EE2012 Tutorial 5

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Location:

- E1A-04-01
- Level 6, #06-02, T-Lab Building

Derivatives

Basic Properties/Formulas/Rules

$$\frac{d}{dx}\left(cf(x)\right) = cf'(x)$$, c is any constant.

$$\frac{d}{dx}\left(f(x) \pm g(x)\right) = f'(x) \pm g'(x)$$

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$, n is any number.

$$\frac{d}{dx}\left(c\right) = 0$$, c is any constant.

$$\left(f(x)\,g(x)\right)' = f'(x)\,g(x) + f(x)\,g'(x)$$ – **Product Rule**

$$\frac{d}{dx}\left(\mathbf{e}^{g(x)}\right) = g'(x)\mathbf{e}^{g(x)}$$

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)\,g(x) - f(x)\,g'(x)}{\left(g(x)\right)^2}$$ – **Quotient Rule**

$$\frac{d}{dx}\left[\ln(g(x))\right] = \frac{g'(x)}{g(x)}$$

$$\frac{d}{dx}\left[f\left(g(x)\right)\right] = f'\left(g(x)\right)\,g'(x)$$ – **Chain Rule**

https://tutorial.math.lamar.edu/pdf/common_derivatives_integrals.pdf

Derivatives

Trig Functions

$$\boxed{\frac{d}{dx}\left[\sin(x)\right] = \cos(x) \qquad \frac{d}{dx}\left[\cos(x)\right] = -\sin(x) \qquad \frac{d}{dx}\left[\tan(x)\right] = \sec^2(x)}$$

$$\frac{d}{dx}\left[\csc(x)\right] = -\csc(x)\cot(x) \qquad \frac{d}{dx}\left[\sec(x)\right] = \sec(x)\tan(x) \qquad \frac{d}{dx}\left[\cot(x)\right] = -\csc^2(x)$$

Inverse Trig Functions

$$\boxed{\frac{d}{dx}\left[\sin^{-1}(x)\right] = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\left[\cos^{-1}(x)\right] = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\left[\tan^{-1}(x)\right] = \frac{1}{1+x^2}}$$

$$\frac{d}{dx}\left[\csc^{-1}(x)\right] = -\frac{1}{|x|\sqrt{x^2-1}} \qquad \frac{d}{dx}\left[\sec^{-1}(x)\right] = \frac{1}{|x|\sqrt{x^2-1}} \qquad \frac{d}{dx}\left[\cot^{-1}(x)\right] = -\frac{1}{1+x^2}$$

Exponential & Logarithm Functions

$$\frac{d}{dx}\left[a^x\right] = a^x \ln(a) \qquad \boxed{\frac{d}{dx}\left[\mathbf{e}^x\right] = \mathbf{e}^x}$$

$$\frac{d}{dx}\left[\ln(x)\right] = \frac{1}{x}, \quad x > 0 \qquad \frac{d}{dx}\left[\ln|x|\right] = \frac{1}{x}, \quad x \neq 0 \qquad \frac{d}{dx}\left[\log_a(x)\right] = \frac{1}{x\ln(a)}, \quad x > 0$$

Integration

Basic Properties/Formulas/Rules

$\int cf(x)\,dx = c\int f(x)\,dx$, c is a constant.

$\int f(x) \pm g(x)\,dx = \int f(x)dx \pm \int g(x)\,dx$

$\int_a^b f(x)\,dx = f(x)\Big|_a^b = F(b) - F(a)$ where $f(x) = \int f(x)\,dx$

$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, c is a constant.

$\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

$\int_a^a f(x)\,dx = 0$

$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

$\int_a^b c\,dx = c(b-a)$, c is a constant.

If $f(x) \geq 0$ on $a \leq x \leq b$ then $\int_a^b f(x)\,dx \geq 0$

If $f(x) \geq g(x)$ on $a \leq x \leq b$ then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

Integration

Polynomials

$$\int dx = x + c \qquad \int k\,dx = kx + c \qquad \int x^n dx = \frac{1}{n+1}x^{n+1} + c,\ n \neq -1$$

$$\int \frac{1}{x}\,dx = \ln|x| + c \qquad \int x^{-1}\,dx = \ln|x| + c \qquad \int x^{-n} dx = \frac{1}{-n+1}x^{-n+1} + c,\ n \neq 1$$

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + c \qquad \int x^{\frac{p}{q}}\,dx = \frac{1}{\frac{p}{q}+1}x^{\frac{p}{q}+1} + c = \frac{q}{p+q}x^{\frac{p+q}{q}} + c$$

Trig Functions

$$\int \cos(u)\,du = \sin(u) + c \qquad \int \sin(u)\,du = -\cos(u) + c \qquad \int \sec^2 u\,du = \tan(u) + c$$

$$\int \sec(u)\tan(u)\,du = \sec(u) + c \qquad \int \csc(u)\cot(u)\,du = -\csc(u) + c \qquad \int \csc^2 u\,du = -\cot(u) + c$$

$$\int \tan(u)\,du = -\ln|\cos(u)| + c = \ln|\sec(u)| + c \qquad \int \cot(u)\,du = \ln|\sin(u)| + c = -\ln|\csc(u)| + c$$

Integration

Exponential & Logarithm Functions

$$\boxed{\int e^u \, du = e^u + c}$$

$$\int a^u \, du = \frac{a^u}{\ln(a)} + c$$

$$\int \ln(u) \, du = u \ln(u) - u + c$$

$$\int e^{au} \sin(bu) \, du = \frac{e^{au}}{a^2 + b^2} \Big(a \sin(bu) - b \cos(bu) \Big) + c$$

$$\int u e^u du = (u - 1) e^u + c$$

$$\int e^{au} \cos(bu) \, du = \frac{e^{au}}{a^2 + b^2} \Big(a \cos(bu) + b \sin(bu) \Big) + c$$

$$\int \frac{1}{u \ln(u)} \, du = \ln \big| \ln(u) \big| + c$$

Inverse Trig Functions

$$\int \frac{1}{\sqrt{a^2 - u^2}} \, du = \sin^{-1}\left(\frac{u}{a}\right) + c$$

$$\int \sin^{-1}(u) \, du = u \sin^{-1}(u) + \sqrt{1 - u^2} + c$$

$$\int \frac{1}{a^2 + u^2} \, du = \frac{1}{a} \tan^{-1}\left(\frac{u}{a}\right) + c$$

$$\int \tan^{-1}(u) \, du = u \tan^{-1}(u) - \frac{1}{2} \ln\left(1 + u^2\right) + c$$

$$\int \frac{1}{u\sqrt{u^2 - a^2}} \, du = \frac{1}{a} \sec^{-1}\left(\frac{u}{a}\right) + c$$

$$\int \cos^{-1}(u) \, du = u \cos^{-1}(u) - \sqrt{1 - u^2} + c$$

Q1

PDF, CDF

Q1. A random variable X has the PDF

Exponential distribution

$$f_X(t) = \frac{1}{4}e^{-t/4}, t > 0.$$

(a) Find the CDF of X.

(b) Using the CDF, find $P(1 < X \leq 2)$ and $P(X > 4)$.

(c) Find $P(X > 2)$ and $P(X > 4|X > 2)$?

Q1(a) Find the CDF of X.

By the relation between the continuous RV PDF and CDF, we have

$$
\begin{aligned}
F_X(x) &= P(X \le x) \\
&= \int_{-\infty}^{x} f_X(t)dt \\
&= \int_{0}^{x} \frac{1}{4}e^{-t/4}dt, \quad \text{for} \quad x \ge 0 \\
&= 1 - e^{-x/4}, \quad \text{for} \quad x \ge 0
\end{aligned}
$$

If $x < 0$, $F_X(x) = P(X \le x) = 0.$

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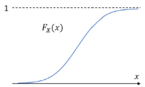
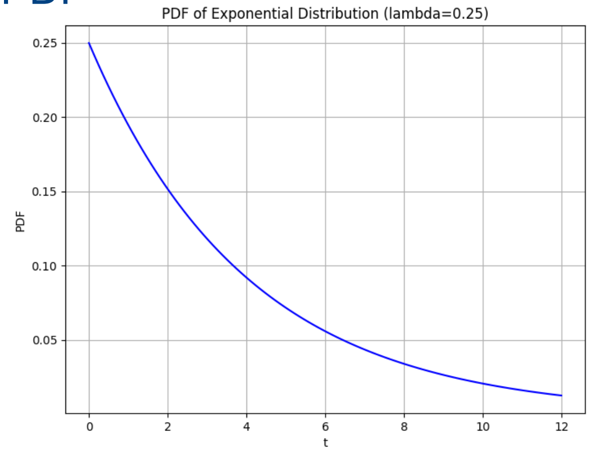
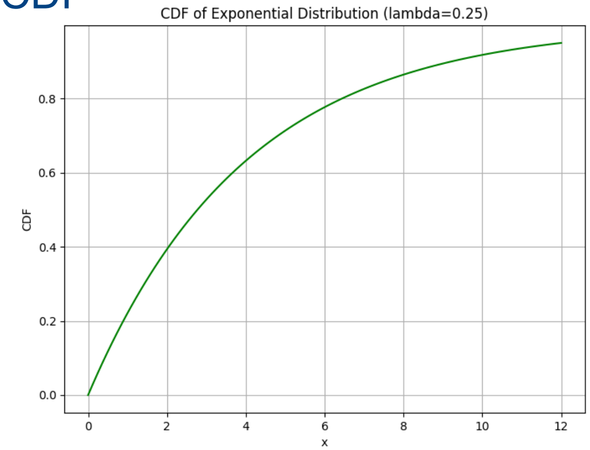
- Cumulative distribution function (CDF).
  - CDF is defined as $F_X(x) \triangleq P(X \le x)$
  - Discrete RV: CDF is a staircase shape and right continuous. It is clumsy when used for probability calculation.
  - Continuous RV X: $F_X(x) = \int_{-\infty}^{x} f(t)\, dt$ for any real x.
  - CDF is much easier to use for continuous RVs.
  - $F_X(x)$ is continuous and non-decreasing for continuous RV X.



PDF



CDF



Q1(a) Find the CDF of X.

- This continuous RV with this PDF or CDF is an exponential distributed RV. This type of RVs are closely linked with Poisson distribution as explained in this short article (https://neurophysics.ucsd.edu/courses/physics_171/exponential.pdf).
- Note that the CDF $F_X(x)$ for a continuous RV X does not have the right-continuity problem as in the CDF of a discrete RV. Also, since $P(X = x_1) = 0$ for any x_1 if X is a continuous RV X, we have $P(X \leq x_1) = P(X < x_1)$.

Q1 (b) Using the CDF, find $P(1 < X \leq 2)$ and $P(X > 4)$.

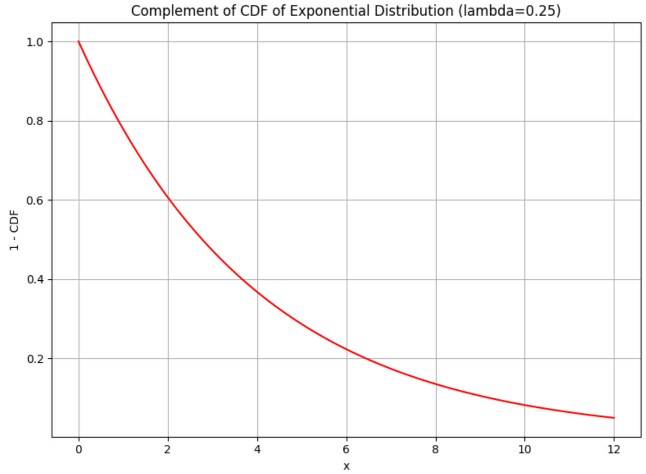
By definition of CDF,

$$F_X(x) = 1 - e^{-\frac{x}{4}}, \text{ for } x \geq 0$$

$$\begin{aligned} P(1 < X \leq 2) &= P(X \leq 2) - P(X \leq 1) \\ &= F_X(2) - F_X(1) \\ &= e^{-1/4} - e^{-1/2}. \end{aligned}$$

By the same approach, $P(X > 4) = 1 - P(X \leq 4) = 1 - F_X(4) = e^{-1}$.

$$P(X > t) = e^{-\lambda t}$$



We define $P(X > t)$ as the complement of the CDF $F_X(t)$

$$P(X > t) = 1 - F_X(t) = e^{-\lambda t}$$

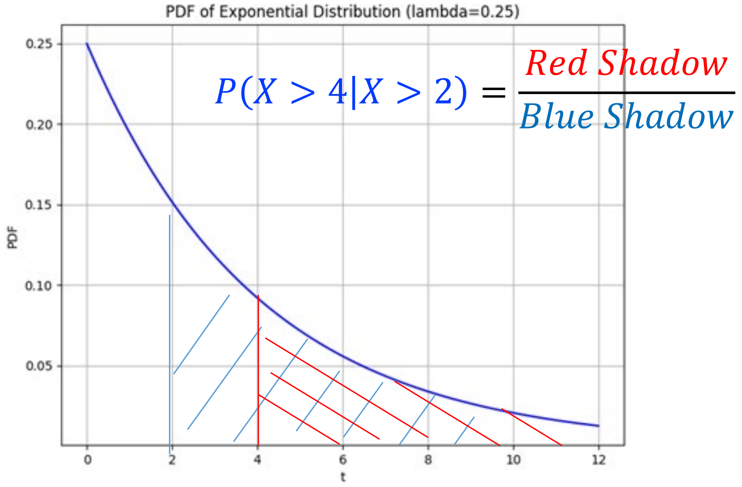
Q1(c) Find $P(X > 2)$ and $P(X > 4|X > 2)$?

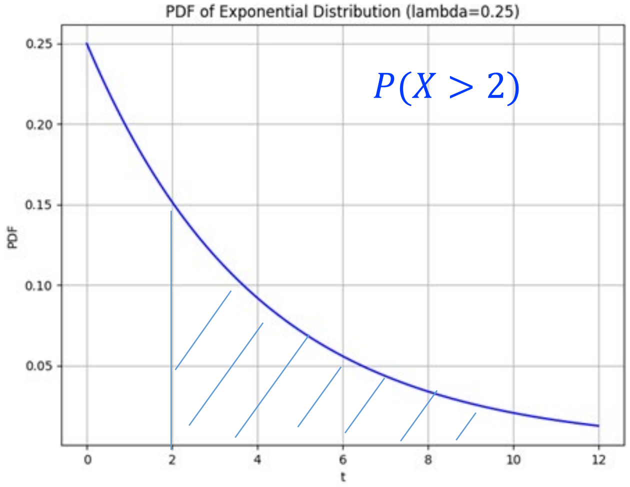
$$P(X > 2) = 1 - F_X(2) = 1 - (1 - e^{-2/4}) = e^{-1/2} = 0.6065.$$

$$P(X > 4|X > 2) = \frac{P(X>4, X>2)}{P(X>2)} = \frac{P(X>4)}{P(X>2)} = \frac{1-(1-e^{-4/4})}{1-(1-e^{-2/4})} = \frac{e^{-1}}{e^{-0.5}} = e^{-0.5} = P(X > 2).$$

Is this coincident?

This is not a coincidence but a property of the memoryless nature of the exponential distribution. For an exponential random variable X with rate λ, the probability of an event occurring after a certain time is independent of how much time has already passed.





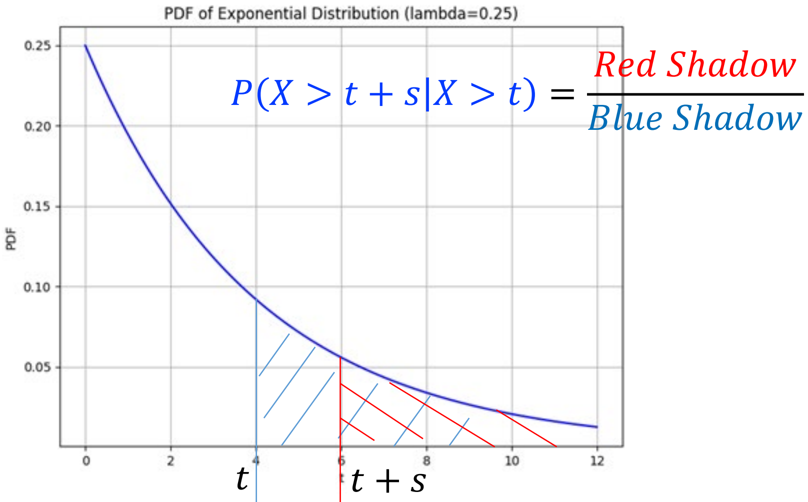
Q1(c) Find $P(X > 2)$ and $P(X > 4|X > 2)$?

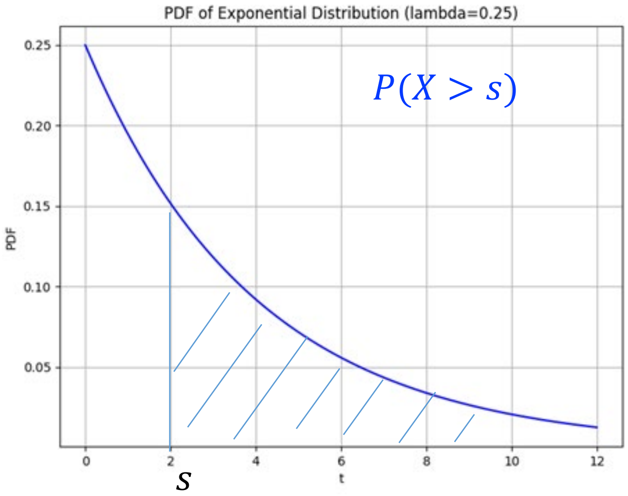
Let's derive the memoryless property for an exponential distribution with rate λ. The probability that X takes more than t units of time is given by the survival function (the complement of the CDF)

$$P(X > t) = 1 - F_X(t) = e^{-\lambda t}$$

Now, we want to show that $P(X > t + s \mid X > t) = P(X > s)$. Using the definition of conditional probability:

$$P(X > t + s \mid X > t) = \frac{P(X > t + s \text{ and } X > t)}{P(X > t)} = \frac{P(X > t + s)}{P(X > t)} = \frac{e^{-\lambda(t+s)}}{e^{-\lambda t}} = e^{-\lambda s} = P(X > s).$$





Q2

PDF, CDF

Consider the PDF $f_X(x) \; = \; cx(1-x), 0 < x < 1,$ where c is constant.

(a) Evaluate c.

(b) Find the CDF $F_X(x)$.

(c) Sketch both the PDF and the CDF.

Q2(a) Evaluate c

To solve this question, we need to know the property of PDF $f_X(x)$, which is

$$\int_{-\infty}^{\infty} f_X(x)dx = 1.$$

Applying this property, we have

$$c\int_{o}^{1} x - x^2 dx = 1.$$

Solve this equation, we have $c = 6$.

Q2 (b) Find the CDF $F_X(x)$.

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt$$

$$= \begin{cases} 0 & x \le 0 \\ \int_0^x 6(t - t^2)dt & 0 < x < 1 \\ 1 & x \ge 1 \end{cases}$$

$$= \begin{cases} 0 & x \le 0 \\ 6(\frac{x^2}{2} - \frac{x^3}{3}) & 0 < x < 1 \\ 1 & x \ge 1 \end{cases}$$

Q2(b) Find the CDF $F_X(x)$.

Since $f_X(t)$ is nonzero only when $0 < t < 1$, we need to consider the cases $x \geq 0$, $0 < x < 1$ and $x \geq 1$ in the integration above. Note that the integration

$$\int_{-\infty}^{x} 6t(1-t)dt = 3t^2 - 2t^3|_{-\infty}^{x}$$

does not converge and it is not a valid $F_X(x)$.

Q2 (c) Sketch both the PDF and the CDF.

Since $6x(1-x) = -6\left(x^2 - x + \frac{1}{4}\right) + \frac{3}{2} = -6\left(x - \frac{1}{2}\right)^2 + \frac{3}{2}$, we can see it is a parabola with maximum value $\frac{3}{2}$ when $x = \frac{1}{2}$. But the $f_X(x)$ is only nonzero for $0 < x < 1$. That is only the red portion in the parabolic plot in Figure 1 below is the plot of the PDF. For the rest value of x, it is zero value.

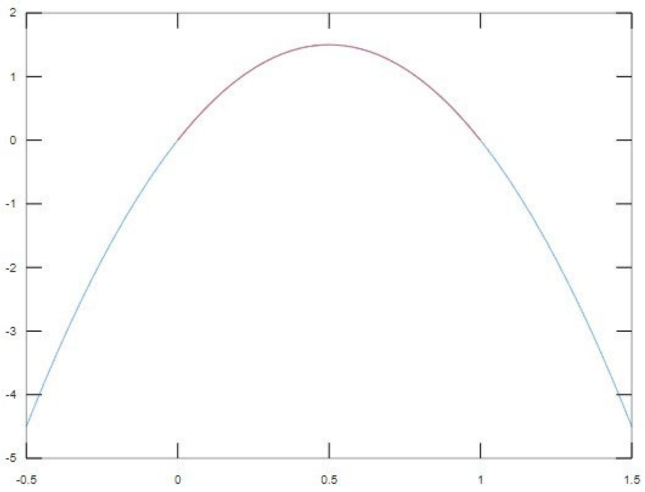


FIG. 1. Plot of the PDF function (red color portion).

Q2 (c) Sketch both the PDF and the CDF.

The CDF plot is defined over the whole x-axis. That is for all the x values. Based on the result in (b), we can plot it as shown in Figure 2. Note that $F_X(x) = 0$ for $x < 0$ and $F_X(x) = 1$ for $x > 1$.

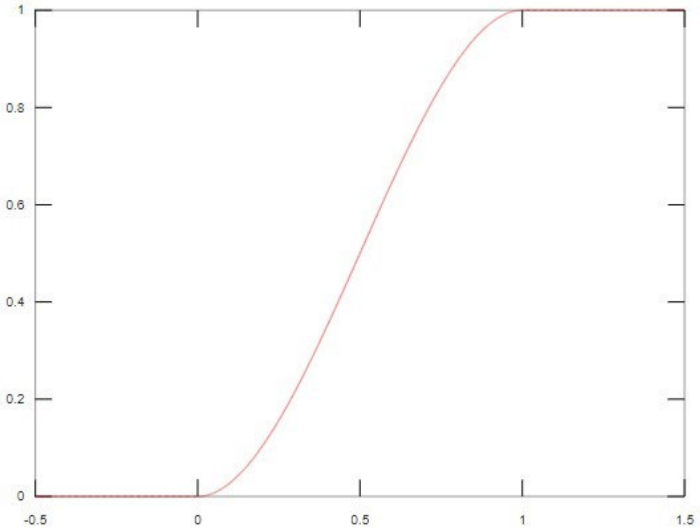


FIG. 2. Plot of the CDF function.

Q3

(PDF and CDF relationship, uniform distribution)

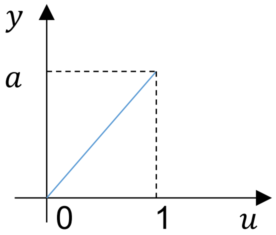
Let U be a uniform (0, 1) random variable, and a be a positive constant.

(a) Show that aU is uniformly distributed on $(0, a)$.

(b) Show that $\min(U, 1 - U)$ is a uniform (0, 1/2) random variable.

(c) Show that $\max(U, 1 - U)$ is a uniform (1/2, 1) random variable.

Q3 (a) Show that aU is uniformly distributed on $(0, a)$.



Since U is uniformly distributed in $(0,1)$, its CDF is $F_U(u) = P(U \leq u) = u$ for $0 \leq u \leq 1$, and 0 otherwise.

Let $Y = aU$. Then $Y \in (0, a)$ and

Compute CDF of Y using CDF of U

$$F_Y(y) = P(Y \leq y) = P(aU \leq y) = P(U \leq \frac{y}{a}) = \frac{y}{a}.$$

To get the PDF, we differentiate the CDF

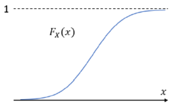
Therefore $f_Y(y) = \frac{dF_Y(y)}{du} = \frac{1}{a}$ for $Y \in (0, a)$ and Y is uniformly distributed in $(0, a)$.

- Relationship between the PDF and CDF of X.

  $f_X(x) = \frac{dF_X(x)}{dx}$ and $F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$.

  – $P(X \leq x) = F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$ is the area under the curve $f_X(x)$ from $-\infty$ to x.

  – $P(a < X \leq b) = F_X(b) - F_X(a) = \int_a^b f_X(t)\,dt$.



PDF and CDF relationship
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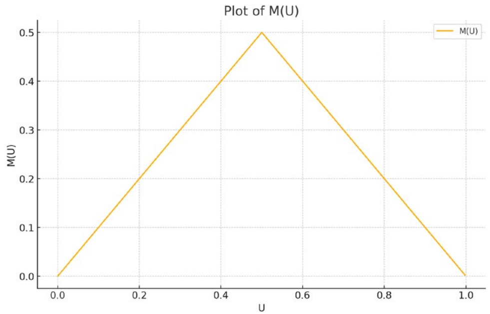
Q3 (b) Show that $min(U, 1 - U)$ is a uniform $(0, 1/2)$ random variable.

Let $M = \min(U, 1 - U)$.



1. **Range of M**

$$M = \begin{cases} U & \text{if } 0 < U < 1/2 \\ 1 - U & \text{if } 0.5 \leq U < 1 \end{cases}$$

We have $M \in (0,1/2)$

2. **Compute CDF of M**

$$\begin{cases} U > m \\ U < m - 1 \end{cases}$$

$$P(M > m) = P(U > m, 1 - U > m) = P(m < U < 1 - m) = 1 - 2m.$$

U be a uniform (0, 1) random variable

We have

$$F_M(m) = 1 - P(M > m) = 1 - (1 - 2m) = 2m$$

3. **Compute PDF of M**

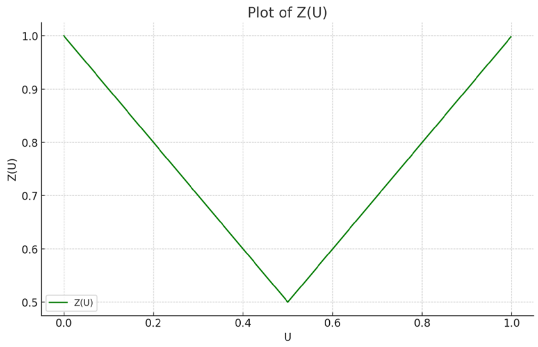
The PDF is obtained by differentiating the CDF:

$$f_M(m) = 2 \text{ for } M \in (0, 1/2).$$

Therefore, M is uniformly distributed in $(0, 1/2)$.

Q3 (c) Show that $max(U, 1 - U)$ is a uniform (1/2, 1) random variable.

Let $Z = \max(U, 1 - U)$.



1. **Range of Z**

$$Z = \begin{cases} 1 - U & \text{if } 0 < U < 1/2 \\ U & \text{if } 0.5 \leq U < 1 \end{cases}$$

We have $Z \in (1/2, 1)$

2. **Compute CDF of Z**

$$P(Z \leq z) = P(U \leq z, 1 - U \leq z) = P(U \leq z, U \geq 1 - z) = P(1 - z \leq u \leq z) = 2z - 1$$

We have

$$F_Z(z) = P(Z \leq z) = P(1 - z \leq U \leq z) = 2z - 1$$

U be a uniform (0, 1) random variable

3. **Compute PDF of Z**

The PDF is obtained by differentiating the CDF:

$$f_Z(z) = 2$$

Therefore Z is uniformly distributed in $(1/2, 1)$.

Q4

(Properties of Gaussian RV, CDF, Q Function)

Let X be a Gaussian random variable with expectation μ and variance σ^2.

(a) Find $P(X \leq \mu)$.

(b) Find $P(|X - \mu| < k\sigma), for\ k = 1, 2, 3, 4$.

(c) Find the value of k for which $P(X > \mu + k\sigma) = 10^{-j}$, for $j = 1, 2, 3, 4, 5, 6$.

Q4(a) Find $P(X \leq \mu)$.

Denote the PDF of X by $f_X(x)$. The probability $P(x \leq \mu) = \int_{-\infty}^{\mu} f_X(x)dx$. It is the areas under the curve of $f_X(x)$ from $-\infty$ to μ. As shown by Figure 3, since $f_X(x)$ has even symmetry about $x = m$, $P(X \leq m) = 0.5$.

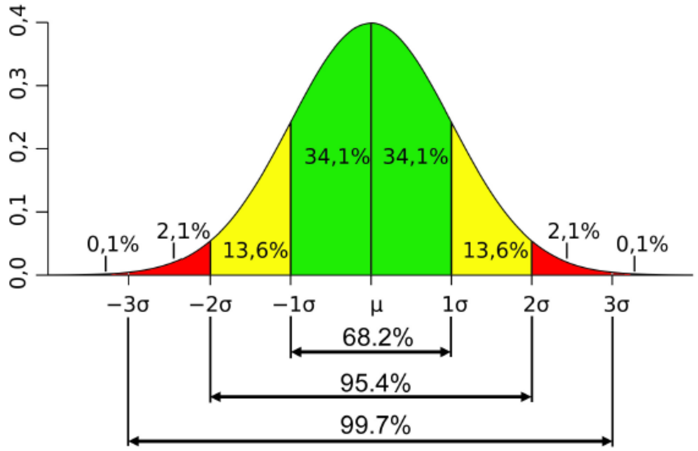
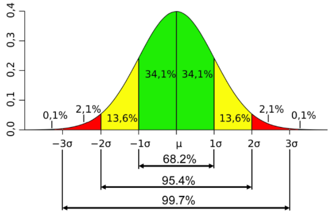


FIG. 3. PDF of $X \sim N(\mu, \sigma^2)$.

Q4(b) Find $P(|X - \mu| < k\sigma)$, $for\ k$ = 1, 2, 3, 4.



Note that $|X - \mu| < k\sigma$ is equivalent to $-k\sigma + \mu < X < k\sigma + \mu$. Since $f_X(x)$ is even symmetry about $x = \mu$ as illustrated in Figure 3, Therefore,

$$P(|X-\mu| < k\sigma) = P(-k\sigma + \mu < X < k\sigma + \mu)$$

$$= 1 - 2P(X > k\sigma + \mu)$$

Even symmetry around $x = \mu$

$$= 1 - 2P\left(\frac{X-\mu}{\sigma} > k\right)$$

$$1 - 2P(\tilde{X} > k)$$

$$1 - 2Q(k)$$

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- Probabilities for standard Gaussian RV Z ? ← Solved!
- Find the probabilities related to X by transforming X to Z.

$$P(X > a) = P\left(\frac{X-\mu}{\sigma} > \frac{a-\mu}{\sigma}\right) = P\left(Z > \frac{a-\mu}{\sigma}\right) = Q\left(\frac{a-\mu}{\sigma}\right)$$

$$P(X < a) = P\left(\frac{X-\mu}{\sigma} < \frac{a-\mu}{\sigma}\right) = P\left(Z < \frac{a-\mu}{\sigma}\right) = 1 - Q\left(\frac{a-\mu}{\sigma}\right)$$

$$P(a < X < b) = P\left(\frac{a-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{b-\mu}{\sigma}\right) = Q\left(\frac{a-\mu}{\sigma}\right) - Q\left(\frac{b-\mu}{\sigma}\right)$$

Page 29, Chapter 5, Linear transform of $X \sim N(\mu, \sigma^2)$ to $Z \sim N(0,1)$

- $Y = aX + b$ is also Gaussian distributed for any $a, b \in R$ and $a \neq 0$.
- $E[Y] = aE[X] + b = a\mu + b$
- $\text{Var}[Y] = a^2\sigma^2$
- Transform X to standard Gaussian RV Z.

  Let $a = \frac{1}{\sigma}$ and $b = -\frac{\mu}{\sigma}$. We have $Z = \frac{X-\mu}{\sigma} \sim N(0,1)$.

Q4(b) Find $P(|X - \mu| < k\sigma), for\ k = 1, 2, 3, 4.$

The RV $\tilde{X} = \frac{X-\mu}{\sigma}$ is standard Gaussian $N(0, 1)$. The Q-function is defined based on standard Gaussian RV $\tilde{X}$ as

$$Q(x) = P(\tilde{X} > k) = \int_k^\infty f_{\tilde{X}}(\tilde{x})d\tilde{x}.$$

The Q-function $Q(x)$ has some important properties such as $Q(-x) + Q(x) = 1$ and $Q(0) = 0.5$.

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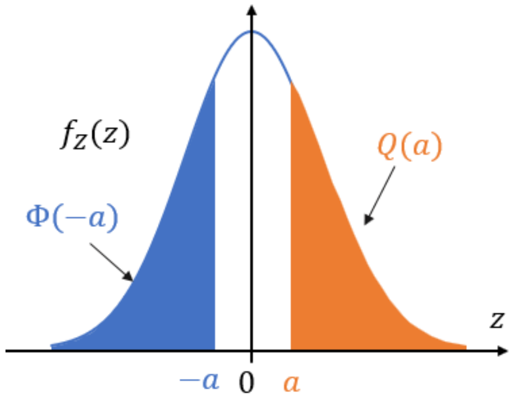
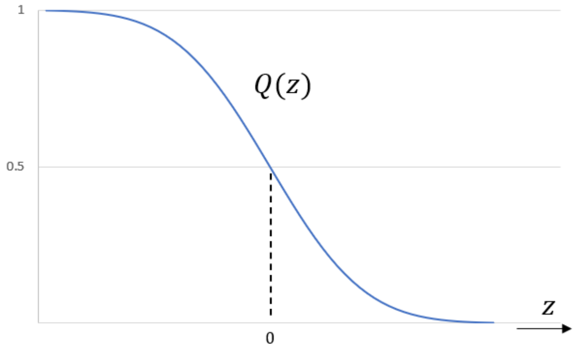
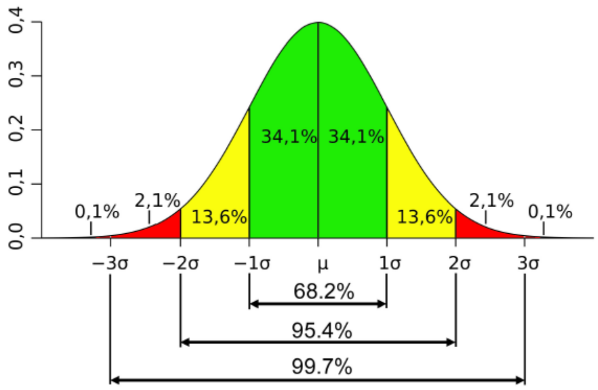
Properties of Q function





Q4(b)

$$P(|X-\mu|<k\sigma)=1-2Q(k)$$



Based on above derivation, we can then find

$$P(|X-\mu|<k\sigma)=\begin{cases}0.682 & k=1\\ 0.9544 & k=2\\ 0.9973 & k=3\\ 0.9999 & k=4\end{cases}$$

Above result means for a relatively larger value of k, the probability of a Gaussian distributed RV X deviating from $\mathrm{E}[X]$ by more than $k\sigma$ is very small. This also means the majority of the X values are near to $\mathrm{E}[X]$. The probabilities for those extremely small and extremely large values are very small. It is a good distribution used for grading.

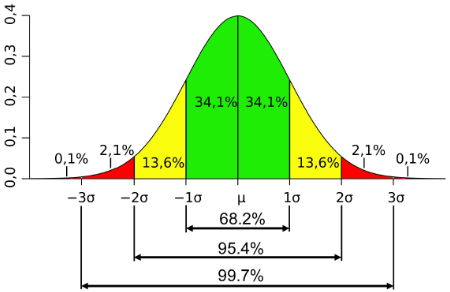
x	$Q(x)$	x	$Q(x)$
0	5.00E-01	2.7	3.47E-03
0.1	4.60E-01	2.8	2.56E-03
0.2	4.21E-01	2.9	1.87E-03
0.3	3.82E-01	3.0	1.35E-03
0.4	3.45E-01	3.1	9.68E-04
0.5	3.09E-01	3.2	6.87E-04
0.6	2.74E-01	3.3	4.83E-04
0.7	2.42E-01	3.4	3.37E-04
0.8	2.12E-01	3.5	2.33E-04
0.9	1.84E-01	3.6	1.59E-04
1.0	1.59E-01	3.7	1.08E-04
1.1	1.36E-01	3.8	7.24E-05
1.2	1.15E-01	3.9	4.81E-05
1.3	9.68E-02	4.0	3.17E-05
1.4	8.08E-02	4.5	3.40E-06
1.5	6.68E-02	5.0	2.87E-07
1.6	5.48E-02	5.5	1.90E-08
1.7	4.46E-02	6.0	9.87E-10
1.8	3.59E-02	6.5	4.02E-11
1.9	2.87E-02	7.0	1.28E-12
2.0	2.28E-02	7.5	3.19E-14
2.1	1.79E-02	8.0	6.22E-16
2.2	1.39E-02	8.5	9.48E-18
2.3	1.07E-02	9.0	1.13E-19
2.4	8.20E-03	9.5	1.05E-21
2.5	6.21E-03	10.0	7.62E-24
2.6	4.66E-03		

Q4(c) Find the value of k for which $P(X > \mu + k\sigma) = 10^{-j}$

This question shows further how hard to get a value far away from $\mathrm{E}[X]$. Recall that $P(X > \mu + k\sigma) = Q(k)$. We want to basically solve the value of k from $Q(k) = 10^{-j}$ for each given j value. Using the Q-function table, we have

$$P(X > \mu + k\sigma) = P\left(\frac{X-\mu}{\sigma} > k\right)$$
$$= P(\tilde{X} > k) = Q(k)$$

$$k = \begin{cases} 1.2815 & j = 1 \\ 2.3263 & j = 2 \\ 3.0902 & j = 3 \\ 3.7190 & j = 4 \\ 4.2649 & j = 5 \\ 4.7535 & j = 6 \end{cases}$$



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1.9	2.87E-02	7.0	1.28E-12
2.0	2.28E-02	7.5	3.19E-14
2.1	1.79E-02	8.0	6.22E-16
2.2	1.39E-02	8.5	9.48E-18
2.3	1.07E-02	9.0	1.13E-19
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This list shows the probability for X exceeding $\mathrm{E}[X]$ by 4.75σ is only 1 in a million, which is a low chance event.

Q5

(Properties of Gaussian RV, Q Function)

Two chips are being considered for use in a certain system.

- The lifetime of chip 1 is modelled by a Gaussian RV with expectation 20,000 hours and standard deviation 5,000 hours.
- The lifetime of chip 2 is also Gaussian, with expectation 22,000 hours and standard deviation 1,000 hours.

Which chip is preferred if the target lifetime is 20,000 hours? What if the target is 24,000 hours?

Q5

Denote the lifetime of Chip 1 and Chip 2 by the random variables X_1 and X_2 respectively. We have $X_1 \sim N(20000, 5000^2)$ and $X_2 \sim N(22000, 1000^2)$. Their PDFs is illustrated in Figure 4. If the target lifetime of the chip is 20,000 hours, we need to compare $P(X_1 > 20000)$ against $P(X_2 > 20000)$ and pick the one with the higher probability.

x	$Q(x)$	x	$Q(x)$
0	5.00E-01	2.7	3.47E-03
0.1	4.60E-01	2.8	2.56E-03
0.2	4.21E-01	2.9	1.87E-03
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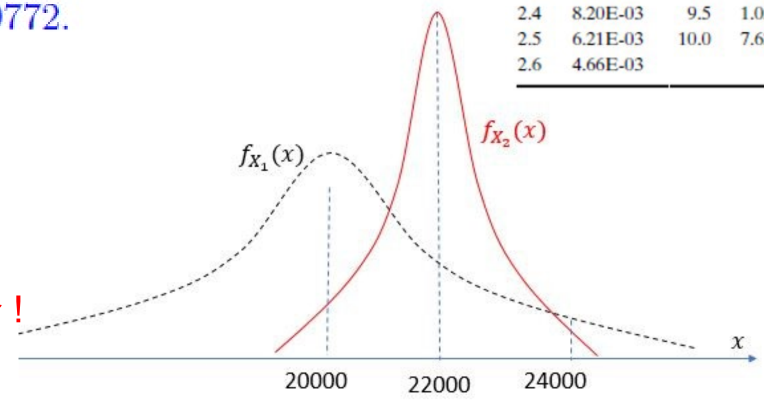
$$P(X_1 > 20000) = Q\left(\frac{20000-20000}{5000}\right) = Q(0) = 0.5$$

$$P(X_2 > 20000) = Q\left(\frac{20000-22000}{1000}\right) = 1 - Q(2) = 0.9772.$$

Therefore Chip 2 is preferred.



- The Q-function table only list the probabilities for $z \geq 0$.
  - Make use of $Q(-z) = 1 - Q(z)$. Slide 28, Chapter 5, Negative z !
  - No need to tabulate values of $Q(x)$ for $x < 0$.

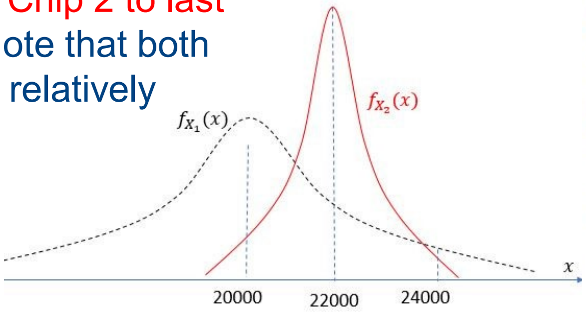
Q5

If the target lifetime is 24,000 hours, we need to compare following two probabilities and choose the chip with relatively higher probability.

$$P(X_1 > 24000) = Q\left(\frac{24000 - 20000}{5000}\right) = Q\left(\frac{4000}{5000}\right) = Q(0.8) = 0.212$$

$$P(X_2 > 24000) = Q(\frac{24000 - 22000}{1000}) = Q\left(\frac{2000}{1000}\right) = Q(2) = 0.0228.$$

Thus Chip 1 is preferred this time, which is surprising. The reason is Chip 1's lifetime has a much larger standard deviation than Chip 2's. Although on average Chip 2 lasts longer than Chip 1, it is more likely for Chip 1 to last more than 4000 hours beyond its average than for Chip 2 to last more than 2000 hours beyond its average. Note that both probabilities in this case are quite small. The relatively larger one in Figure 4 should be selected.



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2.1	1.79E-02	8.0	6.22E-16
2.2	1.39E-02	8.5	9.48E-18
2.3	1.07E-02	9.0	1.13E-19
2.4	8.20E-03	9.5	1.05E-21
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2.6	4.66E-03		

Q6

(Mixed Random Variable)

It is found that the 3G network is available with probability 0.8; independently, the 4G network is available with probability 0.8. A smartphone first tries to connect to the 4G network, then if that is not available, it connects to the 3G network. If neither network is available then there is no connectivity. Let X be some performance measure of the network, with the following conditional distributions:

$$f_X(x|\text{“connected to 4G”}) = 0.1[u(x-10) - u(x-20)]$$

$$f_X(x|\text{“connected to 3G”}) = 0.2[u(x-7) - u(x-12)]$$

$$f_X(x|\text{“neither”}) = \delta(x).$$

Find the PDF of X.

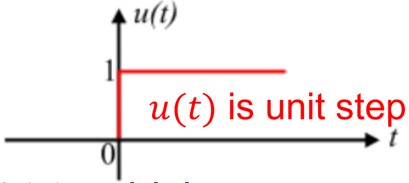
Q6

Problem Understanding:

- Let A_3 and A_4 denote the events “3G network is available” and “4G network is available”,
- $P(A_3) = 0.8$
- $P(A_4) = 0.8$, and it is independent of A_3.
- The smartphone tries to connect to the 4G network first, and if 4G is unavailable, it tries to connect to 3G.
- X is performance measure of the network with the following conditional distributions

$$f_X(x \mid A_4) = 0.1[u(x-10) - u(x-20)]$$

$$f_X(x \mid A_3) = 0.2[u(x-7) - u(x-12)]$$



$u(t)$ is unit step signal

- If neither 4G nor 3G is available, the performance measure is $\delta(x)$, which represents a Dirac delta function centered at $x = 0$.

Q6

Step 1: Compute the Probabilities

- **Connected to 4G**: This happens when the 4G network is available. The probability of this event is

$$P(A_4) = 0.8$$

- **Connected to 3G but not 4G**: This happens when the 4G network is unavailable (0.2) and the 3G network is available (0.8). Thus, the probability of this event is

$$P(A_4^c \cap A_3) = 0.2 \times 0.8 = 0.16$$

- **Neither network is available**: This happens when both 4G and 3G are unavailable, with a probability of

$$P(A_4^c \cap A_3^c) = 0.2 \times 0.2 = 0.04$$

Q6

Step 2: Use the Law of Total Probability

We can find the overall PDF of X by combining the conditional PDFs weighted by the respective probabilities of each event. The law of total probability tells us:

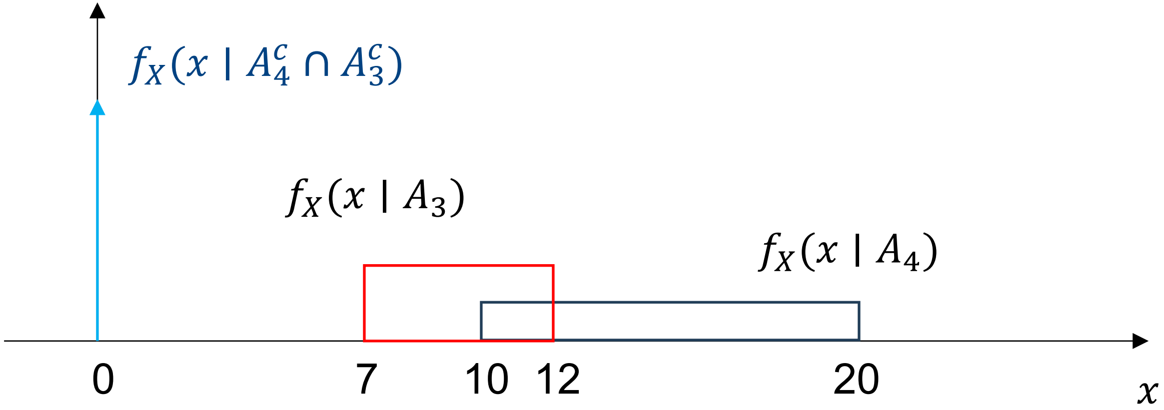
$$f_X(x) = f_X(x|A_4)P(A_4) + f_X(x|A_4^c \cap A_3)P(A_4^c \cap A_3) + f_X(x|A_4^c \cap A_3^c)P(A_4^c \cap A_3^c)$$

Where

$f_X(x \mid A_4) = 0.1[u(x-10) - u(x-20)]$ A uniform distribution between 10 and 20, with height 0.1

$f_X(x \mid A_3) = 0.2[u(x-7) - u(x-12)]$ A uniform distribution between 7 and 12, with height 0.2

$f_X(x \mid A_4^c \cap P_3^c) = \delta(x)$ A Dirac delta function at $x = 0$



Q6

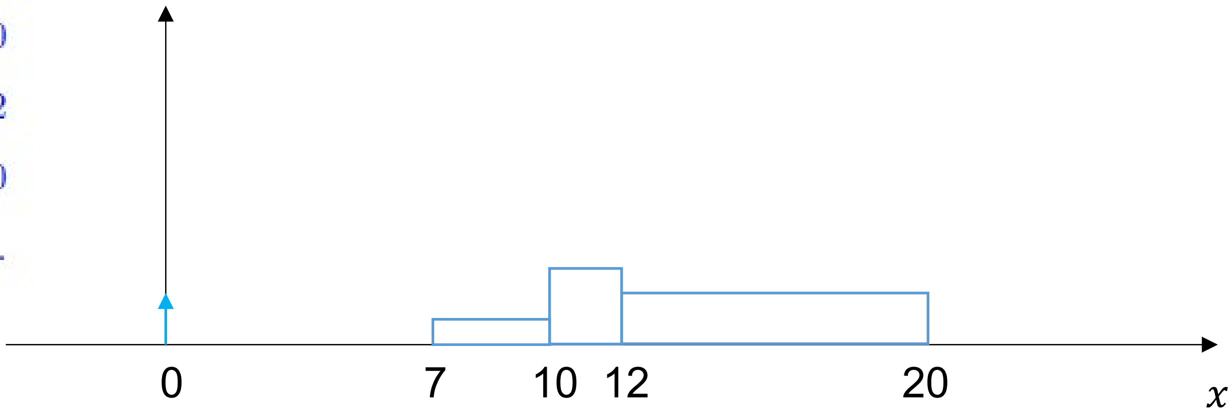
Step 3: Compute Final PDF of X

By substituting the conditional PDFs and the respective probabilities into the law of total probability expression, the final PDF of X is

$$f_X(x) = f_X(x|A_4)P(A_4) + f_X(x|A_4^c \cap A_3)P(A_4^c \cap A_3) + f_X(x|A_4^c \cap A_3^c)P(A_4^c \cap A_3^c)$$

$$= 0.08[u(x - 10) - u(x - 20)] + 0.032[u(x - 7) - u(x - 12)] + 0.04\delta(x)$$

$$= \begin{cases} 0.04\delta(x) & x = 0 \\ 0.032 & 7 \le x < 10 \\ 0.112 & 10 \le x < 12 \\ 0.08 & 12 \le x < 20 \\ 0 & \text{otherwise.} \end{cases}$$



THANK YOU