Exercise 5.1 Let $X = L^2((0,1),\mathbb{R})$. On $D_A := C_c^\infty((0,1),\mathbb{R}) \subset X$ consider the derivative operator

$$A\colon D_A \to X, \quad A(f) = f'.$$

Recall that A is closable. Show that the domain $D_{\overline{A}}$ of its closure is contained in

$$C_0^0([0,1],\mathbb{R}) = \{f \in C^0([0,1],\mathbb{R}) \mid f(0) = 0 = f(1)\}.$$

Note: Do not forget that L^2-convergence does *not* imply pointwise convergence.

Exercise 5.2 Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces. Let $A\colon D_A \subset X \to Y$ be a linear operator with closed graph. Show that the following statements are equivalent:

(i) A is injective and its range $W_A := A(D_A)$ is closed in $(Y, \|\cdot\|_Y)$.

(ii) There exists $C > 0$ so that $\|x\|_X \leq C\|Ax\|_Y$ for every $x \in D_A$.

Exercise 5.3 *(Hörmander).* Let $(X_0, \|\cdot\|_{X_0})$, $(X_1, \|\cdot\|_{X_1})$ and $(X_2, \|\cdot\|_{X_2})$ be Banach spaces and let

$$T_1\colon D_1 \subset X_0 \to X_1, \quad \text{and} \quad T_2\colon D_2 \subset X_0 \to X_2$$

be linear operators with closed graphs such that $D_1 \subset D_2$. Prove that there exists a constant $C > 0$ so that

$$\|T_2 x\|_{X_2} \leq C\Big(\|T_1 x\|_{X_1} + \|x\|_{X_0}\Big) \quad \text{for every } x \in D_1.$$

Exercise 5.4 Let $(H, (\cdot,\cdot))$ be a Hilbert space and let $A\colon H \to H$ be a symmetric linear operator that is *coercive*, i.e. such that there exists $\lambda > 0$ so that

$$(Ax, x) \geq \lambda \|x\|^2 \quad \text{for every } x \in H.$$

Show that A is an isomorphism of normed spaces and $\|A^{-1}\| \leq \lambda^{-1}$.

Hints to Exercises.

5.1 Given $f \in D_{\overline{A}}$ consider a sequence $(f_n)_{n\in\mathbb{N}}$ in D_A which converges to f in X. Compare $f_n(t)$ to $g(t) := \int_0^t \overline{A}f \, \mathrm{d}x$.

5.2 One implication follows from the Inverse Mapping Theorem.

5.3 Recall that, if $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces and $A\colon D_A \subset X \to Y$ is a linear operator with closed graph, then $(D_A, \|\cdot\|_{\Gamma_A})$ is a Banach space, where $\|x\|_{\Gamma_A} = \|x\|_X + \|Ax\|_Y$ is the graph norm.

5.4 To prove surjectivity, i. e. $W_A := A(H) = H$, consider an element $x \in W_A^\perp$ and recall that $(W_A^\perp)^\perp = \overline{W_A}$.