Exercise 8.1 Let $(X, \|\cdot\|_X)$ be a normed space and let $\emptyset \neq Q \subset X$ be an open, convex subset containing the origin. Prove that Q can be written as intersection of open affine half-spaces, namely that there exists a subset $\Upsilon \subset X^*$ such that

$$Q = \bigcap_{f \in \Upsilon} \{x \in X \mid f(x) < 1\}.$$

Exercise 8.2 Let $(H, (\cdot,\cdot)_H)$ be a Hilbert space and let $\emptyset \neq K \subset H$ be a closed, convex subset. Denote by $P\colon H \to K$ be the projection operator onto K which maps $x \in H$ to the unique point $P(x) \in K$ with $\|x - Px\|_H = \operatorname{dist}(x, K)$ constructed in Exercise 7.7 (ii).

(i) For every $x_1, x_2 \in H$ prove the inequality

$$\|P(x_1) - P(x_2)\|_H \leq \|x_1 - x_2\|_H.$$

(ii) Prove that

$$K = \bigcap_{x \in H} \{y \in H \mid (x - P(x), P(x) - y)_H \geq 0\}.$$

Exercise 8.3 A normed space $(X, \|\cdot\|_X)$ is called *strictly convex* if, for every $x, y \in X$ with $x \neq y$ and $\|x\|_X = 1 = \|y\|_X$, there holds

$$\|\lambda x + (1-\lambda)y\|_X < 1 \quad \text{for every } \lambda \in (0,1)$$

Let $(X, \|\cdot\|_X)$ be a normed space.

(i) Prove that if X^* is strictly convex, then for all $x \in X$ there exists a *unique* $x^* \in X^*$ with $\|x^*\|_{X^*}^2 = x^*(x) = \|x\|_X^2$.

(ii) Find a counterexample to (i) when X^* is not strictly convex.

Exercise 8.4

(i) Let $K \subset \mathbb{R}^2$ be a closed, convex subset. Prove that the set E of all extremal points of K is closed.

(ii) Is the statement of (i) also true in $\mathbb{R}^3$?

Exercise 8.5 Let X be vector space and let $K \subset X$ be a convex subset with more than one element.

(i) Given an extremal subset $M \subset K$ of K, prove that $K \setminus M$ is convex.

(ii) If $N \subset K$ and $K \setminus N$ are both convex, does it follow that N is extremal?

(iii) Prove that $y \in K$ is an extremal point of K if and only if $K \setminus \{y\}$ is convex.

Hints to Exercises.

8.1 Define Υ by means of the Minkowski functional for Q (used in the proof of Satz 4.5.1) $p\colon X \to \mathbb{R}$ given by

$$p(x) = \inf\{\lambda > 0 \mid \tfrac{1}{\lambda}x \in Q\}.$$

Note that p is sublinear and combine this with the Hahn-Banach theorem.

8.2 Use that for any $y \in K$ any $x_0 \in H$ there holds

$$(Px_0 - x_0, y - Px_0)_H \geq 0,$$

which is special case of the an inequality shown in Exercise 7.4 (ii) with $a(\cdot,\cdot) = (\cdot,\cdot)_H$.

8.4 For (i), show that E is a subset of the boundary ∂K. Argue by contradiction supposing E is not closed. For (ii), draw a picture.