Exercise 2

According to the French Flag model, morphogens form a gradient across a field of cells, and cells determine their fate according to the local concentration of the morphogen. In this exercise, you will solve two standard 1-component models for morphogen gradients that lead to either a linear or an exponential gradient.

1. **Review Material from the Lecture**

   - Explain how the diffusion equation is derived!
   - Describe the French Flag model!
   - Why is it important to include a degradation term in the PDE?
   - Why is it important to know the time to steady-state? What does this time depend on?

2. **Steady-state Gradient profiles.** Morphogen gradients are often considered to be stationary, i.e. their concentration profile is assumed to be in steady-state. We will now derive two standard steady-state gradient models.

   Derive the (analytic) steady-state solutions of

$$\begin{aligned} \frac{\partial c}{\partial t} &= D\Delta c \\ c(x=0,t) &= c_0, \qquad c(x=L,t)=0 \end{aligned} \tag{1}$$

   and

$$\begin{aligned} \partial c/\partial t &= D\Delta c - k\cdot c \\ c(x=0,t) &= c_0, \qquad c(x\to\infty)=0 \end{aligned} \tag{2}$$

   Hint. The first case gives a linear, the second an exponential gradient. You can transform the second order ODEs into a set of two first order ODEs by writing $z = dc/dx$. You can then eliminate dx and solve the resulting equations by standard methods.

3. **Time-dependent Solution**. Use separation of variables to solve the 1D diffusion problem

$$\begin{aligned} \partial c/\partial t &= D\Delta c - k\cdot c \\ c(x=0,t) &= c_0, \qquad c(x=L,t)=0 \\ c(x=0,0) &= c_0, \qquad c(x>0,0)=0. \end{aligned} \tag{3}$$

   Separation of variables is a powerful technique to solve initial-boundary-value problems (IBVPs), provided that

   - the PDE is linear and homogeneous (coefficients do not necessarily have to be constant)

- the boundary conditions are linear and homogeneous.

Accordingly, you need to first transform the inhomogeneous boundary conditions into homogenous boundary conditions. Proceed as follows:

i Determine the steady-state solution, c_s.

ii Solve the PDE for the function $u(x,t) = c(x,t) - c_s(x)$. Here, use the ansatz $u(x,t) = X(x) \cdot T(t)$ to separate the variables on both sides of the equation. Since the two equal expressions depend on different variables they have to be constant, say λ. This gives two linear, homogeneous ordinary differential equations for X and T that can be easily solved. Use the boundary and initial conditions to fix λ and the integration constants from the ODEs.

ii Determine $c(x,t) = c_s(x) + u(x,t)$.

iv What is the characteristic time to steady state?

4. **Numerical Solution**. Solve Eq. 3 numerically for $c_0 = 1$, $L = 50$, $D = 0.1$, $k = 10^{-5}$ and compare your results to the exact solution. Repeat the simulation for $k = 1$, for $D = 100$, and for $L = 500$. How do the solutions differ? Why? Now repeat your simulations with the initial parameterisation and $k = 0$, and discuss your solution. Finally, repeat your simulations for flux boundary conditions

$$\left.\frac{\partial c}{\partial x}\right|_{x=0} = -1, \quad \left.\frac{\partial c}{\partial x}\right|_{x=L} = 0, \tag{4}$$

with $k = 10^{-5}$ and $k = 0$. How do the solutions differ? Why?

Hint: Use the function `pdepe` in Matlab.