CG2023 Tutorial 6

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Table of Laplace Transforms and Properties

LAPLACE TRANSFORMS	$f(t)$	$\tilde{F}(s)$
Unit Impulse	$\delta(t)$	1
Unit Step	$u(t)$	$1/s$
Ramp	$t\,u(t)$	$1/s^2$
n^{th} order Ramp	$t^n u(t)$	$\frac{n!}{s^{n+1}}$
Damped Ramp	$t e^{-\alpha t} u(t)$	$1/(s+\alpha)^2$
Exponential	$e^{-\alpha t} u(t)$	$1/(s+\alpha)$
Cosine	$\cos(\omega_o t) u(t)$	$s/(s^2+\omega_o^2)$
Sine	$\sin(\omega_o t) u(t)$	$\omega_o/(s^2+\omega_o^2)$
Damped Cosine	$e^{-\alpha t}\cos(\omega_o t) u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega_o^2}$
Damped Sine	$e^{-\alpha t}\sin(\omega_o t) u(t)$	$\frac{\omega_o}{(s+\alpha)^2+\omega_o^2}$

LAPLACE TRANSFORM PROPERTIES	Time-domain	s-domain
Linearity	$\alpha f_1(t)+\beta f_2(t)$	$\alpha\tilde{F}_1(s)+\beta\tilde{F}_2(s)$
Time shifting	$f(t-t_o)$	$e^{-st_o}\tilde{F}(s)$
Shifting in the s-domain	$e^{s_o t} f(t)$	$\tilde{F}(s-s_o)$
Time scaling	$f(\alpha t)$	$\frac{1}{\lvert\alpha\rvert}\tilde{F}\left(\frac{s}{\alpha}\right)$
Integration in the time-domain	$\int_0^t f(\zeta)\,d\zeta$	$\frac{1}{s}\tilde{F}(s)$
Differentiation in the time-domain	$\frac{df(t)}{dt}$	$s\tilde{F}(s)-f(0^-)$
	$\frac{d^n f(t)}{dt^n}$	$s^n\tilde{F}(s)-\sum_{k=0}^{n-1} s^{n-1-k}\left.\frac{d^k f(t)}{dt^k}\right\rvert_{t=0^-}$
Differentiation in the s-domain	$-t f(t)$	$\frac{d\tilde{F}(s)}{ds}$
	$(-t)^n f(t)$	$\frac{d^n\tilde{F}(s)}{ds^n}$
Convolution in the time-domain	$\int_{-\infty}^{\infty} f_1(\zeta) f_2(t-\zeta)\,d\zeta$	$\tilde{F}_1(s)\tilde{F}_2(s)$

Initial value theorem: $f(0)=\lim_{s\to\infty} s\tilde{F}(s)$

Final value theorem: $\lim_{t\to\infty} f(t)=\lim_{s\to 0} s\tilde{F}(s)$

Review of LTI Systems

Impulse Response

$h(t)$

$$x(t) \rightarrow \boxed{h(t)} \rightarrow y(t) = x(t) * h(t)$$

Step Response

$$o(t) = \int_{-\infty}^{t} h(\tau)d\tau$$

$$\tilde{H}(s) = \mathcal{L}\{h(t)\} = \int_{0}^{\infty} h(t)\, e^{-jst} dt$$

$$H(f) = \mathfrak{F}\{h(t)\} = \int_{-\infty}^{\infty} h(t)\, e^{-j2\pi f t} dt$$

Differential Equations

$$\sum_{n=0}^{N} a_n \frac{d^n y(t)}{dt^n} = \sum_{m=0}^{M} b_m \frac{d^m y(t)}{dt^m} \xrightarrow{\mathcal{L}}$$

Transfer Function

$$\tilde{H}(s) \xrightarrow{s = j\omega} \tilde{H}(j\omega)$$

$$\tilde{X}(s) \rightarrow \boxed{\tilde{H}(s)} \rightarrow \tilde{Y}(s) = \tilde{X}(s) \cdot \tilde{H}(s)$$

$$\tilde{H}(s) = \frac{\tilde{Y}(s)}{\tilde{X}(s)} = \frac{b_M s^M + b_{M-1} s^{M-1} + \cdots + b_0}{a_N s^N + a_{N-1} s^{N-1} + \cdots + a_0}$$

Frequency Response

$$\tilde{H}(j\omega) \xrightarrow{\omega = 2\pi f} H(f)$$

$$\tilde{X}(j\omega) \rightarrow \boxed{\tilde{H}(j\omega)} \rightarrow \tilde{Y}(j\omega) = \tilde{X}(j\omega) \cdot \tilde{H}(j\omega)$$

$$\tilde{H}(j\omega) = \left|\tilde{H}(j\omega)\right| e^{j\angle\tilde{H}(j\omega)}$$

$\left|\tilde{H}(j\omega)\right|$

Magintude response

$e^{j\angle\tilde{H}(j\omega)}$

Phase response

Transient Response

- System Stability
  - Locations of System Poles
- Typical Systems
  - First order system
  - Second order system

Sinusoidal Response at Steady-State

$$x(t) = Ae^{j(\omega_0 t + \psi)} \rightarrow \boxed{\tilde{H}(j\omega)} \rightarrow y(t) = A\left|\tilde{H}(j\omega_0)\right| e^{j(\omega_0 t + \psi + \angle\tilde{H}(j\omega_0))}$$

System Stability

	Transfer Function	**Impulse Response**
BIBO Stable	**All** system poles lying on the left-half s-plane	$h(t)$ will converge to zero as t tends to infinity, i.e. $$\lim_{t\to\infty} h(t) = 0$$
Marginally Stable	**One or more** system poles lying on the **imaginary axis** of the s-plane and no system pole lying on the right half s-plane. System poles lying on the imaginary axis must be **distinct**.	$h(t)$ will not "blow up" and become unbounded, but neither will it converge to zero as t tends to infinity, i.e $$\lim_{t\to\infty} \lvert h(t)\rvert \neq \infty \quad \text{and} \quad \lim_{t\to\infty} h(t) \neq 0$$
Unstable (Case 1)	One or more system poles lying on **the right-half s-plane.**	$h(t)$ will "blow up" and become unbounded as t tends to infinity, i.e $$\lim_{t\to\infty} \lvert h(t)\rvert = \infty$$
Unstable (Case 2)	One or more **repeated system poles** lying on the **imaginary axis**.	

Review of LTI Systems

Impulse Response $h(t)$

Step Response

$$o(t) = \int_{-\infty}^{t} h(\tau)d\tau$$

$$\tilde{H}(s) = \mathcal{L}\{h(t)\} = \int_{0}^{\infty} h(t)\, e^{-jst}dt$$

$$H(f) = \mathfrak{I}\{h(t)\} = \int_{-\infty}^{\infty} h(t)\, e^{-j2\pi ft}dt$$

Transfer Function

$$\tilde{H}(s) \xrightarrow{s = j\omega} \tilde{H}(j\omega)$$

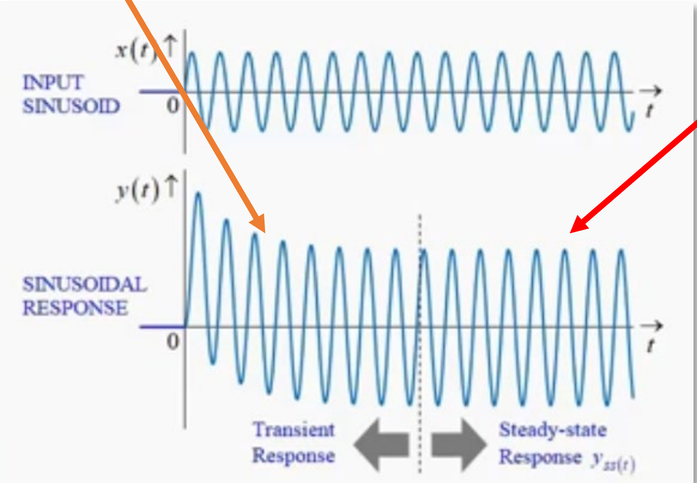
Frequency Response

$$\tilde{H}(j\omega) \xrightarrow{\omega = 2\pi f} H(f)$$

Analyse transient response

Analyse steady state response

The transient response of a system refers to the behaviour of the system from the moment an input or disturbance occurs until it reaches its steady-state condition.



Steady-state response of a system is defined as the system response as $t \to \infty$.

The start of steady-state is usually determined as the time when the system response is estimated to be sufficiently close to its steady-state response

Q1

For the following systems, determine the order of the system and the pole-zero excess, and sketch the pole-zero map.

(a) $\tilde{H}_1(s) = \dfrac{s^2 + 1.5s + 0.5}{s\left(s^2 + 0.36\right)}$

(b) $\tilde{H}_2(s) = \dfrac{\mathbf{1} - \mathbf{2}s}{\left(0.5s + 0.75\right)^2 \left(2s + \mathbf{1}\right)}$

(c) $\tilde{H}_3(s) = \dfrac{40s^2}{\left(s + 6\right)\left(s^2 + 8s + 25\right)}$

(a)

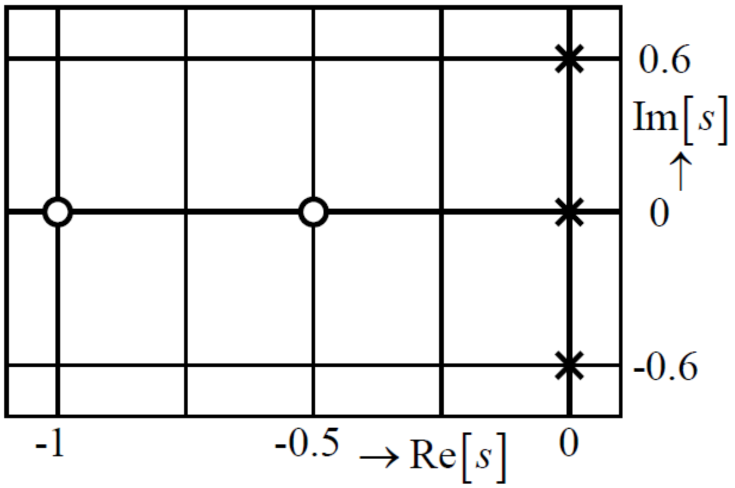
$$\tilde{H}_1(s) = \frac{s^2 + 1.5s + 0.5}{s\left(s^2 + 0.36\right)} = \frac{(s+1)(s+0.5)}{s(s+j0.6)(s-j0.6)}$$

Degree of denominator polynomial $= 3$. Therefore this is a 3rd-order system

There are 3 poles and 2 zeros. Therefore pole-zero excess $= 3 - 2 = 1$

Locations of Zeros: $s = -1, \quad s = -0.5$

Locations of Poles: $s = 0, \quad s = j0.6, \quad s = -j0.6$



(b)

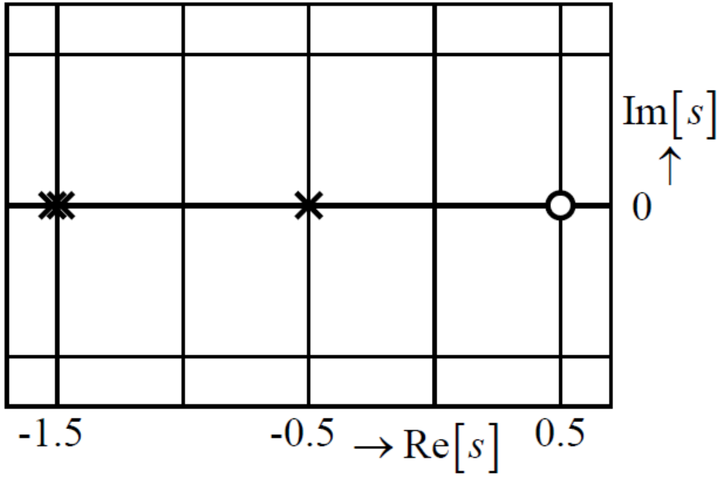
$$\tilde{H}_2(s) = \frac{1-2s}{(0.5s+0.75)^2(2s+1)} = \frac{1-2s}{(0.5s+0.75)(0.5s+0.75)(2s+1)}$$

Degree of denominator polynomial $= 3$. Therefore this is a 3^{rd}-order system

There are 3 poles and 1 zero. Therefore pole-zero excess $= 3 - 1 = 2$

Locations of Zeros: $s = 0.5$

Locations of Poles: $\underbrace{s = -1.5, \quad s = -1.5}_{\text{double pole}}, \quad s = -0.5$



(c)

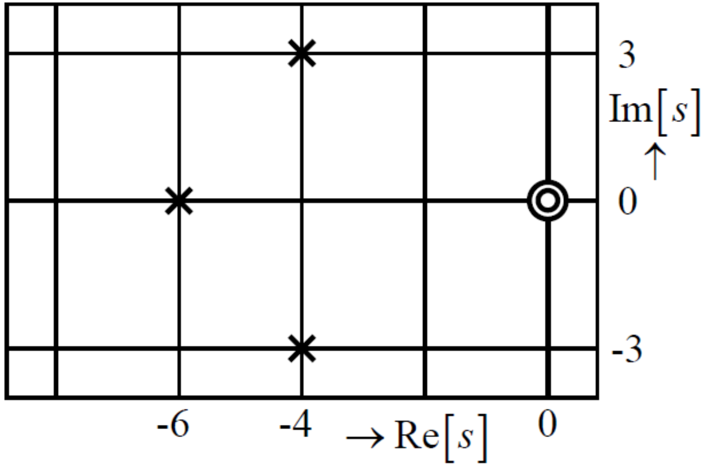
$$\tilde{H}_3(s) = \frac{40s^2}{(s+6)\left(s^2+8s+25\right)} = \frac{40s^2}{(s+6)(s+4-j3)(s+4+j3)}$$

Degree of denominator polynomial $=3$. Therefore this is a 3rd-order system

There are 3 poles and 2 zero. Therefore pole-zero excess $=3-2=1$

Locations of Zeros: $\underbrace{s=0, \quad s=0}_{\text{double zero}}$

Locations of Poles: $s=-6, \quad s=-4-j3, \quad s=-4+j3$



Q2

Q.2 For the following linear time-invariant continuous time systems, determine if the system is BIBO stable, marginally stable or unstable.

(a) Impulse response is $h(t)=\left[e^{-t}+e^{2t}\right]u(t)$.

(b) Impulse response is $h(t)=\sin(2t)u(t)$.

(c) Impulse response is $h(t)=e^{-t}\sin(2t)u(t)$.

(d) Transfer function is $\tilde{H}(s)=\dfrac{s+9}{s^2+9}$.

(e) Transfer function is $\tilde{H}(s)=\dfrac{4}{\left(s^2+4\right)^2}$.

(f) Transfer function is $\tilde{H}(s)=\dfrac{\mathbf{1-2s}}{s^2+2s+\mathbf{5}}$.

(g) System output is $y(t)=2t-\frac{2}{5}+\frac{2}{5}e^{-5t}$ when the input signal is the ramp function $x(t)=t$.

(a) Impulse response is $h(t) = \left[e^{-t} + e^{2t}\right]u(t)$.

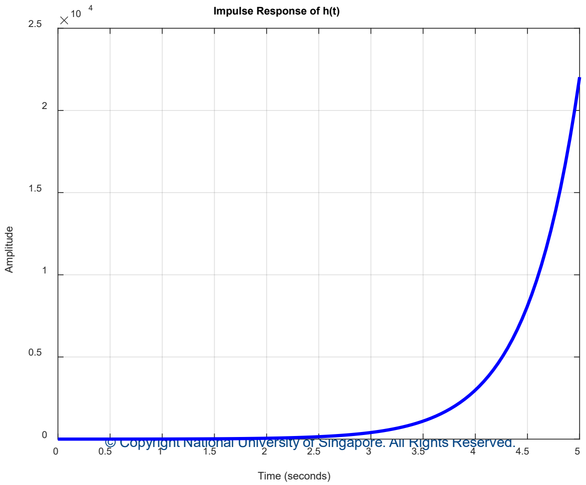
The system is **unstable** because the presence of $\exp(2t)$ causes the impulse response to grow without bound when $t \to \infty$.

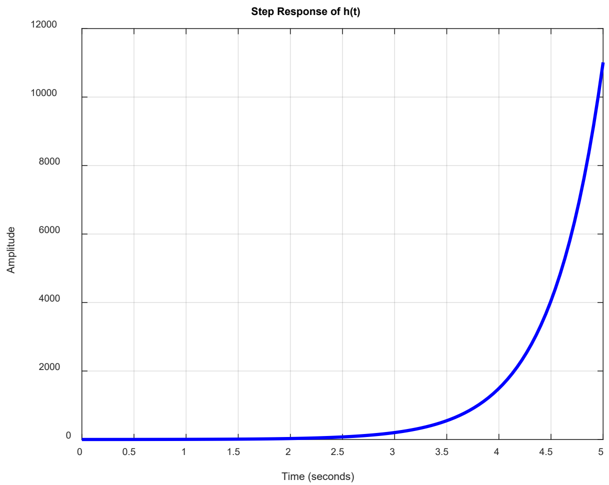
Taking the Laplace transform

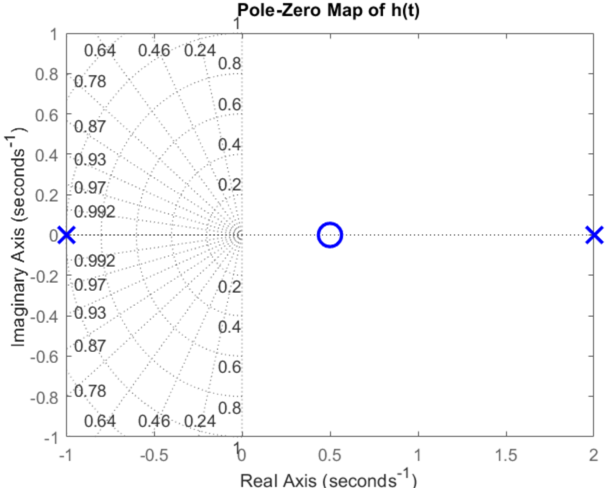
$$\tilde{H}(s) = \mathcal{L}\{e^{-t}u(t)\} + \mathcal{L}\{e^{2t}u(t)\}$$
$$= \frac{1}{s+1} + \frac{1}{s-2} = \frac{2s-1}{(s+1)(s-2)}$$

$p_1 = -1, p_2 = 2, z_1 = 0.5$

One system pole is located in the right-half s-plane.







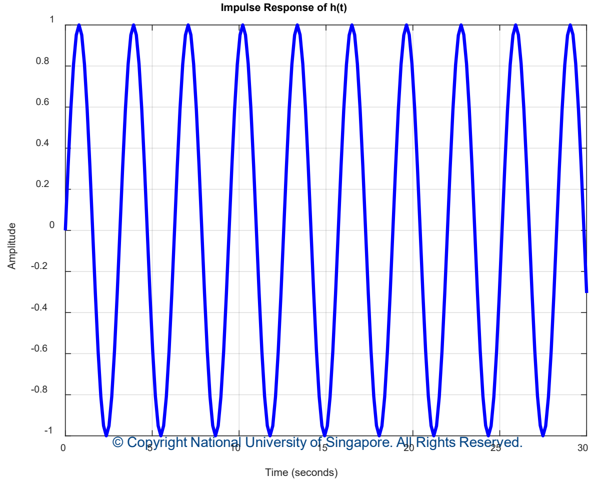
(b) Impulse response is $h(t) = \sin(2t)u(t)$.

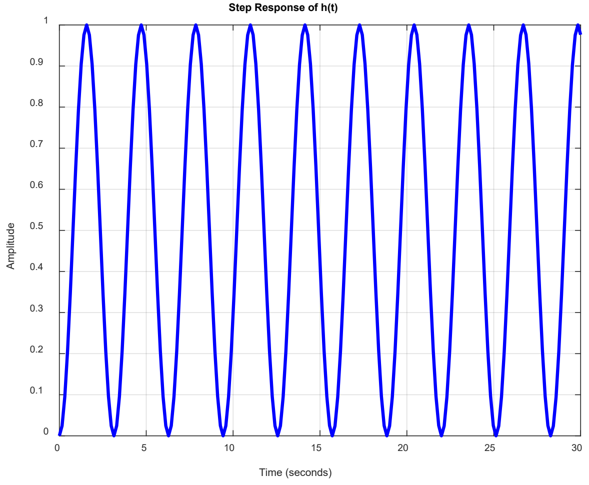
System is **marginally stable** because the impulse response oscillate with constant amplitude.

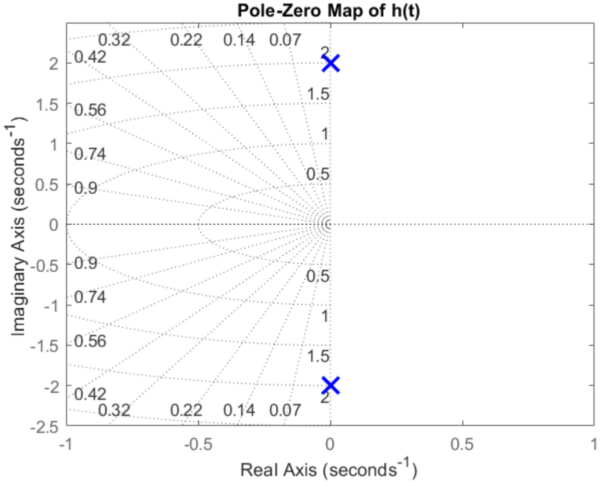
Taking the Laplace transform

$$\widetilde{H}(s) = \mathcal{L}\{\sin(2t)u(t)\} = \frac{2}{s^2+4} \qquad p_{1,2} = \pm 2j$$

Two system poles lying on the **imaginary axis** of the s-plane and no system pole lying on the right half s-plane.







(c) Impulse response is $h(t) = e^{-t}\sin(2t)u(t)$.

System is **stable** because the impulse response decays to zero when $t \to \infty$.

To make use of

$$\mathcal{L}\{e^{-\alpha t}\sin(\omega_0 t)\,u(t)\} = \frac{\omega_0}{(s+\alpha)^2+\omega_0^2}$$

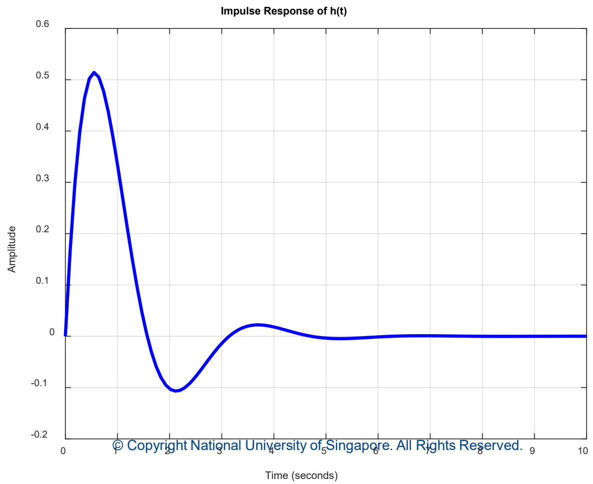
With $\alpha = 1$ and $\omega_0 = 2$

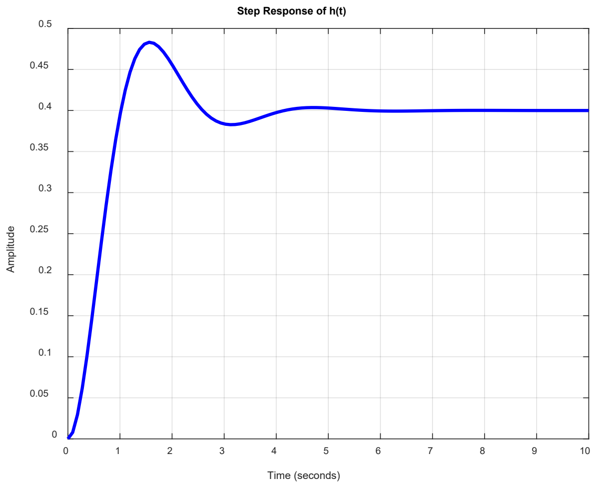
$$\tilde{H}(s) = \frac{2}{(s+1)^2+4} = \frac{2}{s^2+2s+5}$$

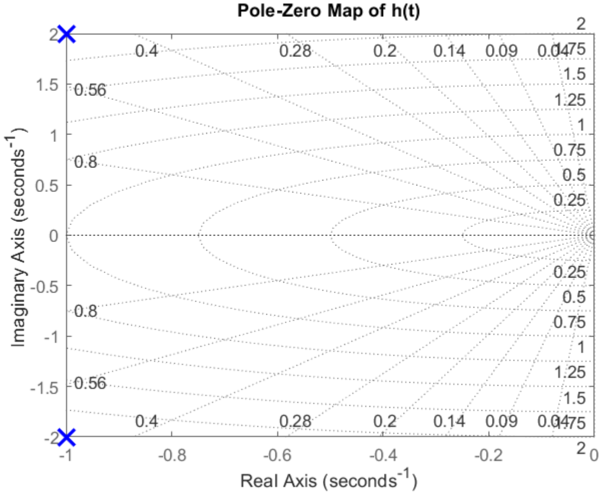
Poles are:

$$p_1 = -1+2j, \qquad p_2 = -1-2j$$

All system poles lying on the left-half s-plane.







(d) Transfer function is $\tilde{H}(s) = \dfrac{s+9}{s^2+9}$.

System poles are located at $s = \pm j3$. Since these are simple poles lying on the imaginary axis and there are no other poles, the system is **marginally stable**.

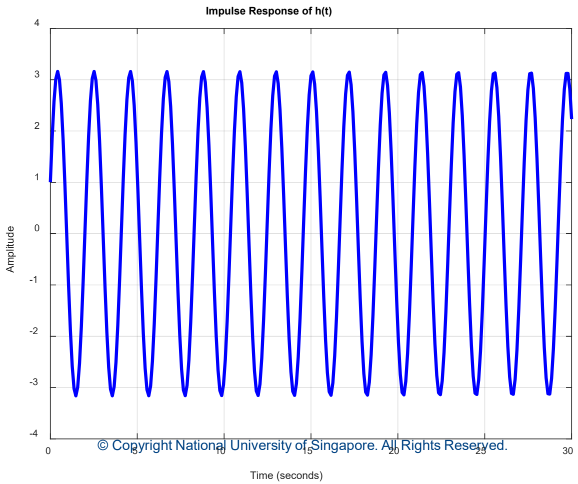
Two system poles lying on the **imaginary axis** of the s-plane and no system pole lying on the right half s-plane.

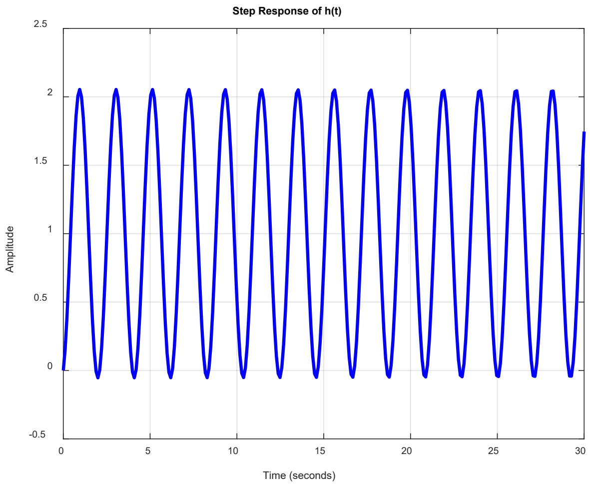
Apply partial fraction decomposition

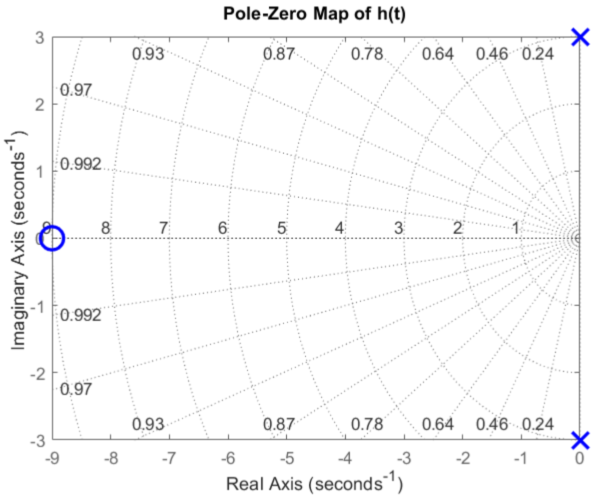
$$\tilde{H}(s) = \frac{s}{s^2+9} + \frac{9}{s^2+9}$$

Using Laplace transform table

$$h(t) = (\cos(3t) + 3\sin(3t))u(t)$$







(e) Transfer function is $\tilde{H}(s) = \dfrac{4}{\left(s^2+4\right)^2}$.

System poles are located at $s = \pm j2, \pm j2$. Since these are repeated poles lying on the imaginary axis, the system is **unstable**.

Four repeated system poles lying on the **imaginary axis**.

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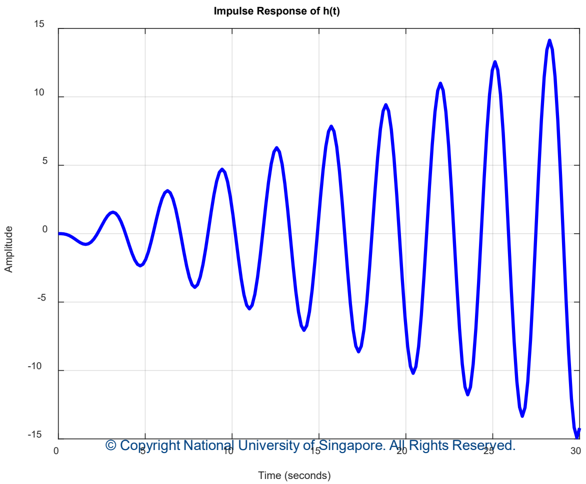
$$\tilde{H}(s) = \frac{\omega_o^2}{\left(s^2+\omega_o^2\right)^2}$$

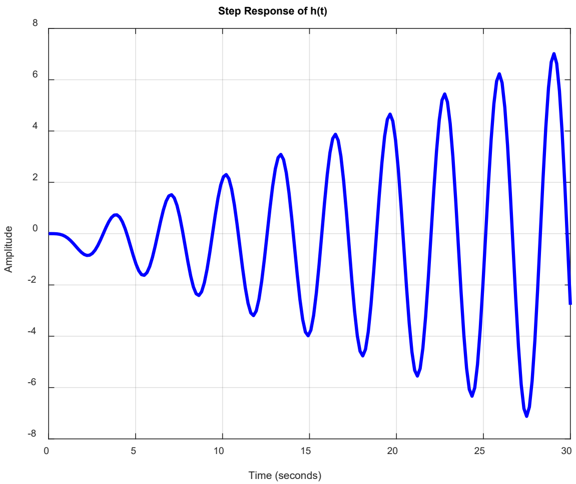
Poles : $s_{1,2,3,4} = \pm j\omega_o, \pm j\omega_o$

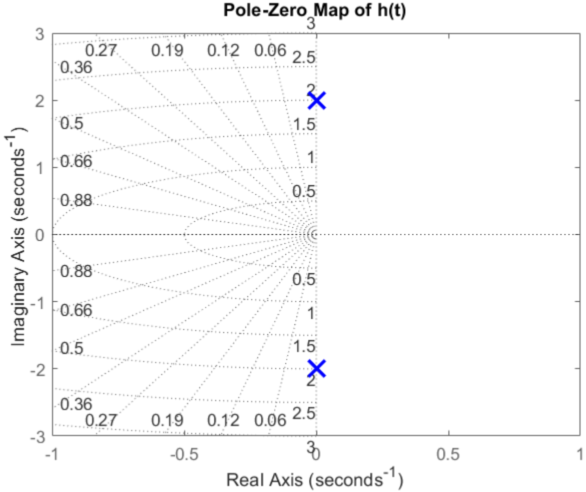
$$h(t) = \frac{1}{2}\left[\omega_o^{-1}\sin(\omega_o t) - t\cos(\omega_o t)\right]u(t)$$

Making use of Laplace transformation pair, we can obtain

$$h(t) = \frac{1}{2}\left[\omega_0^{-1}\sin(\omega_0 t) - t\cos(\omega_0 t)\right] = \frac{1}{4}\sin(2t) - \frac{t}{2}\cos(2t)$$







$$\left[\begin{array}{l} \tilde{H}(s) = \dfrac{\omega_o^2}{\left(s^2+\omega_o^2\right)^2} \\ \text{Poles}: s_{1,2,3,4} = \pm j\omega_o, \pm j\omega_o \\ h(t) = \dfrac{1}{2}\left[\begin{array}{l} \omega_o^{-1}\sin(\omega_o t) - \\ t\cos(\omega_o t) \end{array}\right] u(t) \end{array}\right]$$

We know the Laplace transform

$$\mathcal{L}\{\sin(\omega t)\} = \frac{\omega}{s^2+\omega^2}$$

$$\mathcal{L}\{t\cos(\omega t)\} = \frac{s^2-\omega^2}{(s^2+\omega^2)^2}$$

We define a linear combination of transforms involving $\sin(\omega t)$ and $t\cos(\omega t)$ as

$$f(t) = \frac{1}{2\omega^3}[\sin(\omega t) - \omega\, t\cos(\omega t)]$$

We take Laplace transform based on linearity

$$\mathcal{L}\{f(t)\} = \frac{1}{2\omega^3}\,[\,\mathcal{L}\{\sin(\omega t)\} - \omega\,\mathcal{L}\{t\cos(\omega t)\}\,] = \frac{1}{2\omega^3}\left[\frac{\omega}{s^2+\omega^2} - \omega\,\frac{s^2-\omega^2}{(s^2+\omega^2)^2}\right]$$

$$= \frac{1}{2\omega^3}\left[\frac{\omega s^2+\omega^3}{(s^2+\omega^2)^2} - \frac{\omega s^2-\omega^3}{(s^2+\omega^2)^2}\right] = \frac{1}{(s^2+\omega^2)^2}$$

Thus we have

$$\mathcal{L}^{-1}\left\{\frac{\omega^2}{(s^2+\omega^2)^2}\right\} = \omega^2 f(t) = \frac{1}{2}[\omega^{-1}\sin(\omega t) - t\cos(\omega t)]$$

(f) Transfer function is $\tilde{H}(s) = \dfrac{1-2s}{s^2+2s+5}$.

All system poles lying on the left-half s-plane

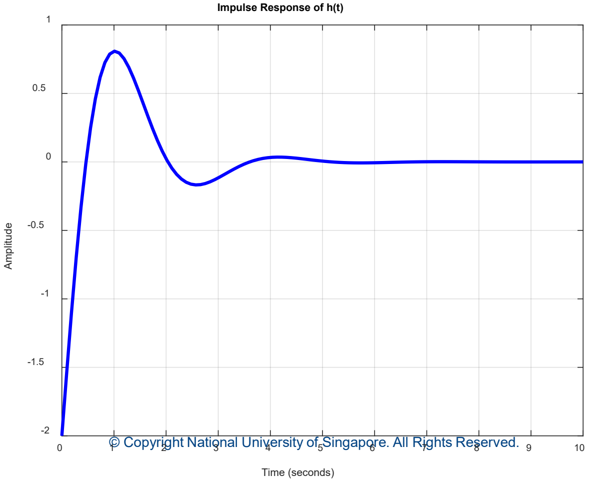
System poles are located at $s = -1 \pm j2$. Since these two poles lie on the left-half of the s-plane and there are no other poles, the system is **stable**. Note that system zeros does not influence stability.

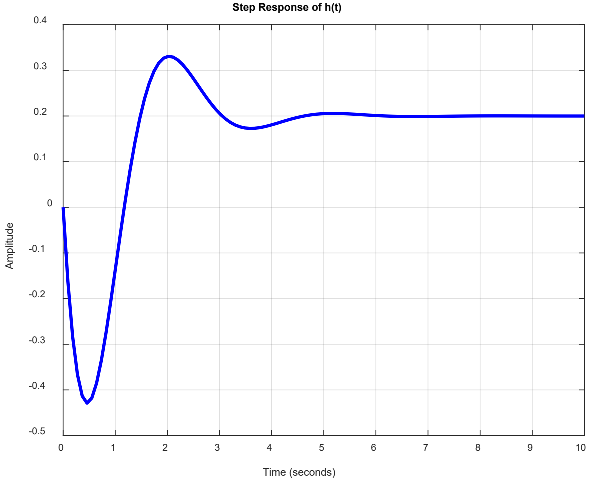
Apply partial fraction decomposition

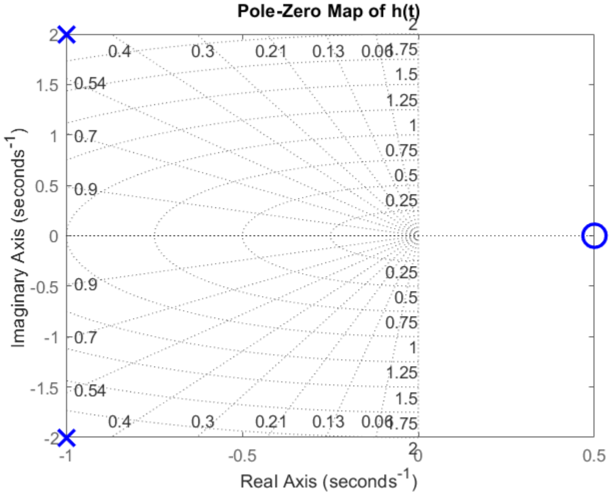
$$\tilde{H}(s) = \frac{1-2s}{(s+1)^2+4} = \frac{-2(s+1)}{(s+1)^2+4} + \frac{3}{(s+1)^2+4}$$

Using Laplace transform table

$$h(t) = e^{-t}\left(-2\cos(2t) + \frac{3}{2}\sin(2t)\right)u(t)$$







(g) System output is $y(t) = 2t - \frac{2}{5} + \frac{2}{5}e^{-5t}$ when the input signal is the ramp function $x(t) = t$.

Although the output signal is unbounded, conclusions about stability cannot be made directly because the input is also unbounded. In cases where 'rules' cannot be applied directly, it is best to revert to first principle by examining the locations of the system poles.

$$X(s) = \mathcal{L}\{x(t)\} = \frac{1}{s^2}$$

$$Y(s) = \mathcal{L}\{y(t)\} = \frac{2}{s^2} - \frac{2}{5s} + \frac{2}{5(s+5)} = \frac{10}{s^2(s+5)}$$

$$\tilde{H}(s) = \frac{Y(s)}{X(s)} = \frac{10}{s^2(s+5)} \div \frac{1}{s^2} = \frac{10}{s+5}$$

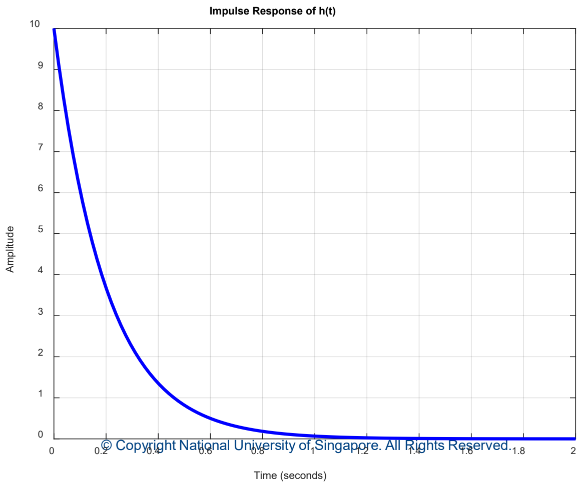
All system poles lying on the left-half s-plane

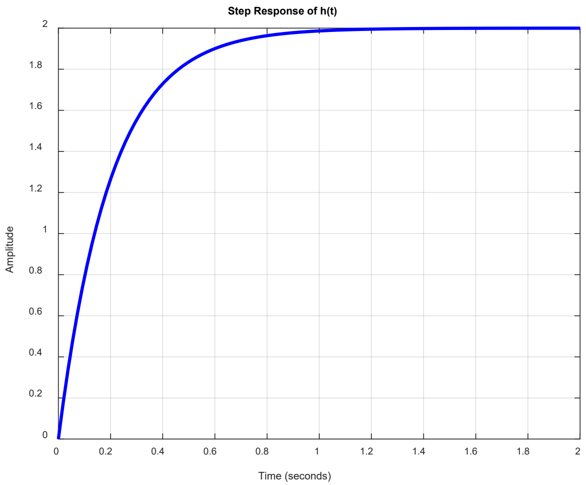
System pole is located at $s = -5$. Since this pole lies on the left-half of the s-plane and there are no other poles, the system is **stable**.

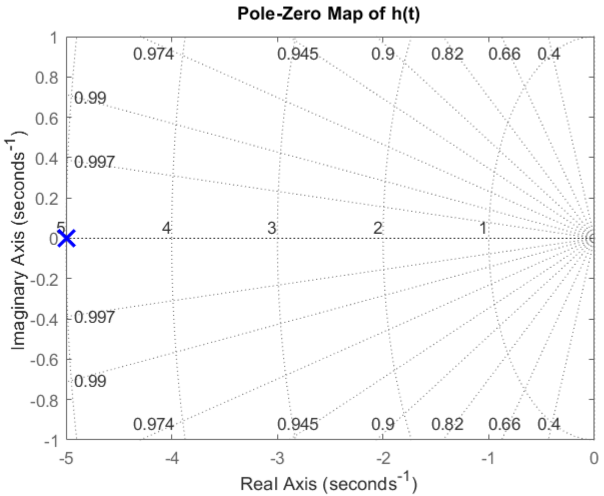
$$\tilde{H}(s) = \frac{Y(s)}{X(s)} = \frac{10}{s^2(s+5)} \div \frac{1}{s^2} = \frac{10}{s+5}$$

Using Laplace transform table to compute the impulse response:

$$h(t) = 10e^{-5t}u(t)$$







Q3

At steady-state, the input to a 1st-order system $\tilde{H}(s) = \dfrac{K}{sT+1}$ is $x(t) = 0.4 + 2\sin(0.25t)$, and $y(t)$ is the corresponding system output.

(a) Find an expression for $y(t)$ in terms K and T.

(b) If $y(t) = 2 + 8\sin\big(0.25(t - 2.574)\big)$, find the values of K and T.

(a)

Sinusoidal Response at Steady-State

$$x(t) = Ae^{j(\omega_0 t+\psi)} \rightarrow \boxed{\tilde{H}(j\omega)} \rightarrow y(t) = A\left|\tilde{H}(j\omega_0)\right|e^{j(\omega_0 t+\psi+\angle\tilde{H}(j\omega_0))}$$

Frequency response of the system $\tilde{H}(s) = \dfrac{K}{sT+1}$ is given by

$$\tilde{H}(j\omega) = \tilde{H}(s)\Big|_{s=j\omega} = \frac{K}{j\omega T+1} \rightarrow \begin{cases} \left|\tilde{H}(j\omega)\right| = \dfrac{K}{\left[\omega^2 T^2+1\right]^{1/2}} & \cdots\cdots \text{ Magnitude response} \\ \angle\tilde{H}(j\omega) = -\tan^{-1}\left(\dfrac{\omega T}{1}\right) & \cdots\cdots\cdots\cdots \text{ Phase response} \end{cases}$$

The system input $x(t) = 0.4 + 2\sin(0.25t)$ consists of a DC component which has a frequency of $0\ \text{rad/s}$ and a sine wave of frequency $0.25\ \text{rad/s}$. The sinusoidal responses of the system at these frequencies are:

$$@\ 0\ \text{rad/s} \rightarrow \begin{cases} \left|\tilde{H}(j0)\right| = K \\ \angle\tilde{H}(j\omega) = 0 \end{cases} \quad \text{and} \quad @\ 0.25\ \text{rad/s} \rightarrow \begin{cases} \left|\tilde{H}(j0.25)\right| = \dfrac{K}{\left[0.25^2T^2+1\right]^{1/2}} \\ \angle\tilde{H}(j0.25) = -\tan^{-1}\left(\dfrac{0.25T}{1}\right) \end{cases}$$

Hence,

$$\begin{aligned} y(t) &= 0.4\left|\tilde{H}(j0)\right| + 2\left|\tilde{H}(j0.25)\right|\sin\left(0.25t + \angle\tilde{H}(j0.25)\right) \\ &= 0.4K + \frac{2K}{\left(0.25^2T^2+1\right)^{1/2}}\sin\left(0.25t - \tan^{-1}\left(\frac{0.25T}{1}\right)\right) \qquad \cdots\cdots\cdots \quad (\clubsuit) \end{aligned}$$

(b)

$$y(t) = 0.4\left|\tilde{H}(j0)\right| + 2\left|\tilde{H}(j0.25)\right|\sin\left(0.25t + \angle\tilde{H}(j0.25)\right)$$

$$= 0.4K + \frac{2K}{\left(0.25^2T^2+1\right)^{1/2}}\sin\left(0.25t - \tan^{-1}\left(\frac{0.25T}{1}\right)\right) \qquad \cdots\cdots\cdots \ (\clubsuit)$$

Comparing $(\clubsuit)$ with the given output $y(t) = 2 + 8\sin\left(0.25(t-2.574)\right)$, we get

$$0.4K = 2 \quad \rightarrow \quad K = 5$$

$$\left.\begin{array}{l} \text{Using } \dfrac{2K}{\sqrt{0.25^2T^2+1}} = 8 \ \text{ or} \\ \tan^{-1}(0.25T) = 0.25 \times 2.574 \end{array}\right\} \quad \rightarrow \quad T = 3$$

Q4

At steady-state, $y(t) = 5 + 4\cos(3t + 0.3142)$ is the output of a 1st-order system $\tilde{H}(s) = \dfrac{20}{s+4}$. Derive the corresponding system input $x(t)$.

$$\left.\begin{array}{l}
x(t)=Ae^{j(\omega_o t+\psi)} \longrightarrow \boxed{\tilde{H}(j\omega)} \longrightarrow y(t)=A\left|\tilde{H}(j\omega_o)\right|e^{j(\omega_o t+\psi+\angle\tilde{H}(j\omega_o))} \\
x(t)=A\cos(\omega_o t+\psi) \longrightarrow \boxed{\tilde{H}(j\omega)} \longrightarrow y(t)=A\left|\tilde{H}(j\omega_o)\right|\cos(\omega_o t+\psi+\angle\tilde{H}(j\omega_o)) \\
x(t)=A\sin(\omega_o t+\psi) \longrightarrow \boxed{\tilde{H}(j\omega)} \longrightarrow y(t)=A\left|\tilde{H}(j\omega_o)\right|\sin(\omega_o t+\psi+\angle\tilde{H}(j\omega_o))
\end{array}\right\} \quad (7.20)$$

Steady-state Sinusoidal Response of a LTI System in ω-domain

Frequency response of the system $\tilde{H}(s)=\dfrac{20}{s+4}$ is given by

$$\tilde{H}(j\omega)=\tilde{H}(s)\Big|_{s=j\omega}=\frac{20}{j\omega+4} \rightarrow \begin{cases} \left|\tilde{H}(j\omega)\right|=\dfrac{20}{\left[\omega^2+16\right]^{1/2}} & \cdots\cdots \text{ Magnitude response} \\ \angle\tilde{H}(j\omega)=-\tan^{-1}\left(\dfrac{\omega}{4}\right) & \cdots\cdots\cdots\cdots \text{ Phase response} \end{cases}$$

The system output $y(t)=5+4\cos(3t+0.3142)$ consists of a DC component which has a frequency of 0 rad/s and a cosine wave of frequency 3 rad/s. The sinusoidal responses of the system at these frequencies are:

$$@\ 0\text{ rad/s} \rightarrow \begin{cases} \left|\tilde{H}(j0)\right|=5 \\ \angle\tilde{H}(j\omega)=0 \end{cases} \qquad \text{and} \qquad @\ 3\text{ rad/s} \rightarrow \begin{cases} \left|\tilde{H}(j3)\right|=\dfrac{20}{[9+16]^{1/2}}=4 \\ \angle\tilde{H}(j3)=-\tan^{-1}\left(\dfrac{3}{4}\right)=-0.6435 \end{cases}$$

Hence, the input at steady-state is

$$\begin{aligned} x(t)&=\frac{5}{\left|\tilde{H}(j0)\right|}+\frac{4}{\left|\tilde{H}(j3)\right|}\cos\left(3t+0.3142-\angle\tilde{H}(j3)\right) \\ &=\frac{5}{5}+\frac{4}{4}\cos\left(3t+0.3142-\left[-\tan\frac{3}{4}\right]\right) \\ &=1+\cos(3t+0.9577) \end{aligned}$$

Q5

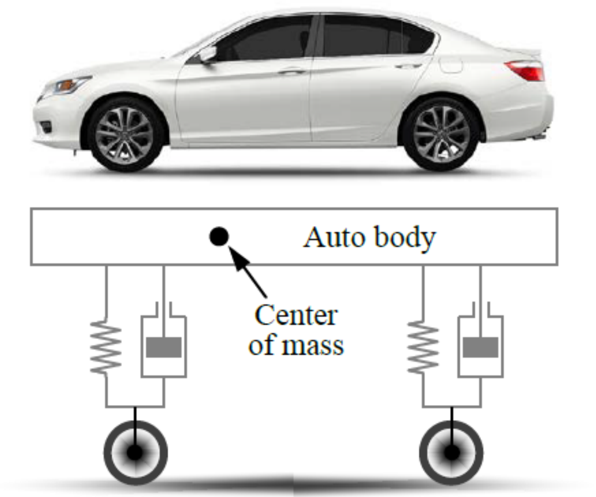
A car suspension system and a very simplifed version of the system are shown in Figure 5(a) and 5(b) respectively.

The transfer function of the simplifed car suspension system is

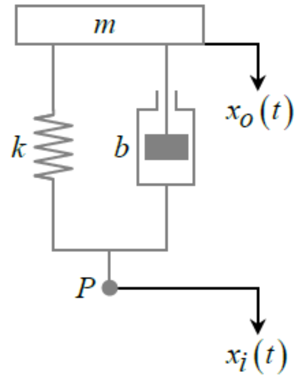
$$\frac{X_o(s)}{X_i(s)} = \frac{bs + k}{ms^2 + bs + k}$$

Suppose a car $\left(m = 4\ kg,\ k = 1\ Nm^{-1},\ b = 1\ Nm^{-1}s\right)$ is travelling on a road that has speed reducing stripes and the input to the simplifed car suspension system, $x_i(t)$, may be modelled by the periodic square wave, of frequency $\omega = 1\ \text{rad/s}$, shown in Figure 5(c).

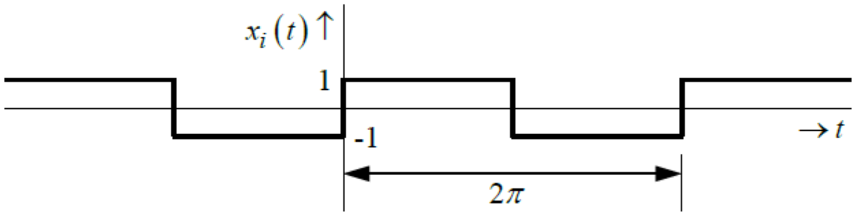
Q5



(a) Automobile suspension system



(b) Simplified suspension system



(c) Input waveform, $x_i(t)$

Figure 5

Q5

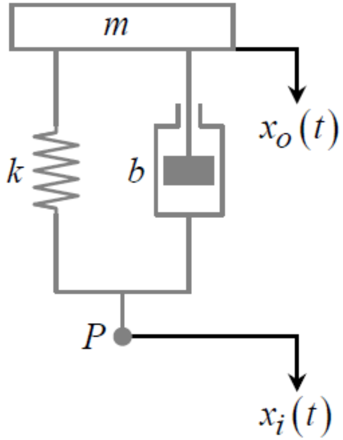
Determine the displacement of the car body, $x_o(t)$.

Hint : The Fourier Series representation of the periodic square wave shown in Figure 5(c) is

$$x_i(t) = \frac{4}{\pi}\left[\sin(t) + \frac{1}{3}\sin(3t) + \frac{1}{5}\sin(5t) + \cdots\cdots\right]$$

Transfer function of suspension system:

$$\tilde{H}(s)=\frac{\tilde{X}_o(s)}{\tilde{X}_i(s)}=\frac{bs+k}{ms^2+bs+k}$$



Substututing $m=4\ kg,\ \ k=1\ Nm^{-1},\ \ b=1\ Nm^{-1}s$, we get

$$\tilde{H}(s)=\frac{s+1}{4s^2+s+1}$$

Frequency response of suspension system:

$$\tilde{H}(j\omega)=\tilde{H}(s)\Big|_{s=j\omega}=\frac{s+1}{4s^2+s+1}\Bigg|_{s=j\omega}=\frac{1+j\omega}{1-4\omega^2+j\omega}$$

$$\rightarrow\begin{cases}\text{Magnitude Response:} & \left|\tilde{H}(j\omega)\right|=\left(\dfrac{1+\omega^2}{\left(1-4\omega^2\right)^2+\omega^2}\right)^{1/2}\\[2ex] \text{Phase Response:} & \angle\tilde{H}(j\omega)=\tan^{-1}\left(\dfrac{\omega}{1}\right)-\tan^{-1}\left(\dfrac{\omega}{1-4\omega^2}\right)\end{cases}$$

$$\left.\begin{aligned}
x(t) = Ae^{j(\omega_o t+\psi)} &\longrightarrow \boxed{\tilde{H}(j\omega)} \longrightarrow y(t) = A\left|\tilde{H}(j\omega_o)\right| e^{j\left(\omega_o t+\psi+\angle\tilde{H}(j\omega_o)\right)} \\
x(t) = A\cos(\omega_o t+\psi) &\longrightarrow \boxed{\tilde{H}(j\omega)} \longrightarrow y(t) = A\left|\tilde{H}(j\omega_o)\right| \cos\left(\omega_o t+\psi+\angle\tilde{H}(j\omega_o)\right) \\
x(t) = A\sin(\omega_o t+\psi) &\longrightarrow \boxed{\tilde{H}(j\omega)} \longrightarrow y(t) = A\left|\tilde{H}(j\omega_o)\right| \sin\left(\omega_o t+\psi+\angle\tilde{H}(j\omega_o)\right)
\end{aligned}\right\} \quad (7.20)$$

Steady-state Sinusoidal Response of a LTI System in ω-domain

Fourier series expansion of input: $x_i(t) = \frac{4}{\pi}\left[\sin(t) + \frac{1}{3}\sin(3t) + \frac{1}{5}\sin(5t) + \cdots\cdots\right]$

- Response of system due to sinusoidal input $\sin(t)$ is given by

$$\left|\tilde{H}(j1)\right| \cdot \sin\left(t + \angle\tilde{H}(j1)\right) = 0.4472\sin(t - 2.0344)$$

- Response of system due to sinusoidal input $\sin(3t)$ is given by

$$\left|\tilde{H}(j3)\right| \cdot \sin\left(3t + \angle\tilde{H}(j3)\right) = 0.09\sin(3t - 1.807)$$

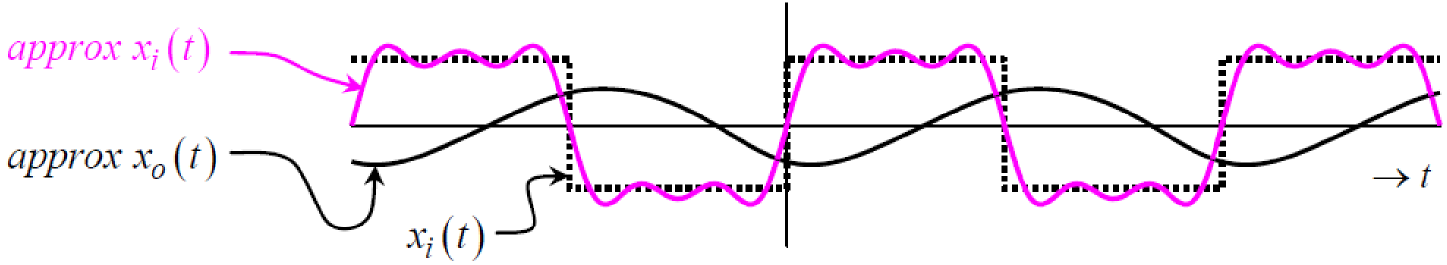
- Response of system due to sinusoidal input $\sin(5t)$ is given by

$$\left|\tilde{H}(j5)\right| \cdot \sin\left(5t + \angle\tilde{H}(j5)\right) = 0.0514\sin(5t - 1.7177)$$

Since system is linear, the output of the system can be obtained by superposition. Hence,

$$x_o(t) = \frac{4}{\pi}\left[0.4472\sin(t-2.0344) + \frac{1}{3}\times 0.09\sin(3t-1.807) + \frac{1}{5}\times 0.0514\sin(5t-1.7177) + \cdots\right]$$

$$= 0.5694\sin(t-2.0344) + 0.0382\sin(3t-1.807) + 0.0131\sin(5t-1.7177) + \cdots\cdots\cdots\cdots$$



Thanks!