Lecture29

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Lecture29

Linear Algebra

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Review: Inner products on real linear space

An inner product on V is a function $\langle , \rangle : V \times V \to \mathbb{R}$ such that

1. $\langle v, v \rangle \geqslant 0$ for all $v \in V$.
2. $\langle v, v \rangle = 0$ if and only if $v = 0$.
3. $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ for all $u, v, w \in V$.
4. $\langle cu, w \rangle = c \langle u, w \rangle$ for all $u, w \in V$ and $c \in \mathbb{R}$.
5. $\langle v, w \rangle = \langle w, v \rangle$. متقارن

Review: Inner products on linear space

- The definition of the above inner product is not useful for complex vector spaces V.

$u \neq 0 \quad \langle u,u \rangle > 0$

- Let $0 \neq u \in V$ and $i \in \mathbb{C}$.

$$\underbrace{\langle iu, iu\rangle} = \underbrace{i^2}_{-1} \underbrace{\langle u, u\rangle}_{+} < 0.$$

$$\langle iu, iu \rangle = i \langle u, iu \rangle$$

$$= i \bar{i} \langle u, u \rangle$$

$$= +1 \langle u, u \rangle$$

Review: Inner products on complex linear space

مقادير $\mathbb{C}$

An inner product on V is a function $\langle , \rangle : V \times V \to \mathbb{R}$ such that

$\mathbb{C}$

1. $\langle v, v\rangle \geqslant 0$ for all $v \in V$.

$\langle v, v \rangle \in \mathbb{R}$

2. $\langle v, v\rangle = 0$ if and only if $v = 0$.
3. $\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$ for all $u, v, w \in V$.
4. $\langle cu, w\rangle = c\langle u, w\rangle$ for all $u, w \in V$ and $c \in \mathbb{R}$.
5. $\langle v, w\rangle = \overline{\langle w, v\rangle}$.

$\langle u, w \rangle = a + ib$

$\langle w, u \rangle = a - ib = \overline{a + ib}$

Review: Notes

$\langle \, , \, \rangle : V \times V \to \mathbb{C}$

Let V be an inner product. Then

1. $\langle w, \underbrace{u + v}\rangle = \langle w, u\rangle + \langle w, v\rangle$ for all $u, v, w \in V$.
2. $\langle u, cw\rangle = c\langle u, w\rangle$ for all $u, w \in V$ and $c \in \mathbb{R}$.

$\langle cw, v\rangle = c\langle w, v\rangle = \bar{c}\langle u, w\rangle$

Review: Symmetric matrices

Definition

A symmetric matrix is a square matrix that is equal to its transpose. $A = A^T$

$A \in M_n(F)$

Let $A \in M_n(\mathbb{R})$, then there is a matrix $B \in M_n(\mathbb{R})$ such that $\langle Ax, y\rangle = \langle x, By\rangle$ for each $x, y \in \mathbb{R}^n$. A^T

$\langle Ax, y\rangle = \langle x, A^T y\rangle$

$\langle Ax, y\rangle = \langle x, Ay\rangle$

IF $\dim V$

$B = \{v_1, \dots, v_n\}$

نمایش برای v در [illegible]

$w, v \in V$ $\quad T : V \to V$

$[v]_B = x$ $\quad A = [T]_B$

$[w]_B = y$ $\quad \langle Tv, w\rangle = \langle v, ?w\rangle$

$\langle Ax, y\rangle = \langle x, By\rangle$

$\langle A[v_i]_B, [v_j]_B\rangle = a_{ji}$

$\langle Ae_i, e_j\rangle = a_{ji}$ t_1, t_2

$\langle e_i, Be_j\rangle = b_{ij}$

$T : V \to V$

$[T]_B = [\quad]$

$[Tv_i]_B$

$A[v_i]_B =$ ستون iام $A = [T]_B$

Review: Hermitian matrices

Definition

A hermitian matrix is a square matrix, which is equal to its conjugate transpose matrix.

Let $A \in M_n(\mathbb{C})$, then there is a matrix $B \in M_n(\mathbb{C})$ such that $\langle Ax, y\rangle = \langle x, By\rangle$ for each $x, y \in \mathbb{C}^n$.

$\langle Ax, y\rangle = \langle x, A^H y\rangle \Rightarrow \langle Ax, y\rangle = \langle x, Ay\rangle$

$\langle Ax, y \rangle = \langle x, A^H y \rangle$; $A^H = A$ $\Rightarrow \langle Ax, y \rangle = \langle x, Ay \rangle$

هر ماتریسی که با هرمیتی خودش برابر باشد، آن را هرمیتی گویند

Review: Self-adjoint matrices

$A \in M_n(\mathbb{F})$, $\mathbb{F} = \mathbb{C} , \mathbb{R}$

Definition

A matrix $A \in \mathbb{F}$ is self-adjoint if $A^* = A$.

$\langle Ax, y \rangle = \langle x, Ay \rangle$

$A \in M_n(\mathbb{R})$

Definition

A matrix $A \in \mathbb{R}$ is symmetric if $A^T = A$.

$\langle Ax, y \rangle = \langle x, Ay \rangle$

$A \in M_n(\mathbb{C})$

Definition

A matrix $A \in \mathbb{C}$ is Hermitian if $A^H = A$.

$\langle Ax, y \rangle = \langle x, Ay \rangle$

Review: Unitary matrices

$U \in M_n(\mathbb{F})$

Definition

A matrix $U \in \mathbb{F}$ is unitary if $U^*U = UU^* = I$.

$U : \mathbb{F}^n \longrightarrow \mathbb{F}^n$

$(x, y) \xrightarrow{U} Ux, Uy$

$\langle x, y \rangle = \langle Ux, Uy \rangle$

- For each $x, y \in \mathbb{F}^n$,

$$\langle Ux, Uy \rangle = \langle x, \underbrace{U^*U}_{I} y \rangle . = \langle x, y \rangle$$

- If U is unitary, then

$$\langle Ux, Uy \rangle = \langle x, y \rangle .$$

That means U preserves inner product.

Review: Inner product on finite-dimensional linear spaces

1. Suppose that V is finite-dimensional linear space where $B = \{v_1, \ldots, v_n\}$ is an ordered basis for V.
2. We are given a particular inner product on V.
3. The inner product is completely determined by the entries of matrix G where
$$G_{ij} = \langle v_j, v_i \rangle.$$
4. Let $v, w \in V$. If $x = [v]_B$ and $y = [w]_B$, then
$$\langle v, w \rangle = y^* G x.$$
5. If $V = \mathbb{F}^n$. Then for each $x, y \in V$,
$$\langle x, y \rangle = y^* x,$$
if we consider standard basis for V.

$\langle x, y \rangle$ s

Inner product on V and change basis

- Suppose that $B = \{v_1, \ldots, v_n\}$ and $B' = \{v'_1, \ldots, v'_n\}$ are two bases for a linear space V.
- For each $w \in V$, $[w]_B = P[w]_{B'}$, where the i-th column of P is $[v'_i]_B$. So $v'_i = \sum_{r=1}^{n} P_{ri} v_r$.
- A matrix H as the inner product matrix respect to B':

$$H_{ij} = \langle v'_j, v'_i \rangle = \left\langle \sum_{r=1}^{n} P_{rj} v_r, \sum_{k=1}^{n} P_{ki} v_r \right\rangle$$
$$= \sum_{r=1}^{n} \sum_{k=1}^{n} P_{rj} P_{ki} \underbrace{\langle v_r, v_k \rangle}_{G_{kr}} G_{kr} = \sum_{r=1}^{n} \sum_{k=1}^{n} P_{rj} \bar{P}_{ki} G_{kr}$$
$$= (P^* G P)_{ij}$$

where G is the inner product respect to B.

- Consequently, $H = P^* G P$.

$\langle v, w \rangle = [w]_B G [v]_B$

$G_{ji} = \langle v_i, v_j \rangle$

G ماتریس ضرب داخلی نسبت به B

A ماتریس ضرب داخلی نسبت به B'

یادآوری: ستون i: $[v'_i]_B = P_{ri}$

$P = \begin{bmatrix} P_{1i} \\ \vdots \\ P_{ni} \end{bmatrix}$

$v'_i = \sum_{r=1}^{n} P_{ri} v_r$

$\langle x, x \rangle = x^* H x = x^* P^* G P x$

Review: The properties of G

1. $G_{ii} > 0$, for each $1 \leqslant i \leqslant n$.
2. G self-adjoint.
3. G is invertible.
4. $\det G > 0$.

Review: Is the above process reversible?

Let V in a lienar space on $\mathbb{R}$ with dimension n with a basis B.
Question. When a bilinear function $\langle , \rangle : V \times V \to \mathbb{F}$ such that

$$\langle v, w \rangle = y^* G x$$

and $x = [v]_B$ and $y = [w]_B$, is an inner product for $G \in M_n(\mathbb{F})$.

Is the above process reversible?

$x>0$
$x \neq 0 \quad \langle x, x \rangle = x^* G x \geqslant 0$

By the definition of an inner product, we should have

1. $\langle x, x \rangle = \underline{x^* G x} \geqslant 0$ for all $v \in V$ such that $[v]_B = x$.
2. G is self-adjoint $\underline{(G^* = G)}$.

$x -$

$\langle v, w \rangle = \overline{\langle w, v \rangle}$

Definition

A self-adjoint matrix $A \in M_n(\mathbb{F})$ is called

1. **positive definite** if $x^* A x > 0$ for each $0 \neq x \in \mathbb{F}^n$.
2. **positive semi-definite** if $x^T A x \geqslant 0$ for each $x \in \mathbb{F}^n$.

Self-adjoint matrices

$P^*P = PP^* = I$

Theorem

If A is a self-adjoint matrix, then an invertible matrix $P \in M_n(\mathbb{F})$ such that $P^*P = I$ and

$$P^*AP = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}.$$

$P = [p_1 \dots p_n] \rightarrow Ap_i = \lambda_i p_i \qquad \langle p_i, p_i \rangle = 1$

$i \neq j \qquad \langle p_i, p_j \rangle = 0$

Tests for positive definiteness

Theorem

Each of the following tests is a necessary and sufficient condition for the self-adjoint matrix A to be positive definite:

1. *All eigenvalues of A are positive.*
2. *All upper left submatrices A_k have positive determinants.*
3. *All pivots (without row exchanges) are positive.*

- The test brings together three of the most basic ideas in the book:
  1. pivots,
  2. determinants,
  3. eigenvalues.

$\det A_k$

$A =$ $k \times k$

$PA = LU$

Proof.

- First, we show that a self-adjoint matrix A is positive definite if and only if all eigenvalues of A are positive.

(⇐) فرض کنید λ مقدار ویژه A باشد: $0 \neq x$ ، بردار ویژه: $Ax = \lambda x$

$$\Downarrow$$

$$0 < x^* A x = \lambda x^* x = \lambda \underbrace{(\bar{x}_1 x_1 + \cdots + \bar{x}_n x_n)}_{>0} \Rightarrow \lambda > 0$$

$0 \neq x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

(⇒) A خودالحاق است پس $A = P \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} P^*$ ، از طرفی: $x \neq 0$

$P^* P = I$ ، داریم $y = P^* x \neq 0$

$$x^* A x = x^* P \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \underbrace{P^* x}_{y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}} = y^* \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} y = \lambda_1 \bar{y}_1 y_1 + \cdots + \lambda_n \bar{y}_n y_n > 0$$

$\lambda_i \bar{y}_i y_i > 0$

مثبت معین $\overset{(1)}{\Longleftrightarrow}$ $\lambda_i > 0$

$\Downarrow$ (2)

$\det A_k > 0$

$\Downarrow$ (3)

$d_k > 0$

(4) $d_k > 0 \Rightarrow$ مثبت معین

Proof.

- Second, we show that if all eigenvalues of self-adjoint matrix A is positive then all upper left submatrices A_k have positive determinants.

حکم: $\det A_k > 0$ $\equiv$ A_k (همه مقادیر ویژه) > 0 $\equiv$ A_k مثبت معین

(حاصلضرب مقادیر ویژه) > 0

$\equiv$ $0 \neq x_k \in F^k$ ، $0 < x_k^* A_k x_k$

$$0 < \begin{bmatrix} x_k^* & 0 \end{bmatrix} A \begin{bmatrix} x_k \\ 0 \end{bmatrix} = x_k^* A x_k$$

$$\Downarrow$$

$0 < \det A_k \Leftarrow$ مثبت معین A_k

Proof.

- Third, we show that if all upper left submatrices A_k of self-adjoint matrix A are positive then all pivots (without row exchanges) are positive.

$$A = LDU = \left[\begin{array}{c|c} L_k & 0 \\ \hline * & * \end{array}\right]\left[\begin{array}{c|c} D_k & 0 \\ \hline 0 & * \end{array}\right]\left[\begin{array}{c|c} U_k & * \\ \hline 0 & * \end{array}\right]$$

$$= \left[\begin{array}{c|c} L_k D_k U_k & * \\ \hline * & * \end{array}\right] \Longrightarrow A_k = L_k D_k U_k$$

$$\det A_k = \underbrace{\det L_k}_{1} \det D_k \underbrace{\det U_k}_{1}$$

$$\Leftarrow \det A_k = \det D_k = d_1 \cdots d_k > 0$$

$$d_k = \frac{\det A_k}{\det A_{k-1}} > 0$$

$$D = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix} \quad d_i > 0$$

Proof.

- Fourth, we show that if all pivots (without row exchanges) of self-adjoint matrix A are positive then A is positive definite.

$A = LDU$ تجزیه یکتاست

$A = A^* = U^* D^* L^*$

$= U^* D L^*$

$\Downarrow$

$L^* = U$

$A = LDU$ ، فرض $x \neq 0$ ، $x^* A x > 0$

$$x^* A x = x^* L D U x = (L^* x)^* D L^* x > 0$$

$y = L^* x$ چون L وارون‌پذیر است بنابراین

چون L^* وارون‌پذیر است پس $y \neq 0$

وارون‌پذیر

$$x^* A x = y^* D y > 0$$

Example

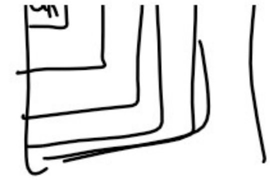
- Let $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$. The matrix A is positive semidefinite, by all three tests. For instance

- The eigenvalues are $\lambda_1 = 0, \lambda_2 = \lambda_3 = 3$ (a zero eigenvalue).

semi-positive

$\det A_k$

a_{11}



Positive definite matrices

$$A = \underline{R^T R}$$

Theorem

A matrix A is positive definite if and only if there is a matrix R, possibly with dependent columns, such that $A = R^* R$.

اشتباه است

ستونهای R

$$\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = y = Rx \neq 0 \Leftarrow x \neq 0 \Leftarrow A = R^* R$$

$$x^* A x = (x^* R^*)(Rx) > 0$$

$$= \sum_{i=1}^{n} \bar{y}_i y_i > 0$$

$$D = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix}, \quad A = L D L^* \qquad (\Leftarrow)$$

$$A = \underbrace{L \begin{bmatrix} \sqrt{d_1} & & \\ & \ddots & \\ & & \sqrt{d_n} \end{bmatrix}}_{R^*} \underbrace{\begin{bmatrix} \sqrt{d_1} & & \\ & \ddots & \\ & & \sqrt{d_n} \end{bmatrix} L^*}_{R} = R^* R$$

Cholesky decomposition

- Every positive definite matrix A can be factored as

$$A = R^* R,$$

where R is upper triangular with positive diagonal elements.

- $R^* R$ is called Cholesky decomposition for A.
- R is called the Cholesky factor of A.

$$\underset{\downarrow \\ \text{بالامثلثی}}{R} = \begin{bmatrix} \sqrt{d_1} & & \\ & \ddots & \\ & & \sqrt{d_n} \end{bmatrix} \underset{\downarrow \\ \text{بالامثلثی}}{L^*}$$

Positive definite square root

$A = P \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} P^*$

- Every positive definite matrix A can be factored as

$$A = \underline{PD}P^* = (P\sqrt{D})(\sqrt{D}P^*),$$

P^*P

where P is invertible matrix.

- So $A = PDP^* = (P\sqrt{D}P^*)(P\sqrt{D}P^*) = I$

- $P\sqrt{D}P^*$ is called **positive definite square root** of A.

جذر متقارن مثبت

$A = R^*R$

$A = (P\sqrt{D}P^*)^2$

Review: Positive definite and positive semi-definite matrices

Definition

A hermitian matrix $A \in M_n(\mathbb{C})$ is called

1. **positive definite** if $x^H A x > 0$ for each $0 \neq x \in \mathbb{C}$.
2. **positive semi-definite** if $x^H A x \geqslant 0$ for each $x \in \mathbb{C}$.

Remark

- **Remark.** The conditions for semidefiniteness could also be deduced from *tests for definiteness* by the following trick:
  1. Add a small multiple of the identity to get a positive definite matrix
     $$A + \epsilon I.$$
  2. Then let ϵ approach zero.
  3. At ϵ they must still be nonnegative.

A نیمه معین

$x^* A x \geq 0$

$A + \epsilon I$

$x \neq 0 \qquad x^*(A + \epsilon I)x > 0$

$\underbrace{x^* A x} + \underbrace{\epsilon x^* x} > 0$

$x^* A x \geq 0$

Tests for positive semi-definiteness

Theorem

Each of the following tests is a necessary and sufficient condition for the self-adjoint matrix A to be positive semi-definite:

1. *All eigenvalues of A are non-negative.*
2. *All upper left submatrices A_k have non-negative determinants.*
3. *All pivots (without row exchanges) are non-negative.*

A useful lemma for self-adjoint matrices

Lemma

Let $A \in M_n(\mathbb{F})$ be a self-adjoint matrix then there is an invertible matrix $P \in M_n(\mathbb{F})$ such that $P^*P = I$ and

$$A = P \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} P^*.$$

ستون‌ها متعامد

قطری

سطرها متعامد است.

where $\lambda_1, \ldots, \lambda_n$ are eigenvalues of A.

- A singular value decomposition (SVD) is a generalization of this where $A \in M_{mn}(\mathbb{F})$ does not have to be self-adjoint or even square.

بله!

$$A = U \begin{bmatrix} \; \end{bmatrix} V^*$$

ستون‌ها متعامد

قطری

سطرها متعامد

$A \in$ self-adjoint

...

$x^* s x$

$x \neq 0 \quad \langle x, x \rangle$

$x \neq 0 \quad x^* A x > 0$

Thank You!

$$f(x) = (x - \lambda_1)^{d_1} \cdots (x - \lambda_k)^{d_k}$$

$A \in M_n(\mathbb{F})$

$$p(x) = (x - \lambda_1)^{r_1} \cdots (x - \lambda_k)^{r_k}$$

$\mathbb{C}$

$$V = W_1 \oplus \cdots \oplus W_k \qquad W_i = N\left((A - \lambda_i I)^{r_i}\right)$$

$$\dim W_i = d_i \geq r_i$$

$\dim W_i = d_i \geq r_i$

$N_i = T|_{W_i} - \lambda_i I \qquad N_i^{r_i} = 0$

if $\dim W_i = d_i = r_i \qquad W_i = Z(v, N_i)$

$B_i = \{v, N_i v, \dots, N_i^{r_i - 1} v\}$

$$[T|_{W_i}] = \begin{bmatrix} \lambda_i & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & \lambda_i \end{bmatrix} \Leftarrow [N_i]_{B_i} = \begin{bmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}$$

ماتریس [illegible] | ماتریس [illegible] $N_i|_{W_i}$

$$W_i = Z(v_{i_1}, N_i) \oplus \cdots \oplus Z(v_{i t_i}, N_i)$$

$p_i \mid p_{i-1} \qquad P_{v_{i t_i}} \mid \cdots \mid P_{v_{i_2}} \mid P_{v_{i_1}} = P_{N_i} = x^{r_i}$

$$[N_i]_{B_i} = \left[\begin{array}{c|c} \begin{matrix} 0 & & \\ 1 & \ddots & \\ & \ddots & 0 \end{matrix} & \\ \hline & \begin{matrix} 0 & & \\ 1 & \ddots & \\ & \ddots & \end{matrix} \\ & & \ddots \\ & & & \begin{matrix} 0 & \\ 1 & \ddots \end{matrix} \end{array}\right] = \begin{bmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}$$

$$[T|_{W_i}]_{B_i} = [T_i]_{B_i} = \begin{bmatrix} \lambda_i & & 0 \\ 1 & \ddots & \\ 0 & 1 & \lambda_i \end{bmatrix}$$

$v \in \underline{W_i} = \qquad (A - \lambda_i I) \qquad T:$

$$V = \underbrace{Z(v_1, A)}_{T_1} \oplus \cdots \oplus \underbrace{Z(v_k, A)}_{T_n}$$

$[T]^{T_i}$

$P \mid \cdots \mid P \mid P_{v_1} = P_A$

T_1

$$P_{v_k} \mid \cdots \mid P_{v_2} \mid P_{v_1} = P_A$$

$$P_A(x) = x^m + a_{m-1}x^{m-1} + \cdots + a_0 \qquad \deg P_A = \dim Z(v, A)$$

$$B = \{v, Av, \ldots, A^{m-2}, A^{m-1}v\}$$

$$[T]_{Z(v_1, A)} = \begin{bmatrix} 0 & 0 & & 0 & -a_0 \\ 1 & 0 & & \vdots & -a_1 \\ 0 & 1 & & \vdots & \\ \vdots & & \ddots & \vdots & \\ 0 & & & 1 & -a_{m-1} \end{bmatrix}$$

$$A(A^{m-1}v) = A^m v$$

$$A^m v - a_{m-1}A^{m-1}v - \cdots = 0$$

$T : V \to V$

$A \in M_n(\mathbb{F})$ ، $p(x)$ مینیمال باشد آنگاه $\frac{\mathbb{F}^n}{V}$ vt تا مورد اولیه

$$P_v(x) = p(x)$$