

Linear Algebra

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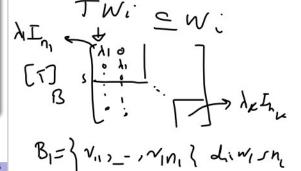
Fall, 2021

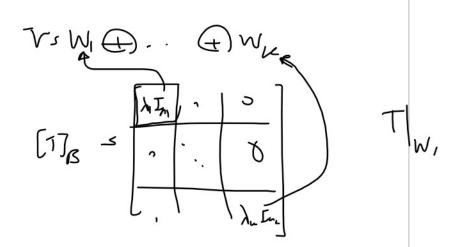
Review: Diagonalizable Linear Function

Theorem

 $Let \underline{T}: V \to V$ be a linear function on a finite dimensional linear space V, and T has different eigenvalues $\lambda_1, \ldots, \lambda_k$. Suppose that W_i is the null space of $T - \lambda_i I$ for each $1 \leq i \leq k$. Then the following statements are equivalent: W: & N(T- X: I)

- i. T is diagonalizable.
- ii. The characteristic polynomial of T is $f(x) = (x \lambda_1)^{n_1} \cdots (x \lambda_k)^{n_k}$, where $n_i = \dim W_i$.
- iii. $\sum_{i=1}^k \dim W_i = \dim V$.





 $T \gamma_{11} = \lambda_1 \gamma_{11}$ $T \gamma_{12} = \lambda_1 \gamma_{12}$

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Lecture #26

3 / 26

Primary Decomposition Theorem

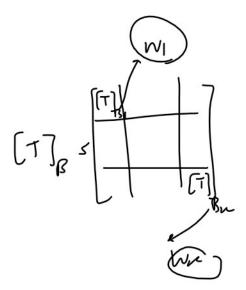
Theorem

Let T be a linear function over a finite dimensional linear space V whose minimal polynomial factorizes as

$$p(x) = p_1^{r_1}(x) \cdots p_k^{r_k}(x),$$

where the p_i 's are monic and mutually coprime (i.e. have no nontrivial common factors). Let $W = N(p_i^{r_i}(T))$ for each $1 \le i \le k$. Then

- The minimal polynomial of $T_i = T \upharpoonright_{W_i}$ is $p_i(x)$.



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Lecture #20

Minimal Polynomials for Vectors

Lemma

Suppose that T is a linear function on linear space V over $\mathbb{F} = \mathbb{R}$ or \mathbb{C} .

- Each vector $v \in V$ has a minimal polynomial with respect to T.
- The minimal polynomial for v with respect to T is unique.
- **3** Take a vector $v \in V$ and assume that f(x) is a polynomial with coefficients in \mathbb{F} such that f(T)v = 0, then $p(x) \mid f(x)$ where p(x)'is the minimal polynomial for v with respect to T.
- Let f(x) and g(x) be two coprime polynomials. Then

 $\bigvee \in N(f(T)) \cap N(g(T)) = \{0\}.$

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Review: Lemma

Lemma

Let T, S be two linear functions on linear space V over $\mathbb{F} = \mathbb{R}$ or \mathbb{C} such that $T \circ S = S \circ T$ and $N(T) \cap N(S) = \{0\}$. Then

- $N(T) + N(S) \subseteq N(T \circ S).$
- If V is finite dimensional, then $\dim N(T \circ S) \leq \dim N(T) + \dim N(S)$ and consequently, $N(T \circ S) = N(T) \oplus N(S)$

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- N(8) n N(TK) = } . {

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Review: Lemma.

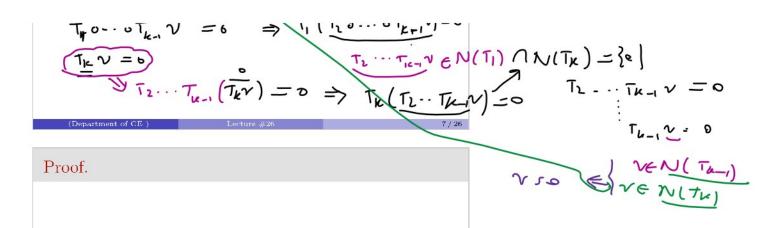
Lemma

Let T_1, \ldots, T_k be linear functions on a finite dimensional linear space V over $\mathbb{F} = \mathbb{R}$ or \mathbb{C} such that



for each $1 \le i < j \le k$. Then $N(T_1 \circ \cdots \circ T_k) = N(T_1) \oplus \cdots \oplus N(T_k)$.

 $V \in N(T_0 \cdots \circ T_{k-1}) \cap N(T_k) = \{\emptyset\}$ $T_{\mu} \circ \cdots \circ T_{k-1} \circ V = \emptyset \implies T_{\mu} (T_{\mu} \circ \cdots \circ T_{k-1} \circ V) = \emptyset$



100 tj s Tj · Ti , N(Ti) / 715) = (.(Lemma. NIFIMITA) = N(TI) D-.. ANTA) Lemma

Let T be a linear functions on a finite dimensional linear space V over $\mathbb{F} = \mathbb{R} \ or \mathbb{C}.$

• Let f(x) be a polynomial with coefficient in \mathbb{F} and $f(x) = f_1(x)^{n_1} \cdots f_k(x)^{n_k}$ such that f_1, \ldots, f_k mutually coprime.

 $N(f(\overline{A})) = N(f_1(\overline{A})^{n_1}) \oplus \cdots \oplus N(f_k(\overline{A})^{n_k}).$

If the minimal polynomial T is factorized as P(1) $\sqrt{=0}$ $p(x) = p_1(x)^{n_1} \cdots p_k(x)^{n_k}$ where p_1, \dots, p_k are mutually coprime,

 $V = N(p_1(\overline{A})^{n_1}) \oplus \cdots \oplus N(p_k(\overline{A})^{n_k}).$ NIFIT) 1 NO(91)) 11.1 (FM) 19m1) = 1

 $T_i \circ T_j = F_i(T) \cap F_j(T)$ $\left(\sum_{k=1}^{\infty} y_k T^k\right)^{h_i} \left(\sum_{j=1}^{n} b_j T^j\right)^{n_j}$ =Ti o Ti

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Proof.
$$T: V \longrightarrow V$$

$$P(X) = P(X)^{-1} \cdot P(X)^{-1}$$

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$$V = N(p(T)) = N(q, q) \oplus \cdots \oplus N(R(q))$$

$$P_{T_i}(x) \mid P_i(x) = \underbrace{f_i(x) - f_e(x)}_{f_i(x)}$$

$$\underline{p(z)} = p_1(z) \cdot \underbrace{p_1(z)}_{r_1(z)} + \underbrace{p_2(z)}_{r_2(z)} + \underbrace{p_2(z)}$$

$$P_{T_i}(x) = P_i^{5}(x) \circ \frac{1}{2} \frac{$$

Primary Decomposition Theorem

 $: W_{i} \longrightarrow T$

Let The linear Mattion over a finite dimensional linear space V whose minimal polynomial factorizes as

where the pi's are monic and mutually coprime (i.e. have no nontrivial common factors). Let $W_i = N(p_i^{r_i}(T))$ for each $1 \le i \le k$. Then

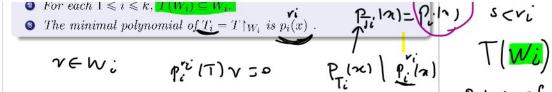
- The minimal polynomial of $T_i = T \upharpoonright_{W_i}$ is $p_i(x)$.

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$$P_{i}^{(1)}(1/\nu)) = T(P_{i}^{(1)}\nu)$$

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Jordan Form

Suppose that T is a linear function on V with the characteristic polynomial is

$$f(x) = (x - \frac{\lambda_1}{\lambda_1})^{d_1} \dots (x - \frac{\lambda_k}{\lambda_k})^{d_k}$$

where $\lambda_1, \ldots, \lambda_k$ are distinct elements and $d_i \ge 1$. Then the minimal polynomial for T will be

$$p(x) = (x - \frac{\lambda_1}{\lambda_1})^{r_1} \dots (x - \frac{\lambda_k}{\lambda_k})^{r_k}$$

where $1 \le r_i \le d_i$ based on the Cayley–Hamilton theorem. If W_i is the null space of $(T - \sum_{i=1}^{n} I)^{r_i}$, then the primary decomposition theorem tells us that

$$V = W_1 \oplus \cdots \oplus W_k$$

such that the linear function $T_i = T \upharpoonright_{W_i} : W_i \to W_i$ has minimal polynomial $(x - \lambda_i)^{r_i}$.

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Locture #26

13 / 26

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$$f_{km}$$
 $\frac{\partial^2 A}{\partial x^2} = \int_{-\lambda_i I}^{\lambda_i} \frac{\partial^2 F_{km}}{\partial x^2}$
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Jordan Form

Thus

$$V = W_1 \oplus \cdots \oplus W_k$$

Suppose that B_i is a basis for W_i . It has been proved that $B = \bigcup_{i=1}^k B_i$ is a basis for V. Based on primary decomposition theorem,

Based on primary decomposition theorem, $[T]_B = \begin{bmatrix} T_1 \\ B_1 \end{bmatrix}$ $[T_k]_{B_k}$

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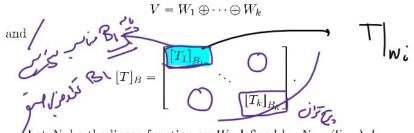
Lecture #2

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TV, = \(\sum_{j=1}^{\infty} \frac{4}{5} \text{V}_{ij}

4B,

Jordan Form



- Let N_i be the linear function on W_i defined by $N_i = T \lambda_i I$.
- Then N_i is nilpotent and has minimal polynomial x^{r_i} .
- Thus, T on W_i acts as N_i plus the scalar λ_i times the identity function I.
- Suppose we choose a basis for the subspace W_i and then find the representation matrix of N_i on W_i .

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Lecture #26

15/20

Nilpotent matrices and Nilpotent linear functions



Definition

A square matrix A is called nilpotent matrix with degree non-negative integer k if A^c is the zero matrix and A^r is the non-zero matrix for each r, $1 \le r \le k$.

Definition

A be a linear function T on V is called nilpotent linear function with degree non-negative integer k if T^k is the zero linear function and T^r is the non-zero one for each $r, 1 \le r \le k$.

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Lecture #2

16/2

 $W_{i} = N((T-\lambda_{i}I)^{v_{i}})$ $T_{i} = T|_{W_{i}}$

だ· = T-1: L

 $N_i^{ri} = 0$

 $\gamma \in W_{i} \quad \left(\underbrace{T - \lambda_{i} I}_{N_{i}^{r_{i}}} \right)^{r_{i}} \gamma = 0$

Yrew:



rck A = 0

A" = a

Example

Let $A \in M_3(\mathbb{R})$ be the following matrix:

A+0, A3=0

Then

$$A^{2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq \mathbf{O}$$

The third power of A is

Nilpotent matrices

$$A = 0 \Rightarrow \lambda = 0$$

Lemma

- Let the $n \times n$ matrix A is nilpotent with degree k, then

 1) The matrix A is nilpotent if and only if all the eigenvalues of A is

2) The matrix Ais nilpotent if and only if
$$A^n = 0$$
.

$$P(z) > 2^k$$

$$F(m) = 2^n$$



Nilpotent matrices

If a linear function T on V with $\dim V = n < \infty$ is nilpotent with degree n, then there is a basis for V such that

$$[T]_{B} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

 $T = 0 , \begin{cases} \frac{n-1}{1+6} + 6 \\ \frac{n-1}{1+6} + 6 \end{cases}$ $T + 6 \Rightarrow T = 0$ $T + 6 \Rightarrow T = 0$

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Lecture #20

19 / 26

Nilpotent matrices

Lemma

Let T is a linear function on V such that $B=\{v,Tv,\dots,T^{n-1}v\}$ is a basis for V where $0\neq v\in V.$ Then

$$[T]_B = \begin{bmatrix} 0 & 0 & \cdots & 0 & -c_0 \\ 1 & 0 & \cdots & 0 & -c_1 \\ 0 & 1 & \ddots & 0 & -c_2 \\ 0 & 0 & \ddots & 0 & \vdots \\ 0 & 0 & \cdots & 1 & -c_{n-1} \end{bmatrix}$$

where $p(x) = x^n + c_{n-1}x^{n-1} + \ldots + c_1x^1 + c_0$ is the minimal polynomial for T.

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Lecture #2

The smallest T-invariant subspace containing v

- Assume V is finite-dimensional linear space over $\mathbb{F} = \mathbb{R}$ or \mathbb{C} and T is a fixed (but arbitrary) linear function on V.
- If W is any subspace of V which is invariant under T and contains v, then W must also contain the vector T(v); hence W must contain $T(Tv) = T^2v$, $T(T^2v) = T^3v$, etc. In other words W must contain g(T)v, for every polynomial g(x) over \mathbb{F} . The set of all polynomial g(x) over \mathbb{F} is denoted by $\mathbb{F}[x]$
- Let $Z(v,T) = \{g(T)v \mid g(x) \in \mathbb{F}[x].\}$
- Z(v,T) is a subspace of V and it is the smallest T-invariant subspace which contains v.

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Lecture #26

21/26

T-cyclic subspace generated by v

Definition

If v is any vector in V, the subspace Z(v,T) is called the T-cyclic subspace generated. If Z(v,T)=V, then v is called a cyclic vector for T.

For any T:

- The *T*-cyclic subspace generated by the zero vector is the zero subspace.
- ② The space Z(v,T) is one-dimensional if and only if v is an eigenvalue vector for T.
- Thus, we shall be interested in linear relations:

$$c_0v + c_1Tv + \cdots, c_kT^kv = 0.$$

between the vectors $T^j v$, that is we shall be interested in the polynomials

$$c_0 + c_1 x + \cdots, c_k x^k = 0$$

which have the property that g(T)v = 0.

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Lecture #26

The dimension of T-cyclic subspace generated by v

Theorem

Assume that T is a linear space on a linear space V. Let v be any non-zero vector in V and let $p_v(x)$ is the minimal polynomial for v respect to T.

- ② If U is the linear function on Z(v,T) induced by T, then the minimal polynomial for U is $p_v(x)$.

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Lecture #20

23 / 20

Minimal and characteristic polynomials of a cyclic vector

Theorem

T has a cyclic vector if and only if the minimal and characteristic polynomials for T are identical.

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Lecture #26

Cyclic Decomposition Theorem

Theorem

Let T be a linear function on a finite-dimensional vector space V. There exist non-zero vectors $v_1, \ldots, v_k \in V$ with minimal polynomial p_{v_1}, \ldots, p_{v_k} such that

- (i) $V = Z(v_1, T) \oplus \cdots \oplus Z(v_k, T)$.
- (ii) $p_{v_i} \mid p_{v_{i-1}}$ for each $i \ge 2$.
- (iii) Furthermore, the integer r and the minimal polynomial p_{v_1}, \ldots, p_{v_k} are uniquely determined by (i), (ii).

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25 / 26

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Thank You!

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