Lecture26

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Lecture26

Linear Algebra

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Review: Diagonalizable Linear Function

Theorem

Let $T : V \to V$ be a linear function on a finite dimensional linear space V, and T has different eigenvalues $\lambda_1, \ldots, \lambda_k$. Suppose that W_i is the null space of $T - \lambda_i I$ for each $1 \leqslant i \leqslant k$. Then the following statements are equivalent:

$W_i = N(T - \lambda_i I)$

i. *T is diagonalizable.*

$\dim W_i = n_i$

ii. *The characteristic polynomial of T is $f(x) = (x - \lambda_1)^{n_1} \cdots (x - \lambda_k)^{n_k}$, where $n_i = \dim W_i$.*

iii. $\sum_{i=1}^{k} \dim W_i = \dim V$.

$B = \cup B_i = B_1 \cup \cdots B_k$

$V = \underbrace{W_1}_{B_1} \oplus \cdots \oplus \underbrace{W_k}_{B_k}$

$W_i \cap W_j = \{0\}$

$T W_i \subseteq W_i$

$[T]_B = \begin{bmatrix} \lambda_1 I_{n_1} & & \\ & \ddots & \\ & & \lambda_k I_{n_k} \end{bmatrix}$

$B_1 = \{v_{11}, \ldots, v_{1n_1}\}$ $\dim W_1 = n_1$

$V = W_1 \oplus \cdots \oplus W_k$

$$[T]_B = \begin{bmatrix} \boxed{\lambda_1 I_{n_1}} & & 0 \\ & \ddots & \\ 0 & & \lambda_k I_{n_k} \end{bmatrix}$$

$T|_{W_1}$

$T v_{11} = \lambda_1 v_{11}$

$T v_{12} = \lambda_1 v_{12}$

Primary Decomposition Theorem

Theorem

Let T be a linear function over a finite dimensional linear space V whose minimal polynomial factorizes as

$$p(x) = p_1^{r_1}(x) \cdots p_k^{r_k}(x),$$

where the p_i's are monic and mutually coprime (i.e. have no nontrivial common factors). Let $W_i = N\left(p_i^{r_i}(T)\right)$ for each $1 \leqslant i \leqslant k$. Then

1. $V = \underbrace{W_1 \oplus \cdots \oplus W_k}$.
2. *For each $1 \leqslant i \leqslant k$, $T(W_i) \subseteq W_i$.*
3. *The minimal polynomial of $T_i = T\restriction_{W_i}$ is $p_i(x)$.*

W_1

$$[T]_B = \begin{bmatrix} [T]_{B_1} & & \\ & \ddots & \\ & & [T]_{B_k} \end{bmatrix}$$

W_k

Minimal Polynomials for Vectors

Lemma

Suppose that T is a linear function on linear space V over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Then

1. *Each vector $v \in V$ has a minimal polynomial with respect to T.*
2. *The minimal polynomial for v with respect to T is unique.*
3. *Take a vector $v \in V$ and assume that $f(x)$ is a polynomial with coefficients in $\mathbb{F}$ such that $f(T)v = 0$, then $p(x) \mid f(x)$ where $p(x)$ is the minimal polynomial for v with respect to T.*
4. *Let $f(x)$ and $g(x)$ be two coprime polynomials. Then*

$$v \in N(f(T)) \cap N(g(T)) = \{0\}.$$

$p(T)v = 0$

$(f(x), g(x)) = 1$

$1 \mid f(x)$

$1 \mid g(x)$

$p(x) = 1$

$v = 0$

Review: Lemma

Lemma

Let T, S be two linear functions on linear space V over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ such that $T \circ S = S \circ T$ and $N(T) \cap N(S) = \{0\}$. Then

1. $N(T) + N(S) \subseteq N(T \circ S)$.
2. *If V is finite dimensional, then* $\dim N(T \circ S) \leqslant \dim N(T) + \dim N(S)$ *and consequently,* $N(T \circ S) = N(T) \oplus N(S)$.

$$N(AB) = N(A) \oplus N(B)$$

$k=2$ با استقرا ✓

$k-1$: درست

Review: Lemma.

Lemma

Let $T_1, \dots, T_k$ be linear functions on a finite dimensional linear space V over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ such that

- $T_i \circ T_j = T_j \circ T_i$
- $N(T_i) \cap N(T_j) = \{0\}$

for each $1 \leqslant i < j \leqslant k$. Then $N(T_1 \circ \cdots \circ T_k) = N(T_1) \oplus \cdots \oplus N(T_k)$.

$N(\underbrace{T_1 \circ \cdots \circ T_{k-1}}_{S}) = \bigoplus_{i=1}^{k-1} N(T_i)$

$N(S T_k) = N(S) \oplus N(T_k)$

$- \ S \circ T_k = T_k \circ S$

$- \ N(S) \cap N(T_k) \overset{?}{=} \{0\}$

$v \in N(T_1 \circ \cdots \circ T_{k-1}) \cap N(T_k) = \{0\}$

$T_1 \circ \cdots \circ T_{k-1} v = 0 \Rightarrow T_1(\underline{T_2 \circ \cdots \circ T_{k-1} v}) = 0$

$T_k v = 0$

$T_2 \cdots T_{k-1} v \in N(T_1) \cap N(T_k) = \{0\}$

$T_1 \circ \cdots \circ T_{k-1} v = 0 \Rightarrow T_1(T_2 \circ \cdots \circ T_{k-1} v) = 0$

$T_k v = 0$

$T_2 \cdots T_{k-1}(\overset{0}{T_k v}) = 0 \Rightarrow T_k(T_2 \cdots T_{k-1} v) = 0$

$T_2 \cdots T_{k-1} v \in N(T_1) \cap N(T_k) = \{0\}$

$T_2 \cdots T_{k-1} v = 0$

$\vdots$

$T_{k-1} v = 0$

$v = 0 \Leftarrow \begin{cases} v \in N(T_{k-1}) \\ v \in N(T_k) \end{cases}$

Proof.

Lemma.

$T_i \circ T_j = T_j \circ T_i$, $N(T_i) \cap N(T_j) = \{0\}$

$N(T_1 \circ \cdots \circ T_k) = N(T_1) \oplus \cdots \oplus N(T_k)$

Lemma

Let T be a linear functions on a finite dimensional linear space V over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

1. *Let $f(x)$ be a polynomial with coefficient in $\mathbb{F}$ and $f(x) = f_1(x)^{n_1} \cdots f_k(x)^{n_k}$ such that $f_1, \ldots, f_k$ mutually coprime. Then*
$$N(f(T)) = N(f_1(T)^{n_1}) \oplus \cdots \oplus N(f_k(T)^{n_k}).$$
$N(p(T)) = V$
2. *If the minimal polynomial T is factorized as $p(x) = p_1(x)^{n_1} \cdots p_k(x)^{n_k}$ where $p_1, \ldots, p_k$ are mutually coprime, then*
$$V = N(p_1(T)^{n_1}) \oplus \cdots \oplus N(p_k(T)^{n_k}).$$
$p(T)v = 0$

$T_i = f_i(T)^{n_i}$

$T_i \circ T_j = f_i(T)^{n_i} f_j(T)^{n_j}$

$\left(\sum_{k=1}^{t} a_k T^k\right)^{n_i} \left(\sum_{\ell=1}^{s} b_\ell T^\ell\right)^{n_j}$

$= T_j \circ T_i$

$\{0\} = N(f_i(T)^{n_i}) \cap N(f_j(T)^{n_j})$

$N(f(T)) \cap N(g(T)) = \{0\}$ $(f(x), g(x)) = 1$

Proof. $T: V \longrightarrow V$ $\qquad p(T)=0 \qquad p(T)v=0$, $v\in V$

$$p(x) = p_1(x)^{r_1} \cdots p_k(x)^{r_k}$$

$$V = N(p(T)) = N(p_1(T)^{r_1}) \oplus \cdots \oplus N(p_k(T)^{r_k})$$

$$V = W_1 \oplus \cdots \oplus W_k$$

Proof. $P_{T_i}(x) = f_1^{s_1}(x) \cdots f_l^{s_l}(x) \qquad \exists j\; s_j < r_{jl}$

$$P_{T_i}(x) \mid p_i^{r_i}(x) = f_1^{r_i}(x) \cdots f_l^{r_i}(x)$$

$$p(x) = p_1^{r_1}(x) \cdot p_i^{r_i}(x) \, p_k^{r_k}(x) \qquad \deg p = \sum_{i=1}^{k} r_i$$

حالت کلی: $p_i(x)$ تجزیه‌ناپذیر است. پس $P_{T_i}(x) = p_i^{s}(x)$.

برهان خلف. فرض کنیم $s < r_i$ باشد.

$$* \; g(x) = p_1^{r_1}(x) \cdots p_{i-1}^{r_{i-1}}(x)\, p_i^{s}(x) \cdots p_k^{r_k}(x)$$

به ازای $v \in V$ ، $g(T)v = 0$ تناقض

$\forall\, v\in V$

$v = w_1 + \cdots + w_k \qquad w_i \in W_i$

$p_i^{r_i}(w_i) = 0 \qquad \forall\, 1\le i\le k,\ i\ne s$

$$g(T)(v) = \sum_{i=1}^{k} g(w_i)$$

$p_1^{r_1}(T) \cdots p_k^{r_k}(T) w_1 =$

$p_2^{r_2}(T) \cdots p_k^{r_k}(T)\, p_1^{r_1}(T) w_1 = 0$

$g(T) w_i = p_1^{r_1}(T) \cdots p_{i-1}^{r_{i-1}}(T)\, p_{i+1}^{r_{i+1}}(T) \cdots p_k^{r_k}(T)\, p_i^{s}(T) w_i = 0$

Primary Decomposition Theorem

$T: \overbrace{W_1 \oplus \cdots \oplus W_k}^{V} \longrightarrow V$

$T_i = T|_{W_i} : W_i \longrightarrow T$

Theorem

Let T be a linear function over a finite dimensional linear space V whose minimal polynomial factorizes as

$$p(x) = p_1^{r_1}(x) \cdots p_k^{r_k}(x),$$

$W_i = N(T - \lambda_i I)$, $x - \lambda_i$

where the p_i's are monic and mutually coprime (i.e. have no nontrivial common factors). Let $W_i = N\left(p_i^{r_i}(T)\right)$ for each $1 \leqslant i \leqslant k$. Then

$(x-c)^{1}$

1. $V = W_1 \oplus \cdots \oplus W_k$.
2. *For each $1 \leqslant i \leqslant k$, $T(W_i) \subseteq W_i$.*
3. *The minimal polynomial of $T_i = T\restriction_{W_i}$ is $p_i^{r_i}(x)$.*

$P_{T_i}(x) = p_i^{s}(x)$

$v \in W_i \Rightarrow p_i^{r_i}(T)v = 0$

$T(v) \overset{?}{\in} W_i$

$p_i^{r_i}(T)\,(T(v)) = T\left(p_i^{r_i}(T)v\right)$

$= T(0) = 0$

$s < r_i$

$T(W_i) \subseteq W_i$

2. For each $1 \leqslant i \leqslant k$, $T(W_i) \subseteq W_i$.
3. The minimal polynomial of $T_i = T\restriction_{W_i}$ is $p_i^{r_i}(x)$.

$p_{T_i}(x) = p_i(x)$ $s < r_i$

$v \in W_i$ $p_i^{r_i}(T)v = 0$ $p_{T_i}(x) \mid p_i^{r_i}(x)$

$T(W_i) \subseteq W_i$

$p_i(x) = f_1(x) \cdots f_k(x)$

Jordan Form

Suppose that T is a linear function on V with the characteristic polynomial is

$$f(x) = (x - \lambda_1)^{d_1} \dots (x - \lambda_k)^{d_k}$$

where $\lambda_1, \dots, \lambda_k$ are distinct elements and $d_i \geqslant 1$.
Then the minimal polynomial for T will be

$$p(x) = (x - \lambda_1)^{r_1} \dots (x - \lambda_k)^{r_k}$$

where $1 \leqslant r_i \leqslant d_i$ based on the Cayley–Hamilton theorem.
If W_i is the null space of $(T - \lambda_i I)^{r_i}$, then the primary decomposition theorem tells us that

$$V = W_1 \oplus \dots \oplus W_k$$

such that the linear function $T_i = T\restriction_{W_i} : W_i \to W_i$ has minimal polynomial $(x - \lambda_i)^{r_i}$.

$Q^{-1}AQ = \begin{bmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_k \end{bmatrix}$ $\frac{n^2-n}{2}$

$W_i = N((T - \lambda_i I)^{r_i})$

$W_i = N(T - \lambda_i I)$

Jordan Form

$$V = \underbrace{W_1}_{B_1} \oplus \dots \oplus \underbrace{W_k}_{B_2}$$

W_i

$B_i = \{v_{i1}, \dots, v_{in_i}\}$

$\dim W_i = n_i$

Suppose that B_i is a basis for W_i. It has been proved that $B = \bigcup_{i=1}^{k} B_i$ is a basis for V. Based on primary decomposition theorem,

$Tv_{i} \in W_i$

$$T(W_i) \subseteq W_i.$$

$Tv_{1i} = \sum_{j=1}^{n_i} d_j v_{ij}$

Thus

$$[T]_B = \begin{bmatrix} [T_1]_{B_1} & & 0 \\ & \ddots & \\ 0 & & [T_k]_{B_k} \end{bmatrix}.$$

$d_1, d_2, \dots, d_{n_i}$

B_1 v_i B_i B_k

B_k

Jordan Form

$$V = W_1 \oplus \cdots \oplus W_k$$

and

نسبت به B_1

B_1

$$[T]_B = \begin{bmatrix} \boxed{[T_1]_{B_1}} & & \bigcirc \\ & \ddots & \\ \bigcirc & & \boxed{[T_k]_{B_k}} \end{bmatrix}.$$

$T|_{W_i}$

جردن

- Let N_i be the linear function on W_i defined by $N_i = T - \lambda_i I$.
- Then N_i is nilpotent and has minimal polynomial x^{r_i}.
- Thus, T on W_i acts as N_i plus the scalar λ_i times the identity function I.
- Suppose we choose a basis for the subspace W_i and then find the representation matrix of N_i on W_i.

$$W_i = N\big(\underbrace{(T - \lambda_i I)}_{N_i}{}^{r_i} \big)$$

$$T_i = T|_{W_i}$$

$$N_i = T - \lambda_i I$$

$$N_i^{r_i} = 0$$

$$v \in W_i \quad (T - \lambda_i I)^{r_i} v = 0$$

$$N_i^{r_i} v = 0$$

$$\forall v \in W_i \quad N_i^{r_i} = 0$$

Nilpotent matrices and Nilpotent linear functions

$$N_i^{r_i} = 0$$

Definition

A square matrix A is called nilpotent matrix with degree non-negative integer k if A^k is the zero matrix and A^r is the non-zero matrix for each r, $1 \leqslant r \leqslant k$.

$$r < k \qquad A^{k} = 0$$

$$A^r \neq 0$$

Definition

A be a linear function T on V is called nilpotent linear function with degree non-negative integer k if T^k is the zero linear function and T^r is the non-zero one for each r, $1 \leqslant r \leqslant k$.

Example

Let $A \in M_3(\mathbb{R})$ be the following matrix:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

$A^2 \neq 0 \quad , A^3 = 0$

Then

$$A^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq \mathbf{O}$$

The third power of A is

$$A^3 = A^2 A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{O}$$

Nilpotent matrices

$A^k = 0 \implies \lambda = 0$

$A^k = 0 \impliedby \lambda = 0$

$x \neq 0 \quad \boxed{Ax = \lambda x \iff A^k x = \lambda^k x}$

Lemma

Let the $n \times n$ matrix A is nilpotent with degree k, then

1) *The matrix A is nilpotent if and only if all the eigenvalues of A is zero.*
2) *The matrix $A$is nilpotent if and only if $A^n = O$.*

$\lambda = 0$

$A^k = 0$

$gh \quad p(x) = x^\alpha$

$A^k = 0 \quad p(x) = x^k \quad x$

$f(x) = x^n$

Nilpotent matrices

$B = \{v, Tv, \ldots, T^{n-1}v\}$ $\quad Tv =$

$T(Tv) = T^2 v$

Lemma

If a linear function T on V with $\dim V = n < \infty$ is nilpotent with degree n, then there is a basis for V such that

$$[T]_B = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & 0 & 0 \\ 0 & 0 & \ddots & 0 & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

$T(T^{n-1}v) = 0$

$T^n v = 0$

$T^n = 0 \quad , \quad T^{n-1} \neq 0 \Rightarrow T^{n-1}v \neq 0$

$\vdots$

$T \neq 0 \Rightarrow Tv \neq 0$

$v_1, \ldots, v_{n-1}$

$\Downarrow$

v

Nilpotent matrices

Lemma

Let T is a linear function on V such that $B = \{v, Tv, \ldots, T^{n-1}v\}$ is a basis for V where $0 \neq v \in V$. Then

$$[T]_B = \begin{bmatrix} 0 & 0 & \cdots & 0 & -c_0 \\ 1 & 0 & \cdots & 0 & -c_1 \\ 0 & 1 & \ddots & 0 & -c_2 \\ 0 & 0 & \ddots & 0 & \vdots \\ 0 & 0 & \cdots & 1 & -c_{n-1} \end{bmatrix}$$

where $p(x) = x^n + c_{n-1}x^{n-1} + \ldots + c_1x^1 + c_0$ is the minimal polynomial for T.

The smallest T-invariant subspace containing v

- Assume V is finite-dimensional linear space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and T is a fixed (but arbitrary) linear function on V.
- If W is any subspace of V which is invariant under T and contains v, then W must also contain the vector $T(v)$; hence W must contain $T(Tv) = T^2v$, $T(T^2v) = T^3v$, etc. In other words W must contain $g(T)v$, for every polynomial $g(x)$ over $\mathbb{F}$. The set of all polynomial $g(x)$ over $\mathbb{F}$ is denoted by $\mathbb{F}[x]$
- Let $Z(v,T) = \{g(T)v \mid g(x) \in \mathbb{F}[x].\}$
- $Z(v,T)$ is a subspace of V and it is the smallest T-invariant subspace which contains v.

T-cyclic subspace generated by v

Definition

If v is any vector in V, the subspace $Z(v,T)$ is called the T-**cyclic subspace generated**. If $Z(v,T) = V$, then v is called a cyclic vector for T.

For any T:

1. The $T-$cyclic subspace generated by the zero vector is the zero subspace.
2. The space $Z(v,T)$ is one-dimensional if and only if v is an eigenvalue vector for T.
3. Thus, we shall be interested in linear relations:
$$c_0v + c_1Tv + \cdots, c_kT^kv = 0.$$
between the vectors T^jv, that is we shall be interested in the polynomials
$$c_0 + c_1x + \cdots, c_kx^k = 0$$
which have the property that $g(T)v = 0$.

The dimension of T-cyclic subspace generated by v

Theorem

Assume that T is a linear space on a linear space V. Let v be any non-zero vector in V and let $p_v(x)$ is the minimal polynomial for v respect to T.

1. $\dim Z(v,T) = deg p_v(x)$.
2. *If U is the linear function on $Z(v,T)$ induced by T, then the minimal polynomial for U is $p_v(x)$.*

Minimal and characteristic polynomials of a cyclic vector

Theorem

T has a cyclic vector if and only if the minimal and characteristic polynomials for T are identical.

Cyclic Decomposition Theorem

Theorem

Let T be a linear function on a finite-dimensional vector space V. There exist non-zero vectors $v_1, \ldots, v_k \in V$ with minimal polynomial $p_{v_1}, \ldots, p_{v_k}$ such that

(i) $V = Z(v_1, T) \oplus \cdots \oplus Z(v_k, T)$.

(ii) $p_{v_i} \mid p_{v_{i-1}}$ *for each* $i \geqslant 2$.

(iii) *Furthermore, the integer r and the minimal polynomial $p_{v_1}, \ldots, p_{v_k}$ are uniquely determined by (i), (ii).*

. . .

Thank You!