Linear Algebra

Samira Hossein Ghorban
s.hosseinghorban@ipm.ir

Fall, 2021

Review: Inner products on real linear space

An inner product on V is a function $\langle , \rangle : V \times V \to \mathbb{R}$ such that

1. $\langle v, v \rangle \geqslant 0$ for all $v \in V$.
2. $\langle v, v \rangle = 0$ if and only if $v = 0$.
3. $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ for all $u, v, w \in V$.
4. $\langle cu, w \rangle = c\langle u, w \rangle$ for all $u, w \in V$ and $c \in \mathbb{R}$.
5. $\langle v, w \rangle = \langle w, v \rangle$.

Review: Inner products on linear space

- The definition of the above inner product is not useful for complex vector spaces V.
- Let $0 \neq u \in V$ and $i \in \mathbb{C}$.

$$\langle iu, iu \rangle = i^2 \langle u, u \rangle < 0.$$

Review: Inner products on complex linear space

An inner product on V is a function $\langle , \rangle : V \times V \to \mathbb{C}$ such that

1. $\langle v, v \rangle \geqslant 0$ for all $v \in V$.
2. $\langle v, v \rangle = 0$ if and only if $v = 0$.
3. $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ for all $u, v, w \in V$.
4. $\langle cu, w \rangle = c \langle u, w \rangle$ for all $u, w \in V$ and $c \in \mathbb{C}$.
5. $\langle v, w \rangle = \overline{\langle w, v \rangle}$.

Review: Notes

Let V be an inner product, Then

1. $\langle w, u+v\rangle = \langle w, u\rangle + \langle w, v\rangle$ for all $u, v, w \in V$.
2. $\langle u, cw\rangle = \bar{c}\langle u, w\rangle$ for all $u, w \in V$ and $c \in \mathbb{C}$.

Review: Symmetric matrices

Definition

A symmetric matrix is a square matrix that is equal to its transpose.

Let $A \in M_n(\mathbb{R})$, then there is a matrix $B \in M_n(\mathbb{R})$ such that $\langle Ax, y\rangle = \langle x, By\rangle$ for each $x, y \in \mathbb{R}^n$.

Review: Hermitian matrices

Definition

A hermitian matrix is a square matrix, which is equal to its conjugate transpose matrix.

Let $A \in M_n(\mathbb{C})$, then there is a matrix $B \in M_n(\mathbb{C})$ such that $\langle Ax, y \rangle = \langle x, By \rangle$ for each $x, y \in \mathbb{C}^n$.

Review: Self-adjoint matrices

Definition

A matrix $A \in M_n(\mathbb{F})$ is self-adjoint if $A^* = A$.

Definition

A matrix $A \in M_n(\mathbb{R})$ is symmetric if $A^T = A$.

Definition

A matrix $A \in M_n(\mathbb{C})$ is Hermitian if $A^H = A$.

Review: Unitary matrices

Definition

A matrix $U \in \mathbb{F}$ is unitary if $U^*U = UU^* = I$.

- For each $x, y \in \mathbb{F}^n$,

$$\langle Ux, Uy \rangle = \langle x, U^*Uy \rangle .$$

- If U is unitary, then

$$\langle Ux, Uy \rangle = \langle x, y \rangle .$$

That means U preserves inner product.

Review: Inner product on finite-dimensional linear spaces

1. Suppose that V is finite-dimensional linear space where $B = \{v_1, \ldots, v_n\}$ is an ordered basis for V.
2. We are given a particular inner product on V.
3. The inner product is completely determined by the entries of matrix G where
$$G_{ij} = \langle v_j, v_i \rangle .$$
4. Let $v, w \in V$. If $x = [v]_B$ and $y = [w]_B$, then
$$\langle v, w \rangle = y^* G x.$$
5. If $V = \mathbb{F}^n$. Then for each $x, y \in V$,
$$\langle x, y \rangle = y^* x,$$
if we consider standard basis for V.

Inner product on V and change basis

- Suppose that $B = \{v_1, \dots, v_n\}$ and $B' = \{v'_1, \dots, v'_n\}$ are two bases for a linear space V.
- For each $w \in V$, $[w]_B = P[w]_{B'}$, where the i-th column of P is $[v'_i]_B$. So $v'_i = \sum_{r=1}^{n} P_{ri} v_r$.
- A matrix H as the inner product matrix respect to B':

$$\begin{aligned} H_{ij} &= \langle v'_j, v'_i \rangle = \left\langle \sum_{r=1}^{n} P_{rj} v_r, \sum_{k=1}^{n} P_{ki} v_r \right\rangle \\ &= \sum_{r=1}^{n} \sum_{k=1}^{n} P_{rj} \bar{P_{ki}} \langle v_r, v_k \rangle G_{kr} \\ &= (P^* G P)_{ij} \end{aligned}$$

  where G is the inner product respect to B.
- Consequently, $H = P^* G P$.

Review: The properties of G

1. $G_{ii} > 0$, for each $1 \leqslant i \leqslant n$.
2. G self-adjoint.
3. G is invertible.
4. $\det G > 0$.

Review: Is the above process reversible?

Let V in a lienar space on $\mathbb{R}$ with dimension n with a basis B.
Question. When a bilinear function $\langle , \rangle : V \times V \to \mathbb{F}$ such that

$$\langle v, w \rangle = y^* G x$$

and $x = [v]_B$ and $y = [w]_B$, is an inner product for $G \in M_n(\mathbb{F})$.

Is the above process reversible?

By the definition of an inner product, we should have

1. $\langle x, x\rangle = x^*Gx \geqslant 0$ for all $v \in V$ such that $[v]_B = x$.
2. G is self-adjoint $(G^* = G)$.

Definition

A self-adjoint matrix $A \in M_n(\mathbb{F})$ is called

1. **positive definite** if $x^T Ax > 0$ for each $0 \neq x \in \mathbb{F}^n$.
2. **positive semi-definite** if $x^T Ax \geqslant 0$ for each $x \in \mathbb{F}^n$.

Theorem

If A is a self-adjoint matrix, then an invertible matrix $P \in M_n(\mathbb{F})$ such that $P^*P = I$ and

$$P^*AP = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}.$$

Tests for positive definiteness

Theorem

Each of the following tests is a necessary and sufficient condition for the self-adjoint matrix A to be positive definite:

1. *All eigenvalues of A are positive.*
2. *All upper left submatrices A_k have positive determinants.*
3. *All pivots (without row exchanges) are positive.*

- The test brings together three of the most basic ideas in the book:
  1. pivots,
  2. determinants,
  3. eigenvalues.

Proof.

- First, we show that a self-adjoint matrix A is positive definite if and only if all eigenvalues of A are positive.

Proof.

- Second, we show that if all eigenvalues of self-adjoint matrix A is positive then all upper left submatrices A_k have positive determinants.

Proof.

- Third, we show that if all upper left submatrices A_k of self-adjoint matrix A are positive then all pivots (without row exchanges) are positive.

Proof.

- Fourth, we show that if all pivots (without row exchanges) of self-adjoint matrix A are positive then A is positive definite.

Example

- Let $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$. The matrix A is positive semidefinite, by all three tests. For instance

- The eigenvalues are $\lambda_1 = 0, \lambda_2 = \lambda_3 = 3$ (a zero eigenvalue).

Positive definite matrices

Theorem

A matrix A is positive definite if and only if there is an inevitable matrix R such that $A = R^* R$.

Cholesky decomposition

- Every positive definite matrix A can be factored as

$$A = R^* R,$$

  where R is upper triangular with positive diagonal elements.

- $R^* R$ is called Cholesky decomposition for A.

- R is called the Cholesky factor of A.

Positive definite square root

- Every positive definite matrix A can be factored as

$$A = PDP^* = (P\sqrt{D})(\sqrt{D}P^*),$$

  where P is invertible matrix.

- So $A = PDP^* = (P\sqrt{D}P^*)(P\sqrt{D}P^*)$

- $P\sqrt{D}P^*$ is called **positive definite square root** of A.

Review: Positive definite and positive semi-definite matrices

Definition

A hermitian matrix $A \in M_n(\mathbb{C})$ is called

1. **positive definite** if $x^H Ax > 0$ for each $0 \neq x \in \mathbb{C}$.
2. **positive semi-definite** if $x^H Ax \geqslant 0$ for each $x \in \mathbb{C}$.

Remark

- **Remark.** The conditions for semidefiniteness could also be deduced from *tests for definiteness* by the following trick:

  1. Add a small multiple of the identity to get a positive definite matrix

  $$A + \epsilon I.$$

  2. Then let ϵ approach zero.
  3. At ϵ they must still be nonnegative.

Tests for positive semi-definiteness

Theorem

Each of the following tests is a necessary and sufficient condition for the self-adjoint matrix A to be positive semi-definite:

1. *All eigenvalues of A are non-negative.*
2. *All upper left submatrices A_k have non-negative determinants.*
3. *All pivots (without row exchanges) are non-negative.*

A useful lemma for self-adjoint matrices

Lemma

Let $A \in M_n(\mathbb{F})$ be a self-adjoint matrix then there is an invertible matrix $P \in M_n(\mathbb{F})$ such that $P^*P = I$ and

$$A = P \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} P^*.$$

where $\lambda_1, \ldots, \lambda_n$ are eigenvalues of A.

- A singular value decomposition (SVD) is a generalization of this where $A \in M_{mn}(\mathbb{F})$ does not have to be self-adjoint or even square.

Thank You!