Linear Algebra

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Review: Diagonalizable linear transformations

Theorem

Let $T : V \to V$ be a linear transformation where V is finite dimensional, and T has different eigenvalues $\lambda_1, \ldots, \lambda_k$. Suppose that W_i is the null space of $\lambda_i I - T$ for each $1 \leqslant i \leqslant k$. Then the following statements are equivalent:

i. *T is diagonalizable.*

ii. *The characteristic polynomial of T is*

$$f(\lambda) = (\lambda - \lambda_1)^{n_1} \cdots (\lambda - \lambda_k)^{n_k},$$

and $\dim W_i = n_i$.

iii. $\sum_{i=1}^{k} \dim W_i = \dim V$.

Review: Cayley-Hamilton's theorem

Theorem

Let $A \in M_n(\mathbb{F})$ where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and $f(x)$ is characteristic polynomial. Then $f(A) = 0$

Corollary

Let $A \in M_n(\mathbb{F})$ where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. For $k > n$, $A^k = g(A)$ where $g(x)$ is a polynomial with coefficients in $\mathbb{F}$ and its degree is less than n.

Review: Minimal polynomial

Definition

Let $A \in M_n(\mathbb{F})$ where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. The minimal polynomial $p(x)$ of A over $\mathbb{F}$ is the monic polynomial over $\mathbb{F}$ of least degree such that $p(A) = 0$.

Corollary

Let $A \in M_n(\mathbb{F})$ where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and $p(x)$ is minimal polynomial. Then $p(x)$ divides the characteristics polynomial $f(x)$.

- Let $A \in M_n(\mathbb{F})$ where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Assume that $p(x)$ is minimal polynomial and $g(x)$ is non-zero polynomial with coefficients in $\mathbb{F}$.
- Note that if $g(A) = 0$, then $p(x)|g(x)$.

Minimal polynomial and characteristic polynomial

Theorem

Let T be a linear function over V with finite dimension $n < \infty$. Then the minimal polynomial and characteristic polynomial have the same roots.

Diagonalizable matrices

Theorem

Let T be a linear function over V with dimension $n < \infty$. The linear function T is diagonalizable then the minimal polynomial of T has no repeated roots.

Triangularizable matrices

Theorem

Let T be a linear function over V with dimension $n < \infty$ over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. The linear function T is triangularizable if and only if the minimal polynomial of T splits in $\mathbb{F}(x)$ into linear factors.

Thank You!