Exercise 4.1 Let $(X, \|\cdot\|_X)$ be a normed space an let $U, V \subset X$ be subspaces. Prove the following.

(i) If U is finite dimensional and V closed, then $U + V$ is a closed subspace of X.

(ii) If V is closed with finite codimension, i. e. $\dim(X/V) < \infty$, then $U + V$ is closed.

Remark. The assumptions on the dimension and codimension above are crucial for the conclusions to hold; see Exercise 2.4.

Exercise 4.2 Let $X = C^0([0,1])$ endowed with the norm $\|\cdot\|_X = \|\cdot\|_{C^0([0,1])}$ and consider

$$U = C_0^0([0,1]) := \{f \in C^0([0,1]) \mid f(0) = 0 = f(1)\}.$$

(i) Show that U is a closed subspace of X

(ii) Compute the dimension of the quotient space X/U and find a basis for X/U.

Exercise 4.3 A subspace $U \subset X$ of a Banach space$(X, \|\cdot\|_X)$ is called *topologically complemented* if there is a subspace $V \subset X$ such that the linear map I given by

$$\begin{aligned} I\colon (U \times V, \|\cdot\|_{U\times V}) &\to (X, \|\cdot\|_X), \qquad \|(u,v)\|_{U\times V} := \|u\|_X + \|v\|_X, \\ (u,v) &\mapsto u + v \end{aligned}$$

is a continuous isomorphism of normed spaces with continuous inverse. In this case V is said to be a *topological complement* of U.

(i) Prove that $U \subset X$ is topologically complemented if and only if there exists a continuous linear map $P\colon X \to X$ with $P \circ P = P$ and image $P(X) = U$.

(ii) Show that a topologically complemented subspace must be closed.

Remarks.
- Clearly, if V is a topological complement of U, then U is a topological complmement of V.
- If X is not isomorphic to a Hilbert space, then X has closed subspaces which are not topologically complemented [Lindenstrauss & Tzafriri. *On the complemented subspaces problem.* (1971)]. An example is $c_0 \subset \ell^\infty$ but this is not easy to prove.

Exercise 4.4 Let $(X\|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be Banach spaces and let $T \in L(X, Y)$ be a surjective, continuous linear map. Prove the equivalence of the following statements:

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(i) The subspace $\ker T$ is topologically complemented.

(ii) There exists a continuous linear map $S \in L(Y, X)$ so that $T \circ S = 1_Y$. S is called *section* of T.

Exercise 4.5 Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ be normed spaces. We consider the space $(X \times Y, \|\cdot\|_{X\times Y})$, where $\|(x,y)\|_{X\times Y} = \|x\|_X + \|y\|_Y$ and a bilinear map $B\colon X \times Y \to Z$.

(i) Show that B is continuous if

$$\exists C > 0 \quad \forall (x,y) \in X \times Y : \quad \|B(x,y)\|_Z \leq C\|x\|_X\|y\|_Y. \tag{†}$$

(ii) Assume that $(X, \|\cdot\|_X)$ is Banach. Assume further that the maps

$$\begin{aligned} X &\to Z \\ x &\mapsto B(x, y') \end{aligned} \qquad\qquad \begin{aligned} Y &\to Z \\ y &\mapsto B(x', y) \end{aligned}$$

are continuous for every $x' \in X$ and $y' \in Y$. Prove that then B is continuous.

Hints to Exercises.

4.1 Is the canonical quotient map $\pi \colon X \to X/V$ continuous? What is $\pi^{-1}(\pi(U))$?

4.3 For (i), consider for one implication the projection map $P_1(u,v) = (u,0)$, and for the other implication the identity $1 = P + (1 - P)$, where 1 denotes the identity map on X.

4.4 For (i) $\Rightarrow$ (ii), let V be the topological complement of $\ker T$ and consider the map $T|_V$. For (ii) $\Rightarrow$ (i), use Exercise 4.3.

4.5 Apply the Theorem of Banach-Steinhaus to a suitable map. Do not forget that the theorem requires completeness of the domain.