Lecture12

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Lecture12

Linear Algebra

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Review: Inner products on real linear space

An inner product on V is a function $\langle , \rangle : V \times V \to \mathbb{R}$ such that

1. $\langle v, v\rangle \geqslant 0$ for all $v \in V$.
2. $\langle v, v\rangle = 0$ if and only if $v = 0$.
3. $\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$ for all $u, v, w \in V$.
4. $\langle cu, w\rangle = c\langle u, w\rangle$ for all $u, w \in V$ and $c \in \mathbb{R}$.
5. $\langle v, w\rangle = \langle w, v\rangle$.

Review: Euclidean inner product

- The Euclidean inner product on $\mathbb{R}^n$:

$$\begin{cases} \langle\, , \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \\ \langle x, y\rangle = y^T x = y_1 x_1 + \cdots + y_n x_n. \end{cases}$$

where $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$.

Review

- Suppose that V is an inner product space. For $v \in V$, we define the norm of v, denoted $\|v\|$, by $\|v\| = \sqrt{\langle v, v\rangle}$.

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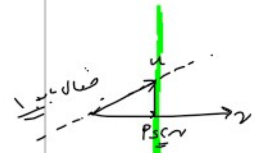
- Two vectors $u, v \in V$ are said to be orthogonal if $\langle u, v \rangle = 0$.



$c = \underbrace{\frac{\langle u, v \rangle}{\|v\|^2}} = \frac{v^T u}{\|v\|^2}$

$P = \frac{v v^T}{\|v\|^2}$

$P(u) = v^T u$

Review

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Review

- **Cauchy-Schwarz Inequality:** Let V be an inner product and $u, v \in V$. Then

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$$| \langle u, v \rangle | \leqslant \|u\| \|v\|$$

- **Triangle Inequality:** Let V be an inner product. If $u, v \in V$, then

$$\|u + v\| \leqslant \|u\| + \|v\|$$

Review

- If nonzero vectors $v_1, \ldots, v_n$ are mutually orthogonal (every vector is perpendicular to every other), then those vectors are linearly independent.

Orthonormal vectors

Definition

q_i, q_j $\quad \|q_i\|=1 \Rightarrow \langle q_i, q_i\rangle = 1$

Vectors $q_1, \ldots, q_n$ are orthonormal if

$$q_i^T q_j = \begin{cases} 0 & \text{whenever} \quad i \neq j \qquad \text{(for orthogonality)} \\ 1 & \text{whenever} \quad i = j \qquad \text{(for normality).} \end{cases}$$

A matrix with orthonormal columns will be denoted by Q.

$$Q = \begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix}$$

- Example. The standard vectors $e_1, \ldots, e_n$.

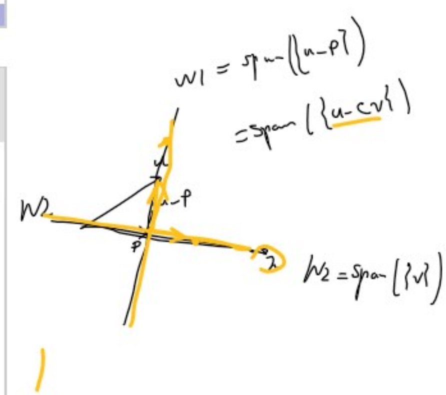
Orthogonal Subspaces

Definition

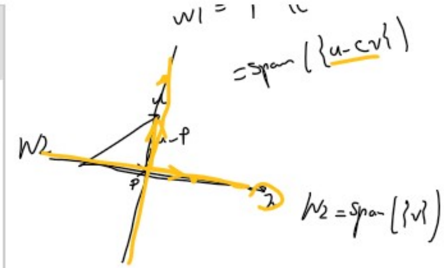
Two subspaces W_1 and W_2 of the same space V are orthogonal, denoted by $W_1 \perp W_2$, if and only if each vector $w_1 \in W_1$ is orthogonal to each vector $w_2 \in W_2$:

$$\langle w_1, w_2 \rangle = 0.$$

for all w_1 and w_2 in W_1 and W_2, respectively.



Orthogonal Subspaces



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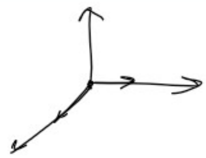

Orthogonal complement of a subspace

$0 \in W^\perp \Leftarrow \forall w \in W \quad \langle w, 0 \rangle = 0$

$x, y \in W^\perp,\ c \in \mathbb{R}$

$cx + y \in W^\perp (?)$

$w \in W$

$\langle cx+y, w \rangle = c\langle x, w\rangle + \langle y, w \rangle$

$\Rightarrow cx+y \in W^\perp$

$W \to W^\perp$

$W_1, W_2 \quad W_1 \perp W_2$

$W^\perp = \{ v \in V \mid \forall w \in W \quad \langle v, w \rangle = 0 \}$

Definition

Given a subspace W in linear space V, the space of all vectors orthogonal to W is called the orthogonal complement of V. It is denoted by $W^\perp$.

- We emphasize that W_1 and W_2 can be orthogonal without being complements.
- $W_1 = \text{span}((1, 0, 0))$ and $W_2 = \text{span}((0, 1, 0))$.

$e_1 \quad e_2$

$V = \text{span}\{\{e_1, e_2\}\}$

$W_1 W_2 \subseteq V$

$W_1, W_2 \subseteq \mathbb{R}^3$

$W_1^\perp \neq W_2$

$W_1 = \text{span}(\{e_1\}) \quad W_1^\perp = W_2$

$W_2 \subsetneq W_1^\perp = \text{span}(\{e_2, e_3\})$

$W = \text{span}\left(\begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \\ -2 \end{bmatrix} \right)$

$A_1^T \quad A_2^T$

$W^\perp \subseteq \mathbb{R}^4$

$A = \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 3 & -2 \end{bmatrix}$

$x \in W^\perp : A_1 x = 0$

$A_2 x = 0$

$0 = \left\langle x, \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\rangle = [-1 \ 0 \ 2 \ 1]\, x$

$0 = \left\langle x, \begin{bmatrix} 0 \\ 1 \\ 3 \\ -2 \end{bmatrix} \right\rangle = [0 \ 1 \ 3 \ -2]\, x$

Fundamental theorem of orthogonality

Fundamental theorem of orthogonality

Let $A \in M_{mn}(\mathbb{R})$.

$A = \begin{bmatrix} A_1 \\ \vdots \\ A_m \end{bmatrix} \qquad W = \text{span}(\{A_1, \dots, A_m\}) = C(A^T)$

$W^\perp = ?$

1. The row space is orthogonal to the nullspace (in $\mathbb{R}^n$).
2. The column space is orthogonal to the left nullspace (in $\mathbb{R}^m$).

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2. The column space is orthogonal to the left nullspace (in $\mathbb{R}^m$).

$x \in W^{\perp} \iff \langle x, A_i \rangle = 0 \quad \forall\ 1 \le i \le m \quad Ax = \begin{bmatrix} A_1 \\ \vdots \\ A_m \end{bmatrix} x = \begin{bmatrix} A_1 x \\ \vdots \\ A_m x \end{bmatrix} = 0$

$\iff x \in N(A) \qquad \dim C(A) + \dim N(A) = n$

$\Rightarrow \quad (C(A^T))^{\perp} = N(A) \qquad T: \mathbb{R}^n \to \mathbb{R}^m,\ T(x) = Ax$

$x \in W^{\perp} \iff x \in N(A)$

$(A \ldots)^{\perp} = W^{\perp} = N(A)$

Fundamental theorem of orthogonality

Let $A \in M_{mn}(\mathbb{R})$.

1. The nullspace is the orthogonal complement of the row space in $\mathbb{R}^n$.
2. The left nullspace is the orthogonal complement of the column space in $\mathbb{R}^m$.

$C(A^T)^{\perp} = N(A)$, $C(A^T) \subseteq \mathbb{R}^n$

$(C(A))^{\perp} = N(A^T)$

$A \in M_{mn} \qquad W = C(A) \Rightarrow W^{\perp} = ?$

$A = [A_1, \ldots, A_n] \quad \forall i \quad \langle A_i, x \rangle = 0 \Rightarrow x^T A_i = 0 \iff x \in N(A^T)$

$A_i^T x = 0$, $\begin{bmatrix} A_1^T \\ \vdots \\ A_n^T \end{bmatrix} x = 0$

$x \in W^{\perp} \Leftarrow$

$T: \mathbb{R}^m \to \mathbb{R}^n,\ T(x) = A^T x$

$\dim N(A^T) + \dim C(A^T) = m$

Column space and row spase of

$A \in M_{mn}(\mathbb{R})$

$\dim C(A) + \dim N(A) = n$

- $N(A) + N(A)^{\perp} = \mathbb{R}^n$.
- $N(A) \cap N(A)^{\perp} = \{0\}$.
- Direct Sum: $\mathbb{R}^n = N(A) \oplus N(A)^{\perp}$.
- $\mathbb{R}^n = N(A) \oplus C(A^T)$.
- Thus, for each $x \in \mathbb{R}^n$, there are $x_r \in C(A^T)$ and $x_n \in N(A)$ such that $x = x_n + x_r$.
- $Ax = Ax_r + \cancel{Ax_n}^{0}$.
  1. The nullspace component goes to zero: $Ax_n = 0$.
  2. The row space component goes to the column space: $Ax = Ax_r$.

$x \in W,\ x \in W^{\perp} \Rightarrow x = 0$

$W = N(A) \qquad \langle x, x \rangle = 0 \Rightarrow x = 0$

$Ax = y$

$V = W_1 \oplus W_2$

$v = w_1 + w_2$

$W_1 \cap W_2 = \{0\}$

$N(A) \oplus C(A^T)$

$T: \mathbb{R}^n \to \mathbb{R}^m$

$T(x) = Ax$

$Ax = b \Rightarrow \begin{cases} A' \\ x_r \quad Ax_r = b, \quad Ax_n = 0 \end{cases} \qquad x_r + x_n$

2. The row space component goes to the column space: $Ax = Ax_r$.

$Ax = b \Rightarrow \begin{cases} A' \\ x_r \quad Ax_r = b, \quad Ax_n = 0 \end{cases} \quad \underline{x_r} + \underline{\underline{x_n}}$

$T(x) = \boxed{Ax}$

$\underline{x} \xmapsto{T} Ax \in C(A)$

$x = x_r + \boxed{x_n}$

$x_r \in C(A^T)$, $x_n \in N(A)$

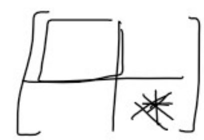
$T|_{C(A^T)} : \underbrace{C(A^T)}_{} \longrightarrow \underbrace{C(A)}_{} \subseteq \mathbb{R}^m$

$\longleftarrow$

$N(A) \xrightarrow{T} 0$

$\longleftarrow$

$T^{-1} = ?$



Column space and row space

$\mathbb{R}^n = N(A) \oplus C(A^T) = C(A^T)$

$\Uparrow$

$N(A) = \{0\}$

$\Uparrow$

Proposition

From the row space to the column space, A is actually invertible. Every vector in the column space comes from exactly one vector in the row space.

$T : \underline{C(A^T) = \mathbb{R}^n} \underset{uA}{\overset{A}{\rightleftarrows}} \underline{C(A)}$

$x = x_r \in C(A^T) \longrightarrow Ax_r$

$\xleftarrow{u}$

$x_r \longleftarrow Ax_r$

$\boxed{Ax = b} \quad x \in \underline{\underline{\mathbb{R}^n}}$

$\mathbb{R}^n = \underline{N(A)} \oplus \underline{C(A^T)}$

$Ax_r = b$

$N(A) = \{x_n \mid \underline{A} x_n = 0\}$

$\underline{\underline{x}} = \underline{\underline{x_n}} + x_r$

$Ax = \underline{\underline{Ax_n}} + Ax_r = 0 + b$

Column space and row space

Corollary

Every matrix transforms its row space onto its column space.

- $A \in M_{mn}(\mathbb{R})$ is invertible on those r-dimensional spaces.
- A on its nullspace is zero.
- Thus A^{-1} exists if and only if $r = m = n$.
- When A^{-1} fails to exist, the best substitute is the pseudoinverse A^+.
- One formula for A^+ depends on the singular value decomposition under some conditions.

Thank You!