Lecture27

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Lecture27

Linear Algebra

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Primary Decomposition Theorem

Theorem

Let T be a linear function over a finite dimensional linear space V whose minimal polynomial factorizes as

$$p(x) = p_1^{r_1}(x) \cdots p_k^{r_k}(x),$$

where the p_i's are monic and mutually coprime (i.e. have no nontrivial common factors). Let $W_i = N\left(p_i^{r_i}(T)\right)$ for each $1 \leqslant i \leqslant k$. Then

1. $V = W_1 \oplus \cdots \oplus W_k$.
2. *For each $1 \leqslant i \leqslant k$, $T(W_i) \subseteq W_i$.*
3. *The minimal polynomial of $T_i = T\restriction_{W_i}$ is $p_i(x)$.*

Primary Decomposition Theorem

$W_i = N(T-\lambda_i I)$

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$W_i = N(T-\lambda_i I)$

- Note that a linear function T is diagonalizable if and only if its minimal polynomial factorizes as

$$p(x) = (x-\lambda_1)\cdots(x-\lambda_k)$$

$V = W_1 \oplus \cdots \oplus W_k$

$f(x) = (x-\lambda_1)^{d_1}\cdots(x-\lambda_k)^{d_k}$

$P(x) = (x-\lambda_1)^{r_1}\cdots(x-\lambda_k)^{r_k}$

$(T-\lambda_i I)v_i = 0$

$Tv_i = \lambda_i v_i$

$[T]_B = \begin{bmatrix} \lambda_1 I_{d_1} & & \\ & \ddots & \\ & & \lambda_k I_{d_k} \end{bmatrix}$

$v_i \in W_i$

$v = v_1 + \cdots + v_k$

$P([T]_B) = 0 \quad \forall v \in V$

$([T]_B - \lambda_1 I)\cdots([T]_B - \lambda_k I)v_i = 0$

$([T]_B - \lambda_1 I)\cdots([T]_B - \lambda_{i-1} I)\cdots([T]_B - \lambda_k I)$

$([T]_B - \lambda_i I)v_i = 0$

Matrix Representation

- Suppose that T is a linear function on V with the characteristic polynomial is

$$f(x) = (x-\lambda_1)^{d_1}\dots(x-\lambda_k)^{d_k}$$

where $\lambda_1, \dots, \lambda_k$ are distinct elements and $d_i \geqslant 1$.
- Then the minimal polynomial for T will be

$$p(x) = (x-\lambda_1)^{r_1}\dots(x-\lambda_k)^{r_k}$$

where $1 \leqslant r_i \leqslant d_i$ based on the Cayley–Hamilton theorem.
- If W_i is the null space of $(T-\lambda_i I)^{r_i}$, then the primary decomposition theorem tells us that

$$V = W_1 \oplus \cdots \oplus W_k$$

$T|_{W_1} = T_1$ $\quad$ $T|_{W_k}$

such that the linear function $T_i = T\restriction_{W_i} : W_i \to W_i$ has minimal polynomial $(x-\lambda_i)^{r_i}$.

x_i

x^i

Matrix Representation

$B = \bigcup B_i$

$V = W_1 \oplus \cdots \oplus W_k$

$T|_{W_i} : W_i \to W_i$

Suppose that B_i is a basis for W_i. It has been proved that $B = \bigcup_{i=1}^{k} B_i$ is a basis for V. Based on primary decomposition theorem,

$$T(W_i) \subseteq W_i.$$

$B_i = \{v_{i1}, \dots, v_{id_i}\}$

$T(v_{i1}) \subseteq W_i$

$T(v_{i1}) =$

Thus

$$[T]_B = \begin{bmatrix} [T_1]_{B_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & [T_k]_{B_k} \end{bmatrix}.$$

Matrix Representation

$T|_{W_i}$

$$V = W_1 \oplus \cdots \oplus W_k$$

$W_i = N\left((T - \lambda_i I)^{r_i}\right)$

and

$$[T]_B = \begin{bmatrix} [T_1]_{B_1} & & \\ & \ddots & \\ & & [T_k]_{B_k} \end{bmatrix}.$$

$T|_{W_i} = T_i : W_i \longrightarrow W_i$

تحدید تابع

$N_i = (T_i - \lambda_i I)$

- Let N_i be the linear function on W_i defined by $N_i = T - \lambda_i I$.
- Then N_i is nilpotent and has minimal polynomial x^{r_i}
- Thus, T on W_i acts as N_i plus the scalar λ_i times the identity function I.
- Suppose we choose a basis for the subspace W_i and then find the representation matrix of N_i on W_i.

$\forall v \in W_i$

$N^{r_i} v = (T - \lambda_i I)^{r_i} v = 0$

$N^{r_i} = 0$

Review: Nilpotent matrices and Nilpotent linear functions

Definition

A square matrix A is called nilpotent matrix with degree non-negative integer k if A^k is the zero matrix and A^r is the non-zero matrix for each r, $1 \leqslant r \leqslant k$.

Definition

A be a linear function T on V is called nilpotent linear function with degree non-negative integer k if T^k is the zero linear function and T^r is the non-zero one for each r, $1 \leqslant r \leqslant k$.

$1 \leq r < k$

Review: Example

Let $A \in M_3(\mathbb{R})$ be the following matrix:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then

$$A^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq \mathbf{O}$$

The third power of A is

$$A^3 = A^2 A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{O}$$

Review: Nilpotent matrices (Revised version)

Lemma

Let $A \in M_n(\mathbb{F})$ where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

1) *The matrix A is nilpotent if and only if all the eigenvalues of A is zero.*
2) *The matrix A is nilpotent if and only if $A^n = O$.*

Nilpotent matrices

$B = \{v, Tv, \dots, T^{n-1}v\}$

$\sum_{i=0}^{n-1} \alpha_i T^i v = 0$ [illegible] $g(x)$ $\deg g(x) \leq n-1$

$g(x) = \sum_{i=0}^{n-1} a_i x^i \qquad g(T)v = 0$

Lemma $\boxed{v \neq 0}$ $T^{n-1}v = 0 \quad T^{n-1} \neq 0 \quad T: V \to V$

Let V be a finite dimensional linear space. If a linear function T on V is nilpotent with degree n where $n = \dim V$, then there is a basis for V such that

$T(v) = \quad T(Tv) = T^2 v$

$$[T]_B = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & 0 & 0 \\ 0 & 0 & \ddots & 0 & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

$k < n \quad T^n = 0 \quad T^k \neq 0 \quad p(x) \mid x^n$

$p(x) = x^r \quad r = n$

$\boxed{p(x) = x^n}$

$p(x) = \underline{x}$ [illegible]

$\alpha_{n-1} T^{n-1} v + \dots + \alpha_1 T v + \alpha_0 I = 0$

$\alpha_1 \neq 0$

$\alpha_\ell \neq 0$

$T^{n-\ell-1}(\alpha_\ell T^\ell v + \dots + \alpha_1 T v + \alpha_0 I) = 0$

$\alpha_\ell T^{n-1} v + \dots + \alpha_1 T^{n-\ell} v + \alpha_0 T^{n-\ell-1} = 0$

$\alpha_\ell T^\ell v + \dots + \alpha T^{n-\ell} v$

$0 = \alpha_\ell T^{n-1} v + \dots + \alpha_{n-1} T^{2n-2-\ell} v$

α_ℓ

Lemma

Let V be a finite dimensional linear space. Then there is a vector $v \in V$ whose minimal polynomial respect to v is minimal polynomial T.

$T: V \to V \qquad \exists v \in V \to P_v(x) = P_T(x)$

$P_T(x) \qquad P_T(T) = 0$

[illegible] T [illegible] | $v \in V \quad P_v(x)$ [illegible] T

$P_T(T) = 0$ | $P_v(T)v = 0$

$\forall w \in V \quad P_T(T)w = 0$

Proof.

$T : V \to V \quad \deg p(x) \le n \quad p(T) = 0 \quad W = N(p^i(T)) \quad f(T) = 0$

$p_T(x) = p_1^{r_1}(x) \cdots p_k^{r_k}(x) \qquad V = W_1 \oplus \cdots \oplus W_k \quad v_i \in W_i = N(p_i^{r_i}(T))$

$v \in V$

$T|_{W_i}$ چندجمله‌ای مینیمال $= p_i^{r_i}(T)$

ادعا

$\exists\, v_i \in W_i$ مینیمال چندجمله‌ای W_i برابر $p_i^{r_i}(T)$

در ضمن فرض کنید برای هر $v_i \in W_i$، مینیمال پلی‌نومیال v_i، $p_{v_i}(x)$

$p_{v_i}(x) \mid p_i^{r_i}(x)$ برابر با $p_i^{r_i}(x)$ نباشد.

$p_{v_i}(x) = p_i^s(x) \qquad s < r_i$

$N(p_i^{r_i-1}(T)) \subsetneq N(p_i^{r_i}(T))$

$\max_{v \in W_i} p_i^s(T)v$

$v = v_1 + \cdots + v_k$, $p_1^{r_1}(x)$, $p_k^{r_k}(x)$

Proof.

$N_i = T_i - \lambda_i I$

$N_i^{r_i} = 0$

$N_i : W_i \longrightarrow W_i$

$\dim W_i = r_i$

$B_i = \{v, N_i v, \dots, N_i^{r_i-1} v\}$

$$[T_i]_{B_i} = \begin{bmatrix} \lambda_i & & 0 \\ 1 & \ddots & \\ 0 & \ddots\ 1 & \lambda_i \end{bmatrix}_{r_i \times r_i}$$

$2r_i - 1$

صفر $r_i^2 - 2r_i + 1$

$$[N_i]_{B_i} = \begin{bmatrix} 0 & & 0 \\ 1 & \ddots & \\ 0 & \ddots\ 1 & 0 \end{bmatrix}$$

Review: Representation Matrix

$r_i < d_i$ اگر d_i

- For the linear function T:
  1. The characteristic polynomial: $f(x) = (x - \lambda_1)^{d_1} \dots (x - \lambda_k)^{d_k}$
  2. The minimal polynomial: $p(x) = (x - \lambda_1)^{r_1} \dots (x - \lambda_k)^{r_k}$.
- $V = W_1 \oplus \cdots \oplus W_k$ where $W_i = N\big((T - \lambda_i)^{r_i}\big)$

$$[T]_B = \begin{bmatrix} [T_1]_{B_1} & & \\ & \boxed{[T_i]_{B}} & \\ & & [T_k]_{B_k} \end{bmatrix}.$$

- Let N_i be the linear function on W_i defined by $N_i = T - \lambda_i I$.
- Then N_i is nilpotent and has minimal polynomial x^{r_i} on W_i.
- But $\dim W_i = d_i \geqslant r_i$ (?)

$$[N_i]_{\bigcup_{j=1} B_{ij}}$$

$$\begin{bmatrix} \lambda_i & 0 \\ \ddots & \ddots \\ 0 & \lambda_i \end{bmatrix}$$

$v, N_i v, \dots$

$\dim N_i \neq r_i$

$[N_i]_{\cancel{B}}$

B_i

$\text{span}(\{v, N_i v, \ldots, N_i^{r_i-1} v\}) \subsetneq W_i$

$\text{Span}(\{v, N_i v, \ldots, N_i^{r_i-1} v\}) = \{g(N_i)v \mid g(x) \in \mathbb{F}(x)\} = Z(v, N_i)$ $\notin \mathbb{R}$

$N_i^{r_i} = 0$ $\quad Z(W, T)$

$\dim W_i = d_i > r_i$

$g(T)v = \alpha_0 v + \alpha_1 T v + \cdots + \alpha_{r_i-1} T^{r_i-1} v$

$g(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_{r_i-1} x^{r_i-1}$

$w \in W_i \setminus \text{Span}(\{v, N_i v, \ldots, N_i^{r_i-1} v\})$

$\{w, N_i w, \ldots, N_i^m w\} = Z(w, N_i)$

$V = W_1 \oplus \cdots \oplus W_k$

$W_i = Z(v_1, N_i) \oplus \cdots \oplus Z(v_k, N_i)$

The smallest T-invariant subspace containing v

- Assume V is finite-dimensional linear space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and T is a fixed (but arbitrary) linear function on V.
- If W is any subspace of V which is invariant under T and contains v, then W must also contain the vector $T(v)$; hence W must contain $T(Tv) = T^2v$, $T(T^2v) = T^3v$, etc. In other words W must contain $g(T)v$, for every polynomial $g(x)$ over $\mathbb{F}$. The set of all polynomial $g(x)$ over $\mathbb{F}$ is denoted by $\mathbb{F}[x]$
- Let $Z(v, T) = \{g(T)v \mid g(x) \in \mathbb{F}[x].\}$
- $Z(v, T)$ is a subspace of V and it is the smallest T-invariant subspace which contains v.

T-cyclic subspace generated by v

Definition

If v is any vector in V, the subspace $Z(v,T)$ is called the **T-cyclic subspace generated**. If $Z(v,T) = V$, then v is called a cyclic vector for T.

$V_i = Z(v_1, T$

$V = Z(v_1, T) \oplus \cdots \oplus Z(v_k, T)$

$\dim Z(v_1, T) = \deg P_v(x)$

For any T:

1. The $T-$cyclic subspace generated by the zero vector is the zero subspace.
2. The space $Z(v,T)$ is one-dimensional if and only if v is an eigenvalue vector for T.
3. Thus, we shall be interested in linear relations:
$$c_0 v + c_1 T v + \cdots, c_k T^k v = 0.$$
between the vectors $T^j v$, that is we shall be interested in the polynomials
$$c_0 + c_1 x + \cdots, c_k x^k = 0$$
which have the property that $g(T)v = 0$.

The dimension of T-cyclic subspace generated by v

Theorem

Assume that T is a linear space on a linear space V. Let v be any non-zero vector in V and let $p_v(x)$ is the minimal polynomial for v respect to T.

1. $\dim Z(v,T) = deg p_v(x)$.
2. *If U is the linear function on $Z(v,T)$ induced by T, then the minimal polynomial for U is $p_v(x)$.*

Minimal and characteristic polynomials of a cyclic vector

Theorem

T has a cyclic vector if and only if the minimal and characteristic polynomials for T are identical.

Cyclic Decomposition Theorem

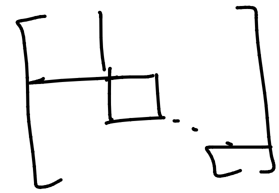
Theorem

Let T be a linear function on a finite-dimensional vector space V. There exist non-zero vectors $v_1, \ldots, v_k \in V$ with minimal polynomial $p_{v_1}, \ldots, p_{v_k}$ such that

(i) $V = Z(v_1, T) \oplus \cdots \oplus Z(v_k, T)$.

(ii) $p_{v_i} \mid p_{v_{i-1}}$ *for each* $i \geqslant 2$.

(iii) *Furthermore, the integer r and the minimal polynomial $p_{v_1}, \ldots, p_{v_k}$ are uniquely determined by (i), (ii).*



$\dim Z(v, T) = \deg p_v(x)$

$= \{v, N_i v, \ldots, N_i^{r_i-1} v\}$

$W_i = Z(v_1, N_i) \oplus \cdots \oplus Z(v_k, N_i)$

$[T_i]_{B_i} = \left[\begin{array}{c|c} \lambda_i & 0 \\ \hline 0 & \lambda_i \end{array}\right]$

$B_i = \{v_1, \ldots, N_i^{r_1-1} v_1, \ldots, v_k, \ldots, N_i^{r_k-1} v_k\}$

$r_k \leq \cdots \leq r_2 \leq r_1$

Proof.

Proof.

Jordan Form

Nilpotent matrices

Lemma $V = Z(v, T)$

Let T is a linear function on V such that $B = \{v, Tv, \dots, T^{n-1}v\}$ is a basis for V where $0 \neq v \in V$. Then

$$[T]_B = \begin{bmatrix} 0 & 0 & \cdots & 0 & -c_0 \\ 1 & 0 & \cdots & 0 & -c_1 \\ 0 & 1 & \ddots & 0 & -c_2 \\ 0 & 0 & \ddots & 0 & \vdots \\ 0 & 0 & \cdots & 1 & -c_{n-1} \end{bmatrix}$$

$n-1$ $\quad$ n

$T(T^{n-1}v) = T^{n-1}v$

$T^n v + c_{n-1}T^{n-1}v + \cdots c_1 T + c_0 I = 0$

where $p(x) = x^n + c_{n-1}x^{n-1} + \ldots + c_1x^1 + c_0$ is the minimal polynomial for T.

$T(T^{n-1}v) = T^n v = -c_{n-1}T^{n-1}v - \cdots - c_1 T - c_0 I v$

W_v

$V = Z(v_1, T) \oplus \cdots \oplus Z(v_k, T)$

$\{v_1, Tv_1, \dots, T^{k_1-1}v_1\}$

$k_1 = \deg p_{v_1}$

$p(T) =$

Rational Form

- By Cyclic Decomposition Theorem: $V = Z(v_1, T) \oplus \cdots \oplus Z(v_k, T)$.
- Matrix representation by diagonal blocks:

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & -c_0 \\ 1 & 0 & \cdots & 0 & -c_1 \\ 0 & 1 & \ddots & 0 & -c_2 \\ 0 & 0 & \ddots & 0 & \vdots \\ 0 & 0 & \cdots & 1 & -c_{n-1} \end{bmatrix}$$

$d_i . w_i = d_i$ (1)

(2)

$[T]_B = \begin{bmatrix} * & & 0 \\ & * & \\ & & \ddots \\ 0 & & * \end{bmatrix}$

$\{v, Tv, \dots, T^{n-1}v\}$

. . .

Thank You!