Robotic Unicycle Loss Function

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The state of the system is given as

$$\boldsymbol{x} = \begin{bmatrix}\dot\theta & \dot\phi & \dot\psi_w & \dot\psi_f & \dot\psi_t & \theta & \phi & \psi_w & \psi_f & \psi_t\end{bmatrix}^\top ,$$

where θ is the roll angle, ϕ is yaw, ψ_w wheel angle, ψ_f pitch angle and ψ_t the disc angle. The *reward* R and *loss* L of being in state $\boldsymbol{x}$ is

$$R(\boldsymbol{x}) \;=\; \exp\big(-\frac{a}{2}d(\boldsymbol{x})^2 - \frac{\phi^2}{2(4\pi)^2}\big), \qquad \text{and} \qquad L(\boldsymbol{x}) \;=\; 1 - R(\boldsymbol{x}),$$

where a is a (positive) constant, and d^2 is the the squared distance between the tip of the unicycle and its ideal location when upright in the centre of the coordinate system.

$$\begin{aligned}
d(\boldsymbol{x})^2 &= d_x^2 + d_y^2 + d_z^2 \qquad \text{with}\\
d_x &= x_c - r\sin\psi_f\\
d_y &= y_c - (r + r_w)\sin\theta\\
d_z &= \big(r_w + r - r_w\cos\theta - r\cos\theta\cos\psi_f\big)^2\\
&= \big(r_w + r - r_w\cos\theta - \tfrac{r}{2}\cos(\theta-\psi_f) - \tfrac{r}{2}\cos(\theta+\psi_f)\big)^2
\end{aligned}$$

Here, r_w is the radius of the wheel and r is the length of the fork; x_c and y_c are coordinates of the laboratory origin in the self-centred coordinate system. Apart from those two latter coordinates, the relevant parts of the state $\boldsymbol{x}$ are the sine and cosine of θ, the sine of ψ_f, and the difference/sum of the angles θ and ψ_f. Therefore, we define $\alpha := \theta - \psi_f$ and $\beta := \theta + \psi_f$. We assume that the joint distribution of these three variables is approximately Gaussian

$$\boldsymbol{z} = \left[\begin{array}{c} x_c \\ \sin\psi_f \\ \hline y_c \\ \sin\theta \\ \hline \cos\theta \\ \cos\alpha \\ \cos\beta \\ \hline \phi \end{array}\right] \quad \text{and} \quad \mathcal{N}(\boldsymbol{m}, S)\,.$$

The *expected reward* is

$$\mathbb{E}[R] \;=\; \langle R(\boldsymbol{x})\rangle_{\boldsymbol{z}} \;=\; \big\langle \exp\big(-\tfrac{1}{2}(\boldsymbol{z}-\boldsymbol{z}_0)^\top T^{-1}(\boldsymbol{z}-\boldsymbol{z}_0)\big)\big\rangle_{\boldsymbol{z}}\,,$$

where we defined

$$z_0 := \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}^\top$$

and

$$\begin{aligned}
\mathbf{T}^{-1} &= a \operatorname{diag}\left\{ \begin{bmatrix} 1 & -r \\ -r & r^2 \end{bmatrix} , \begin{bmatrix} 1 & -(r + r_w) \\ -(r + r_w) & (r + r_w)^2 \end{bmatrix} , \begin{bmatrix} r_w^2 & \frac{r_w r}{2} & \frac{r_w r}{2} \\ \frac{r_w r}{2} & \frac{r^2}{4} & \frac{r^2}{4} \\ \frac{r_w r}{2} & \frac{r^2}{4} & \frac{r^2}{4} \end{bmatrix} , \begin{bmatrix} \frac{1}{16\pi^2} \end{bmatrix} \right\} \\
&= a \operatorname{diag}(\mathbf{C}_x \mathbf{C}_x^\top , \mathbf{C}_y \mathbf{C}_y^\top , \mathbf{C}_z \mathbf{C}_z^\top)
\end{aligned}$$

with

$$\mathbf{C}_x = \begin{bmatrix} 1 \\ -r \end{bmatrix} \qquad \mathbf{C}_y = \begin{bmatrix} 1 \\ -(r + r_w) \end{bmatrix} \qquad \mathbf{C}_z = \begin{bmatrix} r_w \\ r/2 \\ r/2 \end{bmatrix}$$