$$\frac{d}{dz}\log\left(\frac{b}{a^{\xi/D}}\right) = \frac{b'}{b} - \frac{\xi}{D}\frac{a'}{a}$$

which looks like the (1)

Exercise 10

Diffusion form (effect of random

1. **Chemotaxis: Travelling Waves**. Consider the following model for cells that migrate in response to a chemoattractant

$$
\begin{aligned}
\partial b/\partial t &= \frac{\partial}{\partial x}\left[D\frac{\partial b}{\partial x} - \frac{\xi b}{a}\frac{\partial a}{\partial x}\right] \\
\partial a/\partial t &= -kb \qquad (1)
\end{aligned}
$$

chemo movement

consumed by cell

sensitivity

where $b(x,t)$ and $a(x,t)$ are cell density and chemoattractant concentration respectively at a time t and a position x. D, ξ, and k are positive parameters which should be assumed constant.

chemo attractant $\propto -kb$ cell density

$z = x - ct$

$dz = dx$,
$dz = -c\,dt$

i) Briefly explain the biological meaning of each term.

ii) Derive equations satisfied by travelling wave solutions of the form $a(z)$, $b(z)$, $z = x - ct$ (c constant) joining $(a,b) = (0,0)$ at $z \to -\infty$ to $(a,b) = (1,0)$ at $z \to \infty$, and find the relation between $b(z)$ and $a(z)$.

iii) For the case $\xi/D = 2$, find b and a explicitly in terms of z, showing carefully that your solution satisfies the boundary conditions.

Plug in:

$$-c\frac{db(z)}{dz} = \frac{d}{dz}\left[D\frac{db}{dz} - \frac{\xi b(z)}{a(z)}\frac{da(z)}{dz}\right]$$

$$\Rightarrow \frac{a''}{a'} = \frac{b''}{b'}$$

iv) For the case $\xi/D = 2$ and $a(0) = 1/2$, show that $b(z)$ is symmetric about the axis $z = 0$. Sketch a and b as functions of z and briefly describe what is happening biologically.

$$\frac{b''}{b'} = \frac{a''}{a'} \Rightarrow \frac{d}{dz}\log(a')$$

$$\frac{da(z)}{dz} = -ck\,b(z)$$

$$\Rightarrow \frac{d}{dz}\left[cb + Db' - \frac{\xi b}{a}a'\right] = 0$$

Integration

$$\left[cb + Db' - \frac{\xi b}{a}a'\right] = C$$

2. **Chemotaxis: Cell Aggregation**[1]. Consider the following model for cells that secrete a chemoattractant and migrate towards this chemoattractant

$$
\begin{aligned}
\partial n/\partial t &= D_n\frac{\partial^2 n}{\partial x^2} - \xi\frac{\partial}{\partial x}\left(n\frac{\partial a}{\partial x}\right) \\
\partial a/\partial t &= hn - ka + D_a\frac{\partial^2 a}{\partial x^2} \qquad (2)
\end{aligned}
$$

where $n(x,t)$ and $a(x,t)$ are cell density and chemoattractant concentration respectively at a time t and a position x. D_n, D_a, ξ, h and k are positive parameters which should be assumed constant.

i) Non-dimensionalize the system to obtain

$$
\begin{aligned}
\partial n/\partial t &= D_n\frac{\partial^2 n}{\partial x^2} - \xi\frac{\partial}{\partial x}\left(n\frac{\partial a}{\partial x}\right) \\
\partial a/\partial t &= n - a + D_a\frac{\partial^2 a}{\partial x^2} \qquad (3)
\end{aligned}
$$

where the variables and parameters are now non-dimensional.

ii) Consider (a) a finite domain with zero flux boundary conditions, and (b) an infinite domain. Examine the linear stability about the steady state (which introduces a further parameter here), derive the dispersion relation and discuss the role of the various parameter groupings. Hence obtain the conditions on the parameters and domain size for the mechanism to initiate spatially heterogenous solutions.

iii) Using ξ as bifurcation parameter, determine the critical wave length when the system bifurcates to spatially structured solutions in the finite domain. Also examine the bifurcating instability as the domain length increases.

iv) Briefly describe the physical processes operating and explain intuitively how spatial aggregation takes place.

[1] This question is based on problem 2.9 in Murray, Mathematical Biology Vol II, (2003) Springer