

Lecture 17

Linear Algebra

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Lecture #1

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Review: Classification of n-alternating multilinear maps

• For an *n*-alternating multilinear map

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we have

ve
$$\phi(a_1, \dots, a_n) = \sum_{j_1=1}^n \dots \sum_{j_n=1}^n a_{1j_1} \cdots a_{nj_n} \phi(e_{j_1}, \dots, e_{j_n})$$

$$\begin{array}{ll}
& = \sum_{\sigma \in S_n} \left(\prod_{i=1}^n a_{i\sigma(i)} \phi\left(e_{\sigma(1)}, \dots, e_{\sigma(n)}\right) \right) \\
& = \left(\sum_{\sigma \in S_n} \prod_{i=1}^n a_{i\sigma(i)} \right) \operatorname{sgn}(\underline{\sigma}) \phi\left(e_1, \dots, e_n\right)
\end{array}$$

$$= \left(\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^n a_{i\sigma(i)}\right) \underbrace{\phi(e_1, \dots, e_n)}_{\mathbf{J}}$$

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(j,1-1)= (1,-1/m)

$$\Phi\left(\begin{bmatrix} 2 & 1 & -16 \\ 2 & 1 & -16 \\ 2 & 2 & 29 \\ 2 & 2 & 20 \\ 2 & 2 & 2$$

d16, 70)







Determinant









- Let a_i be the *i*-th row of $A = [a_{ij}]$.
- The determinant of A is defined by

$$A = \begin{bmatrix} \frac{q_{13}}{d} \\ \frac{q_{13}}{d} \end{bmatrix} \qquad \det A = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^n \frac{a_{i\sigma(i)}}{6 \cdot 3}.$$

• For 2 by 2 matrix

$$\det \begin{bmatrix} \boxed{a & b} \\ \boxed{c & d} \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

det A = Plan - ~ 6(1) 32 + (e, - -, e_) = 1 6 17111

6: 11,21-11,21

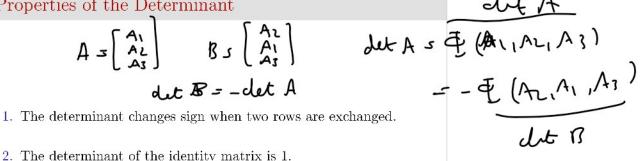
$$G_{1} = \begin{cases} 1 & 2 \\ 1 & 2 \end{cases} \qquad G_{2} = \begin{cases} 1 & 2 \\ 2 & 1 \end{cases} = \begin{cases} 1 & 2 \\ 2 & 1 \end{cases}$$

$$G_{1} = \begin{cases} 1 & 2 \\ 2 & 1 \end{cases} = \begin{cases} 1 & 2 \\ 2 & 1 \end{cases} = \begin{cases} 1 & 2 \\ 2 & 1 \end{cases}$$

6(1) = 1 6(1) 52

Properties of the Determinant

$$A = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} \qquad B \subseteq \begin{bmatrix} A_1 \\ A_1 \\ A_3 \end{bmatrix}$$



- 2. The determinant of the identity matrix is 1.
- 3. The determinant depends linearly on the each row.

Let I = 1

Properties of the Determinant

4. If two rows of A are equal, then $\det A = 0$ (why?)

- 5. Subtracting a multiple of one row from another row leaves the same determinant.(why?)
- 6. If A has a row of zeros, then $\det A = 0$ since the map det is n-multilinear.

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Properties of the Determinant

7. If A is triangular then $\det A = a_{11}a_{22}\cdots a_{nn}$ (why?)



8. If A is singular, then $\det A = 0$. If A is invertible, then $\det A \neq 0$ (why?)

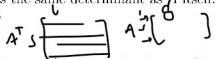
$$A = \begin{bmatrix} A_1 \\ A_n \end{bmatrix}$$

$$A = \left\{ \begin{bmatrix} A_1 \\ A_n \end{bmatrix} - A_n \right\}$$

$$B = \left\{ \begin{bmatrix} A_1 \\ A_1 - CA_1 \end{bmatrix} - A_n \right\}$$

Properties of the Determinant

9. The transpose of A has the same determinant as A itself: $\det A = \det A^T$ (why?)



10. The determinant of \overline{AB} is the product of $\det A$ times $\det B$ (why?)

$$A^{T} = \begin{bmatrix} \alpha_{1j}^{T} \\ \alpha_{1j}^{T} \end{bmatrix} \qquad A^{T} = \begin{bmatrix} \alpha_{11}^{T} & \alpha_{21}^{T} & \alpha_{32}^{T} \\ \alpha_{12}^{T} & \alpha_{22}^{T} & \alpha_{32}^{T} \end{bmatrix} \qquad \text{Left } A^{T}$$

$$PA = \begin{bmatrix} \alpha_{1j}^{T} \\ \alpha_{13}^{T} & \alpha_{22}^{T} \\ \alpha_{13}^{T} & \alpha_{21}^{T} \\ \alpha_{21}^{T} & \alpha_{22}^{T} \end{bmatrix} \qquad \alpha_{13}^{T} \qquad \alpha_{21}^{T} \qquad \alpha_{22}^{T} \qquad \alpha_{22}^{T} \qquad \alpha_{23}^{T} \qquad \alpha_{24}^{T} \qquad \alpha_{25}^{T} \qquad \alpha_{25$$

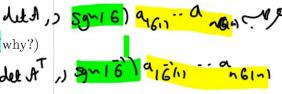
det A = 0 Properties of the Determinant

12. Let $A \in M_r(\mathbb{R}), B \in M_{rs}(\mathbb{R})$ and $C \in M_s(\mathbb{R})$, then

$$\det \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} = \det A \det C.$$

• Proof:

det A = 2 sg-10) a 16(1) n6(1)



det A = > Co Lin A

det A \$ (=) A

Properties of the Determinant

13. Let $A, B, C, D \in M_n(\mathbb{R})$. If CD = DC then

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(AD - BC).$$

- Note that it is also true if AC = CA or AB = BA or BD = DB.
- Proof:

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Properties of the Determinant

14. (Schur formula) Let $A \in M_n(\mathbb{R})$. and

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} =$$

where A_{11} and A_{22} are square matrices. Then

$$\det A = (\det A_{11}) \det (A_{22} - A_2 1 A^{-1} A_1 2).$$

• Proof. The following identity is easy verified:

$$\begin{bmatrix} I & 0 & -A_{21}A_{11}^{-1} & A_{22} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} I & -A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix}$$
$$= \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{bmatrix}$$

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Formulas for the Determinant

- If A is invertible, then PA = LDU
- $\det A = \pm \det L \times \det D \times \det U$.
- $\det L = \det U = 1$.
- $\det D = d_1 \cdots d_n$.
- $\det A = \pm d_1 \cdots d_n$.

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Example

• We obtain:

• Thus,

$$\det A = 2\left(\frac{3}{2}\right)\left(\frac{4}{3}\right)\cdots\left(\frac{n+1}{n}\right) = n+1.$$

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One more formula for the determinant

- Let $A \in M_n(\mathbb{R})$.
- Consider The submatrix A(i|j) that is defined by throwing away row i and column j.
- Let $\phi: \underbrace{V \times \cdots \times V}_{n} \to \mathbb{R}$ be given by

$$\phi(a_1, \dots, a_n) = \sum_{i=1}^n (-1)^{i+j} a_{ij} \det A(i|j).$$

• ϕ is an *n*-alternating multilinear map with $\phi(I) = 1$. Then,

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A(i|j).$$

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Cofactors of A

• Assume that

$$c_{ij} = (-1)^{i+j} \det A(i|j),$$

then c_{ij} is called ij-th cofactor of matrix A.

• Let

$$C = \begin{bmatrix} c_{11} & \cdots & c_{1j} & \cdots & c_{1n} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ c_{n1} & \cdots & c_{nj} & \cdots & c_{nn} \end{bmatrix}$$

- Thus, For each $1 \le j \le n$, inner product of the j-th column of A and the j-th column of C is equal to det A.
- But inner product of the j-th column of A and the k-th column of C is equal to zero for $1 \le j \ne k \le n$.

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Thank You!

det AB; det A det B . I. _ wis

 $A = \begin{bmatrix} \ddots \\ \vdots \\ \ddots \end{bmatrix}$ →: rx -- > 12 --> 1R ₹(A1, - , AL) = det AB

in AB => det A det B so = det AB inÁB REN(AB)

 $N(B) \subseteq N(AB)$ N(B) f ABn = 0 ABn =

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, [A] , VIIR O', (r) A11 -> And

A = []

det AB = Let
$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} B \end{pmatrix} = Let \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} B \end{pmatrix}$$

$$= Let \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} B \end{pmatrix} = Let \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} B \end{pmatrix}$$

$$= Let AB + Let AB$$