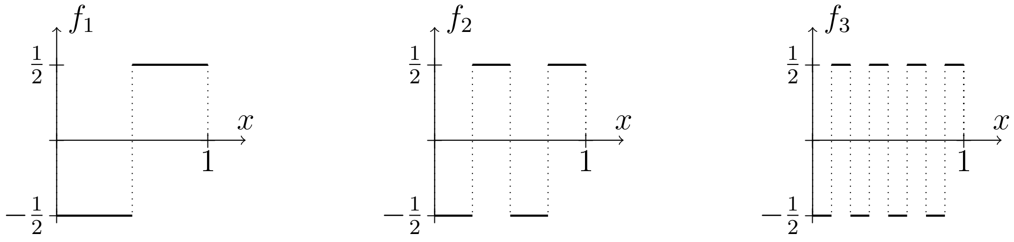
Exercise 10.1 For each of the Banach spaces below (each one endowed with its standard norm), find a sequence which is bounded but does not have a convergent subsequence:

(i) $L^p((0,1),\mathbb{R})$ for $1 \leq p \leq \infty$;

(ii) $c_0 \subset \ell^\infty$, the space of sequences converging to zero.

Solution. (i) Given $n \in \mathbb{N}$, we divide the interval $[0,1]$ into 2^n subintervals $I_1, \ldots, I_{2^n}$ of equal length, and define the function $f_n\colon [0,1] \to \mathbb{R}$ on each I_k to be $-\frac{1}{2}$ if k is odd and $+\frac{1}{2}$ if k is even. More precisely,

$$f_n(x) = \begin{cases} -\frac{1}{2}, & \text{if } \exists k \in \mathbb{N}:\ 2^n x \in [2k-2, 2k-1[\\ \frac{1}{2}, & \text{else.} \end{cases}$$



By construction, $\|f_n\|_{L^p([0,1])} = \frac{1}{2}$ for every $n \in \mathbb{N}$ and every $1 \leq p \leq \infty$. Therefore, the sequence $(f_n)_{n\in\mathbb{N}}$ is bounded in $L^p([0,1])$. However by construction, for *any* pair $n, m \in \mathbb{N}$ with $n \neq m$ the difference $|f_n - f_m|$ is the characteristic function of a union of subintervals whose lengths sum up to $\frac{1}{2}$. In particular, $\|f_n - f_m\|_{L^p([0,1])} = (\frac{1}{2})^{\frac{1}{p}}$ for $1 \leq p < \infty$ and $\|f_n - f_m\|_{L^\infty([0,1])} = 1$. Consequently, $(f_n)_{n\in\mathbb{N}}$ cannot have any convergent subsequence.

(ii) Given $n \in \mathbb{N}$, let $e_n \in c_0$ be given by $e_n = (0, \ldots, 0, 1, 0, \ldots)$, where the 1 is at n-th position. Then the sequence $(e_n)_{n\in\mathbb{N}}$ is bounded in $(c_0, \|\cdot\|_{\ell^\infty})$ since $\|e_n\|_{\ell^\infty} = 1$ for every $n \in \mathbb{N}$. However, for *any* pair $n, m \in \mathbb{N}$ with $n \neq m$ we have $\|e_n - e_m\|_{\ell^\infty} = 1$. Consequently, $(e_n)_{n\in\mathbb{N}}$ cannot have any convergent subsequence. □

Exercise 10.2 Prove that the following statements are equivalent.

(i) $(X, \|\cdot\|_X)$ is separable.

(ii) $B = \{x \in X \mid \|x\|_X \leq 1\}$ is separable.

(iii) $S = \{x \in X \mid \|x\|_X = 1\}$ is separable.

Solution. Since subsets of separable sets are separable (Satz 5.2.1), from $S \subset B \subset X$ we immediately deduce (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

"(ii) $\Rightarrow$ (i)": By assumption, there exists a countable dense subset $D \subset S$. Moreover, as countable union of countable sets,

$$E := \bigcup_{q \in \mathbb{Q}} qD = \{qd \in X \mid q \in \mathbb{Q},\ d \in D\}$$

is countable. We claim is that $E \subset X$ is dense. Let $x \in X$ and $\varepsilon > 0$ be arbitrary. Since $0 \in E$, we may assume $x \neq 0$ and consider the element $\frac{x}{\|x\|_X} \in S$. Since $D \subset S$ is dense, there exists $d \in D$ such that

$$\left\| d - \tfrac{x}{\|x\|_X} \right\|_X < \frac{\varepsilon}{2\|x\|_X}.$$

Moreover, since $\|x\|_X \in \mathbb{R}$ and since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists $q \in \mathbb{Q}$ such that

$$\Big| q - \|x\|_X \Big| < \frac{\varepsilon}{2}.$$

Using $D \subset S \Rightarrow \|d\|_X = 1$ and combining the inequalities, the point $qd \in E$ satisfies

$$\begin{aligned} \|qd - x\|_X &= \Big\| (q - \|x\|_X) d + \|x\|_X d - x \Big\|_X \\ &\leq \Big| q - \|x\|_X \Big| + \Big\| \|x\|_X d - x \Big\|_X < \frac{\varepsilon}{2} + \frac{\varepsilon \|x\|_X}{2\|x\|_X} = \varepsilon, \end{aligned}$$

which proves that $E \subset X$ is dense. Since E is countable, we have shown that X is separable. $\square$

Exercise 10.3 Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ be normed spaces. Recall that if $T \in L(X,Y)$, then its dual operator T^* is in $L(Y^*, X^*)$ and it is characterised by the property

$$\langle T^* y^*, x \rangle_{X^* \times X} = \langle y^*, Tx \rangle_{Y^* \times Y} \quad \text{for every } x \in X \text{ and } y^* \in Y* .$$

Prove the following facts about dual operators.

(i) $(\mathrm{Id}_X)^* = \mathrm{Id}_{X^*}$.

(ii) If $T \in L(X,Y)$ and $S \in L(Y,Z)$, then $(S \circ T)^* = T^* \circ S^*$.

(iii) If $T \in L(X,Y)$ is bijective with inverse $T^{-1} \in L(Y,X)$, then $(T^*)^{-1} = (T^{-1})^*$.

(iv) Let $\mathcal{I}_X \colon X \hookrightarrow X^{**}$ and $\mathcal{I}_Y \colon Y \hookrightarrow Y^{**}$ be the canonical inclusions. Then,

$$\forall\, T \in L(X,Y) : \quad \mathcal{I}_Y \circ T = (T^*)^* \circ \mathcal{I}_X.$$

Solution. (i) Let $x \in X$ and $x^* \in X^*$ be arbitrary. By definition of $(\mathrm{Id}_X)^* \colon X^* \to X^*$, we have

$$\Big\langle (\mathrm{Id}_X)^* x^*, x \Big\rangle_{X^* \times X} = \Big\langle x^*, \mathrm{Id}_X\, x \Big\rangle_{X^* \times X} = \langle x^*, x \rangle_{X^* \times X^*}.$$

Since $x \in X$ is arbitrary, $(\mathrm{Id}_X)^* x^* = x^*$. Since $x^* \in X^*$ is arbitrary, $(\mathrm{Id}_X)^* = \mathrm{Id}_{(X^*)}$.

(ii) Let $z^* \in Z^*$ and $x \in X$ be arbitrary. Then, $(S \circ T)^* = T^* \circ S^*$ follows from

$$\begin{aligned}\Big\langle (S \circ T)^* z^*, x \Big\rangle_{X^* \times X} &= \Big\langle z^*, S(Tx) \Big\rangle_{Z^* \times Z} \\ &= \Big\langle S^* z^*, Tx \Big\rangle_{Y^* \times Y} = \Big\langle T^*(S^* z^*), x \Big\rangle_{X^* \times X}.\end{aligned}$$

(iii) To prove $(T^*)^{-1} = (T^{-1})^*$, we apply the results from i and ii and obtain

$$\begin{aligned}T^* \circ (T^{-1})^* &= (T^{-1} \circ T)^* = (\mathrm{Id}_X)^* = \mathrm{Id}_{X^*}, \\ (T^{-1})^* \circ T^* &= (T \circ T^{-1})^* = (\mathrm{Id}_Y)^* = \mathrm{Id}_{Y^*}\,.\end{aligned}$$

(iv) Let $x \in X$ and $y^* \in Y^*$ be arbitrary. Then, $(\mathcal{I}_Y \circ T) = (T^*)^* \circ \mathcal{I}_X$ follows from

$$\begin{aligned}\Big\langle (\mathcal{I}_Y \circ T)x, y^* \Big\rangle_{Y^{**} \times Y^*} &= \Big\langle y^*, Tx \Big\rangle_{Y^* \times Y} = \Big\langle T^* y^*, x \Big\rangle_{X^* \times X} \\ &= \Big\langle \mathcal{I}_X x, T^* y^* \Big\rangle_{X^{**} \times X^*} = \Big\langle (T^*)^*(\mathcal{I}_X x), y^* \Big\rangle_{Y^{**} \times Y^*}.\end{aligned}$$

□

Exercise 10.4 Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces and $T \in L(X, Y)$. Prove the following.

(i) If T is an isomorphism with $T^{-1} \in L(Y, X)$, then T^* is an isomorphism.

(ii) If T is an isometric isomorphism, then T^* is an isometric isomorphism.

(iii) If X and Y are both reflexive, then the reverse implications of i and ii hold.

(iv) If $(X, \|\cdot\|_X)$ is a reflexive Banach space isomorphic to the normed space $(Y, \|\cdot\|_Y)$, then Y is reflexive.

Solution. (i) The dual operator T^* of any $T \in L(X, Y)$ with $T^{-1} \in L(Y, X)$ is invertible according to Exercise 10.3 (iii) and its inverse is $(T^*)^{-1} = (T^{-1})^*$. Moreover, the assumption $T^{-1} \in L(Y, X)$ implies $(T^{-1})^* \in L(X^*, Y^*)$. Hence, T^* is an isomorphism.

(ii) If T is an isometric isomorphism, then T^* is an isomorphism by (i) and

$$\|T^*y^*\|_{X^*} = \sup_{\|x\|_X \leq 1} \left|\langle T^*y^*, x\rangle_{X^*\times X}\right| = \sup_{\|Tx\|_Y = \|x\|_X \leq 1} \left|\langle y^*, Tx\rangle_{Y^*\times Y}\right| = \|y^*\|_{Y^*}.$$

(iii) If X and Y are reflexive, $\mathcal{I}_X \colon X \to X^{**}$ and $\mathcal{I}_Y \colon Y \to Y^{**}$ are bijective isometries. If T^* is an (isometric) isomorphism, then Exercise 10.3 and (ii) imply that $(T^*)^*$ is an (isometric) isomorphism. Applying Exercise 10.3 (iv), we see that the same holds for

$$T = \mathcal{I}_Y^{-1} \circ (T^*)^* \circ \mathcal{I}_X.$$

(iv) Since X is reflexive by assumption, $\mathcal{I}_X$ is an isomorphism. Suppose, $T\colon X \to Y$ is an isomorphism. Applying part (ii) twice, $(T^*)^*$ is an isomorphism. Moreover,

$$\mathcal{I}_Y = (T^*)^* \circ \mathcal{I}_X \circ T^{-1}$$

according to Exercise 10.3 (iv). Since $\mathcal{I}_Y$ is a composition of isomorphisms, Y is reflexive. □

Exercise 10.5 Let let $\Omega \subset \mathbb{R}^m$ be an open, bounded subset. For fixed $g \in L^2(\mathbb{R}^m)$, we define the map $V\colon L^2(\Omega) \to \mathbb{R}$ by

$$V(f) = \int_\Omega \int_\Omega g(x-y) f(y) f(x)\, dy\, dx,$$

and for fixed $h \in L^2(\Omega)$ we define the map $E\colon L^2(\Omega) \to \mathbb{R}$ by

$$E(f) = \|f-h\|^2_{L^2(\Omega)} + V(f).$$

(i) Check that V is well-defined, namely that the integral is absolutely convergent for every f and g.

(ii) Prove that V is weakly sequentially continuous.

(iii) Under the assumption $g \geq 0$ almost everywhere, prove that E restricted to

$$L^2_+(\Omega) := \{f \in L^2(\Omega) \mid f(x) \geq 0 \text{ for almost every } x \in \Omega\}$$

attains a global minimum.

Solution. (i) Extending f by zero we write

$$\int_\Omega g(x-y)f(y)dy = \int_{\mathbb{R}^m} g(x-y)f(y)dy = g * f\,(x),$$

the convolution between f and g. By Hölder's inequality we see that

$$|(g*f)(x)| \le \|g\|_{L^2(\mathbb{R}^m)}\|f\|_{L^2(\Omega)} \quad \text{for every } x \in \mathbb{R}^m,$$

hence $g*f \in L^\infty(\mathbb{R}^m)$. Since Ω has finite measure then we have the inclusion $L^2(\Omega) \subset L^1(\Omega)$ and so $V(f)$ is well-defined since the integrand is in $L^1(\Omega)$ (one may also note that $g*f$ is a continuous function).

(ii) *Claim 1:* if $(f_k)_{k\in\mathbb{N}}$ is a sequence in $L^2(\Omega)$ such that $f_k \xrightarrow{\mathrm{w}} f$ in $L^2(\Omega)$ as $k\to\infty$, then $g*f_k \to g*f$ in $L^2(\Omega)$ as $k\to\infty$. That is, the operator $f \mapsto g*f$ is *compact* from $L^2(\Omega)$ to $L^2(\Omega)$,

Proof. Note first that $g*f_k$ converges pointwise a.e. to $g*f$. Indeed, for a.e. fixed x we may write

$$(g*f_k)(x) - (g*f)(x) = g*(f_k-f)(x) = \int_\Omega g(x-y)(f_k-f)(y)dy,$$

and, the right-hand side converges to zero as $k\to\infty$ by definition of weak convergence. Note now that, since $(f_k)_{k\in\mathbb{N}}$ is weakly convergent, it is bounded in $L^2(\Omega)$ (Satz 4.6.1), and so as in (i) we have $\|g*f_k\|_{L^2(\Omega)} \le C$, uniformly in $k\in\mathbb{N}$. Thus $g*f_k$ converges to $g*f$ in $L^2(\Omega)$ by the Dominated Convergence Theorem. □

Now let $(f_k)_{k\in\mathbb{N}} \subset L^2(\Omega)$ be weakly convergent to f. We may write

$$\begin{aligned} V(f_k) &= \int_\Omega (g*f_k)(x)f_k(x)dx \\ &= \int_\Omega (g*f)(x)f_k(x)dx + \int_\Omega (g*(f_k-f))(x)f_k(x)dx, \end{aligned}$$

so that, as $k\to\infty$, the first term converges to $V(f)$ by definition of weak convergence, and the second vanishes as a consequence of Claim 1.

(iii) Since $L^2(\Omega)$ is reflexive (being a Hilbert space), we verify all the conditions to apply the direct method (Satz 5.4.1).

Claim 2. $L^2_+(\Omega)$ is non-empty and weakly sequentially closed.

Proof. Clearly, $L^2_+(\Omega) \ni 0$ is non-empty. Let $(f_k)_{k\in\mathbb{N}}$ be a sequence in $L^2_+(\Omega)$ such that $f_k \xrightarrow{\text{w}} f$ for some $f \in L^2(\Omega)$ as $k \to \infty$. Suppose $f \notin L^2_+(\Omega)$. Then there exists $U \subset \Omega$ with positive measure such that $f|_U < 0$. In particular, we can test the weak convergence with the characteristic function χ_U to obtain the contradiction

$$0 > \langle f, \chi_U\rangle_{L^2(\Omega)} = \lim_{k\to\infty} \langle f_k, \chi_U\rangle \geq 0. \qquad \square$$

Claim 3. $E\colon L^2_+(\Omega) \to \mathbb{R}$ is coercive and weakly sequentially lower semi-continuous.

Proof. Since $V(f) \geq 0$ if both $g \geq 0$ and $f \geq 0$ almost everywhere, we have

$$\begin{aligned} E(f) \geq \|f-h\|^2_{L^2(\Omega)} &\geq \|f\|^2_{L^2(\Omega)} - 2\|f\|_{L^2(\Omega)}\|h\|_{L^2(\Omega)} + \|h\|^2_{L^2(\Omega)} \\ &\geq \frac{1}{2}\|f\|^2_{L^2(\Omega)} - \|h\|^2_{L^2(\Omega)} \end{aligned}$$

for every $f \in L^2_+(\Omega)$ using Young's inequality $2ab \leq \frac{1}{2}a^2 + 2b^2$. Since $h \in L^2(\Omega)$ is fixed, E is coercive.

By (ii), $f \mapsto V(f)$ is weakly sequentially lower semi-continuous. Moreover, every term on the right hand side of

$$\|f-h\|^2_{L^2(\Omega)} = \|f\|^2_{L^2(\Omega)} - 2\langle f, h\rangle_{L^2(\Omega)} + \|h\|^2_{L^2(\Omega)}$$

is weakly sequentially lower semi-continuous in f since h is fixed. This proves the claim. $\square$

We may then apply Satz 5.4.1 and deduce the esistence of a minimum for E in $L^2(\Omega)$. $\square$

Hints to Exercises.

10.2 Use Satz 3.4.1.

10.4 Use Exercise 10.3.

10.5 Begin by considering the convolution operator $f \mapsto g * f = \int_\Omega g(x-y)f(y)dy$ from $L^2(\Omega)$ into itself. Prove that is maps weakly convergent sequences into strongly converging subsequences