Linear Algebra

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Review

- Gaussian Elimination

$$
\left[\begin{array}{rrr|r} 1 & 2 & 3 & 6 \\ 2 & -1 & 4 & 8 \\ -1 & 8 & 2 & 12 \end{array}\right] \qquad \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 + R_1 \end{array}
$$

$$
\Rightarrow \quad \left[\begin{array}{rrr|r} 1 & 2 & 3 & 6 \\ 0 & -5 & -2 & -4 \\ 0 & 10 & 5 & 18 \end{array}\right] \qquad \begin{array}{l} \\ \\ R_3 + 2R_2 \end{array}
$$

$$
\Rightarrow \quad \left[\begin{array}{rrr|r} 1 & 2 & 3 & 6 \\ 0 & -5 & -2 & -4 \\ 0 & 0 & 1 & 10 \end{array}\right] \qquad \begin{array}{l} \\ \\ x_3 = 10, \quad x_2 = -\frac{16}{5}, \quad x_1 = -\frac{88}{5} \end{array}
$$

The Breakdown of Elimination

- Under what circumstances could the process break down?
- Example:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 2 & 2 & 5 & b_2 \\ 4 & 6 & 8 & b_3 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|r} 1 & 1 & 1 & b_1 \\ 0 & 0 & 3 & -2b_1 + b_2 \\ 0 & 2 & 4 & -4b_1 + b_3 \end{array}\right].$$

So

$$\Rightarrow \left[\begin{array}{ccc|r} 1 & 1 & 1 & b_1 \\ 0 & 2 & 4 & -4b_1 + b_3 \\ 0 & 0 & 3 & -2b_1 + b_2 \end{array}\right]$$

- The system is now triangular, and back-substitution will solve it. The system is called Nonsingular system.

The Breakdown of Elimination

- Under what circumstances could the process break down?
- Example:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 2 & 2 & 5 & b_2 \\ 4 & 4 & 8 & b_3 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|r} 1 & 1 & 1 & b_1 \\ 0 & 0 & 3 & -2b_1 + b_2 \\ 0 & 0 & 4 & -4b_1 + b_3 \end{array}\right].$$

- The system is called singular system.

Permutation Matrices

- The simplest permutations exchange two rows.

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Other permutations exchange more rows.

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

- A 3 by 3 permutation matrix P with $P^3 = I$:

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}^3}_{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

- Some power of a permutation matrix is the identity.

Permutation Matrices

Definition

P be a square matrix of order n. If P is a binary matrix (every entry in it is either 0 or 1, also called 0-1 matrix or (0, 1) matrix) and there is a unique "1" in its every row and every column, then P is called a permutation matrix .

- Some power of a permutation matrix is the identity.

Problem

- If P_1 and P_2 are permutation matrices, so is P_1P_2. This still has the rows of I in some order. Give examples with $P_1P_2 \neq P_2P_1$ and $P_3P_4 = P_4P_3$.

The Breakdown of Elimination

- If elimination can be completed with the help of row exchanges, then:
  - Those exchanges are done first (by P).
  - The matrix PA will not need row exchanges.
  - PA allows the standard factorization into L times U.

- In the nonsingular case:
  - There is a permutation matrix P that reorders the rows of A to avoid zeros in the pivot positions.
  - Then Ax = b has a unique solution.
  - With the rows reordered in advance, PA can be factored into LU.

- In the singular case, no P can produce a full set of pivots: elimination fails.

Uniqueness

- In the nonsingular case: The LDU factorization and the LU factorization are uniquely determined by A.

Block Matrix Multiplication

- It is often convenient to partition a matrix M into smaller matrices called blocks.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 1 \\ \hline 0 & 1 & 2 & 0 \end{array}\right] = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

- Matrix operations on block matrices can be carried out by treating the blocks as matrix entries.

$$M^2 = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c} A^2 + BC & AB + BD \\ \hline CA + DC & CB + D^2 \end{array}\right]$$

Inverse Matrices

- The matrix A is invertible if there exists a matrix B such that $AB = BA = I$.

- Not all matrices have inverses.
- If $AB = I$ and $CA = I$, then $B = C$ (prove!). Therefore inverse matrix is unique. We denote it by A^{-1}.

- The matrix A is invertible if and only if $AX = b$ has one and only solution for a given b.

- The matrix A is invertible if and only if $A = LU$ where LU is a triangular factorization of A with no zeros on the diagonal of U.

The Calculation of A^{-1}

The inverse of A is written A^{-1} in which $AA^{-1} = I$. Let

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$$

We want to find A^{-1} such that $AA^{-1} = I$, so consider the columns of A^{-1} as x_1, x_2, x_3, that means $A^{-1} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}$.

$$\begin{aligned} AA^{-1} &= I \\ A \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} &= \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} \\ \begin{bmatrix} Ax_1 & Ax_2 & Ax_3 \end{bmatrix} &= \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} \end{aligned}$$

So $\qquad Ax_1 = e_1 \qquad Ax_2 = e_2 \qquad Ax_3 = e_3$

- Thus we have three systems of equations (or n systems).
- They all have the same coefficient matrix A.
- The right-hand sides e_1, e_2, e_3 are different, but elimination is possible on all systems simultaneously.

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 4 & -6 & 0 & 0 & 1 & 0 \\ -2 & 7 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -8 & -2 & -2 & 1 & 0 \\ 0 & 8 & 3 & 1 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -8 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

- We have

$$\left[\begin{array}{c|c} U & L^{-1} \end{array}\right]$$

A^{-1}

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -8 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] \qquad \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 2 & -1 & -1 \\ 0 & -8 & 0 & -4 & 3 & 2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 0 & \frac{12}{8} & -\frac{5}{8} & -\frac{6}{8} \\ 0 & -8 & 0 & -4 & 3 & 2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] \qquad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{12}{16} & -\frac{5}{16} & -\frac{6}{16} \\ 0 & 1 & 0 & \frac{4}{8} & -\frac{3}{8} & -\frac{2}{8} \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

Thank You!