Series 2

1. **Credible intervals**

   Unlike the central credible interval, the highest posterior density credible interval is not invariant to transformations as you will show in the following.

   Assume that X_i i.i.d $\sim \mathcal{N}(0, \sigma^2)$, $i = 1, \ldots, n$ and that σ has the improper prior $\pi(\sigma) \propto \frac{1}{\sigma}$. Show that for the transformed parameter σ^2, the 95% highest posterior density (HPD) credible interval is not the same as the interval obtained when taking the square of the endpoints of the HPD credible interval for σ.

Solution

We prove by contradiction that the two highest posterior density (HPD) credible intervals are not the same.

First, using the change-of-variables formula, we can show that the corresponding prior for σ^2 is $\pi(\sigma^2) \propto \frac{1}{\sigma^2}$.

For the posterior of σ we have

$$\pi(\sigma|x) \propto \sigma^{-n-1} \exp(-s_n/\sigma^2),$$

where $s_n = \frac{1}{n}\sum_{i=1}^n x_i^2$. The posterior of σ^2 is given by

$$\pi(\sigma|x) \propto \left(\sigma^2\right)^{-n/2-1} \exp(-s_n/\sigma^2).$$

We denote by $(\sqrt{a}, \sqrt{b})$ the HPD credible interval for $\pi(\sigma|x)$. One can show that the posterior density is unimodal. Therefore the 95% region of highest density is a single interval determined by

$$a^{-n/2-1/2} \exp(-s_n/a) = b^{-n/2-1/2} \exp(-s_n/b).$$

This is equivalent to

$$(-n/2 - 1/2)\log(a) - s_n/a = (-n/2 - 1/2)\log(b) - s_n/b. \qquad (1)$$

If (a, b) was the HPD credible interval for $\pi(\sigma^2|x)$, it would follow that

$$(-n/2 - 1)\log(a) - s_n/a = (-n/2 - 1)\log(b) - s_n/b. \qquad (2)$$

Combining equations 1 and 2, we would obtain $-1/2\log(a) = -1/2\log(b)$ or $a = b$. However, in this case (a, b) obviously cannot be a 95% HPD credible interval.

2. **Conjugate priors**

   In the lecture, we saw the following examples of conjugate priors for exponential family distributions.

Model	Prior	Posterior
Binomial(n, θ)	Beta(α, β)	Beta$(\alpha + x, \beta + n - x)$
Multinomial $(n, \theta_1, \ldots, \theta_k)$	Dirichlet$(\alpha_1, \ldots, \alpha_k)$	Dirichlet$(\alpha_1 + x_1, \ldots, \alpha_k + x_k)$
i.i.d. Poisson(θ)	Gamma(γ, λ)	Gamma$(\gamma + \sum_i x_i, \lambda + n)$
i.i.d. Normal$(\mu, \frac{1}{\tau})$ $\theta = (\mu, \tau)$	Normal$(\mu_0, \frac{1}{n_0 \tau})$ $\times$ Gamma(γ, λ)	Normal$(\frac{n}{n+n_0}\bar{x} + \frac{n_0}{n+n_0}\mu_0, \frac{1}{(n+n_0)\tau})$ $\times$ Gamma$(\gamma + \frac{n}{2}, \lambda + \frac{1}{2}\sum_i (x_i - \bar{x})^2 + \frac{n n_0}{2(n+n_0)}(\bar{x} - \mu_0)^2)$
Uniform$(0, \theta)$	Pareto(α, σ)	Pareto$(\alpha + n, \max(\sigma, x_1, \ldots, x_n))$

   Show that one indeed obtains the posteriors in the table when using the priors and likelihoods specified in the table.

   ## *Solution*

   For the uniform-pareto case, we have

$$
\begin{aligned}
\pi(\theta|x) &\propto \prod_{i=1}^{n} \frac{1}{n} 1_{[x_i, \infty)}(\theta) \sigma^{\alpha} \theta^{-(\alpha+1)} 1_{[\sigma, \infty)}(\theta) \\
&\propto \theta^{-(\alpha+n+1)} 1_{[\max(x_1, \ldots, x_n \sigma), \infty)}(\theta)
\end{aligned}
$$

   which shows that the poserior has a Pareto$(\alpha + n, \max(\sigma, x_1, \ldots, x_n))$ distribution.

   The other calculations are done analogously. See also the lecture for example solutions.

3. **Improper priors**

   Consider the Poisson model

$$f(x|\theta) = P_\theta(X = x) = \frac{\theta^x}{x!} e^{-\theta}, \quad x \in \mathbb{N}_0, \ \theta > 0,$$

   and the improper prior

$$\pi(\theta) = \frac{1}{\theta}.$$

   Show that the posterior distribution is not well defined for all x.

Solution

For $x = 0$, the normalizing constant is given by

$$\begin{aligned} f(x) &= \int f(x|\theta)\pi(\theta)d\theta \\ &= \int e^{-\theta}\frac{1}{\theta}d\theta. \end{aligned}$$

This integral is not finite.