Bayesian Statistics

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Today's topics

- Hierarchical Bayes models
- Examples of hierarchical Bayes models

Hierarchical Bayes models

Hierarchical Bayes models

- In hierarchical models, the prior $\pi(\theta \mid \xi)$ for θ depends on other parameters ξ, called **hyperparameters**, which also have a prior distribution $\pi(\xi)$
- This leads to a triple of random variables (ξ, θ, x) with joint density
$$\pi(\xi)\pi(\theta \mid \xi)f(x \mid \theta)$$

Hierarchical Bayes models

- By conditioning on the data x, we obtain the posterior $\pi(\xi, \theta|x)$
  - For complex models, $\pi(\xi, \theta|x)$ usually cannot be determined in closed from and some approximation is needed (see later in the course)
- Often, the primary interest is in the parameter θ, i.e. the **marginal posterior** $\pi(\theta \mid x)$
- There are two ways to compute it

Marginal posterior: approach 1

1. In the first approach, one first computes the marginal prior

$$\pi(\theta) = \int \pi(\theta \mid \xi)\pi(\xi) d\xi$$

and then uses Bayes formula

$$\pi(\theta \mid x) \propto \pi(\theta) f(x \mid \theta)$$

Marginal posterior: approach 2

2. In some situations, the approach based on the following formula is computationally easier:

$$\begin{aligned}\pi(\theta \mid x) &= \int \pi(\theta \mid x, \xi)\pi(\xi \mid x)d\xi \\ &\propto \int \pi(\theta \mid x, \xi)\pi(\xi)f(x \mid \xi)d\xi\end{aligned}$$

- One first needs to determine $\pi(\theta \mid x, \xi)$ and $\pi(\xi \mid x)$ or $f(x \mid \xi)$

Marginal posterior: approach 2

- $\pi(\xi \mid x)$ can be calculated either by marginalizing the joint posterior over θ

$$\pi(\xi \mid x) = \int \pi(\theta, \xi \mid x) d\theta$$

or by using

$$\pi(\xi \mid x) = \frac{\pi(\theta, \xi \mid x)}{\pi(\theta \mid x, \xi)}$$

- Analogously, $f(x \mid \xi)$ can be obtained by by marginalizing the joint distribution of x and θ given ξ

$$f(x \mid \xi) = \int \pi(x, \theta \mid \xi) d\theta = \int f(x \mid \theta)\pi(\theta \mid \xi) d\theta$$

Marginal posterior: approach 2

- For conjugate priors, we have an explicit expression not only for $\pi(\theta \mid x, \xi)$ but often also for $f(x \mid \xi)$
- Usually, the final integration over ξ to obtain $\pi(\theta \mid x)$ cannot be done in closed form, and some approximation is needed

Examples of hierarchical models

- Normal means
- Hierarchical Poisson model
- One-way ANOVA model

Normal means example

Assume the following model:

- Likelihood: $X_1, \dots X_n$ i.i.d. $\sim \mathcal{N}(\theta, 1)$
- Hierarchical prior:
  - $\theta \sim \mathcal{N}(\mu, \tau^2)$, μ fix
  - $\tau^{-2} \sim \text{Gamma}(\gamma, \lambda)$

See blackboard for more details

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Hierarchical Poisson model example

Assume the following model:

- Likelihood: X_j independent $\sim$ Poisson(θ_j), $j = 1, 2, \ldots, J$
- Hierarchical prior:
  - θ_j i.i.d. $\sim$ Gamma(γ, λ)
  - Some prior $\pi(\gamma, \lambda)$

See blackboard for more details

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One-way ANOVA model example

Assume the following model:

- Likelihood: $y_{ij} = \theta_i + \varepsilon_{ij}, \quad \varepsilon_{ij}$ i.i.d $\sim \mathcal{N}(0, \sigma_\varepsilon^2)$, $j = 1, \ldots, n_i, i = 1, \ldots, I$
- Hierarchical prior:
  - θ_i i.i.d $\sim \mathcal{N}(\mu, \tau^2)$
  - For simplification, σ_ε is assumed to be known
  - $\pi(\mu, \tau^2) = \pi(\mu)\pi(\tau^2)$ (specified later)

See blackboard for more details

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