CG2023 Selected Tutorial Questions

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Tutorial 1

T1 Q4

Q.4 A signal $y(t)$ is formed by replicating another signal $x(t) = e^{2|t|}\text{rect}\left(\frac{t}{2\ln(3)}\right)$ at uniform intervals of 4 seconds.

(a) Sketch $x(t)$.

(b) Find an expression for $y(t)$ and sketch it.

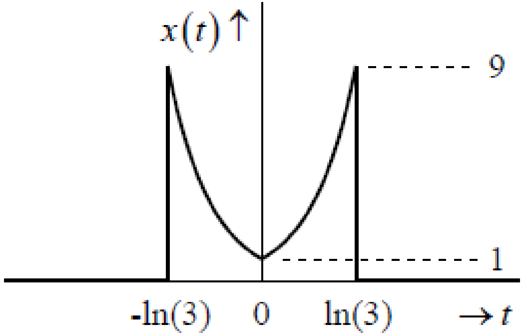
(c) Find the total energy of $x(t)$ and the average power of $y(t)$.

(d) Using the results in part (c), determine the total energy of $w(t) = 3x(-10t+8)$ and average power of $z(t) = \frac{1}{2}y\left(-\frac{t}{3}+5\right)$.

Solution to Q.4

(a) $x(t) = e^{2|t|}\mathrm{rect}\left(\frac{t}{2\ln(3)}\right)$

$T = 2\ln(3)$



- The Dirac-δ Comb function:

$$comb_\lambda(t) = \sum_n \delta(t - n\lambda)$$

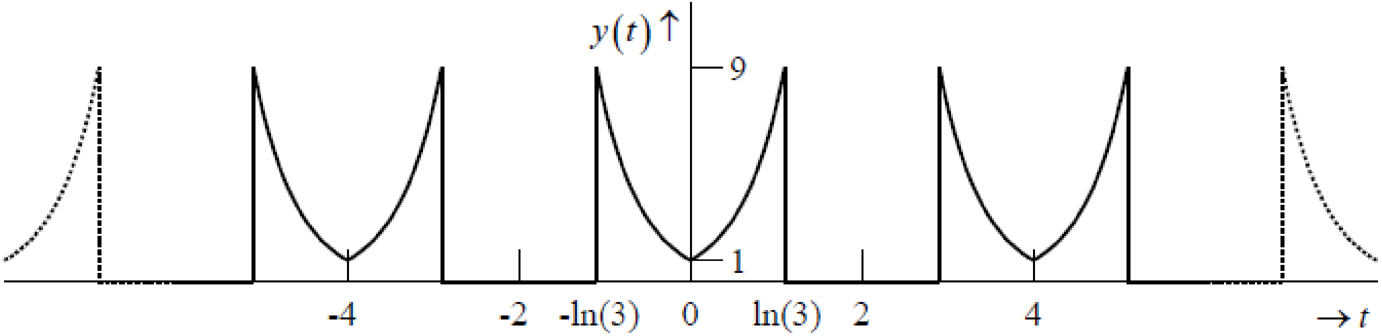
Model a periodic signal:

$$x_p(t) = x(t) * \sum_n \delta(t - n\lambda)$$

Model a sampled signal:

$$x_s(t) = x(t) \times \sum_n \delta(t - n\lambda)$$

(b) $$y(t) = x(t) * \sum_n \delta(t-4n) = \overbrace{\sum_n x(t) * \delta(t-4n)}^{\text{replication property of } \delta(t)} = \sum_n x(t-4n)$$



(c) Total energy of $x(t)$:

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\ln(3)}^{\ln(3)} e^{4|t|} dt = 2\int_0^{\ln(3)} e^{4t} dt = 2 \cdot \frac{e^{4t}}{4}\Big|_0^{\ln(3)} = \frac{1}{2}\left(3^4 - 1\right) = 40 \text{ Joules}$$

Average power of $y(t)$:

$$P_y = \underbrace{\lim_{\tau \to \infty} \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} |y(t)|^2 \, dt = \frac{1}{4}\int_{-2}^{2} |y(t)|^2 \, dt}_{\text{averaged over 1 period because } x(t) \text{ is periodic}} = \frac{1}{4}\int_{-\ln(3)}^{\ln(3)} e^{4|t|} dt = \frac{E_x}{4} = \frac{40}{4} = 10 \text{ Watts}$$

$$\overbrace{\lim_{\tau \to \infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} |x(t)|^2 \, dt}^{\text{original definition of average power}} = \overbrace{\frac{1}{T}\int_0^T |x(t)|^2 \, dt}^{\text{averaged over one period for periodic signals}}$$

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(d) $w(t) = 3x(-10(t-0.8))$ is equal to $x(t)$ amplified by 3 times, time-compressed by 10 times, time-delayed by 0.8 seconds, and time-reversed. Therefore,

Total Energy $E = \int_{-\infty}^{\infty} |x(t)|^2 dt,\ 0 < E < \infty$

$$E = \int_{-\infty}^{\infty} |w(t)|^2 dt = \int_{-\infty}^{\infty} |3x(-10(t-0{,}8))|^2 dt = 3^2 \int_{-\infty}^{\infty} |x(-10(t-0.8))|^2 dt$$

We define $u = -10(t-0.8)$, and obtain $\frac{du}{dt} = -10 \ \rightarrow dt = -\frac{1}{10} du$. The limits of the integration will be $\begin{cases} u \rightarrow -\infty,\ if\ t \rightarrow \infty \\ u \rightarrow \infty, if\ t \rightarrow -\infty \end{cases}$

Total Energy of $x(t)$

$$E = 3^2 \int_{-\infty}^{\infty} |x(u)|^2 dt = 3^2 \int_{\infty}^{-\infty} |x(u)|^2 (\frac{-1}{10}) du = \frac{3^2}{10} \boxed{\int_{\infty}^{-\infty} |x(u)|^2 du}$$

$$\text{Total energy of } w(t) = \frac{3^2}{10} \times [\text{total energy of } x(t)] = \frac{9}{10} \times 40 = 36 \text{ Joules}$$

Time-shifting or time- reversal of an energy signal will not affect its total energy.

$z(t)=\frac{1}{2}y\left(-\frac{1}{3}(t-15)\right)$ is equal to $y(t)$ amplified by 0.5 time, time-expanded by 3 times, time-delayed by 15 seconds, and time-reversed. Therefore,

Assume the period of $y(t)$ is T, and $z(t)$ is equal to $y(t)$ time-expanded by 3 times, therefore the period of $z(t)$ is $3T$

$$\underbrace{\lim_{\tau\to\infty}\frac{1}{2\tau}\int_{-\tau}^{\tau}|x(t)|^2\,dt}_{\text{original definition of average power}}=\underbrace{\frac{1}{T}\int_0^T|x(t)|^2\,dt}_{\text{averaged over one period for periodic signals}}$$

$$P=\frac{1}{3T}\int_{-3T/2}^{3T/2}\left|\frac{1}{2}y\left(-\frac{1}{3}(t-15)\right)\right|^2dt=\frac{1}{2^2}\frac{1}{3T}\int_{-3T/2}^{3T/2}\left|y\left(-\frac{1}{3}(t-15)\right)\right|^2dt$$

We define $u=-\frac{1}{3}(t-15)$, and obtain $\frac{du}{dt}=-\frac{1}{3}\rightarrow dt=-3du$.

The limits of the integration will be $\begin{cases} u=-\frac{T}{2}+5, & if\ t=\frac{3T}{2} \\ u=\frac{T}{2}+5, & if\ t=-\frac{3T}{2} \end{cases}$

Average power of $x(t)$

$$P=\frac{1}{2^2}\frac{1}{3T}\int_0^{3T}\left|y\left(-\frac{1}{3}(t-15)\right)\right|^2dt=\frac{1}{2^2}\frac{1}{3T}\int_{\frac{T}{2}+5}^{-\frac{T}{2}+5}|y(u)|^2(-3)du=\frac{1}{2^2}\frac{3}{3}\boxed{\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}|y(u)|^2du}$$

$$\text{Average power of } z(t)=\frac{1}{2^2}\times\left[\text{average power of } y(t)\right]=\frac{1}{4}\times10=\frac{5}{2}\text{ Watt}$$

Time-scaling, time-shifting or time-reversal of a periodic signal will not affect its average power.

T1 Q5

Q.5 In digital communications, half-cosine or raised-cosine pulses are sometimes used to pulse shape a binary waveform so as to reduce intersymbol interference.

(a) Sketch and label the half-cosine pulse $x(t) = A\cos\left(\frac{\pi t}{T}\right)\text{rect}\left(\frac{t}{T}\right)$ where A and T are positive constants.

(b) Derive the mathematical model for the raised-cosine pulse, $y(t)$, shown in Figure Q5.

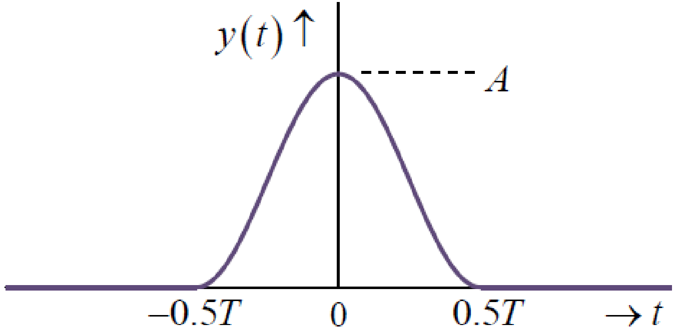


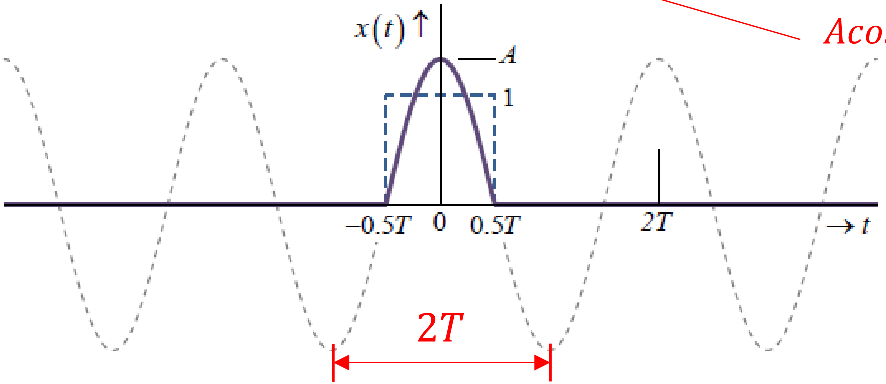
Figure Q5

Solution to Q. 5

(a)

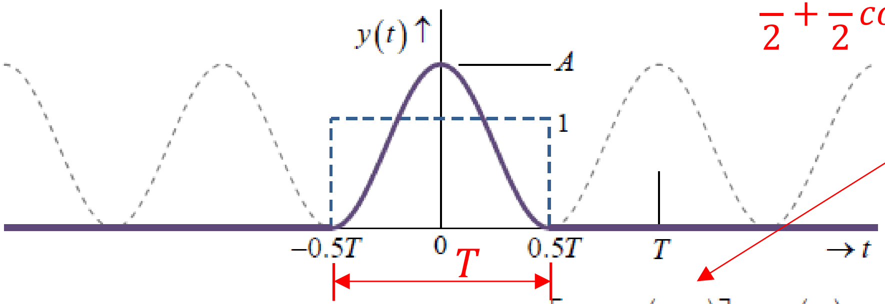
Sketch of half-cosine pulse: $x(t) = A\cos\left(\frac{\pi t}{T}\right)\operatorname{rect}\left(\frac{t}{T}\right)$

$$A\cos\left(\frac{2\pi t}{2T}\right) = A\cos\left(\frac{\pi t}{T}\right)$$



(b)

$$\frac{A}{2} + \frac{A}{2}\cos\left(\frac{2\pi t}{T}\right) = \frac{A}{2}\left[1 + \cos\left(\frac{\pi t}{T}\right)\right]$$



Model for raised-cosine pulse: $y(t) = 0.5A\left[1 + \cos\left(\frac{2\pi t}{T}\right)\right]\cdot\operatorname{rect}\left(\frac{t}{T}\right)$

Tutorial 2

T2 Q1

Q.1 The discrete-frequency spectrum of a signal $x(t)$ is shown in Figure Q1.

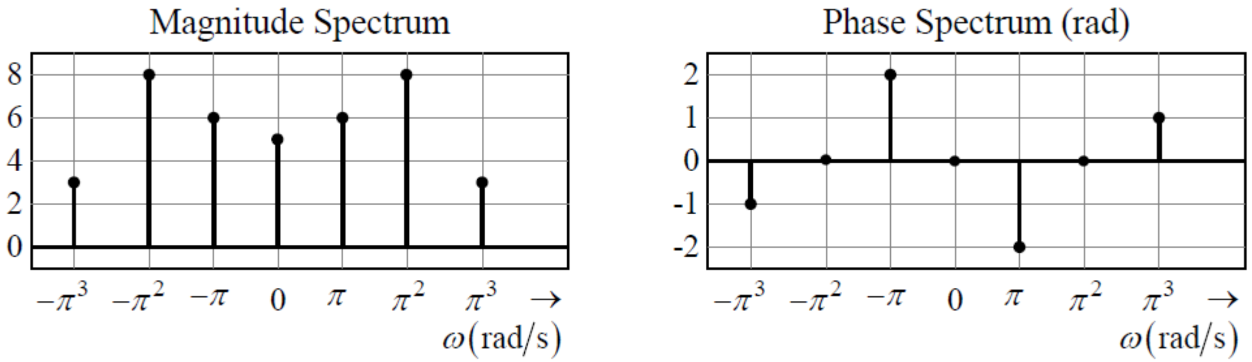


Figure Q1

(a) Express $x(t)$ as a sum of complex sinusoids and show whether or not $x(t)$ is a real signal.

(b) Find the DC value and average power of $x(t)$?

(c) Do the frequencies of the sinusoidal components of $x(t)$ have common factor(s). Hence, what conclusion can you draw about the Fourier series expansion of $x(t)$?

Solution to Q.1

(a) By inspection of the discrete-frequency spectrum of $x(t)$:

$$x(t) = 3e^{-j}e^{-j\pi^3 t} + 8e^{-j\pi^2 t} + 6e^{j2}e^{-j\pi t} + 5 + 6e^{-j2}e^{j\pi t} + 8e^{j\pi^2 t} + 3e^{j}e^{j\pi^3 t}$$
$$= 3e^{-j\left(\pi^3 t+1\right)} + 8e^{-j\pi^2 t} + 6e^{-j(\pi t-2)} + 5 + 6e^{j(\pi t-2)} + 8e^{j\pi^2 t} + 3e^{j\left(\pi^3 t+1\right)}$$

Combining complex conjugate terms, we get

$$x(t) = 5 + 12\cos(\pi t - 2) + 16\cos\left(\pi^2 t\right) + 6\cos\left(\pi^3 t + 1\right)$$

which shows that $x(t)$ is a real signal.

$$Z(t) = a(t) + jb(t)$$

$$Z^*(t) = a(t) - jb(t)$$

$$Z(t) + Z^*(t) = 2a(t)$$ A real signal

(b) Find the DC value and average power of $x(t)$?

$$x(t) = 5 + 12\cos(\pi t - 2) + 16\cos\left(\pi^2 t\right) + 6\cos\left(\pi^3 t + 1\right)$$

The **DC value (Direct Current value)** refers to the constant or average component of a signal over time. It represents the non-varying part of a signal, in contrast to the **AC (Alternating Current) component**, which fluctuates.

For periodic signals:

$$\text{DC Value of } x(t) = c_0 = \frac{1}{T_p}\int_0^{T_p} x(t)dt$$

For aperiodic signals, the DC value is given by the Fourier transform evaluated at $f = 0$

$$\text{DC Value of } x(t) = X(f)\Big|_{f=0} = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi(0)t} dt = \int_{-\infty}^{\infty} x(t)\, dt$$

$$\text{DC value of } x(t) = \left[\overbrace{\text{DC value of } 5}^{5} + \overbrace{\begin{matrix}\text{DC value of}\\ 12\cos(\pi t - 2)\end{matrix}}^{0} + \overbrace{\begin{matrix}\text{DC value of}\\ 16\cos(\pi^2 t)\end{matrix}}^{0} + \overbrace{\begin{matrix}\text{DC value of}\\ 6\cos(\pi^3 t + 1)\end{matrix}}^{0} \right] = 5$$

(b) Find the DC value and average power of $x(t)$?

Method I

For a general sinusoid signal $\alpha cos\left(2\pi\frac{t}{T}\right)$, the average power can be obtained as

$$P=\frac{1}{T}\int_0^T |x(t)|^2 dt=\frac{1}{T}\int_0^T \alpha^2\cos^2\left(2\pi\frac{t}{T}\right)dt=\frac{\alpha^2}{T}\int_0^T\frac{1}{2}\left(1+\cos\left(4\pi\frac{t}{T}\right)\right)dt$$

$$=\frac{\alpha^2}{2T}\left[\int_0^T 1\,dt+\underbrace{\int_0^T\cos\left(4\pi\frac{t}{T}\right)dt}_{=0}\right]=\frac{\alpha^2}{2T}\times T=\frac{\alpha^2}{2}$$

$$\text{Avg Pow of } x(t)=\left[\overbrace{\text{Avg Pow of } 5}^{5^2}+\overbrace{\begin{matrix}\text{Avg Pow of}\\ 12\cos(\pi t-2)\end{matrix}}^{12^2/2}+\overbrace{\begin{matrix}\text{Avg Pow of}\\ 16\cos(\pi^2 t)\end{matrix}}^{16^2/2}+\overbrace{\begin{matrix}\text{Avg Pow of}\\ 6\cos(\pi^3 t+1)\end{matrix}}^{6^2/2}\right]=243\text{ W}$$

Method II

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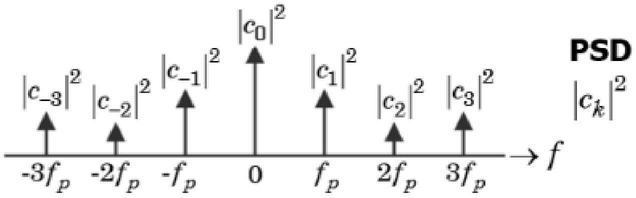
Power Spectral Density of $x_p(t)$:

$$P_x(f) = \sum_{k=-\infty}^{\infty} |c_k|^2 \delta(f - kf_p) \qquad (5.12)$$



Average Power of $x_p(t)$:

$$P = \int_{-\infty}^{\infty} P_x(f)\,df = \sum_{k=-\infty}^{\infty} |c_k|^2 \qquad (5.13)$$

By inspection of the discrete-frequency spectrum of $x(t)$:

$$\begin{aligned} x(t) &= 3e^{-j}e^{-j\pi^3 t} + 8e^{-j\pi^2 t} + 6e^{j2}e^{-j\pi t} + 5 + 6e^{-j2}e^{j\pi t} + 8e^{j\pi^2 t} + 3e^{j}e^{j\pi^3 t} \\ &= 3e^{-j\left(\pi^3 t+1\right)} + 8e^{-j\pi^2 t} + 6e^{-j(\pi t-2)} + 5 + 6e^{j(\pi t-2)} + 8e^{j\pi^2 t} + 3e^{j\left(\pi^3 t+1\right)} \end{aligned}$$

Average Power of $x(t) = 3^2 + 8^2 + 6^2 + 5^2 + 6^2 + 8^2 + 3^2 = 243\ W$

(c) The frequencies $\{\pi^3, \pi^2, \pi, 0, \pi, \pi^2, \pi^3\}$ do not have a common factor, indicating that $x(t)$ is APERIODIC. Therefore, $x(t)$ do not have a Fourier series expansion.

REMARKS: *Summing sinusoids does not necessarily lead to a periodic signal unless the frequencies of the sinusoids are harmonically related, i.e. they have at least one common factor.*

T2 Q4

Q.4 Determine the Fourier series coefficients, c_k, of $x(t) = \sum_{n=-\infty}^{\infty} 2p(t-1.6n)$ where $p(t)$ is shown in Figure Q4.

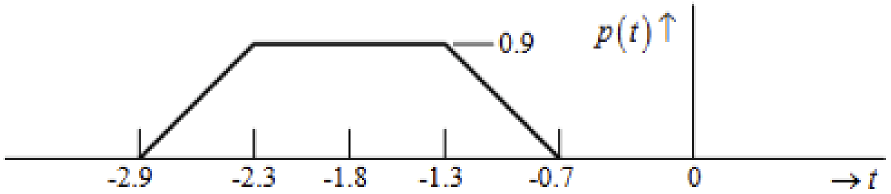
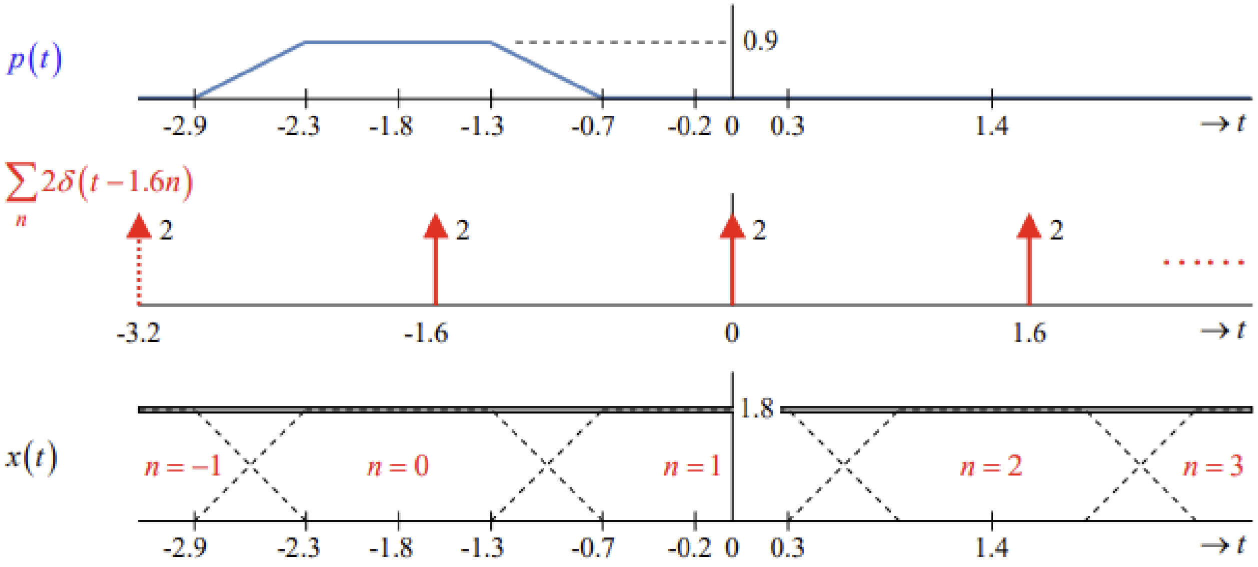


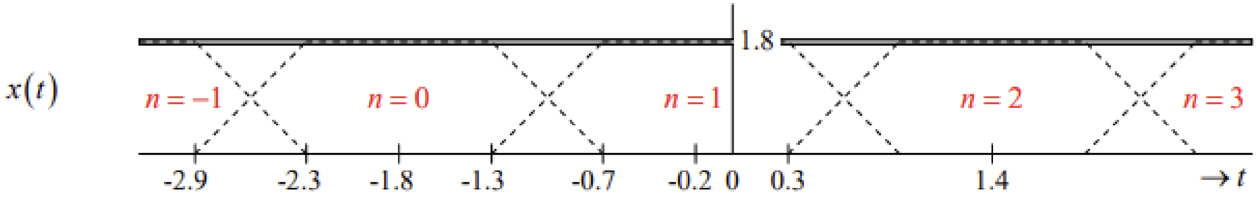
Figure Q4

Solution to Q.4

$$x(t) = \sum_{n=-\infty}^{\infty} 2p(t-1.6n) = p(t) * \sum_{n=-\infty}^{\infty} 2\delta(t-1.6n)$$

$$= \cdots + \underbrace{2p(t+1.6)}_{n=-1} + \underbrace{2p(t)}_{n=0} + \underbrace{2p(t-1.6)}_{n=1} + \underbrace{2p(t-3.2)}_{n=2} + \underbrace{2p(t-4.8)}_{n=3} + \cdots$$





From the above figure, observe that $x(t) = 1.8$

Based on Fourier series expansion

$$x_p(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi kt/T_p}$$

We can rewrite $x(t) = 1.8$ as

$$x(t) = \underbrace{1.8}_{c_0} e^{j2\pi(0)t/T_p} + \sum_{k=-\infty}^{-1} \underbrace{0}_{c_k} e^{j2\pi kt/T_p} + \sum_{k=1}^{\infty} \underbrace{0}_{c_k} e^{j2\pi kt/T_p}$$

Therefore,

- $x(t)$ has one zero-frequency component of value 1.8, which implies that $c_0 = 1.8$.
- $x(t)$ has no non-zero frequency components, which implies that $c_k = 0, \forall k \neq 0$

Tutorial 3

T3 Q3

Q.3 A communication system is shown in Figure Q3 where $x(t)$ is the transmitter input signal and $y(t)$ is the receiver output signal.

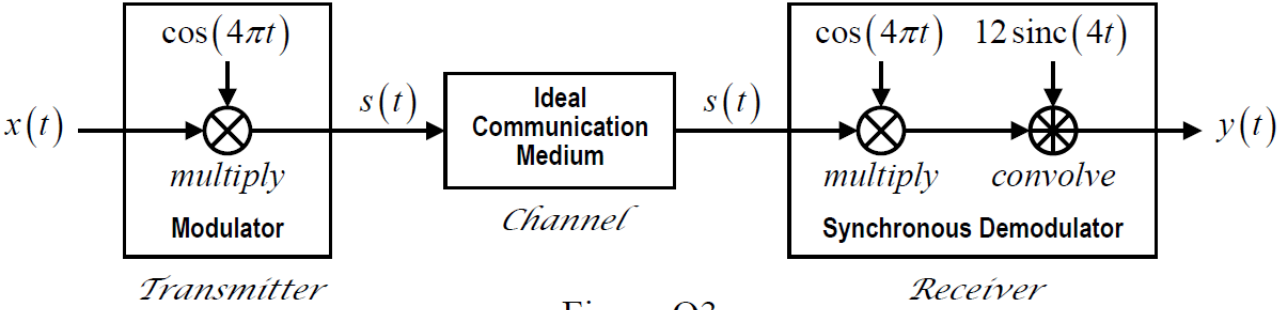


Figure Q3

Find $y(t)$ if $x(t) = 4\,\mathrm{sinc}^2(t)$.

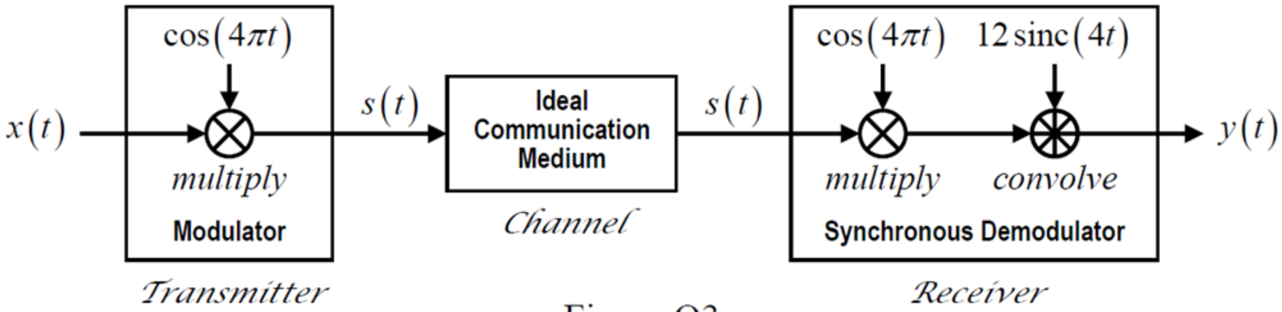


Figure Q3

Solution to Q.3

$$x(t) = 4\,\text{sinc}^2(t)$$

$$\begin{aligned} y(t) &= \left[\left[x(t) \times \cos(4\pi t)\right] \times \cos(4\pi t)\right] * 12\,\text{sinc}(4t) \\ &= \left[4\,\text{sinc}^2(t) \times \cos^2(4\pi t)\right] * 12\,\text{sinc}(4t) \\ &= \left[4\,\text{sinc}^2(t) \times \frac{1}{2}\left[1 + \cos(8\pi t)\right]\right] * 12\,\text{sinc}(4t) \end{aligned}$$

$$\cos^2(\theta) = 0.5[1 + \cos(2\theta)]$$

Rectangle	$\text{rect}\left(\frac{t}{T}\right)$	$T\,\text{sinc}(fT)$
Triangle	$\text{tri}\left(\frac{t}{T}\right)$	$T\,\text{sinc}^2(fT)$
Sine Cardinal	$\text{sinc}\left(\frac{t}{T}\right)$	$T\,\text{rect}(fT)$

Method I

$$Y(f)=\Im\{y(t)\}=\left[\Im\{4\text{sinc}^2(t)\}*\Im\left\{\frac{1}{2}[1+\cos(8\pi t)]\right\}\right]\times\Im\{12\text{sinc}(4t)\}$$

$$=\left[4\,\text{tri}(f)*\left[\frac{1}{2}\delta(f)+\frac{1}{4}\delta(f+4)+\frac{1}{4}\delta(f-4)\right]\right]\times 3\,\text{rect}\left(\frac{f}{4}\right)$$

$$=[2\text{tri}(f)+\text{tri}(f+4)+\text{tri}(f-4)]\times 3\,\text{rect}\left(\frac{f}{4}\right)$$

$$\cos(2\pi f_0 t)\leftrightarrows\frac{1}{2}[\delta(f-f_0)+\delta(f+f_0)]$$

Replication: $x(t)*\delta(t-\lambda)=x(t-\lambda)$

$$\Im\{K\delta(t)\}=K$$
$$\Im\{K\}=K\delta(f)$$
$$\Im\{Ke^{j2\pi f_0}\}=K\delta(f-f_0)$$

Method II

$$Y(f)=\Im\{y(t)\}=\left[\Im\{4sinc^2(t)\}*\Im\left\{\frac{1}{2}[1+\cos(8\pi t)]\right\}\right]\times\Im\{12\text{sinc(4t)}\}$$

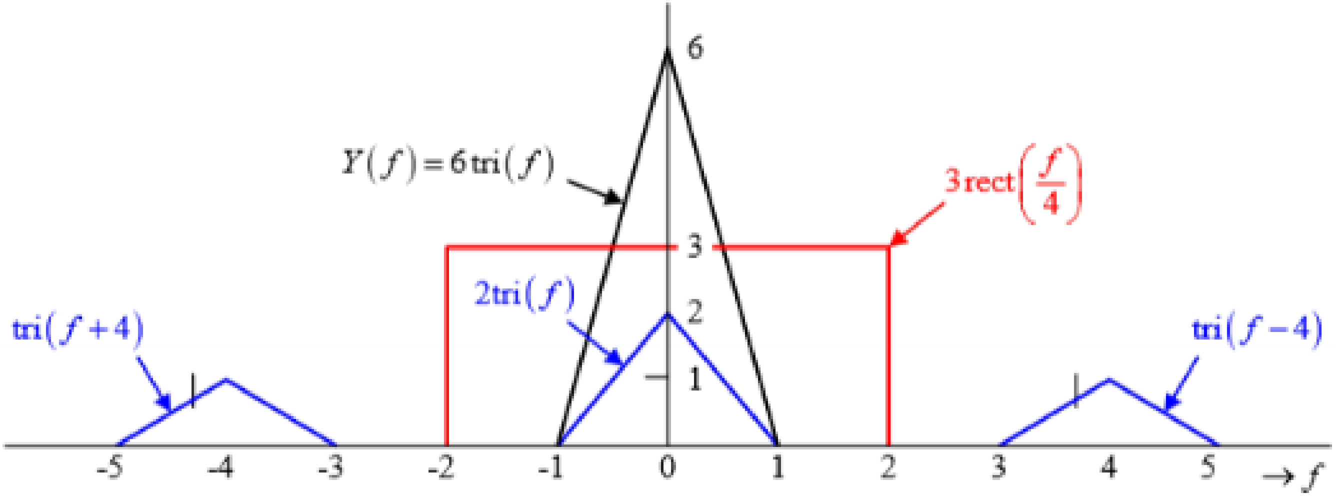
$$=\left[4tri(f)*\Im\left\{\frac{1}{2}\left[1+\frac{1}{2}e^{j8\pi t}-\frac{1}{2}e^{-j8\pi t}\right]\right\}\right]\times 3rect\left(\frac{t}{4}\right)=\left[4tri(f)*\Im\left\{\frac{1}{2}\left[1+\frac{1}{2}e^{j8\pi t}-\frac{1}{2}e^{-j8\pi t}\right]\right\}\right]\times 3rect\left(\frac{t}{4}\right)$$

$$=\left[4tri(f)*\left[\frac{1}{2}\delta(f)+\frac{1}{4}\delta(f+4)+\frac{1}{4}\delta(f-4)\right]\right]\times 3rect\left(\frac{t}{4}\right)=[2tri(f)+tri(f+4)+tri(f-4)]\times 3rect\left(\frac{t}{4}\right)$$

$$Y(f) = [2tri(f) + tri(f+4) + tri(f-4)] \times 3rect\left(\frac{f}{4}\right)$$

… by inspection of the sketch below

$$= 6\ tri(f)$$



Therefore, $y(t) = \mathfrak{F}^{-1}\{6\,\mathrm{tri}(f)\} = 6\,\mathrm{sinc}^2(t) = \frac{3}{2}x(t)$.

T3 Q4

Q.4 (a) Find the Fourier transform of $x(t) = \cos^3(2\pi t)$.

(b) The spectrum $X(f)$ of a periodic signal $x(t)$ is shown in Figure Q4. Express $x(t)$ as a function of real sinusoids.

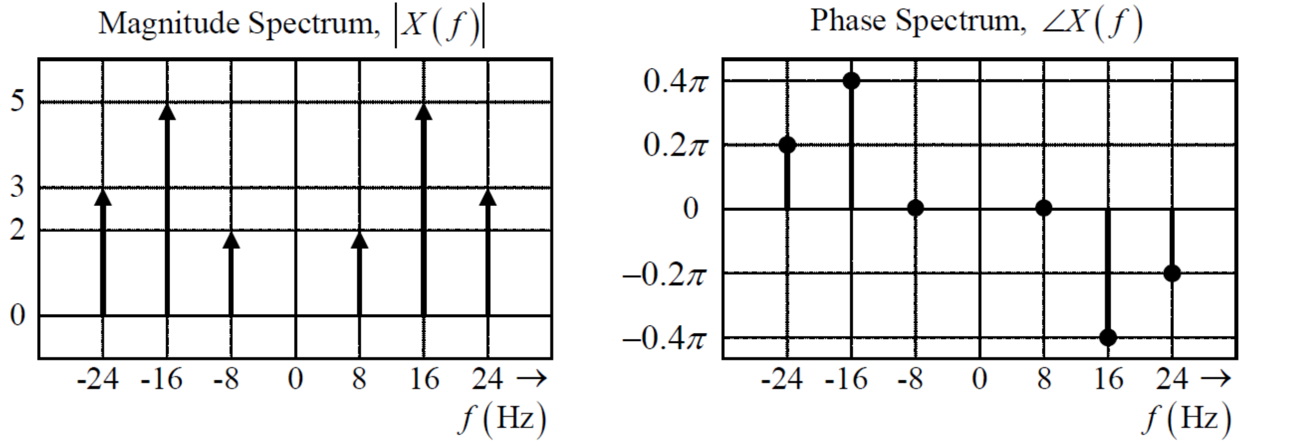


Figure Q4

- Convolution is **Commutative** : $x(t)*y(t)=y(t)*x(t)$.
- Convolution is **Associative** : $[w(t)*x(t)]*y(t)=w(t)*[x(t)*y(t)]$
- Convolution is **Distributive** : $w(t)*[x(t)+y(t)]=w(t)*x(t)+w(t)*y(t)$

Solution to Q.4

(a) $\Im\{\cos(2\pi t)\}=\Im\left\{\frac{1}{2}e^{j2\pi t}+\frac{1}{2}e^{-j2\pi t}\right\}=\frac{1}{2}\delta(f-1)+\frac{1}{2}\delta(f+1)$

Replication: $x(t)*\delta(t-\lambda)=x(t-\lambda)$

$$\begin{aligned}
\Im\{\cos^3(2\pi t)\}&=\Im\{\cos(2\pi t)\}*\Im\{\cos(2\pi t)\}*\Im\{\cos(2\pi t)\}\\
&=\frac{1}{2}[\delta(f-1)+\delta(f+1)]*\frac{1}{2}[\delta(f-1)+\delta(f+1)]*\frac{1}{2}[\delta(f-1)+\delta(f+1)]\\
&=\frac{1}{4}[\delta(f-2)+\delta(f)+\delta(f)+\delta(f+2)]*\frac{1}{2}[\delta(f-1)+\delta(f+1)]\\
&=\frac{1}{8}\begin{bmatrix}\delta(f-1-2)+\delta(f-1)+\delta(f-1)+\delta(f-1+2)\\+\delta(f+1-2)+\delta(f+1)+\delta(f+1)+\delta(f+1+2)\end{bmatrix}\\
&=\frac{1}{8}[\delta(f-3)+3\delta(f-1)+3\delta(f+1)+\delta(f+3)]
\end{aligned}$$

$$\cos(2\pi f_0 t)\ \leftrightarrows\ \frac{1}{2}[\delta(f-f_0)+\delta(f+f_0)]$$

Remarks: Another approach is to use trigonometric identities to first convert $\cos^3(2\pi t)$ *into* $\left[\frac{3}{4}\cos(2\pi t)+\frac{1}{4}\cos(6\pi t)\right]$ *then apply Fourier transform. This will do away with the multiple convolution operations.*

(b) The spectrum $X(f)$ of a periodic signal $x(t)$ is shown in Figure Q4. Express $x(t)$ as a function of real sinusoids.

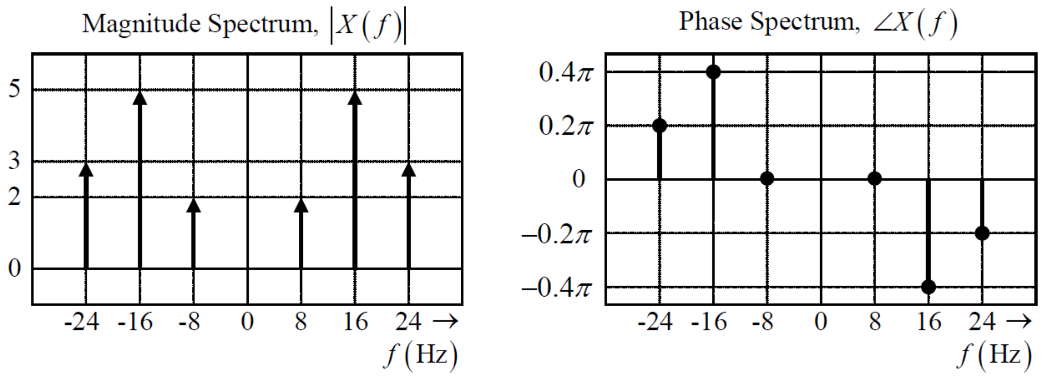


Figure Q4

$$X(f) = 2\left[\delta(f-8)+\delta(f+8)\right] + 5\left[e^{-j0.4\pi}\delta(f-16)+e^{j0.4\pi}\delta(f+16)\right]$$
$$+3\left[e^{-j0.2\pi}\delta(f-24)+e^{j0.2\pi}\delta(f+24)\right]$$

$$x(t) = 2\left[e^{j2\pi(8)t}+e^{j2\pi(-8)t}\right]+5\left[e^{-j0.4\pi}e^{j2\pi(16)t}+e^{j0.4\pi}e^{j2\pi(-16)t}\right]$$
$$+3\left[e^{-j0.2\pi}e^{j2\pi(24)t}+e^{j0.2\pi}e^{j2\pi(-24)t}\right]$$
$$=4\cos(16\pi t)+10\cos(32\pi t-0.4\pi)+6\cos(48\pi t-0.2\pi)$$

T3 Q5

Q.5 A function $g(t)$ is replicated at intervals of 0.5 seconds to generate the periodic signal $x(t)$. The Fourier transform, $G(f)$, of $g(t)$ is shown in Figure Q5.

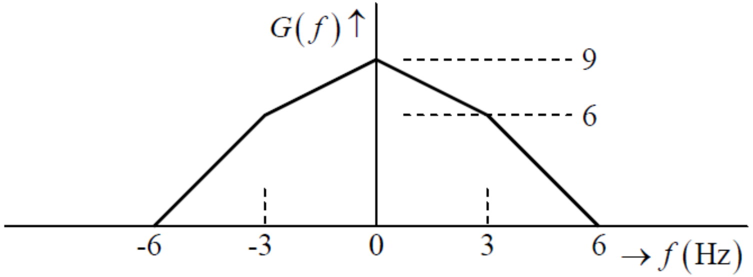


Figure Q5

(a) Derive the Fourier transform, $X(f)$, of $x(t)$ and sketch the continuous-frequency spectrum of $x(t)$.

(b) Derive the Fourier series coefficients, c_k, of $x(t)$ and sketch the discrete-frequency spectrum of $x(t)$.

(c) Show that the inverse Fourier transform of the $X(f)$ found in part (a) is the equal to the Fourier series expansion of $x(t)$.

Solution to Q.5

Model a periodic signal: $x_p(t) = x(t) * \sum_n \delta(t - n\lambda)$

$$\Im\{comb_\lambda(t)\} = \frac{1}{\lambda}\sum_k c_k \delta(f - k/\lambda)$$

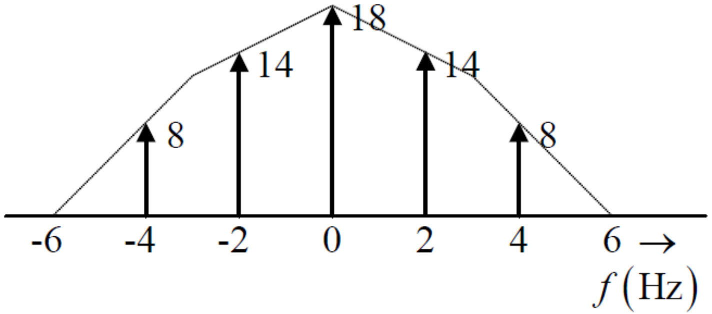
$$\Im\{K\delta(t)\} = K$$
$$\Im\{K\} = K\delta(f)$$
$$\Im\{Ke^{j2\pi f_0}\} = K\delta(f - f_0)$$

(a) Fourier transform, $X(f)$, of $x(t)$:

$$x(t) = g(t) * \sum_n \delta(t - 0.5n)$$

$$X(f) = \Im\left\{g(t) * \sum_n \delta(t - 0.5n)\right\} = \Im\{g(t)\} \times \Im\left\{\sum_n \delta(t - 0.5n)\right\}$$
$$= G(f) \times 2\sum_k \delta(f - 2k) = \sum_k 2G(2k)\delta(f - 2k)$$
$$= 8\delta(f + 4) + 14\delta(f + 2) + 18\delta(f) + 14\delta(f - 2) + 8\delta(f - 4)$$

Continuous-frequency spectrum, $X(f)$:



$$X_p(f) = \sum_{k=-\infty}^{\infty} c_k \delta\left(f - \frac{k}{T_p}\right) = \sum_{k=-\infty}^{\infty} c_k \delta\left(f - kf_p\right) \quad (4.16)$$

(b) Fourier series coefficients, c_k, of $x(t)$:

From part (a):

$$X(f) = \sum_k 2G(2k)\delta(f-2k) \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (1)$$

But, $X(f)$ is also given by:

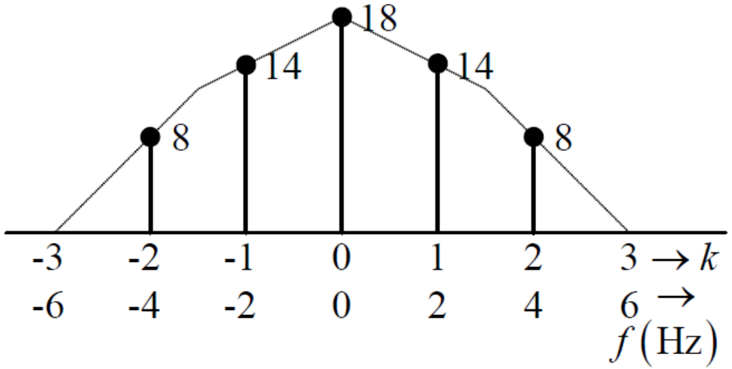
$$X(f) = \sum_k c_k \delta(f-2k) \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (2)$$

Comparing (1) and (2), we obtain $c_k = 2G(2k)$.

Discrete-frequency spectrum, c_k:

$c_0 = 18, \quad c_{\pm 1} = 14, \quad c_{\pm 2} = 8$

$c_k = 0; \; \forall |k| \geq 3$



(c) From part (a): $X(f) = 8\delta(f+4) + 14\delta(f+2) + 18\delta(f) + 14\delta(f-2) + 8\delta(f-4)$

$$x(t) = \Im^{-1}\{X(f)\} = 8e^{-j8\pi t} + 14e^{-j4\pi t} + 18 + 14e^{j4\pi t} + 8e^{j8\pi t}$$

$\Im\{K\delta(t)\} = K$

$\Im\{K\} = K\delta(f)$

$\Im\{Ke^{j2\pi f_0}\} = K\delta(f - f_0)$

With the period of $x(t)$ equal to 0.5 seconds and the c_k's found in part (b), the Fourier series expansion of $x(t)$ is

$$x(t) = \sum_k c_k e^{j4\pi kt} = 8e^{-j8\pi t} + 14e^{-j4\pi t} + 18 + 14e^{j4\pi t} + 8e^{j8\pi t}$$

Fourier Series Coefficients / Discrete-frequency Spectrum

$$c_k = \frac{1}{T_p}\int_{t_0}^{t_0+T_p} x_p(t)e^{-j2\pi kt/T_p}\,dt$$

Continuous-frequency Spectrum

$$X_p(f) = \sum_{k=-\infty}^{\infty} c_k \delta\left(f - \frac{k}{T_p}\right)$$

Periodic Signal

$x(t)$

$$x_p(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi kt/T_p}$$

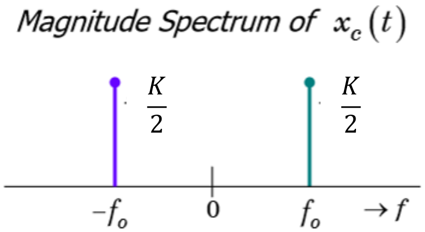
Inverse Fourier Transform

- $\Im\{Ke^{j2\pi f_o t}\} = K\delta(f - f_o)$

Solution Approach

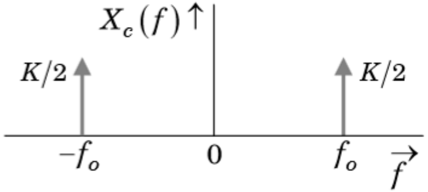
Discrete-frequency Spectrum

$$K\cos(2\pi f_0 t) = \frac{K}{2}e^{j2\pi f_0 t} + \frac{K}{2}e^{-j2\pi f_0 t}$$



Continuous-frequency Spectrum

$$\frac{K}{2}\delta(f - f_0) + \frac{K}{2}\delta(f + f_0)$$



- Properties of $\delta(t)$
  - Symmetry : $\delta(t) = \delta(-t)$
  - Sampling: $x(t)\delta(t-\lambda) = x(\lambda)\delta(t-\lambda)$
  - Sifting: $\int_{-\infty}^{\infty} x(t)\delta(t-\lambda)dt = x(\lambda)$
  - Replication: $x(t) * \delta(t-\lambda) = x(t-\lambda)$

- The Dirac-δ Comb function:

$$comb_\lambda(t) = \sum_n \delta(t - n\lambda)$$

Model a periodic signal:

$$x_p(t) = x(t) * \sum_n \delta(t - n\lambda)$$

Model a sampled signal:

$$x_s(t) = x(t) \times \sum_n \delta(t - n\lambda)$$

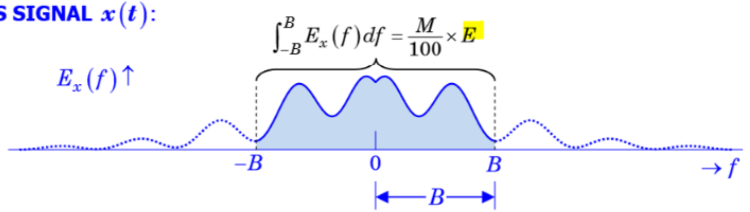
Tutorial 4

T4 Q3

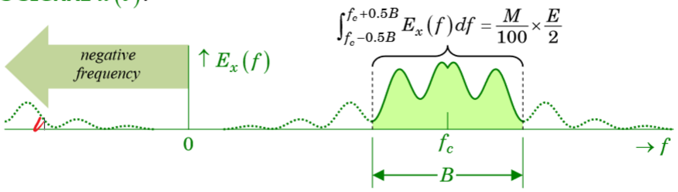
Q.3 The spectrum of a lowpass energy signal $x(t)$ is given by $X(f) = 10^{-0.1|f|}$. Find the 90% energy containment bandwidth of $x(t)$. $\left[\text{Fact: } 10^{\beta} = e^{\beta \ln(10)}\right]$

The $M\%$ energy containment bandwidth, B, of a real energy signal is the smallest bandwidth that contains at least $M\%$ of the total signal energy $E = \int_{-\infty}^{\infty} E_x(f)\,df$.

LOWPASS SIGNAL $x(t)$:



BANDPASS SIGNAL $x(t)$:



Solution to Q.3

Spectrum of $x(t)$: $\quad X(f) = 10^{-0.1|f|} = e^{-[0.1\ln(10)]\cdot|f|}$

ESD of $x(t)$: $\quad E_x(f) = |X(f)|^2 = e^{-[0.2\ln(10)]\cdot|f|}$

Energy of $x(t)$ contained within a bandwidth of B:

$$e(B) = \int_{-B}^{B} E_x(f)\,df = 2\int_0^B e^{-[0.2\ln(10)]\cdot f}\,df = 2\left[\frac{e^{-[0.2\ln(10)]\cdot f}}{-0.2\ln(10)}\right]_0^B$$

$$= \frac{10}{\ln(10)}\left[1 - e^{-[0.2\ln(10)]\cdot B}\right] \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (1)$$

Total energy of $x(t)$ is equal to energy of $x(t)$ contained within a bandwidth of ∞. Hence, substituting $B = \infty$ into (1), we get

$$e(\infty) = \int_{-\infty}^{\infty} E_x(f)\,df = \frac{10}{\ln(10)} \quad \cdots\cdots \quad \text{TOTAL ENERGY}$$

$$e(B) = \frac{10}{\ln(10)}\left[1 - e^{-[0.2\ln(10)]B}\right]$$

The 90% energy containment bandwidth, f_{90}, of $x(t)$ is obtained by solving:

$$\underbrace{e(f_{90})}_{\text{90\% of total energy}} = 0.9 \times \underbrace{e(\infty)}_{\text{Total energy}}$$

$$\rightarrow \frac{10}{\ln(10)}\left[\boxed{1 - e^{-[0.2\ln(10)]\cdot f_{90}}}\right] = 0.9 \times \frac{10}{\ln(10)}$$

$$\rightarrow 1 - e^{-[0.2\ln(10) f_{90}]} = 0.9$$

$$\rightarrow e^{-[0.2\ln(10) f_{90}]} = 1 - 0.9 = 0.1 = 10^{-1}$$

$$\rightarrow e^{[0.2\ln(10) f_{90}]} = 10 = e^{\ln(10)}$$

$$\rightarrow 0.2\ln(10)\, f_{90} = \ln(10)$$

$$\rightarrow e^{[0.2\ln(10)]\cdot f_{90}} = 10$$

$$\rightarrow f_{90} = \frac{\ln(10)}{0.2\ln(10)}$$

$$\rightarrow f_{90} = 5 \text{ Hz}$$

T4 Q4

Q.4 Derive and sketch the spectrum of $72\text{sinc}^2(12t) * \sum_n \delta(t-0.5n)$ where $*$ denotes convolution.

Hence, determine the power spectral density and the average power of $x(t)$?

Solution to Q.4

Signal, $x(t)$:

$$x(t) = 72\,\mathrm{sinc}^2(12t) * \sum_n \delta(t - 0.5n)$$

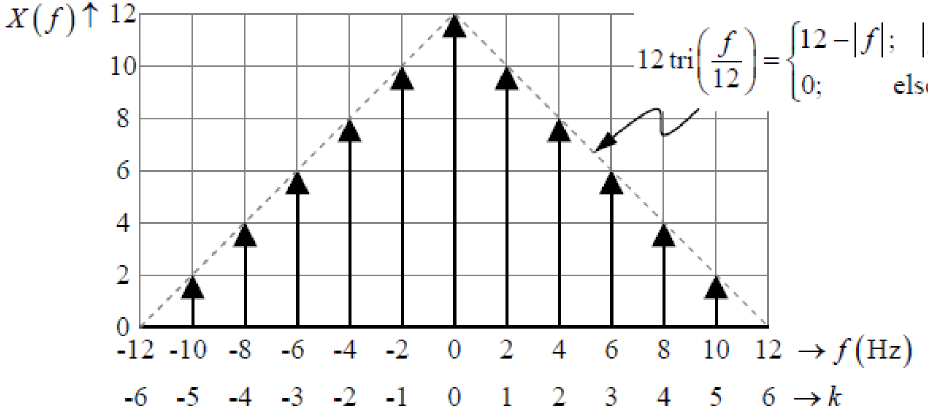
Spectrum, $X(f)$:

$$X(f) = \Im\{x(t)\} = \Im\{72\,\mathrm{sinc}^2(12t)\} \times \Im\left\{\sum_n \delta(t - 0.5n)\right\}$$

$$= 6\,\mathrm{tri}\left(\frac{f}{12}\right) \times \frac{1}{0.5}\sum_k \delta\left(f - \frac{k}{0.5}\right) = 12\,\mathrm{tri}\left(\frac{f}{12}\right) \times \sum_k \delta(f - 2k)$$

$\cdots\cdots$ deducing from the plot of $X(f)$ below

$$= \sum_{k=-5}^{5} (12 - |f|)\,\delta(f - 2k) = \sum_{k=-5}^{5} (12 - 2|k|)\,\delta(f - 2k)$$



$$12\,\mathrm{tri}\left(\frac{f}{12}\right) = \begin{cases} 12 - |f|; & |f| \le 12 \\ 0; & \text{elsewhere} \end{cases}$$

$$\Im\{comb_\lambda(t)\} = \frac{1}{\lambda}\sum_k c_k \delta(f - k/\lambda)$$

- The Dirac-δ Comb function:

$$comb_\lambda(t) = \sum_n \delta(t - n\lambda)$$

Model a periodic signal:

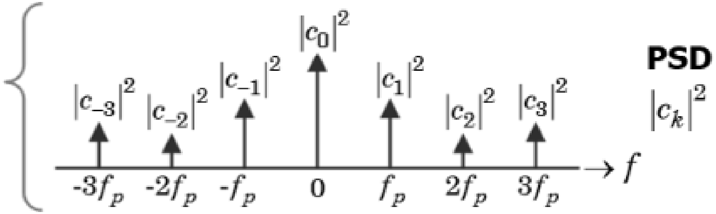
$$x_p(t) = x(t) * \sum_n \delta(t - n\lambda)$$

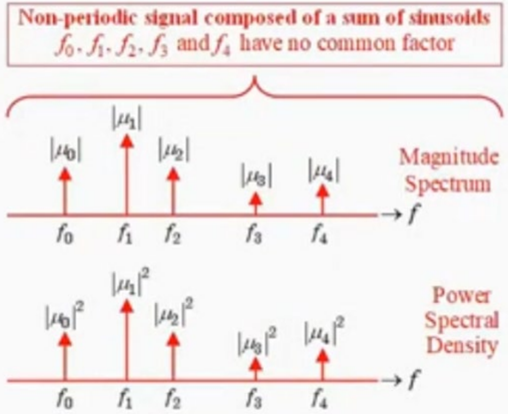
Model a sampled signal:

$$x_s(t) = x(t) \times \sum_n \delta(t - n\lambda)$$

Power Spectral Density of $x_p(t)$:

$$P_x(f) = \sum_{k=-\infty}^{\infty} |c_k|^2 \delta(f - kf_p)$$





Average Power of $x_p(t)$:

$$P = \int_{-\infty}^{\infty} P_x(f)\,df = \sum_{k=-\infty}^{\infty} |c_k|^2$$

$$\int_{-\varepsilon}^{\varepsilon} \delta(t)\,dt = 1, \forall \varepsilon > 0$$

$$P = \int_{-\infty}^{\infty} P_x(f)\,df = \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} |c_k|^2 \delta(f - kf_p)\,df = \sum_{k=-\infty}^{\infty} |c_k|^2 \int_{-\infty}^{\infty} \delta(f - kf_p)\,df = \sum_{k=-\infty}^{\infty} |c_k|^2$$

Power spectral density, $P_x(f)$:

$$P_x(f) = \sum_{k=-5}^{5} (12 - 2|k|)^2 \delta(f - 2k)$$

Average power, P:

$$P = \int_{-\infty}^{\infty} P_x(f)\,df = \sum_{k=-5}^{5} (12 - 2|k|)^2 \underbrace{\int_{-\infty}^{\infty} \delta(f - 2k)\,df}_{=1} = \sum_{k=-5}^{5} (12 - 2|k|)^2$$

$$= 12^2 + 2\left(10^2 + 8^2 + 6^2 + 4^2 + 2^2\right) = 584 \text{ Watts}$$

Thanks!