Linear Algebra

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Review: Inner products on real linear space

An inner product on V is a function $\langle , \rangle : V \times V \to \mathbb{R}$ such that

1. $\langle v, v \rangle \geqslant 0$ for all $v \in V$.
2. $\langle v, v \rangle = 0$ if and only if $v = 0$.
3. $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ for all $u, v, w \in V$.
4. $\langle cu, w \rangle = c\langle u, w \rangle$ for all $u, w \in V$ and $c \in \mathbb{R}$.
5. $\langle v, w \rangle = \langle w, v \rangle$.

Review: Euclidean inner product

- The Euclidean inner product on $\mathbb{R}^n$:

$$\begin{cases} \langle\,,\,\rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \\ \\ \langle x, y\rangle = y^T x = y_1 x_1 + \cdots + y_n x_n. \end{cases}$$

where $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$.

Review

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- **Cauchy-Schwarz Inequality:** Let V be an inner product and $u, v \in V$. Then

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- **Triangle Inequality:** Let V be an inner product. If $u, v \in V$, then

$$\|u + v\| \leqslant \|u\| + \|v\|$$

- If nonzero vectors $v_1, \ldots, v_n$ are mutually orthogonal (every vector is perpendicular to every other), then those vectors are linearly independent.

Orthonormal vectors

Definition

Vectors $q_1, \ldots, q_n$ are orthonormal if

$$q_i^T q_j = \begin{cases} 0 \quad \text{whenever} \quad i \neq j & \text{(for orthogonality)} \\ \\ 1 \quad \text{whenever} \quad i = j & \text{(for normality).} \end{cases}$$

A matrix with orthonormal columns will be denoted by Q.

- Example. The standard vectors $e_1, \ldots, e_n$.

Orthogonal Subspaces

Definition

Two subspaces W_1 and W_2 of the same space V are orthogonal, denoted by $W_1 \perp W_2$, if and only if each vector $w_1 \in W_1$ is orthogonal to each vector $w_2 \in W_2$:

$$\langle w_1, w_2 \rangle = 0.$$

for all w_1 and w_2 in W_1 and W_2, respectively.

Orthogonal complement of a subspace

Definition

Given a subspace W in linear space V, the space of all vectors orthogonal to W is called the orthogonal complement of V. It is denoted by $W^{\perp}$.

- We emphasize that W_1 and W_2 can be orthogonal without being complements.
- $W_1 = \text{span}((1, 0, 0))$ and $W_2 = \text{span}((0, 1, 0))$.

Fundamental theorem of orthogonality

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Let $A \in M_{mn}(\mathbb{R})$.

1. The row space is orthogonal to the nullspace (in $\mathbb{R}^n$).
2. The column space is orthogonal to the left nullspace (in $\mathbb{R}^m$).

Fundamental theorem of orthogonality

Let $A \in M_{mn}(\mathbb{R})$.

1. The nullspace is the orthogonal complement of the row space in $\mathbb{R}^n$.
2. The left nullspace is the orthogonal complement of the column space in $\mathbb{R}^m$.

Column space and row spase of

- $N(A) + N(A)^{\perp} = \mathbb{R}^n$.
- $N(A) \cap N(A)^{\perp} = \{0\}$.
- Direct Sum: $\mathbb{R}^n = N(A) \oplus N(A)^{\perp}$.
- $\mathbb{R}^n = N(A) + C(A^T)$.
- Thus, for each $x \in \mathbb{R}^n$, there are $x_r \in C(A^T)$ and $x_n \in N(A)$ such that $x = x_n + x_r$.
- $Ax = Ax_r + Ax_n$.
  1. The nullspace component goes to zero: $Ax_n = 0$.
  2. The row space component goes to the column space: $Ax = Ax_r$.

Column space and row space

Proposition

From the row space to the column space, A is actually invertible. Every vector in the column space comes from exactly one vector in the row space.

Column space and row space

Corollary

Every matrix transforms its row space onto its column space.

- $A \in M_{mn}(\mathbb{R})$ is invertible on those r-dimensional spaces.
- A on its nullspace is zero.
- Thus A^{-1} exists if and only if $r = m = n$.
- When A^{-1} fails to exist, the best substitute is the pseudoinverse A^{+}.
- One formula for A^{+} depends on the singular value decomposition under some conditions.

Thank You!