In this set of exercises, an *optimal transport map* should be understood as an optimal transport plan that is also a map (so, it must be optimal among all possible transport plans). The quadratic cost is $c(x,y) = \frac{1}{2}|x-y|^2$.

The exercises 2.5, 2.6, 2.7, 2.8 are facultative and are meant to guide you through a full proof of the disintegration theorem (that was stated in class without proof).

Exercise 2.1. Let $S : \mathbb{R}^d \to \mathbb{R}^d$ be the function $S(x) := -x$. Characterize the probabilities $\mu \in \mathcal{P}(\mathbb{R}^d)$ with compact support such that S is an optimal transport map between μ and $S_\#\mu$ with respect to the quadratic cost.

Exercise 2.2. Let $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$ be two compactly supported probability measures invariant under rotations (that is $\mu(L(E)) = \mu(E)$ and $\nu(L(E)) = \nu(E)$ for any Borel set $E \in \mathscr{B}(\mathbb{R}^d)$ and any orthogonal transformation $L \in O(d)$). Show that, if $\mu \ll \mathcal{L}^d$, the optimal transport map from μ to ν with respect to the quadratic cost can be written as $x \to \lambda(|x|)\frac{x}{|x|}$ where $\lambda : [0,\infty) \to [0,\infty)$ is a suitable nondecreasing function.
Hint: The function λ is the monotone transport map between two suitable 1-dimensional measures.

Exercise 2.3 (Middle point)**.** Given two probability measures $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$, let $\mathcal{C}(\mu,\nu)$ be the infimum of the Kantorovich problem with respect to the quadratic cost

$$\mathcal{C}(\mu,\nu) := \inf_{\gamma \in \Gamma(\mu,\nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|x-y|^2}{2} \, \mathrm{d}\gamma(x,y) \,.$$

Let $\mu_0, \mu_1 \in \mathcal{P}(\mathbb{R}^d)$ be two probability measures with compact support. A probability measure $\mu_{\frac{1}{2}}$ is a middle point of μ_0 and μ_1 if $\mathcal{C}(\mu_0, \mu_{\frac{1}{2}}) = \mathcal{C}(\mu_1, \mu_{\frac{1}{2}}) = \frac{1}{4}\mathcal{C}(\mu_0,\mu_1)$.

(a) If $\mu_0 = \delta_{p_0}$ and $\mu_1 = \delta_{p_1}$, show that the middle point is unique and $\mu_{\frac{1}{2}} = \delta_{\frac{p_1+p_2}{2}}$.

(b) Prove that there is always at least one middle point.

(c) Find two probability measures μ_0, μ_1 such that they have more than one middle point.

(d) Show that if the optimal transport plan between μ_0 and μ_1 is unique, then there is a unique middle point.

(e) Prove that if $\mu_0, \mu_1 \ll \mathcal{L}^d$, then the middle point is unique and $\mu_{\frac{1}{2}} \ll \mathcal{L}^d$.

Hint: To solve (d), glue the plans from μ_0 to $\mu_{\frac{1}{2}}$ and from $\mu_{\frac{1}{2}}$ to μ_1 to obtain a plan from μ_0 to μ_1.

To solve (e), use the fact that the gradient of a strongly convex function is bi-Lipschitz (both the gradient and its inverse are Lipschitz-continuous).

Exercise 2.4. Consider n red points $P_1, \dots, P_n$ and n blue points $Q_1, \dots, Q_n$ on the plane. Assume that these $2n$ points are distinct and there are no 3 collinear points.

Show that it is possible to connect each red point to a distinct blue point with a segment in such a way that these segments do not intersect each other. Namely, there exists a permutation $\sigma \in S_n$ such that the segment $P_i Q_{\sigma(i)}$ does not intersect the segment $P_j Q_{\sigma(j)}$ for any $i \neq j$.

Exercise 2.5 (Easy disintegration)**.** Let $\mu \in \mathscr{M}(\mathbb{R}^2)$ be a finite measure on $\mathbb{R}^2$ that is absolutely continuous with respect to the Lebesgue measure with density $\rho : \mathbb{R}^2 \to \mathbb{R}$. Let $\nu \in \mathscr{M}(\mathbb{R})$ be the measure with density $\eta(x) := \int_{\mathbb{R}} \rho(x,y)dy$. For any $x \in \mathbb{R}$ such that $\eta(x) \neq 0$, let μ_x be the measure with density $\rho_x(y) := \frac{\rho(x,y)}{\eta(x)}$. If $\eta(x) = 0$, then simply set $\mu_x := 0$.

Show that for any $g \in L^1(\mu)$ it holds

$$\int_{\mathbb{R}^2} g(x,y)d\mu(x,y) = \int_{\mathbb{R}} \int_{\mathbb{R}} g(x,y)d\mu_x(y)d\nu(x).$$

Exercise 2.6 (Disintegration for product of compact spaces)**.** Let X, Y be two compact spaces and let $\mu \in \mathscr{M}(X \times Y)$ be a finite measure on the product $X \times Y$. Let us denote $\nu := (\pi_1)_\#\mu$ where $\pi_1 : X \times Y \to X$ is the projection on the first coordinate. Prove that there exists a family of probabilities $(\mu_x)_{x\in X} \subseteq \mathcal{P}(Y)$ such that:

(a) For any Borel set $E \in \mathscr{B}(Y)$ the map $x \mapsto \mu_x(E)$ is Borel.

(b) For any $g \in L^1(\mu)$ it holds

$$\int_{X\times Y} g(x,y)d\mu(x,y) = \int_X \int_Y g(x,y)d\mu_x(y)d\nu(x).$$

Hint:

1. Given $\psi \in C^0(Y)$, consider the map $A_\psi : L^1(X,\nu) \to \mathbb{R}$ given by the formula $A_\psi(\phi) := \int_{X\times Y} \phi(x)\psi(y)d\mu(x,y)$. Prove that the said map is linear continuous and therefore $A_\psi \in L^\infty(X,\nu)$.

2. Fix a countable dense subset $S \subseteq C^0(Y)$. Prove that for ν-almost every $x \in X$ the map $\mu_x : S \to \mathbb{R}$ given by $\mu_x(\psi) := A_\psi(x)$ is linear continuous and therefore $\mu_x \in \mathcal{P}(Y)$. Show that the said family $(\mu_x)_{x\in X}$ satisfies *(a)*.

3. Show that *(b)* holds when $g \in L^1(X,\nu) \times S$. Show that this implies that it holds also when $g \in L^1(X,\nu) \times C^0(Y)$. Finally show that this implies *(b)* for any $g \in L^1(\mu)$.

Exercise 2.7 (Disintegration for product of Polish spaces)**.** Show the statement of the previous exercise when X and Y are Polish spaces, i.e. they are complete and separable.
Hint: Use Prokhorov's theorem to find a suitable exhaustion in compact sets that allows to apply the previous exercise.

Exercise 2.8 (Disintegration for fibers of a map)**.** Let X, Y be two Polish spaces (complete and separable), let $f : Y \to X$ be a Borel map and let $\mu \in \mathscr{M}(Y)$ be a finite measure on Y. Let us denote $\nu := f_\#\mu$. Show that there exists a family of probabilities $(\mu_x)_{x\in X} \subseteq \mathcal{P}(Y)$ such that:

- For any Borel set $E \in \mathscr{B}(Y)$ the map $x \mapsto \mu_x(E)$ is Borel.
- For ν-almost every $x \in X$ the measure μ_x is supported on the fiber $f^{-1}(x)$.
- For any $g \in L^1(\mu)$ it holds

$$\int_Y g(y)d\mu(y) = \int_X \int_{f^{-1}(x)} g(y)d\mu_x(y)d\nu(x).$$

Hint: Apply the previous exercise on the measure $(f \times \mathrm{id})_\#\mu$.