Lecture18

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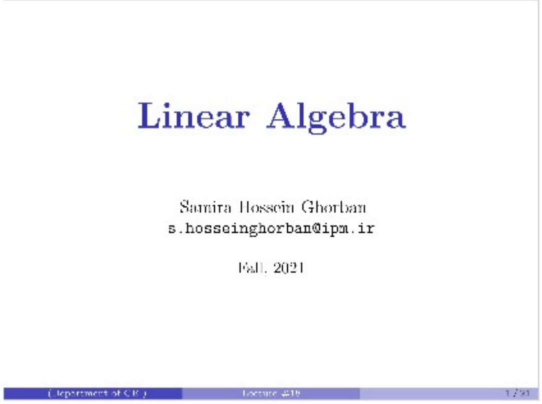

Linear Algebra

Samira Hossein Ghorban
s.hosseinghorban@ipm.ir

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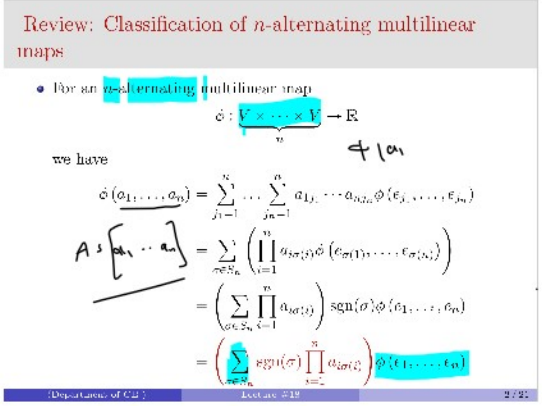


Review: Classification of n-alternating multilinear maps

- For an n-alternating multilinear map

$$\phi : \underbrace{V \times \cdots \times V}_{n} \to \mathbb{R}$$

$\phi(a_1$

we have

$$\phi(\underline{a_1, \ldots, a_n}) = \sum_{j_1=1}^{n} \cdots \sum_{j_n=1}^{n} a_{1j_1} \cdots a_{nj_n} \phi(\epsilon_{j_1}, \ldots, \epsilon_{j_n})$$

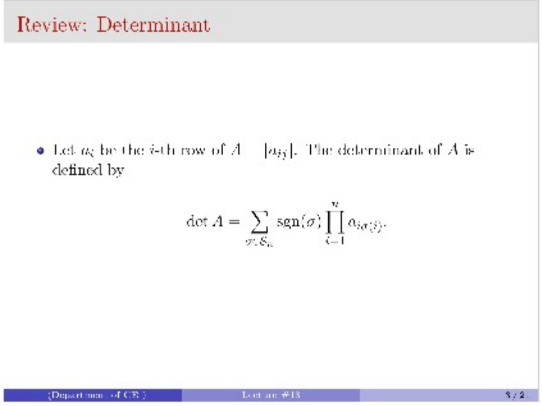
$A = \begin{bmatrix} a_1 \cdots a_n \end{bmatrix}$

$$= \sum_{\sigma \in S_n} \left(\prod_{i=1}^{n} a_{i\sigma(i)} \phi(\epsilon_{\sigma(1)}, \ldots, \epsilon_{\sigma(n)}) \right)$$

$$= \left(\sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)} \right) \operatorname{sgn}(\sigma) \phi(e_1, \ldots, e_n)$$

$$= \left(\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)} \right) \phi(\epsilon_1, \ldots, \epsilon_n)$$

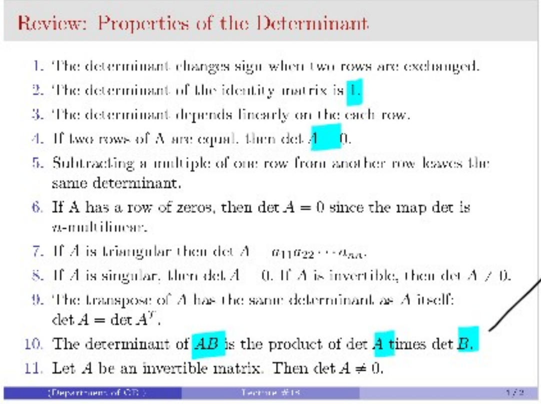
$\phi(e_1, \ldots, e_n) = 1$



Review: Determinant

- Let a_i be the i-th row of $A = [a_{ij}]$. The determinant of A is defined by

$$\det A = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}.$$



Review: Properties of the Determinant

1. The determinant changes sign when two rows are exchanged.
2. The determinant of the identity matrix is 1.
3. The determinant depends linearly on the each row.
4. If two rows of A are equal, then $\det A = 0$.
5. Subtracting a multiple of one row from another row leaves the same determinant.
6. If A has a row of zeros, then $\det A = 0$ since the map det is n-multilinear.
7. If A is triangular then $\det A = a_{11}a_{22} \cdots a_{nn}$.
8. If A is singular, then $\det A = 0$. If A is invertible, then $\det A \neq 0$.
9. The transpose of A has the same determinant as A itself: $\det A = \det A^T$.
10. The determinant of AB is the product of $\det A$ times $\det B$.
11. Let A be an invertible matrix. Then $\det A \neq 0$.

LDU

$\det AB = \det A \det B$

$\det A^T = \det A$

Properties of the Determinant

12. Let $A \in M_r(\mathbb{R})$, $B \in M_{rs}(\mathbb{R})$ and $C \in M_s(\mathbb{R})$, then

$$\det \begin{vmatrix} A & B \\ 0 & C \end{vmatrix} = \det A \det C.$$

- Proof:

Properties of the Determinant

13. Let $A, B, C, D \in M_n(\mathbb{R})$. If $CD = DC$ then

$$\det \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \det(AD - BC).$$

- Note that it is also true if $AC = CA$ or $AB = BA$ or $BD = DB$.
- Proof:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \overset{*}{\begin{bmatrix} I & 0 \\ \underset{-D^{-1}C}{X} & I \end{bmatrix}} = \begin{bmatrix} \overset{A-BD^{-1}C}{A+BX} & B \\ \underset{0}{\underline{CI+DX}} & D \end{bmatrix}$$

$$CI + DX = 0 \Rightarrow DX = -C$$

اگر شرط وارون‌پذیری D را داشته باشیم $\Rightarrow X = -D^{-1}C$

الف. فرض کنید D وارون‌پذیر باشد.

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \overset{*}{\begin{bmatrix} D & 0 \\ -C & I \end{bmatrix}} =$$

$$\begin{bmatrix} AD - BC & B \\ \underset{0}{\underline{CD - DC}} & D \end{bmatrix}$$

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} \cancel{\det D}$$

$$= \det(AD - BC) \cancel{\det D}$$

چون D وارون‌پذیر است پس $\neq \det D$.

ب. فرض کنید D دلخواه است.

ماتریس $D + xI$ را در نظر بگیرید

Properties of the Determinant

14. (**Schur formula**) Let $A \in M_n(\mathbb{R})$, and

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} =$$

where A_{11} and A_{22} are square matrices. Then

$$\det A = (\det A_{11}) \det(A_{22} - A_21 A_{11}^{-1} A_12).$$

$$\det A = (\det A_{11}) \det(A_{22} - A_{21} A_{11}^{-1} A_{12})$$

- Proof. The following identity is easy verified:

$$\begin{bmatrix} I & 0 - A_{21}A_{11}^{-1} & A_{22} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} I & -A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix} \xrightarrow{\det} 1$$

$$\begin{bmatrix} I & 0 \\ -A_{21}A_{11}^{-1} & I \end{bmatrix} \xrightarrow{\det} 1 \qquad = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{bmatrix}$$

$$\det(\underset{D}{D + xI}) =$$

$$\det \begin{bmatrix} d_{11}+x & & d_{1n} \\ d_{21} & d_{22}+x & \\ \vdots & & \ddots \quad d_{n-1\,n} \\ d_{n1} & - & d_{nn}+x \end{bmatrix}$$

$$= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod d_{i\sigma(i)}$$

$$= \underset{1}{\boxed{a_n}} \underline{\underline{x^n}}$$

چندجمله‌ای درجه n.

$\det(D + \underline{x}I) = 0$

S مجموعه ریشه‌های چندجمله‌ای $\det(D+xI)$ [illegible]

$|S| \leq n$

$\forall x \in \mathbb{R} \setminus S$

$$\det(\underline{\underline{D + xI}}) \neq 0$$

Formulas for the Determinant

- If A is invertible, then $PA = LDU$
- $\det A = \pm \det L \times \det D \times \det U$.
- $\det L = \det U = 1$.
- $\det D = d_1 \cdots d_n$.
- $\det A = \pm d_1 \cdots d_n$.

$$\det \begin{bmatrix} A & B \\ C & \underline{\underline{D}} \end{bmatrix} = \det(AD - BC)$$

[illegible]

$$C(D + xI) = CD + xC$$

$$= DC + xC$$

$$= (D + xI)C$$

$\underline{\underline{x \in \mathbb{R} \setminus S}}$ $\overset{|S| < n}{\uparrow}$ [illegible]

$$= \det \begin{bmatrix} A & B \\ C & D + xI \end{bmatrix} =$$

$$\det(A(D + xI) - BC)$$

$$\underline{\underline{f(x)}} = \det \begin{bmatrix} A & B \\ C & D + xI \end{bmatrix}$$

$$= \det(A(D + xI) - BC)$$

Example

- We obtain:

$$\begin{vmatrix} 2 & 1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & \ddots \\ & & & \ddots \end{vmatrix} = L \begin{vmatrix} 2 & & & \\ & \frac{3}{2} & & \\ & & \frac{4}{3} & \\ & & & \ddots \end{vmatrix} U.$$

$$\begin{vmatrix} -1 & 2 & -1 & & & \\ & -1 & 2 & \ddots & & \\ & & & \ddots & & \\ & & & \ddots & & 1 \\ & & & & -1 & 2 \end{vmatrix} = L \begin{vmatrix} 2 & & & & \\ & \frac{3}{2} & & & \\ & & \frac{4}{3} & & \\ & & & \ddots & \\ & & & & \frac{n+1}{n} \end{vmatrix} U.$$

- Thus,

$$\det A = 2\left(\frac{3}{2}\right)\left(\frac{4}{3}\right)\cdots\left(\frac{n+1}{n}\right) = n+1.$$

$$\underline{F(x)} = \det \begin{vmatrix} \cdot & \cdot \\ C & D+xI \end{vmatrix}$$

$$= \det(A(D+xI)-BC)$$

$$\forall\ \underline{\underline{x \in \mathbb{R}}} \setminus S \quad \underline{\underline{F(x)}} = 0$$

$-\varepsilon \quad 0 \quad \varepsilon$

[illegible] $F \equiv 0$

$$\det \begin{bmatrix} A & B \\ C & \end{bmatrix} = \det(AD-BC)$$

One more formula for the determinant

- Let $A \in M_n(\mathbb{R})$.
- Consider The submatrix $A(i|j)$ that is defined by throwing away row i and column j.
- Let $\phi : \underbrace{V \times \cdots \times V}_{n} \to \mathbb{R}$ be given by

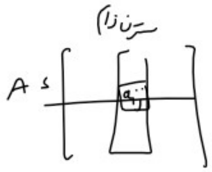
$$\phi(a_1, \ldots, a_n) \overset{?}{=} \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A(i|j).$$

$f(a_1, \ldots) = \det A \quad \phi(e_1, \ldots, e_n) \quad \phi(e_1, \ldots, e_n) = 1$

- ϕ is an n-alternating multilinear map with $\phi(I) = 1$. Then,

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \overset{c_{ij}}{\det A(i|j)}.$$

ستون (j)ام



$\det A$

$$\Phi(a_1, \ldots, a_n) =$$

$$\underbrace{\sum_{\sigma \in S_n} \operatorname{sgn}\sigma \prod a_{\sigma(i)} \phi(e_1, \ldots)}$$

[illegible] $\equiv \sigma(i) = j \qquad a_{ij}$

Cofactors of A

$$\det A = \sum_{i=1}^{n} a_{ij} \cdot c_{ij}$$

A ستون (j)ام

- Assume that

$$c_{ij} = (-1)^{i+j} \det A(i|j).$$

$$\underbrace{[c_{1j} \cdots c_{nj}]}\begin{bmatrix} a_{1j} \\ \vdots \\ a_{nj} \end{bmatrix} = \underline{\underline{\det A}}$$

then c_{ij} is called ij-th cofactor of matrix A.

C ستون (j)ام ، C^T سطر (j)ام

- Let

$$C = \begin{bmatrix} c_{11} & \cdots & c_{1j} & \cdots & c_{1n} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ c_{n1} & \cdots & c_{nj} & \cdots & c_{nn} \end{bmatrix}$$

$\langle$ ستون (j)ام C ، ستون (j)ام $A \rangle$

- Thus, For each $1 \leqslant j \leqslant n$, inner product of the j-th column of A and the j-th column of C is equal to $\det A$.

$[c_{1j} \cdots c_{nj}]$ A ستون (j)ام

- But inner product of the j-th column of A and the k-th column of C is equal to zero for $1 \leqslant j \neq k \leqslant n$.

$$k \neq j \quad \sum_{i=1} a_{ik} c_{ij} = 0$$

$$* \quad \det A = \underbrace{\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod a_{i\sigma(i)}}$$

$$\overset{?}{=} \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A(i|j)$$

[illegible]

ستون j ام فاکتور شده.

[illegible] $*$ [illegible]

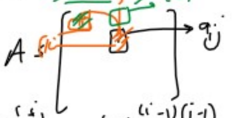
$$\det A = \alpha_{i1} a_{i1} + \cdots + \alpha_{ij} a_{ij}$$

$$+ \cdots + \alpha_{in} a_{in}$$

[illegible]

$$\underline{\alpha_{ij}} \overset{?}{=} (-1)^{i+j} \det A(i|j)$$

= ستون j-1 ; سطر i-1



$$(-1)^{i+j} = (-1)^{(i-1)(j-1)}$$

$\sigma(i) = j$

$A(i|j)$

$$\underbrace{(\ 1 /\ [illegible]\ ) a_{i1}}_{a_{ij}}$$

Adjoint A

$$\det A^T = \det A$$

- We obtain

$$C^T A \quad \begin{bmatrix} c_{11} & \cdots & c_{n1} \\ \vdots & \cdots & \vdots \\ c_{1n} & \cdots & c_{nn} \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \cdots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \quad \begin{bmatrix} \det A & & \\ & \ddots & \\ & & \det A \end{bmatrix}$$

- Thus,

$$C^T A - (\det A) I.$$

- The matrix C^T is called the adjoint of A and is denoted by adj A. So,

$$(\operatorname{adj} A)A = (\det A)I$$

adj A

- By $(\operatorname{adj} A)A = (\det A)I$, we have
  1. $(\operatorname{adj} A^T)A^T = (\det A^T)I = (\det A)I$.
  2. $(\operatorname{adj} A)_{ij} = (-1)^{i+j} \det A(j|i)$.
- It is easy to check that

$$(\operatorname{adj} A^T) = (\operatorname{adj} A)^T.$$

Computation of A^{-1}

- If $A \in M_n(\mathbb{R})$ is invertible, then

$$A\left(\frac{\operatorname{adj} A}{\det A}\right) = \left(\frac{\operatorname{adj} A}{\det A}\right)A = I.$$

- Thus,

$$A^{-1} = \left(\frac{\operatorname{adj} A}{\det A}\right).$$

Cramer's rule

- Let $A \in M_n(\mathbb{R})$ be invertible.
- The solution of $Ax = b$ is $x = A^{-1}b$: just C^Tb divided by $\det A$.
- **Cramer's rule:** The jth component of $x = A^{-1}b$ is the ratio

$$x_j = \frac{\det B_j}{\det A},$$

where

$$B_j = \begin{bmatrix} a_{11} & a_{12} & \cdots & b_1 & \cdots & a_{1n} \\ \vdots & \vdots & \ldots & \vdots & \ldots & \vdots \\ a_{n1} & a_{n2} & \cdots & b_n & \cdots & a_{nn} \end{bmatrix}$$

Chapter5

Eigenvalues and Eigenvectors

Differential equations

- Consider a single equation

$$\frac{du}{dt} = au \quad \text{with} \quad u_0 = u(0).$$

- The solution to this equation is

$$u(t) = e^{at}u_0$$

Differential equation systems

- Consider the differential equation system

$$\begin{cases} \frac{du_1}{dt} = 4u_1 - 5u_2 \\ \frac{du_2}{dt} = 2u_1 - 3u_2 \end{cases} \qquad \text{with} \qquad \begin{cases} x_1 = u_1(0) \\ x_2 = u_2(0) \end{cases}$$

- Similar to a single equation, let

$$\begin{aligned} u_1(t) &= e^{\lambda t}x_1 \\ u_2(t) &= e^{\lambda t}x_2 \end{aligned}$$

Eigenvalue problem

- By substituting and cancellation of $e^{\lambda t}$, we obtain:

$$\begin{aligned} 4x_1 - 5x_2 &= \lambda x_1 \\ 2x_1 - 3x_2 &= \lambda x_2 \end{aligned}$$

or

$$\begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

- Eigenvalue equation:

$$Ax = \lambda x.$$

Eigenvalues and Eigenvectors

- Eigenvectors are the directions along which a linear transformation acts simply by **stretching/compressing** and/or **flipping**.
- We will show how eigenvectors make understanding linear transformations easy.
- We want to have as many linearly independent eigenvectors as possible associated to a single linear transformation.

Thank You!

$C = \begin{bmatrix} c_{ij} \end{bmatrix} \qquad A = \begin{bmatrix} a_{ij} \end{bmatrix}$

$< C$ سطر j ام $,\ A$ ستون k ام $> = [\quad][\quad] \boxed{= 0}\ j \neq k$

$$C^T A \overset{\text{سطر j ام}}{=} \underbrace{\begin{bmatrix} c_{1j} & \cdots & c_{nj} \end{bmatrix}} \underbrace{\begin{bmatrix} & \overset{\text{ستون k ام}}{\boxed{\begin{matrix} a_{1k} \\ \vdots \\ a_{nk} \end{matrix}}} \end{bmatrix}} = \begin{bmatrix} \det A & & 0 \\ & \ddots & \\ 0 & & \det A \end{bmatrix}$$

$$A C^T = C^T A = \underset{\neq 0}{\underline{\det A}}\ I$$

$$A^{-1} = \frac{1}{\det A} C^T$$

$C^T = \underline{\underline{\operatorname{adj}(A)}}$

$$C \qquad A^{-1} = \frac{1}{\det A} \operatorname{adj} A$$

$Ax = b$ $\qquad$ $\underline{\underline{\text{کرامر}}}$

$$Ax = b \Rightarrow \underbrace{(\operatorname{adj} A) A} x = (\operatorname{adj} A) b$$

$$\Rightarrow (\det A) x = (\operatorname{adj} A) b$$

$$\Rightarrow (\det A) \underline{\underline{x_j}} = \sum_{i=1}^{n} (\operatorname{adj})_{ji} b_i$$

$$= \boxed{\sum_{i=1}^{n} (-1)^{i+j} \det A(i|j)\, b_i}$$

$$(\operatorname{adj})_{ji} = c_{ij} = (-1)^{i-j} \det A(i|j)$$

$$x_j = \frac{\det B}{\det A}$$

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} \det \underline{\underline{A}}(i|j)\ a_{ij}$$

$$B = \begin{bmatrix} & \overset{\overset{\text{ستون j ام}}{\downarrow}}{b_1} & \\ & \vdots & \\ & b_n & \end{bmatrix}$$

که به جای ستون j ام از ماتریس A بردار b را گذاشته است