Series 2

1. **Credible intervals**

   Unlike the central credible interval, the highest posterior density credible interval is not invariant to transformations as you will show in the following.

   Assume that X_i i.i.d $\sim \mathcal{N}(0, \sigma^2)$, $i = 1, \dots, n$ and that σ has the improper prior $\pi(\sigma) \propto \frac{1}{\sigma}$. Show that for the transformed parameter σ^2, the 95% highest posterior density (HPD) credible interval is not the same as the interval obtained when taking the square of the endpoints of the HPD credible interval for σ.

2. **Conjugate priors**

   In the lecture, we saw the following examples of conjugate priors for exponential family distributions.

Model	Prior	Posterior
Binomial(n, θ)	Beta(α, β)	Beta$(\alpha + x, \beta + n - x)$
Multinomial $(n, \theta_1, \dots, \theta_k)$	Dirichlet$(\alpha_1, \dots, \alpha_k)$	Dirichlet$(\alpha_1 + x_1, \dots, \alpha_k + x_k)$
i.i.d. Poisson(θ)	Gamma(γ, λ)	Gamma$(\gamma + \sum_i x_i, \lambda + n)$
i.i.d. Normal$(\mu, \frac{1}{\tau})$ $\theta = (\mu, \tau)$	Normal$(\mu_0, \frac{1}{n_0 \tau}) \times$ Gamma(γ, λ)	Normal$(\frac{n}{n+n_0}\bar{x} + \frac{n_0}{n+n_0}\mu_0, \frac{1}{(n+n_0)\tau}) \times$ Gamma$(\gamma + \frac{n}{2}, \lambda + \frac{1}{2}\sum_i (x_i - \bar{x})^2 + \frac{n n_0}{2(n+n_0)}(\bar{x} - \mu_0)^2)$
Uniform$(0, \theta)$	Pareto(α, σ)	Pareto$(\alpha + n, \max(\sigma, x_1, \dots, x_n))$

   Show that one indeed obtains the posteriors in the table when using the priors and likelihoods specified in the table.

3. **Improper priors**

   Consider the Poisson model

$$f(x|\theta) = P_\theta(X = x) = \frac{\theta^x}{x!} e^{-\theta}, \quad x \in \mathbb{N}_0, \ \theta > 0,$$

   and the improper prior

$$\pi(\theta) = \frac{1}{\theta}.$$

   Show that the posterior distribution is not well defined for all x.