Linear Algebra

Samira Hossein Ghorban
s.hosseinghorban@ipm.ir

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Triangularizable matrices

Theorem

Let T be a linear function over V with dimension $n < \infty$ over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. The linear function T is triangularizable if and only if the minimal polynomial of T splits in $\mathbb{F}(x)$ into linear factors.

Proof.

Proof.

Review: Diagonalizable linear transformations

Theorem

Let $T : V \to V$ be a linear transformation where V is finite dimensional, and T has different eigenvalues $\lambda_1, \ldots, \lambda_k$. Suppose that W_i is the null space of $\lambda_i I - T$ for each $1 \leqslant i \leqslant k$. Then the following statements are equivalent:

i. *T is diagonalizable.*

ii. *The characteristic polynomial of T is*

$$f(\lambda) = (\lambda - \lambda_1)^{n_1} \cdots (\lambda - \lambda_k)^{n_k},$$

and $\dim W_i = n_i$.

iii. $\sum_{i=1}^{k} \dim W_i = \dim V$.

Primary decomposition

Theorem

Let T be a linear function over V with dimension $n < \infty$ whose minimal polynomial factorizes:

$$p(x) = p_1^{r_1}(x) \dots p_k^{r_k}(x)$$

where the p_i's are monic and pairwise coprime (that is have no common factors). Let $W_i = N\left(p_i^{r_i}(T)\right)$ for each $1 \leqslant i \leqslant k$. Then

1. $V = W_1 \oplus \cdots \oplus W_k$.
2. *For each $1 \leqslant i \leqslant k$, $T(W_i) \subseteq W_i$*
3. *The minimal polynomial $T_i = T\restriction_{W_i}$ is $p_i(x)$.*

Lemma.

Lemma.

Lemma.

Lemma.

Lemma.

Lemma.

Lemma.

Jordan Form

Suppose that T is a linear function on V with the characteristic polynomial is

$$f(x) = (x - \lambda_1)^{d_1} \dots (x - \lambda_k)^{d_k}$$

where $\lambda_1, \dots, \lambda_k$ are distinct elements and $d_i \geqslant 1$.
Then the minimal polynomial for T will be

$$p(x) = (x - \lambda_1)^{r_1} \dots (x - \lambda_k)^{r_k}$$

where $1 \leqslant r_i \leqslant d_i$ based on the Cayley–Hamilton theorem.
If W_i is the null space of $(T - \lambda_i I)^{r_i}$, then the primary decomposition theorem tells us that

$$V = W_1 \oplus \cdots \oplus W_k$$

such that the linear function $T_i = T\restriction_{W_i} : W_i \to W_i$ has minimal polynomial $(x - \lambda_i)^{r_i}$.

Thank You!