Lecture21

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Linear Algebra

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Review: Diagonalizable matrices

شبیهن $A, B \iff S^{-1}AS = B$

Definition

Assume $A \in M_n(\mathbb{R})$. A is called diagonalizable if it is similar to a diagonal matrix, i.e., if there exists an invertible matrix S and a diagonal matrix D such that

$$S^{-1}AS = D.$$

Review: Diagonalization of a matrix

$\mathbb{R}^2 = \mathrm{span}(\{v_1, v_2\})$ → متعامد

⇓

- **Example.** The eigenvector matrix of the projection $P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

is $S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, and we have

بردار ویژه

$$S^{-1}PS = \begin{bmatrix} \boxed{1} & 0 \\ 0 & \boxed{0} \end{bmatrix}$$

مقادیر ویژه

- The eigenvector matrix S converts A into its eigenvalue matrix which is diagonal.

Diagonalizable linear transformation

$A \in M_n(\mathbb{R}) \to T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$

$x \longmapsto Ax$

T قطری‌شدنی است $\iff$ پایه B پیدا کنیم که $[T]_B$ قطری باشد

$P^{-1}[T]_{B} P = [T]_{B'}$

$[T]_B \in M_n(\mathbb{R}) \leftarrow T: V \longrightarrow V$

Theorem

Let $T: V \to V$ be a linear transformation where the dimension of V is finite with different eigenvalues $\lambda_1, \ldots, \lambda_k$. Suppose that W_i is null space of $T - \lambda_i I$ for each $1 \leqslant i \leqslant k$. The the following statements are equivalent:

$\dim V = n$

$B = \{v_1, \ldots, v_n\}$

$T(v_j)$

$\dim W_1 = 1$

i. *T is diagonalizable.* $f(\lambda) = \det(\lambda I - T)$

ii. *Its eigenvalue vector is $f(\lambda) = (\lambda - d_1)^{n_1} \ldots (\lambda - d_k)^{n_k}$ and* $\dim W_i = n_i$.

iii. $\sum_{i=1}^{k} \dim W_i = \dim V$.

$T(v_j) = c_{1j} v_1 + \cdots + c_{nj} v_n$

$$[Tv_j]_B = \begin{bmatrix} c_{1j} \\ \vdots \\ c_{nj} \end{bmatrix}$$

$$W_i = \underbrace{\{ v \mid Tv = \lambda_i v \}}_{\lambda_i \text{ مجموعه بردارهای ویژه مربوط به}} = N(T - \lambda_i I)$$

Proof: i ⇒ ii

$Tv_1 = \lambda_1 v_1 \qquad Tv_n = \lambda_n v_n$

$v_1, \ldots, v_n$

T is diagonalizable, so there is a basis $B = \{v_1, \ldots, v_n\}$ such that

$f(\lambda)$ چندجمله‌ای مشخصه

$f(\lambda) = \det(\lambda I - T)$

$$[T]_B = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

$\lambda_1 \cdots \lambda_n$

$\underbrace{v_1, \ldots, v_{n_1}}_{\lambda_1 \cdots \lambda_1} \quad \underbrace{v_{n_1+1}, \ldots, v_{n_1+n_2}}_{\lambda_2 \cdots \lambda_2}$

$= (\lambda - \lambda_1)^{n_1} \cdots (\lambda - \lambda_k)^{n_k}$

$\dim W_i = n_i$

بدون از دست دادن کلیت، فرض کنید $\lambda_1, \ldots, \lambda_k$ مقادیر ویژه متمایز T باشند. ... همچنین فرض کنید که $v_1, \ldots, v_{n_1}$ بردارهای ویژه مقدار ویژه λ_1 است. و به طریق مشابه برای هر i ... پایه B'

$$[T]_{B'} = \begin{bmatrix} \lambda_1 I_{n_1} & & \\ & \ddots & \\ & & \lambda_k I_{n_k} \end{bmatrix} \Rightarrow f(\lambda) = (\lambda - \lambda_1)^{n_1} \cdots (\lambda - \lambda_k)^{n_k}.$$

فرض کنید $x \in W_i$، $1 \leq i \leq k$، پس $Tx = \lambda_i x \Leftarrow$

Proof: ii ⇒ iii

$$\begin{bmatrix} \lambda_1 I_{n_1} & & \\ & \ddots & \\ & & \ddots \end{bmatrix} x = \lambda_i x \qquad x = \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix} \quad x_1 \in \mathbb{R}^{n_1}$$

Proof: ii $\Rightarrow$ iii

- The characteristic polynomial of T is
$$f(\lambda) = (\lambda - \lambda_1)^{n_1} \dots (\lambda - \lambda_1)^{n_k}.$$
- $\dim V = \deg f = \sum_{i=1}^{k} n_i = \sum_{i=1}^{k} \dim W_i$

$$\begin{bmatrix} \lambda_1 I_{n_1} & & \\ & \ddots & \\ & & \lambda_k I_{n_k} \end{bmatrix} x = \lambda_i x \qquad x = \begin{bmatrix} x_1 \\ \vdots \\ \boxed{x_i} \\ \vdots \\ x_k \end{bmatrix} \quad x_1 \in \mathbb{R}^{n_1}, \dots, x_k \in \mathbb{R}^{n_k}$$

$$\Rightarrow \left(\lambda_i I - [T]_{B}\right) x = \begin{bmatrix} (\lambda_i - \lambda_1) I_{n_1} & & \\ & \ddots & \\ & & (\lambda_i - \lambda_k) I_{n_k} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$x_1 = 0 \Leftarrow \lambda_i \neq \lambda_1 \Leftarrow (\lambda_i - \lambda_1) x_1 = 0$

$\vdots$

$x_{i-1} = 0 \Leftarrow \lambda_i \neq \lambda_{i-1} \Leftarrow (\lambda_i - \lambda_{i-1}) x_{i-1} = 0$

$x_i \in \mathbb{R}^{n_i} \Leftarrow \underbrace{(\lambda_i - \lambda_i)}_{0} x_i = 0$

$x_{i+1} = 0 \Leftarrow \underbrace{(\lambda_i - \lambda_{i+1})}_{\neq 0} x_{i+1} = 0$

$\vdots$

$x_k = 0 \Leftarrow (\lambda_i - \lambda_k) x_k = 0$

$x_i \in \mathbb{R}^{n_i} \iff x = \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix} \in W_i$

$\Downarrow$

$\dim W_i = n_i$

Proof: iii $\Rightarrow$ i

- $\sum_{i=1}^{k} \dim W_i = \dim V$. $\underline{\underline{T}}$ قطری‌شدنی است.
- We should find a basis that the representation matrix of $[T]_B$ is diagonal.

برهان. فرض کنید که $\dim W_i = n_i$ و W_i پایه $B_i = \{v_{i1}, \dots, v_{in_i}\}$

ادعا می‌کنیم که $\bigcup_{i=1}^{k} B_i$ پایه‌ای برای V است.

نشان دهیم که $\bigcup_{i=1}^{k} B_i$ مستقل است.

W_1

$$\underbrace{(c_{11} v_{11} + \dots + c_{1n_1} v_{1n_1})}_{v_1} + \dots + (c_{i1} v_{i1} + \dots + c_{in_i} v_{in_i}) + \dots + (c_{k1} v_{k1} + \dots + c_{kn_k} v_{kn_k}) = 0$$

$v_1 + \dots + v_k = 0, \quad v_i \in W_i \implies v_1 = \dots = v_k = 0$

$v_1 + \dots + v_k = 0 \qquad v_i = 0$

$v_1, \dots, v$ $\qquad v_1, \dots, v_s \neq 0$ مستقل

$v_{s+1} \neq 0 \qquad v_{s+1} = c_1 v_1 + \dots + c_s v_s$

$T v_{s+1} = c_1 T v_1 + \dots + c_s T v_s$

$\lambda_{s+1} v_{s+1} = c_1 \lambda_1 v_1 + \dots + c_s \lambda_s v_s$

$0 = c_1(\lambda_1 - \lambda_{s+1}) + \dots + c_s(\lambda_s - \lambda_{s+1})$

Lemmas

We needs two lemma for complete the proof of iii $\Rightarrow$ i.

Lemma

Suppose that T is a linear function on V and $Tv = \lambda v$.. If $f(x)$ is a polynomial, then $f(T)v = f(\lambda)v$. $f(T)v = \lambda v$

$Tv = \lambda v$

$T^2 v = TTv = T(\lambda v) = \lambda T v = \lambda^2 v$

$\vdots$

$T^i v = \lambda^i v$

$A^k \leftarrow \lambda_1, \dots, \lambda_n$

Corollary

If $\lambda_1, \dots, \lambda_n$ are eigenvalues of T, then $\lambda_1^k, \dots, \lambda_n^k$ are eigenvalues of T^k.

$f(x) = \sum_{i=0}^{m} a_i x^i$ $\qquad$ $\lambda_k, \dots, \lambda_1$ $A \in M_n(\mathbb{R})$

$f(T) = \sum_{i=0}^{m} a_i T^i$ $\qquad$ $v_1, \dots, v_k$

$f(T) v = \sum_{i=0}^{m} a_i T^i v = \underbrace{\sum a_i \lambda^i}_{f(\lambda)} v$

$A \leftarrow \lambda_1, \dots, \lambda_n$

$\vdots$

$A^k \leftarrow \lambda_1^k, \dots, \lambda_n^k$

برهان خلف $v_1 \neq 0$ $\qquad v_1, \dots, v_s \neq 0 \qquad v_1, \dots, v_t$ مستقل‌اند

Lemma

برهان خلف $v_1 \neq 0$ $\quad v_1, \dots, v_s \neq 0$ $\quad v_1, \dots, v_t$ متناظرند

Lemma

v_{s+}

Lemma

Suppose that T is a linear function on V with different eigenvalues $\lambda_1, \dots, \lambda_k$. Let for each $1 \leqslant i \leqslant k$

$$W_i = \{v \in V \mid Tv = \lambda_i v\}.$$

If $v_1 + \dots + v_k = 0$ for each $v_i \in W_i$, then $v_1 = \dots = v_k = 0$.

$$g_j(x) = \frac{\prod_{i \neq j,\, i=1}^{k} (x - \lambda_i)}{\prod_{i \neq j,\, i=1}^{k} (\lambda_j - \lambda_i)}$$

به ازای $1 \leq j \leq k$

$1 \leq i \leq k \quad v_i \in W_i \Leftrightarrow Tv_i = \lambda_i v_i$

$$g_j(T)v_i = \underbrace{g_j(\lambda_i)}_{0,1} v_i = \begin{cases} 0\, v_i & i \neq j \\ 1\, v_i & i = j \end{cases}$$

$$g_j(T)(\underbrace{v_1 + \dots + v_k}_{0}) = g_j(T)v_1 + \dots + g_j(T)v_k$$

چون $g_j(T)$ تبدیل خطی است پس

$$g_j(T)(v_1 + \dots + v_k) = 0$$

$$\Rightarrow \underbrace{g_j(T)v_1 + \dots + g_j(T)v_k}_{v_j} = 0$$

$v_j = 0$

Proof: iii $\Rightarrow$ i

- $\sum_{i=1}^{k} \dim W_i = \dim V$.
- Let $B_i = \{v_{i1}, \dots, v_{in_i}\}$ be a basis for W_i. By two above lemma, we have

$$\bigcup_{i=1}^{k} B_i$$

is a basis for $W_1 + \dots + W_k$.

$v_1 + \dots + v_k$

$W_1 + \dots + W_k \subseteq V$

$$\dim(W_1 + \dots + W_k) = \sum_{i=1}^{k} \dim W_i = \dim V$$

$i \neq j \quad W_i \cap W_j = \{0\}$

$0 = v_1 + \dots + v_n \in W_1 + \dots + W_k$

$$B = \bigcup_{i=1}^{k} B_i$$

$$[T]_B = \begin{bmatrix} \lambda_1 & & & & \\ & \ddots & & & \\ & & \lambda_1 & & \\ & & & \ddots & \\ & & & & \lambda_k \ddots \lambda_k \end{bmatrix} = \begin{bmatrix} \lambda_1 I_{n_1} & & \\ & \ddots & \\ & & \lambda_k I_{n_k} \end{bmatrix}$$

Diagonalization of A and its powers A^k

- Find A^{555} where

$$A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}.$$

- We obtain

$$\det(\lambda I - A) = (\lambda - 1)(\lambda - 2)^2$$

- So, $\lambda = 1, 2, 2$ are eigenvalues of A and their eigenvectors are as follows, respectively:

$$v_1 = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \qquad v_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

$$S^{-1}AS = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix}$$

$$A = S \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix} S^{-1}$$

$$A^2 = S \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix} S^{-1} S \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix} S^{-1}$$

$$A^2 = \begin{bmatrix} \lambda_1^2 & & \\ & \lambda_2^2 & \\ & & \end{bmatrix} S^{-1}$$

A^{555}

- Let
$$S = \begin{bmatrix} 3 & 2 & 0 \\ -1 & 0 & 0 \\ 3 & -1 & -1 \end{bmatrix}.$$
- Then
$$S^{-1}AS = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad S^{-1}A^{555}S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^{555} & 0 \\ 0 & 0 & 2^{555} \end{bmatrix}$$
- As a result:
$$A^{555} = S \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^{555} & 0 \\ 0 & 0 & 2^{555} \end{bmatrix} S^{-1}$$

$$A^2 = S \begin{bmatrix} \lambda_1^2 & & \\ & \lambda_2^2 & \\ & & \lambda_3^2 \end{bmatrix} S^{-1}$$

$$\underline{\underline{A^{555}}} = S \begin{bmatrix} \lambda_1^{555} & & \\ & \lambda_2^{555} & \\ & & \lambda_3^{555} \end{bmatrix} S^{-1}$$

Diagonalizable matrix A and its characteristic polynomial

- If $A \in M_n(\mathbb{F})$ is diagonalizable, then $f(A) = 0$.
- Since $A \in M_n(\mathbb{F})$ is diagonalizable, its characteristic polynomial is
$$f(\lambda) = (\lambda - \lambda_1)^{n_1} \dots (\lambda - \lambda_k)^{n_k},$$
where $\dim W_i = n_i$.
- Moreover, there is invertible matrix $S \in M_n(\mathbb{R})$ such that
$$S^{-1}AS = \begin{bmatrix} \lambda_1 I_{n_1} & & & \\ & \lambda_2 I_{n_2} & & \\ & & \ddots & \\ & & & \lambda_k I_{n_k} \end{bmatrix}$$

$f(x) = \sum a_i x^i$

$f(A) = \sum a_i A^i = 0$

$f(A) = A^n + \underline{a_{n-1}} A^{n-1} + \dots + \underline{a_1} A + a_0 = 0$

$f(S^{-1}AS) = \begin{bmatrix} 0 & & \\ & & \\ & & \end{bmatrix}^{n_1} \begin{bmatrix} & & \\ & 0 & \\ & & \end{bmatrix} \cdots \begin{bmatrix} & & \\ & & \\ & & 0 \end{bmatrix}$

$0 = f(\underline{S^{-1}AS}) = S^{-1} \cdot \underbrace{f(A)}_{=0} S \Rightarrow f(A) = 0$

Cayley-Hamilton's theorem

Theorem

Let $A \in M_n(\mathbb{F})$ where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and $f(x)$ and $p(x)$ are characteristic polynomial and minimal polynomial, respectively. Then

1. $f(A) = 0$
2. *The minimal polynomial, $p(x)$, divides the characteristic polynomial, $f(x)$.*

$f(A) = 0$

$\underline{\underline{f(A)}} = 0$

$\deg f = n$

Corollary

Let $A \in M_n(\mathbb{F})$ where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. For $k > n$, $A^k = g(A)$ where $g(x)$ is a polynomial with coefficients in $\mathbb{F}$ and its degree is less than n.

$g(A) = 0$ چندجمله‌ای از درجه کمتر از n هست

. . .

Thank You!

[illegible]

$$f(x) = 1x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

$$f(A) = 0$$

$$p(x) = 1x^m + b_{m-1}x^{m-1} + \cdots + b_0$$

$$p(A) = 0$$

$$p(x) \mid f(x)$$

$$f(x) = (x-1)^2 (x-2)^3$$

$$p(x) = (x-1)(x-2)$$

$$p(A) = 0$$

$$f(A) = 0$$