CG2023 Tutorial 8

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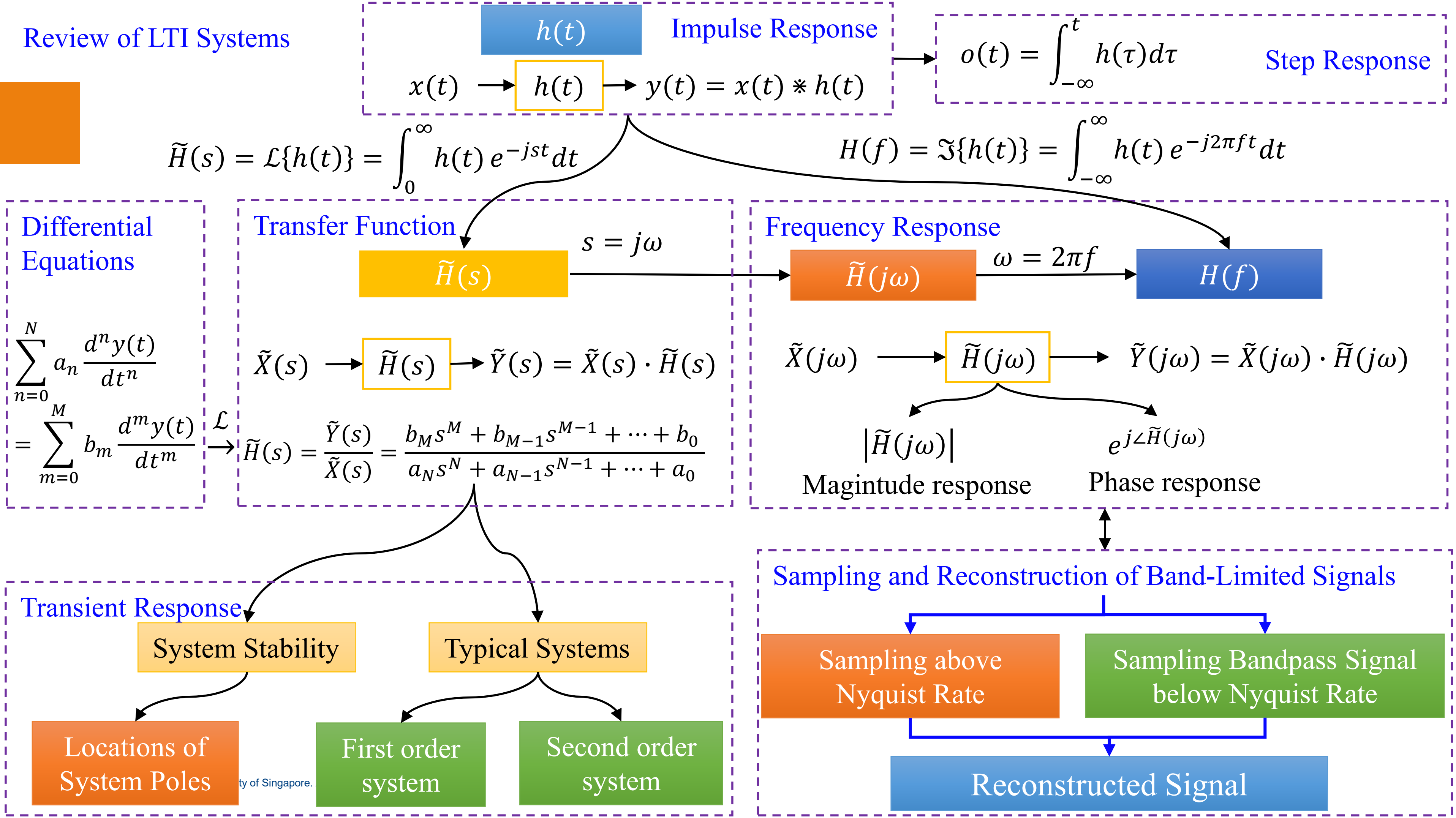
Review of LTI Systems



Review

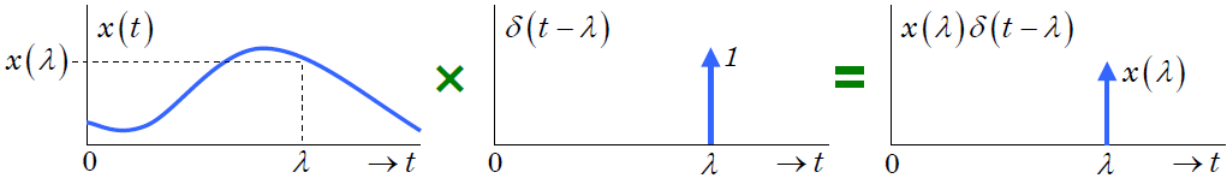
- **Properties of $\delta(t)$**

1. ***Symmetry:***

$$\delta(t) = \delta(-t) \tag{2.3}$$

2. ***Sampling:***

$$x(t)\delta(t-\lambda) = x(\lambda)\delta(t-\lambda) \tag{2.4}$$



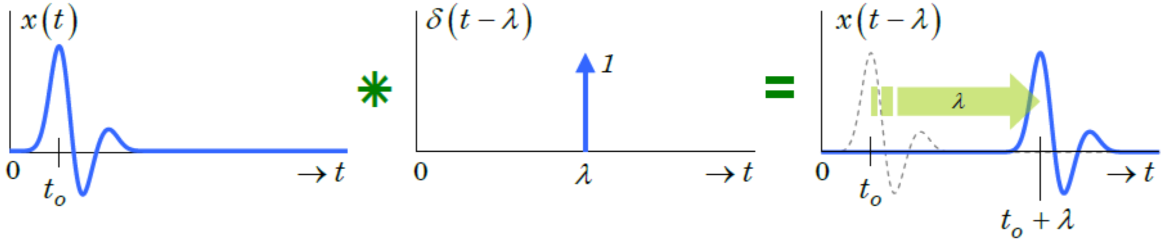
3. ***Sifting:***

$$\underbrace{\int_{-\infty}^{\infty} x(t)\delta(t-\lambda)dt = x(\lambda)\int_{-\infty}^{\infty}\delta(t-\lambda)dt}_{\textit{from the sampling property of } \delta(\bullet)} = x(\lambda) \tag{2.5}$$

4. *Replication:*

$$x(t)*\delta(t-\lambda)=\overbrace{\int_{-\infty}^{\infty}x(\zeta)\delta(t-\zeta-\lambda)\,d\zeta=}^{\textit{apply } \textbf{symmetry} \textit{ property}}\underbrace{\int_{-\infty}^{\infty}x(\zeta)\delta(\zeta-(t-\lambda))\,d\zeta=x(t-\lambda)}_{\textit{apply } \textbf{sifting} \textit{ property}} \tag{2.6}$$

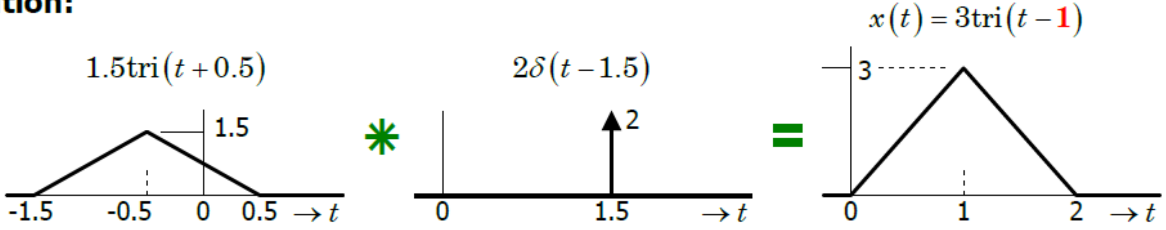
Note: $x(t)*\delta(t)=x(t)$



Example 2-5

Find and sketch $x(t)=\mathbf{1.5tri}(t+0.5)*2\delta(t-1.5)$

Solution:

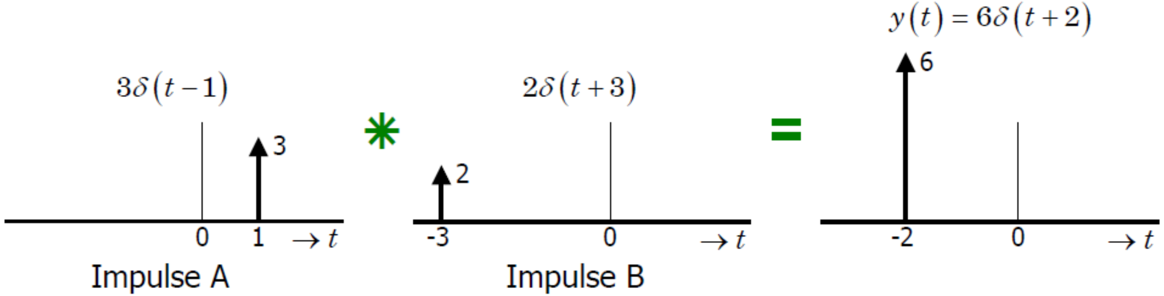


Review

Example 2-6

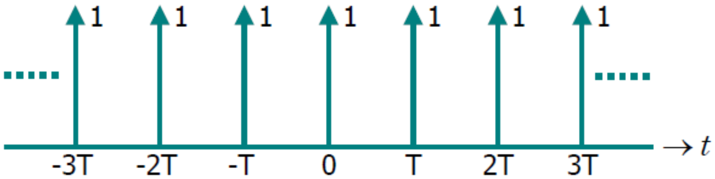
Find and sketch $\boldsymbol{y(t) = 3\delta(t-1) * 2\delta(t+3)}$

Solution:



2.2.1 The Dirac-δ Comb function (a.k.a. Sampling function)

$$\sum_{n=-\infty}^{\infty} \delta(t-nT) = \cdots\cdots + \delta(t+2T) + \delta(t+T) + \delta(t) + \delta(t-T) + \delta(t-2T) + \cdots\cdots \quad (2.7)$$



Review

Example 2-7:

Convolution with a Dirac-δ Comb function *(by applying the replication property of the Dirac-δ function). This is used in modeling periodic signals.*

Signal

$x(t)$

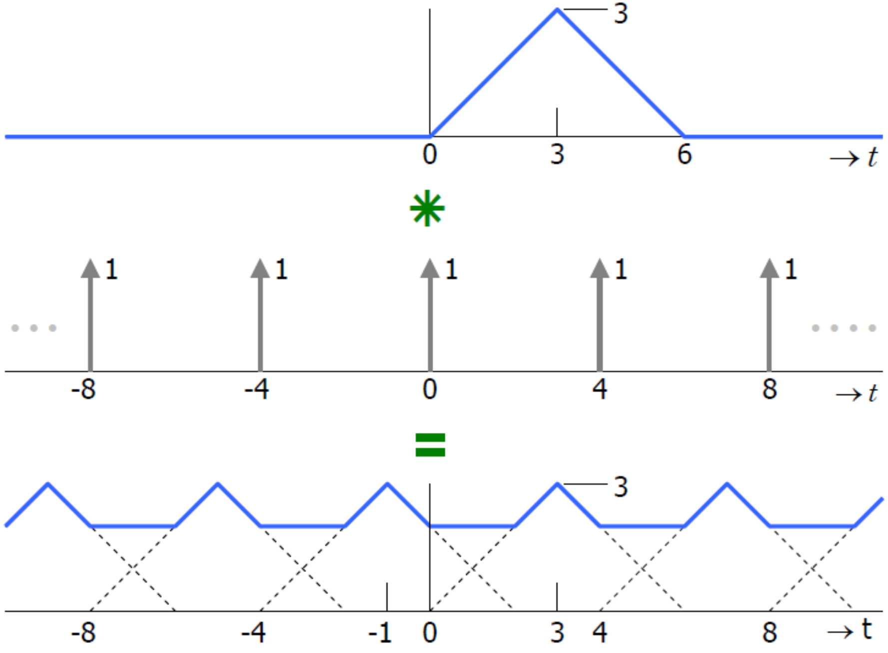
Dirac-δ Comb

$\sum_n \delta(t-4n)$

Periodic Signal

$x_p(t) = x(t) * \sum_n \delta(t-4n)$

$= \sum_n x(t-4n)$



*$x(t)$ is called the **Generating Function** of the periodic signal $y(t)$ whose period is determined by the spacing of the impulses in the Dirac-δ function*

Review

Example 2-8:

Multiplication with a Dirac-δ Comb function *(by applying the sampling property of the Dirac-δ function). This is used in modeling sampled signals.*

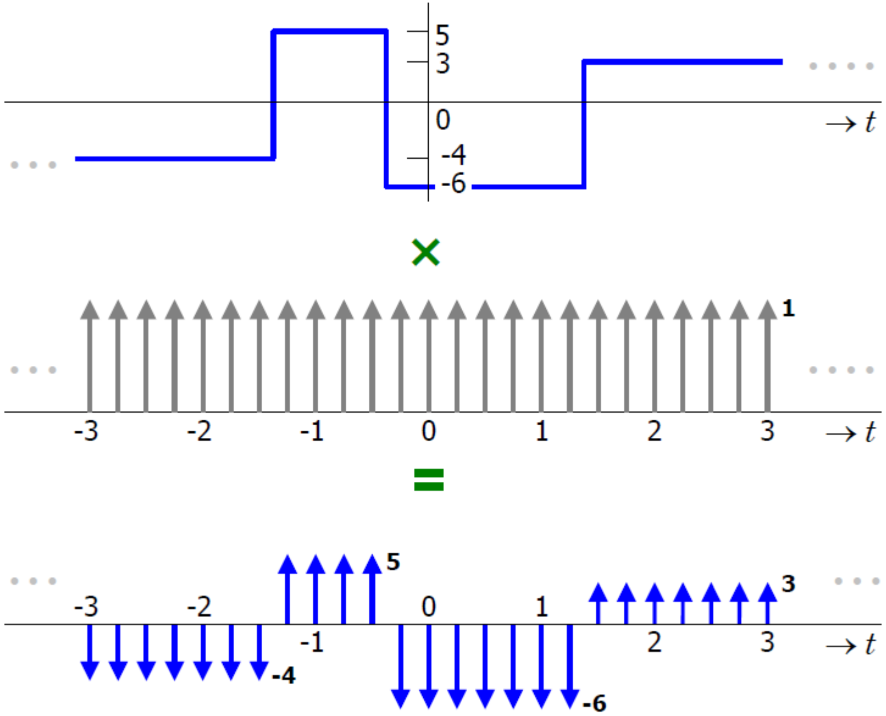
Signal $x(t)$

Dirac-δ Comb $\sum_n \delta\left(t-\frac{n}{4}\right)$

Sampled Signal

$$x_s(t)=x(t)\times\sum_n \delta\left(t-\frac{n}{4}\right)$$

$$=\sum_n x\left(\frac{n}{4}\right)\delta\left(t-\frac{n}{4}\right)$$

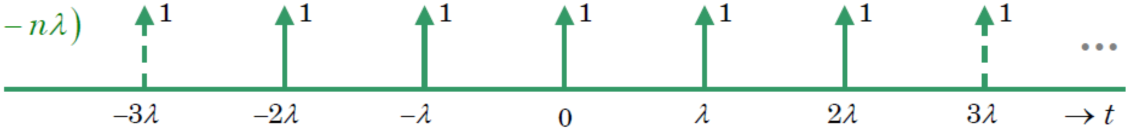


Review

4.4.2.1 Spectrum of Dirac-δ Comb Function

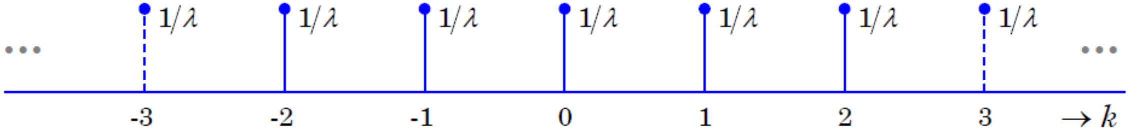
Let the **Dirac-δ Comb** function (a.k.a. impulse train and sampling function in electrical engineering) defined in Chapter 2 be denoted by:

$$\operatorname{comb}_\lambda(t) \triangleq \sum_n \delta(t - n\lambda)$$



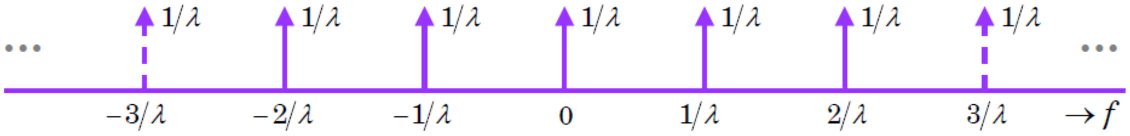
The **discrete-frequency spectrum** (or **Fourier series coefficients**) of $\operatorname{comb}_\lambda(t)$ is given by:

$$c_k = \frac{1}{\lambda}\int_{-\lambda/2}^{\lambda/2} \operatorname{comb}_\lambda(t) e^{-j2\pi kt/\lambda} dt = \frac{1}{\lambda}\int_{-\lambda/2}^{\lambda/2} \delta(t) e^{-j2\pi kt/\lambda}\, dt = \frac{1}{\lambda}$$



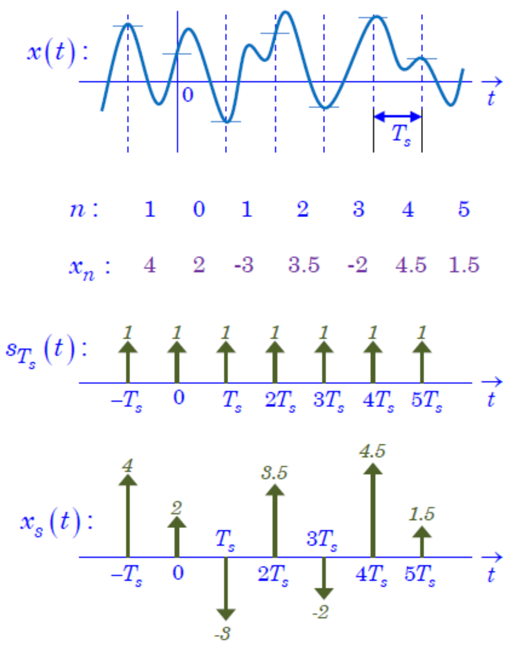
The **continuous-frequency spectrum** (or **Fourier transform**) of $\operatorname{comb}_\lambda(t)$ is given by (4.16):

$$\operatorname{COMB}_\lambda(f) = \sum_k c_k \delta(f - k/\lambda) = \frac{1}{\lambda}\sum_k \delta(f - k/\lambda)$$



9. Sampling & Reconstruction of Signals

Sampling is a process for converting an analog signal into a discrete-time sequence, which (after quantization) can be digitally stored and processed. This is what is done in CD, DVD and Blue Ray recording, and it is also done to your voice signal for transmission over a radio channel by your digital cell Phone.



In practice, when we sample an analog signal $x(t)$, we capture the value of the signal at every T_s seconds to produce a discrete-time sequence, $x_n = x(nT_s)$. The spectrum of x_n can be derived using a tool called *discrete-time Fourier transform* (DTFT).

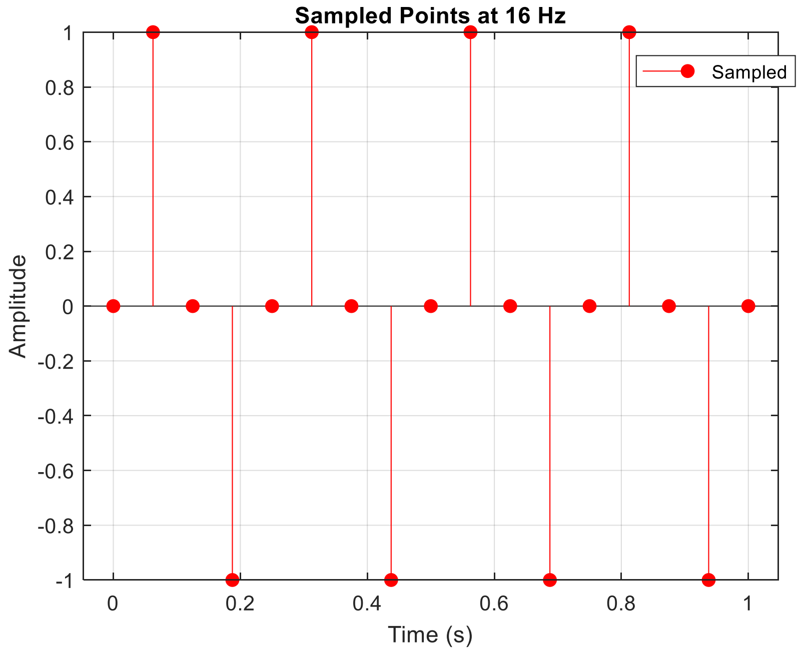
Now suppose we multiply $x(t)$ with the impulse train $s_{T_s}(t)$ to produce

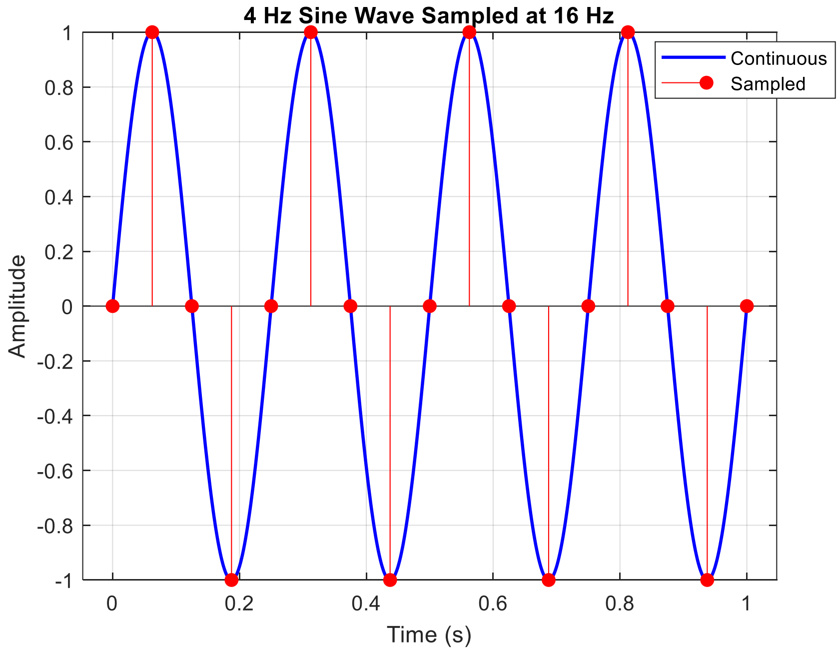
$$x_s(t) = x(t) \cdot \overbrace{\sum_n \delta(t - nT_s)}^{s_{T_s}(t)},$$

which is a continuous-time sampled version of $x(t)$. It can be shown that the spectrum of x_n is equal to the spectrum (or Fourier transform) of $x_s(t)$. This allows us to study the mechanism of signal sampling and reconstruction using the Fourier transform. To aid our discussion, we begin by introducing the notion of ideal filters.

What is an Appropriate Sampling Frequency?

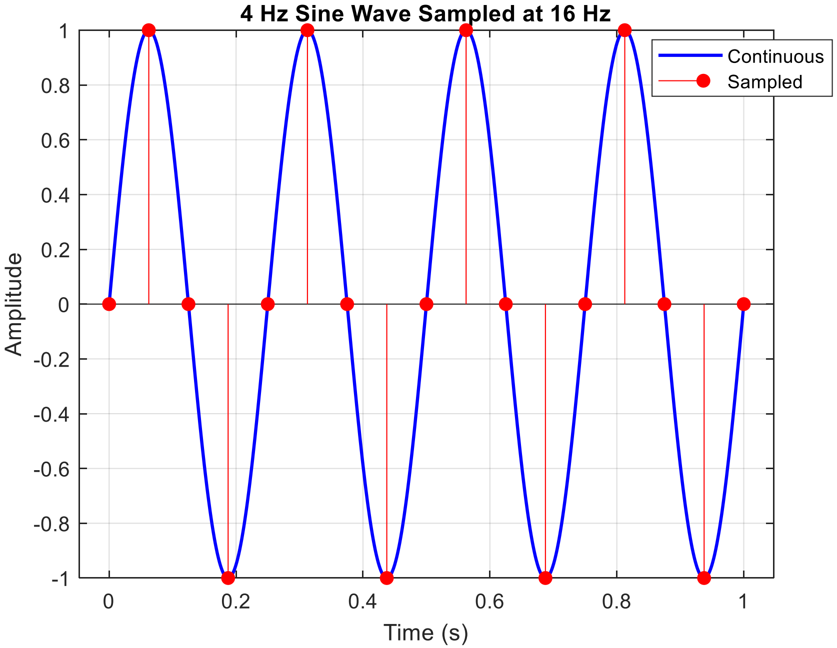
- Red dots represent the discrete samples taken at a 16 Hz sampling rate.
- The sampled points align well with the continuous sine wave in 4 Hz

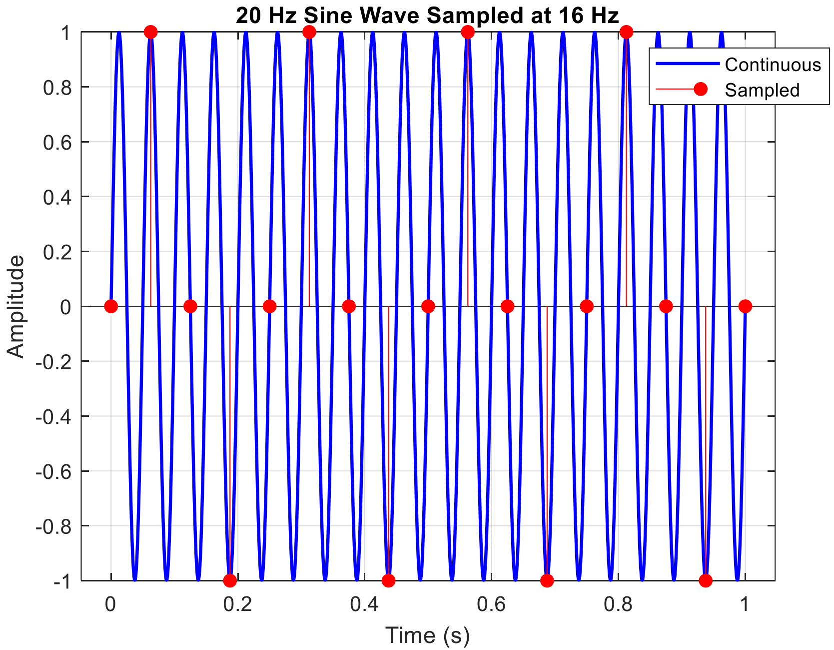




What is an Appropriate Sampling Frequency?

A 4 Hz sine wave sampled at 16 Hz yields the same samples as a 20 Hz sine wave at the same rate





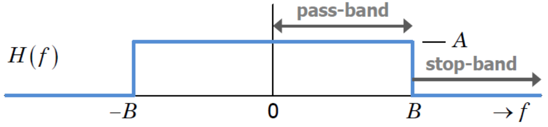
Band-limited Signal

Sampling Band-limited Signal above Nyquist Rate

Conditions for ***perfect reconstruction*** :

$$\left.\begin{array}{l}\begin{pmatrix} x(t) \textit{ must be bandlimited} & : X(f)=0; \quad |f|>f_m \\ \textit{Sampling rate of } x(t) & : f_s \geq 2f_m \end{pmatrix} \\ \textit{Reconstruction LPF}: \quad |H(f)| = \begin{cases} K; & |f|<f_m \\ 0; & |f|>f_m \end{cases}\end{array}\right\}$$

Ideal Low-Pass Filter



Sampling Band-limited Bandpass Signal below Nyquist Rate

(a) **Overlapping spectral images** (*Possible if* $f_c > 0.5B$)

$$\boldsymbol{f_s} = 2f_c/k; \quad \underbrace{k = 1,\ 2,\ \cdots,\ \lfloor 2f_c/B \rfloor}$$

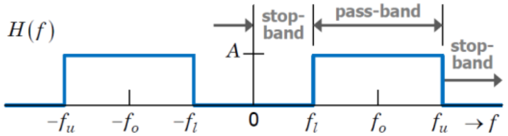
Assumption: The +ve and -ve frequency bands are symmetric about f_c and $-f_c$, respectively

(b) **Un - aliased spectral images** (*Possible if* $f_c > 1.5B$)

$$\frac{2f_c+B}{k+1} \leq \boldsymbol{f_s} \leq \frac{2f_c-B}{k};$$

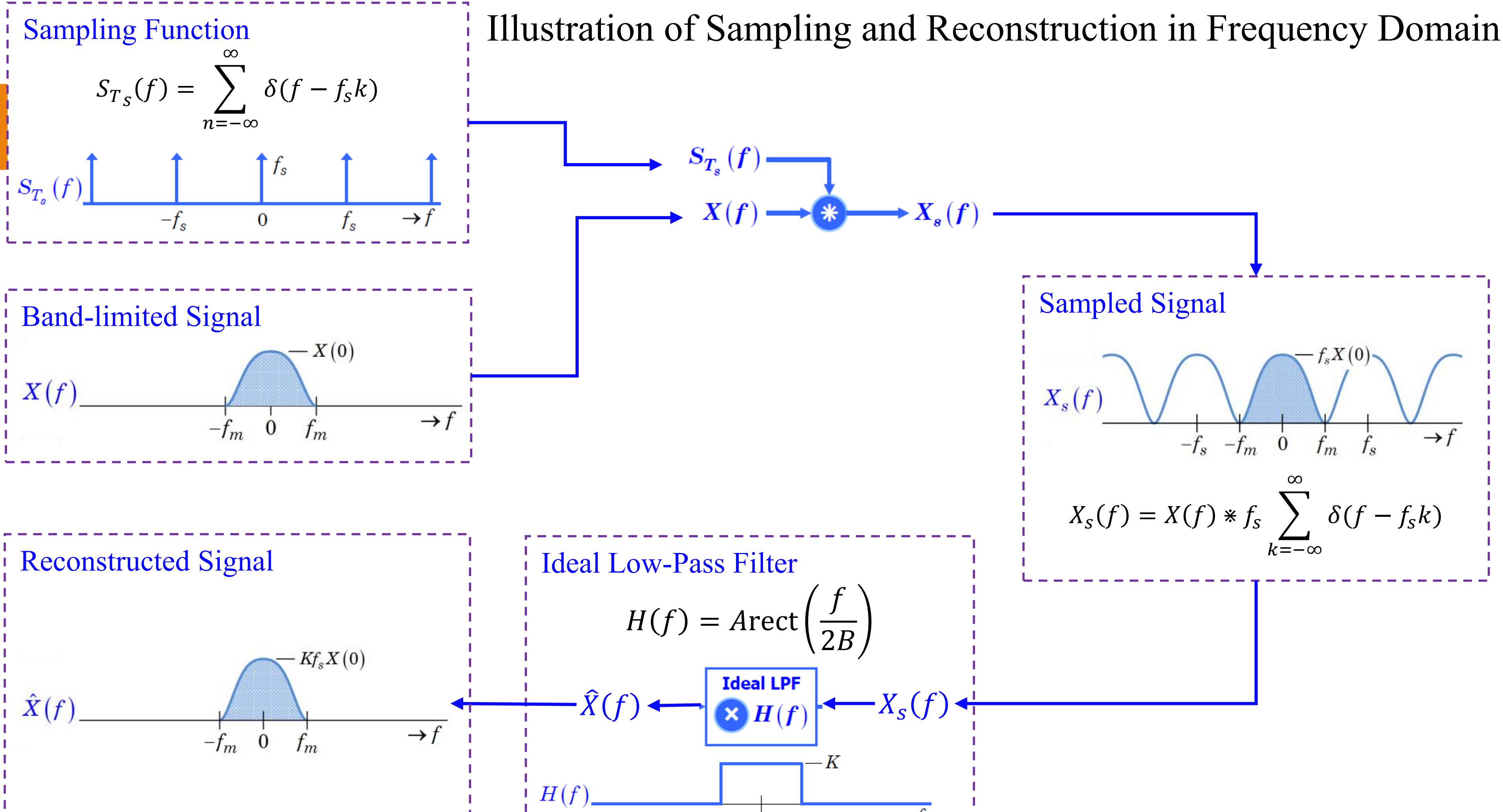
$$k = 1,\ 2,\ \cdots,\ \left\lfloor \frac{2f_c-B}{2B} \right\rfloor$$

Ideal Band-Pass Filter



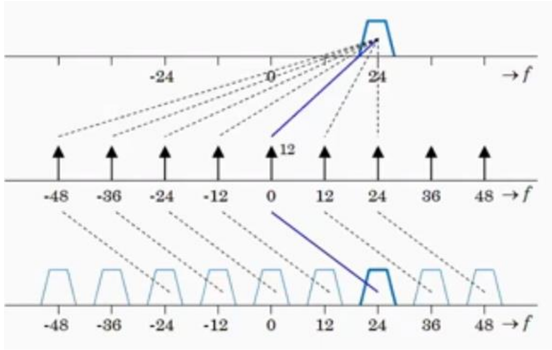
Reconstructed Signal

Illustration of Sampling and Reconstruction in Frequency Domain



Example 9-2(a): Lowest sampling frequency with ***overlapping spectral images***

A signal has center frequency $f_c = 24(\text{Hz})$ *and bandwidth* $B = 8(\text{Hz})$. *What are the possible sampling frequencies (below Nyquist rate) that allow for perfect signal reconstruction?*

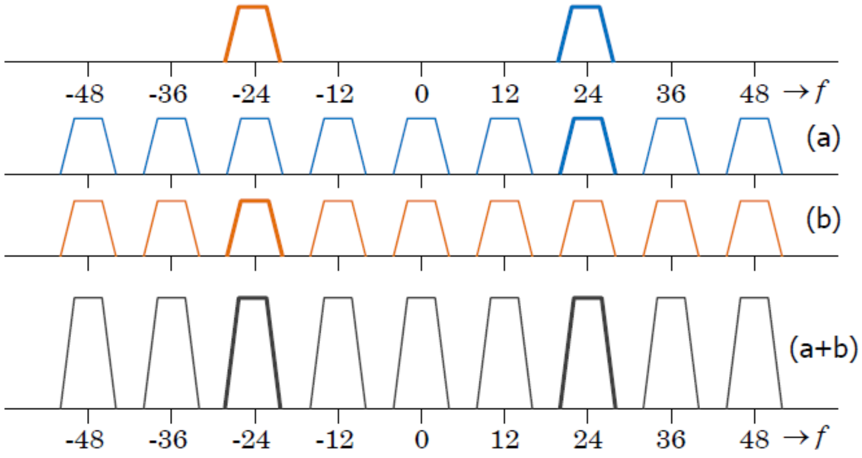


Solution:

Nyquist rate: $(24+4)\times 2 = 56(\text{Hz})$.

From (9.2a):

$$\begin{cases} \boldsymbol{f_s} = \left(\dfrac{2f_c}{k};\; k = 1,\ 2,\ \cdots,\ \left\lfloor \dfrac{2f_c}{B} \right\rfloor \right) = \dfrac{48}{k};\; k = 1,\ 2,\ \cdots,\ 6 \\ \textit{Possible sampling frequencies}:\ \boldsymbol{f_s} = 48,\ 24,\ 16,\ 12,\ 9.6,\ \underset{\text{min}}{8}\ \ Hz \end{cases}$$



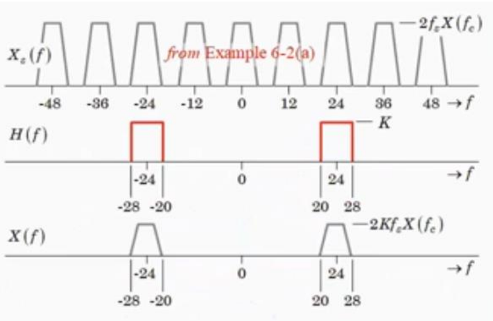
Spectrum of ORIGINAL SIGNAL

Images due to the +ve frequency part of the original spectrum

Images due to the –ve frequency part of the original spectrum

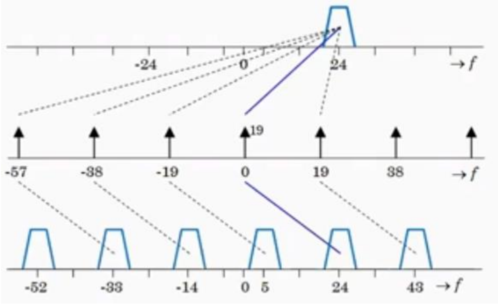
Spectrum of SAMPLED SIGNAL

$\boldsymbol{f_s} = \mathbf{12}\ \boldsymbol{Hz}$



Example 9-2(b): Lowest sampling frequency with __un-aliased spectral images__

A signal has center frequency $f_c = 24\,(\text{Hz})$ *and bandwidth* $B = 8\,(\text{Hz})$. *What are the possible sampling frequencies (below Nyquist rate) that allow for perfect signal reconstruction?*



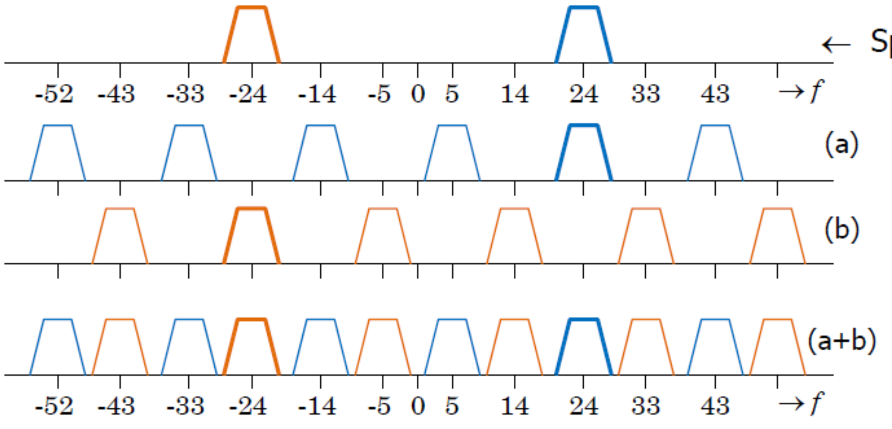
Solution:

Nyquist rate: $(24+4)\times 2 = 56\,(\text{Hz})$.

From (9.2b):

$$\left(\frac{2f_c+B}{k+1} \le \boldsymbol{f_s} \le \frac{2f_c-B}{k};\quad k=1,\ 2,\ \cdots,\ \left\lfloor \frac{2f_c-B}{2B} \right\rfloor\right) \rightarrow \frac{56}{k+1} \le \boldsymbol{f_s} \le \frac{40}{k};\ \ k=1,\ 2$$

Possible sampling frequencies : $\underbrace{28 \le \boldsymbol{f_s} \le 40}_{\text{with } k=1}$ *or* $\underbrace{\overset{\textbf{min}\ \downarrow}{18.67} \le \boldsymbol{f_s} \le 20}_{\text{with } k=2}$ Hz



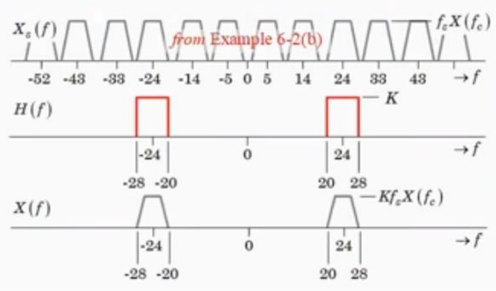
← Spectrum of ORIGINAL SIGNAL

Images due to the +ve frequency part of the original spectrum

Images due to the –ve frequency part of the original spectrum

Spectrum of SAMPLED SIGNAL

$\boldsymbol{f_s} = \mathbf{19}\ \boldsymbol{Hz}$



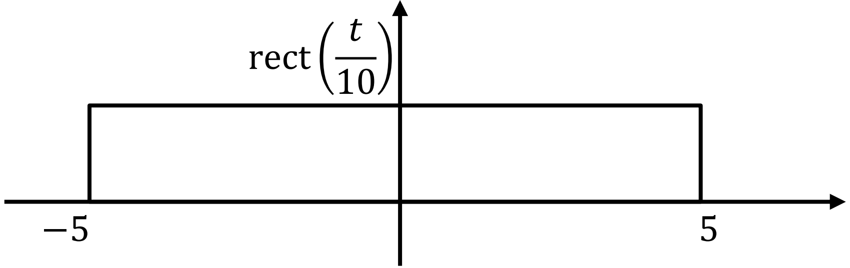
Q.1 A signal $x(t) = 2^{(1-0.5|t|)} \cdot \mathrm{rect}(0.1t)$ is sampled at 0.5 Hz to form $x_s(t)$.

(a) Derive $x_s(t)$, Hence, determine its spectrum $X_s(f)$.

(b) Can $x(t)$ be recovered from $x_s(t)$ without distortion and why?

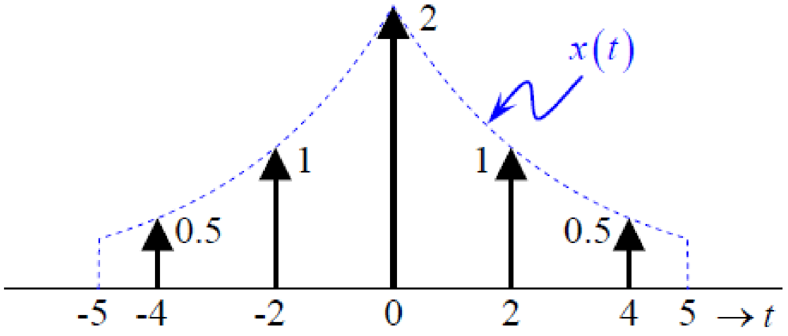
Solution to Q.1

(a) Sampling period $= T_s = \dfrac{1}{0.5} = 2$ s



$$
\begin{aligned}
x_s(t) &= x(t) \times \sum_n \delta(t - nT_s) \\
&= \left[2^{1-0.5|t|} \cdot \mathrm{rect}\left(\frac{t}{10}\right)\right] \times \sum_n \delta(t - 2n) \\
&= \mathrm{rect}\left(\frac{t}{10}\right) \sum_n 2^{1-|n|} \delta(t - 2n) \\
&= \sum_{n=-2}^{2} 2^{1-|n|} \delta(t - 2n) \\
&= \frac{1}{2}\delta(t+4) + \delta(t+2) + 2\delta(t) + \delta(t-2) + \frac{1}{2}\delta(t-4)
\end{aligned}
$$

Sketch of $x_s(t)$



$$
\begin{aligned}
X_s(f) &= \Im\left\{\frac{1}{2}\delta(t+4) + \delta(t+2) + 2\delta(t) + \delta(t-2) + \frac{1}{2}\delta(t-4)\right\} \\
&= \frac{1}{2}e^{j2\pi(4)f} + e^{j2\pi(2)f} + 2 + e^{j2\pi(-2)f} + \frac{1}{2}e^{j2\pi(-4)f} = 2 + 2\cos(4\pi f) + \cos(8\pi f)
\end{aligned}
$$

(b) No, because $x(t)$ is not bandlimited.

Conditions for ***perfect reconstruction*** :

$$\left.\begin{array}{l} \left(\begin{array}{ll} x(t) \ \textit{must be bandlimited} & : X(f)=0; \ \ |f|>f_m \\ \textit{Sampling rate of } x(t) & : f_s \geq 2f_m \end{array}\right) \\ \\ \textit{Reconstruction LPF :} \qquad |H(f)| = \begin{cases} K; & |f|<f_m \\ 0; & |f|>f_m \end{cases} \end{array}\right\} \leftarrow \text{This leads to the Nyquist Sampling Theorem}$$

Q.2 The spectrum, $X(f)$, of a signal $x(t)$ is shown in Figure Q2.

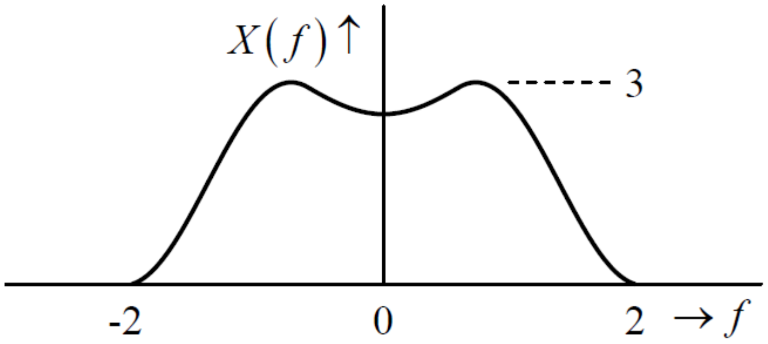


Figure Q2

(a) What is the Nyquist sampling frequency for $x(t)$?

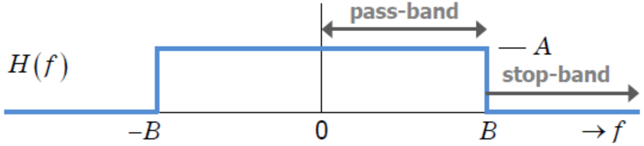
(b) Suppose the signal $x(t)$ is sampled at 6 Hz to obtain the signal $x_s(t)$. Let $X_s(f)$ be the spectrum of $x_s(t)$.

i. Express $x_s(t)$ in terms of $x(t)$.

ii. Find and sketch $X_s(f)$ in terms of $X(f)$.

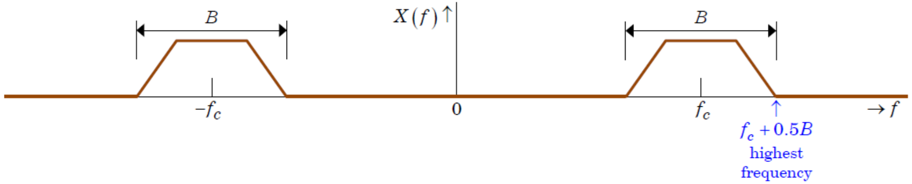
Solution to Q.2

(a) Nyquist sampling frequency for $x(t) = 2 \times 2 = 4$ Hz.



Lowpass Signal
$$\begin{cases} \text{Bandwidth} = B \\ \text{Highest frequency component} = B \\ \text{Nyquist sampling frequency} = 2B \end{cases}$$

Bandpass Signal
$$\begin{cases} \text{Bandwidth} = B \\ \text{Center frequency} = f_c \\ \text{Highest frequency component} = f_c + \frac{B}{2} \\ \text{Nyquist sampling frequency} = 2f_c + B \end{cases}$$



(b) Suppose the signal $x(t)$ is sampled at 6 Hz to obtain the signal $x_s(t)$. Let $X_s(f)$ be the spectrum of $x_s(t)$.

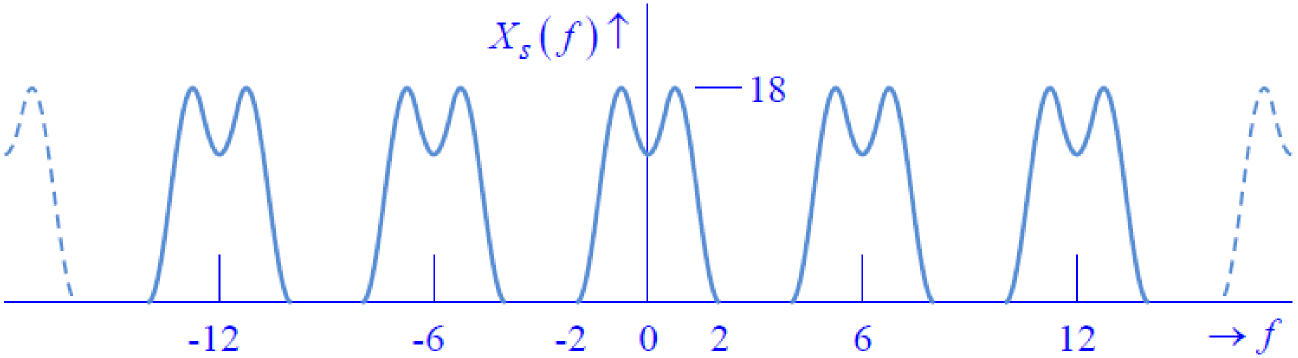
i. Express $x_s(t)$ in terms of $x(t)$.

ii. Find and sketch $X_s(f)$ in terms of $X(f)$.

Sampling frequency $= 6$ Hz

(i) $$x_s(t) = x(t) \times \sum_{n=-\infty}^{\infty} \delta\left(t - \frac{n}{6}\right) = \sum_{n=-\infty}^{\infty} x\left(\frac{n}{6}\right)\delta\left(t - \frac{n}{6}\right)$$

(ii) $$X_s(f) = X(f) * 6\sum_{k=-\infty}^{\infty} \delta(f - 6k) = 6\sum_{k=-\infty}^{\infty} X(f - 6k)$$



Q.3 The spectrums of two real bandpass signals, $x(t)$ and $y(t)$, are shown in Figure Q3.

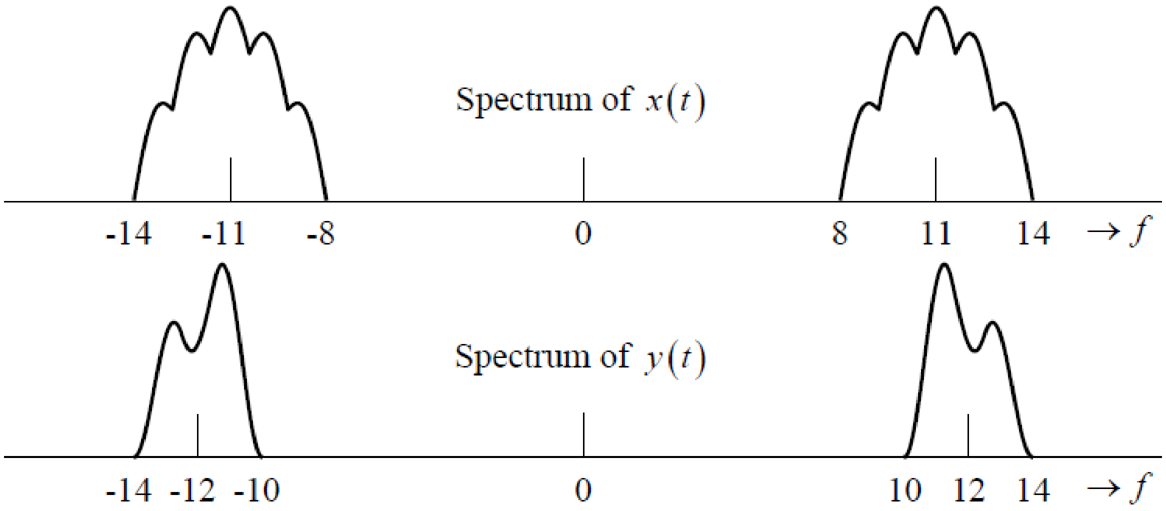


Figure Q3

For each of $x(t)$ and $y(t)$:

(a) What is the Nyquist sampling frequency?

(b) Find the smallest sampling frequency that allows for exact reconstruction of the signal from its sampled version and determine the frequency response, $H(f)$, of the corresponding ideal reconstruction filter.

Q.3 The spectrums of two real bandpass signals, $x(t)$ and $y(t)$, are shown in Figure Q3.

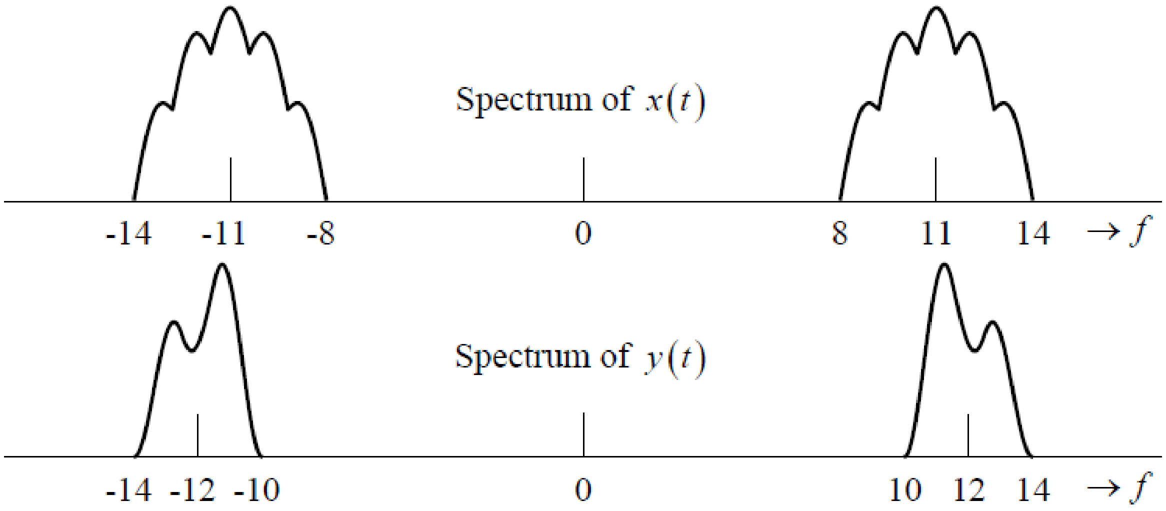


Figure Q3

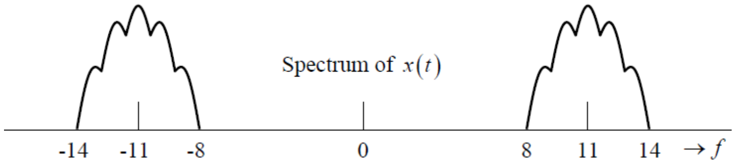
For each of $x(t)$ and $y(t)$:

(a) What is the Nyquist sampling frequency?

Solution to Q.3

(a) Nyquist sampling frequency for both $x(t)$ and $y(t)$ is $2 \times 14 = 28$ Hz.

(b) Find the smallest sampling frequency that allows for exact reconstruction of the signal from its sampled version and determine the frequency response, $H(f)$, of the corresponding ideal reconstruction filter.



SIGNAL $x(t)$:

Due to symmetry of the positive-frequency and negative-frequency spectral images about the center frequencies 11 Hz and -11 Hz, respectively, both the Overlapping Spectral Images formula and the Un-aliased Spectral Images formula are applicable. Of these, the Overlapping Spectral Images formula always yield the lowest sampling frequency.

Center frequency : $f_c = 11 \text{ Hz}$

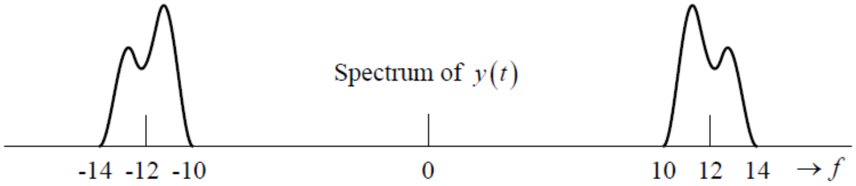
Bandwidth : $B = 6 \text{ Hz}$

Overlapping spectral images formula ($\textit{Possible if } f_c > 0.5B$) :

$$\begin{cases} f_s = \dfrac{2f_c}{k};\ k = 1,\ 2,\ \cdots,\ \left\lfloor \dfrac{2f_c}{B} \right\rfloor \\ \quad = \dfrac{22}{k};\quad k = 1,\ 2,\ 3 \\ \quad = 22,\ \ 11,\ \ \dfrac{22}{3} \text{ Hz} \end{cases}$$

- Therefore, the smallest possible sampling frequency is $f_{s,\min} = \dfrac{22}{3} \text{ Hz}$
- For exact reconstruction of $x(t)$, the frequency response of the ideal reconstruction filter should be:

$$H(f) = \frac{1}{2f_{s,\min}}\left[\text{rect}\left(\frac{f+f_c}{B}\right) + \text{rect}\left(\frac{f-f_c}{B}\right)\right] = \frac{3}{44}\left[\text{rect}\left(\frac{f+11}{6}\right) + \text{rect}\left(\frac{f-11}{6}\right)\right]$$



SIGNAL $y(t)$:

Due to asymmetry of the positive-frequency and negative-frequency spectral images about the center frequencies 12 Hz and -12 Hz, respectively, only the Un-aliased Spectral Images formula is applicable.

Center frequency : $f_c = 12$ Hz

Bandwidth : $B = 4$ Hz

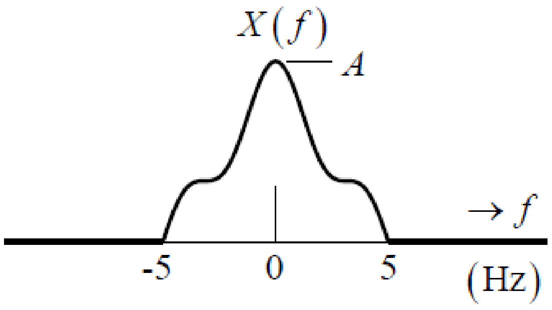
Un-aliased spectral images formula ($Possible\ if\ \ f_c > 1.5B$) :

$$\begin{cases} \dfrac{2f_c + B}{k+1} \le f_s \le \dfrac{2f_c - B}{k}; \quad k = 1,\ 2,\ \cdots,\ \left\lfloor \dfrac{2f_c - B}{2B} \right\rfloor \\ \rightarrow \quad \dfrac{28}{k+1} \le f_s \le \dfrac{20}{k}; \quad k = 1,\ 2 \\ \rightarrow \quad \underbrace{14 \le f_s \le 20}_{k=1} \ \text{and} \ \underbrace{\dfrac{28}{3} \le f_s \le 10}_{k=2} \end{cases}$$

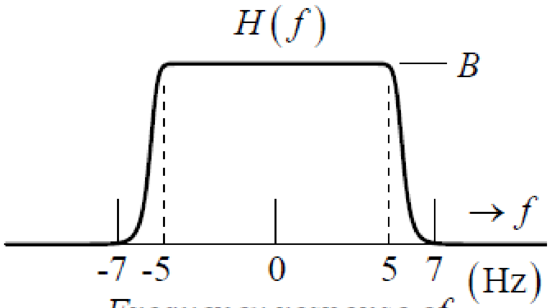
- Therefore, the smallest possible sampling frequency is $f_{s,\min} = \dfrac{28}{3}$ Hz
- For exact reconstruction of $y(t)$, the frequency response of the ideal reconstruction filter should be:

$$H(f) = \frac{1}{f_{s,\min}}\left[\text{rect}\left(\frac{f + f_c}{B}\right) + \text{rect}\left(\frac{f - f_c}{B}\right)\right] = \frac{3}{28}\left[\text{rect}\left(\frac{f+12}{4}\right) + \text{rect}\left(\frac{f-12}{4}\right)\right]$$

Q.4 A signal $x(t)$ is sampled, stored and later reconstructed from the stored samples using a reconstruction filter. The spectrum of $x(t)$ and the frequency response of the reconstruction filter are shown in Figure Q4.



Spectrum of $x(t)$



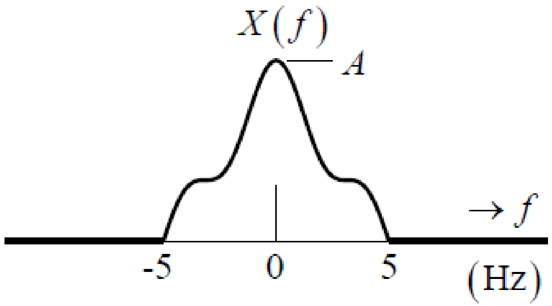
Frequency response of reconstruction filter

Figure Q4

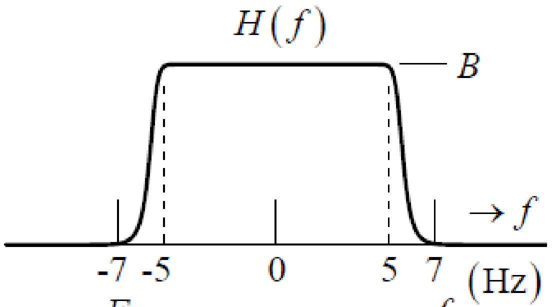
(a) What is the Nyquist sampling frequency for $x(t)$?

(b) What sampling frequency would you recommend so that $x(t)$ can be reconstructed from its samples without distortion, and why? Illustrate your answer.

Q.4 A signal $x(t)$ is sampled, stored and later reconstructed from the stored samples using a reconstruction filter. The spectrum of $x(t)$ and the frequency response of the reconstruction filter are shown in Figure Q4.



Spectrum of $x(t)$



Frequency response of reconstruction filter

Figure Q4

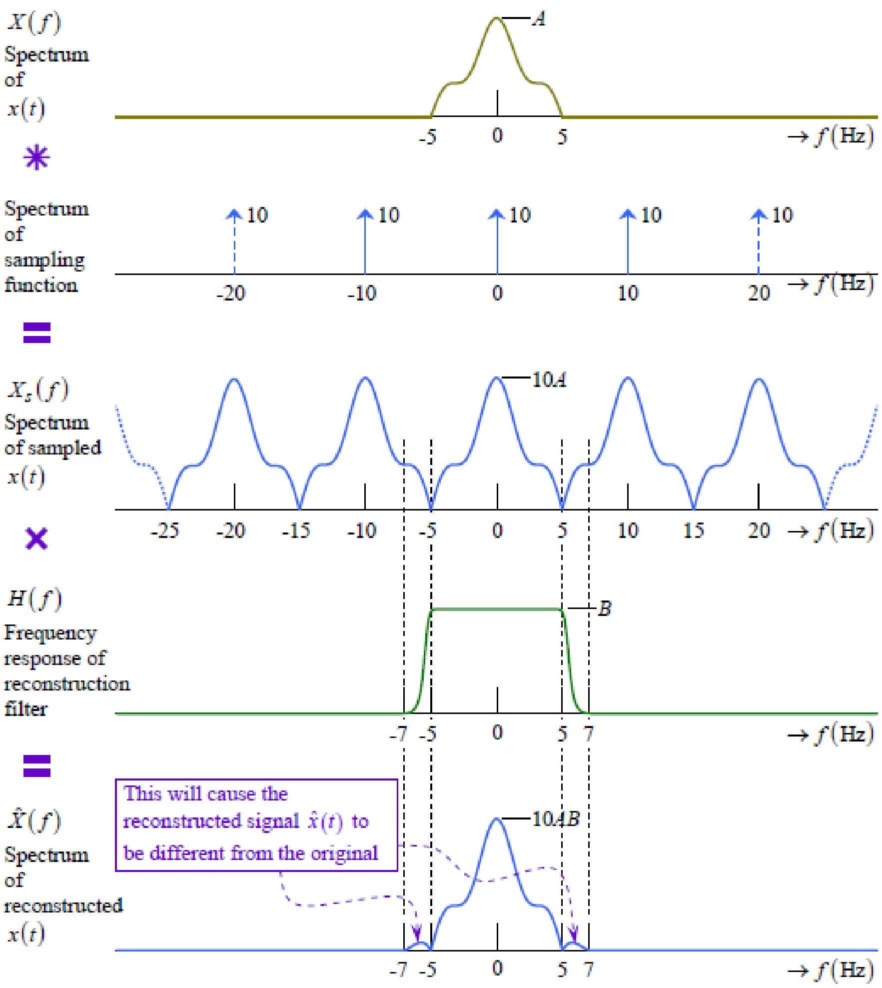
(a) What is the Nyquist sampling frequency for $x(t)$?

Solution to Q.4

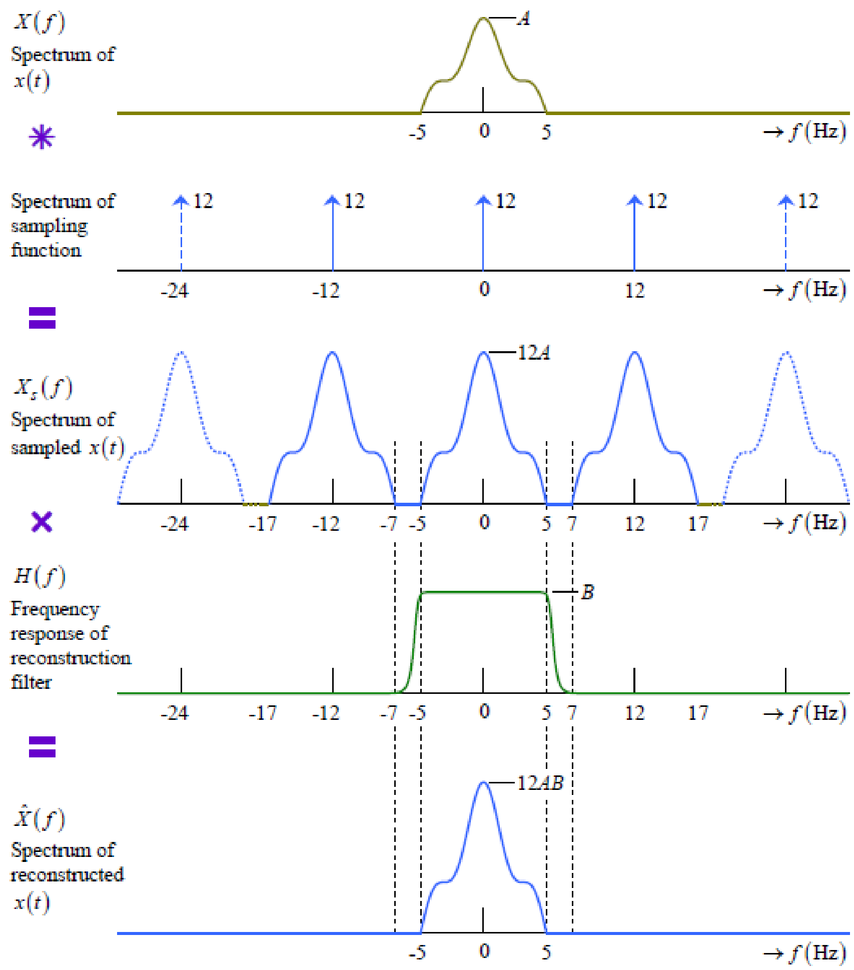
- Nyquist sampling frequency: $2 \times 5 = 10$ Hz
- Recommended sampling frequency: 12 Hz

  The excess 2 Hz is needed to prevent adjacent spectral images from contributing to the reconstruction process. See illustration below.

Sampling frequency: **10 Hz** (Nyquist Rate)



Sampling frequency: **12 Hz**



Remarks: Sampling frequencies higher than 12 Hz may be used to achieve the same result. However, high sampling frequency is usually matched by more costly data acquisition, storage and processing requirement.

Q.5 Speech studies have shown that perceptual cues pertaining to speech intelligibility and speaker identity are mainly found within the first 4 kHz of the speech spectrum. We shall refer to this frequency band as the *TQ-Band*, where *TQ* stands for '*Telephone Quality*'.

The spectrum, $X(f)$, of a speech signal $x(t)$ is shown in Figure Q5 where the *TQ-Band* is indicated.

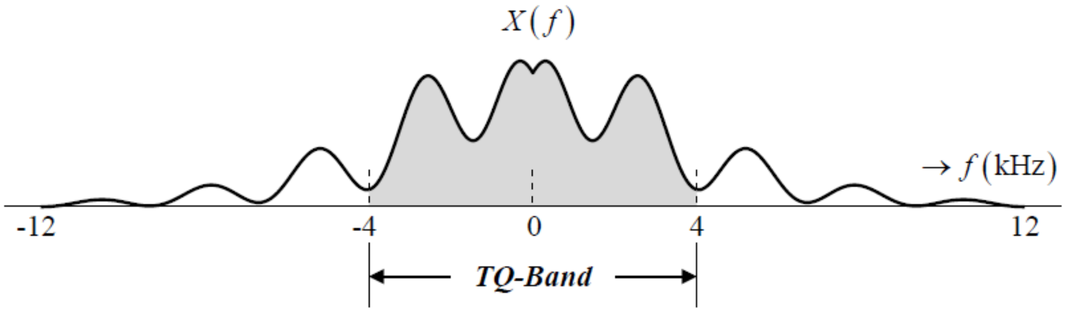


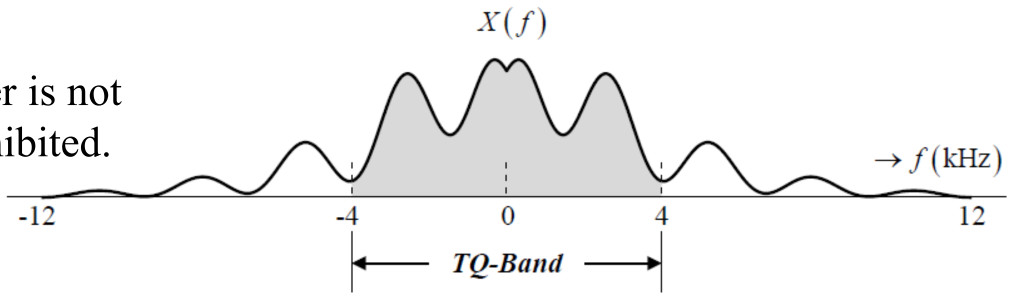
Figure Q5

In order to send $x(t)$ over a digital telephone network, the first step is to sample $x(t)$ without corrupting its *TQ-Band*. Suggest a method for sampling $x(t)$ in each of the following situations.

Situation A:	Anti-aliasing lowpass filter is not available and frequency aliasing is prohibited.
Situation B:	Lowest sampling frequency must be used at all cost.
Situation C:	Anti-aliasing lowpass filter is not available and frequency aliasing is permitted.

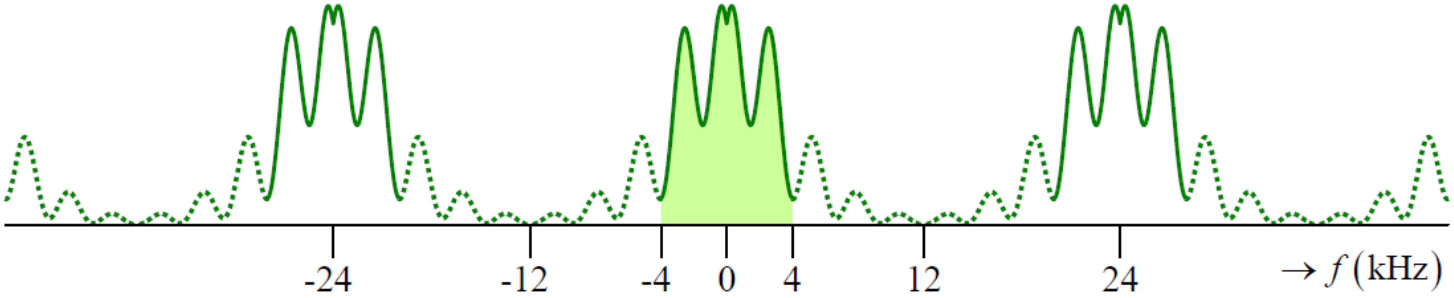
Comment on the advantage and disadvantage of each method.

Situation A: Anti-aliasing lowpass filter is not available and frequency aliasing is prohibited.



Solution to Q.5

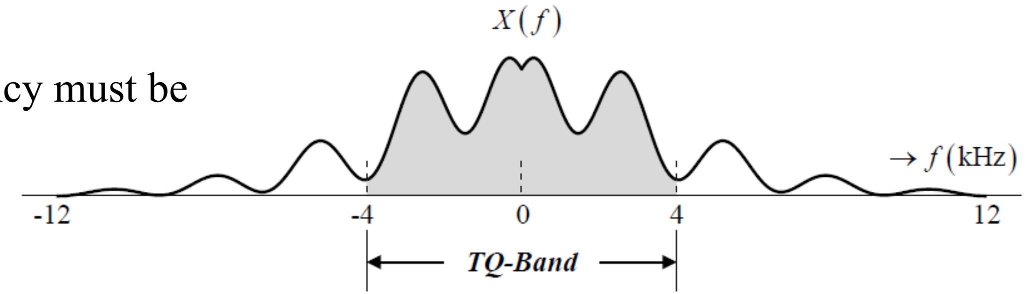
Situation A: $\left(\begin{array}{l}\textbf{No anti-aliasing lowpass filter}\\ \textbf{Frequency aliasing prohibited}\end{array}\right)$: **Sampling frequency** $= \mathbf{2 \times 12 = 24}$ **kHz**



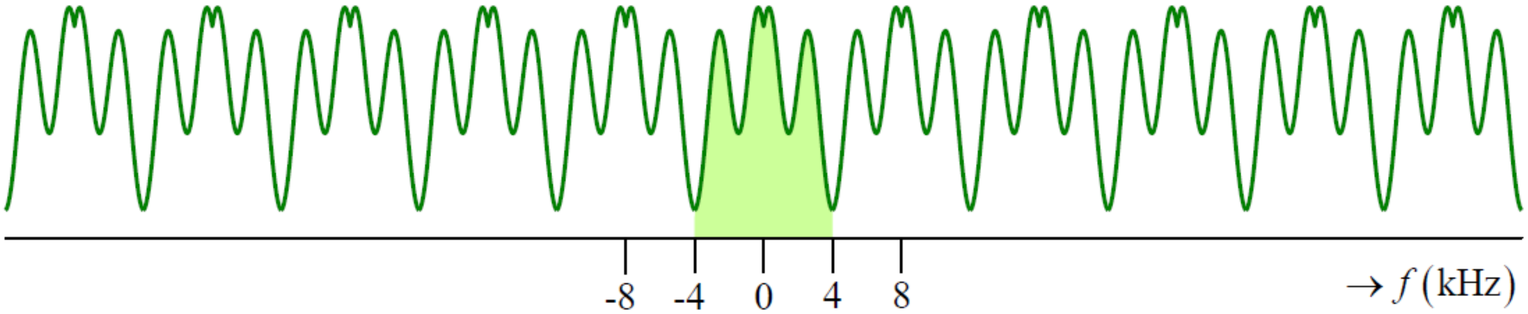
Advantage: No anti-aliasing LPF needed.

Disadvantage: Sampling frequency is higher than necessary to preserve the *TQ-Band*.

Situation B: Lowest sampling frequency must be used at all cost.



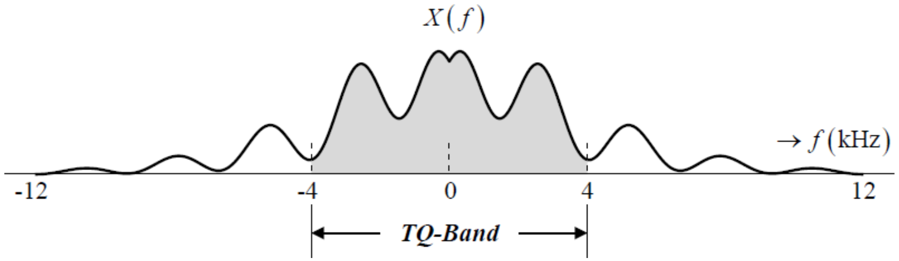
$$\textbf{Situation B: Lowest sampling frequency:} \begin{cases} \textbf{Ideal anti-aliasing filter of bandwidth 4 kHz} \\ \textbf{Sampling frequency} = 2 \times 4 = 8 \textbf{ kHz} \end{cases}$$



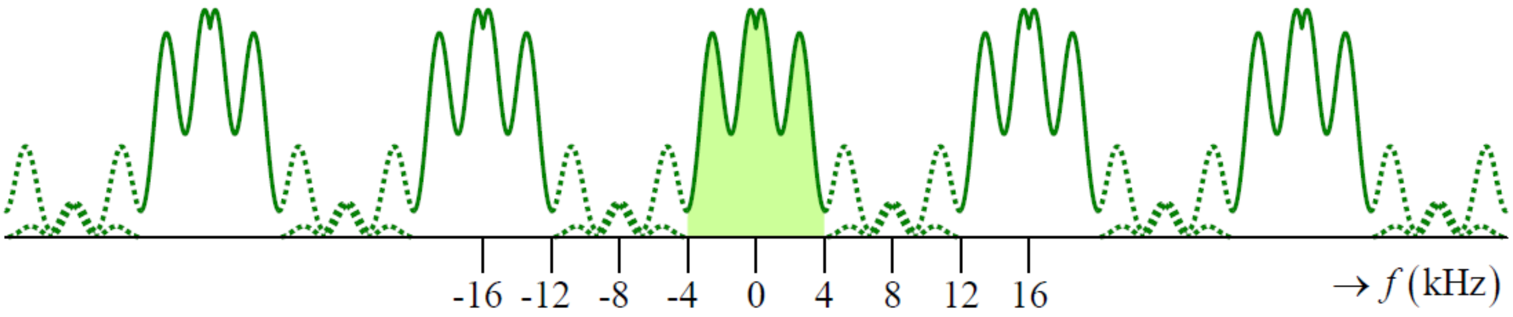
Advantage: Lowest possible sampling frequency to preserve the *TQ-Band*.

Disadvantage: Require anti-aliasing LPF with sharp cutoff.

Situation C: Anti-aliasing lowpass filter is not available and frequency aliasing is permitted.



Situation C: $\left(\begin{array}{c}\textbf{No anti-aliasing lowpass filter}\\ \textbf{Frequency aliasing permitted}\end{array}\right)$: **Sampling frequency = 24 − 8 = 16 kHz**



Advantage: Lowest possible sampling frequency to preserve the *TQ-Band* without requiring anti-aliasing LPF with sharp cutoff.

Disadvantage: Sampling frequency is still higher than the minimum needed to preserve the *TQ-Band*.

This problem illustrates the trade-off between oversampling and the requirement of expensive anti-aliasing LPF with sharp cutoff frequency.

Thanks!