Exercise 5.1 Let $X = L^2((0,1),\mathbb{R})$. On $D_A := C_c^\infty((0,1),\mathbb{R}) \subset X$ consider the derivative operator

$$A\colon D_A \to X, \quad A(f) = f'.$$

Recall that A is closable. Show that the domain $D_{\overline{A}}$ of its closure is contained in

$$C_0^0([0,1],\mathbb{R}) = \{f \in C^0([0,1],\mathbb{R}) \mid f(0) = 0 = f(1)\}.$$

Note: Do not forget that L^2-convergence does *not* imply pointwise convergence.

Solution. Let $f \in D_{\overline{A}}$. Then there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in D_A with

$$\|f_n - f\|_{L^2((0,1))} \to 0 \quad \text{and} \quad \|f_n' - \overline{A}f\|_{L^2((0,1))} \to 0 \tag{1}$$

as $n \to \infty$. Since $f_n \in C_c^\infty((0,1),\mathbb{R})$ there holds $f_n(0) = 0 = f_n(1)$, however L^2-convergence alone is *not* enough to conclude the same for f. Instead from (1) we expect f to be a primitive of $\overline{A}f$. Therefore, we consider the function $g\colon [0,1] \to \mathbb{R}$ given by

$$g(t) := \int_0^t \overline{A}f\,dx.$$

We apply Hölder's inequality to estimate

$$\begin{aligned}
\big|f_n(t) - g(t)\big| &= \left|\int_0^t \Big(f_n' - \overline{A}f\Big)\,dx\right| \\
&\le \int_0^t \Big|f_n' - \overline{A}f\Big|\,dx \\
&\le \left(\int_0^t \Big|f_n' - \overline{A}f\Big|^2\,dx\right)^{\frac{1}{2}} \\
&\le \Big\|f_n' - \overline{A}f\Big\|_{L^2((0,1))}.
\end{aligned}$$

By taking the supremum over $t \in [0,1]$ and then letting $n \to \infty$ we deduce that $(f_n)_{n\in\mathbb{N}}$ converges *uniformly* to g. Since uniform convergence implies L^2-convergence, g must coincide with f and since it is uniform limit of continuous functions, it it also continuous. Finally, uniform convergence implies pointwise convergence, in particular

$$f(0) = \lim_{n\to\infty} \tilde{f}_n(0) = 0, \qquad\qquad f(1) = \lim_{n\to\infty} \tilde{f}_n(1) = 0.$$

□

Exercise 5.2 Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces. Let $A\colon D_A \subset X \to Y$ be a linear operator with closed graph. Show that the following statements are equivalent:

(i) A is injective and its range $W_A := A(D_A)$ is closed in $(Y, \|\cdot\|_Y)$.

(ii) There exists $C > 0$ so that $\|x\|_X \leq C\|Ax\|_Y$ for every $x \in D_A$.

Solution. *"(i) ⇒ (ii)".* As a closed subspace of a complete space, $(W_A, \|\cdot\|_Y)$ is complete. Since $A\colon D_A \subset X \to W_A$ is bijective with closed graph and since X, W_A are Banach spaces and we may apply the Inverse Mapping Theorem to obtain a continuous inverse $A^{-1}\colon W_A \to D_A$. In particular, $\|A^{-1}\| =: C$ is finite and for every $x \in D_A$ we have

$$\|x\|_X = \|A^{-1}Ax\|_X \leq \|A^{-1}\|\|Ax\|_Y = C\|Ax\|_Y.$$

"(ii) ⇒ (i)". Let $x \in D_A$ with $Ax = 0$, the inequality implies $\|x\|_X \leq 0$, hence $x = 0$. This implies that the linear map A is injective.

Let $(y_n)_{n\in\mathbb{N}}$ be a sequence in W_A converging to some $y \in Y$. By definition of W_A there exist $x_n \in D_A$ such that $Ax_n = y_n$. For every $n, m \in \mathbb{N}$, the assumptions implies

$$\|x_n - x_m\|_X \leq C\|Ax_n - Ax_m\|_Y = C\|y_n - y_m\|_Y.$$

From $(y_n)_{n\in\mathbb{N}}$ being Cauchy in $(Y, \|\cdot\|_Y)$, we conclude that $(x_n)_{n\in\mathbb{N}}$ is Cauchy in $(X, \|\cdot\|_X)$. Since $(X, \|\cdot\|_X)$ is complete, there exists $X \ni x = \lim_{n\to\infty} x_n$. Since the graph of A is assumed to be closed, $x \in D_A$ and $Ax = y$. Therefore, $y \in W_A$ and we conclude that W_A is a closed subspace of Y. □

Exercise 5.3 *(Hörmander).* Let $(X_0, \|\cdot\|_{X_0})$, $(X_1, \|\cdot\|_{X_1})$ and $(X_2, \|\cdot\|_{X_2})$ be Banach spaces and let

$$T_1\colon D_1 \subset X_0 \to X_1, \quad \text{and} \quad T_2\colon D_2 \subset X_0 \to X_2$$

be linear operators with closed graphs such that $D_1 \subset D_2$. Prove that there exists a constant $C > 0$ so that

$$\|T_2x\|_{X_2} \leq C\Big(\|T_1x\|_{X_1} + \|x\|_{X_0}\Big) \quad \text{for every } x \in D_1.$$

Solution. Let Γ_1 and Γ_2 be the graphs of T_1 and T_2 respectively. Since T_1 and T_2 have closed graphs by assumption, $(D_1, \|\cdot\|_{\Gamma_1})$ and $(D_2, \|\cdot\|_{\Gamma_2})$ are Banach spaces, where $\|x\|_{\Gamma_i} = \|x\|_{X_0} + \|T_i x\|_{X_i}$, $i = 1, 2$, denote the graph norms. Since $D_1 \subset D_2$, we can consider the identity map $\mathrm{Id}\colon (D_1, \|\cdot\|_{\Gamma_1}) \to (D_2, \|\cdot\|_{\Gamma_2})$ and claim that its graph is closed. Indeed, assume that $x_n \to x$ in $(D_1, \|\cdot\|_{\Gamma_1})$ and $\mathrm{Id}(x_n) = x_n \to y$ in $(D_2, \|\cdot\|_{\Gamma_2})$. Then, the definition of graph norm implies that both, $\|x_n - x\|_{X_0} \to 0$ and $\|x_n - y\|_{X_0} \to 0$ as $n \to \infty$ which implies $x = y$ and proves the claim. The closed graph theorem implies that Id is continuous, which means that there exists $C > 0$ so that

$$\|x\|_{\Gamma_2} \leq C\|x\|_{\Gamma_1} \quad \text{for all } x \in D_1.$$

By definition, this implies $\|T_2 x\|_{X_2} \leq C\Big(\|T_1 x\|_{X_1} + \|x\|_{X_0}\Big) - \|x\|_{X_0}$. □

Exercise 5.4 Let $(H, (\cdot,\cdot))$ be a Hilbert space and let $A\colon H \to H$ be a symmetric linear operator that is *coercive*, i.e. such that there exists $\lambda > 0$ so that

$$(Ax, x) \geq \lambda\|x\|^2 \quad \text{for every } x \in H.$$

Show that A is an isomorphism of normed spaces and $\|A^{-1}\| \leq \lambda^{-1}$.

Solution. If $Ax = 0$, then the assumption implies $\lambda\|x\|^2 \leq (Ax, x) = 0$. Since $\lambda > 0$, we have $x = 0$ which proves that the linear map A is injective.

Let $W_A := A(H)$ be the range of A. To prove that A is surjective, let $x \in W_A^\perp$. Then

$$0 = (Ax, x) \geq \lambda\|x\|^2,$$

which implies $x = 0$. Therefore, $W_A^\perp = \{0\}$ and $\overline{W_A} = (W_A^\perp)^\perp = H$. Surjectivity of A follows if we show that W_A is closed in H because then, $W_A = \overline{W_A} = H$. Let $(y_n)_{n\in\mathbb{N}}$ be a sequence in W_A converging to some $y \in H$. Let $x_n \in H$ such that $Ax_n = y_n$. Then $(x_n)_{n\in\mathbb{N}}$ is a Cauchy-sequence in H, because for every $n, m \in \mathbb{N}$ we have

$$\begin{aligned}\lambda\|x_n - x_m\|^2 &\leq (Ax_n - Ax_m, x_n - x_m) = (y_n - y_m, x_n - x_m)\\ &\leq \|y_n - y_m\|\|x_n - x_m\|\end{aligned}$$

and $(y_n)_{n\in\mathbb{N}}$ is a Cauchy-sequence by assumption. Hence there exists $x = \lim_{n\to\infty} x_n$. The Hellinger–Töplitz theorem (Beispiel 3.3.2) implies that A is continuous. Therefore, $Ax = y$ which implies $y \in W_A$ and proves that W_A is closed in H.

We have shown that A is a continuous, bijective linear operator. The Inverse Mapping Theorem already implies that A has a continuous inverse. What remains to show is the estimate $\|A^{-1}\| \leq \frac{1}{\lambda}$ which follows from the assumption since for every $y \in H$

$$\|A^{-1}y\|^2 \leq \frac{1}{\lambda}\Big(AA^{-1}y, A^{-1}y\Big) \leq \frac{1}{\lambda}\|y\|\|A^{-1}y\|,$$

thus $\|A^{-1}y\| \leq \frac{\|y\|}{\lambda}$. $\square$

Hints to Exercises.

5.1 Given $f \in D_{\overline{A}}$ consider a sequence $(f_n)_{n\in\mathbb{N}}$ in D_A which converges to f in X. Compare $f_n(t)$ to $g(t) := \int_0^t \overline{A}f\,dx$.

5.2 One implication follows from the Inverse Mapping Theorem.

5.3 Recall that, if $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces and $A\colon D_A \subset X \to Y$ is a linear operator with closed graph, then $(D_A, \|\cdot\|_{\Gamma_A})$ is a Banach space, where $\|x\|_{\Gamma_A} = \|x\|_X + \|Ax\|_Y$ is the graph norm.

5.4 To prove surjectivity, i. e. $W_A := A(H) = H$, consider an element $x \in W_A^\perp$ and recall that $(W_A^\perp)^\perp = \overline{W_A}$.