Series 1

1. **Posterior predictive distribution**

   a. Assume the model $X \sim \mathcal{N}(\theta, \sigma^2)$ with the prior $\pi(\theta, \sigma^2) \propto \frac{1}{\sigma^2}$.[1] Consider $Y \sim \mathcal{N}(\rho X, \sigma^2)$ with ρ known and fixed. Derive the density $f(y \mid x)$ of the posterior predictive distribution of Y given X.

   b. Assume the model $X \sim$ Binomial(θ, n) with the prior $\theta \sim$ Beta(α, β). Further, assume that $Y \sim$ Binomial(θ, n) and that conditional on θ, Y is independent of X. Derive the density $f(y \mid x)$ of the posterior predictive distribution of Y given X.

2. **Bayesian decision theory**

   In the lecture, we saw the connection between Bayesian point estimates and Bayesian decision theory.

   a. Show that we obtian the posterior mean if we use a quadratic loss function $L(T, \theta) = (T - \theta)^2$.

   b. Show that we obtian the posterior median if we use $L(T, \theta) = |T - \theta|$.

   c. Show that we obtian the posterior mode if we use $L(T, \theta) = 1_{[-\varepsilon,\varepsilon]^c}(T - \theta)$ and we let ε go to zero.

   Hint:

   - For the median, use the Leibniz integral rule (see, e.g., https://en.wikipedia.org/wiki/Leibniz_integral_rule).

3. **Bayesian testing and Bayes factor**

   Assume the model $X \sim \mathcal{N}(\theta, 1)$ and for θ the prior $\pi(\theta) \propto 1$. Our goal is to test the hypothesis $H_0 : |\theta| \leq c$ versus $H_1 : |\theta| > c$.

   a. Determine the maximal posterior probability of the null hypothesis $\max_x \pi(\Theta_0|x)$ as a function of c.

   b. Determine the values of x and c for which $\pi(\Theta_0|x)$ equals 0.95 and calculate the Bayes factor.

[1] This is an example of a so called improper prior. As we will see later in the lecture, as long as $\pi(\theta)f(x \mid \theta)$ has finite total mass, one is allowed to use improper priors.