Series 3

1. **Jeffreys prior**

   a. Consider the normal model $X \sim \mathcal{N}(0, \sigma^{-2})$ where σ^{-2} is the precision (inverse of the variance). Show that Jeffreys prior for σ^{-2} is proportional to σ^2.

   b. Consider the normal model $X \sim \mathcal{N}(\mu, \sigma)$ with both the mean and the standard deviation unknown. Show that Jeffreys prior for (μ, σ) is proportional to σ^{-2}.

   c. Consider the vector normal means example we saw in the lecture: X_i $1 \leq i \leq 2n$ are independent and normally distributed with variance σ^2 and means $\mathbb{E}(X_{2k-1}) = \mathbb{E}(X_{2k}) = \mu_k$. Show that Jeffreys prior for this model is $\pi(\mu_1, \ldots, \mu_n, \sigma^2) \propto \sigma^{-n-2}$.

   d. Consider the model $X \sim \text{binomial}(n, \theta)$. Show that Jeffreys prior for θ is the Beta$(\frac{1}{2}, \frac{1}{2})$ distribution.

2. **Jeffreys prior and variance stabilizing transformations**

   Find a transform of θ, $\tau = g(\theta)$, such that the Fisher information $I_\tau(\tau)$ is constant for:

   a. The Poisson distribution, $X \sim \text{Poisson}(\theta)$.

   b. The gamma distribution, $X \sim \text{Gamma}(\gamma, \theta)$, with known shape parameter γ.

   c. The binomial distribution, $X \sim \text{Binomial}(n, \theta)$.

   Remark: In frequentist statistics, such a transformation is called a variance stabilizing transformation since for the transformed parameter, the asymptotic variance of the maximum likelihood estimator does not depend on the paramter.

3. **Refference prior**

   In the lecture, we have seen that the reference prior defined using an asymptotic argument leads to the Jeffreys prior, which is equivariant under reparametrizations.

   Show directly that the reference prior is also equivariant under reparametrization when defining it as the maximizer of the mutual information

$$I(X, \theta) = \int_X f(x) \int_\Theta \pi(\theta \mid x) \log \frac{\pi(\theta \mid x)}{\pi(\theta)} d\theta dx.$$

4. **Expert priors**[1]

Suppose that the heights of male students are normally distributed with mean 180 and unknown variance σ^2. We believe that σ^2 is in the range $[22, 41]$ with approximately 95% probability. Thus, we assign an inverse-gamma distribution $\mathrm{IG}(\alpha = 38, \beta = 1110)$ as prior distribution for σ^2.

a. Verify with R that the above choice of parameters for the inverse-gamma distribution leads to a prior probability of approximately 95 % that $\sigma^2 \in [22, 41]$.

b. Derive and plot the posterior density of σ^2 when using the following data:
183, 173, 181, 170, 176, 180, 187, 176, 171, 190, 184, 173, 176, 179,181, 186.

c. Compute the posterior density of the standard deviation σ.

Hint:

- The R package `invgamma` implements the density and cummulative distribution function of the inverse-gamma distribution.

[1] Based on exercise 3 in Section 6.8 of Held and Sabanes Bove (2014).