Linear Algebra

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Linear Spaces

The heart of linear algebra

Spanning subspaces

Definition

Let S be a set of a linear space V. The subspace spanned by S, denoted by $\text{span}(S)$, is the set of all linear combinations of vectors in S.

- $\{0\} = \text{span}(\varnothing)$.
- $\mathbb{R}^3 = \text{span}\left(\{e_1, e_2, e_3\}\right)$,
- $\mathbb{R}^2 = \text{span}\left(\{e_1, e_2, e_1 + e_2\}\right)$.

Spanning subspaces

Theorem

Let W be a subspace of a linear space V such that $S \subseteq W$, then $\text{span}(S) \subseteq W$*.*

- Note that the smallest subspace of V containing S is $\text{span}(S)$.
- $\text{span}(S)$ is the intersection of all subspaces W of V containing S, i.e. $\text{span}(S) = \bigcap_{S \subseteq W \in \text{sub}(V)} W$, where $\text{sub}(V)$ is the set of all subspaces of V.
- The subspace $\text{span}(S)$ contains no proper subspace including S.

Examples

1. Let $W_1, \ldots, W_m$ be subspaces of a linear space V. The sum of them is

$$W_1 + \cdots + W_m = \left\{ w_1 + \cdots + w_m \,\middle|\, w_i \in W_i \text{ for } 1 \leq i \leq m \right\}.$$

   Thus

$$W_1 + \cdots + W_m = \operatorname{span}\Big(\bigcup_{j=1}^{m} W_j \Big).$$

2. For $A \in M_{m,n}(\mathbb{R})$,

$$V = \{Ax \mid x \in \mathbb{R}^n\}$$

   is $C(A)$, as it is generated by all columns of A.

When does $Ax = b$ have a solution?

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}$$

$$C(A) = \left\{ c_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 6 \\ -3 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ 6 \\ -3 \end{bmatrix} + c_4 \begin{bmatrix} 2 \\ 7 \\ 4 \end{bmatrix} \;\middle|\; c_i \in \mathbb{R} \right\}$$

$$= \text{span}\left(\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \\ 4 \end{bmatrix} \right\} \right)$$

When does $Ax = b$ have a solution?

$$\underbrace{c_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + c_2 \underbrace{\begin{bmatrix} 3 \\ 6 \\ -3 \end{bmatrix}}_{3\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}} + c_3 \underbrace{\begin{bmatrix} 3 \\ 9 \\ 3 \end{bmatrix}}_{\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 7 \\ 4 \end{bmatrix}} + c_4 \begin{bmatrix} 2 \\ 7 \\ 4 \end{bmatrix}}_{(c_1+3c_2+c_3)\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + (c_3+c_4)\begin{bmatrix} 2 \\ 7 \\ 4 \end{bmatrix} \Rightarrow C(A) = \text{span}\left(\left\{\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \\ 4 \end{bmatrix}\right\}\right)}$$

Linear dependent vectors

Definition

Let V be a linear space and $S \subseteq V$. We say that the elements of S are linearly dependent if there is some $s \in S$ such that

$$\mathrm{span}(S) = \mathrm{span}(S \setminus \{s\}).$$

If the elements of S are not linearly dependent, then we say that they are linearly independent.

Linear Independence

Fact

The elements $v_1, \ldots, v_m$ of the linear space V are linearly dependent, if there exist scalars $c_1, \ldots, c_m \in \mathbb{R}$ not all zero, such that

$$c_1 v_1 + \cdots + c_m v_m = 0.$$

This implies that at least one of the scalars is nonzero.
Also, the elements $v_1, \ldots, v_m \in V$ is linearly independent if it is not linearly dependent, that is, if the equation

$$c_1 v_1 + \cdots + c_m v_m = 0$$

can only be satisfied by $c_1 = \cdots = c_m = 0$. This implies that no element in the sequence can be represented as a linear combination of the remaining elements in the sequence.

Example

Show that v_1, v_2, v_3 are independent but v_1, v_2, v_3, v_4 are dependent:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad v_4 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

Example

- To show that the set of vectors $\{v_1, v_2, v_3\}$ is linearly independent, we should solve the following equation:

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0$$

Equivalently:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus

$$c_1 = c_2 = c_3 = 0.$$

- The set of vectors $\{v_1, v_2, v_3\}$ is **linearly independent**.

Example

For the set of vectors $\{v_1, v_2, v_3, v_4\}$:

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_4 \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = 0$$

- Equivalently,

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix}}_{A} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- By reduced echelon form, R,

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 4 \end{bmatrix}}_{R} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Checking procedure for linearly independence

Definition

The nullspace of a matrix consists of all vectors x such that $Ax = 0$. It is denoted by $N(A)$.

- To check a set of vectors $\{v_1, \ldots, v_k\}$ is linearly independent:

  1. Put them in the columns of A.
  2. Solve the system $Ac = 0$.
  3. $N(A) = \{\mathbf{0}\}$ and the set of vector is linearly independent.
  4. If there is at least one free variable, then $N(A) \neq \{\mathbf{0}\}$ and the columns are dependent.

Echelon matrix U and Row Reduced matrix R

$$R = \begin{bmatrix} \mathbf{1} & 0 & * & 0 & * & * & * & * & * & 0 \\ 0 & \mathbf{1} & * & 0 & * & * & * & * & * & 0 \\ 0 & 0 & 0 & \mathbf{1} & * & * & * & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Example

$$Ax = \begin{bmatrix} 1 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 1 & 1 & -2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

Echelon matrix U and Row matrix R

$$\begin{bmatrix} A & b \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 & 0 & 1 & 0 & b_1 \\ -1 & -1 & 2 & -3 & 1 & 0 & b_2 \\ 1 & 1 & -2 & 0 & 0 & 2 & b_3 \\ 0 & 0 & 0 & 3 & 1 & -2 & b_4 \end{bmatrix}$$

$$\begin{bmatrix} U & c \end{bmatrix} = \begin{bmatrix} \mathbf{1} & 2 & -1 & 0 & 1 & 0 & b_1 \\ 0 & \mathbf{1} & 1 & -3 & 2 & 0 & b_1 + b_2 \\ 0 & 0 & 0 & \mathbf{-3} & 1 & 2 & b_2 + b_3 \\ 0 & 0 & 0 & 0 & \mathbf{2} & 0 & b_2 + b_3 + b_4 \end{bmatrix}$$

$$\begin{bmatrix} R & d \end{bmatrix} = \begin{bmatrix} \mathbf{1} & 0 & -3 & 0 & 0 & 4 & \frac{2b_1+5b_3+b_4}{2} \\ 0 & \mathbf{1} & 1 & 0 & 0 & -2 & \frac{2b_1+b_2-b_3+b_4}{2} \\ 0 & 0 & 0 & \mathbf{1} & 0 & -2/3 & \frac{3b_2-b_3+b_4}{6} \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 & \frac{b_2+b_3+b_4}{2} \end{bmatrix}$$

$Ax = b \Longleftrightarrow Rx = d.$

$$Rx = \begin{bmatrix} \mathbf{1} & 0 & -3 & 0 & 0 & 4 \\ 0 & \mathbf{1} & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & \mathbf{1} & 0 & -2/3 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} \frac{2b_1+5b_3+b_4}{2} \\ \frac{2b_1+b_2-b_3+b_4}{2} \\ \frac{3b_2-b_3+b_4}{6} \\ \frac{b_2+b_3+b_4}{2} \end{bmatrix}$$

$$\begin{aligned} x_5 &= \frac{b_2+b_3+b_4}{2} \\ x_4 &= \frac{3b_2-b_3+b_4}{6} + \frac{2}{3}x_6 \\ x_2 &= \frac{2b_1+b_2-b_3+b_4}{2} - x_3 + 2x_6 \\ x_1 &= \frac{2b_1+5b_3+b_4}{2} + 3x_3 + 4x_6 \end{aligned}$$

v.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} \frac{2b_1+5b_3+b_4}{2} & +3x_3 & +4x_6 \\ \frac{2b_1+b_2-b_3+b_4}{2} & -x_3 & +2x_6 \\ 0 & +x_3 & \\ \frac{3b_2-b_3+b_4}{6} & & +\frac{2}{3}x_6 \\ \frac{b_2+b_3+b_4}{2} & & \\ 0 & & +x_6 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2b_1+5b_3+b_4}{2} \\ \frac{2b_1+b_2-b_3+b_4}{2} \\ 0 \\ \frac{3b_2-b_3+b_4}{6} \\ \frac{b_2+b_3+b_4}{2} \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} 4 \\ 2 \\ 0 \\ \frac{2}{3} \\ 0 \\ 1 \end{bmatrix}$$

i. For every $b^T = \begin{bmatrix} b_1 & b_2 & b_3 & b_4 \end{bmatrix} \in \mathbb{R}^4$, the equation $Ax = b$ has a solution.

ii. So $C(A) = \mathbb{R}^4$.

iii. Also,

$$\begin{aligned} C(A) &= \left\{ x_1 \underbrace{\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}}_{v_1} + x_2 \underbrace{\begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}}_{v_2} + x_3 \underbrace{\begin{bmatrix} -1 \\ 2 \\ -2 \\ 0 \end{bmatrix}}_{v_3} + x_4 \underbrace{\begin{bmatrix} 0 \\ -3 \\ 0 \\ 3 \end{bmatrix}}_{v_4} + x_5 \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}}_{v_5} + x_6 \underbrace{\begin{bmatrix} 0 \\ 0 \\ 2 \\ -2 \end{bmatrix}}_{v_6} \middle| \, x_i \in \mathbb{R} \right\} \\ &= \operatorname{span}(\{v_1, v_2, v_3, v_4, v_5, v_6\}) \\ &= \operatorname{span}(\{v_1, v_2, v_4, v_5\}). \end{aligned}$$

Thank You!