Linear Algebra

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Orthogonality

- Generalization of the linear structure (addition and scalar multiplication) of $\mathbb{R}^2$ and $\mathbb{R}^2$ leads the definition of a linear space.
- How to generalize the concepts of length and angle?
- These concepts are embedded in the concept we now investigate, **inner products**

Coordinate axes

- Every time we think of the $x - y$ plane or three-dimensional space or $\mathbb{R}^n$, the axes are there.
- They are usually perpendicular!
- The coordinate axes that the imagination constructs are practically always orthogonal.
- In choosing a basis, we tend to choose an orthogonal basis.

Inner products on real linear space

An inner product on V is a function $\langle , \rangle : V \times V \to \mathbb{R}$ such that

1. $\langle v, v \rangle \geqslant 0$ for all $v \in V$.
2. $\langle v, v \rangle = 0$ if and only if $v = 0$.
3. $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ for all $u, v, w \in V$.
4. $\langle cu, w \rangle = c\langle u, w \rangle$ for all $u, w \in V$ and $c \in \mathbb{R}$.
5. $\langle v, w \rangle = \langle w, v \rangle$.

Example

- The Euclidean inner product on $\mathbb{R}^n$:

$$\begin{cases} \langle\, , \,\rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \\ \\ \langle x, y\rangle = y^T x = y_1 x_1 + \cdots + y_n x_n. \end{cases}$$

where $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$.

Example

- The linear space $P_n(x)$ of all polynomials with coefficients in $\mathbb{R}$ and degree at most n:

$$\begin{cases} \langle\, , \,\rangle : P_n(x) \times P_n(x) \to \mathbb{R} \\ \\ \langle f, g \rangle = \int_0^1 f(x) g(x) dx \end{cases}$$

In an inner-product space:

1. In an inner-product space, we have additivity and the homogeneity in the second slot, as well as the first slot:

$$\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle .$$

$$\langle u, cv \rangle = c \langle u, v \rangle .$$

2. For each fixed $u \in V$ the function

$$\langle u, - \rangle : V \to \mathbb{R}$$

is a linear map from $V \to \mathbb{R}$.

3. For each fixed $u \in V$,

$$\langle u, 0 \rangle = \langle 0, u \rangle = 0$$

Bilinear function

- Consider a linear function $T : \mathbb{R}^n \to \mathbb{R}$. Assuming the standard basis for $\mathbb{R}^n$, we have

$$T(x) = x_1 T(e_1) + \cdots + x_n T(e_n).$$

  Let $y_i = T(e_i)$ for $1 \leqslant i \leqslant n$. Thus

$$T(x) = x_1 y_1 + \cdots + x_n y_n = y^T x.$$

- So each the linear function $T : \mathbb{R}^n \to \mathbb{R}$ is determined by a fixed vector $y \in \mathbb{R}^n$.
- Naturally, we can consider

$$\begin{cases} B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \\ B(x, y) = y^T x \end{cases}$$

  which is linear with respect to each of its variables.

Inner product as a bilinear functions

- The Euclidean inner product on $\mathbb{R}^n$, $\langle x, y\rangle$ is naturally defined as a bilinear function:

$$\begin{cases} \langle \, , \, \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \\ \\ \langle x, y\rangle = y^T x. \end{cases}$$

Inner product spaces

Definition

An inner product space is a linear space with a binary operation called an **inner product**.

- The Eucildean inner product on $\mathbb{R}^n$ with the inner product $\langle x, y \rangle = y^T x$ for each $x, y \in \mathbb{R}^n$.

- The linear space $P_n(x)$ of all polynomials with coefficients in $\mathbb{R}$ and degree at most n with an inner product

$$\begin{cases} \langle \, , \, \rangle : P_n(x) \times P_n(x) \to \mathbb{R} \\ \\ \langle f, g \rangle = \int_0^1 f(x)g(x)dx \end{cases}$$

  is an inner product space.

Norms

Definition

For $v \in V$, we define the norm of v, denoted $\|v\|$, by

$$\|v\| = \sqrt{\langle v, v \rangle}.$$

Examples

Example. With the Euclidean inner product:

$$\|x\| = \sqrt{x_1^2 + \ldots + x_n^2}.$$

where $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$.

Example. With inner product on P_n:

$$\|f(x)\| = \sqrt{\int_0^1 f(x)^2 dx}$$

Orthonormal vectors in V

Definition

Two vectors $u, v \in V$ are said to be orthogonal if $\langle u, v \rangle = 0$.

Useful fact

If nonzero vectors $v_1, \ldots, v_n$ are mutually orthogonal (every vector is perpendicular to every other), then those vectors are linearly independent.

Why?

Cauchy-Schwarz Inequality

Proposition

Let V be an inner product and $u, v \in V$. Then

$$| \langle u, v \rangle | \leqslant \|u\| \|v\|$$

Triangle Inequality

Proposition

Let V be an inner product. If $u, v \in V$, then

$$\|u + v\| \leqslant \|u\| + \|v\|$$

Thank You!