Deep Generative Models II

Generative Adversarial Networks – Part I

Machine Perception

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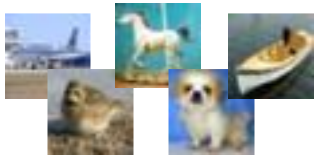
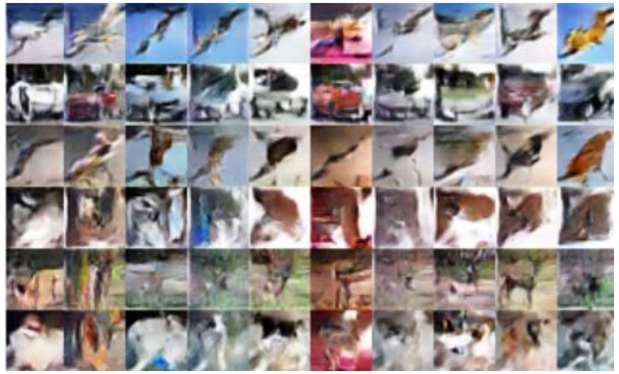
Generative Modelling

Given training data, generate new samples, drawn from "same" distribution



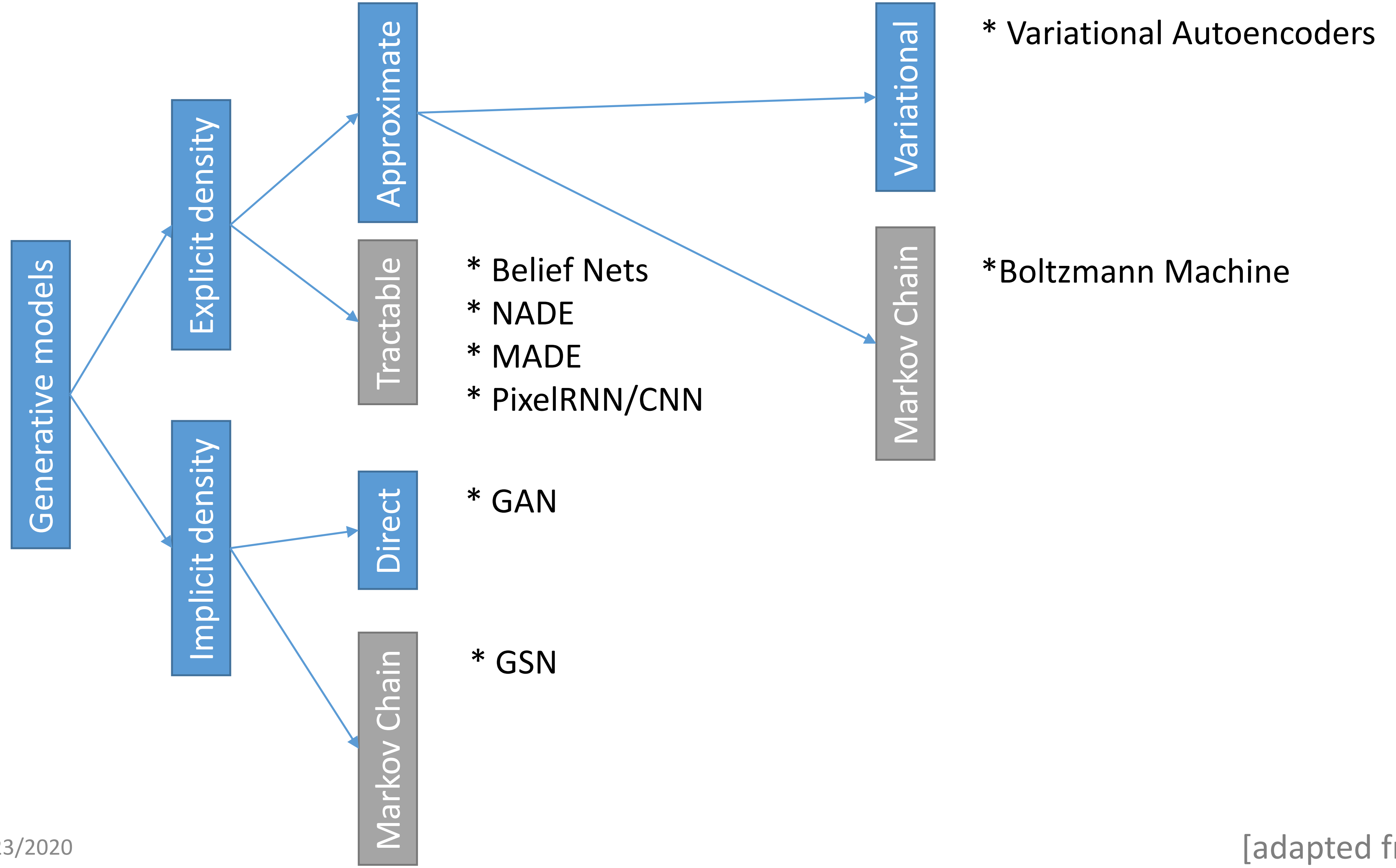
Training samples $\sim p_{data}(x)$



Generated samples $\sim p_{model}(x)$

We want $p_{model}(x)$ to be similar to $p_{data}(x)$

Taxonomy of Generative Models



[adapted from Ian Goodfellow]

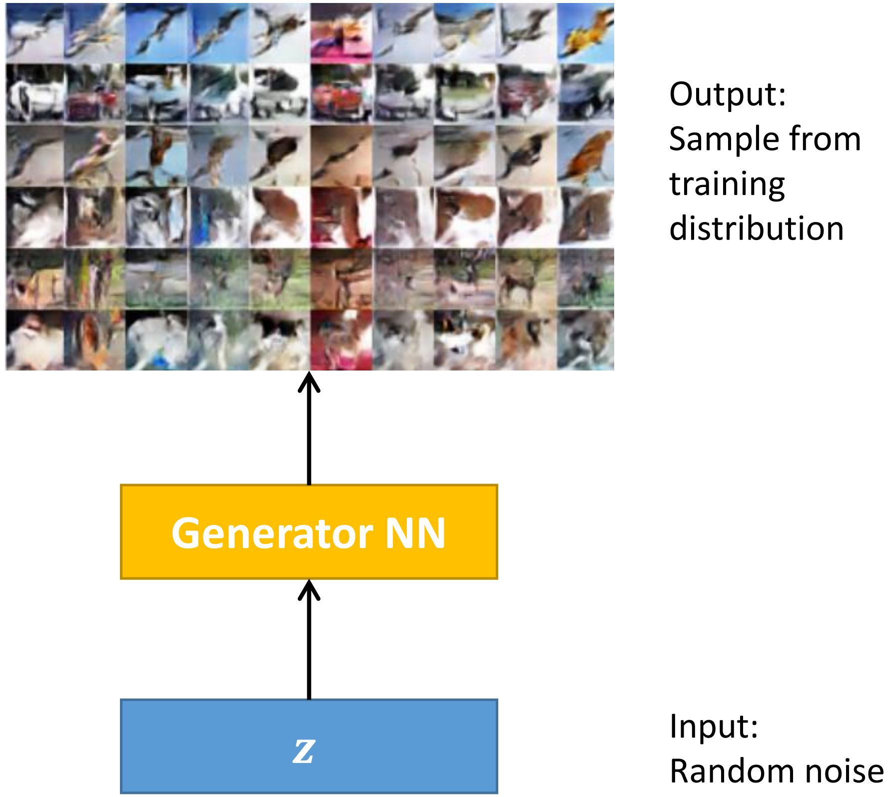
[Ian Goodfellow et al., “Generative Adversarial Nets”, NIPS 2014]

Generative Adversarial Nets: Intuition

Want: Ability to sample high-dimensional data points (e.g., images) from true data distribution (inaccessible).

Issue: No direct way of achieving this

Solution: Draw from simple distribution (i.e., random noise) and use neural network to learn transformation into realistic image

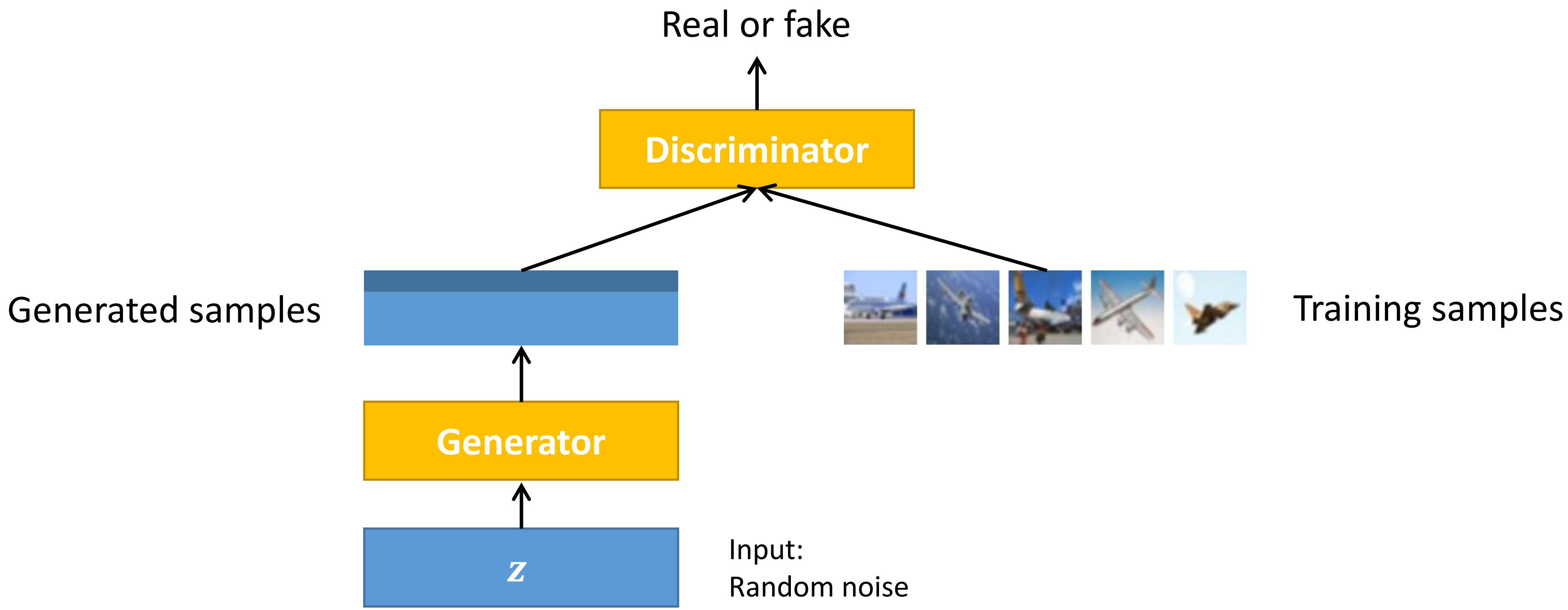


Training GANs: Intuition

General idea is to have two networks play a two-player game

Generator: try to fool the discriminator by generating real-looking images

Discriminator: try to distinguish between real and fake images



Training GANs: Definitions

Let X be the signal space and D the latent space dimension.

Generator:

$$G: \mathbb{R}^D \longrightarrow X$$

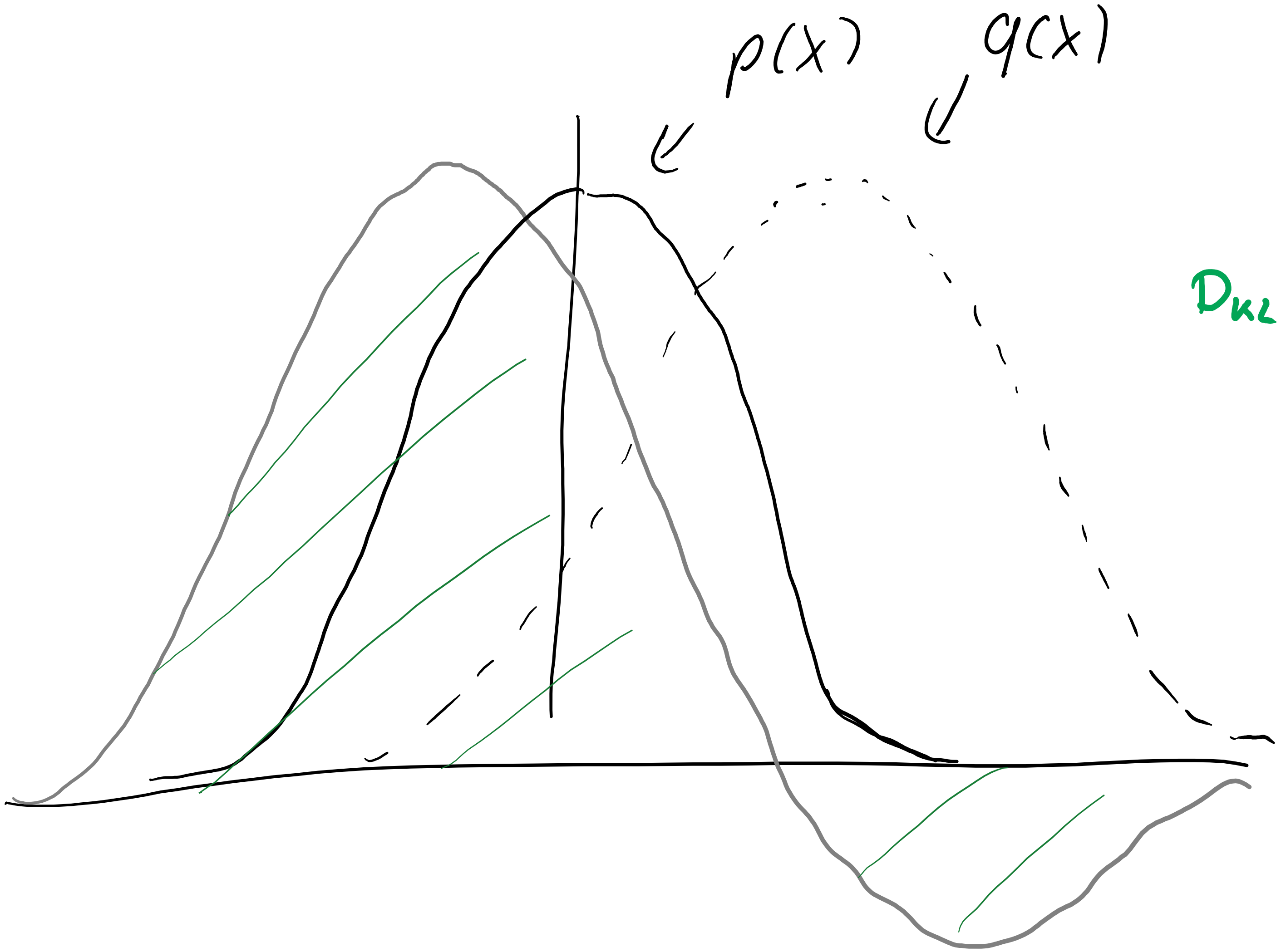
is trained to [ideally] map random normal-distributed inputs, drawn from Z, to a sample following the data distribution as output.

Discriminator:

$$D: X \longrightarrow [0,1]$$

is trained to classify samples as genuine or fake.

Kullback-Leibler divergence



$$D_{KL}(P \| Q) = \int_x p(x) \log \frac{p(x)}{q(x)} dx$$

特点： 1： 非负

2： 不对称

3： un-bounded

JS-Divergence

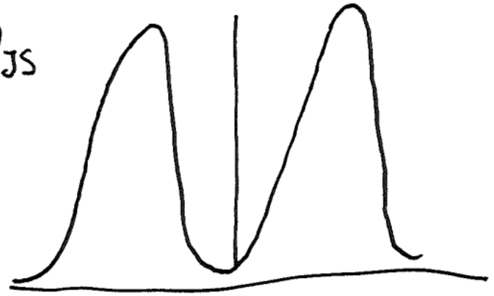
$$D_{KL}(p\|q) = \int_x p(x) \log \frac{p(x)}{q(x)} dx$$

$$D_{JS}(p\|q) = \frac{1}{2} D_{KL}\left(p \,\Big\|\, \frac{p+q}{2}\right) + \frac{1}{2} D_{KL}\left(q \,\Big\|\, \frac{p+q}{2}\right)$$

对称

bounded

$-D_{JS}$



Training GANs: Objective

Assuming G is <u>fixed</u>, to train D given a set of real samples:

$$x^n \sim p_d, \qquad n = 1, \dots, N$$

We generate

$$z^n \sim \mathcal{N}(0,1), \qquad n = 1, \dots, N$$

and form a training dataset

$$\mathbb{D} = \{\left(x^{(1)}, 1\right), \dots, (x^{(n)}, 1), \left(G\left(z^{(1)}\right), 0\right), \dots, G\left(z^{(n)}\right), 0)\}$$

Training GANs: Objective – Part II

Assuming G is <u>fixed</u>, to train D given training data $\mathbb{D}$:

(this will be on the blackboard)

TODO:

- [ ] Derive GAN objective
- [ ] Analyze behavior at optimum
- [ ] Show that objective minimizes JS-Divergence

Part I: Derive the GAN objective

a) Considering only D (G is static):

$$\mathcal{L}(D) = -\frac{1}{2N}\left(\sum_{i}^{N}(y^{(i)}\log(D(x^{(i)})) + \sum_{i}^{N}(1-y^{(i)})\log(1-D(x^{(i)}))\right)$$

for Discriminator only

we know that $\frac{1}{2}$ of samples $y=1$ & $\frac{1}{2}$ $y=0$

b) we leverage this to train D&G jointly

$$= -\frac{1}{2}\left(\mathbb{E}_{x\sim p_d}[\log(D(x))] + \mathbb{E}_{z\sim p_z}[\underline{\log(1-D(G(z)))}]\right)$$

$\Rightarrow$ Good G maximizes $\mathcal{L}(D)$

Part I: Derive the GAN objective

c) to find G we define:

$$V(G,D) = \mathbb{E}_{x \sim p_d}[\log(D(x))] + \mathbb{E}_{z \sim p_z}[\log(1-D(G(z)))]$$

$\Rightarrow G^*$ will fool any D

$$G^* = \arg\min_G \max_D V(G,D)$$

Note the change in sign between $\mathcal{L}(D)$ in (b) and $V(G,D)$ in (c)…

…hence G tries to minimize V and D tries to maximize

Part II: Analyze equilibrium point wrt D

Equilibrium of $V(\cdot, \cdot)$

Optimum of $V(G, D)$ iff $p_d(x) = p_g(x)$

Let $D^* = \arg\max_D V(G, D)$

$$\min_G \max_D V(D, G) = \mathbb{E}_{\boldsymbol{x} \sim p_{\text{data}}(\boldsymbol{x})}[\log D(\boldsymbol{x})] + \mathbb{E}_{\boldsymbol{z} \sim p_{\boldsymbol{z}}(\boldsymbol{z})}[\log(1 - D(G(\boldsymbol{z})))]$$

Proposition: if $p_d = p_g$ then $D^* = \frac{p_d}{p_d + p_g}$

Proof: $V(G, D) = \int_x p_d(x) \log(D(x)) dx + \int_z p_z(z) \log(1 - D(G(z))) dz$

$$= \underline{\underline{\int_x p_d(x) \log(D(x)) + p_g(x) \log(1 - D(x)) dx}}$$

Optimal D: $\forall a, b \setminus \{0, 0\}$ $f(y) = a \log y + b \log(1-y)$

$$f'(y) = 0 \Rightarrow \frac{a}{y} - \frac{b}{1-y} = 0 \Rightarrow y = \frac{a}{a+b} \Leftarrow \text{maximum}$$

$$-\frac{a}{y^2} - \frac{b}{(1-y)^2}\Big|_{y = \frac{a}{a+b}}$$

if $a, b \neq 0$ $\quad f''\left(\frac{a}{a+b}\right) = -\frac{a}{\left(\frac{a}{a+b}\right)^2} - \frac{b}{\left(1 - \frac{a}{a+b}\right)^2} < 0$ with $a, b \in \{0, 1\}$

$$\Rightarrow D^* = \frac{p_d}{p_d + p_g} \quad p_g$$

Note: D is unique but can't be calculated in practice (dep. p_d)

Part III: Analyze equilibrium point w.r.t. G

finding the best G

with D^* the obj. becomes

$$\min_G \max_D V(D,G) = \mathbb{E}_{\boldsymbol{x} \sim p_{\text{data}}(\boldsymbol{x})}[\log D(\boldsymbol{x})] + \mathbb{E}_{\boldsymbol{z} \sim p_{\boldsymbol{z}}(\boldsymbol{z})}[\log(1 - D(G(\boldsymbol{z})))]$$

$$G^* = \arg\min_G V(G, D^*)$$

$$V(G, D^*) = \mathbb{E}_{x \sim p_d}\left[\log \frac{p_d}{p_d + p_g}\right] + \mathbb{E}_{x \sim p_g}\left[\log\left(1 - \frac{p_d}{p_d + p_g}\right)\right]$$

Part III: Analyze equilibrium point w.r.t. G

$G^* = \arg\min_G V(G, D^*)$

$V(G, D^*) = \mathbb{E}_{x\sim p_d}\left[\log \frac{p_d}{p_d+p_g}\right] + \mathbb{E}_{x\sim p_g}\left[\log\left(1-\frac{p_d}{p_d+p_g}\right)\right]$

Theorem: global min is achieved if $p_d = p_g$
and at min $V(\cdot,\cdot) = -\log 4$

Proof: $1 - \frac{p_d}{p_d+p_g} = \frac{p_d+p_g}{p_d+p_g} - \frac{p_d}{p_d+p_g} = \frac{p_g}{p_d+p_g}$

$\Rightarrow \mathbb{E}_{x\sim p_d}\left[\log \frac{p_d}{p_d+p_g}\frac{2}{2}\right] + \mathbb{E}_{x\sim p_g}\left[\log \frac{p_g}{p_d+p_g}\frac{2}{2}\right]$

$= -\log(2) + \mathbb{E}_{x\sim p_d}\left[\log \frac{2p_d}{p_d+p_g}\right] - \log(2) + \mathbb{E}_{x\sim p_g}\left[\log \frac{2p_g}{p_d+p_g}\right]$

$= -\log(4) + D_{KL}\left(p_d \,\|\, \frac{p_d+p_g}{2}\right) + D_{KL}\left(p_g \,\|\, \frac{p_d+p_g}{2}\right)$

$= -\log(4) + 2\underbrace{D_{JS}(p_d \,\|\, p_g)}_{\geq 0;\ 0 \text{ iff } p_d = p_g}$ ☒

JS-Divergence

$D_{KL}(p\|q) = \int_x p(x) \log \frac{p(x)}{q(x)} dx$

$D_{JS}(p\|q) = \frac{1}{2} D_{KL}\left(p \,\|\, \frac{p+q}{2}\right) + \frac{1}{2} D_{KL}\left(q \,\|\, \frac{p+q}{2}\right)$

Next week

Generating high-quality images

Applications