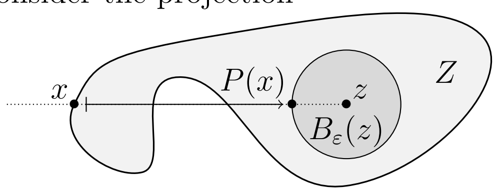
Exercise 3.1 Let $(X, \|\cdot\|)$ be an infinite-dimensional normed vector space an let $Z \subset X$ be a bounded subset with compact boundary. Prove that Z has empty interior.

Solution. If $Z \subset X$ has non-empty interior $Z^\circ \neq \emptyset$, then there exists $z \in Z$ and $\varepsilon > 0$ such that $B_\varepsilon(z) \subset Z^\circ$, where $B_\varepsilon(z)$ denotes the ball of radius ε around z in $(X, \|\cdot\|)$ and $\partial B_\varepsilon(z)$ its boundary. We consider the projection

$$P\colon Z \setminus \{z\} \to \partial B_\varepsilon(z)$$
$$x \mapsto z + \varepsilon \frac{x - z}{\|x - z\|}.$$



For every $y \in \partial B_\varepsilon$ the ray $\gamma = \{z + t(y - z) \mid t > 0\}$ must intersect ∂Z since Z is assumed to be bounded. Therefore, $P(\partial Z) = \partial B_\varepsilon(z)$. Being continuous, P maps compact sets onto compact sets. Since ∂Z is assumed to be compact, we have that the sphere $\partial B_\varepsilon(z)$ is compact. This however contradicts the assumption that the dimension of X is infinite. $\square$

Exercise 3.2 We consider the space $X = C^0([-1,1], \mathbb{R})$ with its usual norm $\|\cdot\|_{C^0([-1,1])}$ and let $\varphi : X \to \mathbb{R}$ be given by

$$\varphi(f) \mapsto \int_0^1 f(t)\,\mathrm{d}t - \int_{-1}^0 f(t)\,\mathrm{d}t.$$

(i) Show that φ is a linear and continuous map with $\|\varphi\|_{L(X,\mathbb{R})} \leq 2$.

(ii) Prove that the norm of φ is exactly 2 by finding a sequence $(f_n)_{n\in\mathbb{N}}$ in X such that $\|f_n\|_{C^0([-1,1])} = 1$ for every $n \in \mathbb{N}$ and such that $\varphi(f_n) \to 2$ as $n \to \infty$.

(iii) Prove that there does not exist $f \in X$ with $\|f\|_{C^0([-1,1])} = 1$ and $|\varphi(f)| = 2$.

Solution. (i) Let $f \in X := C^0([-1,1])$ and $\|\cdot\|_X := \|\cdot\|_{C^0([-1,1])}$. The given map $\varphi\colon X \to \mathbb{R}$ is linear by linearity of the integral. Moreover,

$$|\varphi(f)| \leq \int_0^1 |f(t)|\,\mathrm{d}t + \int_{-1}^0 |f(t)|\,\mathrm{d}t \leq 2\|f\|_{C^0([-1,1])} = 2\|f\|_X$$

implies

$$\|\varphi\|_{L(X,\mathbb{R})} = \sup_{f \in X\setminus\{0\}} \frac{|\varphi(f)|}{\|f\|_X} \leq 2.$$

Thus φ is also bounded and hence, by linearity, continuous.

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(ii) The sign function $f(x) = \frac{x}{|x|}$ is approximated pointwise by the sequence $(f_n)_{n\in\mathbb{N}}$ of functions $f_n \in X$ given by

$$f_n(t) = \begin{cases} -1, & \text{for } \ -1 \leq t < -\frac{1}{n}, \\ nt, & \text{for } \ -\frac{1}{n} \leq t < \frac{1}{n}, \\ 1, & \text{for } \ \ \frac{1}{n} \leq t \leq 1. \end{cases}$$

In particular, $\|f_n\|_X = 1$ for every $n \in \mathbb{N}$. Computing the integrals explicitly, or applying the dominated convergence theorem, we have

$$\lim_{n\to\infty} \varphi(f_n) = 2.$$

(iii) Suppose there exists $f \in X$ with $\|f\|_X = 1$ and $|\varphi(f)| = 2$. Since φ is linear, we may assume $\varphi(f) = 2$, otherwise we replace f by $-f$. Then, the estimates

$$\left|\int_0^1 f(t)\,\mathrm{d}t\right| \leq \max_{x\in[-1,1]} |f(x)| = \|f\|_X = 1, \qquad \left|\int_{-1}^0 f(t)\,\mathrm{d}t\right| \leq 1,$$

imply by definition of φ that

$$\int_0^1 f(t)\,\mathrm{d}t = -\int_{-1}^0 f(t)\,\mathrm{d}t = 1. \tag{$*$}$$

Since f is bounded from above by 1 we can conclude from $(*)$ that $f|_{]0,1]} \equiv 1$. In fact, if $f(t^*) < 1$ for some $t^* \in \,]0,1]$, then $f < 1$ in some neighbourhood of t^* by continuity of f which together with the uniform bound $f \leq 1$ contradicts $(*)$.

Analogously, we conclude $f|_{[-1,0[} \equiv -1$ which (combined with $f|_{]0,1]} \equiv 1$) violates continuity of f at 0. □

Exercise 3.3 Let

$$c_c := \{(x_n)_{n\in\mathbb{N}} \in \ell^\infty \mid \exists N \in \mathbb{N}\ \forall n \geq N :\ x_n = 0\}$$

be the space of compactly supported sequences endowed with the sup-norm $\|\cdot\|_{\ell^\infty}$. Consider the map $T\colon c_c \to c_c$ given by

$$T\left((x_n)_{n\in\mathbb{N}}\right) = (nx_n)_{n\in\mathbb{N}}.$$

(i) Show that T is not continuous.

(ii) Construct continuous linear maps $T_m\colon c_c \to c_c$ such that

$$\forall x \in c_c : \quad T_m x \xrightarrow{m\to\infty} Tx.$$

Solution. (i) The operation $T\colon (x_n)_{n\in\mathbb{N}} \mapsto (nx_n)_{n\in\mathbb{N}}$ is linear in each entry and therefore linear as map $T\colon c_c \to c_c$. For every $k \in \mathbb{N}$ we define the sequence $e_k = (e_{k,n})_{n\in\mathbb{N}} \in c_c$ by

$$e_{k,n} = \begin{cases} 1, & \text{if } n = k, \\ 0, & \text{otherwise.} \end{cases}$$

Then, $\|e_k\|_{\ell^\infty} = 1$ for every $k \in \mathbb{N}$ but $\|Te_k\|_{\ell^\infty} = k$ is unbounded for $k \in \mathbb{N}$. As unbounded linear map, T is not continuous.

(ii) For every $m \in \mathbb{N}$ we define

$$\begin{aligned} T_m\colon c_c &\to c_c \\ (x_n)_{n\in\mathbb{N}} &\mapsto (x_1, 2x_2, 3x_3, \ldots, mx_m, 0, 0, \ldots) \end{aligned}$$

Then T_m is linear. $T_m\colon (c_c, \|\cdot\|_\ell^\infty) \to (c_c, \|\cdot\|_\ell^\infty)$ is also bounded for every (fixed) $m \in \mathbb{N}$ since for every $x = (x_n)_{n\in\mathbb{N}} \in c_c$

$$\|T_m x\| = \sup_{n\in\mathbb{N}} |(T_m x)_n| = \max_{n\in\{1,\ldots,m\}} |nx_n| \leq m\|x\|_{\ell^\infty}.$$

Hence, T_m is continuous.

Let $x = (x_n)_{n\in\mathbb{N}} \in c_c$ be fixed. Then there exists $N \in \mathbb{N}$ such that $x_n = 0$ for all $n \geq N$ which implies $T_m x = Tx$ for all $m \geq N$. In particular,

$$T_m x \xrightarrow{m\to\infty} Tx.$$

$\square$

Exercise 3.4 Let $k\colon [0,1] \times [0,1] \to \mathbb{R}$ be continuous. Show that for every $g \in C^0([0,1])$ there exists a unique $f \in C^0([0,1])$ satisfying

$$\forall t \in [0,1]: \quad f(t) + \int_0^t k(t,s)f(s)\,\mathrm{d}s = g(t).$$

Solution. Let $(X, \|\cdot\|_X) = \big(C^0([0,1]), \|\cdot\|_{C^0([0,1])}\big)$. Since the function k is continuous in both variables, the integral operator $T\colon X \to X$ given by

$$(Tf)(t) = \int_0^t k(t,s)f(s)\,\mathrm{d}s$$

is well-defined. We claim that for every $n \in \mathbb{N}$ and every $f \in X$ and $t \in [0,1]$,

$$|(T^n f)(t)| \leq \frac{t^n}{n!} \|k\|^n_{C^0([0,1]\times[0,1])} \|f\|_X.$$

We prove the claim by induction. For $n = 1$ we have

$$|(Tf)(t)| \leq \int_0^t |k(t,s)||f(s)| \, \mathrm{d}s \leq t\|k\|_{C^0([0,1]\times[0,1])} \|f\|_X.$$

Suppose the claim is true for some $n \in \mathbb{N}$. Then,

$$\begin{aligned} |(T^{n+1} f)(t)| &\leq \int_0^t |k(t,s)||(T^n f)(s)| \, \mathrm{d}s \\ &\leq \frac{1}{n!} \|k\|^{n+1}_{C^0} \|f\|_X \int_0^t s^n \, \mathrm{d}s = \frac{t^{n+1}}{(n+1)!} \|k\|^{n+1}_{C^0} \|f\|_X \end{aligned}$$

which proves the claim. Since $0 \leq t \leq 1$, the claim implies

$$r_T := \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}} \leq \lim_{n\to\infty} \frac{\|k\|_{C^0}}{(n!)^{\frac{1}{n}}} = 0.$$

From $r_T = 0$ we conclude that the operator $(1+T) = (1-(-T))$ is invertible with bounded inverse (Satz 2.2.7). The solution to the Volterra equation $f + Tf = g$ is then given by $f = (1+T)^{-1} g$. $\square$

Hints to Exercises.

3.1 Assume that $Z^\circ \neq \emptyset$. Find a continuous functional that projects the boundary ∂Z to the boundary of a ball inside Z. This will contradict the fact that the unit sphere in an infinite-dimensional normed space is non-compact.

3.4 Begin by choosing a space $(X, \|\cdot\|_X)$ and show that the operator $T\colon X \to X$ given by $(Tf)(t) = \int_0^t k(t,s) f(s)\,\mathrm{d}s$ has spectral radius $r_T = 0$.