Linear Algebra

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An Example of Gaussian Elimination

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 6 \\ 2x_1 - 1x_2 + 4x_3 = 8 \\ -x_1 + 8x_2 + 2x_3 = 12 \end{cases}$$

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$$\left[\begin{array}{rrr|r} 1 & 2 & 3 & 6 \\ 2 & -1 & 4 & 8 \\ -1 & 8 & 2 & 12 \end{array}\right] \quad \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 + R_1 \end{array}$$

$$\Rightarrow \left[\begin{array}{rrr|r} 1 & 2 & 3 & 6 \\ 0 & -5 & -2 & -4 \\ 0 & 10 & 5 & 18 \end{array}\right] \quad \begin{array}{l} \\ \\ R_3 + 2R_2 \end{array}$$

$$\Rightarrow \left[\begin{array}{rrr|r} 1 & 2 & 3 & 6 \\ 0 & -5 & -2 & -4 \\ 0 & 0 & 1 & 10 \end{array}\right] \quad x_3 = 10, \quad x_2 = -\frac{16}{5}, \quad x_1 = -\frac{88}{5}$$

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Left Side	$(n^2 - n) + ((n-1)^2 - (n-1)) + \ldots + 1 = \frac{n^3-n}{3}$
Right Side	$(n-1) + (n-2) + \ldots + 1 = \frac{n(n-1)}{2}$
Solution	$n + (n-1) + \ldots + 1 = \frac{n(n+1)}{2}$
Total	$\frac{n^3-n}{3} + n^2 + n \simeq \frac{1}{3}n^3$

$$\begin{cases} x_1 + x_2 + x_3 = b_1 \\ 2x_1 + 2x_2 + 5x_3 = b_2 \\ 4x_1 + 4x_2 + 8x_3 = b_3 \end{cases} \qquad \left[\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 2 & 2 & 5 & b_2 \\ 4 & 4 & 8 & b_3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 0 & 0 & 1 & \frac{b_2-2b_1}{3} \\ 0 & 0 & 1 & \frac{b_3-2b_1}{4} \end{array}\right]$$

Example

Which number q makes this system singular and which right-hand side t gives it infinitely many solutions? Find the solution that has $x_3 = 1$

$$\begin{cases} x_1 + 4x_2 - 2x_3 = 1 \\ x_1 + 7x_2 - 6x_3 = 6 \\ \quad\;\; + 3x_2 + qx_3 = t \end{cases}$$

$$
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The Matrix Form of Elimination Steps

$$\left[\begin{array}{rrr|r} 1 & 2 & 3 & 6 \\ 2 & -1 & 4 & 8 \\ -1 & 8 & 2 & 12 \end{array}\right] \qquad \begin{array}{l} R_2 - 2R_1 \\ R_3 + R_1 \end{array}$$

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$$\Rightarrow \left[\begin{array}{rrr|r} 1 & 2 & 3 & 6 \\ 0 & -5 & -2 & -4 \\ 0 & 0 & 1 & 10 \end{array}\right]$$

Elementary matrix:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

Upper triangular

The product GFE is the true order of elimination. It is the matrix that takes the original A to the upper triangular U.

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ -1 & 8 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 12 \end{bmatrix}$$

$$GFE \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ -1 & 8 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = GFE \begin{bmatrix} 6 \\ 8 \\ 12 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \\ 10 \end{bmatrix}$$

How can we undo the steps of Gaussian elimination?

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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One operation cancels the other. In matrix terms, one matrix is the inverse of the other.

Inverse Matrices of the elementary matrices

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Inverse Matrices:

$$E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$F^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$G^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

Triangular factorization $A = LU$

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$$GFEA = U$$

$$A = \underbrace{E^{-1}F^{-1}G^{-1}}_{\mathbf{L}}\,\mathbf{U}$$

- L is **lower triangular** , with 1s on the diagonal.
- U is the **upper triangular** matrix which appears after forward elimination, the diagonal entries of U are the pivots.

LU factorization

- A serious point about $A = LU$!

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- One Linear System = Two Triangular Systems
- Two triangular systems in $\frac{n^2}{2}$ steps each.
- A three diagonals matrix

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

and solve $Ax = b$.

LU factorization for three diagonals matrix

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{U}$$

$$Ax = b \Rightarrow LUx = b$$

- First, find c s.t $Lc = b$.

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$$Ax = b \Rightarrow LUx = b$$

- First, find c s.t $Lc = b$.
- Second, find X s.t $UX = c$.

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- Factor out of U a diagonal pivot matrix D:

$$U = \underbrace{\begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & \frac{u_{12}}{d_1} & \cdots & \frac{u_{1n}}{d_1} \\ & 1 & \frac{u_{23}}{d_2} & \vdots \\ & & \ddots & \vdots \\ & & & 1 \end{bmatrix}}_{\mathbf{U}}$$

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- $A = LDU$ where L and U have 1s on the diagonal and D is the diagonal matrix of pivots.

Row exchanges and Permutation Matrices

With the rows reordered in advance, A can be factored into LU.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 2 & 5 & 8 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 3 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{bmatrix} = U$$

So, that row exchange with permutation matrix:

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

recovers LU. The matrix P is called **permutation** matrix.

Elimination in a Nutshell: $PA = LU$

- In the nonsingular case, there is a permutation matrix P that reorders the rows of A to avoid zeros in the pivot positions. Then $Ax = b$ has a unique solution.

- In the singular case, no P can produce a full set of pivots: elimination fails.