

Lecture21

Linear Algebra

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(Department of CE)

Lecture #21

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Review: Diagonalizable matrices



Definition

Assume $A \in M_n(\mathbb{R})$. A is called diagonalizable if it is similar to a diagonal matrix, i.e., $\overline{\Pi}$ there exists an invertible matrix S and a diagonal matrix D such that

$$S^{-1}AS = 0.$$

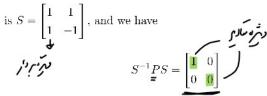
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Lecture #21

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Review: Diagonalization of a matrix

• Example. The eigenvector matrix of the projection $\underline{P} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$



• The eigenvector matrix S converts A into its eigenvalue matrix which is diagonal.

Diagonalizable linear transformation

Theorem

Let $T: V \to V$ be a linear transformation where the dimension of V is finite with different eigenvalues $\lambda_1, \ldots, \lambda_k$. Suppose that W_i is null space of $T = \lambda_i I$ for each $1 \le i \le k$. The the following statements are

- i. T is diagonalizable.
- F(x)= bet (LI -. T)
- ightharpoonup ii. Its eigenvalue vector is $\underline{f(\lambda)} = (\lambda \frac{1}{2})^{\frac{n_1}{n_1}} \dots (\lambda \frac{1}{2})^{\frac{n_k}{n_k}}$ and $\dim \underline{W_i} = n_i.$
- \rightarrow iii. $\sum_{i=1}^k \dim W_i = \dim V$.

Proof: $i \Rightarrow ii$

T is diagonalizable, so there is a basis $B = \{v_1, \dots, v_n\}$ such that

Figure 1. So there is a basis
$$B = \{v_1, \dots, v_n\}$$
 such that
$$[T]_B = \begin{bmatrix} \lambda_1 & & & & \\ \lambda_2 & & & \\ & & \lambda_3 & & \\ & & & \lambda_4 & & \\ & & &$$

AE Mn(IR) > T: R -> R

Thy) = c, v, +.. + e, v, $[\tau_{\gamma_i}]_{g}$ W; ={ 1 | Th= 1, n} = N(t-1, [)

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FIN = det (LT - T)

$$= (\lambda - \lambda_1)^{n_1} - (\lambda - \lambda_{\mu})^{n_{\mu}}$$

$$dimWi = ni$$

$$\begin{bmatrix} \lambda_1 \xi_1 \\ \vdots \end{bmatrix} \chi = \lambda_i \chi \qquad \chi = \begin{bmatrix} \chi_1 \\ \vdots \end{bmatrix} \quad \eta_i \in \mathbb{R}^{n_i}$$

Proof: ii ⇒ iii



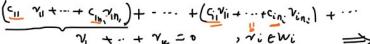
 \bullet The characteristic polynomial of T is

$$f(\lambda) = (\lambda - \frac{1}{\lambda_1})^{\frac{1}{\lambda_1}} \dots (\lambda - \frac{1}{\lambda_1})^{\frac{1}{\lambda_1}}$$

•
$$\dim V = \deg f = \sum_{i=1}^k n_i = \sum_{i=1}^k \dim W_i$$

Proof: iii ⇒ i

- $\bullet \sum_{i=1}^k \dim W_i = \dim V.$
- \bullet We should find a basis that the representation matrix of $[T]_B$ is



Lemmas

We needs two lemma for complete the proof of iii \Rightarrow i.

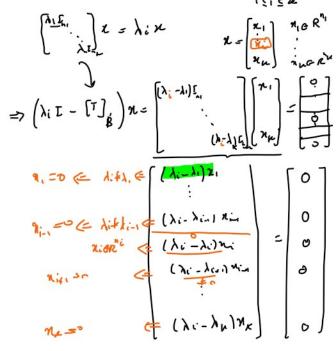
Suppose that T is a linear function on V and $Tv = \lambda v$. If f(x) is a polynomial, then $f(T)v = f(\lambda)v$.

$$\tau^{i}_{\gamma} = \lambda^{i} \nu$$

Corollary

If $\lambda_1, \ldots, \lambda_n$ are eigenvalues of T, then $\lambda_1^k, \ldots, \lambda_n^k$ are eigenvalues of

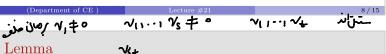




ricir & refine Wi $W_{i} = \frac{1}{2} \left\{ v_{i1}, \dots, v_{in_{i}} \right\}$ $\int dim W_{i} = \frac{1}{2} \left\{ v_{i1}, \dots, v_{in_{i}} \right\}$ $\int dim W_{i} = \frac{1}{2} \left\{ v_{i1}, \dots, v_{in_{i}} \right\}$ $\int dim W_{i} = \frac{1}{2} \left\{ v_{i1}, \dots, v_{in_{i}} \right\}$ $\int dim W_{i} = \frac{1}{2} \left\{ v_{i1}, \dots, v_{in_{i}} \right\}$ $\int v_{i1} + \dots + v_{in_{i}} \left\{ v_{in_{i}} + \dots + v_{in_{i}} \right\}$ $\int v_{in_{i}} + \dots + v_{in_{i}} \left\{ v_{in_{i}} + \dots + v_{in_{i}} \right\}$ $\int v_{in_{i}} + \dots + v_{in_{i}} \left\{ v_{in_{i}} + \dots + v_{in_{i}} \right\}$ $\int v_{in_{i}} + \dots + v_{in_{i}} \left\{ v_{in_{i}} + \dots + v_{in_{i}} \right\}$ $\int v_{in_{i}} + \dots + v_{in_{i}} \left\{ v_{in_{i}} + \dots + v_{in_{i}} \right\}$ $\int v_{in_{i}} + \dots + v_{in_{i}} \left\{ v_{in_{i}} + \dots + v_{in_{i}} \right\}$ $\int v_{in_{i}} + \dots + v_{in_{i}} \left\{ v_{in_{i}} + \dots + v_{in_{i}} \right\}$ $\int v_{in_{i}} + \dots + v_{in_{i}} \left\{ v_{in_{i}} + \dots + v_{in_{i}} \right\}$ $\int v_{in_{i}} + \dots + v_{in_{i}} \left\{ v_{in_{i}} + \dots + v_{in_{i}} \right\}$ $\int v_{in_{i}} + \dots + v_{in_{i}} \left\{ v_{in_{i}} + \dots + v_{in_{i}} \right\}$ $\int v_{in_{i}} + \dots + v_{in_{i}} \left\{ v_{in_{i}} + \dots + v_{in_{i}} \right\}$ $\int v_{in_{i}} + \dots + v_{in_{i}} \left\{ v_{in_{i}} + \dots + v_{in_{i}} \right\}$ $\int v_{in_{i}} + \dots + v_{in_{i}} \left\{ v_{in_{i}} + \dots + v_{in_{i}} \right\}$ $\int v_{in_{i}} + \dots + v_{in_{i}} \left\{ v_{in_{i}} + \dots + v_{in_{i}} \right\}$ $\int v_{in_{i}} + \dots + v_{in_{i}} \left\{ v_{in_{i}} + \dots + v_{in_{i}} \right\}$ $\int v_{in_{i}} + \dots + v_{in_{i}} \left\{ v_{in_{i}} + \dots + v_{in_{i}} \right\}$ $\int v_{in_{i}} + \dots + v_{in_{i}} \left\{ v_{in_{i}} + \dots + v_{in_{i}} \right\}$ $\int v_{in_{i}} + \dots + v_{in_{i}} \left\{ v_{in_{i}} + \dots + v_{in_{i}} \right\}$ $\int v_{in_{i}} + \dots + v_{in_{i}} \left\{ v_{in_{i}} + \dots + v_{in_{i}} \right\}$ $\int v_{in_{i}} + \dots + v_{in_{i}} \left\{ v_{in_{i}} + \dots + v_{in_{i}} \right\}$ $\int v_{in_{i}} + \dots + v_{in_{i}} \left\{ v_{in_{i}} + \dots + v_{in_{i}} \right\}$ $\int v_{in_{i}} + \dots + v_{in_{i}} \left\{ v_{in_{i}} + \dots + v_{in_{i}} \right\}$ $\int v_{in_{i}} + \dots + v_{in_{i}} \left\{ v_{in_{i}} + \dots + v_{in_{i}} \right\}$ $\int v_{in_{i}} + \dots + v_{in_{i}} \left\{ v_{in_{i}} + \dots + v_{in_{i}} \right\}$ $\int v_{in_{i}} + \dots + v_{in_{i}} \left\{ v_{in_{i}} + \dots + v_{in_{i}} \right\}$ $\int v_{in_{i}} + \dots + v_{in_{i}} \left\{ v_{in_{i}} + \dots + v_{in_{i}} \right\}$ $\int v_{in_{i}} + \dots + v_{in_{i}} \left\{ v_{in_{i}} + \dots$

$$f(\tau) v = \sum_{i,s} \alpha_i T^i v = \sum_{i,s} \alpha_i \lambda^i v$$

$$A \leftarrow \lambda_{1,1} - 1\lambda_{2}$$



Lemma

Suppose that T is a linear function on V with different eigenvalues $\lambda_1, \ldots, \lambda_k$. Let for each $1 \leq i \leq k$

$$W_i = \{ v \in V \mid Tv = \lambda_i v \}.$$

If $v_1 + \cdots + v_k = 0$ for each $v_i \in W_i$, then $v_1 = \cdots = v_k = 0$.

$$\mathbf{J}_{j}[n] = \frac{1}{\prod_{i \neq j, j \neq 1}^{k} (x - \lambda_{i})} \\
+ \frac{1}{(\lambda_{j} - \lambda_{i})}$$

$$|\{x \neq j, j \neq 1\}| \quad |\{x \neq j\}| \quad |\{x \neq j\}|$$

Proof: $iii \Rightarrow i$

- $\sum_{i=1}^k \dim W_i = \dim V$.
- Let $\underline{B_i} = \{v_{i1}, \dots, v_{in_i}\}$ be a basis for W_i . By two above lemma,

is a basis for
$$W_1 + \cdots + W_k$$
.

 $W_1 + \cdots + W_k = V$

Diagonalization of A and its powers A^k

• Find A^{555} where

$$A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}.$$

• We obtain

$$\det(\lambda I - A) = (\lambda - 1)(\lambda - 2)^2$$

• So, $\lambda = 1, 2, 2$ are eigenvalues of A and their eigenvectors are as follows, respectively:

$$v_1 = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

(isk view; TV = 1, Vi $g_{i}(T)\gamma = g_{i}(\lambda_{i}) \gamma_{i} = \begin{cases} 0 & \text{if } i \neq j \\ 0 & \text{if } i \neq j \end{cases}$

g: (T) [x, + -+xe) = g: (T) x, + .. + g: (T) V/K حین (17) بسیل فقهاست می 9; (۱۲) ۱۲، + ۱۲۰ (۱۲) ع

=> 9;(1) x, + - + 9; (+) x, 50

→Bs UBi [] B =] ... y. =

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 $A = S \left[\frac{y}{y} \right] \frac{S}{2}$ 4=8 / 41 / 5 / 5 / 41 / 5] 12 - [hi 2] =1

A^{555}

• Let

$$S = \begin{bmatrix} 3 & 2 & 0 \\ -1 & 0 & 0 \\ 3 & -1 & -1 \end{bmatrix}.$$

• Then

$$S^{-1}AS = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad \text{and} \qquad S^{-1}A^{555}S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^{555} & 0 \\ 0 & 0 & 2^{555} \end{bmatrix}$$

• As a result:

Diagonalizable matrix A and its characteristic polynomial

- If $A \in M_n(\mathbb{F})$ is diagonalizable, then f(A) = 0.

$$f(\lambda) = (\lambda - \lambda_1)^{n_1} \dots (\lambda - \lambda_k)^{n_k}$$

where dim $W_i = n_i$.

• Moreover, there is invertible matrix $S \in M_n(\mathbb{R})$ such that

$$S^{-1}\underline{A}S = \begin{bmatrix} \lambda_1 I_{n_1} & & & \\ & \mathbf{1}_{n_2} & & \\ & & \ddots & \\ & & & \lambda_k I_{n_k} \end{bmatrix}$$

 $A^2 = S \left[\begin{array}{c} \lambda_1 \\ \lambda_2 \\ \end{array} \right] \overline{S}^{1}$

 $\frac{A}{2} = S \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = 1$

- Since $A \in M_n(\mathbb{F})$ is diagonalizable, its characteristic polynomial is $f(\lambda) = (\lambda \lambda_1)^{n_1} \dots (\lambda \lambda_r)^{n_k}$.

 - = 0 0=f(s\AS) = s'.f(A) S

Cayley-Hamilton's theorem

Theorem

Let $A \in M_n(\mathbb{F})$ where $\mathbb{F} = \mathbb{R}$ or \mathbb{C} and f(x) and p(x) are characteristic polynomial and minimal polynomial, respectively. Then

- **1** f(A) = 0
- 2 The minimal polynomial, p(x), divides the characteristic polynomial, f(x).

Corollary

Let $A \in M_n(\mathbb{F})$ where $\mathbb{F} = \mathbb{R}$ or \mathbb{C} . For k > n, $A^k = g(A)$ where g(x) is a polynomial with coefficients in \mathbb{F} and its degree is less than n.

 $\frac{f(A)}{m} = 0$ $\frac{f(A)}{m} = 0$ $\int e^{n} f(A) =$

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$$P(x) = ix + b_{m-1}x^{m-1} + \cdots + b_{n}$$