Exercise 12.1 The right shift map on the space ℓ^2 is given by

$$S\colon \ell^2 \to \ell^2$$
$$(x_1, x_2, \ldots) \mapsto (0, x_1, x_2, \ldots).$$

(i) Show that that the map S is a continuous linear operator with norm $\|S\| = 1$.

(ii) Compute the eigenvalues and the spectral radius of S.

(iii) Show that S has a left inverse which is not a right inverse, i.e. there exists $T\colon \ell^2 \to \ell^2$ with $T \circ S = \mathrm{id}_{\ell^2}$ but $S \circ T \neq \mathrm{id}_{\ell^2}$. Is it possible to find a right inverse of S, i.e. $Q : \ell^2 \to \ell^2$ so that $S \circ Q = \mathrm{id}_{\ell^2}$?

Exercise 12.2 Let $(H, \langle\cdot,\cdot\rangle_H)$ be a Hilbert space over $\mathbb{C}$. Recall two definitions:

- A linear operator $T \in L(H)$ is called an *isometry* if $\|Tx\|_H = \|x\|_H$ for every $x \in H$;
- An invertible linear operator $T \in L(H)$ is *unitary* if $T^* = T^{-1}$.

With these definitions,

(i) Prove that T is an isometry if and only if it preserves the scalar product, that is

$$\langle Tx, Ty\rangle_H = \langle x, y\rangle_H \quad \text{for every } x, y \in H.$$

(ii) Prove that $T \in L(H)$ is unitary if and only if T is a bijective isometry.

(iii) Prove that if $T \in L(H)$ is unitary, then its spectrum lies on the unit circle:

$$\sigma(T) \subset \mathbb{S}^1 := \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}.$$

Exercise 12.3 Let $\Omega \subset \mathbb{R}^m$ be an open bounded subset. Given $k \in L^2(\Omega \times \Omega, \mathbb{C})$ such that $k(x, y) = \overline{k(y, x)}$ for almost every $(x, y) \in \Omega \times \Omega$, consider the operator $K\colon L^2(\Omega, \mathbb{C}) \to L^2(\Omega, \mathbb{C})$ defined by

$$(Kf)(x) = \int_\Omega k(x, y) f(y)\, dy,$$

and the operator $A\colon L^2(\Omega, \mathbb{C}) \to L^2(\Omega, \mathbb{C})$ defined by

$$A(f)(x) = f(x) - Kf(x).$$

Prove that injectivity of A and surjectivity of A are equivalent.

Exercise 12.4 Let $(H, \langle\cdot,\cdot\rangle_H)$ be a Hilbert space over $\mathbb{C}$.

(i) Let $A \in L(H)$ be a self-adjoint operator and let $\lambda \in \rho(A)$ be an element in its resolvent set. Show that the resolvent $R_\lambda := (\lambda - A)^{-1}$ is a normal operator, that is $R_\lambda R_\lambda^* = R_\lambda^* R_\lambda$.

(ii) Let $A, B \in L(H)$ be self-adjoint operators. The *Hausdorff distance* of their spectra $\sigma(A), \sigma(B) \subset \mathbb{C}$ is defined to be

$$d\Big(\sigma(A), \sigma(B)\Big) := \max\bigg\{ \sup_{\alpha\in\sigma(A)} \Big(\inf_{\beta\in\sigma(B)} |\alpha - \beta| \Big), \sup_{\beta\in\sigma(B)} \Big(\inf_{\alpha\in\sigma(A)} |\alpha - \beta| \Big) \bigg\}.$$

Prove that

$$d\Big(\sigma(A), \sigma(B)\Big) \leq \|A - B\|_{L(H)}.$$

Remark. The Hausdorff distance d is in fact a distance on compact subsets of $\mathbb{C}$. In particular, it restricts to an actual distance function on the spectra of bounded linear operators.

Exercise 12.5 *(Heisenberg's Uncertainty Principle)* Let $(H, \langle\cdot,\cdot\rangle_H)$ be a Hilbert space over $\mathbb{C}$. Let $D_A, D_B \subset H$ be dense subspaces and let $A\colon D_A \subset H \to H$ and $B\colon D_B \subset H \to H$ be symmetric linear operators. Assume that

$$A(D_A \cap D_B) \subset D_B \quad \text{and} \quad B(D_A \cap D_B) \subset D_A,$$

and define the *commutator* of A and B as

$$[A, B]\colon D_{[A,B]} \subset H \to H, \qquad [A, B](x) \mapsto A(Bx) - B(Ax),$$

where $D_{[A,B]} := D_A \cap D_B$.

(i) Prove that

$$\Big|\langle x, [A, B]x\rangle_H\Big| \leq 2\|Ax\|_H \|Bx\|_H \quad \text{for every } x \in D_{[A,B]}.$$

(ii) Define now the *standard deviation* of A

$$\varsigma(A, x) := \sqrt{\langle Ax, Ax\rangle_H - \langle x, Ax\rangle_H^2}$$

at each $x \in D_A$ with $\|x\|_H = 1$. Verify that $\varsigma(A, x)$ is well-defined for every x (i.e. that the radicand is real and non-negative) and prove that for every $x \in D_{[A,B]}$ with $\|x\|_H = 1$ there holds

$$\Big|\langle x, [A, B]x\rangle_H\Big| \leq 2\varsigma(A, x)\,\varsigma(B, x).$$

Remark. The possible *states* of a quantum mechanical system are given by elements $x \in H$ with $\|x\|_H = 1$. Each *observable* is given by a symmetric linear operator $A\colon D_A \subset H \to H$. If the system is in state $x \in D_A$, we measure the observable A with uncertainty $\varsigma(A, x)$.

(iii) Let $A\colon D_A \to H$ and $B\colon D_B \to H$ be as above. A, B is called *Heisenberg pair* if

$$[A, B] = i \,\mathrm{Id}\,.$$

Show that, if A, B is a Heisenberg pair with B continuous (and $D_B = H$), then A cannot be continuous.

(iv) Consider the Hilbert space $(H, \langle\cdot,\cdot\rangle_H) = \left(L^2([0,1],\mathbb{C}), \langle\cdot,\cdot\rangle_{L^2}\right)$ and the subspace

$$C^1_0([0,1],\mathbb{C}) := \{f \in C^1([0,1],\mathbb{C}) \mid f(0) = 0 = f(1)\}.$$

Recall that $C^1_0([0,1],\mathbb{C}) \subset L^2([0,1],\mathbb{C})$ is a dense subspace. The operators

$$\begin{aligned} P\colon C^1_0([0,1],\mathbb{C}) &\to L^2([0,1],\mathbb{C}), & \qquad Q\colon L^2([0,1],\mathbb{C}) &\to L^2([0,1],\mathbb{C}) \\ f(s) &\mapsto if'(s) & f(s) &\mapsto sf(s) \end{aligned}$$

correspond to the observables *momentum* and *position*. Check that P and Q are well-defined, symmetric operators. Check that $[P, Q]\colon C^1_0([0,1],\mathbb{C}) \to L^2([0,1],\mathbb{C})$ is well-defined.

Show that P and Q form a Heisenberg pair and conclude that the *uncertainty principle* holds: for every $f \in C^1_0([0,1],\mathbb{C})$ with $\|f\|_{L^2([0,1],\mathbb{C})} = 1$ there holds

$$\varsigma(P, f)\,\varsigma(Q, f) \geq \tfrac{1}{2}.$$

Thus we conclude: *The more precisely the momentum of some particle is known, the less precisely its position can be known, and vice versa.*

Hints to Exercises.

12.2 For (i), use the the complex polarization identity

$$\langle x, y\rangle_H = \frac{1}{4}\Big(\|x+y\|^2 - \|x-y\|^2\Big) + \frac{i}{4}\Big(\|x+iy\|_H^2 - \|x-iy\|_H^2\Big).$$

For (iii), use Satz 6.5.3 and Satz 2.2.7.

12.3 Use Exercise 11.3.

12.4 Prove that $R_\lambda^* = R_{\overline{\lambda}}$ and use that resolvents to different values commute (Satz 6.5.2). Argue that it suffices to show the following implication for any $\alpha \in \mathbb{C}$:

$$\inf_{\beta\in\sigma(B)} |\alpha - \beta| > \|A - B\|_{L(H)} \qquad \Rightarrow \ \alpha \in \rho(A).$$

Given $f_\alpha(z) = (\alpha - z)^{-1}$, the spectral mapping theorem implies $f_\alpha(\sigma(B)) = \sigma(f_\alpha(B))$. Show that normal operators R have spectral radius $r_R = \|R\|$. Apply Satz 2.2.7.

12.5 For (ii): in order to apply (i), find symmetric operators $\tilde{A} = A - \lambda$ and $\tilde{B} = B - \mu$ satisfying

$$[A, B] = [\tilde{A}, \tilde{B}], \qquad \varsigma(A, x) = \|\tilde{A}x\|_H, \qquad \varsigma(B, x) = \|\tilde{B}x\|_H.$$

For (iii), begin by checking that $[A, B^n]$ is well-defined and prove $[A, B^n] = niB^{n-1}$ for every $n \in \mathbb{N}$.