Some information on the Exam

Here is a list with useful information on the exam. If you have any question, please mail the course organizer.

- The exam will take place on Monday 27th January 2020, and it will be 3 hours long. For the time and place, please check mystudies in due time.
- The exam will consists in questions concerning the material presented in the lectures and on the exercise sheets. You will be asked to give definitions and statements seen in the course and to solve some exercises. Some of the exercises will ask for (reasonably short) proofs already seen in the lecture or in the exercise sheets.
- No aiding material is allowed during the exam. This includes: lecture notes, personal notes, exercise sheets, mobile phones or tablets, calculators. A **compendium** listing some theorems and useful results will be given with the exam.
- While solving the exercises, you may invoke the main results seen in the course (e.g. Hahn-Banach, the Closed Graph Theorem), their main corollaries and the results quoted in the compendium. Aside from this, every step in your argument must be proved.
- You may bring your own blank paper (A4 format).
- In order to obtain the maximal grade, it will not be necessary to solve all problems.

This Exercise sheet is thought as a mock exam. We recommend to approcach it after a thorough study of the whole course material, Exercise Sheets 1 to 12 included. The exercises are taken, with some adjustments, from past exams.

As in a regular exam, you may want to:

- Avoid the use any aid material such as notes, textbooks or friends (but you may consult the compendium given on the last page).
- Face all exercises without interruption,
- Keep track of the time.

The solution will be released, as usual, after some time. If you have any question you may always write to your assistant or to the course organizer.

Exercise 13.1 Let $(X, \|\cdot\|)$ be a Banach space over $\mathbb{C}$ and let $T \in L(X)$.

(i) Give the definition of spectral radius, of resolvent and of spectrum of T.

(ii) Assume that T satisfies $T^2 = T$. Compute the spectral radius and the spectrum of T.

Solution. (i) We refer to Satz 2.2.6 and Definition 6.5.1.

(ii) If $T = 0$, then $r_T = 0$ and $\sigma(T) = \{0\}$.

From now on we suppose $T \neq 0$. Since $T^2 = T$, it follows by induction that

$$T^n = T \quad \text{for every } n \in \mathbb{N},$$

consequently the spectral radius of T is

$$r_T = \lim_{n\to\infty} \|T^n\|_X^{1/n} = \lim_{n\to\infty} \|T\|_X^{1/n} = 1.$$

As for the spectrum of T, from Satz 6.5.4, since $T^2 - T = 0$, we get

$$\{0\} = \sigma(T^2 - T) = \{\lambda^2 - \lambda : \lambda \in \sigma(T)\},$$

consequently

$$\sigma(T) \subseteq \{0, 1\}.$$

We claim that $1 \in \sigma(T)$. Indeed, if this were not the case, $\operatorname{Id} - T$ would be invertible, but

$$(\operatorname{Id} - T)T = T^2 - T = 0, \tag{1}$$

from which it would follow that $T = 0$, which we are excluding.

We claim that $0 \in \sigma(T)$ if and only if $T \neq \operatorname{Id}$. Indeed, if $0 \notin \sigma(T)$, we have $0 \in \rho(T)$ and T would be invertible. Then by (1) it would follow that $T = \operatorname{Id}$. Conversely, if $T = \operatorname{Id}$ clearly $\sigma(T) = \{1\}$ and $0 \notin \sigma(T)$. To sum up:

$$r_T = \begin{cases} 0 & \text{if } T = 0, \\ 1 & \text{if } T \neq 0, \end{cases} \qquad \sigma(T) = \begin{cases} \{0\} & \text{if } T = 0, \\ \{1\} & \text{if } T = \operatorname{Id}, \\ \{0,1\} & \text{otherwise.} \end{cases}$$

$\square$

Exercise 13.2 Let $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ be normed spaces.

(i) Give the definition of weakly convergent sequence in X and prove that every weakly convergent sequence is bounded.

(ii) Let $L : X \to Y$ be a linear operator. Prove that L is continuous if and only if it is weak-weak sequentially continuous, namely if and only if, for every sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \xrightarrow{w} x$ in X, there holds $Lx_n \xrightarrow{w} Lx$ in Y.

(iii) Let $T \in L(X, Y)$. Use (ii) to prove that, if X is reflexive, $T\left(\overline{B_1(0)}\right)$ is closed.

Solution. (i) We refer to Definition 4.6.1 and Satz 4.6.1.

(ii) "($\Rightarrow$)": Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X such that $x_n \xrightarrow{\mathrm{w}} x$ for some $x \in X$. Let $f \in Y^*$ be arbitrary. If $T\colon X \to Y$ is a continuous linear operator, then $f \circ T \in X^*$ and weak convergence of $(x_n)_{n\in\mathbb{N}}$ implies

$$\lim_{n\to\infty} f(Tx_n) = \lim_{n\to\infty} (f \circ T)(x_n) = (f \circ T)(x) = f(Tx)$$

which proves weak convergence of $(Tx_n)_{n\in\mathbb{N}}$ in Y.

"($\Leftarrow$)" If the linear operator $T\colon X \to Y$ is not continuous, then there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that $\|x_n\|_X \le 1$ and $\|Tx_n\|_Y \ge n^2$ for every $n \in \mathbb{N}$. Then $\frac{1}{n}x_n \to 0$ in X (in particular weakly) but $(T(\frac{1}{n}x_n))_{n\in\mathbb{N}}$ is unbounded in Y and therefore cannot be weakly convergent. $\square$

(iii) Let $(y_n)_{n\in\mathbb{N}} \subset T\left(\overline{B_1(0)}\right)$ be so that $y_n \to y \in Y$, and chosen for every $n \in \mathbb{N}$ x_n, $x_n \in \overline{B_1(0)}$ with $Tx_n = y_n$.

Since $(x_n)_{n\in\mathbb{N}}$ is bounded and X is reflexive, by Eberlein-Šmulian's Theorem and the weak lower-semicontinuity of the norm (Satz 4.6.1) we may assume, up to extracting a subsequence, that $x_n \xrightarrow{w} x$ for some $x \in \overline{B_1(0)}$. Since T is contiuous, by (ii) we have $Tx_n \xrightarrow{w} Tx$ in Y.

Since we also have $Tx_n = y_n \to y$ as $n \to \infty$, we conclude $Tx = y$, and so $y \in T\left(\overline{B_1(0)}\right)$, and so $T\left(\overline{B_1(0)}\right)$ is closed.

Exercise 13.3 Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces.

(i) Prove the Banach-Steinhaus theorem: if $(T_\lambda)_{\lambda\in\Lambda}$ is a family of linear, continuous operators from X to Y so that, for every fixed $x \in X$, there holds

$$\sup_{\lambda\in\Lambda} \|T_\lambda x\|_Y < \infty.$$

then

$$\sup_{\lambda\in\Lambda} \|T_\lambda\|_{L(X,Y)} < \infty.$$

Note: You are required to prove this result without resorting to the more general Uniform Boundedness Principle seen in the course.

(ii) Let X^* be the dual of X. Let $S \subseteq X$ be a subset of X so that, for every $x^* \in X^*$,

$$x^*(S) = \{x^*(x) : x \in S\}$$

is a bounded set in $\mathbb{R}$. Prove that then S is bounded.

Solution. (i) For every $n \in \mathbb{N}$ let

$$X_n = \{x \in X : \|T_\lambda x\|_Y \leq n \text{ for every } \lambda \in \Lambda\}.$$

By assumption there then holds

$$X = \bigcup_{n=1}^{\infty} X_n,$$

and thus, since X is complete, by the Baire Category Theorem there must be some n_0 for which X_{n_0} has nonempty interior.

Let then $\varepsilon > 0$ and x_0 be so that $B_\varepsilon(x_0) \subset X_{n_0}$. It then follows that

$$\|T_\lambda(B_\varepsilon(x_0))\|_Y \leq n_0 \quad \text{for every } \lambda \in \Lambda.$$

Since the maps are linear, this implies that for every $x \in X$ with $\|x\|_X < 1$ we can estimate, for every $\lambda \in \Lambda$,

$$\|T_\lambda(x)\|_Y = \frac{1}{\varepsilon}\|T_\lambda(\varepsilon x)\|_Y \leq \frac{1}{\varepsilon}\Big(\|T_\lambda(x_0+\varepsilon x)\|_Y + \|T_\lambda(x_0)\|_Y\Big) \leq \frac{2n_0}{\varepsilon},$$

from which the thesis follows.

(ii) Consider the family of linear maps $(\delta_s)_{s\in S} \subset X^{**} = L(X^*, \mathbb{R})$ given by:

$$\delta_s : X^* \to \mathbb{R}, \qquad \delta_s(x^*) = x^*(s).$$

Since the hypothesis implies that, for every fixed x^*, there holds

$$\sup_{s\in S} |\delta_s(x^*)| < \infty,$$

by the Banach-Steinhaus theorem we deduce that

$$\sup_{s\in S} \|\delta_s\|_{X^{**}} \leq C,$$

and consequently that

$$|x^*(s)| \leq C\|s\|_X,$$

and so by the characterization of the norms in term of the dual (see the Compendium) it follows that $\|s\|_X \leq C$ for every $s \in S$, which is to say that S is bounded. □

Exercise 13.4 Let $(X, \|\cdot\|_X)$ be a normed space.

(i) Give the definition of weak sequentially closed subset $A \subseteq X$. Which of the following assertion is true (give a proof), which is false (give a counterexample)?

(a) A is weakly sequentially closed $\Longrightarrow$ A is closed.

(b) A is closed $\Longrightarrow$ A is weakly sequentially closed.

(ii) Let X, Y be Banach and let $T : X \to Y$ be a linear map so that

$$y^* \circ T \in X^* \quad \text{for every } y \in Y^*,$$

where X^*, Y^* denote the duals of X and Y respectively. Prove that the graph of T is weakly closed.

(iii) Let X, Y be Banach and let $T \in L(X, Y)$ be a linear and continuous map. Prove that $\ker(T)$ is weakly sequentially closed.

Solution. (i) We refer to Definition 4.6.3, Lemma 4.6.2 and Beispiel 4.6.2.

(ii) Let $(x_n, Tx_n)_{n\in\mathbb{N}} \subset \Gamma(T)$ be a sequence with $(x_n, Tx_n) \xrightarrow{w} (x, y)$ in $X \times Y$, and suppose that $Tx \neq y$. As a consequence of Hahn-Banach theorem (Satz 4.2.3) it is possible to find $y^* \in Y^*$ so that $y^*(Tx) \neq y^*(y)$. But at the same time

$$y^*(y) = \lim_{n\to\infty} y^*(Tx_n) = \lim_{n\to\infty} (y^* \circ T)(x_n) = (y^* \circ T)(x),$$

which is a contradiction.

(iii) Let $(x_n)_{n\in\mathbb{N}} \subset \ker(T)$ be so that $x_n \overset{w}{\rightharpoonup} x$ in X. Then for every $y^* \in Y^*$, $y^* \circ T \in X^*$ and so

$$0 = y^*(Tx_n) \quad\Longrightarrow\quad 0 = \lim_{n\to\infty}(y^* \circ T)(x_n) = (y^* \circ T)(x).$$

Since y^* is arbitrary, again as a consequence of Hahn-Banach, it follows that $Tx = 0$ i.e. $x \in \ker(T)$. □

Exercise 13.5

(i) State the Principle of Calculus of Variations ("Variationsprinzip").

Let now $(H, \langle\cdot,\cdot\rangle)$ be a (nonempty) Hilbert space, let $p \in (1,\infty)$ be fixed and define the function

$$F : H \to \mathbb{R}, \qquad F(u) = \frac{\|u\|_H^p}{p}.$$

(ii) Prove that, for every fixed $v \in H$, the quantity

$$G(v) = \sup_{u\in H}\Big(\langle u, v\rangle - F(u)\Big)$$

is finite and there always exists some $u \in H$ that attains the supremum on the right-hand side.

(iii) Find an explicit expression for G involving only p and the norm of v.

Solution. (i) We refer to Satz 5.4.1.

(ii) Since

$$\sup_{u\in H}\Big(\langle u, v\rangle - F(u)\Big) = \inf_{u\in H}\Big(F(u) - \langle u, v\rangle\Big),$$

the plan is to apply the Principle of Calculus of Variations to the function

$$J : H \to \mathbb{R}, \qquad J(u) = J_v(u) = F(u) - \langle u, v\rangle.$$

Clearly H is reflexive, nonempty and weakly sequentially closed.

Let us prove that J is coercive: with Cauchy-Schwarz we get

$$\begin{aligned} J(u) &= F(u) - \langle u, v\rangle \\ &\geq \frac{\|u\|_H^p}{p} - \|u\|_H\|v\|_H \\ &= \|u\|_H\left(\frac{\|u\|_H^{p-1}}{p} - \|v\|_H\right) \to \infty \quad \text{as } \|u\|_H \to \infty, \end{aligned}$$

which proves coercivity.

Let us prove that J is weakly sequentially lower semi-continuous. Since J is continuous (as composition of continuous functions), and H trivially convex, it is enough to prove that J is convex (Beispiel 5.4.1). The function $\langle \cdot, v\rangle$ is linear and so trivially convex, so it is enough to prove that F is convex.

To this aim, it suffices to note that $F(u) = f(\|u\|_H)$ where

$$f : \mathbb{R}_+ \to \mathbb{R}, \qquad f(x) = x^p,$$

which is inceasing ($f'(x) \geq 0$) and convex ($f''(x) \geq 0$). Then for every $u_1, u_2 \in H$ and every $\lambda \in [0,1]$ we have

$$\begin{aligned} F(\lambda u_1 + (1-\lambda)v_1) &= \frac{1}{p} f\left(\|\lambda u_1 + (1-\lambda)u_2\|_H\right) \\ &\leq \frac{1}{p} f\left(\lambda \|u_1\|_H + (1-\lambda)\|u_2\|_H\right) \\ &\leq \lambda \frac{f(\|u_1\|_H)}{p} + (1-\lambda)\frac{f(\|u_2\|_H)}{p} \\ &= \lambda F(u_1) + (1-\lambda)F(u_2), \end{aligned}$$

which proves the convexity of F. Consequently, J is convex, and thus w.l.s.c. In particular, by the Principle of Calculus of Variations J attains its minimums in H, which proves at once that $G(v)$ is finite for every fixed v and that the supremum in the given expression is attained by the minimum of $J = J_v$.

(iii) Denote by u_v the minimum for $J = J_v$ computed in (ii), so that there holds

$$G(v) = J_v(u_v) = \inf_{u \in H} J_v(u),$$

We then consider function, defined for every fixed $w \in H$,

$$\varphi : \mathbb{R} \to \mathbb{R}, \qquad \varphi(t) = J_v(u_v + tw),$$

which satisfies $\varphi(0) = J_v(u_v)$ and has a minimum in $t = 0$. It will consequently satisfy $\varphi'(0) = 0$ provided it is differentiable at 0, and this will give an equation for u_v. By writing

$$\varphi(u_v + tw) = \frac{(\|u_v + tw\|_H^2)^{\frac{p}{2}}}{p} - (u_v + tw, v)_H$$

we see that φ is differentiable with

$$0 = \frac{d}{dt}\varphi(u_v + tw)\bigg|_{t=0} = \|u_v\|_H^{p-2}(u_v, w) - (w, v)_H \tag{2}$$

for every $w \in H$, and so

$$v = u_v \|u_v\|_H^{p-2} \quad \Longrightarrow \quad \|u_v\|_H = \|v\|_H^{\frac{1}{p-1}}.$$

So we have deduced

$$u_v = v\|v\|_H^{\frac{2-p}{p-1}}.$$

Setting $w = u_v$ in (2) gives

$$(u_v, v) = \|u\|_H^{p-2}\|u\|_H^2 = \|u\|_H^p,$$

whence

$$G(v) = (u_v, v) - F(u_v) = \|u\|_H^p - \frac{\|u\|_H^p}{p} = \frac{(p-1)\|u_v\|_H^p}{p} = \frac{p-1}{p}\|v\|_H^{\frac{p}{p-1}}.$$

□

Exercise 13.6 Let X, Y be Banach spaces and let $T \in L(X, Y)$ be a linear continuous operator with closed image.

(i) Prove that every topologically complemented (see the Compendium) subspace is closed.

(ii) Prove the equivalence of the following statements:

(a) $\mathrm{im}(T)$ and $\ker(T)$ are topologically complemented,

(b) There exists $S \in L(Y, X)$ with

$$STS = S \quad \text{and} \quad TST = T.$$

Solution. (i) We refer to Exercise 4.3.

(ii) $(a) \Rightarrow (b)$: Since $\ker(T)$ is topologically complemented, there is a subspace $X_0 \subseteq X$ with $X = \ker(T) \oplus X_0$, and X_0 is closed. From Exercise 4.3 there exists a continuous linear map $P : Y \to Y$ with $P^2 = P$ and $\mathrm{im}(P) = \mathrm{im}(T)$.

The restriction

$$T|_{X_0} : X_0 \to \mathrm{im}(T)$$

is then bijective and continuous; since X_0 and $\mathrm{im}(T)$ are Banach spaces, by the Open Mapping Theorem (Satz 3.2.1) its inverse

$$L := (T|_{X_0})^{-1} : \mathrm{im}(T) \to X_0$$

is then continuous. We claim that the required map is $S := LP : Y \to X$. Indeed, on the one hand for every $y \in Y$ there holds

$$STS(y) = SP(y) = LP^2(y) = LP(y) = S(y).$$

on the other hand, for every $x \in X$

$$TST(x) = T(T^{-1}|_{X_0})PT(x) = T(x).$$

$(b) \Rightarrow (a)$: We prove first that $\ker(T)$ is topologically complemented. By proving the existence of the projection operator as in Exercise 4.3. We claim that such operator is

$$P : X \to X, \qquad P := 1 - ST.$$

There holds

$$P^2 = (1 - ST)^2 = 1 - 2ST + (STS)T = 1 - ST = P,$$

and for every $x \in \ker(T)$ there holds $P(x) = x - S(0) = x$. Consequently $\ker(T) \subseteq \mathrm{im}(P)$. On the other hand, for every $x \in X$ we have

$$TP(x) = T(x) - TST(x) = T(x) - T(x) = 0,$$

and so $\mathrm{im}(P) \subseteq \ker(T)$ which gives $\mathrm{im}(P) = \ker(T)$. This proves that $\ker(T)$ is topologically complemented.

Let us prove that $\mathrm{im}(T)$ is topologically complemented in the same way. We set

$$Q : Y \to Y, \qquad Q := TS$$

There holds

$$Q^2 = (TS)^2 = T(STS) = TS = Q,$$

and by definition there holds $\mathrm{im}(Q) \subseteq \mathrm{im}(T)$. On the other hand for every $x \in X$ there holds

$$Q(T(x)) = TST(x) = T(x),$$

hence $\mathrm{im}(T) = \mathrm{im}(Q)$. This gives that $\mathrm{im}(T)$ is topologically complemented. □

A Compendium of Functional Analysis

1. *Topological complements* A linear subspace of a normed space $V \subset X$ is complemented if and only if there exists a linear, continuous map $P : X \to X$ so that $P(X) = V$ and $P^2 = P$.

2. *Baire Theorem* Let (M, d) be a complete metric space. If $(D_n)_{n\in\mathbb{N}}$ is a sequence of closed subsets of M so that $\bigcup_{n\in\mathbb{N}} D_n$ has nonempty interior, then at least one of the D_n's has nontempty interior.

3. *Open Mapping Theorem* Let X, Y be Banach spaces and let $L \in L(X, Y)$ be surjective. Then L is an open mapping. In particular, if L is bijective, then $L^{-1} \in L(Y, X)$.

4. *Hahn-Banach Theorem* Let X be a (real) normed space, let $V \subset X$ be a linear subspace and let $L : V \to \mathbb{R}$ be linear with $L(x) \leq p(x)\ \forall x \in V$, where $p : X \to \mathbb{R}$ is a sublinear functional. Then there exists an extension of L, $L_{\text{ext}} : X \to \mathbb{R}$ so that $L_{\text{ext}}(x) \leq p(x)\ \forall x \in X$.

5. *Norms and Duality* Let X be a Banach space and let X^* be its dual. Then for every $x \in X$ there holds $\|x\|_X = \sup\{|x^*(x)| : x \in X^*, \|x^*\|_{X^*} \leq 1\}$.

6. *Eberlein-Šmulian Theorem* Let X be a reflexive Banach space and let $(x_n)_{n\in\mathbb{N}} \subset X$ be a bounded sequence. Then there exists a subsequence $(x_n)_{n\in\Lambda}$, $\Lambda \subset \mathbb{N}$ which is weakly convergent: $x_n \xrightarrow{w} x$ as $n \to \infty$, $n \in \Lambda$.

7. *Weak s.l.s.c. of the Norm* Let X be a Banach space and and let $(x_n)_{n\in\mathbb{N}} \subset X$ be so that $x_n \xrightarrow{w} x$. Then $\|x\|_X \leq \liminf_{n\to\infty} \|x_n\|_X$.