Exercise 3.1 Let $(X, \|\cdot\|)$ be an infinite-dimensional normed vector space an let $Z \subset X$ be a bounded subset with compact boundary. Prove that Z has empty interior.

Exercise 3.2 We consider the space $X = C^0([-1,1],\mathbb{R})$ with its usual norm $\|\cdot\|_{C^0([-1,1])}$ and let $\varphi : X \to \mathbb{R}$ be given by

$$\varphi(f) \mapsto \int_0^1 f(t)\,\mathrm{d}t - \int_{-1}^0 f(t)\,\mathrm{d}t.$$

(i) Show that φ is a linear and continuous map with $\|\varphi\|_{L(X,\mathbb{R})} \leq 2$.

(ii) Prove that the norm of φ is exactly 2 by finding a sequence $(f_n)_{n\in\mathbb{N}}$ in X such that $\|f_n\|_{C^0([-1,1])} = 1$ for every $n \in \mathbb{N}$ and such that $\varphi(f_n) \to 2$ as $n \to \infty$.

(iii) Prove that there does not exist $f \in X$ with $\|f\|_{C^0([-1,1])} = 1$ and $|\varphi(f)| = 2$.

Exercise 3.3 Let

$$c_c := \{(x_n)_{n\in\mathbb{N}} \in \ell^\infty \mid \exists N \in \mathbb{N}\ \forall n \geq N : \ x_n = 0\}$$

be the space of compactly supported sequences endowed with the sup-norm $\|\cdot\|_{\ell^\infty}$. Consider the map $T\colon c_c \to c_c$ given by

$$T\left((x_n)_{n\in\mathbb{N}}\right) = (nx_n)_{n\in\mathbb{N}}.$$

(i) Show that T is not continuous.

(ii) Construct continuous linear maps $T_m\colon c_c \to c_c$ such that

$$\forall x \in c_c : \quad T_m x \xrightarrow{m\to\infty} Tx.$$

Exercise 3.4 Let $k\colon [0,1]\times[0,1] \to \mathbb{R}$ be continuous. Show that for every $g \in C^0([0,1])$ there exists a unique $f \in C^0([0,1])$ satisfying

$$\forall t \in [0,1] : \quad f(t) + \int_0^t k(t,s)f(s)\,\mathrm{d}s = g(t).$$

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Hints to Exercises.

3.1 Assume that $Z^\circ \neq \emptyset$. Find a continuous functional that projects the boundary ∂Z to the boundary of a ball inside Z. This will contradict the fact that the unit sphere in an infinite-dimensional normed space is non-compact.

3.4 Begin by choosing a space $(X, \|\cdot\|_X)$ and show that the operator $T\colon X \to X$ given by $(Tf)(t) = \int_0^t k(t,s) f(s)\, \mathrm{d}s$ has spectral radius $r_T = 0$.