Bayesian Statistics

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Today's topics

- Non-informative priors
- Improper priors
- Jeffreys prior

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Non-informative priors

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Prior distributions Non-informative priors

Non-informative priors

- ► Goal: reduce the subjective element in Bayesian analysis
- ► (Too) simple solution: define a uniform prior on Θ
- This has two potential drawbacks:
 - The uniform distribution on Θ is a probability distribution only if Θ has finite volume
 - 2. The uniform distribution is not invariant under reparametrizations

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Uniform distribution is not invariant under reparametrizations

- Assume $\tau = g(\theta)$ where g is invertible and differentiable
- If θ has density π , then by the change-of-variables formula $\tau = g(\theta)$ has density

$$\lambda(\tau) = \pi\left(g^{-1}(\tau)\right) \left| \det Dg^{-1}(\tau) \right|,$$

where Dg^{-1} is the Jacobi matrix whose (ij)-th entry is $\partial g_i^{-1}/\partial \theta_j$

- ▶ Hence, if π is constant and g is not linear, then λ is not constant
- See example on blackboard

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Improper priors

► We call a (σ -finite) measure π on Θ which is **not a probability** measure, i.e.,

$$\int_{\Theta} \pi(\theta) d\theta = \infty,$$

an improper prior

- ▶ Depending on the likelihood, $\pi(\theta)f(x \mid \theta)$ can have both finite or infinite total mass if $\pi(\theta)$ has infinite mass
- If the total mass is finite, then we have by formal analogy the posterior density

$$\pi(\theta \mid \mathbf{x}) = \frac{\pi(\theta)f(\mathbf{x} \mid \theta)}{\int \pi(\theta')f(\mathbf{x} \mid \theta')d\theta'}$$

and we can construct Bayesian point estimates, tests and credible intervals

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Improper priors

- Often, improper priors with proper posteriors can be justified as follows:
 - 1. Approximate an improper prior by a sequence of proper priors π_k
 - 2. Show that the associated sequence of posteriors $\pi_k(\theta \mid x)$ converges to $\pi(\theta \mid x)$
- In complicated models, it can be difficult to check whether $\pi(\theta)f(x \mid \theta)$ has finite total mass

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Jeffreys prior

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Jeffreys prior

For **Jeffreys prior**, we take

$$\pi(\theta) \propto \det(I(\theta))^{1/2}$$

 \triangleright $I(\theta)$ is the Fisher information matrix

$$I(\theta) = \mathbb{E}_{\theta} \left(\frac{\partial}{\partial \theta} \log f(X \mid \theta) \left(\frac{\partial}{\partial \theta} \log f(X \mid \theta) \right)^T \right)$$

or, equivalently,*

$$I(\theta) = -\mathbb{E}_{\theta} \left(\frac{\partial^2}{\partial \theta \partial \theta^T} \log f(X \mid \theta) \right)$$

See examples on blackboard

^{*}Assuming regularity conditions

Intuition for Jeffreys prior

- ▶ $I(\theta)^{-1}$ is the asymptotic variance of the MLE. Hence, $I(\theta)$ can be seen as an indicator of the amount of information of the data about θ
- ▶ Jeffreys prior $\pi(\theta) \propto \det(I(\theta))^{1/2}$ gives larger (smaller) weight to values of θ for which the information in the data $I(\theta)$ is larger (smaller)
- ➤ The influence of the prior on the posterior is therefore small and the prior is considered as non-informative

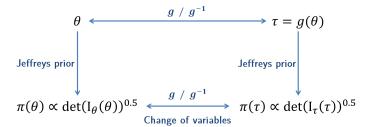
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Equivariance of Jeffreys prior

Jeffreys prior is equivariant under different parametrizations

▶ I.e., for a mapping $\tau = g(\theta)$ that is one-to-one and differentiable the following holds true:

First transforming the parameter $\tau = g(\theta)$ and then calculating the Jeffreys prior for τ is equivalent to first calculating the Jeffreys prior for θ and then transforming the corresponding density



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Equivariance of Jeffreys prior: 1d example

Example:

Assume $X \sim \mathcal{N}(0, \sigma^2)$ with the four parametrizations:

- $\bullet \theta = \sigma \text{ (standard deviation)}$
- $\theta = \sigma^2$ (variance)
- $ightharpoonup heta = \sigma^{-2}$ (precision)
- $\theta = \log(\sigma)$

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Equivariance of Jeffreys prior: 1d example

We obtain the following results:

Parameter	$\frac{\partial^2 \log f(x \theta)}{\partial \theta^2}$	$I(\theta)$	Jeffreys prior	$\frac{{\sf d}\sigma}{{\sf d}\theta}$
$\theta = \sigma$	$\frac{\theta^2-3x^2}{\theta^4}$	$\frac{2}{\theta^2}$	$\propto rac{1}{ heta} = rac{1}{\sigma}$	1
$\theta = \sigma^2$	$\frac{\theta-2x^2}{2\theta^3}$	$\frac{1}{2\theta^2}$	$\propto rac{1}{ heta} = rac{1}{\sigma^2}$	$\frac{1}{2\sqrt{\theta}}$
$\theta = \sigma^{-2}$	$-\frac{1}{2\theta^2}$	$\frac{1}{2\theta^2}$	$\propto rac{1}{ heta} = \sigma^2$	$-\frac{1}{2\theta^{3/2}}$
$\theta = \log(\sigma)$	$-2e^{-2\theta}x^2$	2	∝ 1	$e^{ heta}$

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Prior distributions Non-informative priors

Equivariance of Jeffreys prior for the multiparameter case

- ▶ Denote by Dg^{-1} the Jacobi matrix of g^{-1}
- We can show that
 - 1. directly calculating Jeffreys prior for au

$$\pi(au) \propto \det(I_{ au}(au))^{1/2}$$

2. and first calculating Jeffreys prior $\pi(\theta)$ for θ and then applying the change-of-variables formula

$$\pi(\tau) = \pi(g^{-1}(\tau))|\det Dg^{-1}(\tau)|$$

yield the same result

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Prior distributions Non-informative priors

Jeffreys prior: concluding remarks

- Jeffreys prior is usually a good choice for scalar parameters
- It often leads to improper priors
- However, it violates the likelihood principle because the Fisher information contains an integral over X
- For vector parameters, it can have undesirable features

See example on blackboard

Clicker question

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