Series 5

1. **Gibbs sampler**

The goal of this exercise is to use the Gibbs sampler to simulate random variables from a bivariate normal distribution[1]

$$(X_1, X_2) \sim N\left(0, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right), \quad -1 < \rho < 1.$$

a) Use the Cholesky decomposition to simulate $N = 1000$ samples from this bivariate normal distribution. Visualize your samples (X_1, X_2) in a two-dimensional scatter plot.

b) Construct a Gibbs sampler to simulate $N = 1000$ samples from this bivariate normal distribution. Use the point $(0, 0)$ as deterministic initial value.

- Plot the simulated values X_1 (or X_2) versus the iteration number. This produces so-called trace plots.
- Plot the autocorrelation function (ACF) for one of the simulated values (e.g. X_1).
- Visualize your samples (X_1, X_2) in a two-dimensional scatter plot and create a Q-Q plot to verify that the marginal distributions are indeed normal.
- Investigate the impact of varying ρ on the autocorrelation in the samples. At least, try $\rho = 0.8$, $\rho = 0.95$, and $\rho = 0.99$. What do you observe?
- Investigate the impact of varying the initial value. At least, also try $(20, 20)$ as initial value with $\rho = 0.95$ and $\rho = 0.99$. Visually determine the approximate length of the burn-in phase, i.e., the number of samples to be discarded before the stationary distribution is reached.

Hints:

- You can use the R function `chol` to calculate the Cholesky decomposition. This function returns a matrix R such that $R^T R = \Sigma$ where Σ is the original matrix. See the corresponding help entry (`?chol`) for more information about how R calculates the Cholesky decomposition.

[1] In practice, one would not use MCMC to sample from a bivariate normal distribution. Here, we use this example to learn the basic principles and understand how the algorithms work in various situations.

2. **Random walk Metropolis algorithm**

   The goal of this exercise is to use the random walk Metropolis (RWM) algorithm to simulate random variables from the above bivariate normal distribution.

   a) Use the random walk Metropolis algorithm with a normal proposal densitiy with proposal covariance matrix

   $$\sigma^2 \cdot I_2$$

   to simulate $N = 1000$ samples from the bivariate normal distribution as target distribution.

   - Plot the simulated values X_1 (or X_2) versus the iteration number.
   - Calculate the percentage of the accepted values.
   - Plot the autocorrelation function (ACF) for one of the simulated values (e.g. X_1).
   - Visualize your samples (X_1, X_2) in a two-dimensional scatter plot and create a Q-Q plot to verify that the marginal distributions are indeed normal.

   b) Investigate the impact of varying ρ on the acceptance percentage and on the autocorrelation in the samples. At least, try $\rho = 0.5$, $\rho = 0.8$, $\rho = 0.95$, and $\rho = 0.99$. What do you observe?

   c) Investigate the impact of varying the proposal standard deviation σ on the acceptance rate and on the autocorrelation in the samples. Use $\sigma = 1$ as default choice and also try $\sigma = 0.2$ and $\sigma = 5$.

   d) Investigate the impact of varying the initial value. At least, try $(20, 20)$ as initial value with $\rho = 0.95$ and $\rho = 0.99$. Visually determine the approximate length of the burn-in phase, i.e., the number of samples to be discarded before the stationary distribution is reached.