Linear Algebra

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Review: Diagonalizable Linear Function

Theorem

Let $T : V \to V$ be a linear function on a finite dimensional linear space V, and T has different eigenvalues $\lambda_1, \ldots, \lambda_k$. Suppose that W_i is the null space of $T - \lambda_i I$ for each $1 \leqslant i \leqslant k$. Then the following statements are equivalent:

i. *T is diagonalizable.*

ii. *The characteristic polynomial of T is $f(x) = (x - \lambda_1)^{n_1} \cdots (x - \lambda_k)^{n_k}$, where $n_i = \dim W_i$.*

iii. $\sum_{i=1}^{k} \dim W_i = \dim V$.

Primary Decomposition Theorem

Theorem

Let T be a linear function over a finite dimensional linear space V whose minimal polynomial factorizes as

$$p(x) = p_1^{r_1}(x) \cdots p_k^{r_k}(x),$$

where the p_i's are monic and mutually coprime (i.e. have no nontrivial common factors). Let $W_i = N\left(p_i^{r_i}(T)\right)$ for each $1 \leqslant i \leqslant k$. Then

1. $V = W_1 \oplus \cdots \oplus W_k$.
2. *For each $1 \leqslant i \leqslant k$, $T(W_i) \subseteq W_i$.*
3. *The minimal polynomial of $T_i = T\restriction_{W_i}$ is $p_i(x)$.*

Lemma

Suppose that T is a linear function on linear space V over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Then

1. *Each vector $v \in V$ has a minimal polynomial with respect to T.*
2. *The minimal polynomial for v with respect to T is unique.*
3. *Take a vector $v \in V$ and assume that $f(x)$ is a polynomial with coefficients in $\mathbb{F}$ such that $f(T)v = 0$, then $p(x) \mid f(x)$ where $p(x)$ is the minimal polynomial for v with respect to T.*
4. *Let $f(x)$ and $g(x)$ be two coprime polynomials. Then*

$$N(f(T)) \cap N(g(T)) = \{0\}.$$

Review: Lemma

Lemma

Let T, S be two linear functions on linear space V over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ such that $T \circ S = S \circ T$ and $N(T) \cap N(S) = \{0\}$. Then

1. $N(T) + N(S) \subseteq N(T \circ S)$.
2. *If V is finite dimensional, then* $\dim N(T \circ S) \leqslant \dim N(T) + \dim N(S)$ *and consequently,* $N(T \circ S) = N(T) \oplus N(S)$.

Review: Lemma.

Lemma

Let $T_1, \ldots, T_k$ *be linear functions on a finite dimensional linear space* V *over* $\mathbb{F} = \mathbb{R}$ *or* $\mathbb{C}$ *such that*

- $T_i \circ T_j = T_j \circ T_i$,
- $N(T_i) \cap N(T_j) = \{0\}$

for each $1 \leqslant i < j \leqslant k$. *Then* $N(T_1 \circ \cdots \circ T_k) = N(T_1) \oplus \cdots \oplus N(T_k)$.

Lemma

Let T be a linear functions on a finite dimensional linear space V over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

1. *Let $f(x)$ be a polynomial with coefficient in $\mathbb{F}$ and $f(x) = f_1(x)^{n_1} \cdots f_k(x)^{n_k}$ such that $f_1, \ldots, f_k$ mutually coprime. Then*
$$N(f(T)) = N(f_1(T)^{n_1}) \oplus \cdots \oplus N(f_k(T)^{n_k}).$$
2. *If the minimal polynomial T is factorized as $p(x) = p_1(x)^{n_1} \cdots p_k(x)^{n_k}$ where $p_1, \ldots, p_k$ are mutually coprime, then*
$$V = N(p_1(T)^{n_1}) \oplus \cdots \oplus N(p_k(T)^{n_k}).$$

Primary Decomposition Theorem

Theorem

Let T be a linear function over a finite dimensional linear space V whose minimal polynomial factorizes as

$$p(x) = p_1^{r_1}(x) \cdots p_k^{r_k}(x),$$

where the p_i's are monic and mutually coprime (i.e. have no nontrivial common factors). Let $W_i = N\left(p_i^{r_i}(T)\right)$ for each $1 \leqslant i \leqslant k$. Then

1. $V = W_1 \oplus \cdots \oplus W_k$.
2. *For each $1 \leqslant i \leqslant k$, $T(W_i) \subseteq W_i$.*
3. *The minimal polynomial of $T_i = T\restriction_{W_i}$ is $p_i(x)$.*

Jordan Form

Suppose that T is a linear function on V with the characteristic polynomial is

$$f(x) = (x - \lambda_1)^{d_1} \dots (x - \lambda_k)^{d_k}$$

where $\lambda_1, \dots, \lambda_k$ are distinct elements and $d_i \geqslant 1$.
Then the minimal polynomial for T will be

$$p(x) = (x - \lambda_1)^{r_1} \dots (x - \lambda_k)^{r_k}$$

where $1 \leqslant r_i \leqslant d_i$ based on the Cayley–Hamilton theorem.
If W_i is the null space of $(T - \lambda_i I)^{r_i}$, then the primary decomposition theorem tells us that

$$V = W_1 \oplus \cdots \oplus W_k$$

such that the linear function $T_i = T\restriction_{W_i} : W_i \to W_i$ has minimal polynomial $(x - \lambda_i)^{r_i}$.

Jordan Form

$$V = W_1 \oplus \cdots \oplus W_k$$

Suppose that B_i is a basis for W_i. It has been proved that $B = \bigcup_{i=1}^{k} B_i$ is a basis for V. Based on primary decomposition theorem,

$$T(W_i) \subseteq W_i.$$

Thus

$$[T]_B = \begin{bmatrix} [T_1]_{B_1} & & \\ & \ddots & \\ & & [T_k]_{B_k} \end{bmatrix}.$$

Jordan Form

$$V = W_1 \oplus \cdots \oplus W_k$$

and

$$[T]_B = \begin{bmatrix} [T_1]_{B_1} & & \\ & \ddots & \\ & & [T_k]_{B_k} \end{bmatrix}.$$

- Let N_i be the linear function on W_i defined by $N_i = T - \lambda_i I$.
- Then N_i is nilpotent and has minimal polynomial x^{r_i}.
- Thus, T on W_i acts as N_i plus the scalar λ_i times the identity function I.
- Suppose we choose a basis for the subspace W_i and then find the representation matrix of N_i on W_i.

Nilpotent matrices and Nilpotent linear functions

Definition

A square matrix A is called nilpotent matrix with degree non-negative integer k if A^k is the zero matrix and A^r is the non-zero matrix for each r, $1 \leqslant r \leqslant k$.

Definition

A be a linear function T on V is called nilpotent linear function with degree non-negative integer k if T^k is the zero linear function and T^r is the non-zero one for each r, $1 \leqslant r \leqslant k$.

Example

Let $A \in M_3(\mathbb{R})$ be the following matrix:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then

$$A^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq \mathbf{O}$$

The third power of A is

$$A^3 = A^2 A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{O}$$

Nilpotent matrices

Lemma

Let the $n \times n$ matrix A is nilpotent with degree k, then

1) *The matrix A is nilpotent if and only if all the eigenvalues of A is zero.*
2) *The matrix $A$is nilpotent if and only if $A^n = O$.*

Lemma

If a linear function T on V with $\dim V = n < \infty$ is nilpotent with degree n, then there is a basis for V such that

$$[T]_B = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & 0 & 0 \\ 0 & 0 & \ddots & 0 & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

Thank You!