Linear Algebra

Samira Hossein Ghorban
s.hosseinghorban@ipm.ir

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Primary Decomposition Theorem

Theorem

Let T be a linear function over a finite dimensional linear space V whose minimal polynomial factorizes as

$$p(x) = p_1^{r_1}(x) \cdots p_k^{r_k}(x),$$

where the p_i's are monic and mutually coprime (i.e. have no nontrivial common factors). Let $W_i = N\left(p_i^{r_i}(T)\right)$ for each $1 \leqslant i \leqslant k$. Then

1. $V = W_1 \oplus \cdots \oplus W_k$.
2. *For each $1 \leqslant i \leqslant k$, $T(W_i) \subseteq W_i$.*
3. *The minimal polynomial of $T_i = T\restriction_{W_i}$ is $p_i(x)$.*

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- Note that a linear function T is diagonalizable if and only if its minimal polynomial factorizes as

$$p(x) = (x - \lambda_1) \cdots (x - \lambda_k).$$

Matrix Representation

- Suppose that T is a linear function on V with the characteristic polynomial is

$$f(x) = (x - \lambda_1)^{d_1} \dots (x - \lambda_k)^{d_k}$$

where $\lambda_1, \dots, \lambda_k$ are distinct elements and $d_i \geqslant 1$.

- Then the minimal polynomial for T will be

$$p(x) = (x - \lambda_1)^{r_1} \dots (x - \lambda_k)^{r_k}$$

where $1 \leqslant r_i \leqslant d_i$ based on the Cayley–Hamilton theorem.

- If W_i is the null space of $(T - \lambda_i I)^{r_i}$, then the primary decomposition theorem tells us that

$$V = W_1 \oplus \cdots \oplus W_k$$

such that the linear function $T_i = T\restriction_{W_i} : W_i \to W_i$ has minimal polynomial $(x - \lambda_i)^{r_i}$.

Matrix Representation

Suppose that B_i is a basis for W_i. It has been proved that $B = \bigcup_{i=1}^{k} B_i$ is a basis for V. Based on primary decomposition theorem,

$$T(W_i) \subseteq W_i.$$

Thus

$$[T]_B = \begin{bmatrix} [T_1]_{B_1} & & \\ & \ddots & \\ & & [T_k]_{B_k} \end{bmatrix}.$$

Matrix Representation

$$V = W_1 \oplus \cdots \oplus W_k$$

and

$$[T]_B = \begin{bmatrix} [T_1]_{B_1} & & \\ & \ddots & \\ & & [T_k]_{B_k} \end{bmatrix}.$$

- Let N_i be the linear function on W_i defined by $N_i = T - \lambda_i I$.
- Then N_i is nilpotent and has minimal polynomial x^{r_i}.
- Thus, T on W_i acts as N_i plus the scalar λ_i times the identity function I.
- Suppose we choose a basis for the subspace W_i and then find the representation matrix of N_i on W_i.

Review: Nilpotent matrices and Nilpotent linear functions

Definition

A square matrix A is called nilpotent matrix with degree non-negative integer k if A^k is the zero matrix and A^r is the non-zero matrix for each r, $1 \leqslant r \leqslant k$.

Definition

A be a linear function T on V is called nilpotent linear function with degree non-negative integer k if T^k is the zero linear function and T^r is the non-zero one for each r, $1 \leqslant r \leqslant k$.

Review: Example

Let $A \in M_3(\mathbb{R})$ be the following matrix:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then

$$A^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq \mathbf{O}$$

The third power of A is

$$A^3 = A^2 A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{O}$$

Review: Nilpotent matrices (Revised version)

Lemma

Let $A \in M_n(\mathbb{F})$ *where* $\mathbb{F} = \mathbb{R}$ *or* $\mathbb{C}$.

1) *The matrix* A *is nilpotent if and only if all the eigenvalues of* A *is zero.*
2) *The matrix* A *is nilpotent if and only if* $A^n = O$.

Nilpotent matrices

Lemma

Let V be a finite dimensional linear space. If a linear function T on V is nilpotent with degree n where $n = \dim V$, then there is a basis for V such that

$$[T]_B = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & 0 & 0 \\ 0 & 0 & \ddots & 0 & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

Lemma

Let V be a finite dimensional linear space. Then there is a vector $v in V$ whose minimal polynomial respect to v is minimal polynomial T.

Proof.

Review: Representation Matrix

- For the linear function T:
  1. The characteristic polynomial: $f(x) = (x-\lambda_1)^{d_1} \dots (x-\lambda_k)^{d_k}$
  2. The minimal polynomial: $p(x) = (x-\lambda_1)^{r_1} \dots (x-\lambda_k)^{r_k}$.
- $V = W_1 \oplus \cdots \oplus W_k$ where $W_i = N\big((T-\lambda_i)^{r_i}\big)$

$$[T]_B = \begin{bmatrix} [T_1]_{B_1} & & \\ & \ddots & \\ & & [T_k]_{B_k} \end{bmatrix}.$$

- Let N_i be the linear function on W_i defined by $N_i = T - \lambda_i I$.
- Then N_i is nilpotent and has minimal polynomial x^{r_i} on W_i.
- But $\dim W_i = d_i \geqslant r_i$ (?)

The smallest T-invariant subspace containing v

- Assume V is finite-dimensional linear space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and T is a fixed (but arbitrary) linear function on V.
- If W is any subspace of V which is invariant under T and contains v, then W must also contain the vector $T(v)$; hence W must contain $T(Tv) = T^2v$, $T(T^2v) = T^3v$, etc. In other words W must contain $g(T)v$, for every polynomial $g(x)$ over $\mathbb{F}$. The set of all polynomial $g(x)$ over $\mathbb{F}$ is denoted by $\mathbb{F}[x]$
- Let $Z(v, T) = \{g(T)v \mid g(x) \in \mathbb{F}[x].\}$
- $Z(v, T)$ is a subspace of V and it is the smallest T-invariant subspace which contains v.

T-cyclic subspace generated by v

Definition

If v is any vector in V, the subspace $Z(v, T)$ is called the T-**cyclic subspace generated**. If $Z(v, T) = V$, then v is called a cyclic vector for T.

For any T:

1. The $T-$cyclic subspace generated by the zero vector is the zero subspace.
2. The space $Z(v, T)$ is one-dimensional if and only if v is an eigenvalue vector for T.
3. Thus, we shall be interested in linear relations:

$$c_0 v + c_1 T v + \cdots, c_k T^k v = 0.$$

between the vectors $T^j v$, that is we shall be interested in the polynomials

$$c_0 + c_1 x + \cdots, c_k x^k = 0$$

which have the property that $g(T)v = 0$.

Theorem

Assume that T is a linear space on a linear space V. Let v be any non-zero vector in V and let $p_v(x)$ is the minimal polynomial for v respect to T.

1. $\dim Z(v, T) = degp_v(x)$.
2. *If U is the linear function on $Z(v, T)$ induced by T, then the minimal polynomial for U is $p_v(x)$.*

Minimal and characteristic polynomials of a cyclic vector

Theorem

T has a cyclic vector if and only if the minimal and characteristic polynomials for T are identical.

Cyclic Decomposition Theorem

Theorem

Let T be a linear function on a finite-dimensional vector space V. There exist non-zero vectors $v_1, \ldots, v_k \in V$ with minimal polynomial $p_{v_1}, \ldots, p_{v_k}$ such that

(i) $V = Z(v_1, T) \oplus \cdots \oplus Z(v_k, T)$.

(ii) $p_{v_i} \mid p_{v_{i-1}}$ *for each* $i \geqslant 2$.

(iii) *Furthermore, the integer r and the minimal polynomial $p_{v_1}, \ldots, p_{v_k}$ are uniquely determined by (i), (ii).*

Jordan Form

T-cyclic linear space

Lemma

Let T is a linear function on V such that $B = \{v, Tv, \ldots, T^{n-1}v\}$ is a basis for V where $0 \neq v \in V$. Then

$$[T]_B = \begin{bmatrix} 0 & 0 & \cdots & 0 & -c_0 \\ 1 & 0 & \cdots & 0 & -c_1 \\ 0 & 1 & \ddots & 0 & -c_2 \\ 0 & 0 & \ddots & 0 & \vdots \\ 0 & 0 & \cdots & 1 & -c_{n-1} \end{bmatrix}$$

where $p(x) = x^n + c_{n-1}x^{n-1} + \ldots + c_1x^1 + c_0$ is the minimal polynomial for T.

Rational Form

- By Cyclic Decomposition Theorem: $V = Z(v_1, T) \oplus \cdots \oplus Z(v_k, T)$.
- Matrix representation by diagonal blocks:

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & -c_0 \\ 1 & 0 & \cdots & 0 & -c_1 \\ 0 & 1 & \ddots & 0 & -c_2 \\ 0 & 0 & \ddots & 0 & \vdots \\ 0 & 0 & \cdots & 1 & -c_{n-1} \end{bmatrix}$$

Thank You!