Exercise 5

During development, a large number of symmetry breaks occur, and most of these are precise, reliable, and wired in the DNA of the organism. One mechanism that can lead to deterministic symmetry breaks is the Turing patterning mechanism. In this exercise, we will explore under what conditions Turing patterns can emerge and what properties they have.

1. **Ligand-receptor based Turing Pattern**. Consider the following set of equations in one spatial dimension x

$$\begin{aligned} \partial u/\partial t &= \Delta u + \gamma(a - u + u^2 v) \\ \partial v/\partial t &= d\Delta v + \gamma(b - u^2 v) \end{aligned} \tag{1}$$

where u and v are variables that describe two interacting species respectively and the parameters are all positive constants. In the following, only consider non-zero steady states.

   i) Assume that u represents a receptor and v represents a ligand. Explain the biological meaning of the reaction terms.

   ii) What are the steady-state solutions if you neglect diffusion?

   iii) Calculate the Jacobian of the reaction term and evaluate it at the steady-state.

   iv) Write down the conditions for a diffusion-driven instability and derive the inequalities that define the Turing space. As long as those inequalities are fulfilled, will you always observe a Turing pattern?

   iv) What impact does domain size have on the Turing pattern? Explain this in terms of the implicit equation of the real parts of the eigenvalues of the Jacobian and the wavenumber k.

2. **Numerical solutions of the Turing Model**.

   i) Choose a, b and d such that they fulfill the inequalities in problem 1 iii) and solve Eq. 1 numerically on the domain $x \in [0, 1]$ with zero-flux boundary conditions. Recall that you need to initiate the simulations with noisy initial conditions. What effect does γ have on the pattern?

   ii) Confirm your calculated Turing space by playing with your parameters close to the boundaries of the Turing space. Show that even small parameter changes close to such a boundary will destroy the pattern.

   iii) Choose γ such that it is slightly too small to see a mode. Can you 'rescue' the patterning by increasing d?

iv) Visualize the Turing space dependence on a and b for $\gamma = 10$ and $d = 50$ by checking for each parameter pair a,b if the real parts of all eigenvalues are negative in the absence of diffusion and become positive in the presence of diffusion. What happens when you change γ or d? Discuss your observations.

3. **Substrate-Depletion Model**. Consider the following set of equations in one spatial dimension x

$$\begin{aligned} \partial u/\partial t &= D_u \Delta u + k_1 - k_2 u - h(u,v) \\ \partial v/\partial t &= D_v \Delta v + k_3 - k_4 v - h(u,v) \\ h(u,v) &= \frac{k_5 uv}{k_6 + k_7 u + k_8 u^2} \end{aligned} \tag{2}$$

where u and v are variables that describe the substrate and the enzyme concentrations respectively. The parameter values are all positive constants.

i) Show that the set of equations can be non-dimensionalized to obtain

$$\begin{aligned} \partial u/\partial t &= \Delta u + \gamma(a - u - h(u,v)) \\ \partial v/\partial t &= d\Delta v + \gamma\alpha(b - v - h(u,v)). \\ h(u,v) &= \frac{\rho uv}{(1 + u + Ku^2)} \end{aligned} \tag{3}$$

ii) What are the conditions for a diffusion-driven instability?

iii) Set $a = 150$, $b = 150$, $\alpha = 1.5$, $\rho = 13$, $K = 0.05$ and solve the non-dimensionalized set of equations on the domain $x \in [0, 1]$ for different d. What is the critical d for the formation of Turing pattern? Does this fit with your analytical result? What happens as you double the size of the domain?