Linear Algebra

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Review: Determinants by Expansion

- Let $A \in M_n(\mathbb{R})$. Consider the submatrix $A(i|j)$ that is defined by throwing away row i and column j.

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A(i|j).$$

- Assume that

$$c_{ij} = (-1)^{i+j} \det A(i|j),$$

then c_{ij} is called ij-th cofactor of matrix A.

- Let

$$C = \begin{bmatrix} c_{11} & \cdots & c_{1j} & \cdots & c_{1n} \\ \vdots & \ldots & \vdots & \ldots & \vdots \\ c_{n1} & \cdots & c_{nj} & \cdots & c_{nn} \end{bmatrix}.$$

Review: Cofactors of A

- Thus, For each $1 \leqslant j \leqslant n$, inner product of the j-th column of A and the j-th column of C is equal to $\det A$.

$$\sum_{i=1}^{n} a_{ij} c_{ij} = \det A.$$

- But inner product of the j-th column of A and the k-th column of C is equal to zero for $1 \leqslant j \neq k \leqslant n$.

$$B_j = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} & \cdots & a_{1k} & \cdots & a_{1n} \\ \vdots & \vdots & \ldots & \vdots & \ldots & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nk} & \cdots & a_{nk} & \cdots & a_{nn} \end{bmatrix}$$

So

$$0 = \det B_j = \sum_{i=1}^{n} a_{ik} c_{ij}.$$

Adjoint A

- We obtain

$$C^T A = \begin{bmatrix} c_{11} & \cdots & c_{n1} \\ \vdots & \dots & \vdots \\ c_{1n} & \cdots & c_{nn} \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \dots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} \det A & & \\ & \ddots & \\ & & \det A \end{bmatrix}$$

- Thus,

$$C^T A = (\det A)\, I.$$

- The matrix C^T is called the adjoint of A and is denoted by adj A. So,

$$(\operatorname{adj} A)A = (\det A)I$$

adj A

- By $(\operatorname{adj} A)A = (\det A)I$, we have
  1. $(\operatorname{adj} A^T)A^T = (\det A^T)I = (\det A)I$.
  2. $(\operatorname{adj} A)_{ij} = (-1)^{i+j} \det A(j|i)$.
- It is easy to check that

$$(\operatorname{adj} A^T) = (\operatorname{adj} A)^T.$$

Computation of A^{-1}

- If $A \in M_n(\mathbb{R})$ is invertible, then

$$A\left(\frac{\operatorname{adj} A}{\det A}\right) = \left(\frac{\operatorname{adj} A}{\det A}\right)A = I.$$

- Thus,

$$A^{-1} = \left(\frac{\operatorname{adj} A}{\det A}\right).$$

Cramer's rule

- Let $A \in M_n(\mathbb{R})$ be invertible.
- The solution of $Ax = b$ is $x = A^{-1}b$: just $C^T b$ divided by $\det A$.
- **Cramer's rule:** The jth component of $x = A^{-1}b$ is the ratio

$$x_j = \frac{\det B_j}{\det A},$$

where

$$B_j = \begin{bmatrix} a_{11} & a_{12} & \cdots & b_1 & \cdots & a_{1n} \\ \vdots & \vdots & \ldots & \vdots & \ldots & \vdots \\ a_{n1} & a_{n2} & \cdots & b_n & \cdots & a_{nn} \end{bmatrix}$$

Determinant and linear functions

- Let T be a linear function on a finite dimensional linear space V. Then the determinant of T is defined the determinant of its representation matrix. That means if B is a basis for V, then

$$\det T = \det[T]_B.$$

- Note that the definition is well defined (Why?).

Eigenvalues and Eigenvectors

Differential equations

- Consider a single equation

$$\frac{du}{dt} = au \quad \text{with} \quad u_0 = u(0).$$

- The solution to this equation is

$$u(t) = e^{at} u_0$$

Differential equation systems

- Consider the differential equation system

$$\begin{cases} \frac{du_1}{dt} = 4u_1 - 5u_2 \\ \frac{du_2}{dt} = 2u_1 - 3u_2 \end{cases} \quad \text{with} \quad \begin{cases} x_1 = u_1(0) \\ x_2 = u_2(0) \end{cases}$$

- Similar to a single equation, let

$$u_1(t) = e^{\lambda t} x_1$$
$$u_2(t) = e^{\lambda t} x_2$$

Eigenvalue problem

- By substituting and cancellation of $e^{\lambda t}$, we obtain:

$$4x_1 - 5x_2 = \lambda x_1$$
$$2x_1 - 3x_2 = \lambda x_2$$

or

$$\begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

- Eigenvalue equation:

$$Ax = \lambda x.$$

Eigenvectors and eigenvectors

Definition

An eigenvector or characteristic vector v for a square matrix A is a nonzero vector that changes at most by a scalar factor when that this matrix is applied to it, i.e., $Av = \lambda v$. The corresponding eigenvalue λ is denoted by as an eigenvalue of A.

Definition

Suppose that $T : V \to V$ is a linear function on V. Then an eigenvector or characteristic vector v for the linear function T is a nonzero vector that changes at most by a scalar factor when that this linear function is applied to it, i.e., $Tv = \lambda v$. The corresponding eigenvalue λ is denoted by as an eigenvalue of T.

The steps in solving $Ax = \lambda x$

- **For each eigenvalue solve the equation** $(\lambda I - A)x = 0$.
- The vector x is in the nullspace of $\lambda I - A$.
- The number λ is chosen so that $\lambda I - A$ has a nullspace with nonzero dimension.
- In short, $\lambda I - A$ must be singular.
- **Compute the determinant of** $(A - \lambda I)$.
- **Find the roots of the polynomial** $\det(A - \lambda I) = 0$**.** This roots are the eigenvalues of A.

Sum and product of eigenvalues

- The sum of the n eigenvalues equals the sum of the n diagonal entries:

$$\text{Trace of} \quad A := a_{11} + \cdots + a_{nn} = \lambda_1 + \cdots + \lambda_n.$$

- The product of the n eigenvalues equals the determinant of A, that is

$$\lambda_1 \times \cdots \times \lambda_n = \det A.$$

- Note that, some (or even all) of eigenvalues may be complex numbers.

Example

- Find the eigenvalues and eigenvectors for

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

Example

- Find the eigenvalues and eigenvectors for

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example

- Find the eigenvalues and eigenvectors for diagonal matrices.

$$\begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix}$$

Diagonalization of a Matrix

- **Example.** The eigenvector matrix of the projection $P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ is $S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, and we have

$$S^{-1}PS = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

- The eigenvector matrix S converts A into its eigenvalue matrix which is diagonal.

Thank You!