Linear Algebra

Samira Hossein Ghorban
s.hosseinghorban@ipm.ir

Fall, 2021

The solution of Linear Equations

$$
\begin{cases}
a_{11}x_1 & + & \cdots & + & a_{1n}x_n & = & b_1 \\
\vdots & & \vdots & & \vdots & = & \vdots \\
a_{n1}x_1 & + & \cdots & + & a_{nn}x_n & = & b_n
\end{cases}
$$

n equation in n unknowns.

- Matrix form
- Row picture
- Column picture

Example for $n = 2$

$$\begin{cases} x_1 & - & x_2 & = & 1 \\ x_1 & + & 2x_2 & = & 4 \end{cases}$$

Example for $n = 2$

$$\begin{cases} x_1 - x_2 = 1 \\ x_1 + 2x_2 = 4 \end{cases}$$

- Matrix form

$$\underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 1 \\ 4 \end{bmatrix}}_{b}$$

$$Ax = b$$

Example for $n = 2$

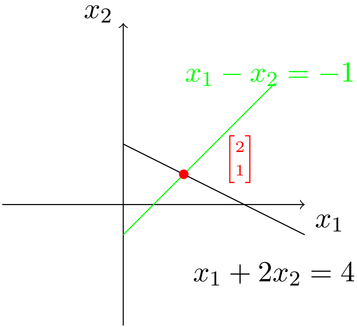
$$\begin{cases} x_1 - x_2 = 1 \\ x_1 + 2x_2 = 4 \end{cases}$$

- Matrix form

$$\underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 1 \\ 4 \end{bmatrix}}_{b}$$

$$Ax = b$$

- Row picture



Example for $n = 2$

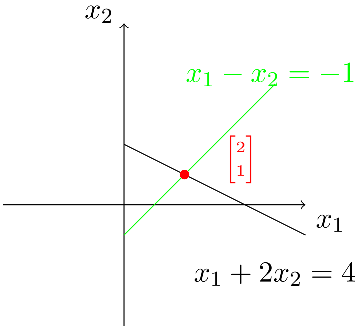
$$\begin{cases} x_1 - x_2 = 1 \\ x_1 + 2x_2 = 4 \end{cases}$$

- Matrix form

$$\underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 1 \\ 4 \end{bmatrix}}_{b}$$

$$Ax = b$$

- Row picture



- Column picture

$$\underbrace{x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix}} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

a linear combination of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

Example for $n = 3$

$$\begin{cases} 2x_1 - x_2 = 0 \\ -x_1 + 2x_2 - x_3 = -1 \\ 3x_2 + 4x_3 = 4 \end{cases}$$

- Matrix form

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

$$Ax = b$$

- Row picture

- Column picture

$$x_1 \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

$x_1 = 0 \quad x_2 = 0 \quad x_3 = 1$

Example for $n = 3$

$$\begin{cases} 2x_1 - x_2 = 0 \\ -x_1 + 2x_2 - x_3 = -1 \\ 3x_2 + 4x_3 = 4 \end{cases}$$

- Matrix form

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

$$Ax = b$$

- Row picture

- Column picture

$$x_1 \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

$$x_1 = 0 \qquad x_2 = 0 \qquad x_3 = 1$$

Example for $n = 3$

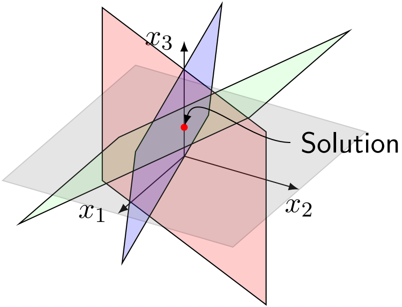
$$\begin{cases} 2x_1 - x_2 = 0 \\ -x_1 + 2x_2 - x_3 = -1 \\ 3x_2 + 4x_3 = 4 \end{cases}$$

- Matrix form

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

$$Ax = b$$

- Row picture



- Column picture

$$x_1 \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

$x_1 = 0 \quad x_2 = 0 \quad x_3 = 1$

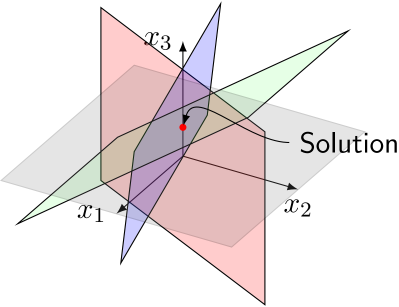
$$\begin{cases} 2x_1 - x_2 = 0 \\ -x_1 + 2x_2 - x_3 = -1 \\ 3x_2 + 4x_3 = 4 \end{cases}$$

- Matrix form

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

$$Ax = b$$

- Row picture



- Column picture

$$x_1 \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

$x_1 = 0$ $\quad x_2 = 0$ $\quad x_3 = 1$

Example for $n = 3$

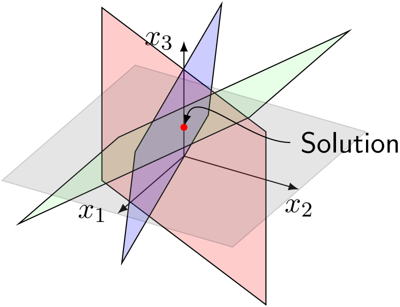
$$\begin{cases} 2x_1 - x_2 = 0 \\ -x_1 + 2x_2 - x_3 = -1 \\ 3x_2 + 4x_3 = 4 \end{cases}$$

- Matrix form

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

$$Ax = b$$

- Row picture



- Column picture

$$x_1 \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

$x_1 = 0 \qquad x_2 = 0 \qquad x_3 = 1$

For another given $b \in \mathbb{R}^3$

$$x_1 \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$$

$$x_1 = 1 \qquad x_2 = 1 \qquad x_3 = 0$$

For another given $b \in \mathbb{R}^3$

$$x_1 \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$$

$$x_1 = 1 \qquad x_2 = 1 \qquad x_3 = 0$$

The equation system $Ax = b$ has a solution.

For every $b \in \mathbb{R}^3$

- Can we solve $Ax = b$ for every $b \in \mathbb{R}^3$?

For every $b \in \mathbb{R}^3$

- Can we solve $Ax = b$ for every $b \in \mathbb{R}^3$?

- Is there a linear combination of $\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$ to be equal to b?

For every $b \in \mathbb{R}^3$

- Can we solve $Ax = b$ for every $b \in \mathbb{R}^3$?

- Is there a linear combination of $\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$ to be equal to b?

- Do all linear combinations of $\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$ fill $\mathbb{R}^3$?

Linear Independence

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & 3 & 4 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 4 & 0 \\ -1 & -2 & -1 \\ 0 & 0 & 4 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Linear function picture of the linear equation system

$$\begin{cases} 2x_1 - x_2 = 0 \\ -x_1 + 2x_2 - x_3 = -1 \\ 3x_2 + 4x_3 = 4 \end{cases}$$

Linear function picture of the linear equation system

$$\begin{cases} \begin{array}{rcrcrcr} 2x_1 & - & x_2 & & & = & 0 \\ -x_1 & + & 2x_2 & - & x_3 & = & -1 \\ & & 3x_2 & + & 4x_3 & = & 4 \end{array} \end{cases}$$

$$T : \mathbb{R}^3 \to \mathbb{R}^3$$
$$T(x) = Ax$$

Linear function picture of the linear equation system

$$\begin{cases} 2x_1 - x_2 = 0 \\ -x_1 + 2x_2 - x_3 = -1 \\ 3x_2 + 4x_3 = 4 \end{cases}$$

$$T : \mathbb{R}^3 \to \mathbb{R}^3$$
$$T(x) = Ax$$

$$range(T) = \{T(x) \mid x \in \mathbb{R}^3\}$$

Linear function picture of the linear equation system

$$\begin{cases} 2x_1 - x_2 = 0 \\ -x_1 + 2x_2 - x_3 = -1 \\ 3x_2 + 4x_3 = 4 \end{cases}$$

$$T : \mathbb{R}^3 \to \mathbb{R}^3$$
$$T(x) = Ax$$

$$range(T) = \{T(x) \mid x \in \mathbb{R}^3\}$$

Two following questions are equivalent:

Can $Ax = b$ be solved for a given $b \in \mathbb{R}^3$?

$b \in range(T)$?

Linear functions

The properties of

$$T : \mathbb{R}^3 \to \mathbb{R}^3$$
$$T(x) = Ax$$

1. $T(u + v) = T(u) + T(v)$ for all $u, v \in \mathbb{R}^3$.
2. $T(cu) = cT(u)$ for all $c \in \mathbb{R}$.

Every function from $\mathbb{R}^n$ to $\mathbb{R}^n$ with two above properties is called linear function.

An Example of a linear function and its representation

$$T : \mathbb{R}^3 \to \mathbb{R}^3$$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_2 + x_3 \\ 0 \end{bmatrix}$$

An Example of a linear function and its representation

$$T : \mathbb{R}^3 \to \mathbb{R}^3$$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_2 + x_3 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = xT\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) + yT\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) + zT\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$$

$$xT\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) + yT\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right) + zT\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Matrix Representations of Linear Function

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear function and $u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \in \mathbb{R}^n$.

In the n-dimensional Euclidean space $\mathbb{R}^n$, the standard basis consists of n distinct vectors

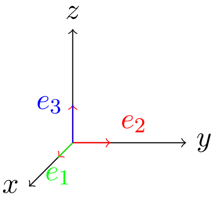
$$E = \{e_i \mid 1 \leq i \leq n\}$$

where e_i denotes the vector with a 1 in the i-th coordinate and 0's elsewhere. Since T is a linear function, we have

$$\begin{aligned} T(u) = T(u_1 e_1 + \ldots + u_n e_n) &= u_1 T(e_1) + \ldots + u_n T(e_n) \\ &= \begin{bmatrix} T(e_1) & \cdots & T(e_n) \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \end{aligned}$$

Matrix Representations of Linear Function

The matrix $\begin{bmatrix} T(e_1) & \cdots & T(e_n) \end{bmatrix}$ is called the matrix representation of linear function (transformation)T which is denoted by $[T]_E$.



$$E = \{e_1, e_2, e_3\}$$

Matrix Representations of Linear Function

The matrix $\begin{bmatrix} T(e_1) & \cdots & T(e_n) \end{bmatrix}$ is called the matrix representation of linear function (transformation)T which is denoted by $[T]_E$.

$E = \{e_1, e_2, e_3\}$ B=$\{B_1, B_2, B_3\}$

What is the relation between $[T]_B$ and $[T]_E$?

Change of basis

$$
\begin{tikzcd}
\mathbb{R}^n \arrow[r, "{[T]_B}"] \arrow[d, "P"'] & \mathbb{R}^n \\
\mathbb{R}^n \arrow[r, "{[T]_E}"'] & \mathbb{R}^n \arrow[u, "P^{-1}"']
\end{tikzcd}
$$

Change of basis

$$
\begin{tikzcd}
\mathbb{R}^n \arrow[r, "{[T]_B}"] \arrow[d, "P"'] & \mathbb{R}^n \\
\mathbb{R}^n \arrow[r, "{[T]_E}"'] & \mathbb{R}^n \arrow[u, "P^{-1}"']
\end{tikzcd}
$$

$$[T]_B = P^{-1}[T]_E P$$

Linear Spaces

- $\mathbb{R}^n$

Linear Spaces

- $\mathbb{R}^n$
- The space of all polynomials of the degree $\leq n-1$.

$$P_{n-1}(x) = \{a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 \mid a_i \in \mathbb{R} \text{ for all } 0 \leq i \leq n-1\}$$

- $\mathbb{R}^n$
- The space of all polynomials of the degree $\leq n-1$.

  $P_{n-1}(x) = \{a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 \mid a_i \in \mathbb{R} \text{ for all } 0 \leq i \leq n-1\}$

- Is the operation of **differentiation**

$$\frac{d}{dt} : P_3(x) \to P_2(x)$$
$$\frac{d}{dt}(a_3x^3 + a_2x^2 + a_1x + a_0) = 3a_3x^2 + 2a_2x + a_1$$

  is a linear transformation?

The matrix representation

- We have $B = \{x^3, x^2, x, 1\}$ and $B' = \{x^2, x, 1\}$ are bases for $P_3(x)$ and $P_2(x)$, respectively.

The matrix representation

- We have $B = \{x^3, x^2, x, 1\}$ and $B' = \{x^2, x, 1\}$ are bases for $P_3(x)$ and $P_2(x)$, respectively.

- Since $\begin{bmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{bmatrix}$ the vector representation of $a_3x^3 + a_2x^2 + a_1x + a_0 \in P_3(x)$, we have

The matrix representation

- We have $B = \{x^3, x^2, x, 1\}$ and $B' = \{x^2, x, 1\}$ are bases for $P_3(x)$ and $P_2(x)$, respectively.

- Since $\begin{bmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{bmatrix}$ the vector representation of $a_3x^3 + a_2x^2 + a_1x + a_0 \in P_3(x)$, we have

$$\left[\frac{d}{dt}\right]_{\{B,B'\}} = \begin{bmatrix} \frac{d}{dt}(x^3) & \frac{d}{dt}(x^2) & \frac{d}{dt}(x) & \frac{d}{dt}(1) \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The operation of differentiation

$$\frac{d}{dt} : \mathbb{R}^4 \to \mathbb{R}^3$$

$$\frac{d}{dt}\begin{bmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{bmatrix} = \begin{bmatrix} 3a_3 \\ 2a_2 \\ a_1 \end{bmatrix}$$

Note. We can suppose $\frac{d}{dt} : P_3(x) \to P_3(x)$ where $range(\frac{d}{dt}) = P_2(x)$ and we have

$$\left[\frac{d}{dt}\right]_B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Subspaces

A subspace U of a linear space V satisfies the following properties

- additive identity
- closed under addition
- closed under scalar multiplication

Example. $U = P_2(x)$ is a subspace of $V = P_3(x)$.

Invariant Subspaces

- Let $T : V \to V$.
  A subspace of $U \subset V$ is *invariant* under T if for each $u \in U$, $T(u) \in U$.

Invariant Subspaces

- Let $T : V \to V$.
  A subspace of $U \subset V$ is *invariant* under T if for each $u \in U$, $T(u) \in U$.

- If we can find invariant subspace such that

$$V = U_1 \oplus \cdots \oplus U_m$$

then to understand the behaviour if T, we need only understand

the behaviour of T on each U_i, that means $T|_{U_i}$

Invariant Subspaces

- Let $T : V \to V$.
  A subspace of $U \subset V$ is *invariant* under T if $u \in U$, then $T(u) \in U$.

Invariant Subspaces

- Let $T : V \to V$.
  A subspace of $U \subset V$ is *invariant* under T if $u \in U$, then $T(u) \in U$.

- If we can find invariant subspace such that

$$V = U_1 \oplus \cdots \oplus U_m$$

then to understand the behaviour if T, we need only understand

the behaviour of T on each U_i, that means $T|_{U_i}$

Invariant Subspaces

- Let $T : V \to V$.
  A subspace of $U \subset V$ is *invariant* under T if $u \in U$, then $T(u) \in U$.

- If we can find invariant subspace such that

$$V = U_1 \oplus \cdots \oplus U_m$$

  then to understand the behaviour if T, we need only understand

  the behaviour of T on each U_i, that means $T|_{U_i}$
- Think about the Matrix representation of T!

Eigenvalues and eigenvectors

How does a linear function $T : V \to V$ behave on an invariant subspace of dimension 1 U where

$$U = \{cu \mid c \in \mathbb{R}\}$$

where $u \neq 0$.

Eigenvalues and eigenvectors

How does a linear function $T : V \to V$ behave on an invariant subspace of dimension 1 U where

$$U = \{cu \mid c \in \mathbb{R}\}$$

where $u \neq 0$.

The subspace U is invariant under T, if there is a scalre $\lambda \in \mathbb{R}$ such that $T(u) = \lambda u$.

Eigenvalues and eigenvectors

How does a linear function $T : V \to V$ behave on an invariant subspace of dimension 1 U where

$$U = \{cu \mid c \in \mathbb{R}\}$$

where $u \neq 0$.

The subspace U is invariant under T, if there is a scalre $\lambda \in \mathbb{R}$ such that $T(u) = \lambda u$.

The scaler λ is called an ***eigenvalue*** of T and the vector u is called an ***eigenvector***

Upper Triangular

Denote $\text{span}(v_1, \cdots, v_k) = \{c_1 v_1 + \cdots + c_k v_k \mid c_i \in \mathbb{R} \text{ for all } 1 \leq i \leq k\}$.

Upper Triangular

Denote $\mathrm{span}(v_1, \cdots, v_k) = \{c_1 v_1 + \cdots + c_k v_k \mid c_i \in \mathbb{R} \text{ for all } 1 \leq i \leq k\}$.

Let $T : V \to V$ be a linear function.
Assume $B = \{v_1, \ldots, v_n\}$ is a basis for V such that for each $k = 1, \ldots, n$

$$T(v_k) \in \mathrm{span}(v_1, \cdots, v_k).$$

Upper Triangular

Denote span$(v_1, \cdots, v_k) = \{c_1 v_1 + \cdots + c_k v_k \mid c_i \in \mathbb{R} \text{ for all } 1 \le i \le k\}$.

Let $T : V \to V$ be a linear function.
Assume $B = \{v_1, \ldots, v_n\}$ is a basis for V such that for each $k = 1, \ldots, n$

$$T(v_k) \in \text{span}(v_1, \cdots, v_k).$$

We have

$$\begin{aligned}
T(v_1) &= a_{11} v_1 \\
T(v_2) &= a_{12} v_1 + a_{22} v_2 \\
T(v_3) &= a_{13} v_1 + a_{23} v_2 + a_{33} v_3 \\
&\vdots \\
T(v_n) &= a_{1n} v_1 + \ldots + a_{nn} v_n
\end{aligned}$$

Its matrix representation:

$$\begin{bmatrix}
a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\
 & a_{22} & a_{23} & \cdots & a_{2n} \\
 & & a_{33} & \cdots & \vdots \\
 & 0 & & \ddots & \vdots \\
 & & & & a_{nn}
\end{bmatrix}$$

Inner product spaces

Now, we want to add an algebraic structure on linear space to generalize the idea of the angel between two vectors in $\mathbb{R}^2$.

Inner product spaces

Now, we want to add an algebraic structure on linear space to generalize the idea of the angel between two vectors in $\mathbb{R}^2$.

An inner product on V is a function $\langle , \rangle : V \times V \to \mathbb{R}$ such that

- $\langle v, v \rangle \geq 0$ for all $v \in V$.
- $\langle v, v \rangle = 0$ if and only if $v = 0$.
- $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ for all $u, v, w \in V$.
- $\langle cu, w \rangle = c \langle u, w \rangle$ for all $u, w \in V$ and $c \in \mathbb{R}$.
- $\langle v, w \rangle = \langle w, v \rangle$.

Norms

Example of an inner product on $\mathbb{R}^n$:

$$\left\langle \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}, \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \right\rangle = u_1 v_1 + \ldots + u_n v_n.$$

Norms

Example of an inner product on $\mathbb{R}^n$:

$$\left\langle \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}, \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \right\rangle = u_1 v_1 + \ldots + u_n v_n.$$

For $v \in V$, the ***norm*** of v is denoted by $\|v\|$ and

$$\|v\| = \sqrt{\langle v, v \rangle}$$

Thank You