Linear Algebra

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Linear Spaces

The heart of linear algebra

Linear Spaces

$$\mathbb{R}^n = \left\{ \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \mid v_i \in \mathbb{R} \ \text{for} \ 0 \leq i \leq n \right\}.$$

a set of vectors together with rules for vector addition and multiplication by real numbers such that

1. **commutativity** $u + v = v + u$ for all $u, v \in \mathbb{R}^n$;
2. **associativity** $(u + v) + w = u + (v + w)$ and $(ab)v = a(bv)$ for all $u, v, w \in \mathbb{R}^n$ and all $a, b \in \mathbb{R}$;
3. **additive identity** there exists an element $0 \in \mathbb{R}^n$ such that $v + 0 = v$ for all $v \in \mathbb{R}^n$;
4. **additive inverse** for every $v \in \mathbb{R}^n$, there exists $w \in \mathbb{R}^n$ such that $v + w = 0$;
5. **multiplicative identity** $1v = v$ for all $v \in \mathbb{R}^n$;
6. **distributive properties** $a(u + v) = au + av$ and $(a + b)u = au + bu$ for all $a, b \in \mathbb{R}$ and all $u, v \in \mathbb{R}^n$.

Linear Spaces

$$M_{m,n}(\mathbb{R}) = \left\{ \begin{bmatrix} v_{11} & \cdots & v_{1n} \\ \vdots & \dots & \vdots \\ v_{m1} & \cdots & v_{mn} \end{bmatrix} \mid v_{ij} \in \mathbb{R} \right\},$$

$(M_{m,n}(\mathbb{R}), +, \cdot)$ has the following properties:

1. **commutativity** $u + v = v + u$ for all $u, v \in M_{m,n}$;
2. **associativity** $(u + v) + w = u + (v + w)$ and $(ab)v = a(bv)$ for all $u, v, w \in M_{m,n}$ and all $a, b \in \mathbb{R}$;
3. **additive identity** there exists an element $0 \in M_{m,n}$ such that $v + 0 = v$ for all $v \in M_{m,n}$;
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Linear Spaces

The infinite-dimensional space $\mathbb{R}^\infty$ whose vectors have infinitely many components, as in $v = (1, 2, 1, 2, \ldots)$ has the following properties:

1. **commutativity** $u + v = v + u$ for all $u, v \in \mathbb{R}^\infty$;
2. **associativity** $(u + v) + w = u + (v + w)$ and $(ab)v = a(bv)$ for all $u, v, w \in \mathbb{R}^\infty$ and all $a, b \in \mathbb{R}$;
3. **additive identity** there exists an element $0 \in \mathbb{R}^\infty$ such that $v + 0 = v$ for all $v \in \mathbb{R}^\infty$;
4. **additive inverse** for every $v \in \mathbb{R}^\infty$, there exists $w \in \mathbb{R}^\infty$ such that $v + w = 0$;
5. **multiplicative identity** $1v = v$ for all $v \in \mathbb{R}^\infty$;
6. **distributive properties** $a(u + v) = au + av$ and $(a + b)u = au + bu$ for all $a, b \in \mathbb{R}$ and all $u, v \in \mathbb{R}^\infty$.

Linear Spaces

The space of functions V that consists of all functions $f : [0, 1] \to \mathbb{R}$ has the following properties :

1. **commutativity** $u + v = v + u$ for all $u, v \in V$;
2. **associativity** $(u + v) + w = u + (v + w)$ and $(ab)v = a(bv)$ for all $u, v, w \in V$ and all $a, b \in \mathbb{R}$;
3. **additive identity** there exists an element $0 \in V$ such that $v + 0 = v$ for all $v \in V$;
4. **additive inverse** for every $v \in \mathbb{R}^n$, there exists $w \in V$ such that $v + w = 0$;
5. **multiplicative identity** $1v = v$ for all $v \in V$;
6. **distributive properties** $a(u + v) = au + av$ and $(a + b)u = au + bu$ for all $a, b \in \mathbb{R}$ and all $u, v \in V$.

$$P_n(x) = \{a_n x^n + \cdots + a_1 x + a_0 \mid a_i \in \mathbb{R} \text{ for all } 0 \leq i \leq n\}$$

has the following properties:

1. **commutativity** $u + v = v + u$ for all $u, v \in P_n(x)$;
2. **associativity** $(u + v) + w = u + (v + w)$ and $(ab)v = a(bv)$ for all $u, v, w \in P_n(x)$ and all $a, b \in \mathbb{R}$;
3. **additive identity** there exists an element $0 \in P_n(x)$ such that $v + 0 = v$ for all $v \in P_n(x)$;
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Linear Spaces

Definition

A set V with an **addition** and a **scalar multiplication**, $(V, +, \cdot)$, is a linear space if it has the following properties:

1. **commutativity** $u + v = v + u$ for all $u, v \in V$;
2. **associativity** $(u + v) + w = u + (v + w)$ and $(ab)v = a(bv)$ for all $u, v, w \in V$ and all $a, b \in \mathbb{R}$;
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Remark

Why for every $v \in V$, we have

- $0 \cdot v = \mathbf{0}$ for every $v \in V$.
- $c \cdot 0 = \mathbf{0}$ for every $c \in \mathbb{R}$.
- $(-1) \cdot v = -v$ for every $v \in V$.

Linear combinations

Definition

Let V be a real linear space. An element $w \in V$ is a linear combination of $v_1, \cdots, v_m \in V$ if and only if there exit scalars $c_1, \cdots, c_m \in \mathbb{R}$ as coefficients such that

$$w = c_1 v_1 + \ldots + c_m v_m.$$

Let $A = \begin{bmatrix} 1 & 2 \\ 5 & 8 \\ 4 & 9 \end{bmatrix}$.

Consider vectors A_1 and A_2, the first and the second column of A. Let $V = \left\{ c_1 A_1 + c_2 A_2 \mid c_1, c_2 \in \mathbb{R} \right\}$.

Is V a linear space? The linear space V is called the column space of A.

Definition of Linear sub-Spaces

Definition

Let a set V is a linear space along with an **addition** on V and a **scalar multiplication** on V

$$\Big(V, +, \cdot\Big).$$

A non-empty subset $W \subseteq V$ is called a linear sub-space if it is a linear space under the addition and the scalar multiplication of V.

Note. The zero vector belongs to every subspace (Why?).

Note. The smallest subspace contains only one vector 0.

Example. Construct a subset of $\mathbb{R}^2$ that is

i. **closed under vector addition** and subtraction, but **not scalar multiplication** on $\mathbb{R}$.

Examples

Example. Construct a subset of $\mathbb{R}^2$ that is

i. **closed under vector addition** and subtraction, but **not scalar multiplication** on $\mathbb{R}$.

ii. **closed under scalar multiplication** but **not under vector addition**.

Example.

- What is the **smallest subspace** of $\mathbb{R}^2$ which contains $\boldsymbol{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{e_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$?

Example.

- What is the **smallest subspace** of $\mathbb{R}^2$ which contains $\boldsymbol{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ <u>and</u> $\boldsymbol{e_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$?

- What is the **smallest subspace** of $\mathbb{R}^3$ which contains $\boldsymbol{e_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ <u>and</u> $\boldsymbol{e_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ <u>and</u> $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$?

Linear Subspace

Theorem

A non-empty subset W of a real linear space V is a sub-space if and only if $cv_1 + v_2 \in W$ for every $v_1, v_2 \in W$ and $c \in \mathbb{R}$.

Linear Subspace

Theorem

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$$\underbrace{\text{Linear combinations stay in the subspace}}_{\text{``closed'' under addition and scalar multiplication.}}.$$

Theorem

The intersection of each family of sub-spaces of a linear space V is a subspace of V?

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Definition

Let $W_1, \ldots, W_m$ be sub-spaces of a linear space V. The sum of them is defined as

$$W_1 + \cdots + W_m = \Big\{ w_1 + \cdots + w_m \Big| w_i \in W_i \text{ for } 1 \le i \le m \Big\}.$$

- Note that $W_1 + \ldots + W_m$ is a subspace of V which contains W_i for each $1 \le i \le m$.

Thank You!