Lecture17

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Linear Algebra

Samira Hossein Ghorban
s.hosseinghorban@ipm.ir

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Review: Classification of n-alternating multilinear maps

- For an n-alternating multilinear map
$$\phi : \underbrace{V \times \cdots \times V}_{n} \to \mathbb{R}$$
we have

n^n $\subseteq n$

$$\phi(a_1, \ldots, a_n) = \sum_{j_1=1}^{n} \cdots \sum_{j_n=1}^{n} a_{1j_1} \cdots a_{nj_n} \phi(e_{j_1}, \ldots, e_{j_n})$$

$n!$

$$= \sum_{\sigma \in S_n} \left(\prod_{i=1}^{n} a_{i\sigma(i)} \phi(e_{\sigma(1)}, \ldots, e_{\sigma(n)}) \right)$$

$$= \left(\sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)} \right) \mathrm{sgn}(\sigma) \phi(e_1, \ldots, e_n)$$

$$= \underbrace{\left(\sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)} \right)}_{\det A} \phi(e_1, \ldots, e_n)$$

$\phi(a_1, \ldots, v, a_{i+1}, \ldots, v, a_{j+1}, \ldots, a_n) = 0$

$\{j_1, \ldots, j_n\} = \{1, \ldots, n\}$

$\sigma = \begin{pmatrix} 1 & \cdots & n \\ j_1 & \cdots & j_n \end{pmatrix}$

$\phi(e_{\sigma(1)}, \ldots, e_{\sigma(n)}) = \mp$

$\phi(e_1, \ldots, e_n)$

Determinant

$\sigma \in S_3$

$[a_{11} \quad a_{12} \quad a_{13}]$

Determinant

$\det A$

- Let a_i be the i-th row of $A = [a_{ij}]$.
- The determinant of A is defined by

$A^{-1} = \begin{bmatrix} & a_{13} & \\ & & \\ & & \end{bmatrix}$

$$\det A = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}.$$

تغیر [illegible]

$\sigma \quad \begin{matrix} 1 \\ 3 \end{matrix}$

- For 2 by 2 matrix

$$\det \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{A} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

$\sigma \in S_3$

$\sigma(1) = 1, 2, 3$

$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$A = [a_1, \dots, a_n]$

$\det A = \Phi(a_1, \dots, a_n)$

$\sigma(1) = 2$
$\sigma(2) = 3$
$\sigma(3) = 1$

$\Phi(e_1, \dots, e_n) = 1$

$\sigma : \{1, 2\} \to \{1, 2\}$

$\sigma_1(1) = 1 \qquad \sigma_2(1) = 2$

$\sigma_1 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = (1\,2)$

$\det A = +ad \qquad +(-bc)$

Properties of the Determinant

$A = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} \qquad B = \begin{bmatrix} A_2 \\ A_1 \\ A_3 \end{bmatrix}$

$\det B = -\det A$

$\underline{\det A}$

$\det A = \Phi(A_1, A_2, A_3)$

$= -\underbrace{\Phi(A_2, A_1, A_3)}_{\det B}$

1. The determinant changes sign when two rows are exchanged.
2. The determinant of the identity matrix is 1.
3. The determinant depends linearly on the each row.

$\det I = 1$

$\Phi(e_1, \dots, e_n) = 1$

$\det A = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod a_{1\sigma(1)} \cdots a_{n\sigma(n)}$

$\det A = \Phi(A_1, \dots, A_n)$

$\det I = \Phi(e_1, \dots, e_n)$

Properties of the Determinant

4. If two rows of A are equal, then $\det A = 0$ (why?)

   $\det A = 0$

5. Subtracting a multiple of one row from another row leaves the same determinant.(why?)

6. If A has a row of zeros, then $\det A = 0$ since the map det is n-multilinear.

$$A = \begin{bmatrix} 0 & \cdots & 0 \\ a_{21} & & a_{2n} \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \Rightarrow \det A = 0$$

$$A = \begin{bmatrix} A_1 \\ \vdots \\ A_n \end{bmatrix}$$

$$\det A = \Phi(A_1, \ldots, A_n)$$

$$B = \begin{bmatrix} A_1 \\ \vdots \\ A_i - cA_j \\ A_{i+1} \\ \vdots \\ A_n \end{bmatrix}$$

$$\det B = \Phi(A_1, \ldots, A_i - cA_j, \ldots)$$
$$= \Phi(A_1, \ldots, A_n)$$
$$- c\,\Phi(A_1, \ldots, A_j, \ldots, A_j, \ldots) \to 0$$
$$= \det A$$

Properties of the Determinant

7. If A is triangular then $\det A = a_{11}a_{22}\cdots a_{nn}$ (why?)

$$A = \begin{bmatrix} a_{11} & & * \\ & \ddots & \\ 0 & & a_{nn} \end{bmatrix}$$

$\det A = a_{11}a_{22} \ldots a_{nn}$

8. If A is singular, then $\det A = 0$. If A is invertible, then $\det A \neq 0$ (why?)

$$P_k \cdots P_1 A = R = \begin{bmatrix} a'_{11} & & * \\ 0 & \ddots & \\ & & a'_{nn} \end{bmatrix}$$

$$PA = R$$

$$\det PA = \det R$$

$$\Phi(A_1, \ldots, A_n)$$

Properties of the Determinant

9. The transpose of A has the same determinant as A itself: $\det A = \det A^T$ (why?)

$A^T =$ $A =$

10. The determinant of AB is the product of $\det A$ times $\det B$ (why?)

$A^T = [a^T_{ij}]$ $\quad A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$ $\quad a_{13}\, a_{21}\, a_{32}$

$PA = [\;]$

11. Let A be an invertible matrix. Then $\det A \neq 0$.

$\det P \det A \quad a_1 \dots a_n \neq 0$

$$A = \begin{bmatrix} a_{11} & \boxed{a_{12}} & a_{13} \\ a_{21} & a_{22} & \boxed{a_{23}} \\ \boxed{a_{31}} & a_{32} & a_{33} \end{bmatrix} \quad \sigma(1) = 2,\ \sigma(2) = 1 \quad \sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

$$\sigma^{-1}(2) = 1 \quad \sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

$AA^{-1} = I \qquad \det A \det A^{-1} = 1$

$\det A \neq 0$

$\det A = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, a_{1\sigma(1)} \cdots a_{n\sigma(n)}$

$\det A$,) $\operatorname{sgn}(\sigma)\, a_{\sigma(1)} \cdots a_{n\sigma(n)}$ ضرب

$\det A^T$,) $\operatorname{sgn}(\sigma^{-1})\, a_{1\sigma^{-1}(1)} \cdots a_{n\sigma^{-1}(n)}$

$\operatorname{sgn}(\sigma) = \operatorname{sgn}(\sigma^{-1})$

$\sigma = \tau_1 \cdots \tau_m$ ترانهش m

$\sigma^{-1} = \tau_m^{-1} \cdots \tau_1^{-1}$ ترانهش m

$1 \to 2 \to 3$

$\det A = 0 \Leftarrow$ تکین A

$\det A \neq 0 \Leftrightarrow A$ وارون‌پذیر

Properties of the Determinant

12. Let $A \in M_r(\mathbb{R})$, $B \in M_{rs}(\mathbb{R})$ and $C \in M_s(\mathbb{R})$, then

$$\det \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} = \det A \det C.$$

- Proof:

Properties of the Determinant

13. Let $A, B, C, D \in M_n(\mathbb{R})$. If $CD = DC$ then

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(AD - BC).$$

- Note that it is also true if $AC = CA$ or $AB = BA$ or $BD = DB$.

- Proof:

Properties of the Determinant

14. (**Schur formula**) Let $A \in M_n(\mathbb{R})$. and

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} =$$

where A_{11} and A_{22} are square matrices. Then

$$\det A = (\det A_{11}) \det(A_{22} - A_21A^{-1}A_12).$$

- Proof. The following identity is easy verified:

$$\begin{bmatrix} I & 0 & -A_{21}A_{11}^{-1} & A_{22} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} I & -A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{bmatrix}$$

Formulas for the Determinant

- If A is invertible, then $PA = LDU$
- $\det A = \pm \det L \times \det D \times \det U$.
- $\det L = \det U = 1$.
- $\det D = d_1 \cdots d_n$.
- $\det A = \pm d_1 \cdots d_n$.

Example

- We obtain:

$$\begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & \ddots & & \\ & & & \ddots & & \\ & & & \ddots & & -1 \\ & & & & -1 & 2 \end{bmatrix} = L \begin{bmatrix} 2 & & & & \\ & \frac{3}{2} & & & \\ & & \frac{4}{3} & & \\ & & & \ddots & \\ & & & & \frac{n+1}{n} \end{bmatrix} U.$$

- Thus,

$$\det A = 2\left(\frac{3}{2}\right)\left(\frac{4}{3}\right)\cdots\left(\frac{n+1}{n}\right) = n+1.$$

One more formula for the determinant

- Let $A \in M_n(\mathbb{R})$.
- Consider The submatrix $A(i|j)$ that is defined by throwing away row i and column j.
- Let $\phi : \underbrace{V \times \cdots \times V}_{n} \to \mathbb{R}$ be given by

$$\phi(a_1, \ldots, a_n) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A(i|j).$$

- ϕ is an n-alternating multilinear map with $\phi(I) = 1$. Then,

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A(i|j).$$

Cofactors of A

- Assume that
$$c_{ij} = (-1)^{i+j} \det A(i|j),$$
then c_{ij} is called ij-th cofactor of matrix A.
- Let
$$C = \begin{bmatrix} c_{11} & \cdots & c_{1j} & \cdots & c_{1n} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ c_{n1} & \cdots & c_{nj} & \cdots & c_{nn} \end{bmatrix}$$
- Thus, For each $1 \leqslant j \leqslant n$, inner product of the j-th column of A and the j-th column of C is equal to $\det A$.
- But inner product of the j-th column of A and the k-th column of C is equal to zero for $1 \leqslant j \neq k \leqslant n$.

Thank You!

خاصیت ۱. $\det AB = \det A \det B$

$$A = \begin{bmatrix} A_1 \\ \vdots \\ A_n \end{bmatrix}$$

$$\Phi : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \longrightarrow \mathbb{R}$$

$$\Phi(A_1, \dots, A_n) = \det AB$$

۱- Φ ، n-خطی است.

B ، A تکین $\Rightarrow \det A \det B = 0 = \det AB$

AB تکین $\quad x \in N(AB)$

$ABx = 0 \Rightarrow Bx \in N(A)$

B تکین $\Rightarrow AB$

$N(B) \subseteq N(AB)$

$N(B) \neq$

$ABx = 0$

A تکین $\rightarrow A^T$ تکین $\qquad \det A^T = 0$

$\det A^T \det B^T = \det B^T A^T$

$= \det AB$

$$A = \begin{bmatrix} A_1 \\ \vdots \\ A_n \end{bmatrix}$$

فرض کنیم A و B هر دو وارون پذیر باشند

$\{A_1, \dots, A_n\}$ پایه‌ای برای $\mathbb{R}^n = V$ است.

$\{A_1, \dots, A_n\}$ سطر (۲) برای $\forall \in \mathbb{R}^n$.

$$\det AB = \det\left(\begin{bmatrix} A_1 \\ \vdots \\ A_n \end{bmatrix} B \right) = \underline{\det \begin{bmatrix} A_1 B \\ \vdots \\ A_n B \end{bmatrix}}$$

$$\phi : \mathbb{R}^n \times \dots \times \mathbb{R}^n$$

$$\phi(A_1, \dots, A_n) = \det A$$

$$1 = \phi(e_1, \dots, e_n)$$

$$\phi : \mathbb{R}^n \times \dots \times \mathbb{R}^n$$

$$\phi(A_1, \dots, A_n) = \det \begin{bmatrix} A_1 B \\ \vdots \\ A_n B \end{bmatrix}$$

$$\Phi^* : \mathbb{R}^n \dots \mathbb{R}^n \longrightarrow \mathbb{R}$$

$\det A$

$$\Phi^*(A_1, \dots, A_n) = \det A$$

$$\Phi^*(e_1, \dots, e_n) = 1$$

$$\phi(\underline{e_1, \dots, e_n}) = \det B$$

$$\Phi : \mathbb{R}^n \times \dots \times \mathbb{R}^n \longrightarrow \mathbb{R}$$

$$\Phi(e_1, \dots, e_n) = \det B$$

$$\Phi(A_1, \dots, A_n) = \underline{\det A}\ \phi\overbrace{\underline{(e_1, \dots, e_n)}}^{\det B} = \det$$

اثبات خاصیت ۱۵ :

$$\Phi^* : \mathbb{R}^n \times \dots \times \mathbb{R}^n$$

$$\Phi^*(A_1, \dots, A_n) = \underline{\det} AB$$

ادعا: Φ^* تابع n خطی، و alternating است. به

به ازای هر $1 \le i \le n$.

$$\Phi^*(A_1, \dots, cA_i + A_i', A_{i+1}, \dots, A_n) =$$

$$\det\left(\begin{bmatrix} A_1 \\ \vdots \\ A_{i-1} \\ cA_i + A_i' \\ A_{i+1} \\ \vdots \\ A_n \end{bmatrix} B \right) = \det\left(\begin{bmatrix} A_1 B \\ \vdots \\ (cA_i + A_i')B \\ \vdots \\ A_n B \end{bmatrix} \right) \overset{\text{خطی}}{=} c \underbrace{\det \begin{bmatrix} A_1 B \\ \vdots \\ A_i B \\ \vdots \\ A_n B \end{bmatrix}}_{\phi^*(A_1, \dots, A_n)}$$

$$\det \begin{bmatrix} A_1 B \\ A_i' B \end{bmatrix} = c\,\Phi^*(A_1, \dots, A_i, \dots, A_n)$$

$$+\ \underbrace{\det\begin{bmatrix} A_1B \\ A_i'B \\ A_nB \end{bmatrix}} = c\,\Phi^*(A_1,\dots,A_i,\dots,A_n) \qquad \Phi^*(A_1,\dots,A_n)$$

$$+\ \Phi^*(A_1,\dots,A_i',\dots,A_n)$$

پس Φ^* نسبت به هر i ، n تایی خطی است پس Φ^* n-خطی است .

تابع Φ^* alternating است زیرا اگر $A_i = A_j = A$

$$\Phi^*(A_1,\dots,\overset{(i)}{A_i},\dots,\overset{(j)}{A_i},\dots,A_n) = \det\begin{bmatrix} A_1B \\ A_iB \\ A_iB \\ A_nB \end{bmatrix} \overset{[illegible]}{=} 0$$

$$\Phi^*(A_1,\dots,A_n) = \underbrace{\left(\sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)\, a_{1\sigma(1)}\cdots a_{n\sigma(n)}\right)\Phi^*(e_1,\dots,e_n)}_{\det A\ \Phi^*(e_1,\dots,e_n)}$$

$$= \det A\ \Phi^*(e_1,\dots,e_n)$$

$$\Phi^*(e_1,\dots,e_n) = \det\begin{bmatrix} e_1B \\ \vdots \\ e_nB \end{bmatrix} = \det B$$

$$\underbrace{\Phi^*(A_1,\dots,A_n)}_{\det AB} = \det A \det B$$

$$\Longrightarrow \quad \det AB = \det A \det B$$