Exercise 4.1 Let $(X, \|\cdot\|_X)$ be a normed space an let $U, V \subset X$ be subspaces. Prove the following.

(i) If U is finite dimensional and V closed, then $U + V$ is a closed subspace of X.

(ii) If V is closed with finite codimension, i. e. $\dim(X/V) < \infty$, then $U + V$ is closed.

Remark. The assumptions on the dimension and codimension above are crucial for the conclusions to hold; see Exercise 2.4.

Solution. Recall first that the canonical quotient map $\pi \colon X \to X/V$ is continuous whenever a subspace $V \subset X$ is closed (Satz 2.3.1).

(i) $\dim \pi(U) \leq \dim U < \infty$ implies that $\pi(U) \subset X/V$ is closed (Satz 2.1.3). Since π is continuous, $\pi^{-1}(\pi(U)) = U + V \subset X$ is also closed.

(ii) Since $\dim \pi(U) \leq \dim(X/V) < \infty$, we can argue the same way as in (i). $\square$

Exercise 4.2 Let $X = C^0([0,1])$ endowed with the norm $\|\cdot\|_X = \|\cdot\|_{C^0([0,1])}$ and consider

$$U = C^0_0([0,1]) := \{f \in C^0([0,1]) \mid f(0) = 0 = f(1)\}.$$

(i) Show that U is a closed subspace of X

(ii) Compute the dimension of the quotient space X/U and find a basis for X/U.

Solution. (i) Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in U which converges to f in $(X, \|\cdot\|_X)$. Then, since $f_n(0) = 0 = f_n(1)$, we can colclude $f(0) = 0 = f(1)$, i. e. $f \in U$ by passing to the limit $n \to \infty$ in the following inequalities:

$$|f(0)| = |f_n(0) - f(0)| \leq \sup_{x\in[0,1]} |f_n(x) - f(x)| = \|f_n - f\|_X,$$
$$|f(1)| = |f_n(1) - f(1)| \leq \sup_{x\in[0,1]} |f_n(x) - f(x)| = \|f_n - f\|_X.$$

Alternative: It suffices to notice that $U = \Phi^{-1}(\{0, 0\})$ where $\Phi : X \to \mathbb{R}^2$ is the continuous function given by $\Phi(f) = (f(0), f(1))$.

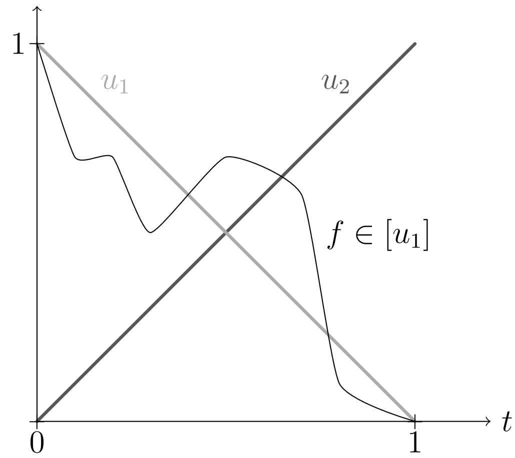

Figure 1: The functions $u_1, u_2 \in X$ and some $f \in [u_1]$.

(ii) Let $u_1, u_2 \in X$ be given by $u_1(t) = 1 - t$ and $u_2(t) = t$. We claim that the equivalence classes $[u_1], [u_2] \in X/U$ form a basis for X/U.

To prove linear independence, let $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $\lambda_1[u_1] + \lambda_2[u_2] = 0 \in X/U$ which means $\lambda_1 u_1 + \lambda_2 u_2 \in U$. This implies by definition

$$\lambda_1 = \lambda_1 u_1(0) + \lambda_2 u_2(0) = 0 = \lambda_1 u_1(1) + \lambda_2 u_2(1) = \lambda_2.$$

To show that $[u_1]$ and $[u_2]$ span X/U, let $[h] \in X/U$ with representative $h \in X$. By evaluation at $t = 0$ and $t = 1$, we conclude

$$\Big(t \mapsto h(t) - h(0)u_1(t) - h(1)u_2(t)\Big) \in U.$$

This implies $[h] = h(0)[u_1] + h(1)[u_2]$ in X/U which proves the claim. $\square$

Remark. The components of $[h]$ in this basis are unique since every representative $\tilde{h} \in [h]$ must have the same boundary values $\tilde{h}(0) = h(0)$ and $\tilde{h}(1) = h(1)$.

Exercise 4.3 A subspace $U \subset X$ of a Banach space$(X, \|\cdot\|_X)$ is called *topologically complemented* if there is a subspace $V \subset X$ such that the linear map I given by

$$\begin{aligned} I\colon (U \times V, \|\cdot\|_{U\times V}) &\to (X, \|\cdot\|_X), \qquad\qquad \|(u,v)\|_{U\times V} := \|u\|_X + \|v\|_X, \\ (u,v) &\mapsto u + v \end{aligned}$$

is a continuous isomorphism of normed spaces with continuous inverse. In this case V is said to be a *topological complement* of U.

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(i) Prove that $U \subset X$ is topologically complemented if and only if there exists a continuous linear map $P\colon X \to X$ with $P \circ P = P$ and image $P(X) = U$.

(ii) Show that a topologically complemented subspace must be closed.

Remarks.

- Clearly, if V is a topological complement of U, then U is a topological complmement of V.
- If X is not isomorphic to a Hilbert space, then X has closed subspaces which are not topologically complemented [Lindenstrauss & Tzafriri. *On the complemented subspaces problem.* (1971)]. An example is $c_0 \subset \ell^\infty$ but this is not easy to prove.

Solution. (i) Suppose $U \subset X$ is topologically complemented by $V \subset X$. Then, $I\colon U \times V \to X$ with $(u, v) \mapsto u+v$ is an continuous isomorphism with continuous inverse. We define

$$\begin{aligned} P_1\colon U \times V &\to U \times V, \qquad\qquad P := I \circ P_1 \circ I^{-1}\colon X \to X. \\ (u, v) &\mapsto (u, 0) \end{aligned}$$

P_1 is linear, bounded since $\|P_1(u, v)\|_{U\times V} = \|u\|_U \leq \|(u, v)\|_{U\times V}$ and hence continuous. As composition of linear continuous maps, P is linear and continuous. Moreover,

$$\begin{aligned} &P \circ P = (I \circ P_1 \circ I^{-1}) \circ (I \circ P_1 \circ I^{-1}) = I \circ P_1 \circ P_1 \circ I^{-1} = I \circ P_1 \circ I^{-1} = P, \\ &P(X) = I(U \times \{0\}) = U. \end{aligned}$$

Conversely, suppose $U \subset X$ allows a continuous linear map $P\colon X \to X$ with $P \circ P = P$ and $P(X) = U$. Let $V := \ker(P)$. Then

$$P \circ (1 - P) = P - P = 0 \qquad \Rightarrow (1 - P)(X) \subseteq \ker(P) = V. \tag{1}$$

In fact, $(1 - P)(X) = V$ since given $v \in V$ we have $v = (1 - P)v$. Analogously,

$$(1 - P) \circ P = P - P = 0 \qquad \Rightarrow\ U = P(X) \subseteq \ker(1 - P). \tag{2}$$

In fact, $U = \ker(1 - P)$ since $x - Px = 0$ implies $x = Px \in U$. We now claim that the map

$$I\colon U \times V \to X, \quad I(u, v) = u + v$$

is continuous and has a continuous inverse. Continuity of I follows directly from

$$\|I(u, v)\|_X = \|u + v\|_X \leq \|u\|_X + \|v\|_X = \|(u, v)\|_{U\times V}.$$

By the assumptions on P, especially (1), the map

$$\Phi\colon X \to U \times V, \quad \Phi(x) = \big(Px, (1-P)x\big)$$

is well-defined and continuous. Since $Pu = u$ for all $u \in U$ by (2) we have

$$\begin{aligned}(\Phi \circ I)(u,v) = \Phi(u+v) &= \big(Pu + Pv, u - Pu + v - Pv\big) = (u,v),\\ (I \circ \Phi)(x) = I(Px, (1-P)x) &= Px + (1-P)x = x,\end{aligned}$$

so Φ is inverse to I. Consequently, U is topologically complemented.

(ii) If $U \subset X$ is topologically complemented, then (i) implies existence of a continuous map $P\colon X \to X$ with $\ker(1-P) = U$. Thus, U must be closed as the kernel of the continuous map $1 - P$. $\square$

Exercise 4.4 Let $(X\|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be Banach spaces and let $T \in L(X, Y)$ be a surjective, continuous linear map. Prove the equivalence of the following statements:

(i) The subspace $\ker T$ is topologically complemented.

(ii) There exists a continuous linear map $S \in L(Y, X)$ so that $T \circ S = 1_Y$. S is called *section* of T.

Solution. *(i) ⇒ (ii).* Let $\ker(T)$ be topologically complemented and let V be a topological complement. Then the map

$$I : \ker(T) \times V \to X, \quad I(u,v) = u + v$$

is a continuous isomorphism with continuous inverse. By construction, the restriction $T|_V : V \to X$ of T to V is bijective and linear, hence its inverse $S := (T|_V)^{-1} : Y \to V$ is linear. Since V is a closed subspace of a Banach space, it is Banach as well. Thus from the Open Mapping Theorem S is also continuous.

(ii) ⇐ (i). We define

$$\Pi := S \circ T : X \to X$$

and $V := \Pi(V)$. The map Π is linear and continuous with

$$\Pi^2 x = S \circ (T \circ S) \circ Tx = (S \circ T)x = \Pi x \quad \forall x \in X.$$

Claim: $\ker(\Pi) = \ker(T)$.

Proof. The inclusion "$\supseteq$" is obvious. For "$\subseteq$", let $x \in \ker(\Pi)$. then $S(T(x)) = 0$ and hence $T(x) \in \ker S$, but since S is has a left inverse, it must be injective. Thus $T(x) = 0$. □

The conclusion now follows using Exercise 4.3 (i) by using $P = 1 - \Pi$. □

Exercise 4.5 Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ be normed spaces. We consider the space $(X \times Y, \|\cdot\|_{X\times Y})$, where $\|(x,y)\|_{X\times Y} = \|x\|_X + \|y\|_Y$ and a bilinear map $B\colon X \times Y \to Z$.

(i) Show that B is continuous if

$$\exists C > 0 \quad \forall (x,y) \in X \times Y: \quad \|B(x,y)\|_Z \leq C\|x\|_X\|y\|_Y. \tag{†}$$

(ii) Assume that $(X, \|\cdot\|_X)$ is Banach. Assume further that the maps

$$\begin{aligned} X &\to Z \\ x &\mapsto B(x, y') \end{aligned} \qquad\qquad \begin{aligned} Y &\to Z \\ y &\mapsto B(x', y) \end{aligned}$$

are continuous for every $x' \in X$ and $y' \in Y$. Prove that then B is continuous.

Solution. (i) Let $((x_k, y_k))_{k\in\mathbb{N}}$ be a sequence in $X \times Y$ converging to (x,y) in $(X \times Y, \|\cdot\|_{X\times Y})$. By definition,

$$\|x_k - x\|_X + \|y_k - y\|_Y = \|(x_k - x, y_k - y)\|_{X\times Y} = \|(x_k, y_k) - (x,y)\|_{X\times Y}$$

which yields convergence $x_k \to x$ in X and $y_k \to y$ in Y. Since $B\colon X \times Y \to Z$ is bilinear, we have

$$\begin{aligned} \|B(x_k, y_k) - B(x,y)\|_Z &= \|B(x_k, y_k) - B(x, y_k) + B(x, y_k) - B(x,y)\|_Z \\ &= \|B(x_k - x, y_k) - B(x, y_k - y)\|_Z \\ &\leq \|B(x_k - x, y_k)\|_Z + \|B(x, y_k - y)\|_Z. \end{aligned}$$

Using the assumption $\|B(x,y)\|_Z \leq C\|x\|_X\|y\|_Y$ and the fact that convergence of $(y_k)_{k\in\mathbb{N}}$ in $(Y, \|\cdot\|_Y)$ implies that $\|y_k\|_Y$ is bounded uniformly for all $k \in \mathbb{N}$, we conclude

$$\|B(x_k, y_k) - B(x,y)\|_Z \leq C\|x - x_k\|_X\|y_k\|_Y + C\|x\|_X\|y - y_k\|_Y \xrightarrow{k\to\infty} 0.$$

(ii) Let $B_1^Y \subset Y$ be the unit ball around the origin in $(Y, \|\cdot\|_Y)$. For every $x \in X$ we have by assumption

$$\sup_{y' \in B_1^Y} \|B(x, y')\|_Z \leq \sup_{y' \in B_1^Y} \|y'\|_Y \|B(x, \cdot)\|_{L(Y,Z)} \leq \|B(x, \cdot)\|_{L(Y,Z)} < \infty,$$

which means that the maps $(B(\cdot, y'))_{y' \in B_1^Y} \in L(X, Z)$ are pointwise bounded. Since X is assumed to be complete, the Theorem of Banach-Steinhaus implies that $(B(\cdot, y'))_{y' \in B_1^Y} \in L(X, Z)$ are uniformly bounded, i. e.

$$C := \sup_{y' \in B_1^Y} \|B(\cdot, y')\|_{L(X,Z)} < \infty.$$

From this we conclude

$$\begin{aligned} \|B(x, y)\|_Z &= \|y\|_Y \left\| B\left(x, \frac{y}{\|y\|_Y}\right) \right\|_Z \\ &\leq \|y\|_Y \|x\|_X \left\| B\left(\cdot, \frac{y}{\|y\|_Y}\right) \right\|_{L(X,Z)} \leq C\|x\|_X \|y\|_Y, \end{aligned}$$

so B is continuous by (i). $\square$

Hints to Exercises.

4.1 Is the canonical quotient map $\pi\colon X \to X/V$ continuous? What is $\pi^{-1}(\pi(U))$?

4.3 For (i), consider for one implication the projection map $P_1(u,v) = (u,0)$, and for the other implication the identity $1 = P + (1-P)$, where 1 denotes the identity map on X.

4.4 For (i) $\Rightarrow$ (ii), let V be the topological complement of $\ker T$ and consider the map $T|_V$. For (ii) $\Rightarrow$ (i), use Exercise 4.3.

4.5 Apply the Theorem of Banach-Steinhaus to a suitable map. Do not forget that the theorem requires completeness of the domain.