Exercise 6.1 Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces and let

$$A\colon D_A \subset X \to Y, \qquad B\colon D_B \subset X \to Y$$

be linear operators with $D_A \subset D_B$. Suppose that there exist $a \in (0,1)$ and $b \geq 0$ such that

$$\|Bx\|_Y \leq a\|Ax\|_Y + b\|x\|_X \quad \text{for every } x \in D_A.$$

Show that if A has closed graph, then $(A+B)\colon D_A \to Y$ has closed graph.

Exercise 6.2 Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces. Let $A\colon D_A \subset X \to Y$ be a closable linear operator. Assume that its closure $\overline{A}$ is injective. Show that then the inverse operator A^{-1} is closable and $\overline{A^{-1}} = (\overline{A})^{-1}$.

Exercise 6.3 Let $(X, \|\cdot\|_X)$ be a Banach space and $U \subset X$ a closed subspace. Recall the notion of topologically complemented space introduced in Exercise 4.3. Prove that:

(i) If $\dim(U) < \infty$, then U is topologically complemented.

(ii) If $\dim(X/U) < \infty$, then U is topologically complemented.

Exercise 6.4 Let $(X, \|\cdot\|_X)$ be a normed space and let $f\colon X \to \mathbb{R}$ be linear and not identically zero.

(i) Show that f is *not* continuous if and only if $\ker(f)$ is dense in X.

(ii) Find an explicit example for X and f as in (i).

Exercise 6.5 Let $(X, \|\cdot\|_X)$ be a normed space and let $\varphi\colon X \to \mathbb{R}$ be a continuous linear functional. Assume $N := \ker(\varphi) \subsetneq X$ and let $x_0 \in X \setminus N$. Prove that the following are equivalent.

(i) There exists $y_0 \in N$ with $\|x_0 - y_0\|_X = \operatorname{dist}(x_0, N)$.

(ii) There exists $x_1 \in X$ with $\|x_1\|_X = 1$ and $\|\varphi\| = |\varphi(x_1)|$.

Exercise 6.6 Let $(X, \|\cdot\|_X) = \Big(C^0([-1,1]), \|\cdot\|_{C^0([-1,1])}\Big)$. Recall the map $\varphi\colon X \to \mathbb{R}$ given by

$$\varphi(f) = \int_0^1 f(t)\,\mathrm{d}t - \int_{-1}^0 f(t)\,\mathrm{d}t$$

from Exercise 3.2. Consider its kernel $N := \{f \in X \mid \varphi(f) = 0\}$ and some $x_0 \in X \setminus N$. Show that N is closed. Show that there does not exist any $y_0 \in N$ such that

$$\|x_0 - y_0\| = \operatorname{dist}(x_0, N).$$

Hints to Exercises.

6.1 Start by proving that $(Ax_n)_{n\in\mathbb{N}}$ is Cauchy.

6.3 When working with finite dimensional vector spaces, you may always argue with bases and dual bases.

6.5 Start by recalling the first isomorphism theorem for linear maps.