy \rangle = \langle x, y \rangle .$$

That means U preserves inner product.

Inner product on finite-dimensional linear spaces

1. Suppose that V is finite-dimensional linear space where $B = \{v_1, \ldots, v_n\}$ is an ordered basis for V.
2. We are given a particular inner product on V.
3. The inner product is completely determined by the entries of matrix G where
$$G_{ij} = \langle v_j, v_i \rangle.$$
4. Let $v, w \in V$. If $x = [v]_B$ and $y = [w]_B$, then
$$\langle v, w \rangle = y^* G x.$$
5. If $V = \mathbb{F}^n$. Then for each $x, y \in V$,
$$\langle x, y \rangle = y^* x,$$
if we consider standard basis for V.

The properties of G

1. $G_{ii} > 0$, for each $1 \leqslant i \leqslant n$.
2. G self-adjoint.
3. G is invertible.
4. $\det G > 0$.

Is the above process reversible?

Let V in a lienar space on $\mathbb{R}$ with dimension n with a basis B.
Question. When a bilinear function $\langle , \rangle : V \times V \to \mathbb{F}$ such that

$$\langle v, w \rangle = y^* G x$$

and $x = [v]_B$ and $y = [w]_B$, is an inner product for $G \in M_n(\mathbb{F})$.

Is the above process reversible?

By the definition of an inner product, we should have

1. $\langle x, x\rangle = x^* G x \geqslant 0$ for all $v \in V$ such that $[v]_B = x$.
2. G is self-adjoint ($G^* = G$).

Definition

A self-adjoint matrix $A \in M_n(\mathbb{F})$ is called

1. **positive definite** if $x^T A x > 0$ for each $0 \neq x \in \mathbb{F}^n$.
2. **positive semi-definite** if $x^T A x \geqslant 0$ for each $x \in \mathbb{F}^n$.

Gradient vector and Hessian matrix for f

- *Gradient* of real-valued differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ is

$$\nabla f(x) = \begin{bmatrix} \frac{\partial f(x)}{\partial x_1} & \cdots & \frac{\partial f(x)}{\partial x_n} \end{bmatrix}$$

- Hessian matrix of a real-valued function $f : \mathbb{R}^n \to \mathbb{R}$ is

$$H(x) = \nabla^2 f(x) = \begin{bmatrix} \frac{\partial^2 f(x)}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 f(x)}{\partial x_n \partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f(x)}{\partial x_1 \partial x_n} & \cdots & \frac{\partial^2 f(x)}{\partial x_n \partial x_n} \end{bmatrix}$$

The second derivative test

Theorem

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a real-valued differentiable function with continuous second partial derivatives. Then

1. *f has a local minimum at x_* if and only if $x^T \nabla^2 f(x_*) x > 0$ for each $0 \neq x \in \mathbb{R}$.*
2. *f has a local maximum at x_* if and only if $x^T \nabla^2 f(x_*) x < 0$ for each $0 \neq x \in \mathbb{R}$.*
3. *Otherwise f has a saddle point at x_*.*

Self-adjoint Matrices

Theorem

If A is a self-adjoint matrix, then an invertible matrix $P \in M_n(\mathbb{F})$ such that $P^*P = I$ and

$$P^*AP = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}.$$

Tests for Positive Definiteness

Theorem

Each of the following tests is a necessary and sufficient condition for the Hermitian matrix A to be positive definite:

1. *All eigenvalues of A are positive.*
2. *All upper left submatrices A_k have positive determinants.*
3. *All pivots (without row exchanges) are positive.*

- The test brings together three of the most basic ideas in the book:
  1. pivots,
  2. determinants,
  3. eigenvalues.

Thank You!