Linear Algebra

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Eigenvalues and Eigenvectors

Review: Eigenvectors and eigenvectors

Definition

An eigenvector or characteristic vector v for a square matrix A is a nonzero vector that changes at most by a scalar factor when that this matrix is applied to it, i.e., $Av = \lambda v$. The corresponding eigenvalue λ is denoted by as an eigenvalue of A.

Review: solving $Ax = \lambda x$

- **Find the roots of the polynomial** $\det(A - \lambda I) = 0$**.** This roots are the eigenvalues of A.
- The sum of the n eigenvalues equals the sum of the n diagonal entries:

$$\text{Trace of} \quad A := a_{11} + \cdots + a_{nn} = \lambda_1 + \cdots + \lambda_n.$$

- The product of the n eigenvalues equals the determinant of A, that is

$$\lambda_1 \times \cdots \times \lambda_n = \det A.$$

Review: Example

- Find the eigenvalues and eigenvectors for diagonal matrices.

$$\begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix}$$

Example

Suppose that P is a projection matrix. Find the eigenvalues of this matrix.

Diagonalization of a Matrix

- **Example.** The eigenvector matrix of the projection $P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ is $S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, and we have

$$S^{-1}PS = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

- The eigenvector matrix S converts A into its eigenvalue matrix which is diagonal.

90° rotation

- The eigenvalues themselves are not so clear for a rotation:

$$K = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

  has $\det(K - \lambda I) = \lambda^2 + 1$.
- The eigenvalues of K are imaginary numbers $\lambda = \pm i$. with eigenvectors $\begin{bmatrix} 1 \\ -i \end{bmatrix}$ and $\begin{bmatrix} 1 \\ i \end{bmatrix}$.
- The eigenvectors are also not real.
- The eigenvalues are distinct, even if imaginary, and the eigenvectors are independent. They go into the columns of S:

$$S = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \qquad S^{-1}KS = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}.$$

Linear Spaces on $\mathbb{C}$

A Linear space is a set V along with an **addition** on V and a **scalar multiplication** on $\mathbb{C}$

$$\Big(V, +, \cdot\Big)$$

such that the following properties hold:

1. **commutativity** $u + v = v + u$ for all $u, v \in V$;
2. **associativity** $(u + v) + w = u + (v + w)$ and $(ab)v = a(bv)$ for all $u, v, w \in V$ and all $a, b \in \mathbb{C}$;
3. **additive identity** there exists an element $0 \in V$ such that $v + 0 = v$ for all $v \in V$;
4. **additive inverse** for every $v \in V$, there exists $w \in V$ such that $v + w = 0$;
5. **multiplicative identity** $1v = v$ for all $v \in V$;
6. **distributive properties** $a(u + v) = au + av$ and $(a + b)u = au + bu$ for all $a, b \in \mathbb{C}$ and all $u, v \in V$.

Diagonalizable matrices

Definition

Similarity. Let $A, B \in M_n(\mathbb{F})$, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. We say that A and B are *similar* over $\mathbb{F}$, if there exists an invertible matrix $S \in \mathbb{F}$ such that $S^{-1}AS = B$.

Definition

Assume $A \in M_n(\mathbb{R})$. A is called diagonalizable if it is similar to a diagonal matrix, i.e., if there exists an invertible matrix S and a diagonal matrix D such that

$$S^{-1}AS = D.$$

Diagonalization of a Matrix

Theorem

The Diagonalization Theorem. *Let $A \in M_n(\mathbb{F})$ where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. The matrix A is diagonalizable if and only if A has n linearly independent eigenvector.*

A corollary of diagonalization theorem

Fact

If the eigenvalues of A are distinct from each other, then A is diagonalizable.

Remarks

- Not all matrices are diagonalizable.

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

- The diagonalizing matrix S is not unique.

Diagonalizable linear transformation

Definition

A linear transformation $T : V \to V$ where the dimension of V is finite, is said to be diagonalizable if there exists a basis B such that $[T]_B$ is diagonal matrix.

Diagonalizable linear transformation

Theorem

Let $T : V \to V$ be a linear transformation where the dimension of V is finite with different eigenvalues $\lambda_1, \ldots, \lambda_k$. Suppose that W_i is null space of $T - \lambda_i I$ for each $1 \leqslant i \leqslant k$. The the following statements are equivalent:

1. *T is diagonalizable.*
2. *Its eigenvalue vector is $f(x) = (x - d_1)^{n_1} \ldots (x - d_1)^{n_1}$ and $\dim W_i = n_i$.*
3. $\sum_{i=1}^{k} \dim W_i = \dim V$.

Thank You!