CG2023 Laplace Transform (Review)

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Outline

- Laplace Transform
- Properties of Laplace Transform
- Inverse Laplace Transform and Partial Factorization
- Using Laplace Transform to solve Ordinary Differential Equation

The slides are prepared based on the document of "Math Laplace Transform of EE2023 Signals & Systems Math Refresher (Professor A. P. Loh)" which can be found in CANVAS.

Laplace Transform

Laplace Transforms (LT) are an integral part of systems and control. It is used widely in solving ordinary differential equations (ODE) by transforming them into algebraic equations involving the complex variable s.

Via the LT, the time domain ODEs are converted into algebraic equations in frequency domain where s is the frequency variable. This also gives the fundamental link between time and frequency domains of signals and systems. Via the frequency domain, many properties of linear time invariant systems can be described and characterised without the need to solve the original ODEs. This leads to a generalization of system behaviour for this class of systems.

Definitions of Laplace Transform

The Laplace transform $\mathcal{L}$ of a time-domain function $f(t)$ is defined by

$$\tilde{F}(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} f(t)e^{-st}dt \qquad (7.1)$$

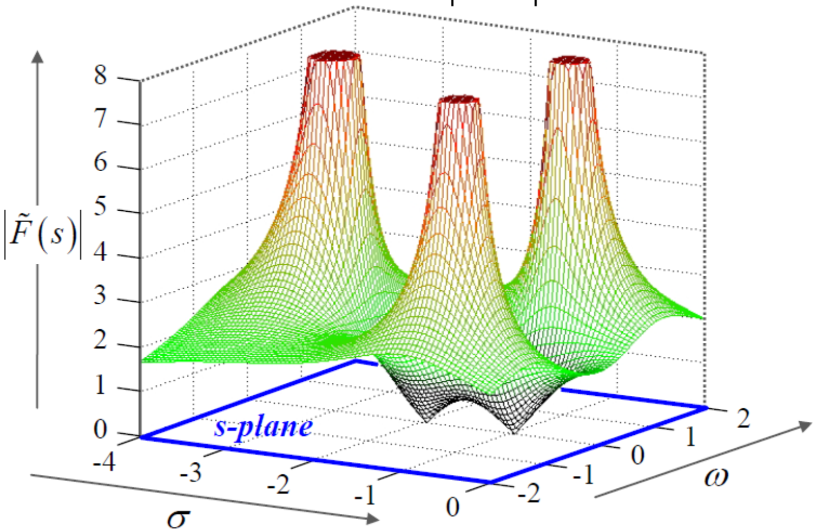
where $s = \sigma + j\omega$ **is a complex variable** in which $\sigma = \mathrm{Re}[s]$ and $\omega = \mathrm{Im}[s]$ are the real and imaginary parts of s, respectively.

Because s is a 2-dimensional domain, the plot for $\tilde{F}(s)$ is a 3-dimensional xyz-plot.

An example of a plot for $\left|\tilde{F}(s)\right|$ is shown on the right.

The s-domain is essentially a complex plane with σ as the x-axis and ω as the y-axis. The $\left|\tilde{F}(s)\right|$ axis is the z-axis.

3D-Plot for $\left|\tilde{F}(s)\right|$



Example 7-1:

Find the Laplace transform of $(\mathrm{a})\ u(t)$ *and* $(\mathrm{b})\ e^{-\alpha t}u(t)$.

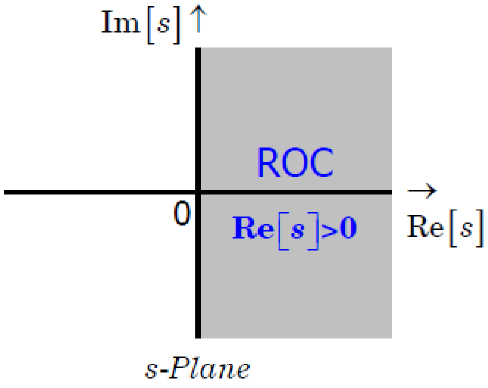
(a)

$$\mathcal{L}\{u(t)\} = \int_0^{\infty} 1 e^{-st}\,dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^{\infty} = \left[\frac{e^{-\sigma t}e^{-j\omega t}}{-s}\right]_0^{\infty}$$

$$= \begin{cases} \dfrac{1}{s}; & \text{if } \sigma > 0 \\ \text{Indefinite}; & \text{if } \sigma \le 0 \end{cases}$$

$\sigma = \mathrm{Re}[s] > 0$ is the Region of Convergence (ROC) of this transform



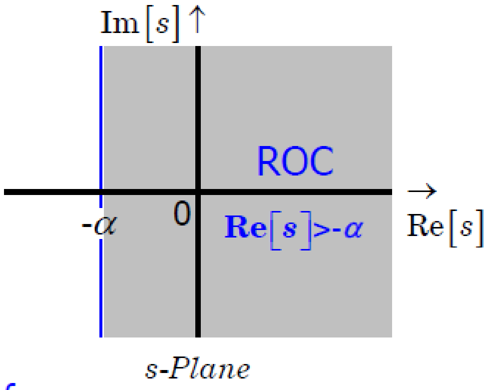
(b)

$$\mathcal{L}\{e^{-\alpha t}u(t)\} = \int_0^{\infty} e^{-\alpha t}e^{-st}\,dt = \int_0^{\infty} e^{-(s+\alpha)t}\,dt$$

$$= \left[-\frac{e^{-(s+\alpha)t}}{s+\alpha}\right]_0^{\infty} = \left[-\frac{e^{-(\sigma+\alpha)t}e^{-j\omega t}}{s+\alpha}\right]_0^{\infty}$$

$$= \begin{cases} \dfrac{1}{s+\alpha}; & \text{if } \sigma > -\alpha \\ \text{Indefinite}; & \text{if } \sigma \le -\alpha \end{cases}$$

$\sigma = \mathrm{Re}[s] > -\alpha$ is the Region of Convergence (ROC) of this transform



Laplace Transform (Optional)

If $f(t)$ is not integrable, we multiply it by the decay function $e^{-\sigma t}$ as shown below:

$$\mathcal{L}\{f(t)\} = \Im\{f(t)e^{-\sigma t}\} = \int_{-\infty}^{\infty} f(t)e^{-\sigma t}e^{-j\omega t}dt = \int_{-\infty}^{\infty} f(t)e^{-(\sigma+j\omega)t}dt$$

Let $\sigma + j\omega = s$, then we obtain

$$\mathcal{L}\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{-st}dt$$

In most engineering and physics applications, we deal with causal systems, where a function $f(t)$ is defined only for $t \geq 0$. Therefore:

Laplace Transform $$\tilde{F}(s) = \mathcal{L}\{f(t)\} = \int_{0}^{\infty} f(t)e^{-st}dt$$

Laplace Transform Table

Fortunately, we do not need to evaluate the LT integral each time we want to find the Laplace Transform of a function $f(t)$, at least for the kind of $f(t)$ that are generally encountered in systems and control.

Table of Laplace Transforms and Properties

LAPLACE TRANSFORMS		
	$f(t)$	$\tilde{F}(s)$
Unit Impulse	$\delta(t)$	1
Unit Step	$u(t)$	$1/s$
Ramp	$t\,u(t)$	$1/s^2$
n$^{\text{th}}$ order Ramp	$t^n u(t)$	$\dfrac{n!}{s^{n+1}}$
Damped Ramp	$t\,e^{-\alpha t}u(t)$	$1/(s+\alpha)^2$
Exponential	$e^{-\alpha t}u(t)$	$1/(s+\alpha)$
Cosine	$\cos(\omega_o t)u(t)$	$s/(s^2+\omega_o^2)$
Sine	$\sin(\omega_o t)u(t)$	$\omega_o/(s^2+\omega_o^2)$
Damped Cosine	$e^{-\alpha t}\cos(\omega_o t)u(t)$	$\dfrac{s+\alpha}{(s+\alpha)^2+\omega_o^2}$
Damped Sine	$e^{-\alpha t}\sin(\omega_o t)u(t)$	$\dfrac{\omega_o}{(s+\alpha)^2+\omega_o^2}$

LAPLACE TRANSFORM PROPERTIES		
	Time-domain	s-domain
Linearity	$\alpha f_1(t)+\beta f_2(t)$	$\alpha\tilde{F}_1(s)+\beta\tilde{F}_2(s)$
Time shifting	$f(t-t_o)$	$e^{-st_o}\tilde{F}(s)$
Shifting in the s-domain	$e^{s_o t}f(t)$	$\tilde{F}(s-s_o)$
Time scaling	$f(\alpha t)$	$\dfrac{1}{\lvert\alpha\rvert}\tilde{F}\left(\dfrac{s}{\alpha}\right)$
Integration in the time-domain	$\int_0^t f(\zeta)d\zeta$	$\dfrac{1}{s}\tilde{F}(s)$
Differentiation in the time-domain	$\dfrac{df(t)}{dt}$	$s\tilde{F}(s)-f(0^-)$
	$\dfrac{d^n f(t)}{dt^n}$	$s^n\tilde{F}(s)-\sum_{k=0}^{n-1}s^{n-1-k}\left.\dfrac{d^k f(t)}{dt^k}\right\rvert_{t=0^-}$
Differentiation in the s-domain	$-t\,f(t)$	$\dfrac{d\tilde{F}(s)}{ds}$
	$(-t)^n f(t)$	$\dfrac{d^n\tilde{F}(s)}{ds^n}$
Convolution in the time-domain	$\int_{-\infty}^{\infty}f_1(\zeta)f_2(t-\zeta)d\zeta$	$\tilde{F}_1(s)\tilde{F}_2(s)$

Initial value theorem: $f(0)=\lim_{s\to\infty}s\tilde{F}(s)$	**Final value theorem:** $\lim_{t\to\infty}f(t)=\lim_{s\to 0}s\tilde{F}(s)$

Properties of Laplace Transform

Laplace transforms are linear operators which has the following properties. Note that $\tilde{F}(s)$ and $\tilde{G}(s)$ are used to denote the Laplace Transforms of $f(t)$ and $g(t)$.

A. Linearity

$$\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha \tilde{F}(s) + \beta \tilde{G}(s), \alpha \text{ and } \beta \text{ are contants.}$$

Example

Given $\mathcal{L}\{t\} = \frac{1}{s^2}$, and $\mathcal{L}\{\sin(2t)\} = \frac{2}{s^2+4}$

Then

$$\mathcal{L}\{3t + 5\sin(2t)\} = \frac{3}{s^2} + \frac{10}{s^2+4}$$

Properties of Laplace Transform

B. Time shifting

$$\mathcal{L}\{f(t-t_0)\} = e^{-st_0}\tilde{F}(s)$$

Proof:

$$\mathcal{L}\{f(t-t_0)\} = \int_0^\infty f(t-t_0)\, e^{-st} dt = \int_0^\infty f(t-t_0)\, e^{-s(t-t_0)} e^{-st_0} dt$$

$$= e^{-st_0} \int_0^\infty f(t-t_0)\, e^{-s(t-t_0)} dt$$

$$= e^{-st_0}\tilde{F}(s)$$ after making the substitution $\lambda = t - t_0$

where $\tilde{F}(s) = \mathcal{L}\{f(t)\}$

Properties of Laplace Transform

Example

Given

$$f(t) = t^3 \quad \overset{\mathcal{L}}{\Leftrightarrow} \quad \tilde{F}(s) = \frac{6}{s^4}$$

Compute the Laplace transform of

$$g(t) = f(t-2)$$

Thus,

$$\mathcal{L}\{g(t)\} = \mathcal{L}\{f(t-2)\} = e^{-2s}\tilde{F}(s) = e^{-2s}\frac{6}{s^4}$$

Properties of Laplace Transform

C. Shifting in the s-Domain

$$\mathcal{L}\{e^{-\alpha t}f(t)\} = \tilde{F}(s+\alpha)$$

Proof:

$$\mathcal{L}\{e^{-\alpha t}f(t)\} = \int_0^\infty e^{-\alpha t}f(t)e^{-st}dt = \int_0^\infty e^{-(s+\alpha)t}f(t)dt$$

Comparing with the definition of the Laplace transform:

$$\mathcal{L}\{e^{-\alpha t}f(t)\} = \int_0^\infty e^{-(s+\alpha)t}f(t)dt = \tilde{F}(s+\alpha)$$

Properties of Laplace Transform

C. Time scaling

$$\mathcal{L}\{\alpha t\} = \frac{1}{|\alpha|}\tilde{F}\left(\frac{s}{\alpha}\right)$$

Example

Given

$$f(t) = t^3 \quad \overset{\mathcal{L}}{\Leftrightarrow} \quad \tilde{F}(s) = \frac{6}{s^4}$$

Compute the Laplace transform of

$$g(t) = 8t^3$$

Thus,

$$\mathcal{L}\{g(t)\} = \mathcal{L}\{(2t)^3\} = \frac{1}{2}\tilde{F}\left(\frac{s}{2}\right) = \frac{1}{2}\frac{6}{\left(\frac{s}{2}\right)^4} = \frac{48}{s^4}$$

Properties of Laplace Transform

D. Integration in the Time-Domain

$$\mathcal{L}\left\{\int_0^t f(\zeta)\,d\zeta\right\} = \frac{1}{s}\tilde{F}(s)$$

Example

Given

$$f(t) = e^{-2t} \quad \overset{\mathcal{L}}{\Leftrightarrow} \quad \tilde{F}(s) = \frac{1}{s+2}$$

Compute the Laplace transform of

$$g(t) = \int_0^t e^{-2\zeta}\,d\zeta$$

Thus,

$$\mathcal{L}\{g(t)\} = \frac{1}{s(s+2)}$$

Properties of Laplace Transform

E. Differentiation in the Time-Domain

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = s\tilde{F}(s) - f(0^-)$$

where $f(0^-)$ denotes the initial condition of $f(t)$. In general,

$$\mathcal{L}\left\{\frac{d^n f(t)}{dt^n}\right\} = s^n\tilde{F}(s) - \sum_{k=0}^{n-1} s^{n-1-k} \left.\frac{d^k f(t)}{dt^k}\right|_{t=0^-}$$

Example

$$\mathcal{L}\left\{\frac{d^2 f(t)}{dt^2}\right\} = s^2\tilde{F}(s) - s^{2-1-0} f(0^-) - s^{2-1-1} \left.\frac{df(t)}{dt}\right|_{t=0^-}$$

$$= s^2\tilde{F}(s) - sf(0^-) - \left.\frac{df(t)}{dt}\right|_{t=0^-}$$

Properties of Laplace Transform

G. Differentiation in the s-Domain

$$\mathcal{L}\{-tf(t)\} = \frac{d\tilde{F}(s)}{ds}$$

In general,

$$(-1)^n \mathcal{L}\{t^n f(t)\} = \frac{d^n \tilde{F}(s)}{ds^n}$$

Example

Since $\mathcal{L}\{\sin(\omega t)\} = \frac{\omega}{s^2+\omega^2}$, then according to the formula

$$\mathcal{L}\{t\sin(\omega t)\} = -\frac{d}{ds}\left\{\frac{\omega}{s^2+\omega^2}\right\} = \frac{2\omega s}{(s^2+\omega^2)^2}$$

Properties of Laplace Transform

F. Convolution

$$\mathcal{L}\left\{\int_{-\infty}^{\infty} f_1(\zeta) f_2(t-\zeta)\, d\zeta\right\} = \tilde{F}_1(s)\tilde{F}_2(s)$$

Example

Given

$$f(t) = e^{-t} \overset{\mathcal{L}}{\Leftrightarrow} \tilde{F}(s) = \frac{1}{s+1}$$

$$g(t) = \sin(t) \overset{\mathcal{L}}{\Leftrightarrow} \frac{1}{s^2+1}$$

Compute the Laplace transform of $f(t) \circledast g(t)$

$$\mathcal{L}\{f(t) \circledast g(t)\} = \frac{1}{s+1} \times \frac{1}{s^2+1} = \frac{1}{(s+1)(s^2+1)}$$

Properties of Laplace Transform

- **Final Value Theorem**

Final Value Theorem: For a time domain function which has a finite steady state value, the final value theorem is give as:

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} s\tilde{F}(t)$$

This is a convenient formula to use when you want to determine the steady state value of a system's output response. You can obtain this steady state value from $\tilde{F}(s)$ instead of from $f(t)$. Applicable only when the $f(t)$ has a constant steady state value. Not applicable when $f(t)$ is always changing with time. Examples where the final value theorem fails is when $f(t) = \sin(\omega t)$, $f(t) = t$, etc.

Properties of Laplace Transform

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s).$$

Proof :

$$\begin{aligned}
\mathcal{L}\{\frac{df(t)}{dt}\} &= \int_0^\infty \frac{df(t)}{dt} e^{-st} dt = sF(s) - f(0) \\
\lim_{s\to 0} \int_0^\infty \frac{df(t)}{dt} e^{-st} dt &= \lim_{s\to 0} [sF(s) - f(0)] \\
\lim_{s\to 0} \int_0^\infty \frac{df(t)}{dt} dt &= \lim_{s\to 0} [sF(s) - f(0)] \\
f(\infty) - f(0) &= \lim_{s\to 0} [sF(s) - f(0)] \\
\lim_{t\to\infty} f(t) &= \lim_{s\to 0} sF(s)
\end{aligned}$$

Inverse Laplace Transform

The inverse Laplace transform $\mathcal{L}^{-1}$ of $\tilde{F}(s)$ is defined by

$$f(t) = \mathcal{L}^{-1}\left\{\tilde{F}(s)\right\} = \frac{1}{2\pi j}\int_{\gamma-j\infty}^{\gamma+j\infty} \tilde{F}(s)e^{st}ds \qquad (7.2)$$

where the integration is done along the vertical line $\mathrm{Re}[s] = \gamma$ in the complex plane such that γ is greater than the real part of all singularities of $\tilde{F}(s)$ and $\tilde{F}(s)$ is bounded on the line, for example if the contour path is in the region of convergence.

For cases where $\tilde{F}(s)$ is a rational function of the form $\dfrac{\tilde{C}(s)}{\tilde{D}(s)}$ where $\tilde{C}(s)$ and $\tilde{D}(s)$ are polynomials of s, we do not need to solve (7.2) to find the inverse Laplace transform of $\tilde{F}(s)$. Instead, we expand $\tilde{F}(s)$ into a sum of partial fractions and make use of the standard Laplace transform table to obtain the inverse transform.

Inverse Laplace Transform (Optional)

Inverse Laplace Transform

$$f(t)e^{-\sigma t} = \mathfrak{I}^{-1}\{\tilde{F}(s)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{F}(s)e^{j\omega t}d\omega$$

Hence:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{F}(s)e^{\sigma t}e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{F}(s)e^{st}d\omega = \frac{1}{2\pi}\int_{\sigma-j\infty}^{\sigma+j\infty} \tilde{F}(s)e^{st}ds$$

Partial-Fraction Expansion

Let

$$\tilde{F}(s) = \frac{\tilde{C}(s)}{\tilde{D}(s)} = \mu \frac{(s+z_1)(s+z_2)\cdots\cdots(s+z_M)}{(s+p_1)(s+p_2)\cdots\cdots(s+p_N)} \tag{7.3}$$

where

μ is a constant

$\{-z_m;\ m = 1, 2, \cdots, M\}$ are roots of $C(s)$

$\{-p_n;\ n = 1, 2, \cdots, N\}$ are roots of $D(s)$.

$\tilde{F}(s)$ is called a **proper rational function** if $M < N$, and **improper rational function** if $M \geq N$.

In the following, we shall examine how may ***partial-fraction expansion*** be used to determine the inverse Laplace transform of $\tilde{F}(s)$.

(A) $(M < N)$: All Roots of $\tilde{D}(s)$ are DISTINCT

If all the roots $(-p_n)$ of $\tilde{D}(s)$ are distinct, then the partial fraction expansion of $\tilde{F}(s)$ has the form

$$\tilde{F}(s) = \frac{\tilde{C}(s)}{\tilde{D}(s)} = \frac{\alpha_1}{s+p_1} + \frac{\alpha_2}{s+p_2} + \cdots + \frac{\alpha_n}{s+p_n} + \cdots + \frac{\alpha_N}{s+p_N} \tag{7.4}$$

where

$$\alpha_n = (s+p_n)\tilde{F}(s)\Big|_{s=-p_n} . \tag{7.5}$$

$$(s+p_n)F(s) = \frac{\alpha_1(s+p_n)}{s+p_1} + \frac{\alpha_2(s+p_n)}{s+p_2} + \cdots + \frac{\alpha_n\cancel{(s+p_n)}}{\cancel{s+p_n}} + \cdots + \frac{\alpha_N(s+p_n)}{s+p_N}$$

$$\left[(s+p_n)F(s)\right]_{s=-p_n} = \underbrace{\frac{\alpha_1(-p_n+p_n)}{s+p_1}}_{=0} + \underbrace{\frac{\alpha_2(-p_n+p_n)}{s+p_2}}_{=0} + \cdots + \alpha_n + \cdots + \underbrace{\frac{\alpha_N(-p_n+p_n)}{s+p_N}}_{=0} = \alpha_n$$

Example 7-2:

$$\tilde{F}(s) = \frac{2s+4}{(s+1)(s+3)} = \frac{\alpha_1}{(s+1)} + \frac{\alpha_2}{(s+3)}$$

$$\alpha_1 = (s+1)F(s)\Big|_{s=-1} = 1$$

$$\alpha_2 = (s+3)F(s)\Big|_{s=-3} = 1$$

$$\therefore \tilde{F}(s) = \frac{1}{(s+1)} + \frac{1}{(s+3)} \Rightarrow \overbrace{\left[\mathcal{L}^{-1}\left\{\tilde{F}(s)\right\} = \left(e^{-t} + e^{-3t}\right)u(t)\right]}^{\text{using Laplace transform table}}$$

(B) $(M < N)$: One or more Roots of $\tilde{D}(s)$ are REPEATED

If $\tilde{D}(s)$ has a repeated root $(-\tilde{p})$ of multiplicity r, then $\tilde{D}(s)$ will contain a factor of the form $(s+\tilde{p})^r$. The partial-fraction expansion of $\tilde{F}(s)$ is then written as

$$\tilde{F}(s) = \frac{\alpha_1}{s+p_1} + \frac{\alpha_2}{s+p_2} + \cdots + \frac{\alpha_{N-r}}{s+p_{N-r}} + \left[\frac{\gamma_1}{s+\tilde{p}} + \frac{\gamma_2}{(s+\tilde{p})^2} + \cdots + \frac{\gamma_r}{(s+\tilde{p})^r}\right]; \quad r \geq 2 \qquad (7.6)$$

where the α's are determined as in Case (A) and

$$\gamma_i = \frac{1}{(r-i)!}\left[\frac{d^{r-i}}{ds^{r-i}}(s+\tilde{p})^r \tilde{F}(s)\right]\Bigg|_{s=-\tilde{p}}; \quad i = 1, 2, \cdots, r. \qquad (7.7)$$

Example 7-3:

$$\tilde{F}(s) = \frac{s^2+2s+5}{(s+3)(s+5)^2} = \frac{\alpha_1}{(s+3)} + \frac{\gamma_1}{(s+5)} + \frac{\gamma_2}{(s+5)^2}$$

Using (7.5) : $\alpha_1 = (s+3)\tilde{F}(s)\big|_{s=-3} = 2$

Using (7.7) with $r=2$: $\gamma_1 = \frac{1}{1!}\frac{d}{ds}\left[(s+5)^2 \tilde{F}(s)\right]\Big|_{s=-5} = \frac{d}{ds}\left[\frac{s^2+2s+5}{s+3}\right]\Big|_{s=-5} = \frac{s^2+6s+1}{(s+3)^2}\Big|_{s=-5} = -1$

Using (7.7) with $r=2$: $\gamma_2 = \frac{1}{0!}(s+5)^2 \tilde{F}(s)\big|_{s=-5} = -10$

$$\therefore \tilde{F}(s) = \frac{2}{(s+3)} - \frac{1}{(s+5)} - \frac{10}{(s+5)^2} \rightarrow \overbrace{\left[\mathcal{L}^{-1}\{\tilde{F}(s)\} = \left(2e^{-3t} - e^{-5t} - 10te^{-5t}\right)u(t)\right]}^{\text{using Laplace transform table}}$$

ALTERNATE APPROACH:

We may also express $\tilde{F}(s)$ as

$$\tilde{F}(s)=\frac{\alpha_1}{s+p_1}+\frac{\alpha_2}{s+p_2}+\cdots+\frac{\alpha_{N-r}}{s+p_{N-r}}+\left[\frac{\beta_1 s^{r-1}+\beta_2 s^{r-2}+\cdots+\beta_r}{(s+\tilde{p})^r}\right]; \quad r\geq 2$$

and work this example out by recombining the partial fractions as follows:

$$\tilde{F}(s)=\frac{s^2+2s+5}{(s+3)(s+5)^2}=\frac{\overset{\overset{\alpha_1}{\downarrow}}{A}}{(s+3)}+\frac{\overset{\overset{\beta_1}{\downarrow}}{B}s+\overset{\overset{\beta_2}{\downarrow}}{C}}{(s+5)^2}=\frac{A(s+5)^2+(Bs+C)(s+3)}{(s+3)(s+5)^2}$$

$$=\frac{A\left(s^2+10s+25\right)+\left(Bs^2+Cs+3Bs+3C\right)}{(s+3)(s+5)^2}=\frac{(A+B)s^2+(10A+C+3B)s+(25A+3C)}{(s+3)(s+5)^2}$$

$\cdots$ equating coefficients of the numerator polynomials

$$\left\{\begin{aligned} A+B&=1\\ 10A+C+3B&=2\\ 25A+3C&=5\end{aligned}\right\}\rightarrow \text{ solving } \rightarrow \; A=2,\; B=-1 \text{ and } C=-15$$

Therefore, $\tilde{F}(s)=\dfrac{2}{(s+3)}-\dfrac{s+15}{(s+5)^2}=\dfrac{2}{(s+3)}-\dfrac{1}{(s+5)}-\dfrac{10}{(s+5)^2}$

$$\mathcal{L}^{-1}\left\{\tilde{F}(s)\right\}=\left(2e^{-3t}-e^{-5t}-10te^{-5t}\right)u(t)$$

(C) $(M \geq N)$. $\tilde{F}(s)$ is an Improper Rational Function

If $M \geq N$, we can apply long division to express $\tilde{F}(s)$ in the form

$$\tilde{F}(s) = \frac{\tilde{C}(s)}{\tilde{D}(s)} = \tilde{Q}(s) + \frac{\tilde{R}(s)}{\tilde{D}(s)} \tag{7.8}$$

such that the $\begin{cases} \text{Quotient} \quad : \tilde{Q}(s) \text{ is a polynomial of } s \text{ with degree } (M-N), \\ \text{Remainder} : \tilde{R}(s) \text{ is a polynomial of } s \text{ with degree strictly less than } N. \end{cases}$

The inverse Laplace transform of $\tilde{R}(s)/\tilde{D}(s)$, which is now a ***proper rational function***, can be computed by first expanding it into partial fraction form using method (A) or (B) discussed above.

The inverse Laplace transform of $\tilde{Q}(s)$ can be computed by using

$$\mathcal{L}^{-1}\left\{s^k\right\} = \frac{d^k}{dt^k}\delta(t); \quad k = 0, 1, 2, \cdots\cdots\cdots. \tag{7.9}$$

Example 7-4:

$$\underbrace{\tilde{F}(s) = \frac{2s^2+10s+10}{(s+1)(s+3)} = 2 + \frac{2s+4}{(s+1)(s+3)}}_{\text{by long division}}$$

$$\therefore \tilde{F}(s) = 2 + \frac{1}{s+1} + \frac{1}{s+3} \rightarrow \underbrace{\left[\mathcal{L}^{-1}\left\{\tilde{F}(s)\right\} = 2\delta(t) + e^{-t}u(t) + e^{-3t}u(t)\right]}_{\text{From } \mathcal{LT} \text{ Table}}$$

$$F(s) = \underbrace{\left[\begin{array}{r|l} & \quad\quad\quad 2 \\ \hline s^2+4s+3 & 2s^2+10s+10 \\ & \underline{2s^2+\ 8s\ +6} \\ & \quad\quad\ \ 2s\ +4 \end{array}\right]}_{\text{Long Division}}$$

$$= 2 + \frac{2s+4}{s^2+4s+3}$$

$$= 2 + \frac{2s+4}{(s+1)(s+3)}$$

Using Laplace Transform to solve Ordinary Differential Equation

Consider the first order RC circuit given in Figure 2. Assume a general input voltage $v(t)$

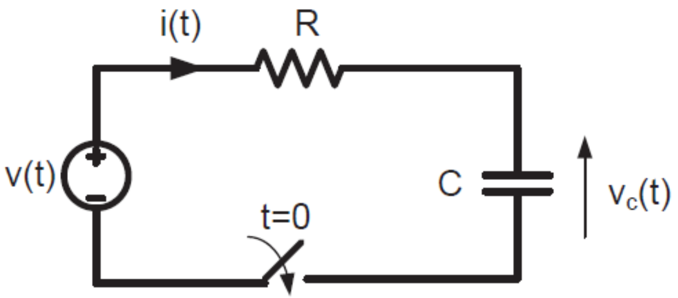


Fig. 2: First order RC circuit.

and an output voltage $v_c(t)$ across the capacitor. Deriving the model in time domain using differential equation and by applying circuit laws :

Using Laplace Transform to solve Ordinary Differential Equation

$$
\begin{aligned}
v(t) &= i(t)R + v_c(t) \\
i(t) &= C\frac{dv_c(t)}{dt} \\
v(t) &= RC\frac{dv_c(t)}{dt} + v_c(t) \qquad (1)
\end{aligned}
$$

Equation (1) may be solved using mathematical techniques and it can also be solved using Laplace Transform. Using the LT approach, since the input voltage is a general input $v(t)$, the LT of $v(t)$ is written generally as $V(s)$ while the LT of the output voltage $v_c(t)$ is written as $V_c(s)$. Taking the LT of (1),

Using Laplace Transform to solve Ordinary Differential Equation

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = s\tilde{F}(s) - f(0^-)$$

$$\begin{aligned}
V(s) &= RCsV_c(s) - RCv_c(0) + V_c(s) \\
&= RCsV_c(s) - RCv_c(0) + V_c(s) \\
V_c(s) &= \frac{V(s)}{RCs+1} + \frac{RCv_c(0)}{RCs+1} \qquad (2)
\end{aligned}$$

If the input voltage $v(t)$ is a constant DC source with value $v(t) = V$, then $V(s) = \frac{V}{s}$ and (2) becomes

$$V_c(s) = \frac{V}{s(RCs+1)} + \frac{RCv_c(0)}{RCs+1}$$

Using Laplace Transform to solve Ordinary Differential Equation

Then $v_c(t) = \mathcal{L}^{-1}\{V_c(s)\}$ can be found as follows :

$$\begin{aligned}
v_c(t) &= \mathcal{L}^{-1}\left\{\frac{V}{s(RCs+1)} + \frac{RCv_c(0)}{RCs+1}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{V}{s} - \frac{VRC}{RCs+1}\right\} + \mathcal{L}^{-1}\left\{\frac{v_c(0)}{s+\frac{1}{RC}}\right\} \\
&= V - Ve^{-\frac{t}{RC}} + v_c(0)e^{-\frac{t}{RC}} \\
&= V + \left[v_c(0) - V\right]e^{-\frac{t}{RC}} \qquad (3)
\end{aligned}$$

Using Laplace Transform to solve Ordinary Differential Equation

From (3), it is clear that $\lim_{t\to\infty} v_c(t) = V$ which is the final value which the capacitor will be charged up to when a DC voltage of V is applied. This same result can be obtained by applying the Final Value Theorem to $V_c(s)$ as follows :

$$\lim_{t\to\infty} v_c(t) = \lim_{s\to 0} s\left\{\frac{V}{s(RCs+1)} + \frac{RCv_c(0)}{RCs+1}\right\} = V.$$

The output voltage in (3) can also be rewritten as :

$$v_c(t) = v_c(0)e^{-\frac{t}{RC}} + V\left(1 - e^{-\frac{t}{RC}}\right)$$

The different components in $v_c(t)$ can be seen in Figure 3.

Using Laplace Transform to solve Ordinary Differential Equation

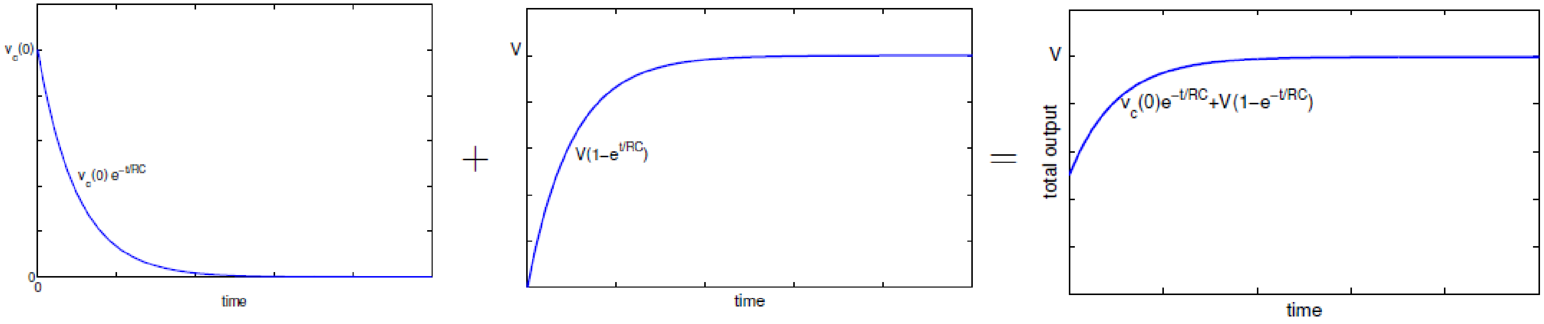


Fig. 3: Plotting individual components of $v_c(t)$.

Notice how the initial condition $v_c(0)$ decays over time. The plot in the middle is the typical charging characteristic of a capacitor with zero initial condition. Finally, the sum total of the first two plots gives the actual charging trajectory of the capacitor, including the non-zero initial condition.

Comparison of Fourier Series, Fourier Transform, and Laplace Transform

	Fourier Series	Fourier Transform	Laplace Transform
Purpose	Represents periodic signals as sums of sinusoids	Analyses signals in terms of frequency components	Transforms functions into algebraic expressions for easier manipulation
Applicable To	Periodic signals/functions	Aperiodic and periodic signals	General functions, including unstable and transient signals, such as exponentially growing ones
Mathematical Form	$x_p(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi kt/T_p}$	$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$	$\tilde{F}(s) = \int_{0}^{\infty} f(t)e^{-st}dt$
Variable Used	Frequency index n (discrete)	Frequency f (continuous)	Complex frequency $s = \sigma + j\omega$

Thanks!