Lecture08

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Linear Algebra

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Chapter 2

Linear Spaces

The heart of linear algebra

Review: Basis for a linear space

Definition

Let V be a linear space and $S \subseteq V$. The set S is a basis for V if

1. $V = \text{span}(S)$,
2. $V \neq \text{span}(T)$ for all $T \subsetneq S$.

- Trivially, a basis for a linear space is a linear independent set.
- A basis is a "minimal" spanning set for the linear space, in the sense that it has no "redundant" vector. At the same time, it is a "maximal" linearly independent set, in the sense that putting up a new vector makes it linearly dependent.
- A linear space may have more than one basis.

Review: Finite basis for a linear space

Theorem

If $V = span(\{v_1, \ldots, v_n\})$, then there is a subset of $\{v_1, \ldots, v_n\}$ which is a basis for V.

Theorem

Suppose that $V = span(\{v_1, \ldots, v_n\})$. Then each independent set of V has at most n elements.

Review: Dimension

Theorem

If $\{v_1, \dots, v_m\}$ and $\{w_1, \dots, w_n\}$ are both bases for a linear space V, then $m = n$.

Definition

Suppose that V has a finite basis. Then ***dimension*** of V denoted by $\dim V$ is the number of elements of any basis of V.

- Example. Assume the linear space $P_2(x) = \{a_2x^2 + a_1x + a_0 \mid a_i \in \mathbb{R} \text{ for all } 0 \le i \le 2\}$.
  1. The sets $\{1, x, x^2\}$ is a basis for $P_2(x)$.
  2. $\dim(P_2(x)) = 3$.

$B = \{x^2, x, 1\}$

$p(x) = 3x^2 - x + ۲$

$$[p(x)]_B = \begin{array}{cc} \begin{bmatrix} 3 \\ -1 \\ ۲ \end{bmatrix} & \begin{matrix} x^2 \\ x \\ 1 \end{matrix} \end{array}$$

ضرایب

$p(x) = x^2$ $\quad \begin{matrix} 1 \\ 0 \\ 0 \end{matrix}$

$-x+2$ $\quad \begin{bmatrix} 0 \\ -1 \\ ۲ \end{bmatrix}$

Review: Coordinates

Now suppose V is a finite-dimensional linear space and that $B = \{v_1, \cdots, v_n\}$ is an ordered basis for V. Given $v \in V$, there is a unique n-tuple $c = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ of scalars such that $v = \sum_{i=1}^{n} c_i v_i$. The vector c is called the coordinate vector of v relative to the ordered basis B and denoted by $[v]_B$.

$c_1 \to v_1$, $c_n \to v_n$

Theorem

Let v be a linear space. Suppose that $B = \{v_1, \dots, v_n\}$ and $B' = \{v'_1, \dots, v'_n\}$ are two bases of V. Then $[v]_B = P[v]_{B'}$ where the columns of P are the coordinates of the vectors $v'_1, \dots, v'_n$ in the basis B.

$[v]_{\text{قدیم } B} = [\ \uparrow\ ]\,[v]_{\text{جدید } B'}$

ستون ها

مختصات v'_i در B

$[v'_i]_B$

Row Reduced Form R

$$
\begin{array}{cccccccc}
 & A_1 & A_2 & & A_3 & A_4 & \\
A = & \multicolumn{6}{c}{\begin{bmatrix} 1 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 1 & 1 & -2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 3 & 1 & -2 \end{bmatrix}}
\end{array}
$$

ستون‌ها

$$
\begin{array}{ccccccc}
R = & \multicolumn{6}{c}{\begin{bmatrix} 1 & 0 & -3 & 0 & 0 & 4 \\ 0 & 1 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 & -2/3 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}} \\
 & R_1 & R_2 & & R_3 & R_4 &
\end{array}
$$

$Ax = b$

$\underline{x_1 A_1} + \cdots + \underline{x_n A_n} = b$

$\underline{C(A)}$

The column space of A

Lemma

Let $A \in M_{mn}(\mathbb{R})$, then $\underline{\dim C(A)} = \underline{\dim C(R)}$.

$A \longrightarrow R$

$EA = R$

$\dim C(A) = \dim C(\underbrace{EA}_{R})$

The column space of A

Lemma

Let $A \in M_{mn}(\mathbb{R})$, then $\dim C(A) = \dim C(R)$.

$\underline{E}A = R$

Lemma

Let $P \in M_m(\mathbb{R})$ is invertible and $A \in M_{mn}(\mathbb{R})$, then
$\dim C(A) = \dim C(PA)$.

The column space of A

Lemma

Let $P \in M_m(\mathbb{R})$ is invertible and $A \in M_{mn}(\mathbb{R})$, then
$\dim(C(A)) = \dim(C(PA))$.

$\dim C(PA) = r \leq \dim C(A)$

Sketch of the proof: Let $(\dim(C(A)) = r$ and denote the i-th

column of A by A_i. Without loss of generality, assume that the first r columns of A are independent. We have

$$PA = P\begin{bmatrix} A_1 \cdots A_n \end{bmatrix} = \begin{bmatrix} PA_1 \cdots PA_n \end{bmatrix}$$

i_j

We show that the first r columns of B are independent. Consider

$$c_1PA_1 + \cdots + c_nPA_r = 0$$
$$P(c_1A_1 + \cdots + c_nA_r) = 0$$
$$\Rightarrow c_1 = \cdots = c_r = 0.$$

$S = PA$

$S_{i_1}, \dots, S_{i_r}$ هم مستقل خطی باشند

$\dim C(PA) = r$

اگر: $A_{i_1}, \dots, A_{i_r}$ مستقل اند

$C(A)$

$$\sum_{j=1}^{r} c_j A_{i_j} = 0$$

$$\underbrace{P\left(\sum_{j=1}^{r} c_j A_{i_j}\right)}\ = 0$$

$$\sum_{j=1}^{r} c_j \underbrace{PA_{i_j}} = 0$$

$$\sum_{j=1}^{r} c_j S_{i_j} = 0$$

$$\Rightarrow c_j = \cdots = c_r = 0$$

فرض کنیم $A_{i_1}, \dots, A_{i_k}$ پایه $C(A)$

اگر $S_{i_1}, \dots, S_{i_k}$ مستقل اند:

$(\dim C(A) \leq \dim C(PA))$

What about column spaces of A?

- For invertible $P \in M_m(\mathbb{R})$, $A \in M_{mn}(\mathbb{R})$ and PA, the column spaces of A and PA might not be the same.
- **Example.**

$$\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}}_{B} = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}}_{A}$$

$$C(A) = \text{span}\left(\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}\right), \qquad C(PA) = \text{span}\left(\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}\right)$$

Thus $C(A) \neq C(PA)$.

$(\dim C(A) \leq \dim C(PA))$

$\sum_{j=1}^{t} d_j S_{ij} = 0$

$P^{-1}\left(\sum d_j \overbrace{S_{ij}}^{PA_{ij}}\right) = 0$

$\sum d_j A_{ij} = 0$

$\Rightarrow d_1 = \cdots = d_t = 0$

The row spaces of A

تعریف: $A \in M_{mn}(\mathbb{R})$. فضای تولید شده توسط سطرهای A را فضای سطری A می‌نامیم.

Lemma

Let $P \in M_m(\mathbb{R})$ is invertible and $A \in M_{mn}(\mathbb{R})$, then the row spaces of A and PA are the same.

فضای سطری PA $= \text{Span}(\{ p_{i1}A_1 + \cdots + p_{im}A_m \mid 1 \leq i \leq m\}) \subseteq \underbrace{\text{Span}(\{A_1, \ldots, A_m\})}_{\text{فضای سطری A}}$ ①

Sketch of the proof:

$A = P^{-1}B = \begin{bmatrix} q_{11} & \cdots & q_{1m} \\ q_{m1} & \cdots & q_{mm} \end{bmatrix}\begin{bmatrix} B_1 \\ \vdots \\ B_m \end{bmatrix}$

$$B = PA = \begin{bmatrix} p_{11} & \cdots & p_{1m} \\ \vdots & \ddots & \vdots \\ p_{m1} & \cdots & p_{mm} \end{bmatrix}\begin{bmatrix} A_1 \\ \vdots \\ A_m \end{bmatrix} = \begin{bmatrix} p_{11}A_1 + \cdots + p_{1m}A_m \\ \vdots \\ p_{m1}A_1 + \cdots + p_{mm}A_m \end{bmatrix}$$

فضای سطری A $= \text{Span}(\{ q_{i1}B_1 + \cdots + q_{im}B_m \mid 1 \leq i \leq m\}) \subseteq \underbrace{\text{Span}(\{B_1, \ldots, B_m\})}_{\text{فضای سطری PA}}$ ②

① و ② $\Rightarrow$ فضای سطری A = فضای سطری PA

The row space and the column space

$A \in M_{mn}(\mathbb{R})$

$A = [\quad]_{m \times n}$

فضای سطری $\subseteq \mathbb{R}^n$

فضای ستونی $\subseteq \mathbb{R}^m$

$\dim($فضای سطری A$) = \dim($فضای ستونی A$)$

Theorem

Let $A \in M_{mn}(\mathbb{R})$, then the dimension of row space and dimension of column space are the same.

ستون محوری

$EA = [\quad]_{m \times n}$

$N(A) = \{x \mid Ax = 0\} \subseteq \mathbb{R}^n$

$\dim($فضای سطری R$) = \dim($فضای ستونی R$) =$ تعداد ستون‌های محوری

$\dim($فضای سطری A$) = \dim($فضای ستونی A$)$

$\dim(A$ ... $) = \dim($ ... $)$

The row space and the column space

Theorem

Let $A \in M_{mn}(\mathbb{R})$, then the dimension of row space and dimension of column space are the same.

Definition

Let $A \in M_{mn}(\mathbb{R})$. The number of independent columns (or rows) is called the rank of A and denoted by $\operatorname{rank}(A)$.

The Rank Theorem

$N(A) = \{x \in \mathbb{R}^n \mid Ax = 0\} \subseteq \mathbb{R}^n$

$C(A) = \{Ax \mid x \in \mathbb{R}^n\}$

Theorem

Let $A \in M_{mn}(\mathbb{R})$, then

$$\dim C(A) + \dim N(A) = n$$

$\dim N(A) = n - \dim C(A)$

برهان. چون $N(A) \subseteq \mathbb{R}^n$، یک زیرفضای با بعد متناهی است. ($\dim v = n$ ، $W \subseteq v$ زیرفضا $\dim W \leqslant \dim v$). پس

$\dim N(A) = r$

فرض کنید $\{v_1, \dots, v_r\} \subseteq \mathbb{R}^n$ پایه‌ای برای $N(A)$ و آن را به یک پایه برای $\mathbb{R}^n$ گسترش دهیم. $\{\underbrace{v_1, \dots, v_r}_{N(A)}, v_{r+1}, \dots, v_n\}$

$\dim C(A) = n - r$

ادعا: $\{Av_{r+1}, Av_{r+2}, \dots, Av_n\}$ پایه‌ای برای $C(A)$ است.

$\sum_{j=1}^{n-r} c_j v_{r+j} \in N(A) \Leftarrow A\left(\sum_{j=1}^{n-r} c_j v_{r+j}\right) = 0 \Leftarrow \sum_{j=1}^{n-r} c_j A v_{r+j} = 0$ زیرا: مستقل خطی اند -

$\Rightarrow \sum_{j=1}^{n-r} c_j v_{r+j} = 0 \xrightarrow[\text{مستقل}]{\{v_{r+1}, \dots, v_n\}} c_1 = \dots = c_{n-r} = 0$

$C(A) = \operatorname{span}(\{Av_{r+1}, \dots, Av_n\})$ ، $C(A)$ تولید کنند -

$\overset{?}{\subseteq}$

فرض کنید $Av \in C(A)$.

$v = \sum_{j=1}^{n} c_j v_j$

$Av = \sum_{j=1}^{n} c_j A v_j$

The Four Fundamental Subspaces

$A \in M_{mn}$

$\underbrace{\dim C(A)}_{\operatorname{rank}(A)} + \dim N(A) = n$

$A = [\ \ ]$

$\underbrace{\dim C(A)}_{\text{rank}(A)} + \dim N(A) = n$

1. The column space of A : $C(A)$
2. The nullspace of A : $N(A)$

   $A^T \in M_{nm}$
3. The row space of A : $C(A^T)$
4. The left nullspace of A : $N(A^T)$

$A =$

$A^T =$

$Av = \sum_{j=1}^{n} c_j A v_j$

$= c_1 Av_1 + \cdots + c_r Av_r +$

$c_{r+1} A v_{r+1} + \cdots + c_n A v_n$

$= c_{r+1} A v_{r+1} + \cdots + c_n A v_n$

$N(A^T) = \underline{\underline{A^T \text{ فضای پوچ}}} = \{ y \mid A^T y = 0 \} = \{ y \mid y^T A = 0 \}$

فضای پوچ چپ

The rank Theorem

رتبه + بعد فضای پوچ $= \underline{\underline{n}}$

Suppose that $A \in M_{mn}(\mathbb{R})$ and $\text{rank}(A) = r$.

- $C(A)$ = column space of A; dimension r.
- $N(A)$ = nullspace of A; dimension $n - r$.
- $C(A^T)$=row space of A; dimension r.
- $N(A^T)$= left nullspace of A; dimension $m - r$.

$C \subseteq \mathbb{R}^n$

$\underbrace{\dim C(A)}_{r} + \dim N(A) = n$

$\underbrace{\dim C(A^T)}_{r} + \dim N(A^T) = m$

Example

- $A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}$ 3×4

$\dim C(R) = 2$

$\dim C(A) = \text{rank}(A)$

- The Row Reduced matrix $R = \begin{bmatrix} \boxed{1} & 3 & 0 & -1 \\ 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 + 3x_2 - x_4 \\ x_3 + x_4 \\ 0 \end{bmatrix}$

- $\text{rank}(A) = r =$ the number of pivot variables $= 2$.

$x_1 = -3x_2 + x_4$

$x_3 = -x_4$

$$\dim(C(A)) = r = 2 \qquad \dim(N(A)) = n - r = 4 - 2 = 2$$
$$\dim(C(A^T)) = r = 2 \qquad \dim(N(A^T)) = m - r = 3 - 2 = 1$$

$N(A) = N(R) = \left\{ \begin{bmatrix} -3x_2 + x_4 \\ x_2 \\ -x_4 \\ x_4 \end{bmatrix} \right\} = x_2 \underbrace{\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{w_1} + x_4 \underbrace{\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}}_{w_2} = \text{Span}(\{w_1, w_2\})$

$\dim N(A) = 2$

$N(A) \subseteq \mathbb{R}^4$

$\underbrace{\text{rank}(A)}_{r} + \dim N(A) = 4$

$N(A^T) \subseteq \mathbb{R}^3$

$\underbrace{\text{rank}(A)}_{r} + \dim N(A^T) = 3$

$\dim N(A) = 4 - r = 2$

$\dim N(A^T) = 3 - r = 1$

$N(A) = N(R)$

$x \in N(A) : Ax = 0$

$\Updownarrow \; Rx = EAx = 0$

$x \in N(R)$

$c_1 w_1 + c_2 w_2 = 0 \Rightarrow c_1 = c_2 = 0$

Matrices of Rank 1

- Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 2 & 2 \\ 8 & 4 & 4 \\ -2 & -1 & -1 \end{bmatrix}$
- Every row is a multiple of the first row, so the row space is one-dimensional.
- The columns are all multiples of the same column vector; the column space shares the dimension 1.
- We have

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 2 & 2 \\ 8 & 4 & 4 \\ -2 & -1 & -1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 2 \\ 4 \\ -1 \end{bmatrix}}_{u} \underbrace{\begin{bmatrix} 2 & 1 & 1 \end{bmatrix}}_{v^T}$$

$u \in \mathbb{R}^n$

$A \in M_{mn}(\mathbb{R})$

Matrices of Rank 1

Every matrix of rank 1 has the simple form $A = uv^T$, column times row.

Thank You!