CG2023 Tutorial 8

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Review of LTI Systems

$h(t)$ — **Impulse Response**

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = x(t) * h(t)$$

$$o(t) = \int_{-\infty}^{t} h(\tau)d\tau$$ **Step Response**

$$\tilde{H}(s) = \mathcal{L}\{h(t)\} = \int_{0}^{\infty} h(t)\,e^{-jst}dt$$

$$H(f) = \Im\{h(t)\} = \int_{-\infty}^{\infty} h(t)\,e^{-j2\pi ft}dt$$

Differential Equations

$$\sum_{n=0}^{N} a_n \frac{d^n y(t)}{dt^n}$$

$$= \sum_{m=0}^{M} b_m \frac{d^m y(t)}{dt^m} \xrightarrow{\mathcal{L}}$$

Transfer Function

$\tilde{H}(s)$ $\qquad s = j\omega$

$$\tilde{X}(s) \longrightarrow \boxed{\tilde{H}(s)} \longrightarrow \tilde{Y}(s) = \tilde{X}(s) \cdot \tilde{H}(s)$$

$$\tilde{H}(s) = \frac{\tilde{Y}(s)}{\tilde{X}(s)} = \frac{b_M s^M + b_{M-1} s^{M-1} + \cdots + b_0}{a_N s^N + a_{N-1} s^{N-1} + \cdots + a_0}$$

Frequency Response

$\tilde{H}(j\omega)$ $\qquad \omega = 2\pi f$ $\qquad H(f)$

$$\tilde{X}(j\omega) \longrightarrow \boxed{\tilde{H}(j\omega)} \longrightarrow \tilde{Y}(j\omega) = \tilde{X}(j\omega) \cdot \tilde{H}(j\omega)$$

$$\left|\tilde{H}(j\omega)\right| \qquad e^{j\angle \tilde{H}(j\omega)}$$

Magintude response $\qquad$ Phase response

Transient Response

System Stability $\qquad$ Typical Systems

Locations of System Poles $\qquad$ First order system $\qquad$ Second order system

ty of Singapore.

Sampling and Reconstruction of Band-Limited Signals

Sampling above Nyquist Rate $\qquad$ Sampling Bandpass Signal below Nyquist Rate
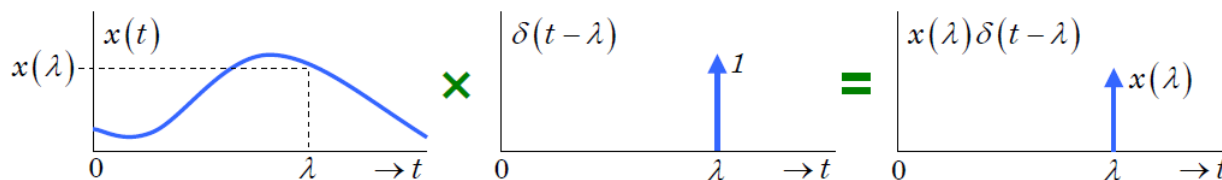
Reconstructed Signal

- **Properties of $\delta(t)$**

1. **Symmetry:**

$$\delta(t) = \delta(-t) \tag{2.3}$$

2. **Sampling:**

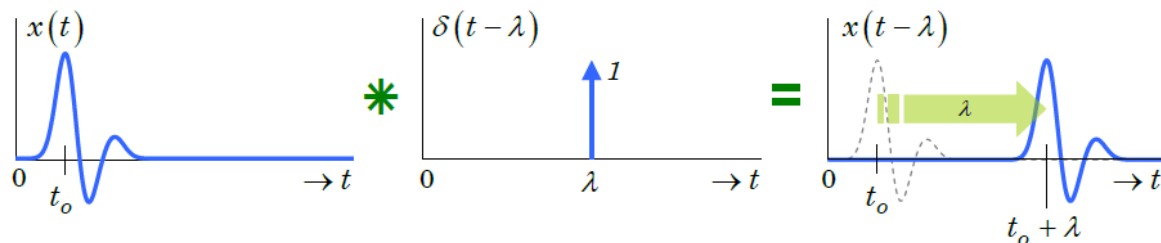$$x(t)\delta(t-\lambda) = x(\lambda)\delta(t-\lambda) \tag{2.4}$$

3. **Sifting:**

$$\underbrace{\int_{-\infty}^{\infty} x(t)\delta(t-\lambda)\,dt = x(\lambda)\int_{-\infty}^{\infty} \delta(t-\lambda)\,dt}_{\text{from the sampling property of }\delta(\bullet)} = x(\lambda) \tag{2.5}$$

© Cc

4. Replication:

apply **symmetry** property

$$x(t) * \delta(t - \lambda) = \int_{-\infty}^{\infty} x(\zeta) \delta(t - \zeta - \lambda) d\zeta = \int_{-\infty}^{\infty} x(\zeta) \delta(\zeta - (t - \lambda)) d\zeta = x(t - \lambda)$$

(2.6)

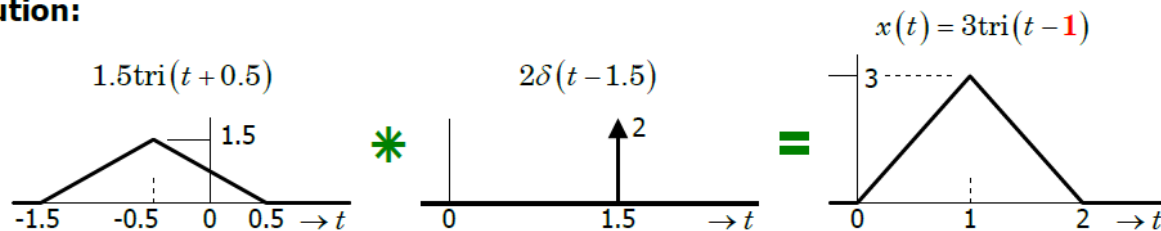apply **sifting** property

Note : $x(t) * \delta(t) = x(t)$

Example 2-5

Find and sketch $x(t) = 1.5\text{tri}(t + 0.5) * 2\delta(t - 1.5)$
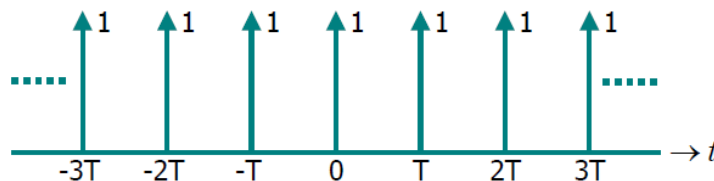
Solution:

Example 2-6

Find and sketch $y(t) = 3\delta(t-1) * 2\delta(t+3)$

Solution:

$3\delta(t-1)$ ***** $2\delta(t+3)$ **=** $y(t) = 6\delta(t+2)$
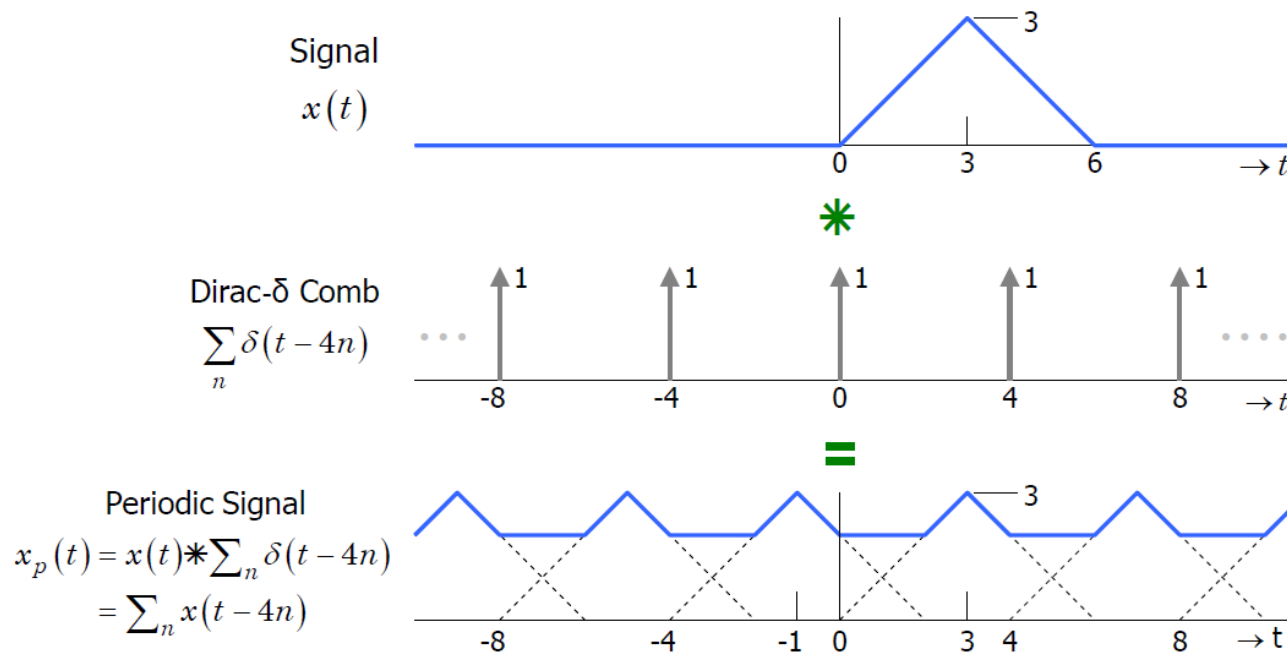
Impulse A   Impulse B
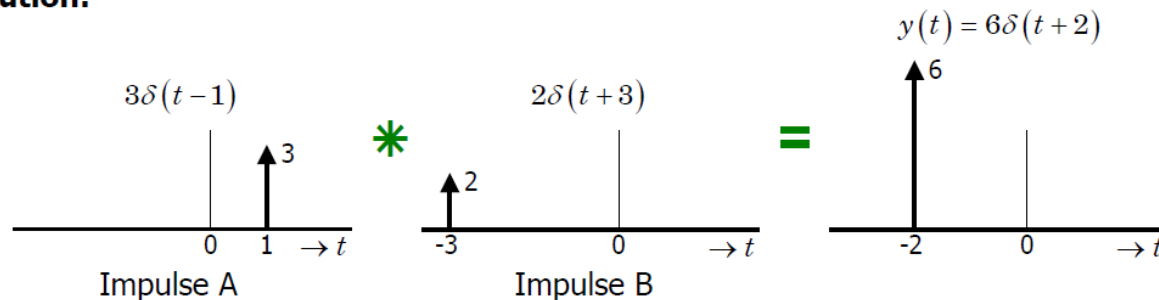
2.2.1 The Dirac-δ Comb function (a.k.a. Sampling function)

$$\sum_{n=-\infty}^{\infty} \delta(t-nT) = \cdots\cdots + \delta(t+2T) + \delta(t+T) + \delta(t) + \delta(t-T) + \delta(t-2T) + \cdots\cdots \qquad (2.7)$$
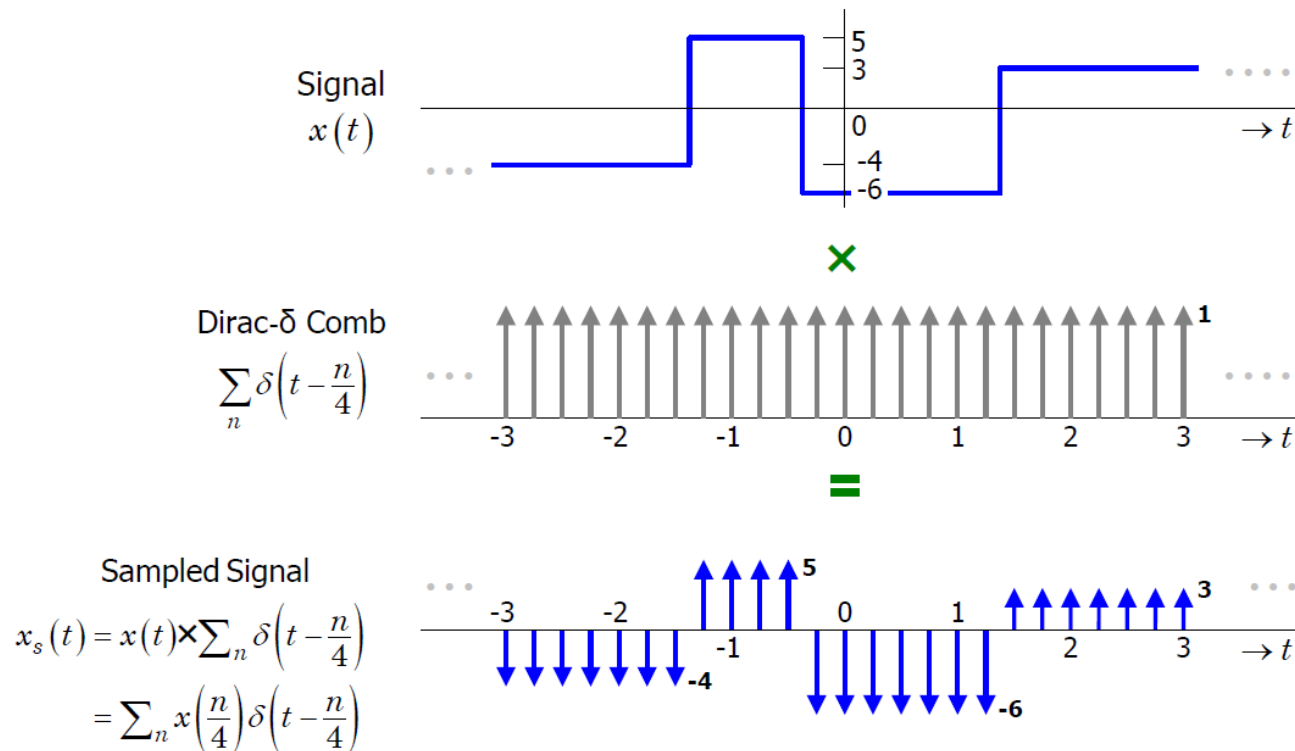
Review

Example 2-7:

Convolution with a Dirac-δ Comb function *(by applying the replication property of the Dirac-δ function). This is used in modeling periodic signals.*

Signal
$x(t)$

Dirac-δ Comb
$$\sum_{n} \delta(t - 4n)$$

Periodic Signal
$$x_p(t) = x(t) * \sum_{n} \delta(t - 4n)$$
$$= \sum_{n} x(t - 4n)$$

$x(t)$ is called the **Generating Function** of the periodic signal $y(t)$ whose period is determined by the spacing of the impulses in the Dirac-δ function

Example 2-8:

Multiplication with a Dirac-δ Comb function (by applying the sampling property of the Dirac-δ function). This is used in modeling sampled signals.

Signal $x(t)$

Dirac-δ Comb
$$\sum_n \delta\left(t - \frac{n}{4}\right)$$

Sampled Signal
$$x_s(t) = x(t) \times \sum_n \delta\left(t - \frac{n}{4}\right)$$
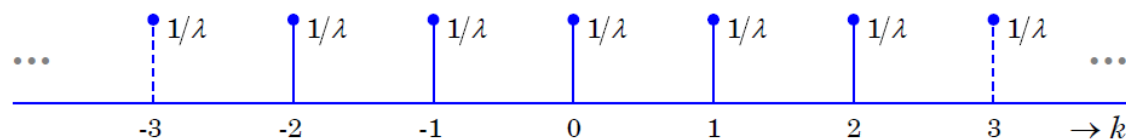$$= \sum_n x\left(\frac{n}{4}\right)\delta\left(t - \frac{n}{4}\right)$$

4.4.2.1 Spectrum of Dirac-δ Comb Function

Let the **Dirac-δ Comb** function (a.k.a. <u>impulse train</u> and <u>sampling function</u> in electrical engineering) defined in Chapter 2 be denoted by:
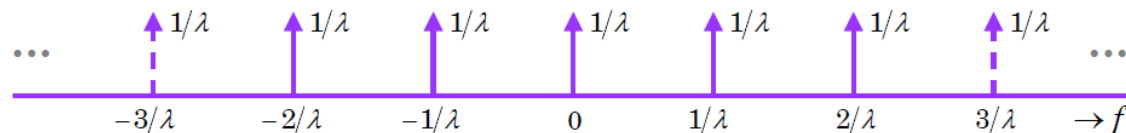
$$\text{comb}_\lambda(t) \triangleq \sum_n \delta(t - n\lambda)$$
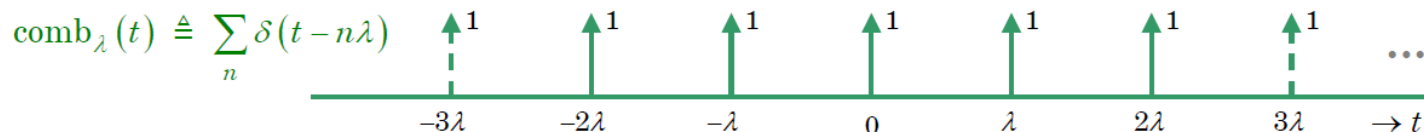
The **discrete-frequency spectrum** (or **Fourier series coefficients**) of $\text{comb}_\lambda(t)$ is given by:

$$c_k = \frac{1}{\lambda}\int_{-\lambda/2}^{\lambda/2} \text{comb}_\lambda(t)e^{-j2\pi kt/\lambda}dt = \frac{1}{\lambda}\int_{-\lambda/2}^{\lambda/2} \delta(t)e^{-j2\pi kt/\lambda}\,dt = \frac{1}{\lambda}$$

The **continuous-frequency spectrum** (or **Fourier transform**) of $\text{comb}_\lambda(t)$ is given by (4.16):

$$\text{COMB}_\lambda(f) = \sum_k c_k \delta(f - k/\lambda) = \frac{1}{\lambda}\sum_k \delta(f - k/\lambda)$$

# 9. Sampling & Reconstruction of Signals

Sampling is a process for converting an analog signal into a discrete-time sequence, which (after quantization) can be digitally stored and processed. This is what is done in CD, DVD and Blue Ray recording, and it is also done to your voice signal for transmission over a radio channel by your digital cell Phone.

In practice, when we sample an analog signal $x(t)$, we capture the value of the signal at every T_s seconds to produce a discrete-time sequence, $x_n = x(nT_s)$. The spectrum of x_n can be derived using a tool called *discrete-time Fourier transform* (DTFT).

Now suppose we multiply $x(t)$ with the impulse train $s_{T_s}(t)$ to produce

$$x_s(t) = x(t) \cdot \overbrace{\sum_n \delta(t - nT_s)}^{s_{T_s}(t)},$$

which is a continuous-time sampled version of $x(t)$. It can be shown that the spectrum of x_n is equal to the spectrum (or Fourier transform) of $x_s(t)$. This allows us to study the mechanism of signal sampling and reconstruction using the Fourier transform. To aid our discussion, we begin by introducing the notion of ideal filters.
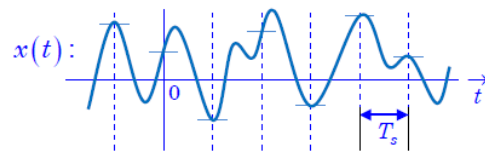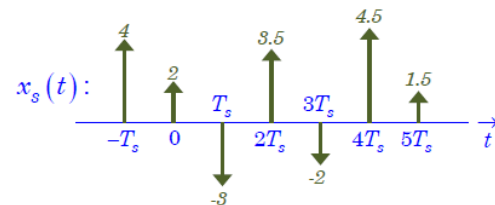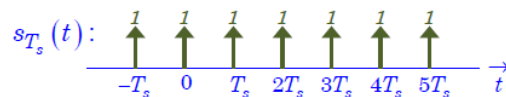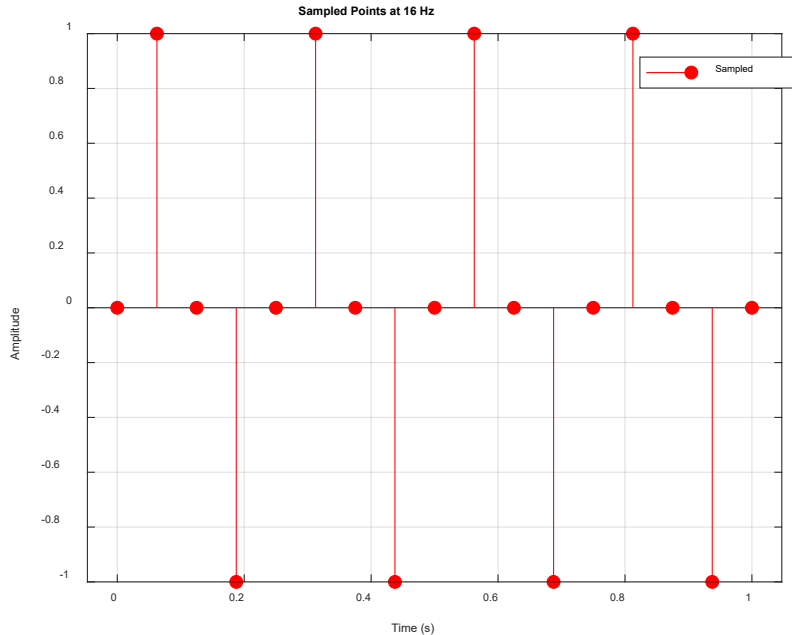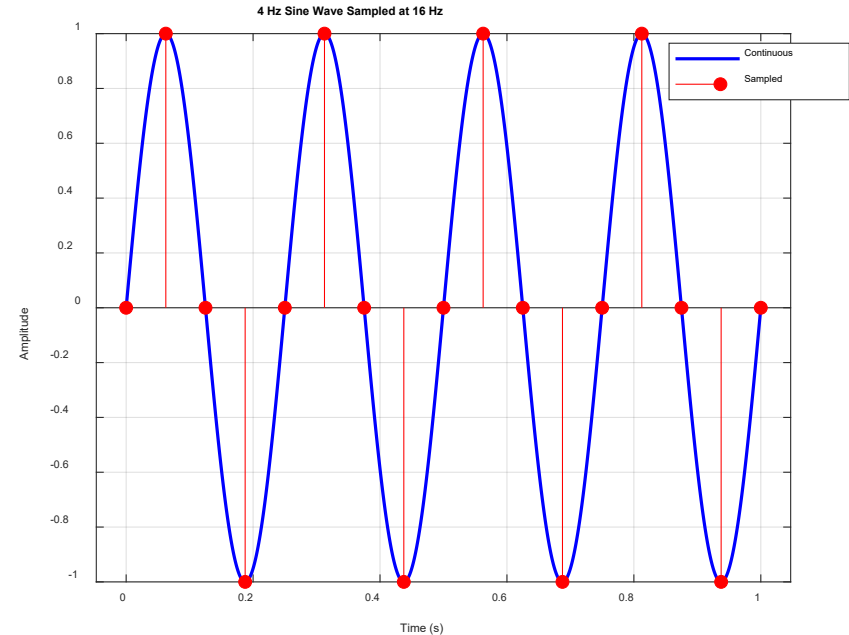
What is an Appropriate Sampling Frequency?

- Red dots represent the discrete samples taken at a 16 Hz sampling rate.
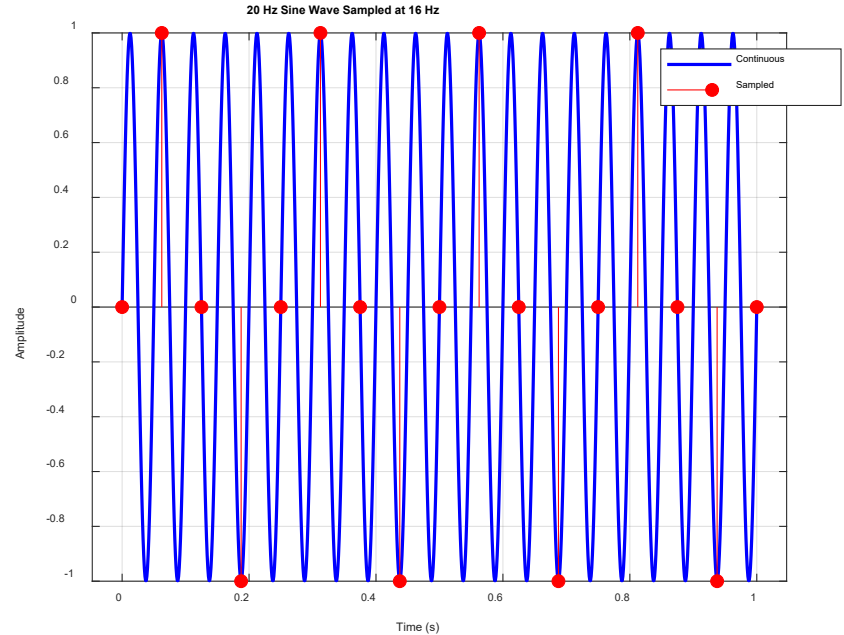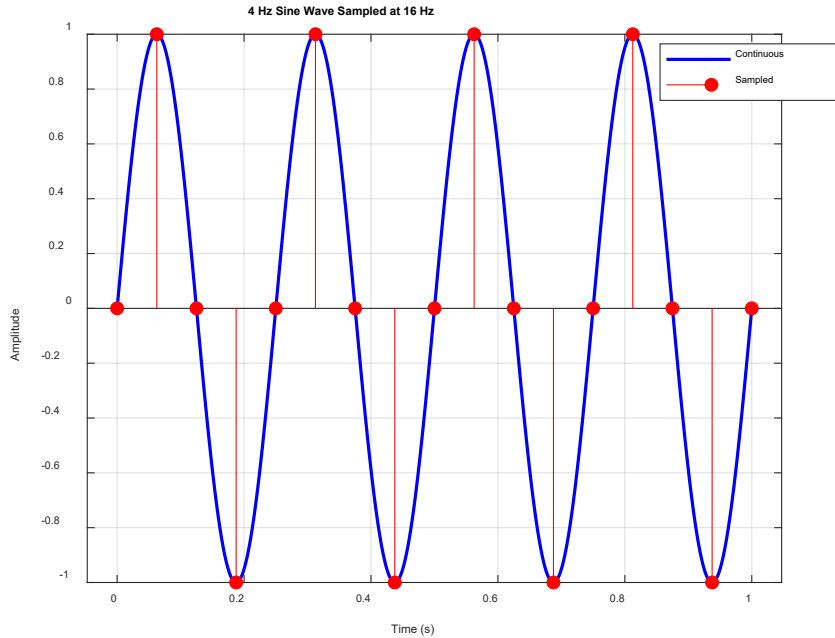
- The sampled points align well with the continuous sine wave in 4 Hz

What is an Appropriate Sampling Frequency?

A 4 Hz sine wave sampled at 16 Hz yields the same samples as a 20 Hz sine wave at the same rate
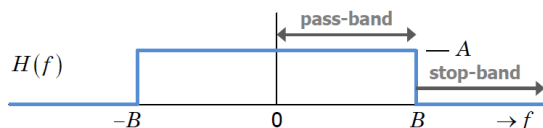
Band-limited Signal

Sampling Band-limited Signal above Nyquist Rate

Sampling Band-limited Bandpass Signal below Nyquist Rate

Conditions for *perfect reconstruction* :

$$\begin{pmatrix} x(t) \text{ must be bandlimited} & : X(f) = 0; \quad |f| > f_m \\ \text{Sampling rate of } x(t) & : f_s \geq 2f_m \end{pmatrix}$$

Reconstruction LPF : $\quad |H(f)| = \begin{cases} K; & |f| < f_m \\ 0; & |f| > f_m \end{cases}$

(a) **Overlapping spectral images** (*Possible if* $f_c > 0.5B$)

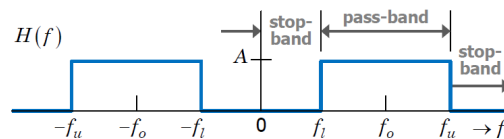$$f_s = 2f_c/k; \quad k = 1, 2, \cdots, \lfloor 2f_c/B \rfloor$$

Assumption: The +ve and -ve frequency bands are symmetric about f_c and $-f_c$, respectively

(b) **Un-aliased spectral images** (*Possible if* $f_c > 1.5B$)

$$\frac{2f_c + B}{k+1} \leq f_s \leq \frac{2f_c - B}{k};$$
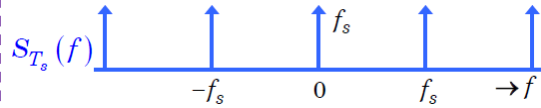
$$k = 1, 2, \cdots, \lfloor \frac{2f_c - B}{2B} \rfloor$$

Ideal Low-Pass Filter

Ideal Band-Pass Filter

Reconstructed Signal

Illustration of Sampling and Reconstruction in Frequency Domain

Sampling Function

$$S_{T_s}(f) = \sum_{n=-\infty}^{\infty} \delta(f - f_s k)$$

$S_{T_s}(f)$ f_s $-f_s$ 0 f_s $\rightarrow f$

$S_{T_s}(f)$

$X(f)$ ✳ $X_s(f)$

Band-limited Signal

$X(f)$ $-X(0)$ $-f_m$ 0 f_m $\rightarrow f$

Sampled Signal

$X_s(f)$ $-f_s X(0)$ $-f_s$ $-f_m$ 0 f_m f_s $\rightarrow f$

$$X_s(f) = X(f) * f_s \sum_{k=-\infty}^{\infty} \delta(f - f_s k)$$

Reconstructed Signal

$\hat{X}(f)$ $-K f_s X(0)$ $-f_m$ 0 f_m $\rightarrow f$

Ideal Low-Pass Filter

$$H(f) = A\,\text{rect}\left(\frac{f}{2B}\right)$$

Ideal LPF
✖ $H(f)$

$\hat{X}(f)$ ← $X_s(f)$

$H(f)$ $-K$ $-f_m$ 0 f_m $\rightarrow f$
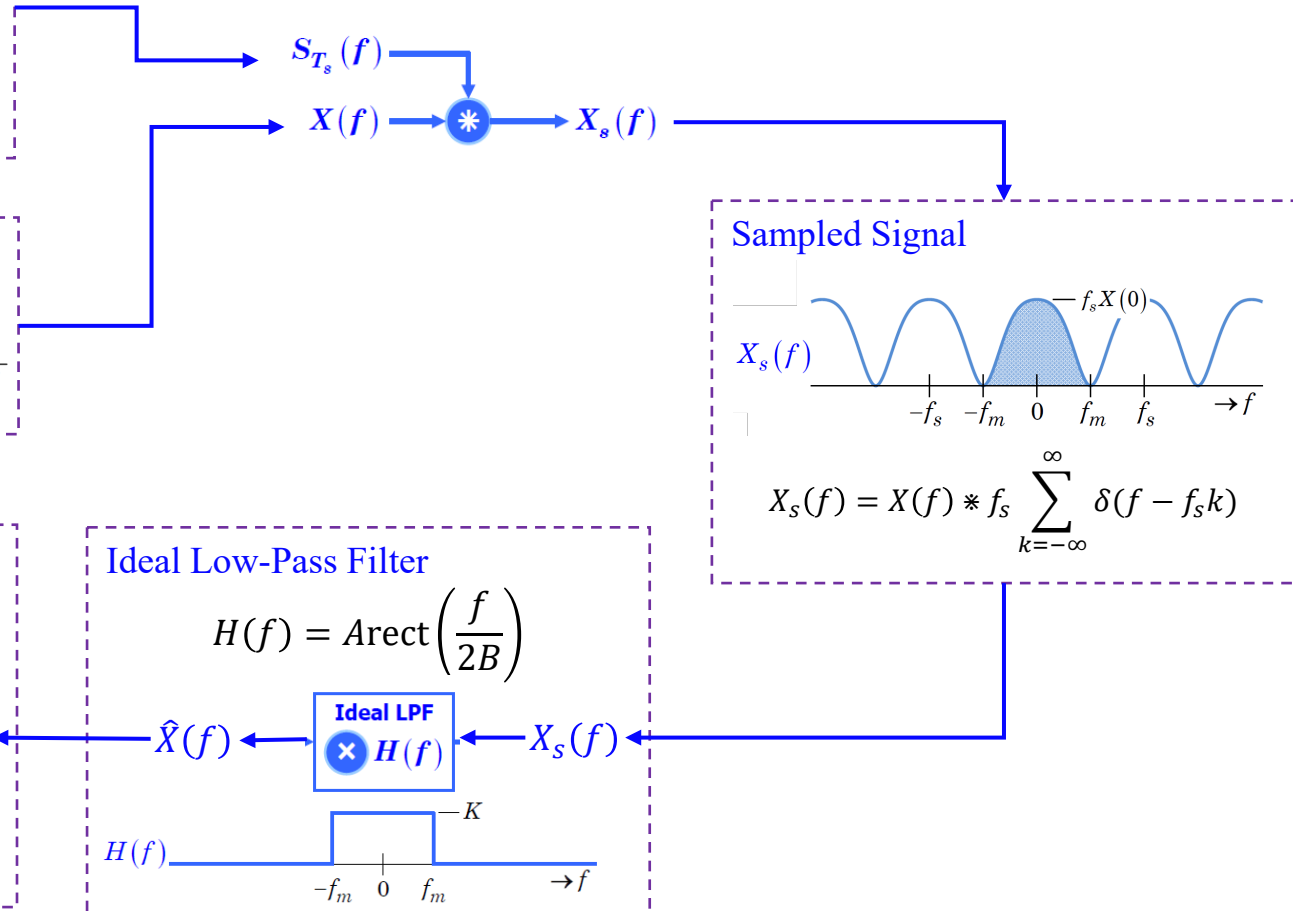
Example 9-2(a): *Lowest sampling frequency with **overlapping spectral images***

A signal has center frequency $f_c = 24\,(\text{Hz})$ and bandwidth $B = 8\,(\text{Hz})$. What are the possible sampling frequencies (below Nyquist rate) that allow for perfect signal reconstruction?

Solution:

Nyquist rate: $(24 + 4) \times 2 = 56\,(\text{Hz})$.

From (9.2a):

$$\begin{cases} f_s = \left(\dfrac{2f_c}{k};\ k = 1,\ 2,\ \cdots,\ \left\lfloor \dfrac{2f_c}{B} \right\rfloor \right) = \dfrac{48}{k};\ k = 1,\ 2,\ \cdots,\ 6 \\[2mm] \text{Possible sampling frequencies :}\ \ f_s = 48,\ 24,\ 16,\ 12,\ 9.6,\ \underset{\substack{\text{min}\\\downarrow}}{8}\ \ Hz \end{cases}$$

Spectrum of ORIGINAL SIGNAL

(a) Images due to the +ve frequency part of the original spectrum

(b) Images due to the −ve frequency part of the original spectrum

(a+b) Spectrum of SAMPLED SIGNAL $f_s = 12\ Hz$
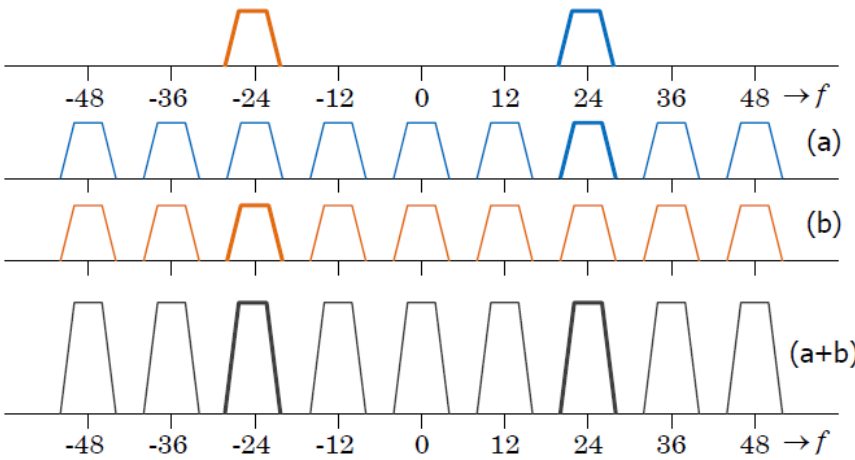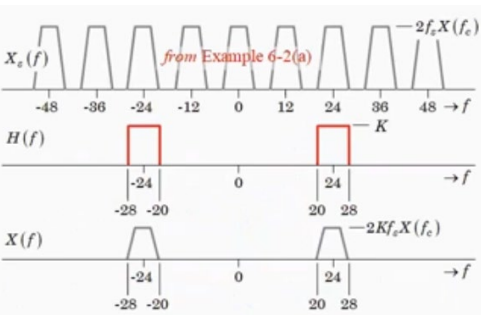
Example 9-2(b): Lowest sampling frequency with <u>un-aliased spectral images</u>

A signal has center frequency $f_c = 24\,(\text{Hz})$ and bandwidth $B = 8\,(\text{Hz})$. What are the possible sampling frequencies (below Nyquist rate) that allow for perfect signal reconstruction?

Solution:

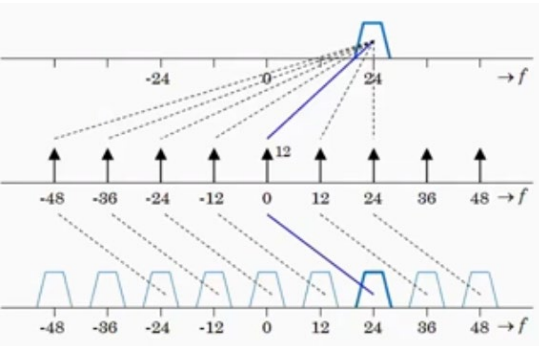Nyquist rate: $(24+4) \times 2 = 56\,(\text{Hz})$.

From (9.2b):

$$\left(\frac{2f_c + B}{k+1} \le f_s \le \frac{2f_c - B}{k}; \quad k = 1,\ 2,\ \cdots,\ \left\lfloor \frac{2f_c - B}{2B} \right\rfloor \right) \ \rightarrow\ \frac{56}{k+1} \le f_s \le \frac{40}{k};\ \ k = 1,\ 2$$

$$\underset{\text{min}}{\downarrow}$$

Possible sampling frequencies : $\underbrace{28 \le f_s \le 40}_{\text{with } k=1}$ or $\underbrace{18.67 \le f_s \le 20}_{\text{with } k=2}$ Hz
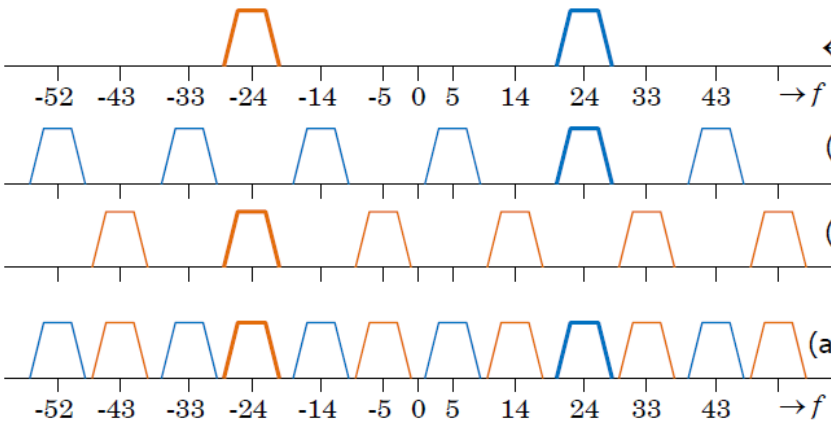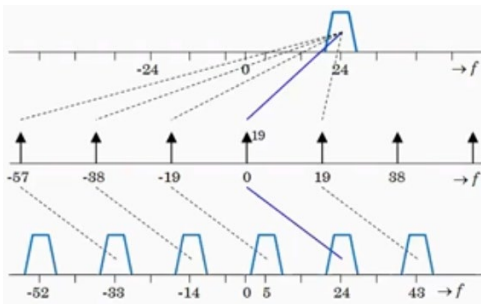

← Spectrum of ORIGINAL SIGNAL

(a) Images due to the +ve frequency part of the original spectrum

(b) Images due to the −ve frequency part of the original spectrum

(a+b) Spectrum of SAMPLED SIGNAL
$f_s = 19\ Hz$
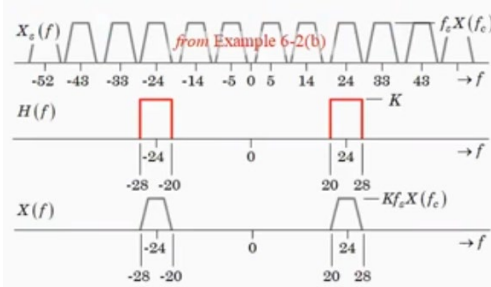
Q.1   A signal $x(t) = 2^{(1-0.5|t|)} \cdot \text{rect}(0.1t)$ is sampled at 0.5 Hz to form $x_s(t)$.

(a)   Derive $x_s(t)$, Hence, determine its spectrum $X_s(f)$.

(b)   Can $x(t)$ be recovered from $x_s(t)$ without distortion and why?

Solution to Q.1

(a) Sampling period $= T_s = \dfrac{1}{0.5} = 2$ s

$$x_s(t) = x(t) \times \sum_n \delta(t - nT_s)$$

$$= \left[2^{1-0.5|t|} \cdot \mathrm{rect}\left(\frac{t}{10}\right) \right] \times \sum_n \delta(t - 2n)$$

$$= \mathrm{rect}\left(\frac{t}{10}\right) \sum_n 2^{1-|n|} \delta(t - 2n)$$

$$= \sum_{n=-2}^{2} 2^{1-|n|} \delta(t - 2n)$$

$$= \frac{1}{2}\delta(t+4) + \delta(t+2) + 2\delta(t) + \delta(t-2) + \frac{1}{2}\delta(t-4)$$

$$X_s(f) = \Im\left\{ \frac{1}{2}\delta(t+4) + \delta(t+2) + 2\delta(t) + \delta(t-2) + \frac{1}{2}\delta(t-4) \right\}$$

$$= \frac{1}{2}e^{j2\pi(4)f} + e^{j2\pi(2)f} + 2 + e^{j2\pi(-2)f} + \frac{1}{2}e^{j2\pi(-4)f} = 2 + 2\cos(4\pi f) + \cos(8\pi f)$$

$\mathrm{rect}\left(\dfrac{t}{10}\right)$

Sketch of $x_s(t)$

$x(t)$
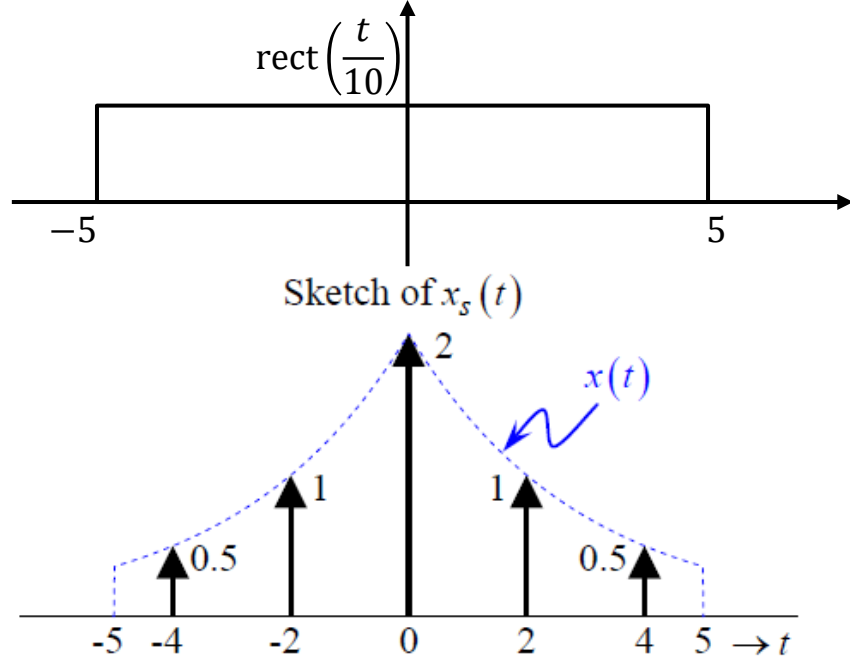
(b)   No, because $x(t)$ is not bandlimited.

Conditions for **perfect reconstruction** :

$$\left(\begin{array}{l} x(t) \ \textit{must be bandlimited} \ : \ X(f) = 0; \quad |f| > f_m \\ \textit{Sampling rate of } x(t) \qquad : \ f_s \geq 2f_m \end{array}\right.$$

$\leftarrow$ This leads to the Nyquist Sampling Theorem

Reconstruction LPF : $|H(f)| = \begin{cases} K; & |f| < f_m \\ 0; & |f| > f_m \end{cases}$

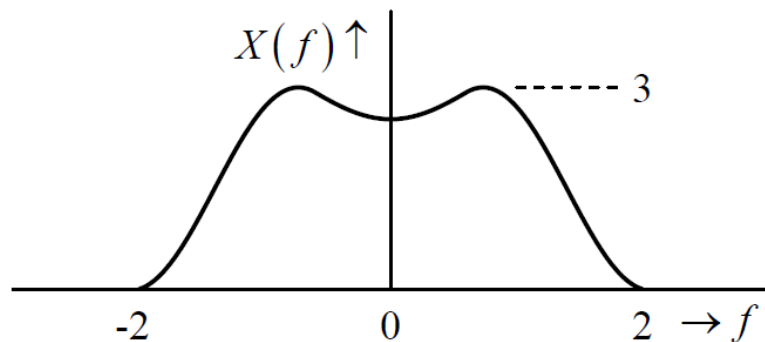Q.2 The spectrum, $X(f)$, of a signal $x(t)$ is shown in Figure Q2.

Figure Q2

(a)   What is the Nyquist sampling frequency for $x(t)$?

(b)   Suppose the signal $x(t)$ is sampled at 6 Hz to obtain the signal $x_s(t)$. Let $X_s(f)$ be the spectrum of $x_s(t)$.

   i.   Express $x_s(t)$ in terms of $x(t)$.

   ii. Find and sketch $X_s(f)$ in terms of $X(f)$.

Solution to Q.2

(a) Nyquist sampling frequency for $x(t) = 2 \times 2 = 4$ Hz.

Lowpass Signal
$$\begin{cases} \text{Bandwidth} = B \\ \text{Highest frequency component} = B \\ \text{Nyquist sampling frequency} = 2B \end{cases}$$
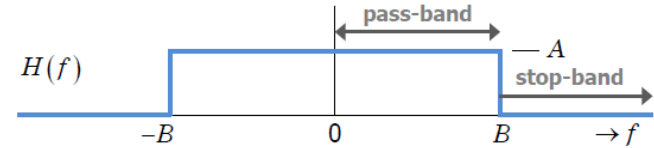
Bandpass Signal
$$\begin{cases} \text{Bandwidth} = B \\ \text{Center frequency} = f_c \\ \text{Highest frequency component} = f_c + \dfrac{B}{2} \\ \text{Nyquist sampling frequency} = 2f_c + B \end{cases}$$
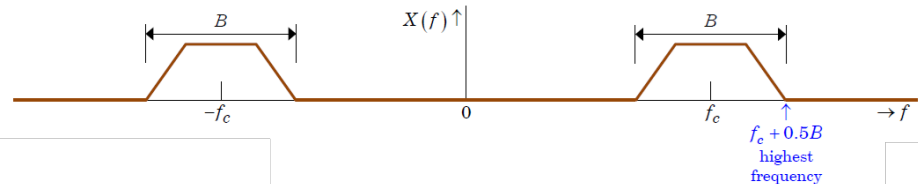
$H(f)$

pass-band

— A

stop-band

$-B$ 0 B $\rightarrow f$

$X(f)$

B B

$-f_c$ 0 f_c $\rightarrow f$

$f_c + 0.5B$
highest frequency

(b) Suppose the signal $x(t)$ is sampled at 6 Hz to obtain the signal $x_s(t)$. Let $X_s(f)$ be the spectrum of $x_s(t)$.

i. Express $x_s(t)$ in terms of $x(t)$.

ii. Find and sketch $X_s(f)$ in terms of $X(f)$.

Sampling frequency $= 6$ Hz

(i) $\quad x_s(t) = x(t) \times \sum_{n=-\infty}^{\infty} \delta\left(t - \frac{n}{6}\right) = \sum_{n=-\infty}^{\infty} x\left(\frac{n}{6}\right)\delta\left(t - \frac{n}{6}\right)$

(ii) $\quad X_s(f) = X(f) * 6\sum_{k=-\infty}^{\infty} \delta(f - 6k) = 6\sum_{k=-\infty}^{\infty} X(f - 6k)$
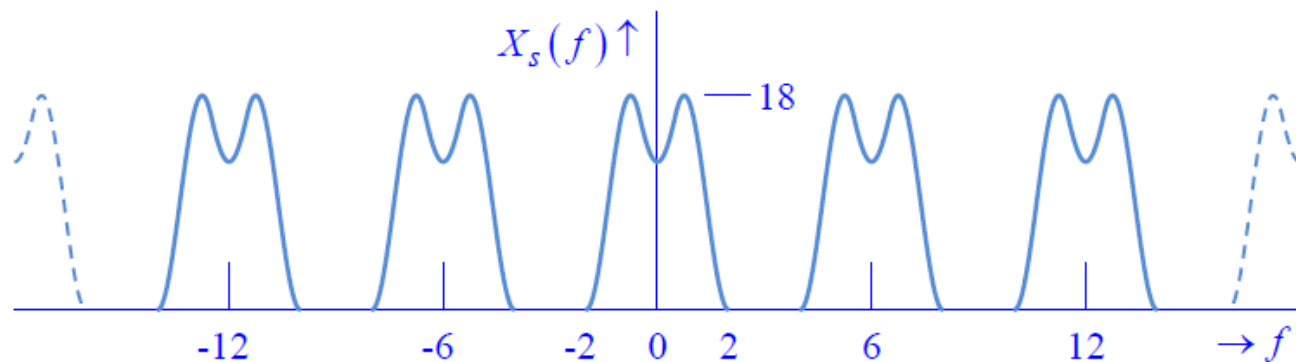
Q.3 The spectrums of two real bandpass signals, $x(t)$ and $y(t)$, are shown in Figure Q3.
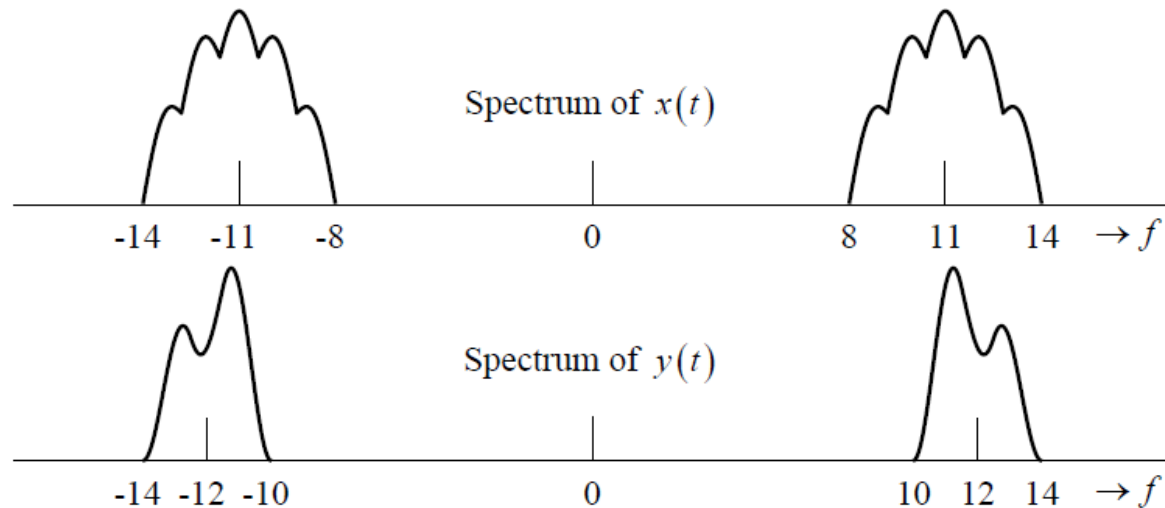
Figure Q3

For each of $x(t)$ and $y(t)$:

(a)   What is the Nyquist sampling frequency?

(b)   Find the smallest sampling frequency that allows for exact reconstruction of the signal from its sampled version and determine the frequency response, $H(f)$, of the corresponding ideal reconstruction filter.

Q.3 The spectrums of two real bandpass signals, $x(t)$ and $y(t)$, are shown in Figure Q3.

Spectrum of $x(t)$

-14   -11   -8   0   8   11   14 $\rightarrow f$

Spectrum of $y(t)$

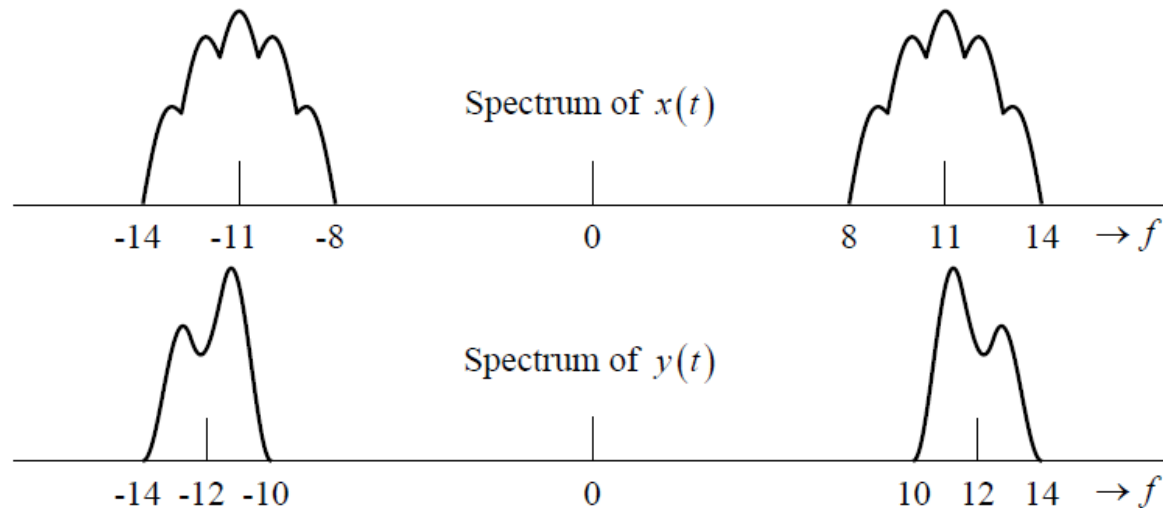-14 -12 -10   0   10 12 14 $\rightarrow f$

Figure Q3

For each of $x(t)$ and $y(t)$:

(a)   What is the Nyquist sampling frequency?

Solution to Q.3

(a)   Nyquist sampling frequency for both $x(t)$ and $y(t)$ is $2 \times 14 = 28$ Hz.

(b) Find the smallest sampling frequency that allows for exact reconstruction of the signal from its sampled version and determine the frequency response, $H(f)$, of the corresponding ideal reconstruction filter.

Spectrum of $x(t)$

SIGNAL $x(t)$:

Due to symmetry of the positive-frequency and negative-frequency spectral images about the center frequencies 11 Hz and -11 Hz, respectively, both the Overlapping Spectral Images formula and the Un-aliased Spectral Images formula are applicable. Of these, the Overlapping Spectral Images formula always yield the lowest sampling frequency.
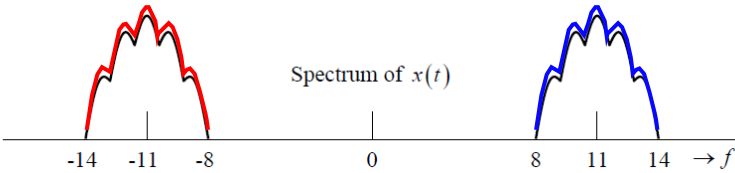
Center frequency : $f_c = 11$ Hz

Bandwidth : $B = 6$ Hz

Overlapping spectral images formula :
$$\begin{cases} f_s = \dfrac{2f_c}{k}; \quad k = 1, 2, \cdots, \left\lfloor \dfrac{2f_c}{B} \right\rfloor \\ = \dfrac{22}{k}; \quad k = 1, 2, 3 \\ = 22, \quad 11, \quad \dfrac{22}{3} \text{ Hz} \end{cases}$$
(*Possible if* $f_c > 0.5B$)

Images due to the positive frequency part

Images due to the negative frequency part

- Therefore, the smallest possible sampling frequency is $f_{s,\min} = \dfrac{22}{3}$ Hz

- For exact reconstruction of $x(t)$, the frequency response of the ideal reconstruction filter should be:

$$H(f) = \frac{1}{2f_{s,\min}}\left[\text{rect}\left(\frac{f+f_c}{B} \right) + \text{rect}\left(\frac{f-f_c}{B} \right) \right] = \frac{3}{44}\left[\text{rect}\left(\frac{f+11}{6} \right) + \text{rect}\left(\frac{f-11}{6} \right) \right]$$

$1/2f_s$

f_s

f_s

$2f_s$
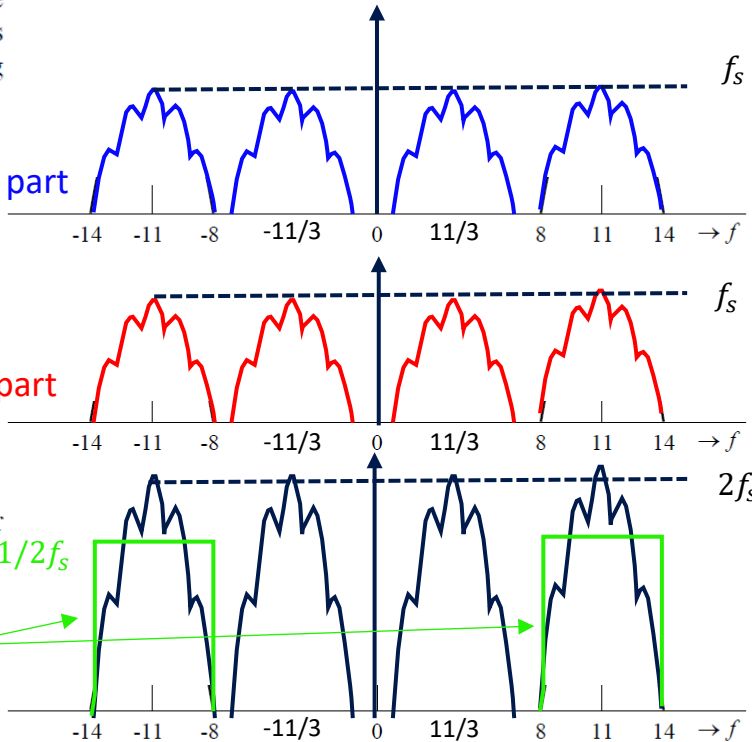
SIGNAL $y(t)$:

Due to asymmetry of the positive-frequency and negative-frequency spectral images about the center frequencies 12 Hz and -12 Hz, respectively, only the Un-aliased Spectral Images formula is applicable.

Center frequency : $f_c = 12$ Hz

Bandwidth : $B = 4$ Hz

Un-aliased spectral images formula
(*Possible if* $f_c > 1.5B$) :
$$\begin{cases} \dfrac{2f_c + B}{k+1} \le f_s \le \dfrac{2f_c - B}{k}; \quad k = 1, 2, \cdots, \left\lfloor \dfrac{2f_c - B}{2B} \right\rfloor \\[2mm] \rightarrow \quad \dfrac{28}{k+1} \le f_s \le \dfrac{20}{k}; \quad k = 1, 2 \\[2mm] \rightarrow \quad \underbrace{14 \le f_s \le 20}_{k=1} \text{ and } \underbrace{\dfrac{28}{3} \le f_s \le 10}_{k=2} \end{cases}$$
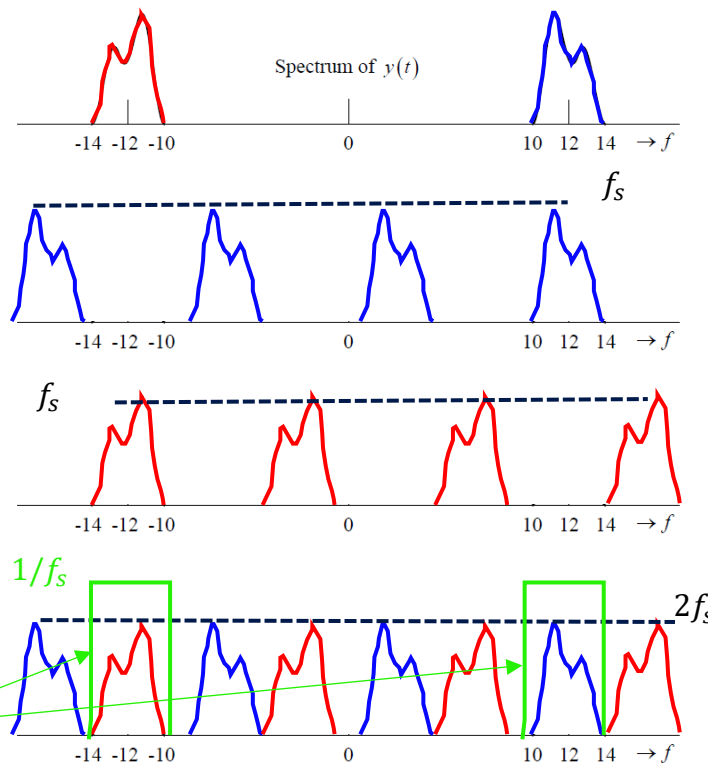
- Therefore, the smallest possible sampling frequency is $f_{s,\min} = \dfrac{28}{3}$ Hz

- For exact reconstruction of $y(t)$, the frequency response of the ideal reconstruction filter should be:
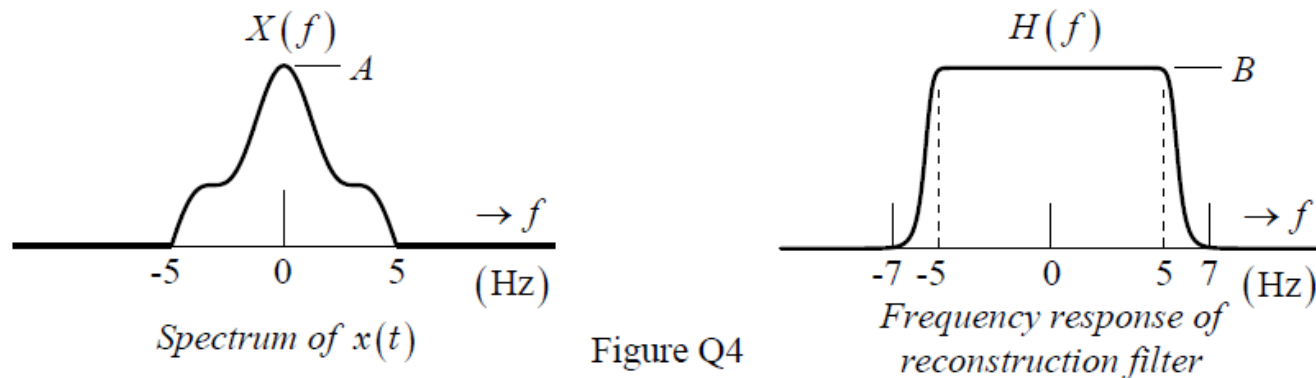
$$H(f) = \frac{1}{f_{s,\min}}\left[\text{rect}\left(\frac{f + f_c}{B}\right) + \text{rect}\left(\frac{f - f_c}{B}\right)\right] = \frac{3}{28}\left[\text{rect}\left(\frac{f + 12}{4}\right) + \text{rect}\left(\frac{f - 12}{4}\right)\right]$$

Images due to the positive frequency part

Images due to the negative frequency part

Spectrum of $y(t)$

Q.4 A signal $x(t)$ is sampled, stored and later reconstructed from the stored samples using a reconstruction filter. The spectrum of $x(t)$ and the frequency response of the reconstruction filter are shown in Figure Q4.

$X(f)$

A

→ f

-5   0   5   (Hz)

Spectrum of $x(t)$

$H(f)$

B

→ f

-7 -5   0   5 7   (Hz)

Frequency response of reconstruction filter

Figure Q4

(a)   What is the Nyquist sampling frequency for $x(t)$?

(b)   What sampling frequency would you recommend so that $x(t)$ can be reconstructed from its samples without distortion, and why? Illustrate your answer.

Q.4 A signal $x(t)$ is sampled, stored and later reconstructed from the stored samples using a reconstruction filter. The spectrum of $x(t)$ and the frequency response of the reconstruction filter are shown in Figure Q4.
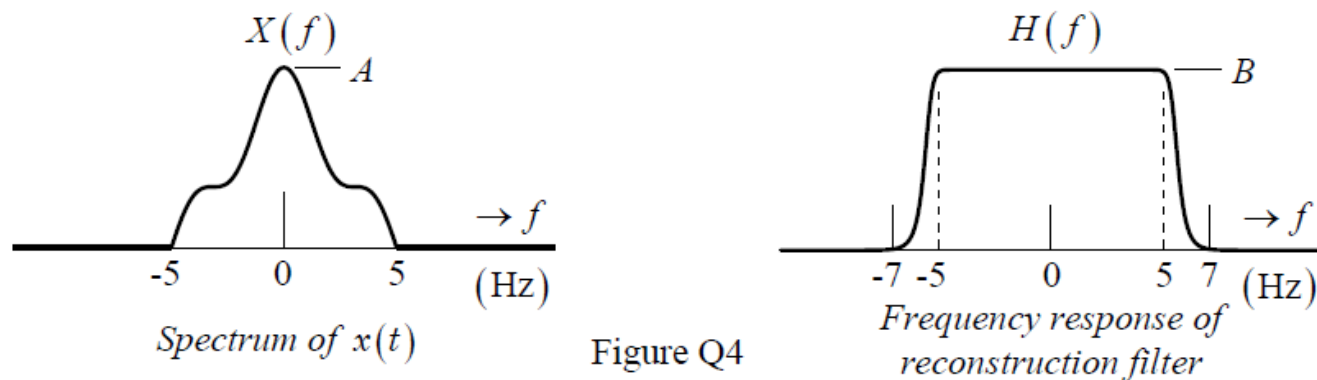

Spectrum of $x(t)$ Figure Q4   Frequency response of reconstruction filter
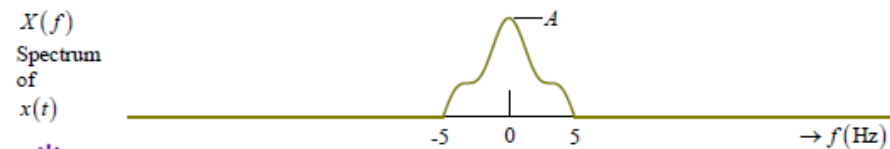
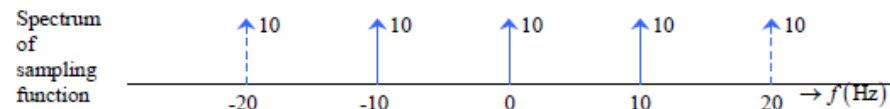(a)   What is the Nyquist sampling frequency for $x(t)$?

Solution to Q.4

- Nyquist sampling frequency: $2 \times 5 = 10$ Hz

- Recommended sampling frequency: 12 Hz

  The excess 2 Hz is needed to prevent adjacent spectral images from contributing to the reconstruction process. See illustration below.
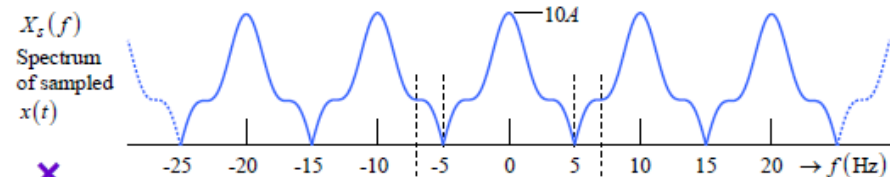
$X(f)$
Spectrum
of
$x(t)$

$-A$

-5   0   5

$\rightarrow f(\text{Hz})$

✳

Spectrum
of
sampling
function

10   10   10   10   10

-20   -10   0   10   20 $\rightarrow f(\text{Hz})$

=

$X_s(f)$
Spectrum
of sampled
$x(t)$

$-10A$

-25 -20 -15 -10 -5   0   5   10   15   20 $\rightarrow f(\text{Hz})$

✕

$H(f)$
Frequency
response of
reconstruction
filter

$-B$

-7 -5   0   5 7 $\rightarrow f(\text{Hz})$

=
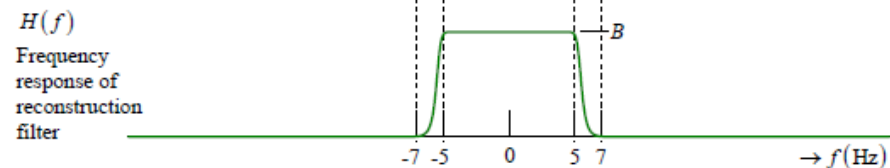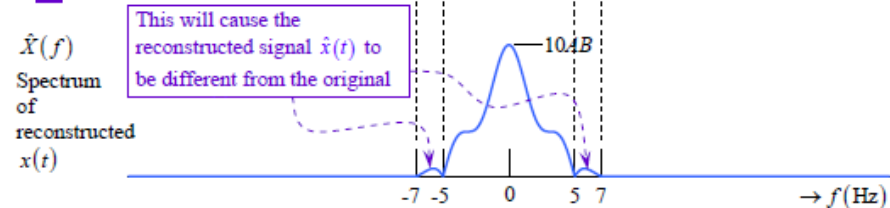
This will cause the
reconstructed signal $\hat{x}(t)$ to
be different from the original

$\hat{X}(f)$
Spectrum
of
reconstructed
$x(t)$

$-10AB$

-7 -5   0   5 7 $\rightarrow f(\text{Hz})$

Sampling frequency: **12 Hz**

$X(f)$ Spectrum of $x(t)$ — A

Spectrum of sampling function

$X_s(f)$ Spectrum of sampled $x(t)$ — 12A

$H(f)$ Frequency response of reconstruction filter — B

$\hat{X}(f)$ Spectrum of reconstructed $x(t)$ — 12AB
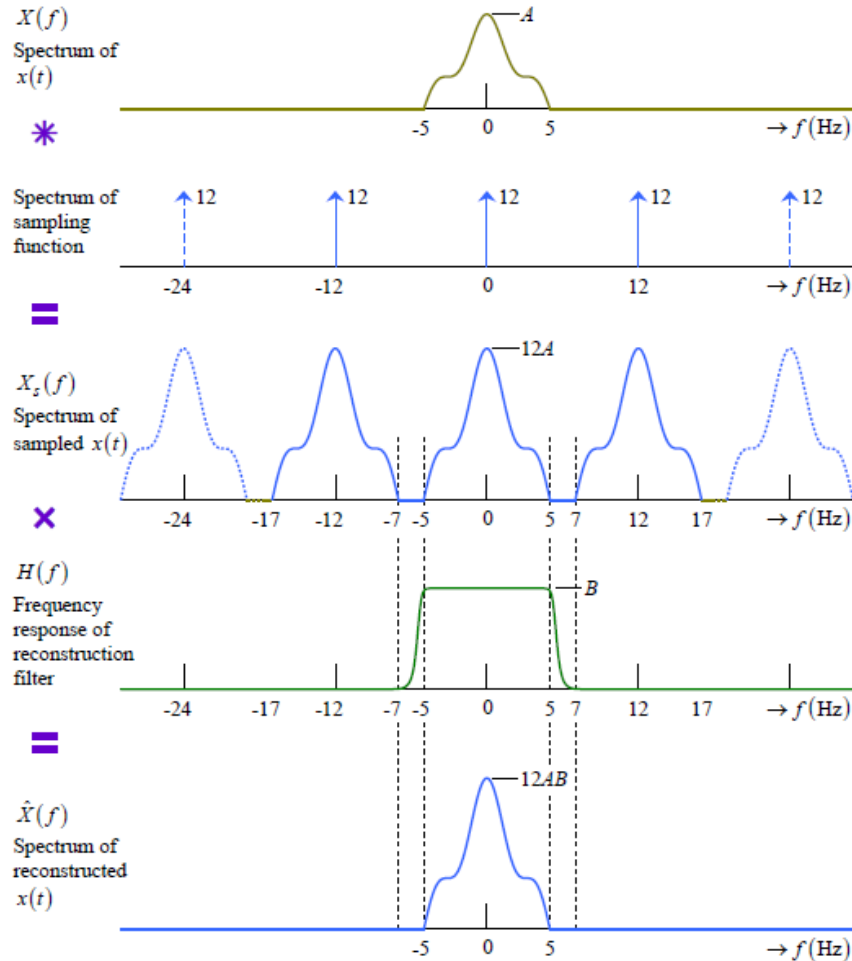
Remarks: Sampling frequencies higher than 12 Hz may be used to achieve the same result. However, high sampling frequency is usually matched by more costly data acquisition, storage and processing requirement.

Q.5 Speech studies have shown that perceptual cues pertaining to speech intelligibility and speaker identity are mainly found within the first 4 kHz of the speech spectrum. We shall refer to this frequency band as the *TQ-Band*, where *TQ* stands for '*Telephone Quality*'.

The spectrum, $X(f)$, of a speech signal $x(t)$ is shown in Figure Q5 where the *TQ-Band* is indicated.
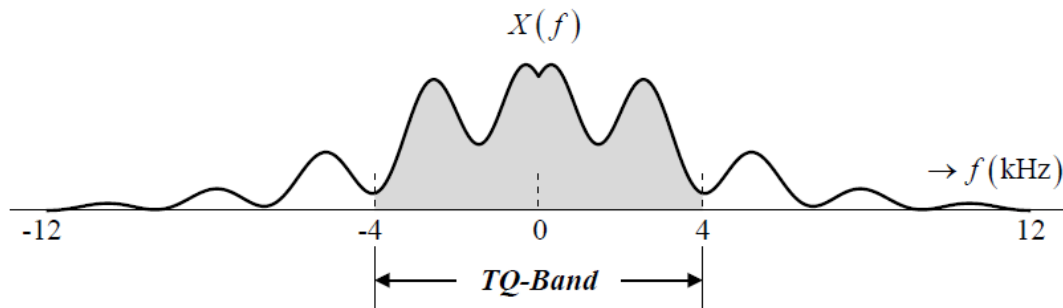

Figure Q5

In order to send $x(t)$ over a digital telephone network, the first step is to sample $x(t)$ without corrupting its *TQ-Band*. Suggest a method for sampling $x(t)$ in each of the following situations.

Situation A: Anti-aliasing lowpass filter is not available and frequency aliasing is prohibited.

Situation B: Lowest sampling frequency must be used at all cost.

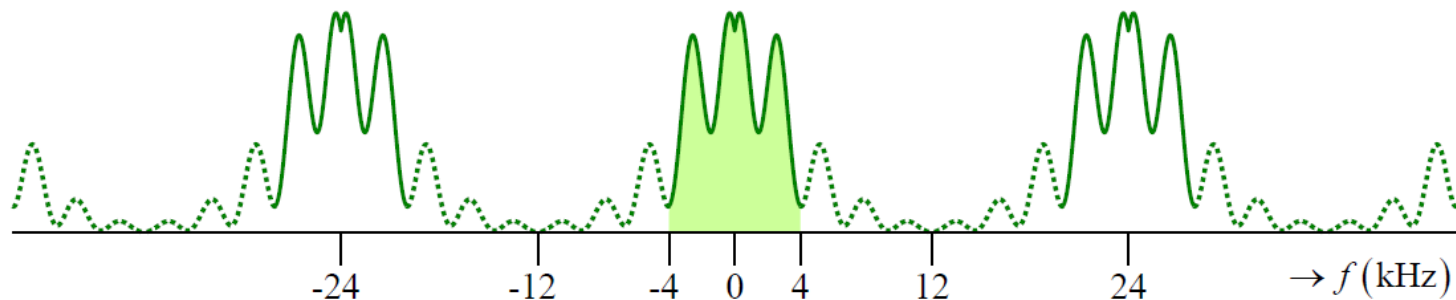Situation C: Anti-aliasing lowpass filter is not available and frequency aliasing is permitted.

Comment on the advantage and disadvantage of each method.

Situation A: Anti-aliasing lowpass filter is not available and frequency aliasing is prohibited.

$X(f)$

$\rightarrow f\,(\text{kHz})$

-12   -4   0   4   12

$\longleftarrow$ *TQ-Band* $\longrightarrow$
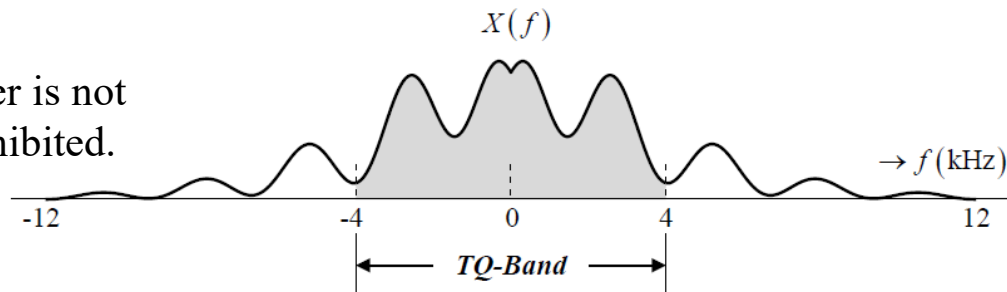
Solution to Q.5

Situation A: $\left(\begin{array}{l}\textbf{No anti-aliasing lowpass filter}\\ \textbf{Frequency aliasing prohibited}\end{array}\right)$: **Sampling frequency** $= 2 \times 12 = 24$ **kHz**

$\rightarrow f\,(\text{kHz})$

-24   -12   -4   0   4   12   24
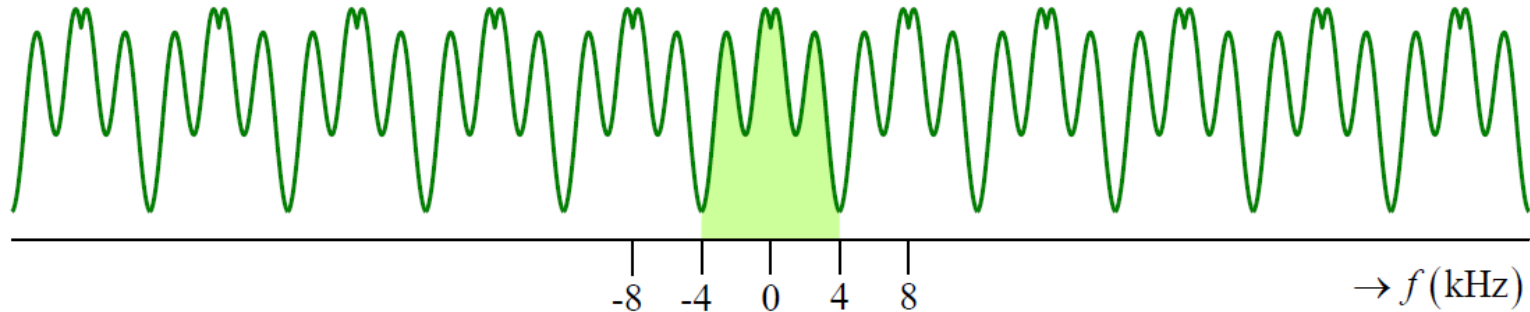
Advantage: No anti-aliasing LPF needed.

Disadvantage: Sampling frequency is higher than necessary to preserve the *TQ-Band*.

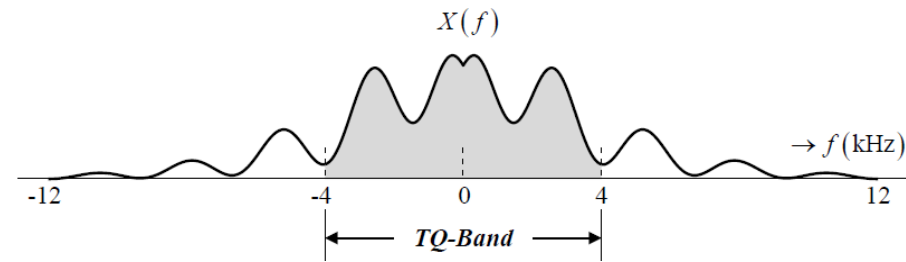**Situation B:** Lowest sampling frequency must be used at all cost.

$X(f)$

$\rightarrow f \text{(kHz)}$

-12   -4   0   4   12
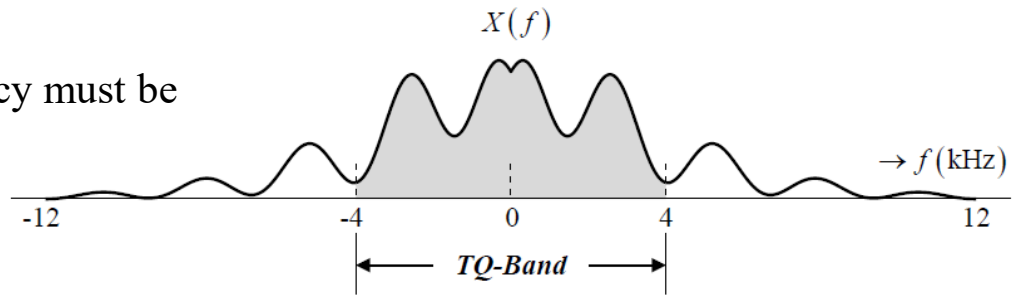
TQ-Band

Situation B: Lowest sampling frequency: $\begin{cases} \textbf{Ideal anti-aliasing filter of bandwidth 4 kHz} \\ \textbf{Sampling frequency} = 2 \times 4 = 8 \textbf{ kHz} \end{cases}$

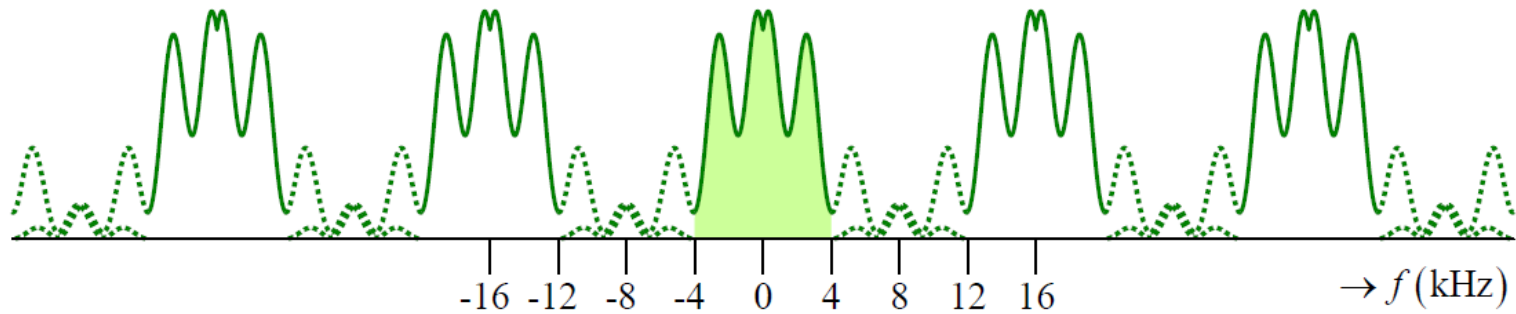
$\rightarrow f \text{(kHz)}$

-8 -4 0 4 8

Advantage: Lowest possible sampling frequency to preserve the *TQ-Band*.

Disadvantage: Require anti-aliasing LPF with sharp cutoff.

Situation C: Anti-aliasing lowpass filter is not available and frequency aliasing is permitted.

$X(f)$

$\rightarrow f \, (\text{kHz})$

-12   -4   0   4   12

TQ-Band

Situation C: $\begin{pmatrix} \textbf{No anti-aliasing lowpass filter} \\ \textbf{Frequency aliasing permitted} \end{pmatrix}$: **Sampling frequency = 24 – 8 = 16 kHz**

$\rightarrow f \, (\text{kHz})$

-16   -12   -8   -4   0   4   8   12   16

Advantage: Lowest possible sampling frequency to preserve the *TQ-Band* without requiring anti-aliasing LPF with sharp cutoff.

Disadvantage: Sampling frequency is still higher than the minimum needed to preserve the *TQ-Band*.

This problem illustrates the trade-off between oversampling and the requirement of expensive anti-aliasing LPF with sharp cutoff frequency.

Thanks!