Linear Algebra

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Linear Spaces

The heart of linear algebra

Definition

Let S be a set of a linear space V. The subspace spanned by S, denoted by $\text{span}(S)$, is the set of all linear combinations of vectors in S.

1. **Example.** For $A \in M_{m,n}(\mathbb{R})$,

$$V = \{Ax \mid x \in \mathbb{R}^n\}$$

is $C(A)$, as it is generated by all columns of A.

Review

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}$$

$$\begin{aligned} C(A) &= \operatorname{span}\left(\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \\ 4 \end{bmatrix} \right\}\right) \\ &= \operatorname{span}\left(\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \\ 4 \end{bmatrix} \right\}\right) \end{aligned}$$

Review

Definition

Let V be a linear space and $S \subseteq V$. We say that the elements of S are linearly dependent if there is some $s \in S$ such that

$$\text{span}(S) = \text{span}(S \setminus \{s\}).$$

If the elements of S are not linearly dependent, then we say that they are linearly independent.

Definition

The nullspace of a matrix consists of all vectors x such that $Ax = 0$. It is denoted by $N(A)$.

Basis for a linear space

Definition

Let V be a linear space and $S \subseteq V$. The set S is a basis for V if

1. $V = \text{span}(S)$,
2. $V \neq \text{span}(T)$ for all $T \subsetneq S$.

- Trivially, a basis for a linear space is a linear independent set.
- A basis is a "minimal" spanning set for the linear space, in the sense that it has no "redundant" vector. At the same time, it is a "maximal" linearly independent set, in the sense that putting up a new vector makes it linearly dependent.
- A linear space may have more than one basis.

A linear space with infinite dimension

- There is no need for a basis to be finite! The linear space $\mathbb{R}[x]$ of all polynomials with real coefficients has no finite basis.

  To show it:

  1. By contradiction, assume that $\{f_1, \ldots, f_n\}$ is a basis for $\mathbb{R}[x]$.
  2. Let $m = \max_{1 \leq i \leq n} \deg f_i$ where $\deg f_i$ is the degree of polynomial f_i for every $1 \leq i \leq n$.
  3. Then $x^{m+1} \notin \operatorname{span}(\{f_1, \ldots, f_n\})$.

Finite basis for a linear space

Theorem

If $V = span(\{v_1, \ldots, v_n\})$, *then there is a subset of* $\{v_1, \ldots, v_n\}$ *which is a basis for* V.

Finite basis for a linear space

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Theorem

Suppose that $V = span(\{v_1, \ldots, v_n\})$. Then each independent set of V has at most n elements.

Dimension

Theorem

If $\{v_1, \ldots, v_m\}$ *and* $\{w_1, \ldots, w_n\}$ *are both bases for a linear space* V*, then* $m = n$*.*

Definition

Suppose that V has a finite basis. Then ***dimension*** of V denoted by $\dim V$ is the number of elements of any basis of V.

- Example. Assume the linear space $P_2(x) = \{a_2x^2 + a_1x + a_0 \mid a_i \in \mathbb{R} \text{ for all } 0 \leq i \leq 2\}$.
  1. The sets $\{1, x, x^2\}$ is a basis for $P_2(x)$.
  2. $\dim(P_2(x)) = 3$.
  3. You can easily check that $\{1, x, x^2 - \frac{1}{3}\}$ is a basis for $P_2(x)$.

Coordinates

Coordinates

Definition

If V is a finite-dimensional linear space, an *ordered basis* for V is a finite sequence of vectors which is linearly independent and spans V.

Now suppose V is a finite-dimensional linear space and that $B = \{v_1, \cdots, v_n\}$ is an ordered basis for V. Given $v \in V$, there is a unique n-tuple $c = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ of scalars such that

$$v = \sum_{i=1}^{n} c_i v_i.$$

The vector c is called the coordinate vector of v relative to the ordered basis B and denoted by $[v]_B$.

Change of basis

- Suppose that V be a linear space with finite dimension.
- Let $B = \{v_1, \ldots, v_n\}$ be a ordered basis for V.
- Let $B' = \{v'_1, \ldots, v'_n\}$ be another ordered basis for V.
- What is relation between $[v]_B$ and $[v]_{B'}$ for any vector $v \in V$?

The change of basis

Theorem

Let v be a linear space. Suppose that $B = \{v_1, \ldots, v_n\}$ and $B' = \{v'_1, \ldots, v'_n\}$ are two bases of V. Then $[v]_B = P[v]_{B'}$ where the columns of P are the coordinates of the vectors $v'_1, \ldots, v'_n$ in the basis B.

Linear subspaces of a finite dimensional linear space

- Let V be a space with finite dimension and $W \subseteq V$. Then every linearly independent subset of W is finite and can be extended to a basis for V.

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Linear subspaces of a finite dimensional linear space

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- If $W \subsetneq V$ and $\dim V < \infty$, then $\dim W < \dim V$.

- Let W_1 and W_2 be two linear subspaces of a linear space V with finite dimension. Then the dimension of $W_1 + W_2$ is finite and

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2).$$

Row Reduced Form R

$$A = \begin{bmatrix} 1 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 1 & 1 & -2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 3 & 1 & -2 \end{bmatrix}$$

$$R = \begin{bmatrix} \mathbf{1} & 0 & -3 & 0 & 0 & 4 \\ 0 & \mathbf{1} & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & \mathbf{1} & 0 & -2/3 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 \end{bmatrix}$$

Thank You!