Lecture20

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Linear Algebra

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Chapter5

Eigenvalues and Eigenvectors

Review: Eigenvectors and eigenvectors

Definition

An eigenvector or characteristic vector v for a square matrix A is a nonzero vector that changes at most by a scalar factor when that this matrix is applied to it, i.e., $Av = \lambda v$. The corresponding eigenvalue λ is denoted by as an eigenvalue of A.

Review: solving $Ax = \lambda x$ $\Leftrightarrow$ $(\lambda I - A)x = 0$ $\Longleftrightarrow$ $N(\lambda I - A) \neq \{0\}$

$\wedge$ $x \neq 0$ $\Rightarrow$ $\underline{\det(\lambda I - A) = 0}$

$\underline{\underline{f(\lambda)}} = \det(\lambda I - A)$

λ_1

- **Find the roots of the polynomial** $\det(A - \lambda I) = 0$. This roots are the eigenvalues of A.
- The sum of the n eigenvalues equals the sum of the n diagonal entries:

$$\text{Trace of } A := a_{11} + \cdots + a_{nn} = \lambda_1 + \cdots + \lambda_n.$$

- The product of the n eigenvalues equals the determinant of A, that is

$$\lambda_1 \times \cdots \times \lambda_n = \det A.$$

Example

- Find the eigenvalues and eigenvectors for diagonal matrices.

$$A = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix}$$

$E = \{e_1, \dots, e_n\}$

$\dim V = n$

$T : V \to V$, B

$A = [T]_B$

$T(x) = \lambda x$

$B = \{v_1, \dots, v_n\}$

$[T]_B = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix}$

$A = \begin{bmatrix} a_{11} & \cdots & * \\ 0 & \ddots & \\ & & a_{nn} \end{bmatrix}$

$\det(\lambda I - A) = 0$

$\|$

$\det \begin{bmatrix} \lambda - a_{11} & \cdots & * \\ 0 & & \lambda - a_{nn} \end{bmatrix}$

$= (\lambda - a_{11}) \cdots (\lambda - a_{nn}) \Rightarrow \lambda_1 = a_{11}, \dots, \lambda = a_{nn}$

Diagonalization of a Matrix

$p(\lambda) = \det(\lambda I - P) = 0 \Rightarrow \det \begin{bmatrix} \lambda - 1/2 & 1/2 \\ 1/2 & \lambda - 1/2 \end{bmatrix}$

- **Example.** The eigenvector matrix of the projection $P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ is $S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, and we have

$(\lambda - 1/2)^2 - 1/4 = 0$

$\lambda = 1 \Leftarrow \lambda - 1/2 = 1/2$

$Pv_1 = 0 \quad \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0$

$$S^{-1}PS = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$\lambda = 0 \Leftarrow \lambda - 1/2 = -1/2$

- The eigenvector matrix S converts A into its eigenvalue matrix which is diagonal.

$Pv_2 = v_2 \quad 1/2 \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$PS = S \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \Rightarrow$

$P[v_1 \; v_2] = [Pv_1 \; Pv_2] = [\lambda v_1 \; \lambda v_2] =$

$P^2 = P \qquad P \in M_n(\mathbb{R})$

ریشه‌ها ۰ و ۱

$f(\lambda) = \lambda^{\alpha} (\lambda - 1)^{\beta}$

$\alpha + \beta = n$

$\text{Rank } P = r$

$f(\lambda) = \lambda^{n-r} (\lambda - 1)^{r}$

$N(P) = \text{Span}\,\underbrace{\{v_1, \dots, v_{n-r}\}}_{\text{پایه‌ای برای } N(P)}$

$\dim N(P) = n - r$

90° rotation

- The eigenvalues themselves are not so clear for a rotation:

$$K = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad A$$

$S^{-1} A S = B$

has $\det(K - \lambda I) = \lambda^2 + 1$.

B, A متشابه

- The eigenvalues of K are imaginary numbers $\lambda = \pm i$, with eigenvectors $\begin{bmatrix} 1 \\ -i \end{bmatrix}$ and $\begin{bmatrix} 1 \\ i \end{bmatrix}$.
- The eigenvectors are also not real.
- The eigenvalues are distinct, even if imaginary, and the eigenvectors are independent. They go into the columns of S:

$\begin{bmatrix} 1 & 1 \end{bmatrix} \quad \begin{bmatrix} i & 0 \end{bmatrix} \quad B$

- The eigenvalues are distinct, even if imaginary; and the eigenvectors are independent. They go into the columns of S:

$$S = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \qquad S^{-1}KS = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}. \quad B$$

Linear Spaces on $\mathbb{C}$

A Linear space is a set V along with an **addition** on V and a **scalar multiplication** on $\mathbb{C}$

$$(V, +, \cdot)$$

such that the following properties hold:

1. **commutativity** $u + v = v + u$ for all $u, v \in V$;
2. **associativity** $(u + v) + w = u + (v + w)$ and $(ab)v = a(bv)$ for all $u, v, w \in V$ and all $a, b \in \mathbb{C}$;
3. **additive identity** there exists an element $0 \in V$ such that $v + 0 = v$ for all $v \in V$;
4. **additive inverse** for every $v \in V$, there exists $w \in V$ such that $v + w = 0$;
5. **multiplicative identity** $1v = v$ for all $v \in V$;
6. **distributive properties** $a(u + v) = au + av$ and $(a + b)u = au + bu$ for all $a, b \in \mathbb{C}$ and all $u, v \in V$.

Diagonalizable matrices

Definition

Similarity. Let $A, B \in M_n(\mathbb{F})$, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. We say that A and B are *similar* over $\mathbb{F}$, if there exists an invertible matrix $S \in \mathbb{F}$ such that $S^{-1}AS = B$.

Definition

Assume $A \in M_n(\mathbb{R})$. A is called diagonalizable if it is similar to a diagonal matrix, i.e., if there exists an invertible matrix S and a diagonal matrix D such that

$$S^{-1}AS = D.$$

Diagonalization of a Matrix

Theorem

The Diagonalization Theorem. *Let $A \in M_n(\mathbb{F})$ where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. The matrix A is diagonalizable if and only if A has n linearly independent eigenvector.*

$$=\ \ \begin{bmatrix}\lambda_1 & \\ \end{bmatrix}$$

برای $N(P)$

$P \in M_n(\mathbb{R})$

$$B = \{\underbrace{v_1, \dots, v_{n-r}}_{N(P)}, v_{n-r+1}, \dots, v_n\}$$

پایه برای $\mathbb{R}^n$

$T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$

$T(x) = Px$

$$[T]_B = \begin{array}{c} n-r \\ \left[\begin{array}{c|c} 0 & C \\ \hline 0 & D \end{array}\right] \end{array}$$

ستون jام $= B[T(v_j)]$ مختصات v_j در B

$1 \le j \le n-r \quad T(v_j) = 0$

$$\det(\lambda I - P) = \det(\lambda I - [T]_B)$$

حداقل $n-r$ تا ریشه صفر داریم

اگر ریشه صفر دیگری داشته باشیم یعنی

$\det(\lambda I - D) = 0$

$Dx = 0$

$$[T]_B \begin{bmatrix} 0 \\ x \end{bmatrix} = \begin{bmatrix} Cx \\ Dx \end{bmatrix} = 0 \Rightarrow Dx = 0$$

$\Rightarrow\ x = 0 \Rightarrow$ صفر ریشه‌ها

$N(P) \notin \{v_1, \dots, v_{n-r}\}$ فرض

صفر ریشه‌ای نیست

جز $n-r$ تا

$$F(\lambda) = \lambda^{n-r}(\lambda - 1)^{\beta}$$

$n - r = \dim N(P) = |\{x \mid Px = 0\}|$

$A \quad \lambda_1, \dots, \lambda_k$

$$W_k = \{x \mid Px = \lambda_k x\}$$

تعریف

$(\lambda - \lambda_k)^{\square}$

The matrix A is diagonalizable if and only if A has n linearly independent eigenvector.

روی قطر

$(\lambda - \lambda_n)$

$$S^{-1}AS = \begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix}$$

(⇐) فرض کنید A قطری‌شدنی باشد پس S ی که ستون‌هایش بردار ویژه‌اند

$S = [v_1 \cdots v_n]$

$$AS = S\begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix} \Rightarrow A[v_1 \cdots v_n] = \underbrace{[v_1 \cdots v_n]\begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix}}_{[\lambda_1 v_1 \cdots \lambda_n v_n]} \Rightarrow \forall i \quad Av_i = \lambda_i v_i$$

v_i بردار ویژه

زیرا $v_i \neq 0$ (S وارون‌پذیر)

λ_i مقدار ویژه

$\{v_1, \cdots, v_n\}$ بردارهای ویژه

A corollary of Diagonalization theorem

$\underline{A}$ $\lambda \neq 0, 1$

Fact

If the eigenvalues of A are distinct from each other, then A is diagonalizable.

برهان: $\lambda_1, \cdots, \lambda_n$ مقادیر ویژه متمایز A باشند.

$v_1, \cdots, v_n$ بردارهای ویژه متناظر

ادعا: مستقل‌اند ⇐ طبق قضیه قطری‌شدنی است

بدون کاسته شدن از کلیت فرض کنید $v_1 \cdots v_k$ مستقل هستند

برهان خلف: فرض کنید $v_1, \cdots, v_n$ مستقل خطی نباشند. فرض کنید

$$v_{k+1} = \sum_{i=1}^{k} c_i v_i \qquad \exists i\ c_i \neq 0$$

فرض کنید $c_r \neq 0$

$$\lambda_{k+1} v_{k+1} = \sum_{i=1}^{k} c_i \lambda_{k+1} v_i$$

$$A v_{k+1} = \sum_{i=1}^{k} c_i A v_i = \sum_{i=1}^{k} c_i \lambda_i v_i$$

$$\Rightarrow \sum_{i=1}^{k} c_i(\lambda_i - \lambda_{k+1}) v_i = 0$$

چون $v_1, \cdots, v_k$ مستقل‌اند

$c_i(\lambda_i - \lambda_{k+1}) = 0$

از جمله r، $c_r \neq 0$

$c_r(\lambda_r - \lambda_{k+1}) = 0$

$\Downarrow$

$\lambda_r - \lambda_{k+1} = 0$

$\Downarrow$

$r < k \quad \underline{\lambda_r} = \underline{\lambda_{k+1}}$ ✗

Remarks

$$f(\lambda) = \det\begin{bmatrix}\lambda & 1 \\ 0 & \lambda\end{bmatrix} = \lambda^2 \Rightarrow$$

مقدار ویژه تکراری صفر با تکرار ۲ است

- Not all matrices are diagonalizable.

$$A = \begin{bmatrix}0 & 1 \\ 0 & 0\end{bmatrix}.$$

$x \neq 0$

$Ax = 0$

$$\begin{bmatrix}0 & 1 \\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} = \begin{bmatrix}x_2 \\ 0\end{bmatrix} = \begin{bmatrix}0 \\ 0\end{bmatrix}$$

- The diagonalizing matrix S is not unique.

$Ax = 0 \Leftrightarrow x_2 = 0$

$$W = \{x \mid Ax = 0\} = \left\{\begin{bmatrix}x_1 \\ 0\end{bmatrix} \,\middle|\, x_1 \in \mathbb{R}\right\} \Rightarrow \dim W = 1$$

Diagonalizable linear transformation

Definition

A linear transformation $T: V \to V$ where the dimension of V is finite, is said to be diagonalizable if there exists a basis B such that $[T]_B$ is diagonal matrix.

$[T]_B = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$

$[T]_B = P^{-1}[T]_E P$

Diagonalizable linear transformation

Theorem

Let $T: V \to V$ be a linear transformation where the dimension of V is finite with different eigenvalues $\lambda_1, \dots, \lambda_k$. Suppose that W_i is null space of $T - \lambda_i I$ for each $1 \leqslant i \leqslant k$. The the following statements are equivalent:

1. *T is diagonalizable.*
2. *Its eigenvalue vector is $f(x) = (x - d_1)^{n_1} \dots (x - d_k)^{n_k}$ and $\dim W_i = n_i$.*
3. $\sum_{i=1}^{k} \dim W_i = \dim V$.

$\sum \dim W_i = \dim V$

$W_i = \{x \mid Ax = d_i x\}$

فضای بردارهای مربوط d_i

مضاعف تری d_i

Thank You!