Linear Algebra

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Review: Diagonalizable linear transformations

Theorem

Let $T : V \to V$ be a linear transformation where V is finite dimensional, and T has different eigenvalues $\lambda_1, \dots, \lambda_k$. Suppose that W_i is the null space of $\lambda_i I - T$ for each $1 \leqslant i \leqslant k$. Then the following statements are equivalent:

i. *T is diagonalizable.*

ii. *The characteristic polynomial of T is*

$$f(\lambda) = (\lambda - \lambda_1)^{n_1} \cdots (\lambda - \lambda_k)^{n_k},$$

and $\dim W_i = n_i$.

iii. $\sum_{i=1}^{k} \dim W_i = \dim V$.

Review: Lemma

Lemma

Suppose that T is a linear function on V with different eigenvalues $\lambda_1, \dots, \lambda_k$, and for each $1 \leqslant i \leqslant k$ let

$$W_i = \{v \in V \mid Tv = \lambda_i v\},$$

which is the null space of $\lambda_i I - T$. If $v_1 + \cdots + v_k = 0$ for each $v_i \in W_i$, then $v_1 = \cdots = v_k = 0$.

Vandermonde matrices

- The following matrix is called **Vandermonde** matrix.

$$V(\lambda_1, \ldots, \lambda_k) = \begin{bmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{k-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_k & \cdots & \lambda_k^{k-1} \end{bmatrix}$$

Vandermonde matrices

- The Vandermonde matrix is invertible; suppose that

$$\begin{bmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{k-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_k & \cdots & \lambda_k^{k-1} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{k-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Vandermonde matrices

We may write

$$\begin{cases} c_0 & + & c_1\lambda_1 & + & \cdots & + & c_{k-1}\lambda_1^{k-1} & = & 0 \\ c_0 & + & c_1\lambda_2 & + & \cdots & + & c_{k-1}\lambda_2^{k-1} & = & 0 \\ \vdots & & \vdots & + & \cdots & + & \vdots & = & \vdots \\ c_0 & + & c_1\lambda_k & + & \cdots & + & c_{k-1}\lambda_k^{k-1} & = & 0 \end{cases}$$

- So $\lambda_1, \ldots, \lambda_k$ are distinct roots of the polynomial

$$c_0 + c_1 x + \cdots + c_{k-1} x^{k-1} = 0.$$

and hence $c_0 = \cdots = c_{k-1} = 0$. As a result,

$$N\big(V(\lambda_1, \ldots, \lambda_k)\big) = \{0\}$$

and the Vandermonde matrix $V(\lambda_1, \ldots, \lambda_k)$ is invertiable.

Proof.

- Since $v_1 + \ldots + v_k = 0$, we have $T^i(v_1 + \cdots + v_k) = 0$ for each $0 \leqslant i \leqslant k-1$.
- So for each $0 \leqslant i \leqslant k-1, \quad \lambda_1^i v_1 + \lambda_2^i v_2 + \cdots + \lambda_k^i v_k = 0$ which may be written in the following form

$$\begin{bmatrix} v_1 & \cdots & v_k \end{bmatrix} V(\lambda_1, \cdots, \lambda_k) = 0.$$

- This shows that $\begin{bmatrix} v_1 & \cdots & v_k \end{bmatrix} = 0$ in turns as $V(\lambda_1, \cdots, \lambda_k)$ is invertible.

Cayley-Hamilton's theorem

Theorem

Let $A \in M_n(\mathbb{F})$ *where* $\mathbb{F} = \mathbb{R}$ *or* $\mathbb{C}$ *and* $f(x)$ *is characteristic polynomial. Then* $f(A) = 0$

A corollary of Cayley-Hamilton's theorem

Corollary

Let $A \in M_n(\mathbb{F})$ *where* $\mathbb{F} = \mathbb{R}$ *or* $\mathbb{C}$*. For* $k > n$*,* $A^k = g(A)$ *where* $g(x)$ *is a polynomial with coefficients in* $\mathbb{F}$ *and its degree is less than* n*.*

Minimal polynomial

Definition

Let $A \in M_n(\mathbb{F})$ where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. The minimal polynomial $p(x)$ of A over $\mathbb{F}$ is the monic polynomial over $\mathbb{F}$ of least degree such that $p(A) = 0$.

- The minimal polynomial is defined for a linear function T.
- The minimal polynomial is unique.

Corollary

Let $A \in M_n(\mathbb{F})$ where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and $p(x)$ is minimal polynomial. Then $p(x)$ divides the characteristics polynomial $f(x)$.

Thank You!