Lecture13-1

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Linear Algebra

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Review: Inner products on real linear space

An inner product on V is a function $\langle , \rangle : V \times V \to \mathbb{R}$ such that

1. $\langle v, v \rangle \geqslant 0$ for all $v \in V$.
2. $\langle v, v \rangle = 0$ if and only if $v = 0$.
3. $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ for all $u, v, w \in V$.
4. $\langle cu, w \rangle = c\langle u, w \rangle$ for all $u, w \in V$ and $c \in \mathbb{R}$.
5. $\langle v, w \rangle = \langle w, v \rangle$.

Review

- Suppose that V is an inner product space. For $v \in V$, we define the norm of v, denoted $\|v\|$, by $\|v\| = \sqrt{\langle v, v \rangle}$.
- Two vectors $u, v \in V$ are said to be orthogonal if $\langle u, v \rangle = 0$.

Review: Orthogonal Subspaces

Definition

Two subspaces W_1 and W_2 of the same space V are orthogonal, denoted by $W_1 \perp W_2$, if and only if each vector $w_1 \in W_1$ is orthogonal to each vector $w_2 \in W_2$:

$$\langle w_1, w_2 \rangle = 0.$$

for all w_1 and w_2 in W_1 and W_2, respectively.

Review: Orthogonal complement of a subspace

$\dim V = n$

$\mathrm{span}(\{x_1, \dots, x_k\}) = W \subseteq V = \mathbb{R}^n \quad , \quad W^\perp = ?$

$A = \begin{bmatrix} v_1^T \\ \vdots \\ v_k^T \end{bmatrix}$

$W =$

$W^\perp = N(A)$

Definition

Given a subspace W in linear space V, the space of all vectors orthogonal to W is called the orthogonal complement of V. It is denoted by $W^\perp$.

- We emphasize that W_1 and W_2 can be orthogonal without being complements.
- $W_1 = \mathrm{span}((1, 0, 0))$ and $W_2 = \mathrm{span}((0, 1, 0))$.

Fundamental theorem of orthogonality

Review: Fundamental theorem of orthogonality

Let $A \in M_{mn}(\mathbb{R})$.

$v_1, \dots, v_m \in \mathbb{R}^n \quad A \cdot$

$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = v \in \mathbb{R}^n$

$A = \begin{bmatrix} v_1^T \\ v_m^T \end{bmatrix}_{m \times n}$

$\{v_1, \dots, v_m\}$

1. The row space is orthogonal to the nullspace (in $\mathbb{R}^n$).
2. The column space is orthogonal to the left nullspace (in $\mathbb{R}^m$).

$v_1, \dots, v_n \in \mathbb{R}^m$

$A = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}_{mn}$

Review: Fundamental theorem of orthogonality

Let $A \in M_{mn}(\mathbb{R})$. $v_1, \dots, v_m \in \mathbb{R}^n$ $W = \mathrm{span}(\{v_1, \dots, v_m\})$

$A = \begin{bmatrix} v_1^T \\ \vdots \\ v_m^T \end{bmatrix}_{mn}$

1. The nullspace ($N(A)$) is the orthogonal complement of the row space in $\mathbb{R}^n$.
2. The left nullspace ($N(A^T)$) is the orthogonal complement of the column space in $\mathbb{R}^m$. (سطر ستون) $v_1, \dots, v_n \in \mathbb{R}^m$ $C(A)$ $A = [v_1 \dots v_n]$
3. Direct Sum: $\mathbb{R}^n = N(A) \oplus N(A)^\perp$. $C(A^T) = W$
4. $\mathbb{R}^n = N(A) \oplus C(A^T)$. $\mathbb{R}^n = W^\perp \oplus W$
5. From the row space to the column space, A is actually invertible. Every vector in the column space comes from exactly one vector in the row space.

$v \to V$ [ ] $T: V \to V$

Matrix Representation of Inner Products

V دلخواه

- Let $B = \{v_1, \dots, v_n\}$ be a basis for $\mathbb{R}^n$.
- Suppose that a bilinear function $\langle , \rangle : V \times V \to \mathbb{R}$ is an inner product for $\mathbb{R}^n$. $G_{ij} = \langle v_j, v_i \rangle$
- We want to investigate a matrix representation of this inner product.

$x, y \in V \quad \langle x, y \rangle = ?$

$x = \sum_{i=1}^n x_i v_i \quad , \quad y = \sum_{j=1}^n y_j v_j$

$$\langle x, y \rangle = \langle \sum x_i v_i , \sum y_j v_j \rangle = \sum_{i=1}^n \sum_{j=1}^n x_i y_j \langle v_i, v_j \rangle = \sum_{j=1}^n y_j \sum_{i=1}^n x_i \langle v_i, v_j \rangle$$

$$= [y_1 \cdots y_n] \underbrace{\begin{bmatrix} \sum_{i=1}^n x_i \langle v_i, v_1 \rangle \\ \vdots \\ \sum_{i=1}^n x_i \langle v_i, v_n \rangle \end{bmatrix}} = [y_1 \cdots y_n] \underbrace{\begin{bmatrix} \langle v_1, v_1 \rangle & \cdots & \langle v_n, v_1 \rangle \\ & \ddots & \\ \langle v_1, v_n \rangle & \cdots & \langle v_n, v_n \rangle \end{bmatrix}}_{G} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = y^T G x$$

ماتریس ضرب داخلی نسبت به پایه B
$B = \{v_1, \dots, v_n\}$

Matrix Representation of Inner Products

ویژگی های G:

$N(G) = \{0\}$
۱- G وارون پذیر است $\Leftrightarrow (Gx = 0 \Rightarrow x = 0)$
برهان: $Gx = 0 \Rightarrow x^T G x = 0 = \langle x, x \rangle \Rightarrow x = 0$
۲- G ماتریس متقارن $G = G^T$
$\langle v_j, v_i \rangle = \langle v_i, v_j \rangle \Rightarrow G_{ij} = G_{ji}$
۳- درایه های قطر اصلی G مثبت است. $v_i \neq 0$

- Let $B = \{v_1, \dots, v_n\}$ be a basis for $\mathbb{R}^n$.
- Suppose that a bilinear function $\langle , \rangle : V \times V \to \mathbb{R}$ is an inner product for $\mathbb{R}^n$.
- We want to investigate a matrix representation of this inner product.
- Orthonormal basis! $E = \{e_1, \dots, e_n\} \xrightarrow{\text{استاندارد}} \mathbb{R}^n$

- We want to investigate a matrix representation of this inner product.
- Orthonormal basis! $E = \{e_1, \dots, e_n\} \xrightarrow{\text{استاندارد}} \mathbb{R}^n$
- Vectors $q_1, \dots, q_n$ are orthonormal if

$$q_i^T q_j = \begin{cases} 0 & \text{whenever } i \neq j \quad \text{(for orthogonality)} \\ 1 & \text{whenever } i = j \quad \text{(for normality).} \end{cases}$$

Change of basis matrix for inner product space

$G = [G_{ij}] = [\langle v_j, v_i \rangle]$ $\qquad$ $H = [H_{ij}] = [\langle v'_j, v'_i \rangle]$

Suppose that $B = \{v_1, \dots, v_n\}$ and $B' = \{v'_1, \dots, v'_n\}$ are two bases for an inner product V. Then for each $v \in V$, we have

$$[v]_B = \underline{P}[v]_{B'}$$

$P = \begin{bmatrix} P_{1j} \\ \vdots \\ P_{nj} \end{bmatrix}$ ستون j ، B روی v'_j مختصات

such that

$$v'_j = \sum_{i=1}^{n} P_{ij} v_r.$$

and P is the change basis matrix.

What is the relationship between the matrix of the inner product relative to the basis B and the basis B'?

۳- درایه‌های روی قطر اصلی G مثبت است. $v_i \neq 0$

$G_{ii} = \langle v_i, v_i \rangle > 0$

نکته: اگر اعضای B متعامد یکه باشند.

$$G = \begin{bmatrix} \langle v_1, v_1 \rangle & & \langle v_i, v_j \rangle \\ & \ddots & \\ & & \langle v_n, v_n \rangle \end{bmatrix} = I$$

ضرب داخلی در فضای اقلیدسی $\mathbb{R}^n$ ، E

$$\langle x, y \rangle = y^T I x = y^T x$$

$$H_{ij} = \langle v'_j, v'_i \rangle = \langle \sum_{r=1}^{n} P_{rj} v_r, \sum_{s=1}^{n} P_{si} v_s \rangle$$

ستون j از P $\qquad$ ستون i از P

$$= \sum_{r=1}^{n} \sum_{s=1}^{n} P_{rj} P_{si} \langle v_r, v_s \rangle$$

$$= [P_{1i} \cdots P_{ni}] \begin{bmatrix} \sum_{r=1}^{n} P_{rj} \langle v_r, v_1 \rangle \\ \vdots \\ \sum_{r=1}^{n} P_{rj} \langle v_r, v_n \rangle \end{bmatrix}$$

$$= \underbrace{[P_{1i} \cdots P_{ni}]}_{i} \underbrace{\begin{bmatrix} \langle v_1, v_1 \rangle & \cdots & \langle v_n, v_1 \rangle \\ & & \\ \langle v_1, v_n \rangle & \cdots & \langle v_n, v_n \rangle \end{bmatrix}}_{G} \underbrace{\begin{bmatrix} P_{1j} \\ \vdots \\ P_{nj} \end{bmatrix}}_{j}$$

$$= (P^T G P)_{ij}$$

$$\Rightarrow H = P^T G P$$

The Gram-Schmidt Process

- Suppose that a, b are independent vectors, but they are not orthogonal.
- Let $V = \text{span}(\{a, b\})$.
- So, $\{a, b\}$ is a basis for V.
- How can we find a way to make an orthogonal basis?

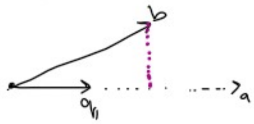
$\text{Span}(\{q_1, q_2\}) = V \qquad \langle q_1, q_2\rangle = 0$

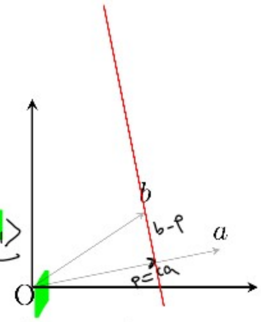
$q_1 = \frac{a}{\|a\|}$

$0 = \langle b - p, a\rangle$

$c = ? \qquad 0 = \langle b - ca, a\rangle = \langle b, a\rangle - c\langle a, a\rangle$

$\langle a, b\rangle = 0 \qquad q_1 = \frac{a}{\|a\|}, \quad q_2 = \frac{b}{\|b\|} \qquad \therefore \langle b, a\rangle = c\langle a, a\rangle$





$b = \langle b, q_1\rangle q_1 + (b - \langle b, q_1\rangle q_1)$

$b = \frac{\langle b, a\rangle}{\|a\|^2} a + \left(b - \frac{\langle b, a\rangle}{\|a\|^2} a\right) \qquad c = \frac{\langle b, a\rangle}{\|a\|^2}$

$Q_2 = b - \langle b, q_1\rangle q_1$

$\langle Q_2, q_1\rangle = 0 \qquad \text{Span}(\{Q_2, q_1\}) = \text{Span}(\{a, b\}) = \text{Span}(\{q_1, q_2\})$

$q_2 = \frac{Q_2}{\|Q_2\|}$

$a = \begin{bmatrix}1\\0\\1\end{bmatrix}, \quad b = \begin{bmatrix}1\\0\\0\end{bmatrix}, \quad c = \begin{bmatrix}2\\1\\0\end{bmatrix}$

$\langle a, b\rangle = 1 \neq 0 \Rightarrow B = \{a, b, c\}$

$q_1 = \frac{1}{\|a\|} a = \frac{1}{\sqrt{2}} a$

$q_2 = ? \qquad b = \langle b, q_1\rangle q_1 + (b - \langle b, q_1\rangle q_1)$

$Q_2 = b - \langle b, q_1\rangle q_1 = \begin{bmatrix}1\\0\\0\end{bmatrix} - \frac{1}{(\sqrt{2})^2}\begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}1\\0\\-1\end{bmatrix}$

$q_2 = \frac{1}{\|Q_2\|} Q_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\-1\end{bmatrix}$

$\text{Span}(\{a, b\}) = \text{Span}(\{q_1, q_2\})$

The Gram-Schmidt Process

$B' = \{q_1, q_2, q_3\}$

- Suppose a, b, c are independent but are not orthogonal vectors.
- Let $V = \text{span}(\{a, b, c\})$. $\quad \dim V = 3$
- So, $\{a, b, c\}$ is a basis for V.
- We want to find a way to make an orthogonal basis:
-

$$q_1 = \frac{1}{\|a\|} a$$

$$q_2 = \frac{1}{\|b - \langle b, q_1\rangle q_1\|}(b - \langle b, q_1\rangle q_1)$$

$$q_3 = \frac{1}{\|c - \langle c, q_1\rangle q_1 - \langle c, q_2\rangle q_2\|}(c - \langle c, q_1\rangle q_1 - \langle c, q_2\rangle q_2)$$

C

$Q_3 = v_3 - \langle v_3, q_1\rangle q_1 - \langle v_3, q_2\rangle q_2 = \begin{bmatrix}2\\1\\0\end{bmatrix} - \sqrt{2}\begin{bmatrix}1/\sqrt{2}\\0\\1/\sqrt{2}\end{bmatrix} - \sqrt{2}\begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\-\frac{1}{\sqrt{2}}\end{bmatrix} = \begin{bmatrix}0\\1\\0\end{bmatrix} = q_3$

Example

$V = \text{Span}(\{a, b, c\}) = \text{Span}(\{q_1, q_2, q_3\})$

- $a = \begin{bmatrix}1\\0\\1\end{bmatrix} \quad b = \begin{bmatrix}1\\0\\0\end{bmatrix} \quad c = \begin{bmatrix}2\\1\\0\end{bmatrix}$
- The Gram-Schmidt Process: $q_1 = \frac{1}{\sqrt{2}} a$

$b - \langle b, q_1\rangle q_1 = \begin{bmatrix}1\\0\\\ldots\end{bmatrix} - \frac{1}{\ldots} \times \frac{1}{\ldots}\begin{bmatrix}1\\0\\\ldots\end{bmatrix} - \frac{1}{\ldots}\begin{bmatrix}1\\0\\\ldots\end{bmatrix} \quad q_2 = \frac{1}{\ldots}\begin{bmatrix}1\\0\\\ldots\end{bmatrix}$

Example

$-V = \mathrm{span}(\{a, b, c\}) = \mathrm{span}(\{q_1, q_2, q_3\})$

- $a = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad c = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$
- The Gram-Schmidt Process: $q_1 = \frac{1}{\sqrt{2}} a$

$$b - \langle b, q_1 \rangle q_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad q_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$c - \langle c, q_1 \rangle q_1 - \langle c, q_2 \rangle q_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \sqrt{2} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} - \sqrt{2} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{q_3}$$

The Gram-Schmidt process

$\{v_2, v_1\}$ $\quad v_2 =$

- The Gram-Schmidt process
  1. starts with independent vectors $v_1, \ldots, v_n$
  2. ends with orthonormal vectors $q_1, \ldots, q_n$.
- At step 1: $q_1 = \frac{1}{\|v_1\|} v_1$.
- At step j $(2 \leqslant j \leqslant n)$:
  1. it subtracts from a_j its components in the directions $q_1, \ldots, q_{j-1}$ that are already settled:

$$Q_j = v_j - \langle v_j, q_1 \rangle q_1 - \cdots - \langle v_j, q_{j-1} \rangle q_{j-1}.$$

  2. $q_j = \frac{1}{\|Q_j\|} Q_j$.

$$Q_{m+1} = v_{m+1} - \langle v_{m+1}, q_1 \rangle q_1 - \cdots + \langle v_{m+1}, q_m \rangle q_m$$

$$q_{m+1} = \frac{1}{\|Q_{m+1}\|} Q_{m+1}$$

قضیه: فرض کنید V یک فضای ضرب داخلی

$v_1, \ldots, v_n$ بردارهای مستقل خطی

در این صورت بردارهای $q_1, \ldots, q_m$ ساخته شده

می‌دهند بطوریکه

$\mathrm{span}(\{v_1, \ldots, v_k\}) = \mathrm{span}(\{q_1, \ldots, q_k\})$

برهان: با استقرا روی n:

$q_1 = \frac{1}{\|v_1\|} v_1$

فرض کنید $q_1, \ldots, q_m$ به شرط قضیه ساخته شده است.

حال q_{m+1} را می‌سازیم:

به ازای هر $1 \le i \le m$

$\langle q_{m+1}, q_i \rangle = \langle v_{m+1}, q_i \rangle - \langle v_{m+1}, q_i \rangle = 0$

بنابر فرض استقرا:

$\mathrm{span}(\{q_1, \ldots, q_m, q_{m+1}\}) \subseteq \mathrm{span}(\{v_1, \ldots, v_m, v_{m+1}\})$

تساوی برقرار است

The factorization $A = QR$

- The Gram-Schmidt process
  1. starts with independent vectors $a_1, \ldots, a_n$, consider these vectors as columns of a matrix A.
  2. ends with orthonormal vectors $q_1, \ldots, q_n$, consider these vectors as columns of a matrix Q.
- What is the relation between these matrices A and Q?
- Think about three vectors A_1, A_2, A_3.

- What is the relation between these matrices A and Q?
- Think about three vectors A_1, A_2, A_3.
- $\text{span}(\{A_1, A_2, A_3\}) = \text{span}(\{q_1, q_2, q_3\})$.
- The idea is to write the a's as combinations of the q's.

$$
\begin{aligned}
a_1 &= \langle a_1, q_1 \rangle q_1 \\
a_2 &= \langle a_2, q_1 \rangle q_1 + \langle a_2, q_2 \rangle q_2 \\
a_3 &= \langle a_3, q_1 \rangle q_1 + \langle a_3, q_2 \rangle q_2 + \langle a_3, q_3 \rangle q_3.
\end{aligned}
$$

$\text{span}(\{v_1, \ldots, v_n, v_{m+1}\})$

با توجه به اینکه

$Q_{m+1} \in \text{Span}(\{v_1, \ldots, v_{m+1}\})$

$\Rightarrow \text{Span}(\{q_1, \ldots, q_{m+1}\}) = \text{Span}(\{v_1, \ldots, v_{m+1}\})$

پس حکم ثابت شد.

The factorization $A = QR$

- By

$$
\begin{aligned}
a_1 &= \langle a_1, q_1 \rangle q_1 && +0q_2 && +0q_3 \\
a_2 &= \langle a_2, q_1 \rangle q_1 && +\langle a_2, q_2 \rangle q_2 && +0q_3 \\
a_3 &= \langle a_3, q_1 \rangle q_1 && +\langle a_3, q_2 \rangle q_2 && +\langle a_3, q_3 \rangle q_3,
\end{aligned}
$$

- we obtain:

$$
\underbrace{\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} \langle a_1, q_1 \rangle & \langle a_2, q_1 \rangle & \langle a_3, q_1 \rangle \\ 0 & \langle a_2, q_2 \rangle & \langle a_3, q_2 \rangle \\ 0 & 0 & \langle a_3, q_3 \rangle \end{bmatrix}}_{R}
$$

QR factorization

- In QR factorization for the matrix A (with independent columns):
  1. the first factor Q has orthonormal columns.
  2. R is upper triangular (The second factor is called R, because the nonzeros are to the right of the diagonal).
- Example:

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} & \sqrt{2} \\ 0 & \frac{1}{\sqrt{2}} & \sqrt{2} \\ 0 & 0 & 1 \end{bmatrix} = QR$$

QR factorization

- At the first step of the Gram-Schmidt process: $a_1 = \|a_1\| q_1$
- At step j $(2 \leqslant j \leqslant n)$
  1. it subtracts from a_j its components in the directions $q_1, \dots, q_{j-1}$ that are already settled:

$$A_j = a_j - \langle a_j, q_1 \rangle q_1 - \cdots - \langle a_j, q_{j-1} \rangle q_{j-1}.$$

  2. $q_j = \frac{1}{\|A_j\|} A_j$.
  3. Thus, for j $(2 \leqslant j \leqslant n)$

$$a_j = \langle a_j, q_1 \rangle q_1 - \cdots - \langle a_j, q_{j-1} \rangle q_{j-1} + \|A_j\| q_j.$$

  4. Therefore, the lengths of $\|A_j\|$ are on the diagonal of R.

QR factorization

- If $A \in M_{mn}(\mathbb{R})$ with independent columns, it can be factored into **A=QR**.
  1. The columns of Q are orthonormal,
  2. R is upper triangular and invertible (Why?)
  3. If $m = n$, then Q is an orthogonal matrix.

. . .

Thank You!

$$Q_{m+1} = v_{m+1} - \langle v_{m+1}, q_1\rangle q_1 - \cdots - \langle v_{m+1}, q_m\rangle q_m$$

$\underline{\underline{q_1, \cdots, q_m}}$ متعامد

$1 \leq i \leq m$

$$q_{m+1} = \frac{1}{\|Q_{m+1}\|} Q_{m+1}$$

$$\forall\ 1 \leq i \leq m \qquad \langle q_{m+1}, q_i\rangle = 0$$

$$\langle q_{m+1}, q_i\rangle = \left\langle \frac{1}{\|Q_{m+1}\|} Q_{m+1}, q_i \right\rangle = \frac{1}{\|Q_{m+1}\|} \langle Q_{m+1}, q_i\rangle$$

$$\cdots - \langle v_{m+1}, q_i\rangle \overset{0}{\cancel{\langle q_{i-1}, q_i\rangle}} - \langle v_{m+1}, q_i\rangle \langle q_i, q_i\rangle$$

$$\langle q_{m+1}, q_i \rangle = \left\langle \frac{1}{\|Q_{m+1}\|} Q_{m+1}, q_i \right\rangle = \quad \|Q_{m+1}\|$$

$$= \langle v_{m+1}, q_i \rangle - \langle v_{m+1}, q_1 \rangle \cancel{\langle q_1, q_i \rangle} - \cdots - \langle v_{m+1}, q_{i-1} \rangle \cancelto{0}{\langle q_{i-1}, q_i \rangle} - \langle v_{m+1}, q_i \rangle \langle q_i, q_i \rangle$$

$$- \langle v_{m+1}, q_{i+1} \rangle \cancelto{0}{\langle q_{i+1}, q_i \rangle} - \cdots - \langle v_{m+1}, q_m \rangle \cancelto{0}{\langle q_m, q_i \rangle}$$

$$= \langle v_{m+1}, q_i \rangle - \langle v_{m+1}, q_i \rangle \overbrace{\langle q_i, q_i \rangle}^{1} = 0$$