Linear Algebra

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Determinant

What is volume?

- Every $n-$dimensional parallelepiped with $\{a_1, \ldots, a_n\}$ as legs is associated with a real number, called its volume which has the following properties:

  1. If we stretch a parallelepiped by multiplying one of its legs by a scalar λ, its volume gets multiplied by λ.
  2. If we add a vector w to i-th legs of a $n-$dimensional parallelepiped with $\{a_1, \ldots, a_i, a_{i+1}, \ldots, a_n\}$, then its volume is the sum of the volume from $\{a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_n\}$ and the volume of $\{a_1, \ldots, a_{i-1}, w, a_{i+1}, \ldots, a_n\}$.
  3. The volume changes sign when two legs are exchanged.
  4. The volume of the parallelepiped with $\{e_1, \ldots, e_n\}$ is one.

Volume map as an n-alternating multilinear map

- More formally, the volume is a n-alternating multilinear map on all n-parallelepipeds such that the volume of the standard unit parallelepiped is one.

- Thus, the volume is n-alternating multilinear map

$$\phi : \underbrace{V \times \cdots \times V}_{n} \to \mathbb{R}$$

such that V is a linear space with dimension n and $\phi(e_1, \ldots, e_n) = 1$.

Classification of n-alternating multilinear maps

- Let $\phi : \underbrace{V \times \cdots \times V}_{n} \to \mathbb{R}$ be an n-alternating multilinear map.
- $\phi(a_1, \ldots, a_{i-1}, v, a_{i+1}, \ldots, a_{j-1}, v, a_{j+1}, \ldots, a_n) = 0$ (why?)
- Since the map is n- multilinear map, we have

$$\begin{aligned}\phi\left(a_1, \ldots, a_n\right) &= \phi\left(\sum_{j_1=1}^{n} a_{1j_1} e_{j_1}, \ldots, \sum_{j_n=1}^{n} a_{nj_n} e_{j_n}\right) \\ &= \sum_{j_1=1}^{n} \ldots \sum_{j_n=1}^{n} a_{1j_1} \cdots a_{nj_n} \phi\left(e_{j_1}, \ldots, e_{j_n}\right)\end{aligned}$$

- Now, we note that if any two of the e_{j_k}'s are equal, then

$$\phi\left(e_{j_1}, \ldots, e_{j_n}\right) = 0.$$

- So, we may remove the corresponding term from the sum.

Classification of n-alternating multilinear maps

$$\begin{aligned}\phi\left(a_1,\ldots,a_n\right) &= \phi\left(\sum_{j_1=1}^{n} a_{1j_1}e_{j_1},\ldots,\sum_{j_n=1}^{n} a_{nj_n}e_{j_n}\right)\\ &= \sum_{j_1=1}^{n}\cdots\sum_{j_n=1}^{n} a_{1j_1}\cdots a_{nj_n}\phi\left(e_{j_1},\ldots,e_{j_n}\right)\end{aligned}$$

- We want all tuples of j_k's such that each pair of j_k's are mutually distinct.
- That is, $(j_1,\ldots,j_n)$ must be a permutation of $(1,\ldots,n)$.
- We denote the set of all permutations of $(1,\ldots,n)$ by S_n.
- $(j_1,\ldots,j_n)$ is a permutation of $(1,\ldots,n)$ means that there is $\sigma\in S_n$ such that $j_k=\sigma(k)$.

Classification of n-alternating multilinear maps

- For an n-alternating multilinear map

$$\phi : \underbrace{V \times \cdots \times V}_{n} \to \mathbb{R}$$

we have

$$\begin{aligned}
\phi\left(a_1, \ldots, a_n\right) &= \sum_{j_1=1}^{n} \ldots \sum_{j_n=1}^{n} a_{1j_1} \cdots a_{nj_n} \phi\left(e_{j_1}, \ldots, e_{j_n}\right) \\
&= \sum_{\sigma \in S_n} \left(\prod_{i=1}^{n} a_{i\sigma(i)} \phi\left(e_{\sigma(1)}, \ldots, e_{\sigma(n)}\right) \right)
\end{aligned}$$

n-alternating multilinear maps

- Let $\phi : \underbrace{V \times \cdots \times V}_{5} \to \mathbb{R}$ be an 5-alternating multilinear map.
- So, $\phi(e_2, e_1, \ldots, e_5) = -\phi(e_1, e_2, \ldots, e_5)$.
- Consider the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 5 & 2 \end{pmatrix}$$

- That means that σ satisfies

$$\sigma(1) = 3 \quad \sigma(2) = 4 \quad \sigma(3) = 1 \quad \sigma(4) = 5 \quad \sigma(5) = 2.$$

- $\phi(e_{\sigma(1)}, e_{\sigma(2)}, e_{\sigma(3)}, e_{\sigma(4)}, e_{\sigma(5)}) = ?\phi(e_1, e_2, e_3, e_4, e_5)$.

n-alternating multilinear maps

- Look at the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 5 & 2 \end{pmatrix}.$$

- 1 is sent to 3. But 3 is sent back to 1.This is given by the transposition $(1,3)$.
- Now look at what is left $\{2,4,5\}$.
  1. Look at 2. Then 2 is sent to 4.
  2. Now, 4 is sent to 5.
  3. Finally 5 is sent to 2.
  4. So another part of the cycle type is given by the 3-cycle $(2,4,5)$.
  5. $(2,4,5) = (2,4)(4,5)$.
- $\sigma = (2,4)(4,5)(1,3)$.
- $\phi(e_{\sigma(1)}, e_{\sigma(2)}, \ldots, e_{\sigma(5)}) = -\phi(e_1, e_2, \ldots, e_5)$.

Permutations

Definition

A permutation of a finite set X is **even** if it can be written as the product of an even number of transpositions, and it is **odd** if it can be written as a product of an odd number of transpositions.

Example

$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 5 & 2 \end{pmatrix} = (2,4)(4,5)(1,3)$ is an odd permutation

The sign of a permutation

Definition

The sign of a permutation σ, denoted by $\operatorname{sgn}(\sigma)$ and

$$\operatorname{sgn}(\sigma) = \begin{cases} +1 & \text{if } \sigma \text{ is even,} \\ -1 & \text{if } \sigma \text{ is odd.} \end{cases}$$

Example

Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 5 & 2 \end{pmatrix} = (2,4)(4,5)(1,3)$. Then $\operatorname{sgn}(\sigma) = -1$ and

$$\phi\left(e_{\sigma(1)}, e_{\sigma(2)}, \ldots, e_{\sigma(5)}\right) = \operatorname{sgn}(\sigma)\phi(e_1, e_2, \ldots, e_5)$$

Classification of n-alternating multilinear maps

- For an n-alternating multilinear map

$$\phi : \underbrace{V \times \cdots \times V}_{n} \to \mathbb{R}$$

we have

$$\begin{aligned}
\phi\left(a_1, \ldots, a_n\right) &= \sum_{j_1=1}^{n} \cdots \sum_{j_n=1}^{n} a_{1j_1} \cdots a_{nj_n} \phi\left(e_{j_1}, \ldots, e_{j_n}\right) \\
&= \sum_{\sigma \in S_n} \left(\prod_{i=1}^{n} a_{i\sigma(i)} \phi\left(e_{\sigma(1)}, \ldots, e_{\sigma(n)}\right) \right) \\
&= \left(\sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)} \right) \operatorname{sgn}(\sigma) \phi\left(e_1, \ldots, e_n\right) \\
&= \left(\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)} \right) \phi\left(e_1, \ldots, e_n\right)
\end{aligned}$$

The result of classification of n-alternating multilinear map

- An n-alternating multilinear map

$$\phi : \underbrace{V \times \cdots \times V}_{n} \to \mathbb{R}$$

is given with

$$\phi\left(a_1, \ldots, a_n\right) = \left(\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)} \right) \phi\left(e_1, \ldots, e_n\right)$$

Determinant

- Let a_i be the i-th row of $A = [a_{ij}]$.
- The determinant of A is defined by

$$\det A = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}.$$

Thank You!