Lecture07

Monday, October 11, 2021 4:25 PM

Lecture07

Linear Algebra

Samira Hossein Ghorban
s.hosseinghorban@ipm.ir

Fall, 2021

Chapter 2

Linear Spaces

The heart of linear algebra

Review

$S \subseteq V \qquad \text{span}(S) =$

Definition

Let S be a set of a linear space V. The subspace spanned by S, denoted by span(S), is the set of all linear combinations of vectors in S.

1. **Example.** For $A \in M_{m,n}(\mathbb{R})$,

$$V = \{Ax \mid x \in \mathbb{R}^n\}$$

is $C(A)$, as it is generated by all columns of A.

Review

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}$$

$$V = C(A) = \text{span}\left(\left\{ \underset{}{\overset{v_1}{\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}}}, \overset{v_2}{\begin{bmatrix} 3 \\ 6 \\ -3 \end{bmatrix}}, \overset{v_3}{\begin{bmatrix} 3 \\ 6 \\ -3 \end{bmatrix}}, \overset{v_4}{\begin{bmatrix} 2 \\ 7 \\ 4 \end{bmatrix}} \right\}\right)$$

$$= \text{span}\left(\left\{ \underset{v_1}{\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}}, \underset{v_4}{\begin{bmatrix} 2 \\ 7 \\ 4 \end{bmatrix}} \right\}\right)$$

Review

Definition

Let V be a linear space and $S \subseteq V$. We say that the elements of S are linearly dependent if there is some $s \in S$ such that

$$\text{span}(S) = \text{span}(S \setminus \{s\}).$$

If the elements of S are not linearly dependent, then we say that they are linearly independent.

Definition

The nullspace of a matrix consists of all vectors x such that $Ax = 0$. It is denoted by $N(A)$.

$N(A) \neq \{0\}$ $\quad A = [v_1 \cdots v_n] \quad Ac = A\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = 0$

$S = \{v_1, \ldots, v_n\}$

$\exists i \; v_i \in S$

$\text{Span}(S) = \text{Span}(S \setminus \{v_i\})$

$v_i = c_1 v_1 + \cdots + c_{i-1} v_{i-1} + c_{i+1} v_{i+1} \cdots$

$\alpha_1 v_1 + \cdots + \alpha_n v_n = 0$

$\exists \alpha_i \neq 0$

$c_1 v_1 + \cdots + c_n v_n = 0$

$\Rightarrow c_1 = \cdots = c_n = 0$

Basis for a linear space

Definition

Let V be a linear space and $S \subseteq V$. The set S is a basis for V if

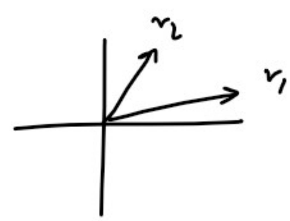
1. $V = \text{span}(S)$. $= \text{Span}(S \setminus \{s\})$
2. $V \neq \text{span}(T)$ for all $T \subsetneq S$.

- Trivially, a basis for a linear space is a linear independent set.
- A basis is a "minimal" spanning set for the linear space, in the sense that it has no "redundant" vector. At the same time, it is a "maximal" linearly independent set, in the sense that putting up a new vector makes it linearly dependent.
- A linear space may have more than one basis.

$V = \text{Span}(S)$

$T \subsetneq S$

$\text{Span}(T) \subsetneq V$



A linear space with infinite dimension

- There is no need for a basis to be finite! The linear space $\mathbb{R}[x]$ of all polynomials with real coefficients has no finite basis.

  To show it:

  1. By contradiction, assume that $\{f_1, \ldots, f_n\}$ is a basis for $\mathbb{R}[x]$.
  2. Let $m = \max_{1 \le i \le n} \deg f_i$ where $\deg f_i$ is the degree of polynomial f_i for every $1 \le i \le n$.
  3. Then $x^{m+1} \notin \text{span}(\{f_1, \ldots, f_n\})$.

$\mathbb{R}[x] = \text{Span}(S)$

$S = \{\underline{1, x, x^2, \ldots}\}$

$\cancel{a_1 x^1 + a_2 x^3 + \cdots}$

$\underline{\underline{p(x)}} = c_1 x^{i_1} + \cdots + c_m x^{i_m}$

Finite basis for a linear space

Theorem

If $V = span(\{v_1, \ldots, v_n\})$, then there is a subset of $\{v_1, \ldots, v_n\}$ which is a basis for V.

- همه زیرمجموعه‌های $S = \{v_1, \ldots, v_n\}$ را لیست کنید.

فرض کنید $T \subseteq S$ است که مینیمم تعداد اعضا را بین همه زیرمجموعه‌های که فضای تولید شده توسط آنها V را تولید می‌کنند.

$T \in \{S' \subseteq S \mid V = \text{Span}(S')\}$

$\underline{T}$ پایه است. $V = \text{Span}(T)$ ✓ فرض کنید $T' \subsetneq T$ و $V \neq \text{Span}(T')$

Finite basis for a linear space

Theorem

If $V = span(\{v_1, \ldots, v_n\})$, then there is a subset of $\{v_1, \ldots, v_n\}$ which is a basis for V.

Theorem

Suppose that $V = span(\{v_1, \ldots, v_n\})$. Then each independent set of V has at most n elements.

$\{w_1, \ldots, w_m\} \subseteq V$, $m > n$ (فرض خلف) $\xRightarrow{\text{تناقض}}$ $m \leq n$

مستقل خطی

$\forall\ 1 \leq j \leq m \qquad w_j = \sum_{i=1}^{n} a_{ij} v_i$

$c_1 w_1 = a_{11} v_1 + \cdots + a_{n1} v_n$

$\vdots$

$c_m w_m = a_{1m} v_1 + \cdots + a_{nm} v_n$

$c_1 w_1 + \cdots + c_m w_m = 0 \Rightarrow 0 = (c_1 a_{11} + \cdots + c_m a_{1m}) v_1 + \cdots + \cdots + (c_1 a_{n1} + \cdots + c_m a_{nm}) v_n = 0$

$$\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & & a_{nm} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ \vdots & & \ddots & \\ & & & 1 \end{bmatrix}_{n \times m} \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix}$$

$n < m$

$m - n$

$Ac = 0$

$c \neq 0$

Dimension

Theorem

If $\{v_1, \ldots, v_m\}$ and $\{w_1, \ldots, w_n\}$ are both bases for a linear space V, then $m = n$.

Definition

Suppose that V has a finite basis. Then ***dimension* of V** denoted by $\dim V$ is the number of elements of any basis of V.

- Example. Assume the linear space $P_2(x) = \{a_2x^2 + a_1x + a_0 \mid a_i \in \mathbb{R} \text{ for all } 0 \leq i \leq 2\}$.
  1. The sets $\{1, x, x^2\}$ is a basis for $P_2(x)$.
  2. $\dim(P_2(x)) = 3$.
  3. You can easily check that $\{1, x, x^2 - \frac{1}{3}\}$ is a basis for $P_2(x)$.

$B = \{v_1, \ldots, v_m\} \Rightarrow$

$V = span(\{v_1, \ldots, v_m\})$

$\{w_1, \ldots, w_n\} \subseteq V \Rightarrow n \leq m$

مستقل خطی

$V = span(\{w_1, \ldots, w_n\})$

$\{v_1, \ldots, v_m\} \subseteq V \Rightarrow m \leq n$

مستقل

Coordinates

$V = \text{span}(\underbrace{\{v_1, \cdots, v_n\}}_{B})$

$v \in V \Rightarrow v = c_1 v_1 + \cdots + c_n v_n$

$v = d_1 v_1 + \cdots + d_n v_n$

$0 = (c_1 - d_1) v_1 + \cdots + (c_n - d_n) v_n$

$\forall i \quad c_i - d_i = 0 \qquad c_i = d_i$

$v \longrightarrow \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

$B \quad v_1, \cdots, v_n \qquad \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \rightarrow \quad v_1 \ \cdots \ v_n$

$P_2(x)$

$x^2 - x + 1 \qquad [\ ]$

Coordinates

Definition

If V is a finite-dimensional linear space, an *ordered basis* for V is a finite sequence of vectors which is linearly independent and spans V.

Now suppose V is a finite-dimensional linear space and that $B = \{v_1, \cdots, v_n\}$ is an ordered basis for V. Given $v \in V$, there is a unique n-tuple $c = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ of scalars such that

$$v = \sum_{i=1}^{n} c_i v_i.$$

The vector c is called the coordinate vector of v relative to the ordered basis B and denoted by $[v]_B$.

$[v]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

$[v]_{B'} = \begin{bmatrix} c_1' \\ \vdots \\ c_n' \end{bmatrix}$

Change of basis

- Suppose that V be a linear space with finite dimension.
- Let $B = \{v_1, \dots, v_n\}$ be a ordered basis for V.
- Let $B' = \{v'_1, \dots, v'_n\}$ be another ordered basis for V.
- What is relation between $[v]_B$ and $[v]_{B'}$ for any vector $v \in V$?

The change of basis

$v \in V$

پایه

Theorem

Let V be a linear space. Suppose that $B = \{v_1, \dots, v_n\}$ and مبنا $B' = \{v'_1, \dots, v'_n\}$ are two bases of V. Then $[v]_B = P[v]_{B'}$ where the columns of P are the coordinates of the vectors $v'_1, \dots, v'_n$ in the basis B.

$[v]_{B'} = \begin{bmatrix} c'_1 \\ \vdots \\ c'_n \end{bmatrix} \Rightarrow v = \sum_{j=1}^{n} c'_j v'_j$, $v'_j = \sum_{i=1}^{n} P_{ij} v_i$ $\quad v'_1 = \sum_{i=1}^{n} P_{i1} v_i$

$\Rightarrow v = \sum_{j=1}^{n} c'_j \cdot \sum_{i=1}^{n} P_{ij} v_i$

$= \left(\sum_{j=1}^{n} c'_j P_{1j}\right) v_1 + \dots + \left(\sum c'_j P_{nj}\right) v_n \Rightarrow [v]_B = \begin{bmatrix} \sum_{j=1}^{n} c'_j P_{1j} \\ \vdots \\ \sum_{j=1}^{n} c'_j P_{nj} \end{bmatrix}$

مختصات v'_1 در B

$= \underbrace{\begin{bmatrix} P_{11} & \cdots & P_{1n} \\ P_{n1} & \cdots & P_{nn} \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} c'_1 \\ \vdots \\ c'_n \end{bmatrix}}_{[v]_{B'}}$

$\Rightarrow [v]_B = P[v]_{B'}$

Linear subspaces of a finite dimensional linear space

- Let V be a space with finite dimension and $W \subseteq V$. Then every linearly independent subset of W is finite and can be extended to a basis for V.

$V = span(S)$ $\quad |S| < \infty$

نشان دهیم $T \cup \{v\}$

$v \in span(T)$ X

$V = \mathrm{span}(S)$ $\quad |S| < \infty$

$T \subseteq V$ $\quad V = \mathrm{span}(T)$

$\mathrm{span}(T) \neq V$

$v \in V \setminus \mathrm{span}(T)$

$T \cup \{v\}$ مستقل است.

Linear subspaces of a finite dimensional linear space

- Let V be a space with finite dimension and $W \subseteq V$. Then every linearly independent subset of W is finite and can be extended to a basis for V.
- If $W \subsetneq V$ and $\dim V < \infty$, then $\dim W < \dim V$.

dim $\quad W \subsetneq V = \mathrm{span}(\{v_1, \ldots, v_m\})$

$\dim W < m$

$v \in \mathrm{span}(T)$ $\quad$ X

$0 \neq v_1 \in V$

$T = \{v_1\}$

$V \neq \mathrm{span}(T)$

$v_2 \in V \setminus \mathrm{span}(T)$

$T = T \cup \{v_2\}$

$T = \{v_1, \ldots, v_m\}$

$V = \mathrm{span}(T)$

Linear subspaces of a finite dimensional linear space

- Let V be a space with finite dimension and $W \subseteq V$. Then every linearly independent subset of W is finite and can be extended to a basis for V.
- If $W \subsetneq V$ and $\dim V < \infty$, then $\dim W < \dim V$.
- Let W_1 and W_2 be two linear subspaces of a linear space V with finite dimension. Then the dimension of $W_1 + W_2$ is finite and

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2).$$

$a \in W_1,\ b \in W_2$ $\quad a = \sum c_i v_i + \sum a_i \rho_i$

$b = \sum d_i v_i + \sum \beta_i s_i$

$a+b = \sum (c_i + d_i) v_i + \sum \ + \sum$

$W_1 \cap W_2 = \mathrm{span}(\{v_1, \dots, v_n\})$

$W_1 \quad W_2$

$\dim \leftarrow W_1 \cap W_2 \subseteq W_1$

$W_1 = \mathrm{span}(\{v_1, \dots, v_n, \rho_1, \dots, \rho_r\})$

$W_2 = \mathrm{span}(\{v_1, \dots, v_n, s_1, \dots, s_t\})$

$W_1 + W_2 \overset{?}{=} \mathrm{span}(\{v_1, \dots, v_n, \rho_1, \dots, \rho_r, s_1, \dots, s_t\})$

$c_1 v_1 + \dots + c_n v_n + d_1 \rho_1 + \dots + d_r \rho_r + \beta_1 s_1 + \dots + \beta_t s_t$

$c_1 v_1 + \dots + c_n v_n = -d_1 \rho_1 - \dots - d_r \rho_r - \beta_1 s_1 - \dots - \beta_t s_t$

$c_1 = \dots = c_n = 0$

$d_1 \rho_1 + \dots + d_r \rho_r = \beta_1 s_1 + \dots + \beta_t s_t = 0$

Row Reduced Form R

$c_1 A_1 + c_2 A_2 + c_3 A_4 + c_4 A_5 = 0$

$c_1 \bar{P}^1 R_1 + c_2 \bar{P}^1 R_2 + c_3 \bar{P}^1 R_4 + c_4 \bar{P}^1 R_5$

$P \bar{P}^1 (c_1 R_1 + c_2 R_2 + c_3 R_4 + c_4 R_5) = 0$

$c_1 = c_2 = c_3 = c_4 = 0$

$$\begin{array}{cccccc} A_1 & A_2 & & A_4 & A_5 & \\ \end{array}$$

$$\begin{bmatrix} 1 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 1 & 1 & -2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 3 & 1 & -2 \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & 0 & -3 & 0 & 0 & 4 \\ 0 & 1 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 & -2/3 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{array}{cccccc} R_1 & R_2 & & R_4 & R_5 & \\ \end{array}$$

$PA = R$

$\dim C(A) = \dim C(R) = r$

$PA = P[A_1 \dots A_6] = R$

$[PA_1 \dots PA_6] = R$

$PA_j = R_j \quad A_j = \bar{P}^1 R_j$

The column space of A

Lemma

Let $A \in M_{mn}(\mathbb{R})$, then $\dim C(A) = \dim C(R)$.

The column space of A

Lemma

Let $A \in M_{mn}(\mathbb{R})$, *then* $\dim C(A) = \dim C(R)$.

The column space of A

Lemma

Let $A \in M_{mn}(\mathbb{R})$, *then* $\dim C(A) = \dim C(R)$.

$PA = R$

Lemma

Let $P \in M_m(\mathbb{R})$ *is invertible and* $A \in M_{mn}(\mathbb{R})$, *then*
$\dim C(A) = \dim C(PA)$.

The column space of A

Lemma

Let $P \in M_m(\mathbb{R})$ is invertible and $A \in M_{mn}(\mathbb{R})$, then $\dim(C(A)) = \dim(C(PA))$.

Sketch of the proof: Let $(\dim(C(A)) = r$ and denote the i-th

column of A by A_i. Without loss of generality, assume that the first r columns of A are independent. We have

$$PA = P\left[A_1 \cdots A_n\right] = \left[PA_1 \cdots PA_n\right]$$

We show that the first r columns of B are independent. Consider

$$\begin{aligned} & c_1 PA_1 + \cdots + c_n PA_r = 0 \\ & P(c_1 A_1 + \cdots + c_n A_r) = 0 \\ & \Rightarrow c_1 = \cdots = c_r = 0. \end{aligned}$$

What about column spaces of A?

- For invertible $P \in M_m(\mathbb{R})$, $A \in M_{mn}(\mathbb{R})$ and PA, the column spaces of A and PA might not be the same.

- **Example.**

$$\underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}}_{B} = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}}_{A}$$

$$C(A) = \operatorname{span}\left(\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}\right), \qquad C(PA) = \operatorname{span}\left(\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}\right)$$

Thus $C(A) \neq C(PA)$.

The row spaces of A

Lemma

Let $P \in M_m(\mathbb{R})$ is invertible and $A \in M_{mn}(\mathbb{R})$, then the row spaces of A and PA are the same.

Sketch of the proof:

$$B = PA = \begin{bmatrix} p_{11} & \cdots & p_{1m} \\ \vdots & \ddots & \vdots \\ p_{m1} & \cdots & p_{mm} \end{bmatrix} \begin{bmatrix} A_1 \\ \vdots \\ A_m \end{bmatrix} = \begin{bmatrix} p_{11}A_1 + \cdots + p_{1m}A_m \\ \vdots \\ p_{m1}A_1 + \cdots + p_{mm}A_m \end{bmatrix}$$

The row space and the column space

Theorem

Let $A \in M_{mn}(\mathbb{R})$, then the dimension of row space and dimension of column space are the same.

The Rank Theorem

Theorem

Let $A \in M_{mn}(\mathbb{R})$, *then*

$$\dim C(A) + \dim N(A) = n$$

...

Thank You!