EE2012 Tutorial 4

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Location:

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Q1

(PMF, Two Discrete RVs)

Let X and Y be random variables that take on values from the set $\{-1, 0, 1\}$, and suppose their PMFs are respectively.

$$p_X(k) = \frac{1}{3}, \qquad k = -1, 0, 1$$

$$p_Y(-1) = 0.5;\ p_Y(0) = 0.2; p_Y(1) = 0.3;$$

a) If X and Y are independent, find $P(X \geq Y)$;

b) Find the joint PMF of X^2 and Y^2 by considering the (X, Y) event equivalent to $\{X^2 = j, Y^2 = k\}$. Hence verify that X^2 and Y^2 are also independent random variables.

Q1 (a) If X and Y are independent, find $P(X \geq Y)$

Step 1: Understand the independence property

Since X and Y are independent, the joint probability $P(X = x, Y = y)$ is the product of their individual probabilities. Their joint PMF can be shown in the table below.

X \ Y	-1	0	1
-1	$(-1,-1)$	$(-1,0)$	$(-1,1)$
0	$(0,-1)$	$(0,0)$	$(0,1)$
1	$(1,-1)$	$(1,0)$	$(1,1)$

Y	-1	0	1
p_Y	0.5	0.2	0.3

X	p_X
-1	$1/3$
0	$1/3$
1	$1/3$

X \ Y	-1	0	1
-1	$1/6$	$1/15$	$1/10$
0	$1/6$	$1/15$	$1/10$
1	$1/6$	$1/15$	$1/10$

Q1 (a) If X and Y are independent, find $P(X \geq Y)$

Step 2: Compute the probabilities for each valid pair

X \ Y	-1	0	1
-1	$(-1,-1)$	$(-1,0)$	$(-1,1)$
0	$(0,-1)$	$(0,0)$	$(0,1)$
1	$(1,-1)$	$(1,0)$	$(1,1)$

X \ Y	-1	0	1
-1	1/6	1/15	1/10
0	1/6	1/15	1/10
1	1/6	1/15	1/10

The event $X \geq Y$ consists of the six pairs of $(X, Y) \in \{(-1,-1),(0,0),(0,-1),(1,1),(1,0),(1,-1)\}$.

For each of these values of (X, Y), their probabilities are shown in red color cells in the table above. Since these outcomes are disjoint events, we have

$$P(X \geq Y) = \frac{1}{6} + \frac{1}{15} + \cdots + \frac{1}{6} = \frac{11}{15}$$

Q1(b) Verify X^2 and Y^2 are independent

To prove X^2 and Y^2 are independent, we need to show they satisfy the sufficient condition

$$p_{X^2,Y^2}(j,k) = p_{X^2}(j)p_{Y^2}(k).$$

Step 1: compute the values of X^2 and Y^2

Both X and Y have limited-size supports and the support for (X^2, Y^2) can be found using the table below, where the values of (X^2, Y^2) are shown in red color in the cells.

X \ Y	-1	0	1
-1	$(-1,-1)$	$(-1,0)$	$(-1,1)$
0	$(0,-1)$	$(0,0)$	$(0,1)$
1	$(1,-1)$	$(1,0)$	$(1,1)$

$\Rightarrow$

X \ Y	-1	0	1
-1	$(1,1)$	$(1,0)$	$(1,1)$
0	$(0,1)$	$(0,0)$	$(0,1)$
1	$(1,1)$	$(1,0)$	$(1,1)$

(X^2, Y^2)

Q1(b) Verify X^2 and Y^2 are independent

Step 2: Compute joint probabilities

Based on this table, the joint PMF of X^2 and Y^2 can be obtained as follows.

X \ Y	-1	0	1
-1	1/6	1/15	1/10
0	1/6	1/15	1/10
1	1/6	1/15	1/10

X \ Y	-1	0	1
-1	(1,1)	(1,0)	(1,1)
0	(0,1)	(0,0)	(0,1)
1	(1,1)	(1,0)	(1,1)

(X^2, Y^2)

$$P(X^2=0,Y^2=0)=P(X=0,Y=0)=p_X(0)p_Y(0)=\frac{1}{15}$$

$$P(X^2=0,Y^2=1)=P(X=0,Y=\pm1)=p_X(0)[p_Y(-1)+p_Y(1)]=\frac{4}{15}$$

$$P(X^2=1,Y^2=0)=P(X=\pm1,Y=0)=p_Y(0)[p_X(-1)+p_X(1)]=\frac{2}{15}$$

$$P(X^2=1,Y^2=1)=P(X=\pm1,Y=\pm1)=[p_X(-1)+p_X(1)][p_Y(-1)+p_Y(1)]=\frac{8}{15}$$

Q1(b) Verify X^2 and Y^2 are independent

Step 3: Compute marginal PMF

We next find their marginal PMF. From the joint PMF, the marginal PMF of X^2 is

X^2 \ Y^2	0	1
0	$\frac{1}{15}$	$\frac{4}{15}$
1	$\frac{2}{15}$	$\frac{8}{15}$

$$p_{X^2}(0) = \frac{1}{15} + \frac{4}{15} = \frac{1}{3}$$

$$p_{X^2}(1) = \frac{2}{15} + \frac{8}{15} = \frac{2}{3}$$

The marginal PMF of Y^2 is

X^2 \ Y^2	0	1
0	$\frac{1}{15}$	$\frac{4}{15}$
1	$\frac{2}{15}$	$\frac{8}{15}$

$$p_{Y^2}(0) = \frac{1}{15} + \frac{2}{15} = \frac{1}{5}$$

$$p_{Y^2}(1) = \frac{4}{15} + \frac{8}{15} = \frac{4}{5}$$

Q1(b) Verify X^2 and Y^2 are independent

Step 4: Verifying independence of X^2 and Y^2

we can verify that $p_{X^2}(j)p_{Y^2}(k) = p_{X^2,Y^2}(j,k)$ for all $j,k \in \{0,1\}$.

For each pair

- $P(X^2 = 0, Y^2 = 0) = \frac{1}{15} = P(X^2 = 0)P(Y^2 = 0) = \frac{1}{3} \times 0.2$
- $P(X^2 = 0, Y^2 = 1) = \frac{8}{30} = P(X^2 = 0)P(Y^2 = 1) = \frac{1}{3} \times 0.8$
- $P(X^2 = 1, Y^2 = 0) = \frac{4}{30} = P(X^2 = 1)P(Y^2 = 0) = \frac{2}{3} \times 0.2$
- $P(X^2 = 1, Y^2 = 1) = \frac{16}{30} = P(X^2 = 1)P(Y^2 = 1) = \frac{2}{3} \times 0.8$

Hence X^2 and Y^2 are independent.

Q2

Roll a 4-sided dice twice with numbers $\{1, 2, 3, 4\}$. Let X be the first roll minus the second roll. Let $Y = X^2$.

a) Find the sample space of X, and Y ;

b) Find PMF of X and use it to find the PMF of Y .

Q2(a) Find the sample space of X, and Y

Possible values of X

Denote the outcomes of two rolls by R_1 and R_2, which both have support $S = \{1, 2, 3, 4\}$.The sample space of X can be determined using the table below. The values of X are shown in red color.

R_2 \ R_1 (X)	1	2	3	4
1	0	1	2	3
2	-1	0	1	2
3	-2	-1	0	1
4	-3	-2	-1	0

From the table, the support of X is $S_X = \{-3, -2, -1, 0, 1, 2, 3\}$.

Possible values of Y

Since $Y = X^2$, the possible values of Y are the squares of the values of X: $Y \in \{0,1,4,9\}$

Q2(b) Find PMF of X and use it to find the PMF of Y

Step 1: List all possible outcomes of dice rolls

The PMF of X is the probability values associated with each element in S_X. Based on the S_X found in Q2(a) and the table used, we have

(R_1, R_2)	X	$p_X(x)$
(1,4)	-3	1/16
(1,3), (2,4)	-2	1/8
(1,2), (2,3), (3,4)	-1	3/16
(1,1), (2,2), (3,3), (4,4)	0	1/4
(2,1), (3,2), (4,3)	1	3/16
(3,1), (4,2)	2	1/8
(4,1)	3	1/16

Note that all the possible values of (R_1, R_2) occur with the same probability $1/16$. The two rightmost columns in the table above show the PMF of X.

Q2(b) Find PMF of X and use it to find the PMF of Y

Step 2: Find the PMF of Y

Since $Y = X^2$, the possible values of Y are 0,1,4,9. Now, let's calculate the probabilities for each value of Y based on the probabilities of X

- $P(Y = 0)$ corresponds to $X = 0$:

$$P(Y = 0) = P(X = 0) = \frac{1}{4}$$

- $P(Y = 1)$ corresponds to $X = \pm 1$:

$$P(Y = 1) = P(X = -1) + P(X = 1) = \frac{3}{16} + \frac{3}{16} = \frac{6}{16} = \frac{3}{8}$$

- $P(Y = 4)$ corresponds to $X = \pm 2$:

$$P(Y = 4) = P(X = -2) + P(X = 2) = \frac{1}{8} + \frac{1}{8} = \frac{2}{8} = \frac{1}{4}$$

- $P(Y = 9)$ corresponds to $X = \pm 3$:

$$P(Y = 9) = P(X = -3) + P(X = 3) = \frac{1}{16} + \frac{1}{16} = \frac{2}{16} = \frac{1}{8}$$

Thus, the PMF of Y is

X	Y	$p_Y(y)$
0	0	1/4
-1,1	1	3/8
-2,2	4	1/4
-3,3	9	1/8

Q3

(PMF, Binomial Distribution)

Let X be the number of active (non-silent) speakers in a group of N non-interacting (i.e., independent) speakers. Suppose that a speaker is active with probability p.

a) Find the PMF of X;

b) If N is eight, find the probability that there are more active speakers than inactive speakers.

Q3(a) Find the PMF of X;

Since all the N speakers are independent and everyone are active with probability p. The number of active speakers is an RV $X \sim$ Binomial(N, p). The PMF of X is

$$p_X(k) = \binom{N}{k} p^k (1-p)^{N-k}$$

for $0 \leq k \leq N$

Q3(b) If N is eight, find the probability that there are more active speakers than inactive speakers.

- The number of active speakers is X and the number of inactive speakers is $N - X$.
- The event “more active speaker than inactive speakers” $=$ $\{X > 8 - X\}$, which is $\{X > 4\}$.
- The probability of this event is

$$P(8 - X < X) = P(X > 4) = p_X(5) + p_X(6) + p_X(7) + p_X(8)$$

Q4

Let X be the number of active speakers in a group of N independent speakers. Let p be the probability that a speaker is active. In Q3, it was shown that X has a binomial distribution with parameters N and p. Suppose that a voice transmission system can transmit up to M voice signals at a time, and that when X exceeds $M, Y = (X - M)$ randomly selected signals are discarded. Let Y be the number of signals discarded, and find the PMF of Y .

Q4

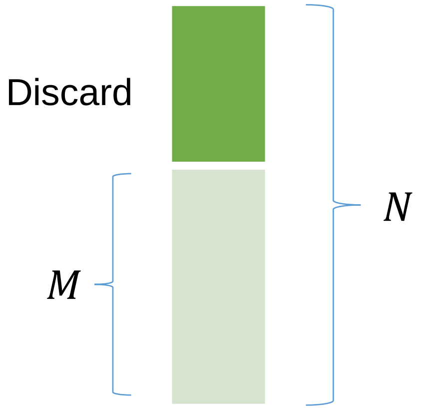
For the transmission of these N speakers, the peak traffic only occurs if all the N speakers are talking. But the probability of such an event is only $p_X(N) = p^N$. Most of the time, the traffic is less than N. To reduce the cost, the transmission system is not designed for the peak traffic, but to ensure the drop rate is less than a certain threshold value.

The number of dropped signals Y takes a value from the set $S_Y = \{0, 1, ..., N - M\}$. We also have $Y = 0$ when $X \leq M$, and $Y = j$ when $X = M + j$ for a positive integer $j \in [1, N - M]$. Denote the PMF of X by $p_X(\cdot)$. We have

$$P(Y = 0) = P(X \in \{0, 1, ..., M\}) = \sum_{i=0}^{M} p_X(i),$$

and

$$P(Y = j) = P(X = M + j) = p_X(M + j) \text{ for } 1 \leq j \leq N - M.$$



Q5

(Expectation)

Suppose a fair coin is tossed 4 times. Each coin toss costs one dollar and the reward for obtaining X heads is $X^2 + X$. Find the PMF of the net reward Y, and the expected net reward $E[Y]$.

Q5

The net reward is the difference between the game reward and the game cost,

$$Y = X^2 + X - 4.$$

When a fair coin is flipped 4 times, the number of heads X is a binomial distributed RV with its PMF

$$p_X(k) = P(X = k) = \binom{4}{k}\left(\frac{1}{2}\right)^k\left(\frac{1}{2}\right)^{4-k} = \binom{4}{k}\left(\frac{1}{2}\right)^4 = \binom{4}{k} \times \frac{1}{16}.$$

Q5

Method I

We can find the mapping from X to Y and the PMF for Y using the table below.

X	0	1	2	3	4
Y	-4	-2	2	8	16
$p_X(k)$	$1/16$	$4\ \times 1/16$	$6\ \times 1/16$	$4\ \times 1/16$	$1/16$
$p_Y(y)$	$1/16$	$1/4$	$3/8$	$1/4$	$1/16$

Therefore, the expected value of Y is,

$$E[Y] = \sum_{y \in S_Y} y \times p_Y(y) = (-4) \times \left(\frac{1}{16}\right) + (-2) \times \left(\frac{4}{16}\right) + 2 \times \left(\frac{6}{16}\right) + 8 \times \left(\frac{4}{16}\right) + 16 \times \left(\frac{1}{16}\right) = \frac{48}{16} = 3.$$

Q5

Method II

Another way to find E[Y]:

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Properties of expectation

$$E[Y] = E[X^2] + E[X] - 4$$

Note that $X \sim \text{Binomial}(4, \frac{1}{2})$. We have

$$E[X] = np = 4 \times \frac{1}{2} = 2$$

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Expectation and variance

- Expectation $m_X = np$
- Variance $\sigma_X^2 = np(1-p)$

and

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Variance of discrete RVs

- $\sigma_X^2 = E[X^2] - m_X^2$

$$\text{Var}[X] = np(1-p) = 4 \times \frac{1}{4} = 1.$$

Since $\boxed{\text{Var}[X] = E[X^2] - (E[X])^2}$ and $E[X^2] = \text{Var}[X] + (E[X])^2 = 1 + 4 = 5$, we have

$$E[Y] = E[X^2] + E[X] - 4 = 5 + 2 - 4 = 3.$$

Q6

A discrete RV X has its support $S_X = \{-1, 0, 1\}$ and its PMF

$$p_X(k) = \frac{1}{3} \text{ for } k \in S_X.$$

- Let $Y = X^2$. Find the PMF of Y and $E[Y]$.

Recall page 21 Chapter 4

- Two approaches to find the expectation of $Y = g(X)$. Assume S_X, S_Y are the respective supports of X and Y.
  - $E[Y] = \sum_{y \in S_Y} y \times p_Y(y)$

    The expectation $E[Y]$ in the previous example is

    $E[Y] = 0 \times \frac{1}{6} + 1 \times \frac{1}{3} + 4 \times \frac{1}{3} + 9 \times \frac{1}{6} = \frac{19}{6}$.

  - $E[Y] = \sum_{x \in S_X} g(x) \times p_X(x)$

    The expectation $E[Y]$ can also be computed as

    $E[Y] =$

    $\frac{1}{6}\left((1-3)^2 + (2-3)^2 + (3-3)^2 + (4-3)^2 + (5-3)^2 + (6-3)^2\right) = \frac{19}{6}$

Q6

Method I

Since the function is $Y = X^2$ and $S_X = \{-1, 0, 1\}$, we can find S_Y and how they are mapped from S_X to S_Y by the table below.

X	-1	0	1
$p_X(k)$	$1/3$	$1/3$	$1/3$
Y	1	0	1

From the third row, we can see $S_Y = \{0, 1\}$. We also have that $Y = 0$ is only from $X = 0$, but $Y = 1$ is from $X = \pm 1$, which means the PMF of Y is

$$P(Y = 0) = P(X = 0) = 1/3$$

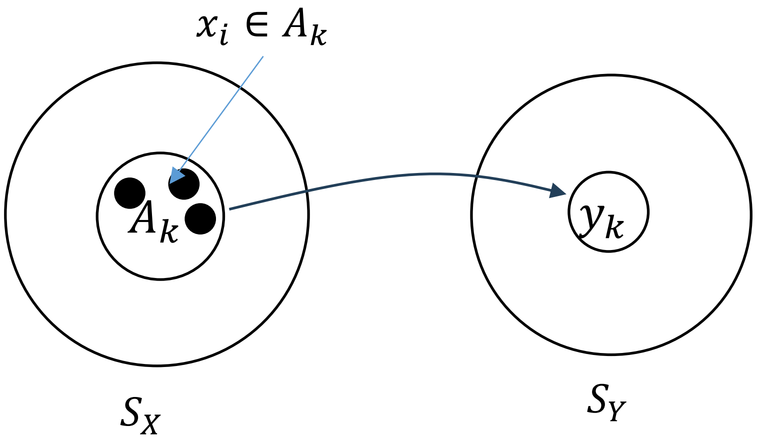
and

$$P(Y = 1) = P(X = -1 \; or \; X = 1) = P(X = -1) + P(X = 1) = 2/3.$$

Based on the support of Y and its PMF, we have $E[Y] = 0 \times 1/3 + 1 \times 2/3 = 2/3$.

Q6

Method II



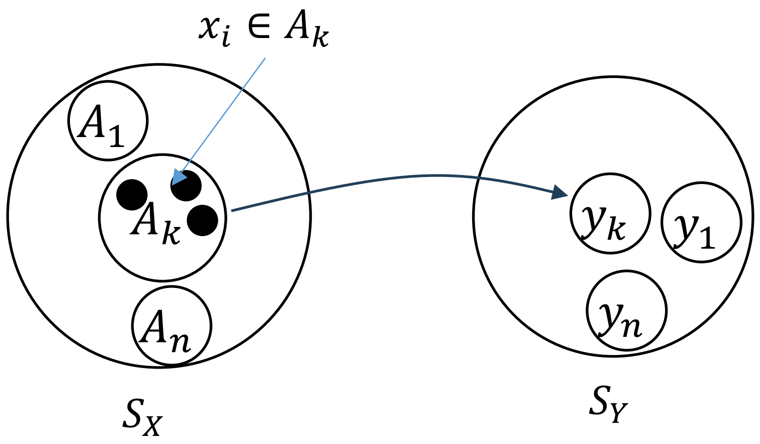
If we denote the function mapping by $\mathbf{F}(\cdot)$, two observations can be generalized to this type of problem.

- All $x_i \in A_k$, where $A_k \subseteq S_X$, may be mapped to the same $y_k \in S_Y$. That is $y_k = \mathbf{F}(x_i)$ for all $x_i \in A_k$.

$$y_k \times P_Y(y_k) = \mathbf{F}(x_i) \times \sum_{x_i \in A_k} p_X(x_i) = \sum_{x_i \in A_k} \mathbf{F}(x_i) \times p_X(x_i)$$

- The PMF of Y can be calculated by $p_Y(y_k) = P(Y = y_k) = \sum_{x_i \in A_k} P(X = x_i) = \sum_{x_i \in A_k} p_X(x_i)$. Therefore, $y_k \times p_Y(y_k) = \sum_{x_i \in A_k} \mathbf{F}(x_i) \times p_X(x_i)$.

Q6



By the expectation definition and the above observations, we have a general result on the expectation for the function of an RV as below.

$$\mathrm{E}[Y] = \sum_{y_k \in S_Y} y_k \times p_Y(y_k) = \boxed{\sum_{y_k \in S_Y} \sum_{x_i \in A_k}} \mathbf{F}(x_i) \times p_X(x_i) = \boxed{\sum_{x_j \in S_X}} \mathbf{F}(x_j) \times p_X(x_j).$$

Applying this result to this problem, we have

$$\mathrm{E}[Y] = \sum_{x_j \in S_X} x_j^2 p_X(x_j) = (-1)^2 \times p_X(-1) + 0^2 \times p_X(0) + 1^2 \times p_X(1) = 2/3.$$

Q7

(PMF, CDF)

A dice is rolled three times with outcomes X_1, X_2, X_3 and $Y = \max(X_1, X_2, X_3)$.

(a) Show the sample space of Y ;

(b) Show that $P(Y \leq j) = P(X_1 \leq j)^3$;

(c) Show the PMF of Y .

Q7(a) Show the sample space of Y

A dice has six sides and with numbers 1 to 6 on each side unless it is indicated as other types. So the value of Y can come from the set $S_Y = \{1, 2, 3, 4, 5, 6\}$

Q7(b) Show that $P(Y \leq j) = P(X_1 \leq j)^3$;

Identical and independent distributed (iid) RVs, page 35, Chapter 4

Since the three rolls are independent, RVs X_1, X_2, X_3 are iid. By the definiton of Y,

$$P(Y \leq j) = P(X_1 \leq j, X_2 \leq j, X_3 \leq j) = P(X_1 \leq j) \times P(X_2 \leq j) \times P(X_3 \leq j) = P(X_1 \leq j)^3.$$

If the dice is fair, $P(X_1 \leq j) = j/6$ for positive integer $j \leq 6$. Therefore, $P(Y \leq j) = (\frac{j}{6})^3$.

Q7(c) Show the PMF of Y .

The probability mass function (PMF) of Y, denoted by $P(Y = j)$, gives the probability that the maximum of the three dice rolls is exactly j. We can find this by calculating:

$$P(Y = j) = P(Y \leq j) - P(Y \leq j - 1) = F_Y(j) - F_Y(j - 1)$$

$F_Y(\cdot)$ is the CDF of Y

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Using the result from part (b), we know that:

$$P(Y \leq j) = \left(\frac{j}{6}\right)^3$$

$$P(Y \leq j - 1) = \left(\frac{j - 1}{6}\right)^3$$

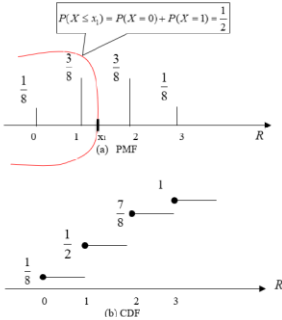
Cumulative distribution functions (CDF)

notation

definition

- CDF definition: $F_X(x) \triangleq P(X \leq x)$ for $x \in R$.
  - Staircase shape for discrete RVs.
  - Compare with PMF.
- Properties of CDF
  - $0 \leq F_X(x) \leq 1$ for all x.
  - $\lim_{x \to -\infty} F_X(x) = 0$
  - $\lim_{x \to \infty} F_X(x) = 1$
  - $F_X(x)$ is non-decreasing.
  - $F_X(x)$ is right-continuous.

$F_X(0.9) = 1/8$
$F_X(1.1) = 4/8$



Thus, the PMF of Y is:

$$P(Y = j) = \left(\frac{j}{6}\right)^3 - \left(\frac{j - 1}{6}\right)^3$$

For $j = 1,2,3,4,5,6$, we can plug in the values of j to compute the exact probabilities for each outcome of Y.

Q8

A transmitter sends data packets to a receiver over a communication link. The receiver uses error detection to identify packets that have been corrupted by radio noise. When a packet is error-free, the receiver sends an acknowledgment (ACK) back to the transmitter. If the packet is corrupted or has errors, the receiver sends a negative acknowledgment (NACK) back to the transmitter. Each time the transmitter receives a NACK, the package is retransmitted. Assume that each packet is independently corrupted by errors with probability q.

(a) Find the PMF of X, the number of times that a packet is transmitted by the source.

(b) Suppose each packet takes 1 millisecond to transmit and that the transmitter waits an additional millisecond to receive the acknowledgment message (ACK or NACK) before retransmitting. Let T equal the time required until the packet is successfully received (at the point when the receiver obtains the correct packet). For example, if $X = 2$, then it takes 3 milliseconds for the packet to be correctly received. What is the PMF of T?

Q8(a) Find PMF of X, the number of times that a packet is transmitted.

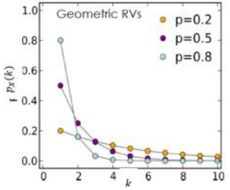
As described by this retransmission protocol, the transmitter repeats the same data packet until it receives the ACK. Hence, the number of times that a packet is transmitted is x if the previous $x-1$ transmissions are unsuccessful and the last transmission is successful. In this case, the PMF of X is a geometric RV with "success" probability $1-q$, i.e.,

$$p_X(x) = \begin{cases} q^{x-1}(1-q), & x = 1, \ldots, \\ 0, & \text{otherwise} \end{cases}$$

The support of X is $S_X = \{1, 2, 3, \ldots\}$.

Geometric RV
Page 29, Chapter 3

- Geometric RV $X \sim \text{Geometric}(p)$.
  - Model: a coin gives head with probability p when it is flipped. The number of flips is counted until the first head appears (including the flip shows the first head). Let's call the final head appearance as "success".
  - PMF $p_X(k) = (1-p)^{k-1}p$ with $S_X = \{1,2,\cdots\}$
  - Expectation $m_X = p^{-1}$.
  - Variance $\sigma_X^2 = \frac{1-p}{p^2}$.



(Refer to the supplementary material for the derivation.)

Q8(b) What is the PMF of T

Each of the packages, ACK, and NACK needs one millisecond to travel across the link, illustrated in Figure 1. The time T is the time duration from sending a new package to the package received correctly by the receiver. This means

$$T = 2X - 1$$

and T is also a discrete RV. By this relation,

$$P(T = t) = P\left(X = \frac{t+1}{2}\right).$$

Since $P(X = \frac{t+1}{2}) = p_X(\frac{t+1}{2})$, we have

$$p_T(t) = P(T = t) = p_X\left(\frac{t+1}{2}\right).$$

Since the support of X is the set {1, 2, 3, ...}, the support of T is $\{1, 3, 5, \ldots\}$. Therefore, the PMF of T is

$$p_T(t) = \begin{cases} q^{(t-1)/2}(1-q), & t = 1, 3, 5, \ldots, \\ 0, & \text{otherwise.} \end{cases}$$

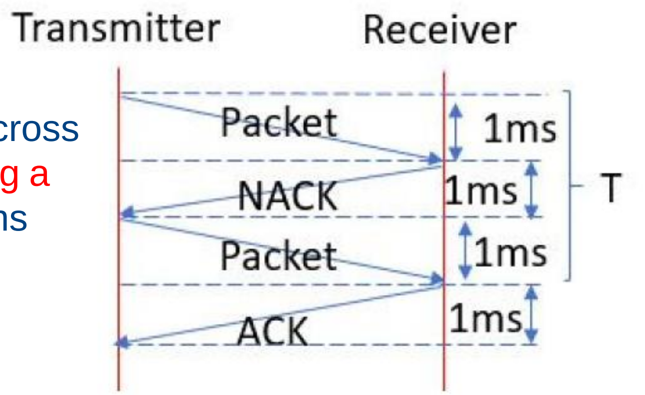


FIG. 1. Timing diagram for Q8(b).

X	T	
1	1	P + A
2	3	P+N+P+A
3	5	P+N+P+N+P+A
4	?	P+N+P+N+P+N+P+A

THANK YOU