Lecture25

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Lecture25

Linear Algebra

* اگر ماتریس قطری‌پذیر باشد، همه‌ی ستون‌های Q را ستون‌های یکسانی کنیم همه‌ی n بردار ویژه را پیدا کنیم؟ بله

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$V = W_1 \oplus W_2$

$B = \{Q_1, \dots, Q_n\}$ مستقل

بردارهای ویژه λ_1 ← B_1

$V = W_1 \oplus \dots \oplus W_k$

ستون‌های آخر بردارهای ویژه λ_k

Fall 2021

$q_1, \dots, q_{n_1} \in W_1 = \mathrm{span}(B_1)$

$\dim W_1 = n_1 \Leftarrow W_1 = N(A - \lambda_1 I) = \{$ بردارهای ویژه‌ی λ_1 و صفر $\}$

$Av = \lambda_k v$

تعدادی n_1 ستون اول Q پایه‌ای برای W_1

Review: Diagonalizable Linear Function

$A = [T]_B$ $\quad T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $T(x) = Ax$

$$Q^{-1} A Q = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} = \begin{bmatrix} \lambda_1 I_{n_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_k I_{n_k} \end{bmatrix}$$

$\lambda_1, \dots, \lambda_k$ مقادیر ویژه متمایز، $n_1, \dots, n_k$

Theorem

$F = \mathbb{R}$ (or $\mathbb{C}$), $\mathbb{R}^n$

Let $T: V \to V$ be a linear function on a finite dimensional linear space V, and T has different eigenvalues $\lambda_1, \dots, \lambda_k$. Suppose that W_i is the null space of $T - \lambda_i I$ for each $1 \leqslant i \leqslant k$. Then the following statements are equivalent:

i. *T is diagonalizable.*

ii. *The characteristic polynomial of T is $f(x) = (x - \lambda_1)^{n_1} \cdots (x - \lambda_k)^{n_k}$, where $n_i = \dim W_i$.*

iii. $\sum_{i=1}^{k} \dim W_i = \dim V$.

ستون‌های Q بردارهای ویژه‌اند

$Q: \ AQ = Q \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$

$A[Q_1 \cdots Q_n] = [Q_1 \cdots Q_n] \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$

$[AQ_1 \cdots AQ_n] = [\lambda_1 Q_1 \cdots \lambda_n Q_n]$

$B = \{Q_1, \dots, Q_n\} \Rightarrow$ پایه‌ای از بردارهای ویژه‌ی A

$\Rightarrow AQ_i = \lambda_i Q_i$

ستون‌های Q بردار ویژه‌اند

Primary Decomposition Theorem

$$Q^{-1} A Q = \begin{bmatrix} \square & 0 & 0 \\ 0 & \square & 0 \\ 0 & 0 & \square \end{bmatrix}$$

Theorem

Let T be a linear function over a finite dimensional linear space V

$A \in M_n(F)$ $\quad \mathbb{R}^n, \mathbb{R}, \mathbb{C}$

$Q^{-1}AQ =$

Theorem

Let T be a linear function over a finite dimensional linear space V whose minimal polynomial factorizes as

$$p(x) = p_1^{r_1}(x) \cdots p_k^{r_k}(x),$$

where the p_i's are monic and mutually coprime (i.e. have no nontrivial common factors). Let $W_i = N\left(p_i^{r_i}(T)\right)$ for each $1 \leqslant i \leqslant k$. Then

1. $V = W_1 \oplus \cdots \oplus W_k$.
2. *For each $1 \leqslant i \leqslant k$, $T(W_i) \subseteq W_i$.*
3. *The minimal polynomial of $T_i = T\restriction_{W_i}$ is $p_i(x)$.*

Minimal Polynomials for Vectors

$P(A)v$

صفر به ازای بردار دلخواه که از آن بیش v مواست.

Definition

Let $A \in M_n(\mathbb{F})$, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and $v \in \mathbb{F}^n$. We say that a monic polynomial $p(x)$ with coefficients in $\mathbb{F}$ is a *minimal polynomial for v with respect to A* if

1. $p(A)v = 0$,
2. $\deg p \leqslant \deg m$ for any non-zero polynomial $m(x)$ with $m(A)v = 0$.

$m(x) =$

$f(A) = 0$

$f(A)v = 0$

$A \in M_n(\mathbb{F})$ $\mathbb{R}^n$, $\mathbb{R}$ یا $\mathbb{C}$

$f(A) = 0$ $\deg f(x) = n$

$p(x)$

$W_i = N(\lambda_i I - T)$

$f(x) = (x-\lambda_1)^{n_1} \cdots (x-\lambda_k)^{n_k}$

$p(x) = (x-\lambda_1)^{r_1} \cdots (x-\lambda_k)^{r_k}$

بلوک

مینیمال A

$P_A(x)$

$P_A(A) = 0$

$P_A(A)v = 0$

Minimal Polynomials for Vectors

Similarly, minimal polynomial may be defined for a vector with respect to a linear function.

Definition

Let T be a linear function on linear space V on $\mathbb{F}$ where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and $v \in V$. We say that a monic polynomial $p(x)$ with coefficients in $\mathbb{F}$ is a *minimal polynomial for v with respect to T* if

1. $p(T)v = 0$,
2. $\deg \cancel{m}\, p \leqslant \deg \cancel{p}\, m$ for any non-zero polynomial $m(x)$ with $m(T)v = 0$.

Lemma.

$f(T)v = 0$ $\quad f \,\big|\, \frac{p}{q(x)}$

$p(T)v = 0$ $\quad R(x)$

$f(x) = q(x)p(x) + R(x)$ $\quad R(x) = 0$

$0 = f(T)v = q(T)\cancel{p(T)}v + R(T)v$

Lemma

Suppose that T is a linear function on linear space V over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Then

1. *Each vector $v \in V$ has a minimal polynomial with respect to T.*
2. *The minimal polynomial for v with respect to T is unique.*
3. *Take a vector $v \in V$ and assume that $f(x)$ is a polynomial with coefficients in $\mathbb{F}$ such that $f(T)v = 0$, then $p(x) \mid f(x)$ where $p(x)$ is the minimal polynomial for v with respect to T.*

$q(x) \mid f(x)$, $q(x) \mid g(x)$

4. *Let $f(x)$ and $g(x)$ be two coprime polynomials. Then*
$$N(f(T)) \cap N(g(T)) = \{0\}.$$

$x^2+1,\ x^2$

$v \in N(f(T)) \cap N(g(T))$

$f(T)v = 0$

$g(T)v = 0$

$p(x)$ مینیمال v نسبت به T باشد

$p(x) \mid f(x)$

$p(x) \mid g(x)$

$v = 0 \Leftarrow 0 = p(T)v = v \leftarrow p(x) = 1 \Leftarrow (f(x), g(x)) = 1$ از طرفی

Lemma.

$AB = BA$ $\quad N(A) \cap N(B) = \{0\}$

$A, B \in M_n(\mathbb{R})$ $\quad N(AB) = N(A) \oplus N(B)$

Lemma

Let T, S be two linear functions on linear space V over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ such that $T \circ S = S \circ T$ and $N(T) \cap N(S) = \{0\}$. Then

1. $N(T) + N(S) \subseteq N(T \circ S)$.
2. *If V ~~is~~ is finite dimensional, then* $\dim N(T \circ S) \leqslant \dim N(T) + \dim N(S)$ *and consequently,* $N(T \circ S) = N(T) \oplus N(S)$.

$A = [T]_{B'}$ $\quad B = [T]_{B'}$

$N(A) \subseteq N(A) + N(B) \subseteq N(AB)$

فرض $v + w \in N(A) + N(B)$

حکم $v + w \in N(AB)$ $\quad T \circ S$

$Av = 0 \quad BAv = 0$

$Bw = 0 \quad ABw = 0$

$AB(v+w) = 0$

$\Rightarrow v + w \in N(AB)$

$v + w \in$

$N(A) \subseteq N(A) + N(B) \subseteq N(AB)$

$N(AB) = N(A) \oplus N(B)$

$v + w \in$

$\dim N(AB) \le \dim N(A) + \dim N(B)$

Proof.

$N(A) + N(B) \subseteq N(AB) \quad (*)$

$\dim N(B)$

$w_i \notin N(A)$

$N(B)$ $\{w_1, \dots, w_s\}$

$N(AB)$ $\{v_1, \dots, v_k, w_1, \dots, w_s\}$

$\dim N(AB) = k + s \le \dim N(A) + \dim N(B)$

$\forall\, 1 \le i \le s \quad ABw_i = 0$

$BAw_i = 0$

$Aw_i \in N(B)$

$\{Aw_1, \dots, Aw_s\} \subseteq N(B) \Leftarrow$

$s \le \dim N(B)$

$c_1 A w_1 + \cdots + c_s A w_s = 0$

$A(c_1 w_1 + \cdots + c_s w_s) = 0$

$\Rightarrow c_1 w_1 + \cdots + c_s w_s \in N(A)$

$c_1 w_1 + \cdots + c_s w_s = \sum d_i v_i$

$d_j \neq 0$

$v_j = \sum d_j v_j + \sum c_i w_i$

$\Rightarrow c_1 w_1 + \cdots + c_s w_s = 0$

$\Rightarrow c_1 = \cdots = c_s = 0$

$AB = BA$

$N(A) \cap N(B) = \{0\}$

Lemma.

Lemma

Let $T_1, \dots, T_k$ be linear functions on a finite dimensional linear space V over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ such that

- $T_i \circ T_j = T_j \circ T_i$,
- $N(T_i) \cap N(T_j) = \{0\}$

for each $1 \leqslant i < j \leqslant k$. Then $N(T_1 \circ \cdots \circ T_k) = N(T_1) \oplus \cdots \oplus N(T_k)$.

$BA(v_j') = 0$

$AB(v_i') = 0$

$N(A) + N(B) \subseteq N(AB)$

$\geq$

$k + l \le \dim N(AB)$

$\dim N(A) + \dim N(B) \le \dim N(AB)$

$N(AB) = N(A) \oplus N(B)$

$\{v_1, \dots, v_k\} \xrightarrow{c_y} N(A)$

$\{v_1', \dots, v_l'\} \xrightarrow{c_i} N(B)$

$\{v_1, \dots, v_k, v_1', \dots, v_l'\} \subseteq N(AB)$

$A(v_i) = 0 \quad AB(v_j') = 0$

$BA(v_i) = 0 \; N(AB)$

$AB(v_i) = 0$

$v_i \in N(AB)$

$v_j' \in N(AB)$

Proof.

$T: V \to V$

$v \in N(AB) \qquad ABv = 0 \Rightarrow Bv \in N(A)$

$v \notin N(B)$

$Bv \neq 0$

$N(AB) \subseteq (C(B) \cap N(A)) + N(B)$

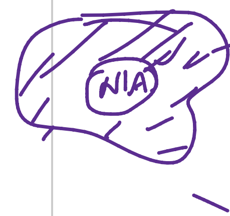
$T|_W$

$N(AB) \subseteq (C(B) \cap N(A)) + N(B)$

$T(x) = Ax$

$T: \mathbb{R}^n \to \mathbb{R}^m$

$N(A) \to$



$T|_W$

$Bv \neq 0$

$\exists\, Bv \in C(B)$

$ABv = 0$

$N(B)$

Lemma.

Lemma

Let T be a linear functions on a finite dimensional linear space V over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

1. *Let $f(x)$ be a polynomial with coefficient in $\mathbb{F}$ and $f(x) = f_1(x)^{n_1} \cdots f_k(x)^{n_k}$ such that $f_1, \ldots, f_k$ mutually coprime. Then*
$$N(f(A)) = N(f_1(A)^{n_1}) \oplus \cdots \oplus N(f_k(A)^{n_k}).$$
2. *If the minimal polynomial T is factorized as $p(x) = p_1(x)^{n_1} \cdots p_k(x)^{n_k}$ where $p_1, \ldots, p_k$ are mutually coprime, then*
$$V = N(p_1(A)^{n_1}) \oplus \cdots \oplus N(p_k(A)^{n_k}).$$

Proof.

Proof.

Primary Decomposition Theorem

Theorem

Let T be a linear function over a finite dimensional linear space V whose minimal polynomial factorizes as

$$p(x) = p_1^{r_1}(x) \cdots p_k^{r_k}(x),$$

where the p_i's are monic and mutually coprime (i.e. have no nontrivial common factors). Let $W_i = N\left(p_i^{r_i}(T)\right)$ for each $1 \leqslant i \leqslant k$. Then

1. $V = W_1 \oplus \cdots \oplus W_k$.
2. *For each $1 \leqslant i \leqslant k$, $T(W_i) \subseteq W_i$.*
3. *The minimal polynomial of $T_i = T\restriction_{W_i}$ is $p_i(x)$.*

...

Thank You!

$V = \underbrace{W_1}\oplus \cdots \oplus \underline{W_k}$

$\mathbb{R}^n =$

$$Q^{-1} A Q = \left[\begin{array}{c|c|c} A_1 & 0 & 0 \\ \hline 0 & \ddots & 0 \\ \hline 0 & 0 & A_k \end{array}\right]$$

$A \in M_n(\mathbb{R})$

$A \in \underline{\underline{M_{mn}}} (\mathbb{R})$