Linear Algebra

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Review: Diagonalizable Linear Function

Theorem

Let $T : V \to V$ be a linear function on a finite dimensional linear space V, and T has different eigenvalues $\lambda_1, \ldots, \lambda_k$. Suppose that W_i is the null space of $T - \lambda_i I$ for each $1 \leqslant i \leqslant k$. Then the following statements are equivalent:

i. *T is diagonalizable.*

ii. *The characteristic polynomial of T is $f(x) = (x - \lambda_1)^{n_1} \cdots (x - \lambda_k)^{n_k}$, where $n_i = \dim W_i$.*

iii. $\sum_{i=1}^{k} \dim W_i = \dim V$.

Primary Decomposition Theorem

Theorem

Let T be a linear function over a finite dimensional linear space V whose minimal polynomial factorizes as

$$p(x) = p_1^{r_1}(x) \cdots p_k^{r_k}(x),$$

where the p_i's are monic and mutually coprime (i.e. have no nontrivial common factors). Let $W_i = N\left(p_i^{r_i}(T)\right)$ for each $1 \leqslant i \leqslant k$. Then

1. $V = W_1 \oplus \cdots \oplus W_k$.
2. *For each* $1 \leqslant i \leqslant k$, $T(W_i) \subseteq W_i$.
3. *The minimal polynomial of* $T_i = T\restriction_{W_i}$ *is* $p_i(x)$.

Minimal Polynomials for Vectors

Definition

Let $A \in M_n(\mathbb{F})$, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and $v \in \mathbb{F}^n$. We say that a monic polynomial $p(x)$ with coefficients in $\mathbb{F}$ is a *minimal polynomial for v with respect to A* if

1. $p(A)v = 0$,
2. $\deg p \leqslant \deg m$ for any non-zero polynomial $m(x)$ with $m(A)v = 0$.

Minimal Polynomials for Vectors

Similarly, minimal polynomial may be defined for a vector with respect to a linear function.

Definition

Let T be a linear function on linear space V on $\mathbb{F}$ where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and $v \in V$. We say that a monic polynomial $p(x)$ with coefficients in $\mathbb{F}$ is a *minimal polynomial for v with respect to T* if

1. $p(T)v = 0$,
2. $\deg p \leqslant \deg m$ for any non-zero polynomial $m(x)$ with $m(T)v = 0$.

Lemma.

Lemma

Suppose that T is a linear function on linear space V over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Then

1. *Each vector $v \in V$ has a minimal polynomial with respect to T.*
2. *The minimal polynomial for v with respect to T is unique.*
3. *Take a vector $v \in V$ and assume that $f(x)$ is a polynomial with coefficients in $\mathbb{F}$ such that $f(T)v = 0$, then $p(x) \mid f(x)$ where $p(x)$ is the minimal polynomial for v with respect to T.*
4. *Let $f(x)$ and $g(x)$ be two coprime polynomials.Then*

$$N(f(T)) \cap N(g(T)) = \{0\}.$$

Lemma.

Lemma

Let T, S be two linear functions on linear space V over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ such that $T \circ S = S \circ T$ and $N(T) \cap N(S) = \{0\}$. Then

1. $N(T) + N(S) \subseteq N(T \circ S)$.
2. *If V is finite dimensional, then* $\dim N(T \circ S) \leqslant \dim N(T) + \dim N(S)$ *and consequently,* $N(T \circ S) = N(T) \oplus N(S)$.

Proof.

Lemma.

Lemma

Let $T_1, \ldots, T_k$ *be linear functions on a finite dimensional linear space* V *over* $\mathbb{F} = \mathbb{R}$ *or* $\mathbb{C}$ *such that*

- $T_i \circ T_j = T_j \circ T_i$,
- $N(T_i) \cap N(T_j) = \{0\}$

for each $1 \leqslant i < j \leqslant k$. *Then* $N(T_1 \circ \cdots \circ T_k) = N(T_1) \oplus \cdots \oplus N(T_k)$.

Proof.

Lemma.

Lemma

Let T be a linear functions on a finite dimensional linear space V over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

1. *Let $f(x)$ be a polynomial with coefficient in $\mathbb{F}$ and $f(x) = f_1(x)^{n_1} \cdots f_k(x)^{n_k}$ such that $f_1, \ldots, f_k$ mutually coprime. Then*
$$N(f(A)) = N(f_1(A)^{n_1}) \oplus \cdots \oplus N(f_k(A)^{n_k}).$$
2. *If the minimal polynomial T is factorized as $p(x) = p_1(x)^{n_1} \cdots p_k(x)^{n_k}$ where $p_1, \ldots, p_k$ are mutually coprime, then*
$$V = N(p_1(A)^{n_1}) \oplus \cdots \oplus N(p_k(A)^{n_k}).$$

Proof.

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1. $V = W_1 \oplus \cdots \oplus W_k$.
2. *For each $1 \leqslant i \leqslant k$, $T(W_i) \subseteq W_i$.*
3. *The minimal polynomial of $T_i = T\restriction_{W_i}$ is $p_i(x)$.*

Thank You!