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Lecture 14
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# Solution 14.1: The Differential Pair

1. The Differential pair has two very nice properties: As input it takes a difference between two voltages. One way to look at this is to say that one of the inputs (*e.g.* $V_1$) is a reference voltage; then $I_2$ will only have appreciable magnitude, when $V_2 \geq V_1$.

   In addition to this the asymptotic behaviour (bounded, approaches a finite value) is 'nice': Even if you accidentally go outside the range you intended for your input ($V_1$, $V_2$), you cannot be terribly far off in your output. (Of course any electrical system can be thermally destroyed if you go sufficiently far outside its operational range.) Biological 'circuits' also tend to have this property. Furthermore, as we will see below, the plotted curves are sigmoids which have convenient properties.

2. Let $I_b$ be the current from node $V_s$ to ground through $M_1$ ($b$ stands for '*bias*'). Using Kirchoff's Law at the node $V_s$ we get

   $$I_1 + I_2 = I_b$$

   Using the saturation approximation we further have

   $$I_1 = I_0 \exp\left(\frac{\kappa V_1 - V_s}{U_T}\right)$$

   $$I_2 = I_0 \exp\left(\frac{\kappa V_2 - V_s}{U_T}\right)$$

   Together this yields

   $$I_b = I_1 + I_2 = I_0 \exp\left(\frac{-V_s}{U_T}\right) \cdot \left(\exp\left(\frac{\kappa V_1}{U_T}\right) + \exp\left(\frac{\kappa V_2}{U_T}\right)\right)$$

   Solve for the $V_s$ term

   $$I_0 \cdot \exp\left(\frac{-V_s}{U_T}\right) = \frac{I_b}{\exp\left(\frac{\kappa V_1}{U_T}\right) + \exp\left(\frac{\kappa V_2}{U_T}\right)}$$

   By inserting this expression it follows that:

   $$I_1 = I_0 \exp\left(\frac{\kappa V_1 - V_s}{U_T}\right) = \exp\left(\frac{\kappa V_1}{U_T}\right) \cdot I_0 \exp\left(\frac{-V_s}{U_T}\right) = I_b \frac{\exp\left(\frac{\kappa V_1}{U_T}\right)}{\exp\left(\frac{\kappa V_1}{U_T}\right) + \exp\left(\frac{\kappa V_2}{U_T}\right)}$$

   $$I_2 = I_0 \exp\left(\frac{\kappa V_2 - V_s}{U_T}\right) = \exp\left(\frac{\kappa V_2}{U_T}\right) \cdot I_0 \exp\left(\frac{-V_s}{U_T}\right) = I_b \frac{\exp\left(\frac{\kappa V_2}{U_T}\right)}{\exp\left(\frac{\kappa V_1}{U_T}\right) + \exp\left(\frac{\kappa V_2}{U_T}\right)}$$

   To transform to the parameter we want (*i.e.* $V_1 - V_2$) expand the fraction by $\exp(-\kappa V_1/U_T)$ to get

   $$I_1 = I_b \frac{\exp\left(\frac{\kappa(V_1 - V_1)}{U_T}\right)}{\exp\left(\frac{\kappa(V_1 - V_1)}{U_T}\right) + \exp\left(\frac{\kappa(V_2 - V_1)}{U_T}\right)} = I_b \frac{1}{\exp\left(\frac{-\kappa(V_1 - V_2)}{U_T}\right) + 1}$$

   and similarly for $I_2$. This is the sigmoid function. Its slope at the inflection point is $\frac{I_b \kappa}{4U_T}$. Hence, besides $\kappa$ and $U_T$ (which are fixed for a given circuit), $I_b$ and thereby $V_{bn}$ set the absolute slope (*i.e.* the conductance, or, more correctly, the '*trans*'-conductance) of the linear part of the given plot. Note that $I_b$ scales the whole curve for the $I_1$ and $I_2$. Hence, it has also a linear effect on the maximal slope.

Two applications of this circuit in neuromorphic engineering are:

- To see how this circuit is applied to emulate synaptic dynamics have a look at the Neural Computation paper 'Synaptic Dynamics in Analog VLSI' by C. Bartolozzi and G. Indiveri (under the heading Diff-Pair Integrator Synapse) from 2007.

- Also, 'The voltage-dependent conductances of biological membranes in steady state have a sigmoidal conductance-voltage relation.' (citation from Mahowald M, Douglas R (1991). A silicon neuron. *Nature*; 354(6354): 515-8). Here the sigmoid property is exploited to replace active membrane conductances with a differential pair.