

Linear Algebra

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Review: Diagonalizable linear transformations

Theorem

Let $T: V \to V$ be a linear transformation where V is finite dimensional, and T has different eigenvalues $\lambda_1, \ldots, \lambda_k$. Suppose that W_i is the null space of $\lambda_i I - T$ for each $1 \leq i \leq k$. Then the following statements are equivalent: W: = {x (Tx = \; x }

- i. T is diagonalizable.
- ii. The characteristic polynomial of T is

$$f(\lambda) = (\lambda - \frac{\lambda_1}{\lambda_1})^{n_1} \cdots (\lambda - \frac{\lambda_k}{\lambda_k})^{n_k},$$

and dim
$$W_i = \underline{n_i}$$

$$W_1 \oplus \ldots \oplus W_{lc} = V$$

iii. $\sum_{i=1}^k \dim W_i = \dim V$.

Review: Lemma

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Lemma

Suppose that T is a linear function on V with different eigenvalues $\lambda_1, \ldots, \lambda_k$, and for each $1 \leq i \leq k$ let

$$W_i = \{ v \in V \mid Tv = \lambda_i v \},\$$

which is the null space of $\lambda_i I - T$. If $v_1 + \dots + v_k = 0$ for each $v_i \in W_i$, then $\underline{v_1} = \cdots = \underline{v_k} = 0.$ VieWi

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Vandermonde matrices

• The following matrix is called Vandermonde matrix.

$$V(\lambda_1, \dots, \lambda_k) = \begin{bmatrix} 1 & \lambda_1 & \dots & \lambda_1^{k-1} \\ 1 & \lambda_2 & \dots & \lambda_2^{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_k & \dots & \lambda_k^{k-1} \end{bmatrix} \qquad \begin{array}{c} \lambda_1 \\ \vdots \\ \lambda_k \end{array}$$

Vandermonde matrices

• The Vandermonde matrix is invertible; suppose that

Vandermonde matrices

We may write

$$\begin{cases} c_0 + c_1 \frac{\lambda_1}{\lambda_1} + \cdots + c_{k-1} \lambda_1^{k-1} = 0 \\ c_0 + c_1 \frac{\lambda_2}{\lambda_2} + \cdots + c_{k-1} \lambda_2^{k-1} = 0 \\ \vdots & \vdots + \cdots + \vdots = \vdots \\ c_0 + c_1 \frac{\lambda_k}{\lambda_k} + \cdots + c_{k-1} \lambda_k^{k-1} = 0 \end{cases}$$

• So $\lambda_1, \ldots, \lambda_k$ are distinct roots of the polynomial

$$c_0 + c_1 x + \dots + c_{k-1} x^{k-1} = 0.$$

and hence $c_0 = \cdots = c_{k-1} = 0$. As a result,

$$N(V(\lambda_1,\ldots,\lambda_k)) = \{0\}$$

and the Vandermonde matrix $V(\lambda_1, \ldots, \lambda_k)$ is invertiable.

(2- 1) 9 m) = C + C1 m -+ .. - C1-1 n (x->1)... | 1 | 1/m | = -

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Proof.

• Since $v_1 + \ldots + v_k = 0$, we have $T^i(v_1 + \cdots + v_k) = 0$ for each

 $0 \leqslant i \leqslant k-1.$ • So for each $0 \leqslant i \leqslant k-1$, $\sum_{i=1}^{k} v_1 + \sum_{i=1}^{k} v_2 + \cdots + \sum_{i=1}^{k} v_k = 0 \text{ which may be written in the following form}$

$$\begin{bmatrix} v_1 & \cdots & v_k \end{bmatrix} V(\lambda_1, \cdots, \lambda_k) = 0$$

Cayley-Hamilton's theorem

Theorem

Let $A \in M_n(\mathbb{F})$ where $\mathbb{F} = \mathbb{R}$ or \mathbb{C} and f(x) is characteristic polynomial. Then f(A) = 0 $f(A) = A + a_{n-1}A^{n-1} + \cdots + a_{n-$

o = f(A)

$$adj(\pi_1 - A) = B_{n-1} \times + B$$

By
$$f(n) = \lambda + \alpha_n n^{-1} + \dots - \alpha_n n + \alpha_n$$

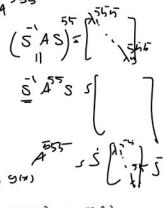
Corollary

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Corollary

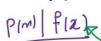
Let $A \in M_{\underline{u}}(\mathbb{F})$ where $\mathbb{F} = \underline{\mathbb{R}}$ or \mathbb{C} . For k > n, $A^k = g(A)$ where g(x) is a polynomial with coefficients in $\overline{\mathbb{F}}$ and its degree is less than n.



Definition

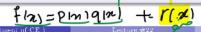
Let $A \in M_n(\mathbb{F})$ where $\mathbb{F} = \mathbb{R}$ or \mathbb{C} . The minimal polynomial p(x) of A over \mathbb{F} is the monic polynomial over \mathbb{F} of least degree such that p(A) = 0.

- ullet The minimal polynomial is defined for a linear function T.
- The minimal polynomial is unique.

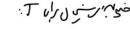


Corollary

Let $A \in M_n(\mathbb{F})$ where $\mathbb{F} = \mathbb{R}$ or \mathbb{C} and p(x) is minimal polynomial. Then p(x) divides the characteristics polynomial f(x).



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P(A) = p(A) + r(A)

Thank You!

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(Department of CE)

Lecture #22

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P(A) = 0 $P(x) = x^{m} + b_{m-1} x^{m-1} + \cdots + b_{m-1}$ P(A) = 0 P(A)