Linear Algebra

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Review: Diagonalizable matrices

Definition

Assume $A \in M_n(\mathbb{R})$. A is called diagonalizable if it is similar to a diagonal matrix, i.e., if there exists an invertible matrix S and a diagonal matrix D such that

$$S^{-1}AS = D.$$

Review: Diagonalization of a matrix

- **Example.** The eigenvector matrix of the projection $P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ is $S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, and we have

$$S^{-1}PS = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

- The eigenvector matrix S converts A into its eigenvalue matrix which is diagonal.

Diagonalizable linear transformation

Theorem

Let $T : V \to V$ be a linear transformation where the dimension of V is finite with different eigenvalues $\lambda_1, \ldots, \lambda_k$. Suppose that W_i is null space of $\lambda_i I - T$ for each $1 \leqslant i \leqslant k$. The the following statements are equivalent:

i. *T is diagonalizable.*

ii. *Its eigenvalue vector is $f(\lambda) = (\lambda - \lambda_1)^{n_1} \ldots (\lambda - \lambda_k)^{n_k}$ and $\dim W_i = n_i$.*

iii. $\sum_{i=1}^{k} \dim W_i = \dim V$.

Proof: i $\Rightarrow$ ii

T is diagonalizable, so there is a basis $B = \{v_1, \ldots, v_n\}$ such that

$$[T]_B = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

Proof: ii $\Rightarrow$ iii

- The characteristic polynomial of T is

$$f(\lambda) = (\lambda - \lambda_1)^{n_1} \ldots (\lambda - \lambda_k)^{n_k}.$$

- $\dim V = \deg f = \sum_{i=1}^{k} n_i = \sum_{i=1}^{k} \dim W_i$

Proof: iii $\Rightarrow$ i

- $\sum_{i=1}^{k} \dim W_i = \dim V$.
- We should find a basis that the representation matrix of $[T]_B$ is diagonal.

Lemmas

We needs two lemma for complete the proof of iii $\Rightarrow$ i.

Lemma

Suppose that T is a linear function on V and $Tv = \lambda v$.. If $f(x)$ is a polynomial, then $f(T)v = f(\lambda)v$.

Corollary

If $\lambda_1, \ldots, \lambda_n$ are eigenvalues of T, then $\lambda_1^k, \ldots, \lambda_n^k$ are eigenvalues of A^k.

Lemma

Lemma

Suppose that T is a linear function on V with different eigenvalues $\lambda_1, \ldots, \lambda_k$. Let for each $1 \leqslant i \leqslant k$

$$W_i = \{v \in V \mid Tv = \lambda_i v\}.$$

If $v_1 + \cdots + v_k = 0$ for each $v_i \in W_i$, then $v_1 = \cdots = v_k = 0$.

Proof: iii $\Rightarrow$ i

- $\sum_{i=1}^{k} \dim W_i = \dim V$.
- Let $B_i = \{v_{i1}, \ldots, v_{in_i}\}$ be a basis for W_i. By two above lemma, we have
$$\bigcup_{i=1}^{k} B_i$$
is a basis for $W_1 + \cdots + W_k$.

Diagonalization of A and its powers A^k

- Find A^{555} where

$$A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}.$$

- We obtain

$$\det(\lambda I - A) = (\lambda - 1)(\lambda - 2)^2$$

- So, $\lambda = 1, 2, 2$ are eigenvalues of A and their eigenvectors are as follows, respectively:

$$v_1 = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

A^{555}

- Let

$$S = \begin{bmatrix} 3 & 2 & 0 \\ -1 & 0 & 0 \\ 3 & -1 & -1 \end{bmatrix}.$$

- Then

$$S^{-1}AS = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad S^{-1}A^{555}S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^{555} & 0 \\ 0 & 0 & 2^{555} \end{bmatrix}$$

- As a result:

$$A^{555} = S \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^{555} & 0 \\ 0 & 0 & 2^{555} \end{bmatrix} S^{-1}$$

Diagonalizable matrix A and its characteristic polynomial

- If $A \in M_n(\mathbb{F})$ is diagonalizable, then $f(A) = 0$.
- Since $A \in M_n(\mathbb{F})$ is diagonalizable, its characteristic polynomial is

$$f(\lambda) = (\lambda - \lambda_1)^{n_1} \dots (\lambda - \lambda_k)^{n_k},$$

where $\dim W_i = n_i$.

- Moreover, there is invertible matrix $S \in M_n(\mathbb{R})$ such that

$$S^{-1}AS = \begin{bmatrix} \lambda_1 I_{n_1} & & & \\ & \lambda_2 I_{n_2} & & \\ & & \ddots & \\ & & & \lambda_k I_{n_k} \end{bmatrix}$$

Cayley-Hamilton's theorem

Theorem

Let $A \in M_n(\mathbb{F})$ where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and $f(x)$ and $p(x)$ are characteristic polynomial and minimal polynomial, respectively. Then

1. $f(A) = 0$
2. *The minimal polynomial, $p(x)$, divides the characteristic polynomial, $f(x)$.*

Corollary

Let $A \in M_n(\mathbb{F})$ where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. For $k > n$, $A^k = g(A)$ where $g(x)$ is a polynomial with coefficients in $\mathbb{F}$ and its degree is less than n.

Thank You!