Linear Algebra

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Fall, 2021

Linear Spaces

The heart of linear algebra

Review: Basis for a linear space

Definition

Let V be a linear space and $S \subseteq V$. The set S is a basis for V if

1. $V = \mathrm{span}(S)$,
2. $V \neq \mathrm{span}(T)$ for all $T \subsetneq S$.

- Trivially, a basis for a linear space is a linear independent set.
- A basis is a "minimal" spanning set for the linear space, in the sense that it has no "redundant" vector. At the same time, it is a "maximal" linearly independent set, in the sense that putting up a new vector makes it linearly dependent.
- A linear space may have more than one basis.

Review: Finite basis for a linear space

Theorem

If $V = span(\{v_1, \ldots, v_n\})$, then there is a subset of $\{v_1, \ldots, v_n\}$ which is a basis for V.

Theorem

Suppose that $V = span(\{v_1, \ldots, v_n\})$. Then each independent set of V has at most n elements.

Review: Dimension

Theorem

If $\{v_1, \ldots, v_m\}$ and $\{w_1, \ldots, w_n\}$ are both bases for a linear space V, then $m = n$.

Definition

Suppose that V has a finite basis. Then ***dimension*** of V denoted by $\dim V$ is the number of elements of any basis of V.

- Example. Assume the linear space $P_2(x) = \{a_2x^2 + a_1x + a_0 \mid a_i \in \mathbb{R} \text{ for all } 0 \leq i \leq 2\}$.
  1. The sets $\{1, x, x^2\}$ is a basis for $P_2(x)$.
  2. $\dim(P_2(x)) = 3$.

Review: Coordinates

Now suppose V is a finite-dimensional linear space and that $B = \{v_1, \cdots, v_n\}$ is an ordered basis for V. Given $v \in V$, there is a unique n-tuple $c = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ of scalars such that $v = \sum_{i=1}^{n} c_i v_i$. The vector c is called the coordinate vector of v relative to the ordered basis B and denoted by $[v]_B$.

Theorem

Let v be a linear space. Suppose that $B = \{v_1, \ldots, v_n\}$ and $B' = \{v'_1, \ldots, v'_n\}$ are two bases of V. Then $[v]_B = P[v]_{B'}$ where the columns of P are the coordinates of the vectors $v'_1, \ldots, v'_n$ in the basis B.

Row Reduced Form R

$$A = \begin{bmatrix} 1 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 1 & 1 & -2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 3 & 1 & -2 \end{bmatrix}$$

$$R = \begin{bmatrix} \mathbf{1} & 0 & -3 & 0 & 0 & 4 \\ 0 & \mathbf{1} & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & \mathbf{1} & 0 & -2/3 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 \end{bmatrix}$$

The column space of A

Lemma

Let $A \in M_{mn}(\mathbb{R})$, *then* $\dim C(A) = \dim C(R)$.

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Let $P \in M_m(\mathbb{R})$ *is invertible and* $A \in M_{mn}(\mathbb{R})$,*then*
$\dim C(A) = \dim C(PA)$.

The column space of A

Lemma

Let $P \in M_m(\mathbb{R})$ is invertible and $A \in M_{mn}(\mathbb{R})$, then $\dim(C(A)) = \dim(C(PA))$.

Sketch of the proof: Let $(\dim(C(A)) = r$ and denote the i-th

column of A by A_i. Without loss of generality, assume that the first r columns of A are independent. We have

$$PA = P\begin{bmatrix}A_1 \cdots A_n\end{bmatrix} = \begin{bmatrix}PA_1 \cdots PA_n\end{bmatrix}$$

We show that the first r columns of B are independent. Consider

$$\begin{aligned} c_1PA_1 + \cdots + c_nPA_r &= 0 \\ P(c_1A_1 + \cdots + c_nA_r) &= 0 \\ \Rightarrow c_1 = \cdots = c_r &= 0. \end{aligned}$$

What about column spaces of A?

- For invertible $P \in M_m(\mathbb{R})$, $A \in M_{mn}(\mathbb{R})$ and PA, the column spaces of A and PA might not be the same.

- **Example.**

$$\underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}}_{B} = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}}_{A}$$

$$C(A) = \operatorname{span}\left(\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}\right), \qquad C(PA) = \operatorname{span}\left(\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}\right)$$

Thus $C(A) \neq C(PA)$.

Lemma

Let $P \in M_m(\mathbb{R})$ is invertible and $A \in M_{mn}(\mathbb{R})$, then the row spaces of A and PA are the same.

Sketch of the proof:

$$B = PA = \begin{bmatrix} p_{11} & \cdots & p_{1m} \\ \vdots & \ddots & \vdots \\ p_{m1} & \cdots & p_{mm} \end{bmatrix} \begin{bmatrix} A_1 \\ \vdots \\ A_m \end{bmatrix} = \begin{bmatrix} p_{11}A_1 + \cdots + p_{1m}A_m \\ \vdots \\ p_{m1}A_1 + \cdots + p_{mm}A_m \end{bmatrix}$$

The row space and the column space

Theorem

Let $A \in M_{mn}(\mathbb{R})$, then the dimension of row space and dimension of column space are the same.

The row space and the column space

Theorem

Let $A \in M_{mn}(\mathbb{R})$, then the dimension of row space and dimension of column space are the same.

Definition

Let $A \in M_{mn}(\mathbb{R})$. The number of independent columns (or rows) is called the rank of A and denoted by $\mathrm{rank}(A)$.

The Rank Theorem

Theorem

Let $A \in M_{mn}(\mathbb{R})$, *then*

$$\dim C(A) + \dim N(A) = n$$

The Four Fundamental Subspaces

1. The column space of A : $C(A)$
2. The nullspace of A : $N(A)$
3. The row space of A : $C(A^T)$
4. The left nullspace of A : $N(A^T)$

The rank Theorem

Suppose that $A \in M_{mn}(\mathbb{R})$ and $\text{rank}(A) = r$.

- $C(A)$ = column space of A; dimension r.
- $N(A)$ = nullspace of A; dimension $n - r$.
- $C(A^T)$=row space of A; dimension r.
- $N(A^T)$= left nullspace of A; dimension $m - r$.

Example

- $A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}$
- The Row Reduced matrix $R = \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
- $\text{rank}(A) = r =$ the number of pivot variables $= 2$.

$$\begin{aligned} \dim(C(A)) &= r = 2 & \dim(N(A)) &= n - r = 4 - 2 = 2 \\ \dim(C(A^T)) &= r = 2 & \dim(N(A^T)) &= m - r = 3 - 2 = 1 \end{aligned}$$

Matrices of Rank 1

- Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 2 & 2 \\ 8 & 4 & 4 \\ -2 & -1 & -1 \end{bmatrix}$
- Every row is a multiple of the first row, so the row space is one-dimensional.
- The columns are all multiples of the same column vector; the column space shares the dimension 1.
- We have

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 2 & 2 \\ 8 & 4 & 4 \\ -2 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \\ -1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \end{bmatrix}$$

Matrices of Rank 1

Every matrix of rank 1 has the simple form $A = uv^T$, column times row.

Thank You!