Lecture19

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Linear Algebra

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Review: Determinants by Expansion

- Let $A \in M_n(\mathbb{R})$. Consider the submatrix $A(i|j)$ that is defined by throwing away row i and column j.

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A(i|j).$$

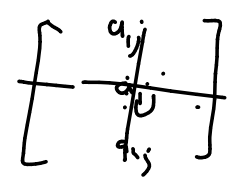
- Assume that

$$c_{ij} = (-1)^{i+j} \det A(i|j),$$

then c_{ij} is called ij-th cofactor of matrix A.

- Let

$$C = \begin{bmatrix} c_{11} & \cdots & c_{1j} & \cdots & c_{1n} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ c_{n1} & \cdots & c_{nj} & \cdots & c_{nn} \end{bmatrix}.$$



$\det A = \det A^T$

$\sum_{i=1}^{n} a_{ij} c_{ij} = \det A$

$\langle$ ستون jام A , ستون jام C $\rangle = \det A$

$\langle$ ستون iام A , ستون jام C $\rangle = 0$

$i \neq j$

Review: Cofactors of A

- Thus, For each $1 \leqslant j \leqslant n$, inner product of the j-th column of A

Review: Cofactors of A

$\det A = \det A^T$

- Thus, For each $1 \leqslant j \leqslant n$, inner product of the j-th column of A and the j-th column of C is equal to $\det A$.

$$\sum_{i=1}^{n} a_{ij} c_{ij} = \det A.$$

$j \neq k$ $\quad$ $k < j$

- But inner product of the j-th column of A and the k-th column of C is equal to zero for $1 \leqslant j \neq k \leqslant n$.

k ستون jام

$$B_j = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} & \cdots & a_{1k} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nk} & \cdots & a_{nk} & \cdots & a_{nn} \end{bmatrix}$$

$0 = \det B_j = \langle \underbrace{\text{ستون jام}}_{B_j,\ a_{ik}}, \underbrace{\text{ستون jام}}_{C,\ c_{ij}} \rangle = \sum_{i=1}^{n} a_{ik} c_{ij}$

So

$$0 = \det B_j = \sum_{i=1}^{n} a_{ik} c_{ij}.$$

Adjoint A

$\langle C$ ستون jام $, A$ ستون jام $\rangle = \det A$

- We obtain $\quad C^T$ سطر jام $\quad$ ستون j

سطر

$$C^T A = \begin{bmatrix} c_{11} & \cdots & c_{n1} \\ \vdots & \cdots & \vdots \\ c_{1n} & \cdots & c_{nn} \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \cdots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} \det A & & \\ & \det A & \\ & & \det A \end{bmatrix}$$

$\operatorname{adj} A := C^T$

$C^T A = (\det A) I$

$\langle C$ ستون iام $,$ ستون jام $A \rangle = \det A$

- Thus,

$$C^T A = (\det A)\, I.$$

- The matrix C^T is called the adjoint of A and is denoted by adj A. So,

$$(\text{adj } A)A = (\det A)I$$

$W(A) = \{X \mid XA = AX\}$

adj A

- By $(\operatorname{adj} A)A = (\det A)I$, we have
  1. $(\operatorname{adj} A^T)A^T = (\det A^T)I = (\det A)I$.
  2. $(\operatorname{adj} A)_{ij} = (-1)^{i-j} \det A(j|i)$.
- It is easy to check that
$$\underline{(\operatorname{adj} A^T) = (\operatorname{adj} A)^T}.$$

$a_{ij} = (-1)^{i+j} \det A(i|j)$

$(\operatorname{adj} A^T) A^T = (\operatorname{adj} A) A$

$(\operatorname{adj} A^T)_{ij} = (-1)^{i+j} \det A^T(j|i)$

$= (-1)^{i+j} \det A(i|j)$

$= (\operatorname{adj} A)_{ji}$

$= (\operatorname{adj} A)^T_{ij}$.

Computation of A^{-1}

$(\operatorname{adj} A)A = A \operatorname{adj} A$

- If $A \in M_n(\mathbb{R})$ is invertible, then
$$A\left(\frac{\operatorname{adj} A}{\det A}\right) = \left(\frac{\operatorname{adj} A}{\det A}\right)A = I.$$
- Thus,
$$A^{-1} = \left(\frac{\operatorname{adj} A}{\det A}\right).$$

$\det A \neq 0$

$A^{-1} = \left(\frac{\operatorname{adj} A}{\det A}\right)$

$AA^{-1} = A^{-1}A$

$\left(\frac{\operatorname{adj} A}{\det A}\right)A = A\left(\frac{\operatorname{adj} A}{\det A}\right) = I$

$(\operatorname{adj} A)A = A \operatorname{adj} A$

Cramer's rule

- Let $A \in M_n(\mathbb{R})$ be invertible.
- The solution of $Ax = b$ is $x = A^{-1}b$: just $C^T b$ divided by $\det A$.
- **Cramer's rule:** The jth component of $x = A^{-1}b$ is the ratio

$$x_j = \frac{\det B_j}{\det A},$$

where

$$B_j = \begin{bmatrix} a_{11} & a_{12} & \cdots & b_1 & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & b_n & \cdots & a_{nn} \end{bmatrix}$$

Determinant and linear functions

- Let T be a linear function on a finite dimensional linear space V. Then the determinant of T is defined the determinant of its representation matrix. That means if B is a basis for V, then

$$\det T = \det [T]_B.$$

لزوماً مستقل از انتخاب پایه B

- Note that the definition is well defined (Why?).

$\dim V = n$

$T : V \longrightarrow V$

$\det T := \det [T]_B$

خوش تعریف

$[T]_B = P [T]_{B'} P^{-1}$

$\det [T]_B =$

$\det P \det [T]_{B'} \det P^{-1}$

$= \det [T]_{B'}$

Eigenvalues and Eigenvectors

Differential equations

$u \in \mathbb{R}^n$ $u(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{bmatrix}$

- Consider a single equation

$$\frac{du}{dt} = au \quad \text{with} \quad u_0 = u(0).$$

$a \in \mathbb{R}$

$\cancel{a = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}}$

$A \in M_n(\mathbb{R})$

- The solution to this equation is

$$u(t) = e^{at} u_0$$

$\frac{du}{dt} = Au$

$\frac{du}{dt} = au(t)$ $a \in \mathbb{R}$

$a \in \mathbb{R}$

$u : \mathbb{R} \to \mathbb{R}$

$u(0) = u_0$

$u(t) = e^{at} u_0$

du

$\frac{du(t)}{dt} = \begin{bmatrix} \frac{du_1(t)}{dt} \\ \vdots \\ \frac{du_n(t)}{dt} \end{bmatrix}$

Differential equation systems

- Consider the differential equation system

$$\begin{cases} \frac{du_1}{dt} = 4u_1 - 5u_2 \\ \frac{du_2}{dt} = 2u_1 - 3u_2 \end{cases} \quad \text{with} \quad \begin{cases} x_1 = u_1(0) \\ x_2 = u_2(0) \end{cases}$$

- Similar to a single equation, let

$$u_1(t) = e^{\lambda t} x_1$$
$$u_2(t) = e^{\lambda t} x_2$$

$e^{\lambda t} \neq 0$

$$\frac{dy}{dt} = \begin{bmatrix} \frac{du_1}{dt} \\ \frac{du_2}{dt} \end{bmatrix} = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_2 \end{bmatrix}$$

$$\begin{cases} \lambda e^{\lambda t} x_1 = ۴ e^{\lambda t} x_1 - ۵ e^{\lambda t} x_2 \\ \lambda e^{\lambda t} x_2 = 2 e^{\lambda t} x_1 - 3 e^{\lambda t} x_2 \end{cases}$$

$\Downarrow$

$$\lambda x_1 = ۴ x_1 - ۵ x_2$$
$$\lambda x_2 = ۲ x_1 - ۳ x_2$$

Eigenvalue problem

- By substituting and cancellation of $e^{\lambda t}$, we obtain:

$$4x_1 - 5x_2 = \lambda x_1$$
$$2x_1 - 3x_2 = \lambda x_2$$

or

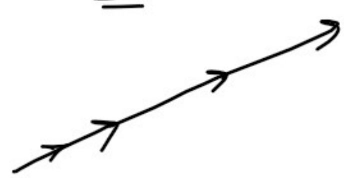
$$\underbrace{\begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} = \lambda \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x}.$$

- Eigenvalue equation:

$$Ax = \lambda x.$$

$$u_1 = e^{\lambda t} x_1 = e^{\lambda t} u_1(0)$$
$$u_2 = e^{\lambda t} x_2 = e^{\lambda t} u_2(0)$$

$$Ax = \lambda x$$



$$Ax = \lambda x$$
$$(A - I\lambda)x = 0$$

$\lambda : \quad x \in N(A - \lambda I)$

$x \neq 0 \quad N(A - \lambda I) \neq \{0\}$

تکین $A - \lambda I$

$\Updownarrow$

$\det(A - \lambda I) = 0$

$A - \lambda I = \begin{bmatrix} ۴-\lambda & -۵ \end{bmatrix}$

Eigenvectors and eigenvectors

$A v = \lambda v \qquad A v = \lambda' v$

$A v = (\lambda - \lambda')$ λ

Definition

An eigenvector or characteristic vector v for a square matrix A is a nonzero vector that changes at most by a scalar factor when that this matrix is applied to it, i.e., $Av = \lambda v$. The corresponding eigenvalue λ is denoted by as an eigenvalue of A.

λ

$v \, , \; Tv$

Definition

Suppose that $T : V \to V$ is a linear function on V. Then an eigenvector or characteristic vector v for the linear function T is a

Definition

Suppose that $T : V \to V$ is a linear function on V. Then an eigenvector or characteristic vector v for the linear function T is a nonzero vector that changes at most by a scalar factor when that this linear function is applied to it, i.e., $Tv = \lambda v$. The corresponding eigenvalue λ is denoted by as an eigenvalue of T.

The steps in solving $Ax = \lambda x$

$\mathbb{R}^n \xrightarrow{T} \mathbb{R}^n$

$Ax = \lambda$

- **Compute the determinant of $(A - \lambda I)$.** $\det(\lambda I - A)$
- **Find the roots of this polynomial.** The n roots are the eigenvalues of A. $(\lambda I - A)x = 0$
- **For each eigenvalue solve the equation $(A - \lambda I)x = 0$.**
- The vector x is in the nullspace of $A - \lambda I$. $x \neq 0$
- The number λ is chosen so that $A - \lambda I$ has a nullspace.
- In short, $A - \lambda I$ must be singular.

$$0 = \det(A - \lambda I) = (-1)^n \det(\lambda I - A)$$

Sum and product of eigenvalues

$f(\lambda) = \det(\lambda I - A) = 0 \Rightarrow$

$f(\lambda) = 1 \cdot \lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_1 \lambda + \alpha_0$

$\alpha_0 = (-1)^n \det A$

- The sum of the n eigenvalues equals the sum of the n diagonal entries:

$$\text{Trace of } A = \lambda_1 + \cdots + \lambda_n = a_{11} + \cdots + a_{nn}.$$

$-\alpha_{n-1} = \text{Trace of } A$

$(\lambda - a_{11})(\lambda - a_{22}) \cdots (\lambda - a_{nn})$

- the product of the n eigenvalues equals the determinant of A, that is

$f(\lambda) = (\lambda - \lambda_1) \cdots (\lambda - \lambda_n) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_0$, $\alpha_{n-1} = -(\lambda_1 + \cdots + \lambda_n)$

$$\lambda_1 \times \cdots \times \lambda_n = \det A.$$

$\alpha_0 = (-1)^n \lambda_1 \cdots \lambda_n$

- Note that, some (or even all) of eigenvalues may be complex numbers.

$A - \lambda I = \begin{bmatrix} ۴-\lambda & -۵ \\ ۲ & -۳-\lambda \end{bmatrix}$

$\det(A - \lambda I) = (۴-\lambda)(-۳-\lambda) + ۱۰$

$(\lambda - 4)(\lambda + 3) + 10$

$\Rightarrow \lambda^2 - \lambda - 2 = 0$

$\Rightarrow \lambda = ۲$ و $\lambda = -1$

$Av = \lambda v$

$\lambda = -1 \Rightarrow \underbrace{\begin{bmatrix} ۵ & -۵ \\ ۲ & -۲ \end{bmatrix}}_{A - \lambda I} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = 0$

$\Rightarrow \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$u(t) = e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = ۲ \Rightarrow \begin{bmatrix} ۲ & -۵ \\ ۲ & -۵ \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = 0$

$\Rightarrow \begin{bmatrix} ۵ \\ ۲ \end{bmatrix}$

$u(t) = e^{۲t}\begin{bmatrix} ۵ \\ ۲ \end{bmatrix}$

$\underbrace{\left\{ c_1 e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{۲t}\begin{bmatrix} ۵ \\ ۲ \end{bmatrix} \right\} c_1, c_2 \in \mathbb{R}}$: $u(t)$ صورت کلی

$\frac{du}{dt} = Au$

Example

rank $P = r$ $\qquad P^2 = P \qquad Px = 0x$

- Find the eigenvalues of the projection matrix $P \in M_n(\mathbb{R})$.

$Px = 0$

$Px = \lambda x \quad x \neq 0 \qquad \lambda I - P =$

$x \in N(P) = \{x \mid Px = 0\}$

$Px = P^2 x = \lambda Px = \lambda^2 x \Rightarrow \lambda^2 x = \lambda x$

λx

$\Rightarrow (\lambda^2 - \lambda)x = 0 \quad x \neq 0$

$\Rightarrow \lambda^2 - \lambda = 0 \Rightarrow \lambda = 1 \text{ , } 0$

$f(\lambda) = \det(\lambda I - P) = (\lambda - 0)^{n-r}(\lambda - 1)^r$

$\det(A - \lambda I) = 0 \rightarrow (-1)^n \lambda^n$

$\lambda I - A$

$\det(\lambda I - A) = 0$

$\lambda I - A = \begin{bmatrix} \lambda - a_{11} & -a_{12} & \cdots & -a_{1n} \\ & \ddots & & \\ -a_{ij} & & \ddots & \lambda - a_{nn} \end{bmatrix}$

$\det(\lambda I - A) = 1\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0$

$(\lambda - a_{11}) \cdots (\lambda - a_{nn})$

Diagonal matrices

- Find the eigenvalues and eigenvectors for

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

$\det(\lambda I - A) = (\lambda - 3)(\lambda - 2)$

$\lambda_i = 3, 2$

$f(\lambda) = \det(\lambda I - P)$

$= \lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_0$

$\lambda(\lambda - 1) = 0$

$f(\lambda) = \lambda^\beta (\lambda - 1)^\gamma$

$\beta + \gamma = n$

$1 \quad \beta$

γ

Example

- Find the eigenvalues and eigenvectors for

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\lambda = 1, 0$

$Px = 0x$

$W(\lambda) = \{x \mid Px = \lambda x\}$

$\{x \mid Px = 0\}$

$= N(P) \quad n - \text{rank} P = n - r$

$\dim \mathcal{N}(P)$ $= n -$ rank P (r)

$\lambda = 1$, 0

Example

- Find the eigenvalues and eigenvectors for diagonal matrices.

$$\begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix} \begin{bmatrix} \vdots \\ 1 \\ \vdots \end{bmatrix} = d_i \begin{bmatrix} \\ \\ \end{bmatrix}$$

e_i e_i

$\lambda = d_1 \to d_n$

$= e_1, \ldots, e_n$

Diagonalization of a Matrix

$f(\lambda) = \det(\lambda I - \cancel{P}) \overset{?}{=} \det(\lambda I - D) = \det(\lambda I - S^{-1}PS)$ ($1, 0$)

$\det(\lambda S^{-1}S - S^{-1}PS)$

$\det(S^{-1}(\lambda I - P)S)$

$\cancel{\det S^{-1}}\ \det(\lambda I - P)$

$1\ \cancel{\det S}$

$= \det(\lambda I - P)$

- **Example.** The eigenvector matrix of the projection $P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ is $S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, and we have

$= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$$S^{-1}PS = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

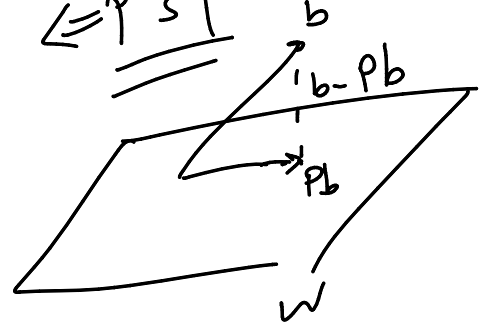
$\boxed{P = \frac{1}{2} J}$

- The eigenvector matrix S converts A into its eigenvalue matrix which is diagonal.

$S^{-1}PS = D$

$\Leftarrow P^2 = P$

$\Leftarrow P^2 = P$



$b - Pb \in W^{\perp}$

$w_1, \ldots, w_m$

$\langle b - Pb, w_i \rangle = 0$

$W(\lambda) = \{x \mid Ax = \lambda x\}$

۱- زیرفضات

$\lambda_1, \ldots, \lambda_m$

$\det(\lambda I - A) = 0$

$P^2 - P$

$f(x) = x(x-1) = 0$

P قطری شدنی است.

$f(\lambda) = (x-\lambda_1)^{n_1} \cdots (x-\lambda_k)^{n_k}$

$\dim W(\lambda_i) = n_i$

$\sum_{i=1}^{k} \dim W_i = n$

Thank You!