Lecture22

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Lecture22

Linear Algebra

Samira Hossein Ghorban
s.hosseinghorban@ipm.ir

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Review: Diagonalizable linear transformations

Theorem

Let $T: V \to V$ be a linear transformation where V is finite dimensional, and T has different eigenvalues $\lambda_1, \ldots, \lambda_k$. Suppose that W_i is the null space of $\lambda_i I - T$ for each $1 \leqslant i \leqslant k$. Then the following statements are equivalent:

$W_i = \{x \mid Tx = \lambda_i x\}$

i. *T is diagonalizable.*

ii. *The characteristic polynomial of T is*

$$f(\lambda) = (\lambda - \lambda_1)^{n_1} \cdots (\lambda - \lambda_k)^{n_k},$$

and $\dim W_i = \underline{n_i} =$ λ_i

iii. $\sum_{i=1}^{k} \dim W_i = \dim V$.

$\underline{W_1 \oplus \ldots \oplus W_k = V}$

Review: Lemma

$$\sum \dim W_i = \dim V \Rightarrow [T]_B = \begin{bmatrix} \lambda_1 I_{n_1} & & \\ & \ddots & \\ & & \lambda_k I_{n_k} \end{bmatrix}$$

Lemma

Suppose that T is a linear function on V with different eigenvalues $\lambda_1, \ldots, \lambda_k$, and for each $1 \leqslant i \leqslant k$ let

$$W_i = \{v \in V \mid Tv = \lambda_i v\},$$

which is the null space of $\lambda_i I - T$. If $v_1 + \cdots + v_k = 0$ for each $v_i \in W_i$, then $v_1 = \cdots = v_k = 0$.

$v_i \in W_i$

۱- درون تیر

۲- وقتی بردارهای مربوط به مقادیر مشخصه متمایز باشند.

$v_i \in W_i$

$\lambda_1, \ldots, \lambda_k$ متمایزند.

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Vandermonde matrices

- The following matrix is called **Vandermonde** matrix.

$$V(\lambda_1, \ldots, \lambda_k) = \begin{array}{c} \text{صفر} \quad \text{یک} \quad \quad k-1 \\ \begin{bmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{k-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_k & \cdots & \lambda_k^{k-1} \end{bmatrix}_{k \times k} \end{array} \quad \begin{array}{c} \lambda_1 \\ \vdots \\ \lambda_k \end{array}$$

تمایز

Vandermonde matrices

- The Vandermonde matrix is invertible; suppose that

$$\begin{bmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{k-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_k & \cdots & \lambda_k^{k-1} \end{bmatrix} \underbrace{\begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{k-1} \end{bmatrix}}_{\text{صفر}} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$c_0 + c_1 \lambda_i + \cdots + c_{k-1} \lambda_i^{k-1} = 0$$

$$N(V(\lambda_1, \ldots, \lambda_n)) = \{0\}$$

Vandermonde matrices

We may write

$$\begin{cases} c_0 + c_1\lambda_1 + \cdots + c_{k-1}\lambda_1^{k-1} = 0 \\ c_0 + c_1\lambda_2 + \cdots + c_{k-1}\lambda_2^{k-1} = 0 \\ \vdots \quad\quad \vdots \quad + \cdots + \quad \vdots \quad = \vdots \\ c_0 + c_1\lambda_k + \cdots + c_{k-1}\lambda_k^{k-1} = 0 \end{cases}$$

- So $\lambda_1, \ldots, \lambda_k$ are distinct roots of the polynomial

$$c_0 + c_1 x + \cdots + c_{k-1}x^{k-1} = 0.$$

$(x-\lambda_1)q(x) = c_0 + c_1 x + \cdots + c_{k-1}x^{k-1}$

$\underbrace{(x-\lambda_1)\cdots(x-\lambda_k)}_{k} \mid q(x) = \underbrace{\quad}_{k-1}$

and hence $c_0 = \cdots = c_{k-1} = 0$. As a result,

$$N(V(\lambda_1, \ldots, \lambda_k)) = \{0\}$$

and the Vandermonde matrix $V(\lambda_1, \ldots, \lambda_k)$ is invertiable.

Proof.

$T^i(v_1) = \lambda_1^i v_1$

$v_1 + \cdots + v_k = 0$

$v_i \in W_i$

$\Downarrow$

$v_1 = \cdots = v_k = 0$

- Since $v_1 + \ldots + v_k = 0$, we have $T^i(v_1 + \cdots + v_k) = 0$ for each $0 \leqslant i \leqslant k-1$.

$T^i(v_1) + \cdots + T^i(v_k) = 0$

- So for each $0 \leqslant i \leqslant k-1$, $\lambda_1^i v_1 + \lambda_2^i v_2 + \cdots + \lambda_k^i v_k = 0$ which may be written in the following form

$$\begin{bmatrix} v_1 & \cdots & v_k \end{bmatrix} V(\lambda_1, \cdots, \lambda_k) = 0.$$

$$\begin{bmatrix} v_1 & \cdots & v_k \end{bmatrix} \begin{bmatrix} 1 & & \lambda_1^i & & \lambda_1^{k-1} \\ 1 & & \vdots & & \\ 1 & & \lambda_k^i & & \lambda_k^{k-1} \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

- This shows that $\begin{vmatrix} v_1 & \cdots & v_k \end{vmatrix} = 0$ in turns as $V(\lambda_1, \cdots, \lambda_k)$ is invertible.

$v_1 = \cdots = v_k = 0$

Cayley-Hamilton's theorem

$f(A) = A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I = 0$

Theorem

Let $A \in M_n(\mathbb{F})$ where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and $f(x)$ is characteristic polynomial. Then $f(A) = 0$

$f(x) = \det(xI - A) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$

$(xI - A)\,\mathrm{adj}(xI - A) = \det(xI - A)\, I$

$xI - A = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} : \mathrm{adj}(xI - A)\,[\;]_{ij}$

$\mathrm{adj}(xI - A) = B_{n-1}x^{n-1} + B_{n-2}x^{n-2} + \cdots + B_1 x + B_0$

$0 \leq i \leq n-1$

$B_i \in M_n(F)$

$(xI - A)(B_{n-1}x^{n-1} + B_{n-2}x^{n-2} + \cdots + B_0) = (x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + a_1 x + a_0)\, I$

$x^n \to B_{n-1} = I$

$x^{n-1} \to B_{n-2} - AB_{n-1} = a_{n-1} I$

$x^{n-2} \to B_{n-3} - AB_{n-2} = a_{n-2} I$

$\vdots$

$x \to B_0 - AB_1 = a_1 I$

$-AB_0 = a_0 I$

$\to A^n B_{n-1} = A^n$

$A^{n-1}B_{n-2} - A^n B_{n-1} = a_{n-1}A^{n-1}$

$A^{n-2}B_{n-3} - A^{n-1}B_{n-2} = a_{n-2}A^{n-2}$

$\vdots$

$AB_0 - A^2 B_1 = a_1 A$

$-AB_0 = a_0 I$

$0 = f(A)$

$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$

$f(B) = B^n + \cdots + a_1 B + a_0 I$

A corollary of Cayley-Hamilton's theorem

A^{555}

Corollary

$f(B) = B^{n} + \dots + a_1 B + a_0 I$

Corollary

Let $A \in M_n(\mathbb{F})$ where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. For $k > n$, $A^k = g(A)$ where $g(x)$ is a polynomial with coefficients in $\mathbb{F}$ and its degree is less than n.

$A \in M_{50}(\mathbb{F})$ $\qquad f(x) = x^{50} + a_{49}x^{49} + \dots + a_1 x + a_0$

A^{5000} $\qquad f(A) = 0$

$g(A) = A^{5000}$

$g(x) \mid f(x)$, quotient $q(x)$, remainder $r(x)$

$g(A) = f(A)q(A) + r(A)$ $\qquad \deg r(x) < \deg g(x)$

$x^{5000} = 50 \quad + r(x)$ $\qquad \deg r(x) < 50$

$g(A) = r(A)$

A^{555}

$(S^{-1}AS)^{55} = \begin{bmatrix} \lambda_1^{555} & & \\ & \ddots & \\ & & \lambda_n^{555} \end{bmatrix}$

$S^{-1}A^{55}S = \begin{bmatrix} & & \\ & & \end{bmatrix}$

$A^{555} = S \begin{bmatrix} \lambda_1^{55} & & \\ & \ddots & \\ & & \lambda_n^{55} \end{bmatrix} S^{-1}$

Minimal polynomial

$f(x) = x^n + a_{n-1}x^{n-1} + \dots + a_0$

$p(x) = 1x^m + \dots \quad f(A)$

Definition

Let $A \in M_n(\mathbb{F})$ where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. The minimal polynomial $p(x)$ of A over $\mathbb{F}$ is the monic polynomial over $\mathbb{F}$ of least degree such that $p(A) = 0$.

- The minimal polynomial is defined for a linear function T.
- The minimal polynomial is unique.

$\deg p(x) \le \deg f(x)$

$p(x) \mid f(x)$

Corollary

Let $A \in M_n(\mathbb{F})$ where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and $p(x)$ is minimal polynomial. Then $p(x)$ divides the characteristics polynomial $f(x)$.

$f(x) = p(x)q(x) + r(x)$ $\qquad \deg r(x) < \deg p(x)$

...

$\cancel{f(A)} = \cancel{p(A)}q(A) + r(A)$

$0 = r(A) \implies r(x) \equiv 0$

Thank You!

$f(A) = 0$

$p(A) = 0$

$p(x) = a_m x^m + \dots + a_1 x + a_0$

$2p(x) = 2a_m x^m + \dots + 2a_1 x + 2a_0$

$T : V \longrightarrow V$

$\det(xI - [T]_B)$

ضرایب ... برای T:

$[T]_B$ ... [illegible]

[illegible]

$[T]_B = S [T]_{B'} S^{-1}$

$p(A) = \sum_{i=1}^{m} c_i x^i \qquad c_m = 1$

$p([T]_B) = 0$

$S^{-1} p([T]_B) S = 0$

$= S^{-1}\left(\sum c_i [T]_B\right) S$

$= \sum c_i S^{-1}[T]_B S$

$= \sum c_i [T]_{B'} = 0$

$p([T]_B) \quad p([T]_{B'})$

چندجمله‌ای مینیمال A یکتاست.

فرض کنید $p(x)$, $p'(x)$ چندجمله‌ای مینیمال A باشند. چون چندجمله‌ای مینیمال کمترین درجه را دارد

$\deg p(x) = \deg p'(x)$

$$P(A) = 0 \qquad p(x) = x^m + b_{m-1}x^{m-1} + \cdots + b_0$$

$$p'(A) = 0 \qquad p'(x) = x^m + c_{m-1}x^{m-1} + \cdots + c_0$$

$$p(x) - p'(x) = (b_{m-1} - c_{m-1})x^{m-1} + \cdots + (b_0 - c_0)$$

$$0 = p(A) - p'(A) = \boxed{\text{درجه کمتر از } m}$$

پس باید صفر باشد

$$b_{m-1} = c_{m-1}, \quad \ldots, \quad b_0 = c_0$$