CG2023 Tutorial 5

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Table of Laplace Transforms and Properties

LAPLACE TRANSFORMS		
	$f(t)$	$\tilde{F}(s)$
Unit Impulse	$\delta(t)$	1
Unit Step	$u(t)$	$1/s$
Ramp	$t\,u(t)$	$1/s^2$
n$^{\text{th}}$ order Ramp	$t^n u(t)$	$\dfrac{n!}{s^{n+1}}$
Damped Ramp	$t\,e^{-\alpha t}u(t)$	$1/(s+\alpha)^2$
Exponential	$e^{-\alpha t}u(t)$	$1/(s+\alpha)$
Cosine	$\cos(\omega_o t)u(t)$	$s/(s^2+\omega_o^2)$
Sine	$\sin(\omega_o t)u(t)$	$\omega_o/(s^2+\omega_o^2)$
Damped Cosine	$e^{-\alpha t}\cos(\omega_o t)u(t)$	$\dfrac{s+\alpha}{(s+\alpha)^2+\omega_o^2}$
Damped Sine	$e^{-\alpha t}\sin(\omega_o t)u(t)$	$\dfrac{\omega_o}{(s+\alpha)^2+\omega_o^2}$

LAPLACE TRANSFORM PROPERTIES		
	Time-domain	s-domain
Linearity	$\alpha f_1(t)+\beta f_2(t)$	$\alpha\tilde{F}_1(s)+\beta\tilde{F}_2(s)$
Time shifting	$f(t-t_o)$	$e^{-st_o}\tilde{F}(s)$
Shifting in the s-domain	$e^{s_o t}f(t)$	$\tilde{F}(s-s_o)$
Time scaling	$f(\alpha t)$	$\dfrac{1}{\lvert\alpha\rvert}\tilde{F}\left(\dfrac{s}{\alpha}\right)$
Integration in the time-domain	$\int_0^t f(\zeta)\,d\zeta$	$\dfrac{1}{s}\tilde{F}(s)$
Differentiation in the time-domain	$\dfrac{df(t)}{dt}$	$s\tilde{F}(s)-f(0^-)$
	$\dfrac{d^n f(t)}{dt^n}$	$s^n\tilde{F}(s)-\sum_{k=0}^{n-1}s^{n-1-k}\left.\dfrac{d^k f(t)}{dt^k}\right\rvert_{t=0^-}$
Differentiation in the s-domain	$-t\,f(t)$	$\dfrac{d\tilde{F}(s)}{ds}$
	$(-t)^n f(t)$	$\dfrac{d^n\tilde{F}(s)}{ds^n}$
Convolution in the time-domain	$\int_{-\infty}^{\infty}f_1(\zeta)f_2(t-\zeta)\,d\zeta$	$\tilde{F}_1(s)\tilde{F}_2(s)$

Initial value theorem: $f(0)=\lim_{s\to\infty}s\tilde{F}(s)$

Final value theorem: $\lim_{t\to\infty}f(t)=\lim_{s\to 0}s\tilde{F}(s)$

Q1

(a)

Q.1 (a) $\mathcal{L}\left\{\cos^2(\omega t)\right\}$

$$\mathcal{L}\{\cos^2(\omega t)\} = \mathcal{L}\left\{\frac{1}{2} + \frac{1}{2}\cos(2\omega t)\right\} = \frac{1}{2}\{1\} + \frac{1}{2}\mathcal{L}\{\cos(2\omega t)\}$$

$$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2 + 4\omega^2}\right]$$

$$\mathcal{L}\{u(t)\} = \frac{1}{s}$$

$$\mathcal{L}\{\cos(\omega_0 t)u(t)\} = \frac{s}{(s^2 + \omega_0^2)}$$

(b) $\mathcal{L}^{-1}\left\{\dfrac{1}{(s-1)(s+2)(s+4)}\right\}$

Functions with distinct linear factors

$$\tilde{F}(s) = \frac{1}{(s-1)(s+2)(s+4)} = \frac{A_1}{s-1} + \frac{A_2}{s+2} + \frac{A_3}{s+3}$$

$$A_1 = (s-1)\tilde{F}(s)\Big|_{s=1} = \frac{1}{(s+2)(s+4)}\Bigg|_{s=1} = \frac{1}{15}$$

$$A_2 = (s+2)\tilde{F}(s)\Big|_{s=-2} = \frac{1}{(s-1)(s+4)}\Bigg|_{s=-2} = -\frac{1}{6}$$

$$A_3 = (s+4)\tilde{F}(s)\Big|_{s=-4} = \frac{1}{(s-1)(s+2)}\Bigg|_{s=-4} = \frac{1}{10}$$

Using Laplace transform table

$$\mathcal{L}\{\tilde{F}(s)\} = \mathcal{L}\left\{\frac{\frac{1}{15}}{s-1} + \frac{-\frac{1}{6}}{s+2} + \frac{\frac{1}{10}}{s+3}\right\} = \left[\frac{1}{15}\mathrm{e}^{t} - \frac{1}{6}\mathrm{e}^{-2t} + \frac{1}{10}\mathrm{e}^{-4t}\right]u(t)$$

Application of the shift in the s-domain function rule : $\mathcal{L}\left\{e^{-\alpha t}f(t)\right\} = \tilde{F}(s+a)$

(c) $\mathcal{L}^{-1}\left\{\dfrac{1}{(s+1)^2}\right\}$

From the Laplace transform table, we recognize the inverse Laplace transform formula

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = tu(t)$$

Comparing with our given function,

$$\frac{1}{(s+1)^2}$$

We notice a shift in the s-domain, meaning we apply the shifting property .

$$\mathcal{L}\{e^{-\alpha t}f(t)\} = \tilde{F}(s+\alpha), \qquad where\ f(t) = t, and\ \alpha = 1$$

Final solution:

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} = te^{-t}u(t)$$

(d) $\mathcal{L}^{-1}\left\{\dfrac{s+9}{s^2+6s+13}\right\}$

To make use of

$$\mathcal{L}\{e^{-\alpha t}\cos(\omega_0 t)\,u(t)\} = \frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}, \qquad \text{and} \quad \mathcal{L}\{e^{-\alpha t}\sin(\omega_0 t)\,u(t)\} = \frac{\omega_0}{(s+\alpha)^2+\omega_0^2}$$

We write

$$\tilde{F}(s) = \frac{s+9}{s^2+6s+13} = \frac{s+9}{(s+3)^2+4}$$

$$= \frac{s+3}{(s+3)^2+4} + 3\,\frac{2}{(s+3)^2+4}$$

Note that

$$s^2+6s+13 = (s+3)^2+4$$

$$(s+\alpha)^2+\omega_0^2$$

Apply shift in s-domain

$$\mathcal{L}\{e^{-\alpha t}f(t)\} = \tilde{F}(s+\alpha)$$

Using Laplace transform table

$$\mathcal{L}^{-1}\{\tilde{F}(s)\} = [\cos(2t) + 3\sin(2t)]e^{-3t}u(t)$$

(e) $\mathcal{L}^{-1}\left\{\dfrac{s+3}{s^2+3s+2}\right\}$

$$\tilde{F}(s) = \frac{s+3}{s^2+3s+2} = \frac{s+3}{(s+1)(s+2)} = \frac{A_1}{s+1} + \frac{A_2}{s+2}$$

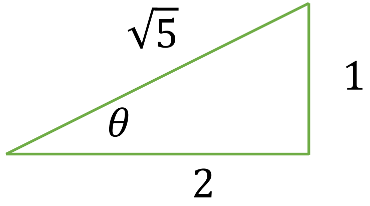
$$A_1 = (s+1)\tilde{F}(s)\Big|_{s=1} = \left.\frac{(s+3)}{(s+2)}\right|_{s=-1} = 2$$

$$A_2 = (s+2)\tilde{F}(s)\Big|_{s=-2} = \left.\frac{s+3}{(s+1)}\right|_{s=-2} = -1$$

Using Laplace transform table

$$\mathcal{L}\{\tilde{F}(s)\} = \mathcal{L}\left\{\frac{2}{s+1} + \frac{-1}{s+2}\right\} = [2\mathrm{e}^{-t} - \mathrm{e}^{-2t}]u(t)$$

(f) $\mathcal{L}\left\{\frac{3}{5}-\frac{\sqrt{45}}{5}e^{-2t}\sin\left(t+\tan^{-1}(0.5)\right)\right\}$



We set $\theta = \tan^{-1}(0.5)$ $\tan(\theta) = 0.5$

$\sin(\theta) = 1/\sqrt{5}$ $\cos(\theta) = 2/\sqrt{5}$

Note that

$$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta)$$

$$f(t) = \frac{3}{5} - \frac{\sqrt{45}}{5}e^{-2t}\sin(t+\tan^{-1}(0.5)) = \frac{3}{5} - \frac{\sqrt{45}}{5}e^{-2t}(\sin(t)\cos(\theta) + \cos(t)\sin(\theta))$$

$$= \frac{3}{5} - \frac{\sqrt{45}}{5}e^{-2t}\left(\sin(t)\frac{2}{\sqrt{5}} + \cos(t)\frac{1}{\sqrt{5}}\right) = \frac{3}{5} - \frac{3\times 2}{5}e^{-2t}\sin(t) - \frac{3}{5}e^{-2t}\cos(t)$$

$$= \frac{3}{5}(1 - 2e^{-2t}\sin(t) - e^{-2t}\cos(t))$$

Apply shift in s-domain

$$\mathcal{L}\{e^{-\alpha t}f(t)\} = \tilde{F}(s+\alpha)$$

Using Laplace transform table

$$\mathcal{L}\{f(t)\} = \frac{3}{5}\left[\frac{1}{s} - 2\frac{1}{(s+2)^2+1} - \frac{s+2}{(s+2)^2+1}\right] = \frac{3}{5}\left[\frac{1}{s} - \frac{s+4}{s^2+4s+1}\right] = \frac{3}{s(s^2+4s+5)}$$

Application of the shift in the times-domain function rule: $\mathcal{L}\left\{f\left(t-t_o\right)u\left(t-t_o\right)\right\}=e^{-st_o}F\left(s\right)$

(g) $\mathcal{L}\left\{\left(t-1\right)^2 u\left(t-1\right)\right\}$

To make use of

$$\mathcal{L}\{t^n u(t)\} = \frac{n!}{s^{n+1}} \rightarrow \mathcal{L}\{t^2 u(t)\} = \frac{2}{s^3}$$

Thus, we obtain

Also apply time shifting

$$\mathcal{L}\{f(t-t_0)u(t-t_0)\} = e^{-st_0}\tilde{F}(s)$$

$$\mathcal{L}\{(t-1)^2 u(t-1)\} = \frac{2}{s^3} \times e^{-s}$$

$t_0 = 1$

(h) $\mathcal{L}\left\{t^2u(t-1)\right\}$

$$f(t) = t^2u(t-1) = (t-1+1)^2u(t-1)$$

$$= (t-1)^2u(t-1) + 2(t-1)u(t-1) + u(t-1)$$

Using Laplace transform table

$$\mathcal{L}\{t^n u(t)\} = \frac{n!}{s^{n+1}}$$

Also apply time shifting

$$\mathcal{L}\{f(t-t_0)u(t-t_0)\} = e^{-st_0}\tilde{F}(s)$$

Thus, we obtain

$$\mathcal{L}\{f(t)\} = \frac{2}{s^3}e^{-s} + \frac{2}{s^2}e^{-s} + \frac{1}{s}e^{-s}$$

$$t_0 = 1$$

(i) $\mathcal{L}^{-1}\left\{\dfrac{se^{-2s}}{s^2+\pi^2}\right\}$

To make use of

$$\mathcal{L}\{\cos(\omega_0 t)\,u(t)\} = \frac{s}{s^2+\omega_0^2}$$

Also apply time shifting

$$\mathcal{L}\{f(t-t_0)u(t-t_0)\} = e^{-st_0}\tilde{F}(s)$$

We define

$$\frac{se^{-2s}}{s^2+\pi^2} = \tilde{F}(s)e^{-t_0 s}$$

where $\tilde{F}(s) = \dfrac{s}{s^2+\omega_0^2}$, $\quad \omega_0 = \pi \quad and \quad t_0 = 2$

Thus, we obtain

$$\mathcal{L}^{-1}\left\{\frac{se^{-2s}}{s^2+\pi^2}\right\} = \mathcal{L}^{-1}\{\tilde{F}(s)e^{-t_0 s}\} = f(t-t_0)u(t-t_0)$$

$$= \cos\big(\pi(t-2)\big)\,u(t-2) = \cos(\pi t)\,u(t-2)$$

Application of the derivative of transforms rule : $\tilde{F}'(s) = \mathcal{L}\{-tf(t)\}$

(j) $\mathcal{L}\{te^{-t}\sin(t)\}$

To make use of

$$\mathcal{L}\{-tf(t)\} = \tilde{F}'(s)$$

$$\mathcal{L}\{e^{-\alpha t}\sin(\omega_0 t)\,u(t)\} = \frac{\omega_0}{(s+\alpha)^2+\omega_0^2}$$

We define

$$f(t) = -e^{-t}\sin(t) \;\rightarrow\; \tilde{F}(s) = -\frac{1}{(s+1)^2+1}$$

Thus, we obtain

$$\mathcal{L}\{-te^{-t}\sin(t)\} = \mathcal{L}\{-tf(t)\} = \tilde{F}'(s) = -\frac{d}{ds}\left[\frac{1}{(s+1)^2+1}\right] = \frac{2(s+1)}{(s^2+2s+2)^2}$$

(k) $\mathcal{L}^{-1}\left\{\dfrac{s}{\left(s^2+9\right)^2}\right\}$

$$\mathcal{L}\{\sin(\omega_0 t)\,u(t)\} = \frac{\omega_0}{s^2+\omega_0^2}$$

To make use of

$$\mathcal{L}\{-tf(t)\} = \tilde{F}'(s)$$

We define

$$\mathcal{L}\{\sin(3t)\,u(t)\} = \frac{3}{s^2+9} \rightarrow \mathcal{L}\{t\sin(3t)\,u(t)\} = -\frac{d}{ds}\left[\frac{3}{s^2+9}\right] = \frac{6s}{(s^2+9)^2}$$

Thus, we obtain

$$\omega_0 = 3$$

$$\mathcal{L}\left\{\frac{s}{(s^2+9)^2}\right\} = \frac{1}{6}t\sin(3t)\,u(t)$$

Q2

Q.2 Solve the following linear second order differential equation using Laplace Transform:

$$\frac{d^2y(t)}{dt^2}+4\frac{dy(t)}{dt}+3y(t)=2u(t)$$

where $y(0^-)=1$ and $y'(0^-)=0$.

Solution to Q.2

Given: $\dfrac{d^2y(t)}{dt^2}+4\dfrac{dy(t)}{dt}+3y(t)=2u(t)$, $\quad y(0^-)=1 \quad$ and $\quad y'(0^-)=0$ $\qquad \mathcal{L}\left\{\dfrac{df(t)}{dt}\right\}=s\tilde{F}(s)-f(0^-)$

Apply Laplace transform: $\qquad \mathcal{L}\left\{\dfrac{d^2f(t)}{dt^2}\right\}=s^2\tilde{F}(s)-sf(0^-)-f'(0^-)$

$$s^2Y(s)-\underbrace{y(0^-)}_{1}s-\underbrace{y'(0^-)}_{0}+4sY(s)-4\underbrace{y(0^-)}_{1}+3Y(s)=\frac{2}{s}$$

$$\left(s^2+4s+3\right)Y(s)=\frac{2}{s}+s+4$$

$$Y(s)=\frac{s^2+4s+2}{s\left(s^2+4s+3\right)}=\frac{s^2+4s+2}{s(s+1)(s+3)}=\frac{2}{3s}+\frac{1}{2(s+1)}-\frac{1}{6(s+3)}$$

Apply inverse Laplace transform:

$$y(t)=\frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}+\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}-\frac{1}{6}\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}=\left[\frac{2}{3}+\frac{1}{2}e^{-t}-\frac{1}{6}e^{-3t}\right]u(t)$$

Thanks!